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# THE DIFFRACTION OF PLANE SOUND WAVES BY A PERFECTLY REFLECTING QUARTER-PLANE

CURTIS SPAULDING MORSE

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THE DIFFRACTION  
OF PLANE SOUND WAVES  
BY A  
PERFECTLY REFLECTING QUARTER-PLANE

by

CURTIS S. MORSE

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ABSTRACT

—  
THE DIFFRACTION  
OF PLANE SOUND WAVES

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The problem considered is that of determining the diffracted field when a plane wave is incident on a perfectly reflecting quarter-plane ( $x_1 \geq 0, x_2 \geq 0, x_3 = 0$ ). A closed-form solution is derived and shown to be unique. The method of solution consists of first showing that the physically motivated problem is equivalent to a function-theoretical problem in the product space of two complex variables. The function-theoretical problem is then solved by a procedure which is in essence a generalization of the Wiener-Hopf technique from one to two complex variables. The solution is shown to reduce in the limiting cases  $x_1 \rightarrow +\infty$  and  $x_2 \rightarrow +\infty$  to the solutions for the perfectly reflecting half-planes  $x_2 \geq 0$  and  $x_1 \geq 0$ , respectively. The field behavior at the corner is also discussed.



## SECTION I

### INTRODUCTION

A mixed boundary value problem is one in which the unknown function must meet a Dirichlet condition on part of the boundary and a Neumann condition on the rest of the boundary. An example is the classical Poincaré-Sommerfeld half-plane problem of diffraction theory. Here a solution of the two-dimensional reduced wave equation is required to satisfy a Dirichlet condition on the positive  $x$ -axis and a Neumann condition on the negative  $x$ -axis. Copson ([4], or see Baker and Copson [1]) showed that the Poincaré-Sommerfeld problem can be reduced to a function-theoretical problem of the Wiener-Hopf type. Such a problem involves determining two unknown functions of a complex-variable from a single equation. The key to the solution is the Wiener-Hopf factorization lemma (Wiener and Hopf [17]), which gives conditions under which a function analytic in a strip can be factored into the product of a function analytic in a right half-plane and a function analytic in a left half-plane. The half-planes intersect in the original strip, and it is then possible to solve for the unknown functions by analytic continuation and Liouville's theorem.

In acoustic terms, the half-plane problem we have just described is the problem of diffraction by the perfectly absorbent half-plane  $x \geq 0$ . It is also of interest to consider the mixed boundary value problem on the half-line corresponding to diffraction by a perfectly reflecting half-plane. Here the two-dimensional wave function must meet a Neumann condition on the positive  $x$ -axis and a Dirichlet condition on the negative  $x$ -axis. There are many other interesting mixed boundary problems on the half-line. We refer to the book by Noble [10] for a survey and account of these.

The diffraction problem for a quarter-plane generalizes the half-plane problem of Poincaré-Sommerfeld. Consider the perfectly absorbent quarter-plane. In this case a solution of the three-dimensional reduced wave equation is required to meet a Dirichlet condition on the quarter-plane ( $x_1 \geq 0, x_2 \geq 0, x_3 = 0$ ) and a Neumann condition on the complementary three-quarter-plane ( $x_1 < 0 \cup x_2 < 0; x_3 = 0$ ). Physically, this is a three-dimensional mixed boundary value problem. It presents considerable mathematical difficulties. It remained an open problem until quite recently. Then it was shown (Radlow [11], [13]) that the problem can be reduced to a function-theoretical problem of a two variable Wiener-Hopf type, and that this problem can be solved by a generalization of the Wiener-Hopf method from one to two complex variables.

Our purpose here is to show that the two-variable Wiener-Hopf method of [11] and [13] can be applied to the problem of diffraction by a perfectly reflecting quarter-plane. In this case the mixed boundary value problem is to find a three-dimensional wave function which meets a Neumann condition on the quarter-plane and a Dirichlet condition on the complementary three-quarter-plane.

Our analysis will be organized as follows. In Section 2 we summarize relevant notions of the theory of two-dimensional Laplace transforms and of functions of two complex variables. In Section 3 we state our mixed boundary value problem. We then apply the ideas of Section 2 to show (Section 4, 5) that the problem is equivalent under two-dimensional Laplace transformation to a function-theoretical problem of solving a transform equation involving four unknown functions of two complex variables.

Probably the most significant feature of the Radlow generalization [13] of the Wiener-Hopf method is that the solution of the transform equation requires not one factorization lemma (as in the one variable case) but two factorization lemmas. The first one is a direct generalization of the factorization encountered in the one variable case. The second lemma is concerned with the analyticity domain of certain products of these factors taken two at a time.

We state and discuss the two factorization lemmas in Section 6. Then we make use of the lemmas to solve our mixed boundary value problem. The theorem proved in Section 7 yields an explicit unique solution to the problem. The uniqueness proof makes use of a recent result by Douglas and Howe [5]. In Section 8 we consider the behavior of the solution in the limits  $x_1 \rightarrow +\infty$  and  $x_2 \rightarrow +\infty$ . Physically, we expect the solution to reduce in these limiting cases to the solutions for the perfectly reflecting half-planes  $x_2 > 0$  and  $x_1 > 0$ , respectively. We show in Section 8 that this expectation is borne out. Finally (Section 9) we obtain results for the behavior of the solution and its normal derivative as  $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \rightarrow 0$ .

## SECTION II

TWO-VARIABLE ANALYTIC FUNCTION THEORY  
AND DOUBLE LAPLACE TRANSFORMS

The analysis used in solving our diffraction problem is carried out in the product space of two complex variables  $s_j = u_j + iv_j$ ,  $j = 1, 2$ . For brevity, we shall frequently write  $x = (x_1, x_2)$ ,  $s = (s_1, s_2)$  etc. In particular, this allows us to make use of the scalar product notation  $s \cdot x = s_1 x_1 + s_2 x_2$ .

Certain definitions and theorems which are pertinent to the function-theoretical method presented in the following sections may now be stated for future reference.

DEFINITION 2.1: Let  $D$  be a domain in the  $u_1 u_2$ -plane. The set

$$T(D) = \{s: u \in D, -\infty < v_j < \infty\}$$

is called a tube with basis  $D$ .

DEFINITION 2.2: Let  $F(s)$  be analytic in the tube  $T(D)$ . For each fixed  $u \in D$ , the  $L_m$ -norm ( $m = 1, 2$ ) of the function  $F(u + iv)$  is defined by

$$\|F(u + iv)\|_m = \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u + iv)|^m dv_1 dv_2 \right]^{1/m}.$$

A function  $F(s)$  is said to be bounded in  $L_m$ -norm for  $u \in S \subseteq D$  provided that

$$\|F(u + iv)\|_m < \infty$$

for each  $u \in S$ . Moreover, a function  $F(s)$  is uniformly bounded in  $L_m$ -norm for  $u \in S \subseteq D$  if there is a constant  $M > 0$  such that

$$\|F(u + iv)\|_m < M$$

for all  $u \in S$ .

We now state a basic theorem due to S. Bochner [2] on which the Wiener-Hopf method for two complex variables is in part based.

**THEOREM 2.1:** Assume that  $F(s)$  is analytic in a domain  $D$  containing the tube  $T(D)$  with basis

$$D = \{u: \alpha_j \leq u_j \leq \beta_j\}$$

and let  $F(s)$  be uniformly bounded in  $L_2$ -norm for  $u \in D$ . Then for  $s \in T(D)$ , Cauchy's integral formula yields the unique additive decomposition

$$F(s) = \sum_{n=1}^4 F_n(s) \quad (2.1)$$

where  $F_n(s)$  is analytic in the tube  $T(D_n)$ , the bases  $D_n$  being given by

$$D_1 = \{u: u_1 > \alpha_1, u_2 > \alpha_2\},$$

$$D_2 = \{u: u_1 < \beta_1, u_2 > \alpha_2\},$$

$$D_3 = \{u: u_1 < \beta_1, u_2 < \beta_2\},$$

$$D_4 = \{u: u_1 > \alpha_1, u_2 < \beta_2\},$$

and where  $F_n(s) \rightarrow 0$  as  $|u_j| \rightarrow \infty$  ( $j = 1$  or  $2$ ) in their respective tubes of analyticity. The functions  $F_n(s)$  are given by

$$F_n(s) = \frac{(-1)^{n+1}}{(2\pi i)^2} \int_{\Gamma_{n2}} \int_{\Gamma_{n1}} \frac{F(z) dz_1 dz_2}{(z_1 - s_1)(z_2 - s_2)} \quad (2.2)$$

where  $\Gamma_{nj}$  denotes a vertical contour from  $\gamma_{nj} - i\infty$  to  $\gamma_{nj} + i\infty$  in the  $z_j$ -plane with

$$\gamma_{n1} = \begin{cases} \alpha_1 & \text{if } n = 1, 4 \\ \beta_1 & \text{if } n = 2, 3 \end{cases} ; \gamma_{n2} = \begin{cases} \alpha_2 & \text{if } n = 1, 2 \\ \beta_2 & \text{if } n = 3, 4 \end{cases} \quad (2.3)$$

Proof: Let  $R_1$  denote the boundary of a positively oriented rectangle with vertices  $\alpha_1 \pm i\rho_1$ ,  $\beta_1 \pm i\rho_1$  ( $\rho_1 > 0$ ). Then for  $s_1$  inside  $R_1$ , Cauchy's formula gives

$$\begin{aligned} 2\pi i F(s) &= \int_{R_1} \frac{F(z_1, s_2)}{(z_1 - s_1)} dz_1 = \\ &= \left( \int_{\beta_1 - i\rho_1}^{\beta_1 + i\rho_1} - \int_{\alpha_1 + i\rho_1}^{\alpha_1 + i\rho_1} - \int_{\alpha_1 - i\rho_1}^{\alpha_1 - i\rho_1} + \int_{\beta_1 - i\rho_1}^{\beta_1 - i\rho_1} \right) \frac{F(z_1, s_2)}{(z_1 - s_1)} dz_1 \\ &= I_1 - I_2 - I_3 + I_4. \end{aligned}$$

We wish to show that

$$\lim_{\rho_1 \rightarrow \infty} I_j(\rho_1) = 0, \quad j = 2, 4.$$

Since the mean value theorem for integrals gives

$$\bar{I}_j(t) = \frac{1}{t} \int_t^{2t} |I_j(q)| dq = |I_j(m)|$$

for some  $m \in (t, 2t)$ , it is sufficient to show that

$$\lim_{t \rightarrow \infty} \bar{I}_j(t) = 0, \quad j = 2, 4.$$

For  $t > 2|v_1|$ , we have that

$$\begin{aligned} |\bar{I}_2(t)| &\leq \int_t^{2t} |I_2(q)| dq \\ &\leq \frac{1}{t} \int_{\alpha_1}^{\beta_1} \int_t^{2t} \frac{|F(p+iq, s_2)|}{|s_1 - p - iq|} dq dp. \end{aligned}$$

An application of Schwarz's inequality to the inner integral yields

$$|\bar{I}_2(t)| \leq \frac{1}{t} \int_{\alpha_1}^{\beta_1} \left\{ \left[ \int_t^{2t} |F(p+iq, s_2)|^2 dq \right]^{\frac{1}{2}} \left[ \int_t^{2t} \frac{dq}{|s_1 - p - iq|^2} \right]^{\frac{1}{2}} \right\} dp.$$

By our assumption on  $F$ , the quantity in the first square bracket is bounded on  $[\alpha_1, \beta_1]$ , say less than  $M$ . Also,

$$\begin{aligned} \frac{1}{|s_1 - p - iq|^2} &= \frac{1}{(u_1 - p)^2 + (v_1 - q)^2} \leq \\ \frac{1}{(q - v_1)^2} &< \frac{1}{(t - t/2)^2} = \frac{4}{t^2}, \end{aligned}$$

and hence

$$|\bar{I}_2(t)| < 2M(\beta_1 - \alpha_1)/t^{3/2}.$$

Consequently,

$$\lim_{t \rightarrow \infty} \bar{I}_j(t) = 0$$

for  $j = 2$ , and a similar argument yields the result for  $j = 4$ . Letting  $\rho_1 \rightarrow \infty$ , we thus obtain the additive decomposition



$$2\pi i F(s) = \left( \int_{\beta_1 - i\infty}^{\beta_1 + i\infty} - \int_{\alpha_1 - i\infty}^{\alpha_1 + i\infty} \right) \frac{F(z_1, s_2)}{(z_1 - s_1)} dz_1 \quad (2.4)$$

$$= G(s) + H(s).$$

Now let  $R_2$  denote a positively oriented rectangle with vertices  $\alpha_2 \pm i\rho_2$ ,  $\beta_2 \pm i\rho_2$  ( $\rho_2 > 0$ ). For  $s_2$  inside  $R_2$ , Cauchy's formula again gives

$$2\pi i G(s) = \int_{R_2} \frac{G(s_1, z_2)}{(z_2 - s_2)} dz_2 =$$

$$\left( \int_{\beta_2 - i\rho_2}^{\beta_2 + i\rho_2} - \int_{\alpha_2 + i\rho_2}^{\beta_2 + i\rho_2} - \int_{\alpha_2 - i\rho_2}^{\alpha_2 + i\rho_2} + \int_{\beta_2 - i\rho_2}^{\alpha_2 - i\rho_2} \right) \frac{G(s_1, z_2)}{(z_2 - s_2)} dz_2$$

$$= J_1 - J_2 - J_3 + J_4.$$

As before we can show that the second and fourth integrals vanish as  $\rho_2 \rightarrow \infty$  and the same procedure applies to the function  $H(s)$  in (2.4). We conclude that  $F(s)$  may be additively decomposed as in (2.1) with the functions  $F_n(s)$  given in (2.2).

Let us check that  $F_1(s)$  is analytic in  $T(D_1)$  as asserted. For  $s \in T(D_1)$ , Schwarz's inequality gives

$$(z_j = \alpha_j + iy_j)$$

$$\begin{aligned}
& \left[ \int_{\Gamma_{12}} \int_{\Gamma_{11}} \frac{F(z) dz_1 dz_2}{(z_1 - s_1)(z_2 - s_2)} \right] \leq \int_{-\infty - i\infty}^{\infty - i\infty} \int_{-\infty - i\infty}^{\infty - i\infty} \frac{|F(z)| dy_1 dy_2}{|(z_1 - s_1)(z_2 - s_2)|} \\
& \leq \|F(\alpha + iy)\|_2 \left[ \int_{-\infty - i\infty}^{\infty - i\infty} \int_{-\infty - i\infty}^{\infty - i\infty} \frac{dy_1 dy_2}{|(z_1 - s_1)(z_2 - s_2)|^2} \right]^{\frac{1}{2}} \\
& = \Pi \|F(\alpha + iy)\|_2 (u_1 - \alpha_1)^{-\frac{1}{2}} (u_2 - \alpha_2)^{-\frac{1}{2}}.
\end{aligned}$$

Since this last quantity is bounded for  $u_j - \alpha_j \geq \epsilon > 0$ , we have that  $F_1(s)$  is analytic in  $T(D_1)$ . A similar argument shows that  $F_n(s)$  is analytic in  $T(D_n)$  for  $n = 2, 3, 4$  as desired.

The uniqueness proof may be found in [7]. It is based on the requirement that  $F_n(s) \rightarrow 0$  as  $|u_j| \rightarrow \infty$  ( $j = 1$  or  $2$ ). Clearly, the functions given in (2.2) satisfy this condition.

Before stating a useful corollary to the above theorem, it is convenient to introduce the notations

$$F_{1+}(s) = F_1(s) + F_4(s), \quad (2.5)$$

$$F_{2+}(s) = F_1(s) + F_2(s).$$

Also, let  $A_j$  and  $B_j$  denote vertical contours from  $\alpha_j - i\infty$  to  $\alpha_j + i\infty$  and from  $\beta_j - i\infty$  to  $\beta_j + i\infty$ , respectively.

**COROLLARY 2.1:** If  $F(s)$  satisfies the conditions of theorem 2.1, then

$$F_{1+}(s) = \frac{1}{2\pi i} \int_{A_1} \frac{F(z_1, s_2)}{(s_1 - z_1)} dz_1, \quad (2.6)$$

$$F_{2^+}(s) = \frac{1}{2\pi i} \int_{B_2} \frac{F(s_1, z_2)}{(s_2 - z_2)} dz_2. \quad (2.7)$$

Furthermore,

$$[F_{1^+}(s)]_{2^+} = [F_{2^+}(s)]_{1^+} = F_1(s). \quad (2.8)$$

Proof: As in the derivation of (2.4), Cauchy's formula gives

$$F(z_1, s_2) = \frac{1}{2\pi i} \left( \int_{A_2} - \int_{B_2} \right) \frac{F(z_1, z_2)}{(s_2 - z_2)} dz_2. \quad (2.9)$$

Now multiply both sides of (2.9) by  $1/2\pi i(s_1 - z_1)$  and integrate over  $A_1$  to obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{A_1} \frac{F(z_1, s_2)}{(s_1 - z_1)} dz_1 = & \quad (2.10) \\ \frac{1}{(2\pi i)^2} \left( \int_{A_1} \int_{A_2} - \int_{A_1} \int_{B_2} \right) \frac{F(z) dz_2 dz_1}{(z_1 - s_1)(z_2 - s_2)}. \end{aligned}$$

By Schwarz's inequality, both of the iterated integrals on the right side of (2.10) converge absolutely. Thus Fubini's theorem (see [9], p. 155) implies that the order of integration may be interchanged. Equation (2.6) now follows from (2.5) and theorem 2.1. The verification of (2.7) is analogous.

To establish (2.8) we note that

$$\begin{aligned} [F_{1^+}(s)]_{2^+} &= \frac{1}{2\pi i} \int_{A_2} \frac{F_{1^+}(s_1, z_2)}{(s_2 - z_2)} dz_2 = \\ \frac{1}{(2\pi i)^2} \int_{A_2} \int_{A_1} \frac{F(z) dz_1 dz_2}{(z_1 - s_1)(z_2 - s_2)} &= F_1(s) \end{aligned}$$

and a similar computation yields the second half of (2.8).

We can put theorem 2.1 in a slightly more general form.

**THEOREM 2.2:** Let  $D$  be a convex domain in the  $u_1u_2$ -plane which contains the origin and assume  $F(s)$  is analytic in  $T(D)$  and uniformly bounded in  $L_2$ -norm on compact subsets of  $D$ . Then for each  $u \in D$ , Cauchy's integral formula gives the unique additive decomposition

$$F(s) = \sum_{n=1}^4 F_n(s) \quad (2.11)$$

where  $F_n(s)$  is analytic in the tube  $T(\langle D \cup q_n \rangle)$  with  $q_n$  denoting the  $n^{\text{th}}$  quadrant of the  $u_1u_2$ -plane and the brackets  $\langle \rangle$  indicating the convex hull of the set enclosed.

**Proof:** We may cover  $D$  with a countable number of closed rectangles

$$D_k = \{u: \alpha_{kj} \leq u_j \leq \beta_{kj}\},$$

$j = 1, 2; k = 1, 2, \dots$ . For each domain  $D_k$ , we may use theorem 2.1 to write

$$F(s) = \sum_{n=1}^4 F_{kn}(s) \quad , \quad s \in T(D_k).$$

Since the functions  $F_{kn}(s)$  are unique, for each fixed  $n$  they constitute analytic continuations of each other.

Hence  $F(s)$  has the representation (2.11) with each  $F_n(s)$  being analytic in a tube  $T_n$  containing  $T(D)$  and clearly  $T_n = T(\langle D \cup q_n \rangle)$ .

We shall have occasion to use the restricted two-dimensional Laplace transformation  $\mathfrak{L}_n$  defined for  $n = 1, 2, 3, 4$  by

$$F_n(s) = \mathfrak{L}_n[f(x)] = \iint_{Q_n} f(x) \exp(-s \cdot x) dx_1 dx_2 \quad (2.12)$$

where  $Q_n$  denotes the  $n^{\text{th}}$  quadrant of the  $x_1 x_2$ -plane (hereafter written  $X$ ) and  $f(x)$  vanishes outside of  $Q_n$ . A basic theorem concerning the existence of  $F_n(s)$  is the following:

THEOREM 2.3: If

$$\iint_{Q_n} |f(x)| \exp(-\gamma_n \cdot x) dx_1 dx_2 < \infty$$

where  $\gamma_n = (\gamma_{n1}, \gamma_{n2})$  (see (2.3) for the definitions of  $\gamma_{nj}$ ), then  $\mathfrak{L}_n[f(x)]$  converges absolutely and uniformly in the tube  $T(D_n)$ . Consequently,  $F_n(s)$  is analytic in  $T(D_n)$ .

Now let us turn our attention to the Laplace transform over the full  $x_1 x_2$ -plane,

$$F(s) = \mathfrak{L}[f(x)] = \iint_X f(x) \exp(-s \cdot x) dx_1 dx_2 .$$

If  $f(x)$  vanishes outside  $Q_n$ , then  $\mathfrak{L}[f(x)] = \mathfrak{L}_n[f(x)]$  and the content of the preceding paragraph applies. In case  $f(x)$  does not vanish on any quadrant  $Q_n$ , we introduce the functions

$$\begin{aligned}
f_1(x) &= f(x)g(x_1)g(x_2), \\
f_2(x) &= f(x)g(-x_1)g(x_2), \\
f_3(x) &= f(x)g(-x_1)g(-x_2), \\
f_4(x) &= f(x)g(x_1)g(-x_2),
\end{aligned}
\tag{2.13}$$

where  $g$  is the Heaviside function

$$g(x_j) = \begin{cases} 0 & \text{if } x_j < 0 \\ 1 & \text{if } x_j > 0. \end{cases}$$

This allows us to write

$$f(x) = \sum_{n=1}^4 f_n(x),$$

and therefore,

$$\mathcal{L}[f(x)] = \sum_{n=1}^4 \mathcal{L}[f_n(x)].
\tag{2.14}$$

The next theorem will aid us in obtaining a function-theoretical equivalent of our mixed boundary value problem.

**THEOREM 2.4:** (see [3], Chapter VI, 8, 9)

Let  $f(x)$  be a measurable function on  $X$ . If, for each  $u \in D$ ,

$$\|f(x)\exp(-u \cdot x)\|_m < \infty$$

for  $m = 1, 2$ , then  $F(s) = \mathcal{L}[f(x)]$  is analytic in  $T(D)$  and bounded in  $L_m$ -norm for  $u \in D$ . Conversely, if  $F(s)$  is analytic in  $T(D)$  and bounded in  $L_m$ -norm for  $u \in D$ , then there exists a unique inverse,

$$f(x) = \mathcal{L}^{-1}[F(s)] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s) \exp(s \cdot x) dv_1 dv_2,$$

which is independent of  $u$ .

Remark: This theorem is also true for  $m = 2$  only.

We now state two convolution theorems.

DEFINITION 2.3: The convolution of the functions  $f$  and  $g$  is the function

$$(f * g)(x) = \iint_X f(\xi) g(x - \xi) d\xi_1 d\xi_2.$$

The operation  $*$  is commutative:

$$f * g = g * f.$$

THEOREM 2.5: (see [8], p. 10) If  $\|f(x) \exp(-u \cdot x)\|_1 < \infty$  and  $\|g(x) \exp(-u \cdot x)\|_m < \infty$  for  $m = 1$  or  $2$  and some  $u = (u_1, u_2)$ , then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \mathcal{L}\{g\}.$$

It is worth noting that additive decomposition (2.1) is a consequence of our second theorem on convolution.

THEOREM 2.6: (see [8], p. 10) If  $\|f(x) \exp(-\xi \cdot x)\|_2 < \infty$  and  $\|g(x) \exp[-(u - \xi) \cdot x]\|_2 < \infty$  for some  $u = (u_1, u_2)$ ,  $\xi = (\xi_1, \xi_2)$ , then

$$\mathcal{L}\{fg\} = \mathcal{L}\{f\} * \mathcal{L}\{g\},$$

that is,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(x)\exp(-s \cdot x)dx_1dx_2 =$$

$$\frac{1}{(2\pi i)^2} \int_{\Gamma_2} \int_{\Gamma_1} F(z)G(s-z)dz_1dz_2$$

where  $\Gamma_j$  denotes a vertical contour from  $\xi_j - i\infty$  to  $\xi_j + i\infty$ .

To see that additive decomposition (2.1) follows immediately from this result, let  $g(x_j)$  be the Heaviside function and define  $f_n(x)$  as in (2.13). Then by (2.14) and theorem 2.6,

$$F(s) = \mathcal{L}[f(x)] = \sum_{n=1}^4 \mathcal{L}[f_n(x)]$$

$$= \sum_{n=1}^4 \frac{(-1)^{n+1}}{(2\pi i)^2} \int_{\Gamma_{n2}} \int_{\Gamma_{n1}} \frac{F(z)dz_1dz_2}{(z_1-s_1)(z_2-s_2)}$$

as desired.

Our last theorem in this section is an asymptotic result.

**THEOREM 2.7:** Let  $f(x)$  be a function of the real variable  $x$  and denote its one variable Laplace transform by  $F(s)$ . If  $F(s)$  has a unique singularity  $s = s_0$  with greatest real part that is either a pole or an isolated essential singularity, then

$$f(x) \sim \operatorname{Res}_{s = s_0} e^{sx}F(s) \quad (2.15)$$

as  $x \rightarrow +\infty$ .



The proof of this result may be found in [15] (pp. 99-100). In particular, if  $s_0$  is a simple pole, then (2.15) may be written as

$$\lim_{x \rightarrow +\infty} e^{-s_0 x} f(x) = \lim_{s \rightarrow s_0} (s-s_0)F(s). \quad (2.16)$$

In the case  $s_0 = 0$ , (2.16) is sometimes called the final value theorem.

## SECTION III

## STATEMENT OF THE PROBLEM

We consider a plane wave which is incident on the quarter-plane  $x_1 \geq 0$ ,  $x_2 \geq 0$ ,  $x_3 = 0$ . The incident wave  $\phi_0$  is given by

$$\phi_0(x_1, x_2, x_3) = \exp(-a_1 x_1 - a_2 x_2 - a_3 x_3) \quad (3.1)$$

where  $a_1 = ik \sin \alpha_0 \cos \beta_0$ ,  $a_2 = ik \cos \alpha_0$ ,  $a_3 = ik \sin \alpha_0 \sin \beta_0$  with  $k = p - iq$  ( $p > 0$ ,  $q > 0$ ) and  $0 \leq \alpha \leq \pi/2$ ,  $0 \leq \beta_0 \leq \pi/2$ .

If the total wave field is  $\phi_t = \phi + \phi_0$ , then the scattered wave  $\phi$  is a function which satisfies the reduced wave equation

$$(\nabla^2 + k^2)\phi = 0 \quad (3.2)$$

for  $-\infty < x_1 < \infty$ ,  $-\infty < x_2 < \infty$ ,  $x_3 > 0$ .

In the case of a perfectly reflecting quarter-plane, the boundary conditions that  $\phi$  must satisfy on the  $x_1 x_2$ -plane are

$$\left. \frac{\partial \phi}{\partial x_3} \right|_{x_3=0} = a_3 \exp(-a_1 x_1 - a_2 x_2) \quad (3.3)$$

for  $(x_1, x_2) \in Q_1$ ;

$$\phi(x_1, x_2, 0) = 0 \quad (3.4)$$

for  $(x_1, x_2) \in X \sim Q_1$ .

In order to obtain a function-theoretical equivalent of our mixed boundary value problem and insure uniqueness, we require that the functions

$$f(x) = \begin{cases} \phi(x, 0) & \text{if } x \in Q_1 \\ 0 & \text{if } x \in X \sim Q_1 \end{cases}$$

and

$$g(x) = \begin{cases} a_3 \exp(-a \cdot x) & \text{if } x \in Q_1 \\ \left. \frac{\partial \phi}{\partial x_3} \right|_{x_3=0} & \text{if } x \in X \sim Q_1 \end{cases}$$

satisfy the conditions

$$\|f(x)\exp(-u \cdot x)\|_m < \infty \quad (3.5)$$

for  $u_j > -\text{Re} a_j$ ,  $m = 1, 2$ ;

$$\|g(x)\exp(-u \cdot x)\|_2 < \infty \quad (3.6)$$

for  $u_j > -\text{Re} a_j \cap u_1^2 + u_2^2 < q^2$ . In particular,  $f(x)$  is absolutely integrable and square integrable while  $g(x)$  is square integrable.

It is customary in diffraction theory to require that the diffracted field satisfy Sommerfeld's radiation condition

$$\lim_{r \rightarrow \infty} r \left( \frac{\partial \phi}{\partial r} + ik\phi \right) = 0,$$

where  $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$ . Since  $k$  is complex, we may

replace (see [1], pp. 154-155) the above condition by the requirement that  $\phi$  be outgoing at infinity. This means that  $\phi$  must have the behavior of  $\exp(-ikr)/r$  for large  $r$ . We note that this requirement is implicit in conditions (3.5), (3.6). Thus we need not make an independent requirement of the radiation condition. The solution we construct will, in fact, be outgoing at infinity. This is verified in Section 7: see the remarks following equations (7.7), (7.8).

## SECTION IV

## SEPARATION OF VARIABLES AND INTEGRAL EQUATIONS

Karp [6] has shown how to deduce one-variable Wiener-Hopf transform equations by the method of separation of variables. The method can be generalized to the two-variable case. To obtain a solution of (3.2) by separation of variables, set

$$\phi(x_1, x_2, x_3) = X_1(x_1)X_2(x_2)X_3(x_3).$$

The wave equation now becomes

$$\frac{1}{X_1} X_1'' + \frac{1}{X_2} X_2'' + \frac{1}{X_3} X_3'' + k^2 = 0$$

which gives

$$X_1 = \exp(s_1 x_1), \quad X_2 = \exp(s_2 x_2), \quad X_3 = \exp(x_3/2K(s))$$

where  $s_j = u_j + iv_j$  are the separation constants and

$$K(s) = \frac{1}{2}(s_1^2 + s_2^2 + k^2)^{-\frac{1}{2}},$$

the branch of  $K(s)$  being determined by the choice

$\text{Re } K(s) < 0$ . The function  $K(s)$  is analytic in the tube  $T(D)$  with basis

$$D = \{u: u_1^2 + u_2^2 < q^2\}$$

and for sufficiently large values of  $|s_1^2 + s_2^2|$  with  $s \in T(D)$ ,

$$K(s) = O[(s_1^2 + s_2^2)^{-\frac{1}{2}}].$$

The separation solutions are thus seen to be

$$\phi(x_1, x_2, x_3) = \exp[s \cdot x + x_3/2K(s)] \quad (4.1)$$

and upon multiplying the right side of (4.1) by  $F(s)/(2\pi)^2$  and applying the principle of superposition, we obtain

$$\begin{aligned} \phi(x_1, x_2, x_3) = & \\ & \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s) \exp[s \cdot x + x_3/2K(s)] dv_1 dv_2. \end{aligned} \quad (4.2)$$

This formal solution of (3.2) is recognizable as the inverse Laplace transform of  $F(s)\exp[x_3/2K(s)]$  and boundary condition (3.3) will be met provided that

$$\begin{aligned} \left. \frac{\partial \phi}{\partial x_3} \right|_{x_3=0} &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F(s)}{2K(s)} \exp(s \cdot x) dv_1 dv_2 \quad (4.3) \\ &= a_3 \exp(-a \cdot x) \end{aligned}$$

for  $x \in Q_1$  whereas condition (3.4) requires that

$$\phi(x, 0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s) \exp(s \cdot x) dv_1 dv_2 \quad (4.4)$$

for  $x \in X \sim Q_1$ . The requirement that  $\phi$  given in (4.2) satisfy boundary conditions (3.3) and (3.4) is thus seen to be formally equivalent to requiring that  $F(s)$  satisfy the dual integral equations (4.3) and (4.4). The dual integral equation formulation of the one-variable Wiener-Hopf problem dates back to Karp [6]. See [13] for the extension to two variables.

## SECTION V

## THE FUNCTION-THEORETICAL PROBLEM

We now show that the mixed boundary value problem of Section 3 is equivalent to the function-theoretical problem of determining the unknown function  $F(s)$ .

THEOREM 5.1: The function  $\phi$  given in (4.2) satisfies requirements (3.1) through (3.6) of our mixed boundary value problem if and only if:

(i)  $F$  is analytic in  $T(A)$  and bounded in  $L_m$ -norm ( $m = 1, 2$ ) for  $u \in A$ .

(ii) The first term in the additive decomposition of  $F/2K$  is

$$[F/2K]_1 = a_3(s_1 + a_1)^{-1}(s_2 + a_2)^{-1}.$$

Proof: Assume that  $\phi$  given in (4.2) meets the requirements of our mixed boundary value problem. Let

$$F(s) = \iint_{Q_1} f(x) \exp(-s \cdot x) dx_1 dx_2.$$

In view of (3.5) and theorem 2.4,  $F(s)$  satisfies condition (i).

Next, we show that  $\partial\phi/\partial x_3$  at  $x_3 = 0$  may be calculated from (4.2) by differentiation under the integral sign. Choose  $\epsilon > 0$  and take  $x_2 \geq 2\epsilon$ . Since  $\operatorname{Re}[K(s)]^{-1} < 0$ , we have

$$|F(s) \exp[iv \cdot x + x_3/2K(s)]| \leq |F(s)| \exp\{\epsilon \operatorname{Re}[K(s)]^{-1}\}.$$

Schwarz's inequality and the boundedness of  $F(s)$  in  $L_2$ -norm now yield

$$\int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} |F(s)| \exp\{\epsilon \operatorname{Re}[K(s)]^{-1}\} dv_1 dv_2 \leq$$

$$\|F(u + iv)\|_2 \|\exp\{\epsilon \operatorname{Re}[K(u + iv)]^{-1}\}\|_2 < \infty.$$

Thus the double integral in (4.2) converges absolutely and uniformly for  $-\infty < x_1 < \infty$ ,  $-\infty < x_2 < \infty$ ,  $x_3 > 0$ . In particular, we conclude that

$$\left. \frac{\partial \phi}{\partial x_3} \right|_{x_3=0} = \frac{1}{(2\pi)^2} \int_{-\infty-\infty}^{\infty} \int_{-\infty-\infty}^{\infty} \frac{F(s)}{2K(s)} \exp(s \cdot x) dv_1 dv_2,$$

which defines  $\partial\phi/\partial x_3$  at  $x_3 = 0$  as the inverse Laplace transform of  $F/2K$ . Thus by (3.6) and theorem 2.4,  $F/2K$  is analytic and bounded in  $L_2$ -norm for  $u \in A \cap D$ . Consequently, theorem 2.2 implies the existence of the unique additive decomposition

$$\frac{F(s)}{2K(s)} = \sum_{n=1}^4 G_n(s) \quad (5.1)$$

with the functions  $G_n(s)$  being analytic in their respective tubes  $T(\langle (A \cap D) \cup q_n \rangle)$ .

Now set

$$g_n(x) = \begin{cases} \left. \frac{\partial \phi}{\partial x_3} \right|_{x_3=0} & \text{if } x \in Q_n \\ 0 & \text{if } x \in X \sim Q_n. \end{cases}$$



Then

$$\frac{F(s)}{2K(s)} = \mathfrak{L} \left[ \left. \frac{\partial \phi}{\partial x_j} \right|_{x_j=0} \right] = \sum_{n=1}^4 \mathfrak{L}_n[g_n(x)] \quad (5.2)$$

and upon comparing (5.1) with (5.2) we have by uniqueness that

$$G_n(s) = \mathfrak{L}_n[g_n(x)]$$

whence

$$[F/2K]_1 = G_1(s) = a_3(s_1 + a_1)^{-1}(s_2 + a_2)^{-1}.$$

Thus condition (ii) is satisfied.

Conversely, assume that  $F(s)$  satisfies conditions (i) and (ii). Again, by dominated convergence, the first and second partial derivatives of  $\phi$  given (4.2) may be computed by differentiation under the integral signs. We conclude that  $\phi$  given in (4.2) is a solution of (3.2).

To show that  $\phi$  satisfies (4.3) we note that the first half follows by dominated convergence. Now consider the function  $G_2(s)$  and fix  $s_2$ . Then  $G_2$  as a function of  $s_1$  is analytic for  $u_1 < q$  and since  $G_2(s_1, s_2) \rightarrow 0$  as  $|s_1| \rightarrow \infty$ , we have

$$G_2(s_1, s_2) = O[1/s_1^\delta]$$

for some  $\delta > 0$ . Upon completing a straight line contour from  $u_1 - ir$  to  $u_1 + ir$  by a semi-circle to the left, applying Cauchy's integral theorem and letting  $r \rightarrow +\infty$  we obtain that

$$\int_{-\infty}^{\infty} G_2(s_1, s_2) \exp(s \cdot x) dv_1 = 0$$

for  $n = 2, 3, 4$  and  $x_j > 0$ .

The residue theorem now gives

$$\begin{aligned} \left. \frac{\partial \phi}{\partial x_3} \right|_{x_3=0} &= \frac{a_3}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp(s \cdot x) dv_1 dv_2}{(s_1 + a_1)(s_2 + a_2)} \\ &= a_3 \exp(-a \cdot x), \end{aligned}$$

for  $x_j > 0$  as desired.

Since  $F(s)$  satisfies condition (i) there exists a measurable function  $f(x)$  such that

$$\begin{aligned} f(x) &= \mathcal{L}^{-1}[F(s)] \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s) \exp(s \cdot x) dv_1 dv_2. \end{aligned} \quad (5.3)$$

But the analyticity of  $F(s)$  in  $T(A)$  together with Cauchy's integral theorem imply that  $f(x) = 0$  for  $x \in X \sim Q_1$ .

Furthermore (5.3) implies that  $f(x) = \phi(x, 0)$  and therefore, (4.4) is satisfied. Requirements (3.5) and (3.6) are also met in view of theorem 2.4.

## SECTION VI

FACTORIZATION OF  $K(s)$ 

Our solution depends on the factorization of  $K(s)$  as

$$K(s) = \frac{1}{2} \prod_{n=1}^4 K_n(s) \quad (6.1)$$

where the functions  $K_n(s)$  are required to be analytic and nonvanishing in their respective tubes  $T(\langle D \cup q_n \rangle)$ . This factorization is precisely the one given by Radlow in [11] and generalizes the usual Wiener-Hopf factorization from one to two complex variables. The result is that

$$K_n(s) = \exp \left[ -\frac{1}{4} \int^k k \mathcal{L}_n[H_o^{(2)}(kr)] dk \right]$$

where  $H_o^{(2)}(kr) = J_o(kr) - iN_o(kr)$  is the Hankel function of the second kind and  $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ . The transform  $\mathcal{L}_1[H_o^{(2)}(kr)]$  is given explicitly in [14] with the result being

$$\mathcal{L}_1[H_o^{(2)}(kr)] = \frac{1}{s_1^2 + s_2^2 + k^2} .$$

$$\left[ -1 + \frac{s_1}{\sqrt{s_2^2 + k^2}} \left( 1 + \frac{2i}{\pi} \log S_2 \right) + \frac{s_2}{\sqrt{s_1^2 + k^2}} \left( 1 + \frac{2i}{\pi} \log S_1 \right) \right]$$

where

$$S_j = \frac{s_j + \sqrt{s_j^2 + k^2}}{k} .$$

If we set  $H_n(s) = \mathcal{L}_n[H_0^{(2)}(kr)]$ , it follows that

$$H_2(s_1, s_2) = H_1(-s_1, s_2), \quad H_3(s_1, s_2) = H_1(-s_1, -s_2),$$

$$H_4(s_1, s_2) = H_1(s_1, -s_2).$$

The fact that  $K_n(s)$  is analytic in  $T(\langle D \cup q_n \rangle)$  is a consequence of theorem 2.6. For example, if we set  $h(x) = H^{(2)}(kr)$  and  $g(x) = 1$  for  $x_j \geq 0$  and zero otherwise, then

$$\mathcal{L}_1[H_0^{(2)}(kr)] = \mathcal{L}[hg] = \frac{1}{(2\pi i)^2} \int_{\Gamma_2} \int_{\Gamma_1} \frac{H(z) dz_1 dz_2}{(z_1 - s_1)(z_2 - s_2)}$$

where (see [11])  $H(s) = \mathcal{L}[h(x)] = -4i/(s_1^2 + s_2^2 + k^2)$  and  $\Gamma_j$  denotes a vertical contour from  $\gamma_j - i\infty$  to  $\gamma_j + i\infty$  with  $\gamma_j > -q$ . Since  $H(s)$  is analytic in  $T(D)$  and bounded in  $L_2$ -norm for  $u \in D$ , it follows that  $\mathcal{L}_1[H_0^{(2)}(kr)]$  is analytic in  $T(\langle D \cup q_n \rangle)$  because it is the first term in the additive decomposition of  $H(s)$ .

The growth estimates given for  $K_n(s)$  in [13] are

$$K_n(s) = O[(s_1^2 + s_2^2)^{-1/8}] \quad (6.2)$$

with validity for large  $|s_1^2 + s_2^2|$  and  $s \in T(\langle D \cup q_n \rangle)$ .

The second major result concerning the factorization of  $K(s)$  is again given in [13]. It is shown that

$$H_2 + H_4 = -2i\mathcal{L}_2[N_0(kr)] = -2i\mathcal{L}_4[N_0(kr)]. \quad (6.3)$$

The arguments given in [13] (see also [12]) establish that  $\mathcal{L}_2[N_0(kr)]$  is analytic in the tube  $T(D_2 \cap q_2)$  with

$$D_2 = \{u: u_2 - u_1 > q\},$$

whereas  $\mathcal{L}_4[N_0(kr)]$  is analytic in the tube  $T(D_4 \cap q_4)$  with

$$D_4 = \{u: u_1 - u_2 > q\}.$$

Observe that the tubes  $T(D_2 \cap q_2)$  and  $T(D_4 \cap q_4)$  are disjoint but both of them have a non-empty intersection with  $T(D)$ . We now denote

$$K_{24}(s) = \exp \left[ -\frac{1}{2} \int^k k \mathcal{L}_2[N_0(kr)] dk \right], \quad (6.4)$$

$$K_{42}(s) = \exp \left[ -\frac{1}{2} \int^k k \mathcal{L}_4[N_0(kr)] dk \right]. \quad (6.5)$$

Clearly  $K_2(s)K_4(s)$  is analytic in  $T(D)$  whereas  $K_{24}(s)$  and  $K_{42}(s)$  are analytic in the disjoint tubes  $T(D_2 \cap q_2)$  and  $T(D_4 \cap q_4)$ , respectively. Moreover, according to (6.3), the same analytical expression represents all three functions. Hence this analytical expression may be considered as a function which is analytic in any one of the tubes  $T(D)$ ,  $T(D_2 \cap q_2)$  and  $T(D_4 \cap q_4)$ . Our use of the respective notations  $K_2(s)K_4(s)$ ,  $K_{24}(s)$ ,  $K_{42}(s)$  will indicate that we are considering the function in the respective tubes  $T(D)$ ,  $T(D_2 \cap q_2)$ ,  $T(D_4 \cap q_4)$ . We are not allowed to continue analytically from one of these tubes to another. But we can, for example, interpret  $K_2(s)K_4(s)$  as either  $K_{24}(s)$  or  $K_{42}(s)$  provided we have made no prior determination of the tube of analyticity in which we are working.

In the next section, we will need to apply the ideas set forth in the preceding paragraph. Specifically, we will need the fact that  $K_2(s_1, -a_2)K_4(s_1, -a_2)$  may be interpreted as  $K_{24}(s_1, +a_2)$ , that is, a function which is analytic for  $u_1 < \text{Re}a_2 - q$ . It is clear from the last paragraph that  $K_2(s_1, -a_2)K_4(s_1, -a_2)$  may be interpreted as  $K_{42}(s_1, -a_2)$ . Moreover, we now show that  $K_{42}(s_1, -a_2)$  (and hence  $K_2(s_1, -a_2)K_4(s_1, -a_2)$ ) may be interpreted as  $K_{24}(s_1, +a_2)$ . Recall that  $K_{42}(s_1, -a_2)$  is defined in (6.5) by means of

$$\iint_{\mathbb{Q}_4} N_0(kr) \exp(-s_1 x_1 + a_2 x_2) dx_1 dx_2$$

while  $K_{24}(s_1, +a_2)$  is defined in (6.4) by means of

$$\iint_{\mathbb{Q}_2} N_0(kr) \exp(-s_1 x_1 - a_2 x_2) dx_1 dx_2.$$

The two double integrals yield precisely the same analytical expression, since  $K_{42}(s_1, -a_2)$  by definition has  $u_1 > q - \text{Re}a_2$ , while  $K_{24}(s_1, +a_2)$  by definition has  $u_1 < \text{Re}a_2 - q$ . This is why  $K_2(s_1, -a_1)K_4(s_1, -a_2)$  can be interpreted either as  $K_{42}(s_1, -a_2)$  or  $K_{24}(s_1, +a_2)$ .

## SECTION VII

## THE SOLUTION

We now make use of the factorization lemmas of the last section to determine the unknown function  $F(s)$ . The result is a unique closed form solution to the mixed boundary value problem of Section 3.

THEOREM 7.1: The unique function  $\phi(x_1, x_2, x_3)$  meeting conditions (3.1) through (3.6) is

$$\phi(x_1, x_2, x_3) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s) \exp[s \cdot x + x_3/2K(s)] dv_1 dv_2 \quad (7.1)$$

where

$$F(s) = ia_3 P(s)/Q_1(s) \quad (7.2)$$

and

$$P(s) = K_1(s)K_2(-a_1, s_2)K_3(-a_1, -a_2)K_4(s_1, -a_2),$$

$$Q_1(s) = (s_1 + a_1)(s_2 + a_2).$$

Proof: It must be shown that  $F(s)$  given by (7.2) satisfies conditions (i) and (ii) of theorem 5.1. First we show that  $F(s)$  is analytic in  $T(A)$ . Clearly  $Q_1(s)$  is analytic in  $T(A)$ . The function  $P(s)$  is analytic in the tube  $T(B)$  with basis

$$B = \{u: u_j > -b_j\}$$

where  $b_1^2 = q^2 - (\text{Re}a_2)^2$ ,  $b_2^2 = q^2 - (\text{Re}a_1)^2$  and  $b_j > 0$ .

To see this, consider the factors on the right side of (7.2.1):  $K_1(s)$  is analytic in  $T(\langle D \cup q_1 \rangle)$ ;  $K_2(-a_1, s_2)$  is analytic for  $u_2 > -b_2$ ;  $K_3(-a_1, -a_2)$  is a constant;  $K_4(s_1, -a_2)$  is analytic for  $u_1 > -b_1$  and the intersection of these analyticity domains in  $T(B)$ . Since  $A \subset B$ , it follows that  $F(s)$  is analytic in  $T(A)$ . In view of growth estimate (6.2),  $F(s)$  is bounded in  $L_m$ -norm ( $m = 1, 2$ ) for  $u \in A$ .

It remains to be shown that the first term in the additive decomposition of  $F/2K$  is  $a_3(s_1 + a_1)^{-1}(s_2 + a_2)^{-1}$ . To this end divide both sides of (7.2) by  $2K(s)$  and make use of factorization (6.1) to obtain

$$\frac{F(s)}{2K(s)} = a_3 \frac{K_2(-a_1, s_2)K_3(-a_1, -a_2)K_4(s_1, -a_2)}{(s_1 + a_1)(s_2 + a_2)K_2(s)K_4(s)K_3(s)}. \quad (7.3)$$

Now multiply both numerator and denominator of the right side of (7.3) by  $K_2(s_1, -a_2)$  to get

$$\frac{F(s)}{2K(s)} = a_3 \frac{K_2(-a_1, s_2)K_3(-a_1, -a_2)K_2(s_1, -a_2)K_4(s_1, -a_2)}{(s_1 + a_1)(s_2 + a_2)K_2(s)K_4(s)K_3(s)K_2(s_1, -a_2)} \quad (7.4)$$

and consider the function  $K_2(s_1, -a_2)K_4(s_1, -a_2)$  which appears in the numerator on the right side of (7.4). In the present argument, we have not yet discussed the analyticity domain of this function. Thus we are at liberty to choose from among three alternatives: (i) we can consider it as  $K_2(s)K_4(s)$  evaluated at  $s_2 = -a_2$ ; (ii)



we can consider it as  $K_{24}(s)$  evaluated at  $s_2 = +a_2$  (see the last paragraph of Section 6); (iii) we can consider it as  $K_{42}(s)$  evaluated at  $s_2 = -a_2$ . In other words, the function can be regarded as having analyticity domain:

(i)  $|u_1| < b_1$ ; (ii)  $u_1 < \text{Re}a_2 - q$  or (iii)  $u_1 > q - \text{Re}a_2$ .

For present purposes, we choose alternative (ii) and write

$$K_2(s_1, -a_2)K_4(s_1, -a_2) = K_{24}(s_1, +a_2) \quad (7.5)$$

meaning that the same analytical expression may be interpreted as the left side or as the right side.

Making use of (7.5), we rewrite (7.4) as

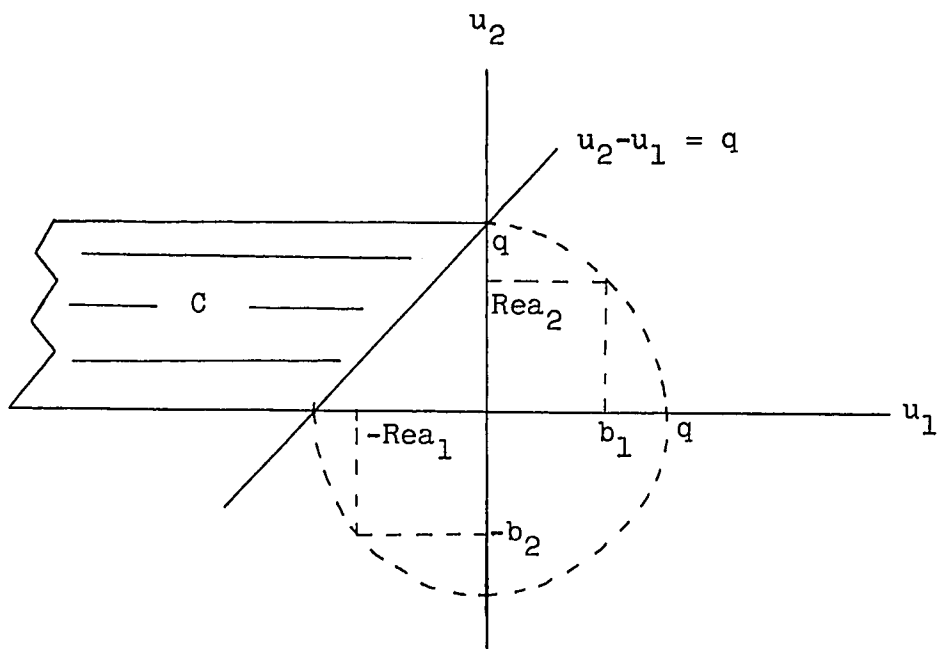
$$\frac{F(s)}{2K(s)} = a_3 \frac{K_2(-a_1, s_2)K_3(-a_1, -a_2)K_{24}(s_1, +a_2)}{(s_1 + a_1)(s_2 + a_2)K_{24}(s)K_3(s)K_2(s_1, -a_2)}. \quad (7.6)$$

The right side of (7.6) is analytic in the tube  $T(C)$  with basis (see Figure 7.1)

$$C = \{u: u_2 - u_1 > q, 0 < u_2 < q\}$$

except for a simple pole at  $s_1 = -a_1$ .

FIGURE 7.1



To see that this is so, consider the factors on the right side of (7.6): the functions  $K_2(-a_1, s_2)$ ,  $K_{24}(s_1, +a_2)$ ,  $K_{24}(s)$ ,  $K_3(s)$ ,  $K_2(s_1, -a_2)$  are analytic in  $u_2 > -b_2$ ,  $u_1 < \text{Re}a_2 - q$ ,  $T(D_2 \cap q_2)$ ,  $T(\langle D \cup q_3 \rangle)$ ,  $u_1 < b_1$ , respectively. The intersection of these analyticity domains is  $T(C)$ .

Let  $\Gamma_1$  denote a vertical contour from  $\gamma_1 - i\infty$  to  $\gamma_1 + i\infty$  with  $\gamma_1 > -\text{Re}a_1$ . Then the residue theorem together with (2.6) and (7.6) yields

$$\begin{aligned}
[F/2K]_{1^+} &= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{F(z_1, s_2) dz_1}{2(s_1 - z_1)K(z_1, s_2)} = \\
&\text{Res}_{z_1 = -a_1} \frac{F(z_1, s_2)}{2(s_1 - z_1)K(z_1, s_2)} = \\
&\lim_{z_1 \rightarrow -a_1} \frac{(z_1 + a_1)F(z_1, s_2)}{2(s_1 - z_1)K(z_1, s_2)} = \\
&a_3 \frac{K_3(-a_1, -a_2)K_2(-a_1, +a_2)}{(s_1 + a_1)(s_2 + a_2)K_2(-a_1, -a_2)K_3(-a_1, s_2)K_4(-a_1, s_2)}.
\end{aligned}$$

This last expression is analytic for  $u_2 < b_2$  with the exception of a simple pole at  $s_2 = -a_2$ . Let  $\Gamma_2$  denote a vertical contour from  $\gamma_2 - i\infty$  to  $\gamma_2 + i\infty$  with  $\gamma_2 > -\text{Re}a_2$ . Then the residue theorem in conjunction with (2.7) and (2.8) gives

$$\begin{aligned}
[F/2K]_1 &= [(F/2K)_{1^+}]_{2^+} \\
&= \frac{1}{2\pi i} \int_{\Gamma_2} \frac{[F(s_1, z_2)/2K(s_1, z_2)]_{1^+}}{(s_2 - z_2)} dz_2 \\
&= a_3(s_1 + a_1)^{-1}(s_2 + a_2)^{-1}.
\end{aligned}$$

We have used the fact that (7.5) holds for  $s_1 = -a_1$ .

We now proceed to show uniqueness. By dominated convergence, the right side of (7.1) may be written as

$$\frac{\partial}{\partial x_3} \left[ \frac{1}{(2\pi i)^2} \int_{-i\infty - i\infty}^{i\infty} \int_{-i\infty}^{i\infty} 2F(s)K(s)\exp[s \cdot x + x_3/2K(s)] ds_1 ds_2 \right]$$

and applying theorem 2.5 to the above bracket, we find that

$$\begin{aligned} \phi(x_1, x_2, x_3) = \\ - \frac{1}{4\pi} \frac{\partial}{\partial x_3} \int_0^\infty \int_0^\infty f(y_1, y_2) \frac{\exp(-ikR)}{R} dy_1 dy_2 \end{aligned} \quad (7.7)$$

where

$$R = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2]^{\frac{1}{2}} \quad (7.8)$$

and  $f = \mathcal{L}_1^{-1}[F]$ . The physical interpretation of (7.7) is that the diffracted field results from a distribution of point sources with density  $f(x)$  over  $Q_1$ . Also, in view of (7.8),  $\phi$  is outgoing at infinity.

If  $x_3 = 0$  and  $x_1 \in Q_1$ , then (7.7) takes the form

$$\iint_{Q_1} f(y) \mathcal{K}(x - y) dy_1 dy_2 = g(x) \quad (7.9)$$

where

$$\mathcal{K}(x) = - \frac{1}{4\pi} \frac{\partial}{\partial x_3} \left[ \frac{\exp(-ikr)}{r} \right] \Big|_{x_3=0} \quad (7.10)$$

with  $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$  and  $g(x) = a_3 \exp(-a \cdot x)$ . Douglas and Howe [5] have recently shown that a Wiener-Hopf operator on the quarter-plane is invertible if its symbol doesn't vanish. The symbol in our case is simply the two-dimensional Laplace transform of the kernel  $\mathcal{K}(x)$  of (7.10). Since

$$\mathfrak{L}\left[-\frac{1}{4\pi} \frac{\partial}{\partial x_3} \frac{\exp(-ikr)}{r}\right] = \exp[x_3/2K(s)],$$

we see that  $\mathfrak{L}[\mathcal{K}(x)]$  does not vanish in  $\mathbb{T}(D)$ .

Now suppose that  $F_0(s)$  is another solution of our transform equation (5.1) and let  $f_0(x)$ , correspondingly, be a second solution of integral equation (7.9). If we set  $h(x) = f(x) - f_0(x)$ , then

$$\iint_{Q_1} h(y)\mathcal{K}(x-y)dy_1dy_2 = 0$$

and since our Wiener-Hopf operator is invertible we conclude that  $h(x) = 0$ . Thus our solution is unique.

## SECTION VIII

## COMPARISON WITH HALF-PLANE SOLUTIONS

The purpose of this section is to show that our diffraction problem for the quarter-plane is asymptotically equivalent (when  $x_2$  approaches infinity) to diffraction by the perfectly reflecting half-plane  $x_1 \geq 0$ . The corresponding result for the half-plane  $x_2 \geq 0$  is also true. We treat the case of the half-plane  $x_1 \geq 0$  in detail and indicate the results for the half-plane  $x_2 \geq 0$ .

Consider a plane wave (3.1) which is incident on the half-plane  $H_1: x_1 \geq 0, -\infty < x_2 < \infty, x_3 = 0$ . The field  $\phi_1$  scattered by this half-plane is clearly of the form

$$\phi_1(x_1, x_2, x_3) = \exp(-a_2 x_2) \psi_1(x_1, x_3)$$

where  $\psi_1$  is the solution of the following boundary value problem:

$$\frac{\partial^2 \psi_1}{\partial x_1^2} + \frac{\partial^2 \psi_1}{\partial x_3^2} + k_1^2 \psi_1 = 0, \quad (8.1)$$

for  $-\infty < x_1 < \infty, x_3 > 0, k_1^2 = k^2 + a_2^2, \text{Im}k_1 = -q_1 < 0;$

$$\left. \frac{\partial \psi_1}{\partial x_3} \right|_{x_3=0} = a_3 \exp(-a_1 x_1) \quad (8.2)$$

for  $x_1 \geq 0;$

$$\psi_1(x_1, 0) = 0 \quad (8.3)$$

for  $x_1 < 0$ ;

$$\int_0^{\infty} |\psi_1(x_1, 0)|^m \exp(-u_1 x_1) dx_1 < \infty \quad (8.4)$$

for  $u_1 > -\operatorname{Re} a_1$ ,  $m = 1, 2$ ;

$$\int_{-\infty}^0 \left| \frac{\partial \psi_1}{\partial x_3}(x_1, 0) \right|^2 \exp(-u_1 x_1) dx_1 < \infty \quad (8.5)$$

for  $u_1 > -\operatorname{Re} a_1 \cap |u_1| < q_1$ .

The Sommerfeld radiation condition for  $\psi_1$  is

$$\lim_{r_1 \rightarrow \infty} \sqrt{r_1} \left( \frac{\partial \psi_1}{\partial r_1} + ik\psi_1 \right) = 0,$$

where  $r_1 = (x_1^2 + x_3^2)^{\frac{1}{2}}$ .

The above boundary problem may be solved using the one variable Wiener-Hopf method. We include some of these details. The wave equation (8.1) possesses separation solutions given by

$$\psi_1(x_1, x_3; s_1) = \exp[s_1 x_1 + x_3/2K(s_1, -a_2)] \quad (8.6)$$

and upon multiplying the right side of (8.6) by  $F_+(s_1)/2\pi$  and using superposition, we obtain

$$\begin{aligned} \psi_1(x_1, x_3) = \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} F_+(s_1) \exp[s_1 x_1 + x_3/2K(s_1, -a_2)] dv_1. \end{aligned} \quad (8.7)$$

This formal solution of (8.1) will satisfy boundary condition (8.2) provided that

$$\begin{aligned} \left. \frac{\partial \psi_1}{\partial x_3} \right|_{x_3=0} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_+(s_1)}{2K(s_1, -a_2)} \exp(s_1 x_1) dv_1 \\ &= a_3 \exp(-a_1 x_1) \end{aligned} \quad (8.8)$$

for  $x_1 > 0$ . Also boundary condition (8.3) will be satisfied if

$$\psi_1(x_1, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_+(s_1) \exp(s_1 x_1) dv_1 = 0 \quad (8.9)$$

for  $x_1 < 0$ .

Since

$$F_+(s_1) = \mathcal{L}_+[\psi_1(x_1, 0)] = \int_0^{\infty} \psi_1(x_1, 0) \exp(-s_1 x_1) dx_1,$$

it follows from condition (8.4) that  $F_+(s_1)$  is analytic and bounded in  $L_m$ -norm for  $u_1 > -\text{Re} a_1$ . Moreover by (8.5),  $F_+(s_1)/2K(s_1, -a_2)$  is analytic and bounded in  $L_2$ -norm for  $u_1 > -\text{Re} a_1 \cap |u_1| < q_1$ . Therefore, Cauchy's formula may be used to obtain the unique additive decomposition

$$\frac{F_+(s_1)}{2K(s_1, -a_2)} = G_+(s_1) + G_-(s_1) \quad (8.10)$$

where  $G_+(s_1)$  is analytic for  $u_1 > -\text{Re} a_1$  and  $G_-(s_1)$  is analytic for  $u_1 < q_1$ . However, by (8.8) and (8.2)



$$\begin{aligned} \frac{F_+(s_1)}{2K(s_1, -a_2)} &= \int_{-\infty}^{\infty} \frac{\partial \psi_1}{\partial x_3} (x_1, 0) \exp(-s_1 x_1) dx_1 \\ &= a_3 (s_1 + a_1)^{-1} + \int_{-\infty}^0 \frac{\partial \psi_1}{\partial x_3} (x_1, 0) \exp(-s_1 x_1) dx_1 \end{aligned}$$

whence

$$G_+(s_1) = a_3 (s_1 + a_1)^{-1}.$$

Thus our transform equation (8.10) becomes

$$\frac{F_+(s_1)}{2K(s_1, -a_2)} = a_3 (s_1 + a_1)^{-1} + G_-(s_1) \quad (8.11)$$

where the unknown functions  $F_+(s_1)$  and  $G_-(s_1)$  are analytic for  $u_1 > -\text{Re} a_1$  and  $u_1 < q_1$ , respectively, and  $F_+(s_1)$  is bounded in  $L_m$ -norm for  $u_1 > -\text{Re} a_1$ .

The factorization of  $K(s)$  needed to solve (8.11) is decidedly simpler than the factorization required in the case of the quarter-plane. In the present problem we factor  $K(s)$  as

$$\begin{aligned} K(s) &= \frac{1}{2} K_+(s) K_-(s) \\ &= \frac{1}{2} [s_1 + i(s_2^2 + k^2)^{\frac{1}{2}}]^{-\frac{1}{2}} [s_1 - i(s_2^2 + k^2)^{\frac{1}{2}}]^{-\frac{1}{2}} \end{aligned} \quad (8.12)$$

whereas in the case of the half-plane  $H_2: -\infty < x_1 < \infty, x_2 \geq 0, x_3 = 0$  we would factor  $K(s)$  as

$$\begin{aligned} K(s) &= \frac{1}{2} K^+(s) K^-(s) \\ &= \frac{1}{2} [s_2 + i(s_1^2 + k^2)^{\frac{1}{2}}]^{-\frac{1}{2}} [s_2 - i(s_1^2 + k^2)^{\frac{1}{2}}]^{-\frac{1}{2}}. \end{aligned} \quad (8.13)$$

In (8.12),  $s_2$  is fixed in the strip  $|u_2| < q$  while in (8.13),  $s_1$  is fixed in the strip  $|u_1| < q$ .

We now use (8.12) to rewrite (8.11) as

$$\frac{F_+(s_1)}{iK_+(s_1, -a_2)} = a_3 \frac{K_-(s_1, -a_2)}{(s_1 + a_1)} + K_-(s_1, -a_2)G_-(s_1). \quad (8.14)$$

The additive decomposition

$$\frac{K_-(s_1, -a_2)}{(s_1 + a_1)} = \frac{K_-(a_1, -a_2)}{(s_1 + a_1)} + \frac{K_-(s_1, -a_2) - K_-(a_1, -a_2)}{(s_1 + a_1)}$$

allows us to rewrite (8.14) as

$$\begin{aligned} E(s_1) &= \frac{F_+(s_1)}{iK_+(s_1, -a_2)} - a_3 \frac{K_-(a_1, -a_2)}{(s_1 + a_1)} \\ &= \frac{K_-(s_1, -a_2) - K_-(a_1, -a_2)}{(s_1 + a_1)} + K_-(s_1, -a_2)G_-(s_1). \end{aligned} \quad (8.15)$$

The second part of (8.15) is analytic for  $u_1 > \alpha_1 = \max\{-q_1, -\text{Re}a_1\}$  while the third part is analytic for  $u_1 < \beta_1 = q_1$  and this equation holds in the strip  $\alpha_1 < u_1 < \beta_1$ . Hence by analytic continuation  $E(s_1)$  is an entire function. Furthermore  $E(s_1)$  is bounded and  $E(s_1) \rightarrow 0$  as  $u_1 \rightarrow \pm\infty$ . By Liouville's theorem,  $E(s_1) = 0$ . Consequently,

$$F_+(s_1) = ia_3 K_+(s_1, -a_2) K_-(a_1, -a_2) (s_1 + a_1)^{-1}. \quad (8.16)$$

The solution for the half-plane  $H_1$  is therefore

$$\begin{aligned} \phi_1(x_1, x_2, x_3) &= \\ &= \frac{\exp(-a_2 x_2)}{2\pi} \int_{-\infty}^{\infty} F_+(s_1) \exp[s_1 x_1 + x_3/2K(s_1, -a_2)] dv_1 \end{aligned} \quad (8.17)$$

with  $F_+(s_1)$  given as in (8.16) and the corresponding solution for the half-plane  $H_2$  is

$$\begin{aligned} \phi_2(x_1, x_2, x_3) = & \\ & \frac{\exp(-a_1 x_1)}{2\pi} \int_{-\infty}^{\infty} F^+(s_2) \exp[s_2 x_2 + x_3/2K(-a_1, s_2)] dv_2 \end{aligned} \quad (8.18)$$

where

$$F^+(s_2) = ia_3 K^+(-a_1, s_2) K^-(-a_1, -a_2) (s_2 + a_2)^{-1}.$$

These solutions of the two half-plane problems may be converted into expressions analogous to (7.7). Since

$$\begin{aligned} -\frac{1}{2} \int_{-\infty}^{\infty} \exp(-s_1 x_1) H_0^{(2)}[k_1(x_1^2 + x_3^2)^{\frac{1}{2}}] dx_1 = \\ 2K(s_1, -a_2) \exp[x_3/2K(s_1, -a_2)] \end{aligned}$$

we have by the one variable convolution theorem that

$$\begin{aligned} \phi_1(x_1, x_2, x_3) = & \\ & \frac{1}{4} \exp(-a_2 x_2) \frac{\partial}{\partial x_3} \int_0^{\infty} f_1(y_1) H_0^{(2)}\{k_1[(y_1 - x_1)^2 + x_3^2]^{\frac{1}{2}}\} dy_1 \end{aligned} \quad (8.19)$$

with  $f_1(y_1)$  denoting the inverse transform of  $F_+(s_1)$ .

The integral representation

$$\begin{aligned} \exp(-a_2 x_2) H_0^{(2)}\{k_1[(y_1 - x_1)^2 + x_3^2]^{\frac{1}{2}}\} = \\ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\exp(a_2 y_2 - ikR)}{R} dy_2 \end{aligned}$$

where  $R$  is given by (7.8) allows us to rewrite (8.19) as

$$\begin{aligned} \phi_1(x_1, x_2, x_3) = \\ - \frac{1}{4\pi} \frac{\partial}{\partial x_3} \int_{-\infty}^{\infty} \int_0^{\infty} g_1(y_1, y_2) \frac{\exp(-ikR)}{R} dy_1 dy_2 \end{aligned}$$

with  $g_1(y_1, y_2) = \exp(-a_2 y_2) f_1(y_1)$ . Thus, as in the case of the quarter-plane, the field produced from scattering by the half-plane  $H_1$  results from a distribution of point sources with density  $g_1(x_1, x_2)$  over  $H_1$ . By symmetry, (8.18) may be written as

$$\begin{aligned} \phi_2(x_1, x_2, x_3) = \\ - \frac{1}{4\pi} \frac{\partial}{\partial x_3} \int_0^{\infty} \int_{-\infty}^{\infty} g_2(y_1, y_2) \frac{\exp(-ikR)}{R} dy_1 dy_2 \end{aligned}$$

where  $g_2(y_1, y_2) = \exp(-a_1 y_1) f_2(y_2)$  and  $f_2(y_2)$  is the inverse transform of  $F^+(s_2)$ .

We are now in a position to show that the field scattered by the quarter-plane  $Q_1$  behaves for large positive  $x_2$  like the field scattered by the half-plane  $H_1$ . That is, we must show that

$$f(x_1, x_2) \sim \exp(-a_2 x_2) f_1(x_1)$$

as  $x_2 \rightarrow +\infty$  which is the same as

$$\lim_{x_2 \rightarrow +\infty} \exp(a_2 x_2) f(x_1, x_2) = f_1(x_1). \quad (8.20)$$

To verify (8.20) we first prove that it is equivalent to

$$\lim_{s_2 \rightarrow -a_2} (s_2 + a_2) F(s) = F_+(s_1). \quad (8.21)$$

Suppose that (8.20) holds and set

$$F(x_1; s_2) = \int_0^{\infty} f(x_1, x_2) \exp(-s_2 x_2) dx_2.$$

By (2.16),

$$\begin{aligned} \lim_{s_2 \rightarrow +\infty} \exp(a_2 x_2) f(x_1, x_2) &= \\ \lim_{s_2 \rightarrow -a_2} (s_2 + a_2) F(x_1; s_2). & \end{aligned} \quad (8.22)$$

Now multiply both sides of (8.22) by  $\exp(-s_1 x_1)$ , integrate from 0 to  $\infty$ , and use (8.20) to obtain

$$\begin{aligned} \lim_{s_2 \rightarrow -a_2} (s_2 + a_2) F(s) &= \\ \int_0^{\infty} f_1(x_1) \exp(-s_1 x_1) dx_1 &= F_+(s_1). \end{aligned}$$

Since the above steps are reversible (take an inverse transform in  $s_1$  and then use (8.22)), the equivalence of (8.20) and (8.21) is established.

To show that (8.21) (and hence (8.20)) holds we observe that straightforward calculations give

$$\begin{aligned} K_1(s)K_4(s) &= K_+(s), \\ K_2(s)K_3(s) &= K_-(s), \\ K_1(s)K_2(s) &= K^+(s), \\ K_3(s)K_4(s) &= K^-(s). \end{aligned} \quad (8.23)$$

Making use of the first two equations in (8.23), we have

$$\lim_{s_2 \rightarrow -a_2} (s_2 + a_2)F(s) =$$

$$ia_3 K_1(s_1, -a_2) K_2(-a_1, -a_2) K_3(-a_1, -a_2) K_4(s_1, -a_2) (s_1 + a_1)^{-1} =$$

$$ia_3 K_+(s_1, -a_2) K_-(-a_1, -a_2) (s_1 + a_1)^{-1} = F_+(s_1)$$

as desired. A similar calculation which makes use of the last two equations in (8.23) gives the corresponding result for the half-plane  $H_2$ :

$$\lim_{x_1 \rightarrow +\infty} \exp(a_1 x_1) f(x_1, x_2) = f_2(x_2).$$

## SECTION IX

## BEHAVIOR AT THE CORNER

In this section, we investigate the radial behavior of our solution (7.1) near the origin. Since the right hand side of (7.1) still represents the solution for  $u_j = 0$ , we may write

$$\begin{aligned} \phi(x_1, x_2, x_3) = \\ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(v) \exp[iv \cdot x - x_3 M(v)] dv_1 dv_2 \end{aligned} \quad (9.1)$$

where  $M(v) = (v_1^2 + v_2^2 - k^2)^{\frac{1}{2}}$  with choice of branch determined by  $\text{Re}M(v) > 0$ . Introducing the spherical coordinates

$$x_1 = r \sin \alpha \cos \beta, \quad x_2 = r \sin \alpha \sin \beta, \quad x_3 = r \cos \alpha$$

( $0 < \alpha < \pi/2$ ,  $0 < \beta < 2\pi$ ) and the polar coordinates

$$v_1 = \rho \cos \gamma, \quad v_2 = \rho \sin \gamma$$

( $0 < \gamma < 2\pi$ ) gives the relations

$$x_1 + ix_2 = r \sin \alpha e^{i\beta}, \quad v_1 + iv_2 = \rho e^{i\gamma}. \quad (9.2)$$

Substitution of (9.2) into (9.1) yields

$$\phi(r, \alpha, \beta) = \frac{1}{(2\pi)^2} \cdot$$

$$\int_0^{\infty} \int_0^{2\pi} F(\rho, \gamma) \exp[i\rho r \sin \alpha \cos(\beta - \gamma) - r \cos \alpha (\rho^2 - k^2)^{\frac{1}{2}}] \rho d\gamma d\rho.$$

Now fix  $\alpha$  and define the mean values

$$\begin{aligned}\phi_m(r, \alpha) &= \frac{1}{2\pi} \int_0^{2\pi} \phi(r, \alpha, \beta) d\beta, \\ F_m(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} F(\rho, \gamma) d\gamma.\end{aligned}\quad (9.3)$$

Using Hansen's integral (see [16], p. 20), that is, the result

$$\frac{1}{2\pi} \int_0^{2\pi} \exp[i\lambda \cos(\beta-\gamma)] d\beta = J_0(\lambda)$$

we have that the two mean values are related by

$$\begin{aligned}\phi_m(r, \alpha) &= \\ &\frac{1}{2\pi} \int_0^\infty F_m(\rho) \exp[-r \cos \alpha (\rho^2 - k^2)^{\frac{1}{2}}] J_0(\rho r \sin \alpha) \rho d\rho.\end{aligned}\quad (9.4)$$

The change of variable  $t = \rho r$  in (9.4) gives

$$\begin{aligned}r^2 \phi_m(r, \alpha) &= \\ &\frac{1}{2\pi} \int_0^\infty F_m\left(\frac{t}{r}\right) \exp[-\cos \alpha (t^2 - r^2 k^2)^{\frac{1}{2}}] J_0(t \sin \alpha) t dt.\end{aligned}\quad (9.5)$$

It is clear from (9.5) that

$$\begin{aligned}r^2 \phi_m(r, \alpha) &\sim \\ &\frac{1}{2\pi} F_m\left(\frac{1}{r}\right) \int_0^\infty t \exp(-t \cos \alpha) J_0(t \sin \alpha) dt\end{aligned}\quad (9.6)$$

as  $r \rightarrow 0$ . But since the one variable Laplace transform of  $t J_0(at)$  is  $s/(s^2 + a^2)^{3/2}$ , (9.6) becomes

$$r^2 \phi_m(r, \alpha) \sim \frac{\cos \alpha}{2\pi} F_m\left(\frac{1}{r}\right) \quad (9.7)$$



as  $r \rightarrow 0$ . If we set

$$\phi_m(r) = \frac{2}{\pi} \int_0^{\pi/2} \phi_m(r, \alpha) d\alpha$$

and average each side of (9.7) over the interval  $0 < \alpha < \pi/2$ , we obtain

$$r^2 \phi_m(r) \sim \frac{1}{\pi^2} F_m\left(\frac{1}{r}\right) \quad (9.8)$$

as  $r \rightarrow 0$ .

Growth estimate (6.2) yields

$$P(\rho, \beta) = o(\rho^{-3/4})$$

and clearly

$$Q_1(\rho, \beta) = o(\rho^2)$$

so that

$$F(\rho, \beta) = o(\rho^{-11/4}) \quad (9.9)$$

for large  $\rho$ . In view of (9.3) and (9.9)

$$F_m\left(\frac{1}{r}\right) = o(r^{11/4}) \quad (9.10)$$

for small  $r$ . Since  $\phi_m(r) \rightarrow \phi$  as  $r \rightarrow 0$ , (9.8) and (9.10) give

$$\phi = o(r^{3/4}) \quad (9.11)$$

and

$$\frac{\partial \phi}{\partial x_3} = o(r^{-1/4}) \quad (9.12)$$

as  $r \rightarrow 0$ .

Estimates (9.11) and (9.12) tell us that the scattered wave remains finite near the corner whereas its normal derivative does not. In contrast, the exponents given in [13] for diffraction by a perfectly absorbent quarter-plane are  $1/4$  for the scattered wave and  $-3/4$  for its normal derivative.

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