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WALLMAN-TYPE COMPACTIFICATIONS

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by

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ABSTRACT

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This thesis investigates the general problem: Can every Hausdorff compactification of a completely regular Hausdorff space be constructed by a certain generalization of Wallman's method? Essentially, chapters I, II and III are concerned with the development and general theory of Wallman compactifications, while chapters IV and V deal with applications to particular compactifications. Chapter VI summarizes the thesis and presents some topics for further investigation. We discuss the chapters more specifically in the following paragraphs.

In chapter I we present a brief historical background along with the needed topological preliminaries.

Chapter II contains an outline of Wallman's method as well as necessary and sufficient conditions in order that a given compactification be Wallman.

Chapter III is concerned with the construction of Wallman bases on compact spaces and then with the restriction of these bases to a given dense subspace. We are, of course, interested in when Wallman's method applied to this restricted Wallman base gives the original compactification. We have settled this question with the theorem: Let T be a compactification of X and L a Wallman base on T. Then $T = wL_X$ iff $\overline{A}^T \cap \overline{B}^T = \emptyset$ whenever $A, B \in L$ and $A \cap B = \emptyset$, where $L_X = \{A \cap X | A \in L\}$. We conclude the chapter with various results that can be derived from this theorem.

Chapter VI is concerned with special kinds of Wallman bases and special kinds of compactifications. We show that every orderable compact space is a Wallman compactification of each of its dense subspaces and that every compact F-space is a Z-compactification of each of its dense subspaces by means of a Wallman base of zero sets. These Wallman bases of zero sets of continuous functions on a compact space are investigated in some detail. We mention the following result: Let T be a compact space and L a Wallman base on T. Then $T = wL_X$ for each dense subset X of T iff $T = wL_Y$ for each dense co-L set Y in T. Various applications are derived from this theorem; for example, we use this theorem to give a Wallman-type characterization of pseudocompact spaces.

Chapter V is a study of a certain kind of Wallman base. In particular, if αX and γX are compactifications of X where $\alpha X = wL$ for some Wallman base L on X and $h:\alpha X \to \gamma X$ is the canonical map, then define $L(h) = \{\overline{A}^{\gamma X} | A \in L.\ h^{-1}[\overline{A}^{\gamma X}] = \overline{A}^{\alpha X}\}$. We investigate when L(h) is a Wallman base on γX and, moreover, when $\gamma X = wL(h)$. Included in this investigation is a new proof of the theorem: Any compactification of the space X having at most a countable number of multiple points with respect to the Stone-Cech compactification of X is a Z-compactification. In particular, any countable point compactification is a Z-compactification. We conclude the chapter with the theorem: Every compact space is regular Wallman iff the Stone-Cech compactification of every locally compact space is regular Wallman.

CHAPTER I

INTRODUCTION

Historical Background and Purpose

The subject of Wallman-type compactifications was first initiated by Henry Wallman in 1938 [32]. His classic contribution is essentially summarized in Kelley's General Topology [19, page 167, problem R]. It would be difficult to attribute the modern interest in this subject to any one person, but a rash of papers on Wallman-type compactifications has appeared since the elegant paper by Frink [10] in 1964. In this paper Frink gives a neat internal characterization of Tychonoff spaces by means of Wallman bases. The Wallman base concept has also been successfully used by Alo and Shapiro [2], Brooks [6] and Steiner [28] to obtain further results. Preceding Frink is the interesting paper by Banaschewski [5] in 1963 which contains a summary of what we might consider the folklore about Wallman compactifications. Another important work is the text Rings of Continuous Functions by Gillman and Jerison [11], published in 1960, which exploits Wallman's technique in constructing the Stone-Cech compactification.

The chief motivation for this paper is to answer the question: Can every Hausdorff compactification of a completely regular Hausdorff space be constructed by Wallman's method? This general question still remains unsolved, but this paper provides some partial solutions and related results.

Topological Preliminaries

All topological spaces in this thesis are assumed to be completely regular and Hausdorff, unless otherwise specified. The reason for this is given in the body of the thesis. We shall be primarily concerned with extensions and compactifications of a given topological space.

Let X be a topological space. The space T is an extension of X (denoted $T \in eX$) means there exists a homeomorphism h from X into T such that h[X] is dense in T. The function h is called an embedding map, or simply an embedding, of X into T. T is a compactification of X (denoted $T \in cX$) means that T is a compact extension of X.

When we refer to an extension T of the space X, we shall simply take X as a subspace of T. This is justified since if T is an extension of X, where h embeds X into T, then there exists a space T' such that X is dense in T' and h can be extended to a homeomorphism from T' onto T. Occasionally in this thesis the necessary embedding maps and homeomorphisms will be explicitly mentioned, but usually they will merely be tacitly assumed.

Let X be a topological space. Let T_1 and T_2 be compactifications of X. T_1 is equivalent to T_2 as a compactification of X (denoted $T_1 = T_2$) means there exists a homeomorphism h between T_1 and T_2 such that $h(x) = x$ for each $x \in X$.

Let X be a topological space and A a subset of X. Then $\overline{A}^X$ denotes the closure of A in X and $\text{int}_X A$ denotes the interior of A in X. The subset A of X is regular closed in X means that A is the closure of its interior.

The notation βX is reserved for the Stone-Cech compactification of X.

CHAPTER II

WALLMAN-TYPE COMPACTIFICATIONS

Introduction

In this section we present the basic concepts which will be used throughout this paper.

Let X be a set. Then L is a lattice on X means L is a collection of subsets of X such that

(i) $\emptyset, X \in L$;

(ii) if $A, B \in L$, then $A \cap B \in L$ and $A \cup B \in L$.

We shall refer to an element of L as an L-set. For any lattice L on X an L-filter is a non-void subset F of L with the following properties:

(i) $\emptyset \notin F$;

(ii) if $A, B \in F$, then $A \cap B \in F$;

(iii) if $A \in F$, $B \in L$ and $A \subset B$, then $B \in F$.

An L-ultrafilter is a maximal (with respect to inclusion) L-filter. The set of all L-ultrafilters is denoted by wL.

Let L be a lattice on the set X. In order to introduce a topology for wL, define $A^* = \{U \in wL | A \in U\}$ for each $A \in L$. It is a straightforward exercise to show that $\emptyset^* = \emptyset$, $X^* = wL$, $(A \cap B)^* = A^* \cap B^*$ and that $(A \cup B)^* = A^* \cup B^*$. Hence, $\{A^* | A \in L\}$ is a base for the closed sets of some (necessarily unique) topology T for wL. We shall denote the space (wL,T) simply as wL (since this is the only topology on wL we shall consider in this paper). The space wL is a compact T_1-space [24]. The space wL has been referred to as the space of L-ultrafilters [6] or as the Wallman space determined by L [28].

Let X be a topological space and L a lattice on X. We now seek conditions under which wL is a compactification of X. The lattice L is a Wallman base on X means

(i) L is a base for the closed subsets of X;

(ii) L is a disjunctive lattice on X (i.e., for each $A \varepsilon L$ and $x \varepsilon X - A$ there exists $B \varepsilon L$ such that $x \varepsilon B$ and $A \cap B = \emptyset$;

(iii) L is a normal lattice on X (i.e., for each $A,B \varepsilon L$, if A and B are disjoint, then there exists $C,D \varepsilon L$ such that $A \subset X - C$, $B \subset X - D$ and $C \cup D = X$).

We observe that conditions (i) and (ii) above are equivalent to the following: each L set is closed and L separates points and closed sets (i.e., if F is a closed subset of X and $x \varepsilon X - F$, then there exists $A,B \varepsilon L$ such that $F \subset A$, $x \varepsilon B$ and $A \cap B = \emptyset$).

Let X be a topological space and L be a lattice on X. For each $x \varepsilon X$ define $\phi(x) = \{A \varepsilon L | x \varepsilon A\}$. It follows that wL is a compactification of X (by means of the embedding map ϕ) iff L is a Wallman base on X (see [6] or [28] for a complete proof). This procedure of constructing compactifications is called Wallman's method of compactification. Hence, a compactification T of a space X is a Wallman compactification of X (denoted by $T \varepsilon wX$) means there exists a Wallman base L on X such that $T = wL$.

Steiner has shown that no generalization of Wallman's method can be obtained by replacing a Wallman base on a topological space X by an arbitrary collection S of closed subsets of X and forming the ultrafilter space determined by S. In fact, if L is the smallest lattice on X which contains S, then wL = wS [28, lemma 1].

Steiner [28] has also shown that a T_1-compactification of a (arbitrary) topological space need not be Wallman. On the other hand, Frink [10] has shown that a (arbitrary) topological space has a Wallman compactification iff it is completely regular and Hausdorff. Accordingly all topological spaces in this paper are assumed to be completely regular and Hausdorff.

Relationships between L and wL

We now consider necessary and sufficient conditions for a compactification to be Wallman. To facilitate the discussion we introduce the following notation. Let X be a topological space, L a lattice on X, T a compactification of X and U and L-ultrafilter. Then U converges to $t \in T$ (denoted $U \to t$) means $\cap\{\overline{A}^T | A \in U\} = \{t\}$. Define $U_t = \{A \in L | t \in \overline{A}^T\}$ for each $t \in T$. Obviously if $U \to t$, then $U \subset U_t$.

2.1 THEOREM. Let $T \in cX$ and L a Wallman base on X. Then $T = wL$ iff $wL = \{U_t | t \in T\}$ and $U_t \to t$ for each $t \in T$.

Proof: We prove this theorem using a series of lemmas.

Lemma A. Let L be a lattice on the space X and $T \in eX$. Then $\{A \in L | x \in A\} = \{A \in L | x \in \overline{A}^T\}$ for each $x \in X$ if each L-set is closed in X.

Proof: The proof is straight-forward.

Lemma B. Let $T \in eX$ and L a lattice on X. Then

(i) $U_t \in$ L-filter for each $t \in T$ iff for each $A,B \in L$ we have $\overline{A \cap B}^T = \overline{A}^T \cap \overline{B}^T$.

(ii) $U_t \in wL$ for each $t \in T$ iff

(a) $U_t \in$ L-filter for each $t \in T$, and

(b) if $A \in L$ and $t \in T - \overline{A}^T$, then there exists $B \in L$ such that $t \in \overline{B}^T$ and $A \cap B = \emptyset$.

(iii) For each $t \in T$, $U_t \in wL$ and $U_t \to t$ iff

(a) $U_t \in wL$ for each $t \in T$, and

(b) if $p,q \in T$ where $p \neq q$, then there exists $A \in L$ such that $p \in \overline{A}^T$ and $q \notin \overline{A}^T$.

Proof of (i): Suppose U_t is an L-filter for each $t \in T$. Let $A,B \in L$. Clearly $\overline{A \cap B}^T \subset \overline{A}^T \cap \overline{B}^T$. We show $\overline{A}^T \cap \overline{B}^T \subset \overline{A \cap B}^T$. If $\overline{A}^T \cap \overline{B}^T = \emptyset$, then obviously $\overline{A \cap B}^T = \overline{A}^T \cap \overline{B}^T$. Suppose $t \in \overline{A}^T \cap \overline{B}^T$. Then $A,B \in U_t$, so $A \cap B \in U_t$ and $t \in \overline{A \cap B}^T$.

Suppose $\overline{A \cap B}^T = \overline{A}^T \cap \overline{B}^T$ for all $A,B \in L$. Let $t \in T$. Now $X \in U_t$, so $U_t \neq \emptyset$. Clearly $\emptyset \notin U_t$ and if $A \in U_t$, $B \in L$ such that $A \subset B$, then $B \in U_t$. If $A,B \in U_t$, then $A \cap B \in U_t$ by assumption. This proves (i).

Proof of (ii): Suppose $U_t \in wL$ for each $t \in T$. Clearly (a) holds. Let $A \in L$ and $t \in T - \overline{A}^T$. Then $A \notin U_t$, so there exists $B \in U_t$ such that $A \cap B = \emptyset$ since $U_t \in wL$.

Suppose conditions (a) and (b) hold. Let $A \in L - U_t$. Then $t \in T - \overline{A}^T$. By (b) there exists $B \in L$ such that $t \in \overline{B}^T$ and $A \cap B = \emptyset$. So $B \in U_t$ and $A \cap B = \emptyset$. Hence, $U_t \in wL$. This proves (ii).

Proof of (iii): Suppose $U_t \in wL$ and $U_t \in t$ for each $t \in T$. Clearly (a) holds. Let $p,q \in T$ where $p \neq q$. Then $U_p \neq U_q$, so there exists $A \in U_p$, $B \in U_q$ such that $A \cap B = \emptyset$. Thus $p \in \overline{A}^T$ and $q \notin \overline{A}^T$.

Suppose conditions (a) and (b) hold. Let $t \in T$. Clearly $U_t \in wL$ for each $t \in T$. Suppose $p \in \cap\{\overline{A}^T | A \in U_t\}$ and $p \neq t$. Then there exists $B \in L$ such that $t \in \overline{B}^T$ and $p \notin \overline{B}^T$ using (b). But then $B \in U_t$ since $t \in \overline{B}^T$ and $p \in \overline{B}^T$ since $p \in \cap\{\overline{A}^T | A \in U_t\}$. This can not happen, so $p = t$ and $U_t \to t$. This proves (iii) and completes the proof of the lemma.

<u>Lemma C</u>. Let $T \in cX$ and L a lattice on X. Then $wL = \{U_t | t \in T\}$ iff $U_t \in wL$ for each $t \in T$.

Proof: Suppose $U_t \in wL$ for each $t \in T$. We show $wL \subset \{U_t | t \in T\}$. Let $U \in wL$. Then $\cap\{\overline{A}^T | A \in U\} \neq \emptyset$ since T is compact. Let $t \in \cap\{\overline{A}^T | A \in U\}$. Then $U \subset U_t$. Hence, $U = U_t$ since U is a maximal L-filter. The converse is obvious. This completes the proof of the lemma.

The proof of the theorem now follows readily. If $T = wL$, then conditions (i) and (ii) of the theorem hold by lemma B (iii). Conversely, suppose that conditions (i) and (ii) hold. Define $h:T \to wL$ by $h(t) = U_t$. Easily h is a bijection and $h^{-1}[A^*] = \overline{A}^T$ for each $A \in L$ (recall that $A^* = \{U \in L | A \in U\}$). Thus, the inverse image of every basic closed set in wL is closed in T; i.e., h is continuous. Since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, h is a homeomorphism. Hence, $T = wL$. This completes the proof of the theorem.

Summarizing, if $T = wL$ where L is a Wallman base on the space X, we then have the following. Define the mapping $\phi:X \to wL$ by $\phi(x) = U_x$ ($= \{A \in L | x \in A\}$). Then $wL \in cX$ by means of the embedding map ϕ. Since $wL = \{U_t | t \in T\}$, define $h:wL \to T$ by $h(U_t) = t$. Then h is a homeomorphism and $h(\phi(x)) = x$ for each $x \in X$.

Theorem 2.1 provides necessary and sufficient conditions for the compactification T of the space X to be Wallman in terms of the ultrafilter space wL. This is in contrast to the following theorem of Brooks [6, p161] which gives necessary and sufficient conditions for the compactification T of the space X to be Wallman in terms of the Wallman base L.

2.2 THEOREM (Brooks). Let $T \in cX$ and L a Wallman base on X. Then $T = wL$ iff

(i) $\overline{A \cap B}^T = \overline{A}^T \cap \overline{B}^T$ for each $A,B \in L$;

(ii) if $A \in L$ and $t \in T - \overline{A}^T$, then there exists $B \in L$ such that $t \in \overline{B}^T$ and $A \cap B = \emptyset$;

(iii) if $p,q \in T$ where $p \neq q$, then there exists $A \in L$ such that $p \in \overline{A}^T$ and $q \notin \overline{A}^T$.

We now establish a proposition which will be useful in applying the theorems of this section.

2.3 THEOREM. Let $T \in cX$ and L a lattice on X. Suppose for each $t \in T$ there exists $U \in wL$ such that $U \to t$. Then $\overline{A}^T \cap \overline{B}^T = \emptyset$ whenever $A,B \in L$ and $A \cap B = \emptyset$ iff $\overline{A \cap B}^T = \overline{A}^T \cap \overline{B}^T$ for each $A,B \in L$.

Proof: Suppose $\overline{A}^T \cap \overline{B}^T = \emptyset$ whenever $A,B \in L$ and $A \cap B = \emptyset$. Let $A,B \in L$. It suffices to show that $\overline{A}^T \cap \overline{B}^T \subset \overline{A \cap B}^T$. This is obvious if $\overline{A}^T \cap \overline{B}^T = \emptyset$. Suppose $t \in \overline{A}^T \cap \overline{B}^T$. Select $U \in wL$ such that $U \to t$. We show that $A \in U$. If not, there exists $C \in U$ such that $A \cap C = \emptyset$ [6, 2.1]. Then $\overline{A}^T \cap \overline{C}^T = \emptyset$ by hypothesis. But $t \in \overline{A}^T \cap \overline{C}^T$ since $C \in U$ and $U \to t$. Hence, we must have $A \in U$. Similarly $B \in U$. So $A \cap B \in U$ and thus we have $t \in \overline{A \cap B}^T$. This completes the proof since the converse is obvious.

CHAPTER III

WALLMAN BASES FOR COMPACT SPACES

Let T be a compactification of X. We will show that the search for a Wallman base L on X such that T = wL may be reduced to consideration of Wallman bases on T. As an auxiliary result we find necessary and sufficient conditions that $T = wL_X$ where $L_X = \{A \cap X | A \in L\}$ for some Wallman base L on T.

As preliminary information we seek a simplified description for Wallman bases on a compact space.

3.1 LEMMA. Let L be a lattice on the compact space T which is a base for the closed sets in T. Let F_1 and F_2 be disjoint closed sets in T. Then there exists disjoint $A, B \in L$ such that $F_1 \subset A$ and $F_2 \subset B$.

Proof: Let $p \in F_2$. Select $A_p \in L$ such that $F_1 \subset A_p$ and $p \in T - A_p$. Clearly $\{T - A_p | p \in F_2\}$ is an open cover for F_2. Since F_2 is compact, select $\{p_1, p_2, \ldots, p_n\} \subset F_2$ such that $\{T - A_{p_k} | k = 1,2,\ldots,n\}$ is a finite open cover for F_2. Clearly $F_2 \subset \bigcup \{T - A_{p_k} | k = 1,2,\ldots,n\} = T - \bigcap \{A_{p_k} | k = 1,2,\ldots,n\}$. Let $A = \bigcap \{A_{p_k} | k = 1,2,\ldots,n\}$. Then $A \in L$, $F_1 \subset A$ and $A \cap F_2 = \emptyset$. Now apply the above procedure to F_2 and A, replacing F_1 by F_2 and F_2 by A. We then obtain $B \in L$ such that $F_1 \subset A$, $F_2 \subset B$ and $A \cap B = \emptyset$. This completes the proof of the lemma.

3.2 THEOREM. Let T be a compact space. Then L is a Wallman base on T iff L is a lattice on T which is a base for the closed subsets of T.

Proof: Suppose L is a lattice on T which is a base for the closed subsets of T.

First, L separates points and closed sets in T. Let F be a closed set in T and $t \in T - F$. Since L is a base for the closed sets in T, there exists $A,B \in L$ such that $F \subset A$, $p \in B$ and $A \cap B = \emptyset$ by lemma 3.1. Hence, L separates points and closed sets in T.

Secondly, L is a normal lattice on T. Let $A,B \in L$ where $A \cap B = \emptyset$. Since T is a normal space, select open sets G,H in T such that $A \subset G$, $B \subset H$ and $\overline{G}^T \cap \overline{H}^T = \emptyset$. Clearly A and $T - G$ are disjoint closed sets in T, as are B and $T - H$. Using 3.1, select $C,D \in L$ such that $T - G \subset C$, $A \cap C = \emptyset$, $T - H \subset D$ and $B \cap D = \emptyset$. So $A \subset T - C$ and $B \subset T - D$. Since $T - C \subset G$, $T - D \subset H$ and $G \cap H = \emptyset$, then $(T - C) \cap (T - D) = \emptyset$. So $C \cup D = T$. Hence, L is a normal lattice on T. This completes the proof of the theorem.

3.3 THEOREM. Let $T \in cX$ and L a Wallman base on T. Then $T = wL_X$ iff $\overline{A}^T \cap \overline{B}^T = \emptyset$ whenever $A,B \in L_X$ and $A \cap B = \emptyset$, where $L_X = \{A \cap X | A \in L\}$.

Proof: Suppose $\overline{A}^T \cap \overline{B}^T = \emptyset$ whenever $A,B \in L_X$ and $A \cap B = \emptyset$.

A. Let $t \in T$. We show there exists $U \in wL_X$ such that $U \to t$. Define $F = \{B \in L | t \in \text{int}_T B\}$. Then F is an L-filter and $t \in \text{int}_T(\overline{B \cap X}^T)$ for each $B \in F$. Let $F_X = \{B \cap X | B \in F\}$. Then F_X has the finite intersection property. Select $U \in wL_X$ such that $F_X \subset U$. Then $\emptyset \neq \bigcap\{\overline{V}^T | V \in U\} \subset \bigcap\{\overline{B}^T | B \in F_X\} = \{t\}$. Hence, $U \in wL_X$ and $U \to t$.

By theorem 2.3, condition (i) of Brooks' theorem 2.2 holds since L_X is clearly a lattice on X. It suffices to show that L_X is a normal lattice on X.

B. L_X is a normal lattice on X. Let $A,B \in L$ where $A \cap B \cap X = \emptyset$. Then $\overline{A \cap X}^T \cap \overline{B \cap X}^T = \emptyset$ by hypothesis. Using 3.1, select $A',B' \in L$ such that $\overline{A \cap X}^T \subset A'$, $\overline{B \cap X}^T \subset B'$ and $A' \cap B' = \emptyset$. Since L is a normal base on T, select $C,D \in L$ such that $A' \subset T - C$, $B' \subset T - D$ and $C \cup D = T$. Then $\overline{A \cap X}^T \subset T - C$, $\overline{B \cap X}^T \subset T - D$ and $C \cup D = T$. Hence, $A \cap X \subset X - (C \cap X)$, $B \cap X \subset X - (D \cap X)$, $(C \cap X) \cup (D \cap X) = X$ and $C \cap X, D \cap X \in L_X$. Hence L_X is a normal lattice on X.

Using the same method as in B, conditions (ii) and (iii) of Brooks' theorem 2.2 are easily verified. Hence, $T = wL_X$. This completes the proof of the theorem.

We can now derive some further remarks. First, for any compactification T of X and any lattice L on T, X is <u>L-dense</u> means if $A \in L$ and $A \neq \emptyset$, then $A \cap X \neq \emptyset$.

3.4 COROLLARY. Let $T \in cX$ and L a Wallman base for T. Suppose X is L-dense. Then $T = wL_X$, where $L_X = \{A \cap X | A \in L\}$.

Proof: We first show that $\overline{A \cap X}^T = A$ for each $A \in L$. Let $A \in L$. It suffices to show that $A \subset \overline{A \cap X}^T$. Let $x \in A$ and G an open neighborhood of x in T. Then $x \notin T - G$ and $T - G$ is closed in T. By lemma 3.1, select $B \in L$ such that $x \in B$ and $B \cap (T - G) = \emptyset$. Clearly $x \in A \cap B$. Since $A \cap B \neq \emptyset$, then $A \cap B \cap X \neq \emptyset$ since X is L-dense. $G \cap A \cap X \neq \emptyset$ since $B \subset G$. Thus $x \in \overline{A \cap X}^T$. Hence, $\overline{A \cap X}^T = A$ for each $A \in L$.

We now apply theorem 3.3. Let $A, B \in L$ and suppose $A \cap B \cap X = \emptyset$. If $\overline{A \cap X}^T \cap \overline{B \cap X}^T \neq \emptyset$, then $A \cap B \neq \emptyset$, and $A \cap B \cap X \neq \emptyset$. Hence, $\overline{A \cap X}^T \cap \overline{B \cap X}^T = \emptyset$. By theorem 3.3, $T = wL_X$. This completes the proof of the corollary.

3.5 COROLLARY. Let $T \in cX$ and B a lattice on T which is a base for the open sets. Suppose for each $A \in B$, if $A \neq T$, then $X \not\subset A$. Then $T \in wX$.

Proof: Let $L = \{T - A | A \in B\}$ and apply corollary 3.4.

3.6 COROLLARY. Let T be a compact space and L a Wallman base of regular closed sets in T. Then T is a Wallman compactification of each of its dense subspaces.

Proof: Let X be a dense subspace of T. Then X is L-dense. By 3.4 $T = wL_X$, where $L_X = \{A \cap X | A \in L\}$. This completes the proof.

Some results related to these corollaries were obtained by Alo and Shapiro [2].

Clearly it is useful for a compact space to have a regular Wallman base (i.e., a Wallman base of regular closed sets) since a compact space with a regular Wallman base is a Wallman compactification of each of its dense subspaces. We thus make the following definitions. The compact space T is <u>regular Wallman</u> means T has a Wallman base of regular closed sets. The compact space T is <u>dense Wallman</u> means that T is a Wallman compactification of each of its dense subspaces. Obviously, every compact space which is regular Wallman is also dense Wallman. It seems to be unknown whether the converse holds.

Let T be a compactification of X. We seek a Wallman base L on X such that T = wL. From corollary 3.4, in order to find a suitable such L, we might find a suitable Wallman base L' on T and then restrict L' to X in order to obtain L; i.e., $L = \{A \cap X \mid A \in L'\}$. That this is, in fact, always the case is the context of the next theorem.

3.7 THEOREM. Let $T \in cX$ and L a Wallman base on X. Let $\overline{L} = \{\overline{A}^T \mid A \in L\}$. Then T = wL iff $\overline{L}$ is a Wallman base on T. Moreover, we have $T = w\overline{L}_X$.

Proof: Suppose $\overline{L}$ is a Wallman base on T. Clearly X is $\overline{L}$-dense. By corollary 3.4, $T = w\overline{L}_X$. Since $\overline{L}_X = L$, then T = wL. Now suppose that T = wL. By theorem 3.2 it suffices to show that $\overline{L}$ is a lattice on T which is a base for the closed subsets of T.

First, $\overline{L}$ is a lattice on T. Since $\emptyset, X \in L$, then $\emptyset, T \in \overline{L}$. Let $\overline{A}^T, \overline{B}^T \in \overline{L}$ where $A, B \in L$. Then $\overline{A}^T \cup \overline{B}^T \in \overline{L}$ since $\overline{A}^T \cup \overline{B}^T = \overline{A \cup B}^T$ and also $A \cup B \in L$. Also $\overline{A}^T \cap \overline{B}^T \in \overline{L}$ since $\overline{A}^T \cap \overline{B}^T = \overline{A \cap B}^T$ by theorem 2.1(i) and since $A \cap B \in L$. So $\overline{L}$ is a lattice on T.

Second, $\overline{L}$ is a base for the closed subsets of T. Let F be closed in T and $t \in T - F$. Let $F^* = \{U_p | p \in F\}$, where $U_p = \{A \in L | p \in \overline{A}^T\}$. Then F^* is closed in wL and $U_t \notin F^*$ since the mapping $q \to U_q$, $q \in T$, is a homeomorphism between T and wL (see theorem 1.1 and sequel). Now $\{A^* | A \in L\}$ where $A^* = \{U \in wL | A \in U\}$ is a base for the closed subsets of wL. So there exists $A \in L$ such that $F^* \subset A^*$ and $U_t \notin A^*$. Thus $F \subset \overline{A}^T$ and $t \notin \overline{A}^T$. Hence, $\overline{L}$ is a base for the closed subsets of T. This completes the proof of the theorem. Steiner also observed this result in [28, theorem 2]; however, he uses different techniques.

3.8 COROLLARY. Let $T \in wX$ and $X \subset Y \subset T$. Then $T \in wY$.

Proof: Let L be a Wallman base on X such that T = wL. Let $\overline{L} = \{\overline{A}^T | A \in L\}$. By theorem 3.7, $\overline{L}$ is a Wallman base on T. Clearly X is $\overline{L}$-dense, so Y is $\overline{L}$-dense. By corollary 3.4, $T = w\overline{L}_Y$ where $\overline{L}_Y = \{A \cap Y | A \in \overline{L}\}$. Hence, $T \in wY$. This completes the proof of the corollary.

3.9 COROLLARY. Let L be a Wallman base of regular closed sets on the space X. Then T = wL is a regular Wallman compactification of X.

Proof: Since T = wL, then $\overline{L} = \{\overline{A}^T | A \in L\}$ is a Wallman base for T by theorem 3.7. It then suffices to show that if $A \in L$, then $\overline{A}^T$ is regular closed in T; i.e., $\overline{A}^T = \overline{\text{int}_T(\overline{A}^T)}^T$. Now $A = \overline{\text{int}_X A}^X$ since A is regular closed in X. We now show that $\text{int}_X A \subset \text{int}_T(\overline{A}^T)$. Let $x \in \text{int}_X A$. Then there exists H open in T such that $x \in H \cap X \subset A$ since $\text{int}_X A$ is open in X. So $x \in \overline{H \cap X}^T \subset \overline{A}^T$. But $\overline{H \cap X}^T = \overline{H}^T$ since H is open in T. Thus we have $x \in H \cap X \subset H \subset \overline{H}^T \subset \overline{A}^T$, and so $x \in \text{int}_T(\overline{A}^T)$. Now $\overline{A}^T = \overline{\overline{\text{int}_X A}^X}^T = \overline{\text{int}_X A}^T \cap X$ $\subset \overline{\text{int}_X A}^T \subset \overline{\text{int}_T(\overline{A}^T)}^T$; i.e., $\overline{A}^T \subset \overline{\text{int}_T(\overline{A}^T)}^T$. This completes the proof of the corollary.

CHAPTER IV

APPLICATIONS

In chapter III we have observed that if T is a compactification of X, then the search for a Wallman base L on X such that T = wL may be replaced by a search among the Wallman bases for T. In this case one searches for a Wallman base L' on T such that $L'_X = \{A \cap X | A \in L'\}$ is a Wallman base on X. It should be noted that, in general, if T is a compactification of X and L is a Wallman base on T, then $L_X = \{A \cap X | A \in L\}$, the restriction of L to X, need not be a Wallman base for X. In particular, although L_X is always a lattice on X which is a base for the closed subsets of X, L_X need not be a normal lattice on X. For example, let X be a non-normal space and T a compactification of X. If L is the lattice of all closed subsets of T, then L is a Wallman base on T. But, L_X is not a normal lattice on X and, hence, L_X is not a Wallman base on X.

On the other hand, it may happen that T is a compactification of X and L is a Wallman base for T such that L_X is a Wallman base for X, and yet, T is not wL_X (see [10]). This is also evident from corollary 4.7.

With this word of caution we now apply the technique suggested in chapter III to obtain some information about Wallman compactifications in various contexts.

Dense Wallman Compactifications

Let T be a topological space and L a lattice on T. For any subset S of T, $L_S = \{S \cap A | A \in L\}$. Also, B is a co-L set in T means that there is an L-set A in T such that B = T − A. We denote $B = co_T A$.

4.1 THEOREM. Let L be a Wallman base for the space T. Then L_X is a Wallman base for each subspace X of T iff L_Y is a Wallman base for each co-L set Y in T.

Proof: Suppose L_Y is a Wallman base for Y for each co-L set Y in T. Let X be any subspace of T. Then clearly L_X is a lattice on X which is a base for the closed subsets of X. It remains to show that L_X is a normal lattice on X. Let $A,B \in L_X$ such that $A \cap B = \emptyset$. Then $A = A_1 \cap X$, $B = B_1 \cap X$ where $A_1, B_1 \in L$. Then $X \subset S = co_T(A_1 \cap B_1)$. If $A_2 = A_1 \cap S$ and $B_2 = B_1 \cap S$, then clearly $A_2 \cap B_2 = \emptyset$. Since L_S is a Wallman base on S, then L_S is a normal lattice on S; i.e., disjoint L_S-sets can be separated by disjoint co-L_S sets. So there exists $C_1, D_1 \in L$ such that $C = C_1 \cap S$, $D = D_1 \cap S \in L_S$, $A_2 \subset co_S C$, $B_2 \subset co_S D$ and $co_S C \cap co_S D = \emptyset$. So we have that $co_T C_1 \cap co_T D_1 \cap S = \emptyset$ since $(co_T F) \cap S = co_T(F \cap S)$ for each $F \in L$. Thus $A \subset co_X(C_1 \cap X)$, $B \subset co_X(D_1 \cap X)$ and $co_X(C_1 \cap X) \cap co_X(D_1 \cap X) = \emptyset$. Hence, L_X is a normal base on X. This completes the proof of the theorem.

Similarly we have the

4.2 THEOREM. Let L be a Wallman base for the space T. Then L_X is a Wallman base for each dense subspace X of T iff L_Y is a Wallman base for each dense co-L set Y in T.

4.3 THEOREM. Let T be a compact space and L a Wallman base on T. Then $T = wL_X$ for each dense subspace X of T iff $T = wL_Y$ for each dense co-L set Y in T.

Proof: Suppose $T = wL_Y$ for each dense co-L set Y in T. Let X be a dense subspace of T. By theorem 3.3 it suffices to show that if $A,B \in L_X$ and $A \cap B = \emptyset$, then $\overline{A}^T \cap \overline{B}^T = \emptyset$. Let $A,B \in L_X$ where $A \cap B = \emptyset$. Select $A_1, B_1 \in L$ such that $A = A_1 \cap X$ and $B = B_1 \cap X$. Thus, $S = co_T(A_1 \cap B_1)$ is a dense co-L set in T. Define $A_2 = A_1 \cap S$ and $B_2 = B_1 \cap S$. Since we have $A_2 \cap B_2 = A_1 \cap B_1 \cap S = \emptyset$ and $T = wL_S$, then $\overline{A_2}^T \cap \overline{B_2}^T = \emptyset$ (from theorem 3.3).

But, $A \subset \overline{A_2}^T$ and $B \subset \overline{B_2}^T$, so that $\overline{A}^T \cap \overline{B}^T = \emptyset$. Again from theorem 3.3, we have that $T = wL_X$. This completes the proof of the theorem.

Z-compactifications

We will need to introduce the following somewhat standard notations for our discussion. The set of all real-valued continuous functions on the space X is denoted by C(X). For any $f \in C(X)$, $Z(f) = \{x \in X | f(x) = 0\}$ is called the zero set of f in X. The complement of a zero set in X is called a cozero set in X (denote $X - Z(f) = \mathrm{coz}_X f$ for each $f \in C(X)$). Whenever $A \subset C(X)$, we define A^* to be the set of all bounded functions in A. In addition $Z[A] = \{Z(f) | f \in A\}$, whereas Z[C(X)] is customarily denoted by Z(X).

A compactification T is a Z-compactification of X (denoted $T \in zX$) means that there is a Wallman base $L \subset Z(X)$ such that $T = wL$.

For any extension T of X, the set of all functions in C(X) that are extendable to functions in C(T) is denoted by E(X,T). Alternatively, $E(X,T) = \{g|X : g \in C(T)\}$. It is well-known and easily verified that E(X,T) is a subalgebra (with pointwise operations) of the algebra C(X), E(X,T) contains the constant functions and E(X,T) separates points and closed sets in X. Whenever X and T are understood, we will denote Z[E(X,T)] by L(E).

We now use the preceding techniques to show if T is a compactification of X, then L(E) is always a Wallman base for X. As far as I know, Frink [10] first observed that L(E) is a Wallman base for X, but no proof was given. Later, Brooks [6] was not able to show that L(E) was a normal lattice. Subsequently, Hager [14, 4.3(a)] supplied a proof based on structure spaces. We offer here a different proof from Hager's.

4.4 THEOREM. Let T be a compact space. Then Z[E(X,T)] is a Wallman base on X for each dense subspace X in T.

Proof:

<u>Lemma</u>: Let S be a cozero set in T. Then a cozero set in S is also a cozero set in T.

Proof of lemma: Let $f \varepsilon C(T)$ such that $S = coz_T f$. Let $h \varepsilon C(S)$. Then $coz_S h \subset coz_T f = S$. Consider $h_1 \varepsilon C(S)$ where $h_1(s) = \inf\{h^2(s),1\}$ for each $s \varepsilon S$. Define f_1 on T by $f_1(t) = h_1(t)f(t)$ for each $t \varepsilon S$ and $f_1(t) = 0$ for each $t \varepsilon T - S$. Clearly, $coz_T f_1 = coz_S h$. Moreover, $f_1 \varepsilon C(T)$ since if $t_\alpha \rightarrow t_o$ represents any net in T which converges to $t_o \varepsilon T - S$, then $f(t_\alpha) \rightarrow f(t_o) = 0$. This proves the lemma.

By 4.2 it suffices to show Z[E(S,T)] is a Wallman base on S for each dense cozero set S in T. Let $f \varepsilon C(T)$ where $S = coz_T f$ is dense in T. In order to show that Z[E(S,T)] is a Wallman base on S, it suffices to show that Z[E(S,T)] is a normal base on S. Let $F_1,F_2 \varepsilon Z(T)$, and let $F_1' = F_1 \cap S$, $F_2' = F_2 \cap S$ and suppose $F_1' \cap F_2' = \emptyset$. Since S is normal, select $h_1',h_2' \varepsilon C(S)$ such that $F_1' \subset coz_S h_1'$, $F_2' \subset coz_S h_2'$ and $coz_S h_1' \cap coz_S h_2' = \emptyset$. By the lemma, select $h_1,h_2 \varepsilon C(T)$ such that $coz_S h_1' = coz_T h_1$ and $coz_S h_2' = coz_T h_2$. Hence, $F_1' \subset coz_T h_1 \cap S$, $F_2' \subset coz_T h_2 \cap S$ and $(coz_T h_1 \cap S) \cap (coz_T h_2 \cap S) = \emptyset$; i.e., Z[E(S,T)] is a normal lattice on S. This completes the proof of the theorem.

Let T be a compactification of X. Since $L(E) = \{A \cap X | A \varepsilon Z(T)\}$ and Z(T) is a Wallman base on T, we have immediately, from theorem 3.3,

4.5 THEOREM. Let $T \varepsilon cX$. Then $T = wL(E)$ iff $\overline{A}^T \cap \overline{B}^T = \emptyset$ whenever $A,B \varepsilon L(E)$ and $A \cap B = \emptyset$.

The following known results (given as corollaries) are now immediate.

4.6 COROLLARY. For any space X, $\beta X = wZ(X)$ (i.e., $\beta X \in zX$).

Proof: It is well-known [11, 6.5] that $E(X,\beta X) = C^*(X)$ and that $\overline{A}^{\beta X} \cap \overline{B}^{\beta X} = \emptyset$ whenever $A,B \in Z(X)$ and $A \cap B = \emptyset$. By 4.5, $\beta X = wZ(X)$. This completes the proof of the corollary. See also [5] and [24].

For any extension T of the space X, X is <u>z-embedded</u> in T means for each $Z \in Z(X)$ there exists $Z' \in Z(T)$ such that $Z = Z' \cap X$ [14]. Clearly, if $T \in cX$ and X is z-embedded in T, then $\{Z(f) \mid f \in E(X,T)\} = Z(X)$. This yields

4.7 COROLLARY. Let $T \in cX$. If X is Lindelof or almost compact, then $wL(E) = \beta X$.

Proof: Suppose that X is Lindelof or almost compact. Then X is z-embedded in T [15, theorem 3]. By corollary 4.6, $wL(E) = wZ(X) = \beta X$. This completes the proof of the corollary.

As another application of the theory developed in this section, we give a characterization of pseudocompact spaces.

4.8 COROLLARY. The space X is pseudocompact iff $T = wZ[E(X,T)]$ for each $T \in cX$.

Proof: Suppose that X is pseudocompact and $T \in cX$. We show that X is Z(T)-dense. Suppose $f \in C(T)$ and $Z(f) \cap X = \emptyset$. Then $1/f \in C(X) = C^*(X)$. Hence, $Z(f) = \emptyset$; otherwise, $1/f \in C(X)$ and $1/f$ is unbounded, contradicting $C(X) = C^*(X)$. Thus X is Z(T)-dense. By corollary 3.4, $T = wZ[E(X,T)]$.

Conversely, suppose X is a space which is not pseudocompact. Then there exists $Z \in Z(\beta X)$ such that Z has at least two distinct points p and q and $Z \subset \beta X - X$ (see [11, 6I and 9.6]). Choose $p_o \notin \beta X$ and define $T = (\beta X - Z) \cup \{p_o\}$. Let T have the quotient topology induced by the function $h: \beta X \to T$ defined by $h(t) = t$, if $t \in \beta X - Z$, and $h(t) = p_o$, if $t \in Z$. Now $T \in cX$ [22]. Define $T_o = T - \{p_o\}$. Then $\beta T_o = \beta X$ ($\neq T$). In βX there exists open sets G and H with disjoint closures which contain p and q respectively. Hence, there exist $Z_1, Z_2 \in Z(\beta X)$ with $\overline{G}^{\beta X} \subset Z_1$, $\overline{H}^{\beta X} \subset Z_2$ and $Z_1 \cap Z_2 = \emptyset$. Let $A = Z_1 \cap T_o$ and $B = Z_2 \cap T_o$. Then $p \in \overline{A \cap X}^{\beta X} = \overline{A \cap X}^{\beta T_o}$ and $q \in \overline{B \cap X}^{\beta X} = \overline{B \cap X}^{\beta T_o}$. So $p_o \in \overline{A \cap X}^{T} \cap \overline{B \cap X}^{T}$ and $A \cap B \cap X = \emptyset$. Now each cozero set in βT_o is Lindelof; hence, T_o is z-embedded in T. So $A, B \in Z[E(T_o,T)]$. Thus $A \cap X$, $B \cap X \in Z[E(X,T)]$. Using 4.5, $T \neq wZ[E(X,T)]$. This completes the proof of the corollary. A different proof of this corollary is given in an unpublished paper by Hager [14, 4.7].

We remark that the previous corollary then shows that any compact space is a Z-compactification of each of its dense pseudocompact subspaces.

We now conclude this section with the following

4.9 COROLLARY. Let $T \in cX$. Suppose $T = wZ[E(X,T)]$. Then $T = wZ[E(Y,T)]$ for each Y where $X \subset Y \subset T$.

Proof: In [14, 4.6], Hager proves that $T = wZ[E(X,T)]$ iff $T = \beta S$ for each cozero set S in T which covers X. Since $T = \beta S$ for each cozero set S in T covering X, then clearly $T = \beta S$ for each cozero set S in T which covers Y for each dense subspace Y of T which covers X. The result now follows immediately from Hager's theorem. This completes the proof of the corollary.

Compact F-spaces

Using the previous theory, we can give a Wallman-type characterization of F-spaces (i.e., disjoint cozero sets are completely separated [11]).

4.10 LEMMA. Let T be a compact space. Then the following are equivalent:

(i) T is an F-space,

(ii) $C^*(Y) = E(Y,T)$ for each cozero set Y in T,

(iii) $\overline{Y}^T = \beta Y$ for each cozero set Y in T,

(iv) $wZ[E(Y,\overline{Y}^T)] = \overline{Y}^T$ for each cozero set Y in T.

Proof:

(i) implies (ii): [11, 14.25(b)].

(ii) implies (iii): [11, 6.5(II)].

(iii) implies (iv): Let Y be a cozero set in T. Clearly $\overline{Y}^T \in cY$. By corollary 4.7, $wZ[E(Y,\overline{Y}^T)] = \beta Y$ since Y is Lindelof. Hence, we have $wZ[E(Y,\overline{Y}^T)] = \overline{Y}^T$ since $\beta Y = \overline{Y}^T$.

(iv) implies (i): Let Y be a cozero set in T. By corollary 4.7, $wZ[E(Y,\overline{Y}^T)] = \beta Y$. Thus $\overline{Y}^T = \beta Y$, so $C^*(Y) = E(Y,T)$ by [11, 6.5]. Hence, T is an F-space [11, 14.25(6)].

This completes the proof of the lemma.

4.11 THEOREM. Let T be a compact F-space. Then T is a Z-compactification of each of its dense subspaces.

Proof: We shall use theorem 4.3. Now Z(T) is a Wallman base on T. Let Y be a dense cozero set of T. Then $T = \beta Y$ by 4.10(iii). Hence, $T = wZ(Y)$ by corollary 4.7. But $Z(Y) = \{Z \cap Y | Z \in Z(T)\}$ since Y is z-embedded in T [14]. By 4.3, $T = wZ(Y)$ for each dense subset Y in T. This completes the proof of the theorem.

We remark that if X is σ-compact and locally compact, then $\beta X - X$ is a compact F-space [11, 14.27]. So $\beta R - R$ and $\beta N - N$ are Z-compactifications of each of their dense subspaces, where R denotes the reals and N denotes the positive integers with their usual topologies. Also, if T is a compactification of X and Y is a dense cozero set in T, then $\beta Y - Y$ is a Z-compactification of each of its dense subspaces [11, 14(O.2)]. If X is an F-space, then βX is a Z-compactification of each of its dense subspaces since βX is also an F-space [11, 14.25(10)].

Orderable Compact Spaces

The purpose of this section is to show that all orderable compact spaces are dense Wallman.

A topological space T is orderable means there exists an order on T such that the order topology on T is the given topology on T. In this case we say that the order is compatible with the topology for T. Suppose T is a compactification for the space X where X is an ordered space [19]. Then T is an ordered compactification for X means there exists an order on T compatible with the topology for T which extends the order on X to T. Clearly any ordered compactification of a space X is orderable; however, the converse need not hold. For example, the two-point compactification of the natural numbers is an orderable, but not ordered, compactification of the natural numbers.

4.12 THEOREM. Let T be an orderable compact space. Then T is dense Wallman.

Proof: Let $\leq$ be an order on T compatible with the topology on T. Let X be a dense subspace of T. The left end-point a of the interval $[a,b]$ in T is admissable means either $a \in X$ or $[a,b]$ is a neighborhood of a in T. The right end-point b is admissable means $b \in X$ or $[a,b]$ is a neighborhood of b in T. Let L(X) be the lattice on T of finite unions of closed intervals with admissable end-points. It is obvious that X is L(X)-dense. By corollary 3.4 it then suffices to show that L(X) is a base for the closed sets in T. Let F be a closed subset of T and $p \in T - F$.

Case 1: $p < t$ for each $t \in F$. Let $t' = \inf_T F$. If $t' \in X$, then $[t',1_T] \in L(X)$ ($1_T = \sup_T T$), $F \subset [t',1_T]$ and $p \notin [t',1_T]$. Suppose $t' \notin X$. If $x \in (p,t') \cap X$, then $[x,1_T] \in L(X)$, $F \subset [x,1_T]$ and $p \notin [x,1_T]$. If $(p,t') \cap X = \emptyset$, then $[t',1_T]$ is a T-neighborhood of t'. So $[t',1_T] \in L(X)$, $F \subset [t',1_T]$ and $p \notin [t',1_T]$.

Case 2: $t < p$ for each $t \in F$. Let $t' = \sup_T F$ and proceed as in case 1 obtaining $q \in [t',p)$ such that $[0_T,q] \in L(X)$ ($0_T = \inf_T T$), $F \subset [0_T,q]$ and $p \notin [0_T,q]$.

Case 3: there exists $q_1,q_2 \in F$ such that $q_1 < p < q_2$. Let $F_1 = \{t \in F | p < t\}$ and $F_2 = \{t \in F | t < p\}$. Then F_1,F_2 are disjoint closed subsets of T and $F = F_1 \cup F_2$. Apply case 1 to p and F_1 obtaining $t_1 \in T$ such that $[t_1,1_T] \in L(X)$, $F_1 \subset [t_1,1_T]$ and $p \notin [t_1,1_T]$. Apply case 2 to F_2 and p obtaining $t_2 \in T$ such that $[0_T,t_2] \in L(X)$, $F_2 \subset [0_T,t_2]$ and $p \notin [0_T,t_2]$. Then $[0_T,t_2] \cup [t_1,1_T] \in L(X)$, $F \subset [0_T,t_2] \cup [t_1,1_T]$ and $p \notin [0_T,t_2] \cup [t_1,1_T]$.

Hence, L(X) is a base for the closed subsets of T. This completes the proof of the theorem.

4.13 THEOREM. Let X be an ordered space and let T be the maximal ordered compactification for X. Then $T \in zX$.

Proof: For the existence of T, see [18]. T has the following property: if $t \in T - X$, then there exists $Y \subset X$ such that either $t = \sup_T Y$ or $t = \inf_T Y$ but never both. Let L be the lattice on T of finite unions of sets of the form $f^{-1}(a)$ where $f:T \to [0,1]$ is continuous and increasing, $a \in [0,1]$ and if $t \in f^{-1}(a) \cap (T - X)$, then $f^{-1}(a)$ is a neighborhood of t in T. Using the method of the previous theorem, it is straight-forward to show that L is a lattice on T, L is a base for the closed sets in T and that X is L-dense. By corollary 3.4, $T = wL_X$. Hence, $T \in zX$. This completes the proof of the theorem.

CHAPTER V

APPLICATIONS OF A CANONICAL LATTICE

Introduction

Let αX and γX be compactifications of X such that $\gamma X \leq \alpha X$. In this chapter we are concerned about what is known about γX if we assume that αX is a Wallman compactification of X. Toward this end we introduce the following notation. If $\gamma X \leq \alpha X$, then there exists a (unique) continuous function $h:\alpha X \to \gamma X$ such that $h(x) = x$ for every $x \in X$. We call h the <u>canonical</u> <u>mapping</u>. It is well-known that $h[\overline{A}^{\alpha X}] = \overline{A}^{\gamma X}$ for each subset A of X. A point $t \in \gamma X$ is a <u>multiple</u> <u>point</u> (with respect to h) means that the cardinality of $h^{-1}(t)$ is not 1. Let M(h) be the set of all multiple points in γX. Then $M(h) \subset \gamma X - X$. Suppose L is a Wallman base on X, $\alpha X = wL$ and $h:\alpha X \to \gamma X$ is the canonical mapping. Define L(h) to be the set $\{\overline{A}^{\gamma X} | A \in L.\ h^{-1}[\overline{A}^{\gamma X}] = \overline{A}^{\alpha X}\}$. We then have the following

5.1 LEMMA.

(i) If $A \in L$ and $h^{-1}(t) \cap \overline{A}^{\alpha X} \neq \emptyset$ implies that $h^{-1}(t) \subset \overline{A}^{\alpha X}$ whenever $t \in M(h)$, then $\overline{A}^{\gamma X} \in L(h)$.

(ii) If $A \in L$ and $M(h) \cap (\text{int}_{\gamma X}\overline{A}^{\gamma X}) = M(h) \cap \overline{A}^{\gamma X}$, then $\overline{A}^{\gamma X} \in L(h)$.

(iii) If $B \subset \gamma X$, $B \cap X \in L$ and $M(h) \cap (\text{int}_{\gamma X}\overline{B \cap X}^{\gamma X}) = M(h) \cap \overline{B \cap X}^{\gamma X}$, then $B \cap X \in L(h)$.

Proof: The proof is straight-forward.

5.2 LEMMA. Let $\alpha X, \gamma X \in cX$ where $\gamma X \leq \alpha X$. Suppose L is a Wallman base on X such that $\alpha X = wL$. Let $h:\alpha X \to \gamma X$ be the canonical mapping. Then L(h) is a lattice on γX and $X \in L(h)$-dense.

Proof: It is obvious that X is L(h)-dense. It is easy to show that $\emptyset$, $\gamma X \in L(h)$ and that if $A,B \in L(h)$, then $A \cup B \in L(h)$. Now suppose $A',B' \in L(h)$. We show that $A' \cap B' \in L(h)$. Select $A,B \in L$ such that $A' = \overline{A}^{\gamma X}$ and $B' = \overline{B}^{\gamma X}$. Since L is a Wallman base on X, then $A \cap B \in L$. It then suffices to show that $h^{-1}[\overline{A \cap B}^{\gamma X}] = \overline{A \cap B}^{\alpha X}$. Since $\overline{A}^{\gamma X}, \overline{B}^{\gamma X} \in L(h)$, we have that $h^{-1}[\overline{A}^{\gamma X}] = \overline{A}^{\alpha X}$ and $h^{-1}[\overline{B}^{\gamma X}] = \overline{B}^{\alpha X}$. Now $\overline{A \cap B}^{\alpha X} = \overline{A}^{\alpha X} \cap \overline{B}^{\alpha X} = h^{-1}[\overline{A}^{\gamma X}] \cap h^{-1}[\overline{B}^{\gamma X}] = h^{-1}[\overline{A}^{\gamma X} \cap \overline{B}^{\gamma X}]$; i.e., $\overline{A \cap B}^{\alpha X} = h^{-1}[\overline{A}^{\gamma X} \cap \overline{B}^{\gamma X}]$. And so $\overline{A \cap B}^{\gamma X} = h[\overline{A \cap B}^{\alpha X}] = h[h^{-1}[\overline{A}^{\gamma X} \cap \overline{B}^{\gamma X}]] = \overline{A}^{\gamma X} \cap \overline{B}^{\gamma X}$; i.e., $\overline{A \cap B}^{\gamma X} = \overline{A}^{\gamma X} \cap \overline{B}^{\gamma X}$. Hence, if $A',B' \in L(h)$, then $A' \cap B' \in L(h)$. This completes the proof of the lemma.

We remark that L(h) need not be a base for the closed sets in γX. Suppose X = N (the positive integers with the discrete topology). Let γN be a compactification of N which is not totally disconnected. Let $\alpha N = \beta N$, the Stone-Cech compactification of N. Now, βN is totally disconnected [11]. If $A \in L(h)_N$, then $N - A \in L(h)_N$. Thus $wL(h)_N$ is totally disconnected. So $\gamma N \neq wL(h)_N$. Hence, L(h) can not be a base for the closed sets in γN. However, if L(h) is a base for the closed subsets of γX, then $\gamma X = wL(h)_X$. Accordingly, we seek conditions under which L(h) is a base for the closed sets in γX.

5.3 THEOREM. Suppose $h^{-1}[M(h)]$ is a locally finite family in αX where $h:\alpha X \to \gamma X$ is the canonical mapping and $\alpha X, \gamma X \in cX$. Then

(i) $\gamma X \in wX$;

(ii) if $\alpha X \in zX$, then $\gamma X \in zX$;

(iii) if αX is regular Wallman, then γX is regular Wallman.

Proof: From 3.3, 3.4 and 5.2, it suffices to show that L(h) is a base for the closed sets in γX. Let F be a closed subset of γX and $p \in \gamma X - F$. Let $F' = h^{-1}[F] \cup (\cup\{h^{-1}(t) \mid t \in M(h).\ t \neq p.\})$. Then F' is closed in αX and $F' \cap h^{-1}(p) = \emptyset$. So there exists $A \in L$ such that $F' \subset \overline{A}^{\alpha X}$ and

$h^{-1}(p) \cap \overline{A}^{\alpha X} = \emptyset$. Thus $\overline{A}^{\gamma X} \in L(h)$, $F \subset \overline{A}^{\gamma X}$ and $p \notin \overline{A}^{\gamma X}$. Now (i) and (ii) follow immediately, while (iii) follows from corollary 3.9. This completes the proof of the theorem.

5.4 COROLLARY. Suppose $M(h)$ is finite. Then

(i) if $\alpha X \in wX$, then $\gamma X \in wX$;

(ii) if $\alpha X \in zX$, then $\gamma X \in zX$;

(iii) if αX is regular Wallman, then γX is regular Wallman.

Proof: Since $M(h)$ is finite, then $h^{-1}[M(h)]$ is a locally finite family in αX. The corollary now follows immediately from the above theorem. This completes the proof of the corollary.

5.5 COROLLARY. Let X be a locally compact space and let γX be an n-point compactification of X. Then $\gamma X \in zX$.

Proof: Let $\alpha X = \beta X$, the Stone-Cech compactification of X, in the above corollary. Now $\beta X \in zX$. Let $h:\beta X \to \gamma X$ be the canonical mapping. Clearly $M(h)$ is finite. The result now follows from the previous corollary. This completes the proof of the corollary.

Some Applications

The above techniques can be used to show that the compactifications of a locally compact space constructed by Magill [22] are Wallman.

5.6 THEOREM. Let X be a locally compact space and $\alpha X \in cX$. Let $\{K_1, K_2, \ldots, K_n\}$ be a finite family of mutually disjoint closed subsets of $\alpha X - X$, where the cardinality of each K_i in the family is greater than 1. For each i, choose distinct $q_i \notin \alpha X$. Let $\gamma X = [\alpha X - \bigcup\{K_i | i = 1,2,\ldots,n\}] \cup \{q_1, q_2, \ldots, q_n\}$. Define $h:\alpha X \to \gamma X$ by $h(p) = p$ if $p \in \alpha X - \bigcup\{K_i | i = 1,2,\ldots,n\}$ and $h(p) = q_i$ if $p \in K_i$. Let γX have the quotient topology determined by h. Then

(i) if $\alpha X \in wX$, then $\gamma X \in wX$;

(ii) if $\alpha X \in zX$, then $\gamma X \in zX$;

(iii) if αX is regular Wallman, then γX is regular Wallman.

Proof: Now $\gamma X \in cX$ [21]. Clearly M(h) is finite, so corollary 5.4 applies. This completes the proof of the theorem.

In corollary 5.4 we show that any compactification of a given space X which has a finite number of multiple points with respect to the Stone-Cech compactfication of X is a Z-compactification of X. We now show that any compactification of X which has a countable number of multiple points with respect to the Stone-Cech compactification is also a Z-compactification of X.

5.7 THEOREM. Let $T \in cX$ and suppose that T has at most a countable number of multiple points with respect to βX. Then $T \in zX$.

Proof: Now $\beta X \in zX$; in fact, $\beta X = wZ(X)$. Let $h:\beta X \to T$ be the canonical mapping and $M = M(h)$. We note the following easily proved facts: if G is open in T, then $\overline{G \cap X}^T = \overline{G}^T$; if G is open in T and $M \cap G = M \cap \overline{G}^T$, then $h^{-1}[\overline{G}^T] = \overline{G \cap X}^{\beta X}$. Now it suffices to show that L(h) is a base for the closed sets in T. Let F be a closed subset of T and $p \in T - F$. There exist G,H open in T such that $F \subset G$, $p \in H$ and $\overline{G}^T \cap \overline{H}^T = \emptyset$. Since T is a normal space, choose $f \in C^*(T)$ such that $f:T \to [0,1]$, $f(t) = 0$ for each $t \in \overline{G}^T$ and $f(t) = 1$ for each $t \in \overline{H}^T$.

Case 1: There exists $r \in [0,1]$ such that $f^{-1}[0,r]$ is both open and closed in T. Clearly $F \subset f^{-1}[0,r]$ and $p \notin f^{-1}[0,r]$. Since we have $f^{-1}[0,r] = Z(\max(|f|, \underline{r})$ [6], then $f^{-1}[0,r] \in Z(T)$ ($\underline{r}$ denoted the constant r-function on T). So $f^{-1}[0,r] \cap X \in Z(X)$. Now $\overline{f^{-1}[0,r] \cap X}^T = \overline{f^{-1}[0,r]}^T = f^{-1}[0,r]$, so $f^{-1}[0,r] \cap M = \overline{f^{-1}[0,r]}^T \cap M$. Hence, $h^{-1}[\overline{f^{-1}[0,r] \cap X}^T] = \overline{f^{-1}[0,r] \cap X}^{\beta X}$. Thus $f^{-1}[0,r] \in L(h)$, $F \subset f^{-1}[0,r]$ and $p \notin f^{-1}[0,r]$.

Case 2: For each $r \in [0,1]$, $f^{-1}[0,r]$ is not open in T. Clearly $f^{-1}(r) \neq \emptyset$ for each $r \in [0,1]$. Now if $r,s \in [0,1]$ where $r < s$, then

$f^{-1}[0,r) \subsetneq f^{-1}[0,r] \subsetneq f^{-1}[0,s) \subsetneq f^{-1}[0,s]$ (*). Also $\overline{f^{-1}[0,r)}^T \subset f^{-1}[0,r]$ for each $r \in [0,1]$. Since $[0,1/4]$ is uncountable, F has an uncountable number of neighborhoods disjoint from $\{p\}$. Let $V_r = f^{-}[0,r]$ for $r \in (0,1/4]$. Clearly V_r is a neighborhood of F in T and $V_r \cap \overline{H}^T = \emptyset$. Rewriting (*) we have that if $r,s \in (0,1/4]$ where $r<s$, then $\text{int}_T V_r \subsetneq V_r \subsetneq \text{int}_T V_s \subsetneq V_s$. So $\text{bdy}_T V_r \cap \text{bdy}_T V_s = \emptyset$ for $0 < r < s \leq 1/4$ (bdy denotes boundary). Since M is countable, there exists $r \in (0,1/4]$ such that $M \cap V_r = M \cap \text{int}_T V_r$. Since $\overline{V_r \cap X}^T \cap M \supset \overline{(\text{int}_T V_r) \cap X}^T \cap M = \overline{\text{int}_T V_r}^T \cap M \supset \text{int}_T V_r \cap M = V_r \cap M$, then $\overline{V_r \cap X}^T \cap M = M \cap V_r$. Similarly $V_r \cap M \supset \text{int}_T(\overline{V_r \cap X}^T) \cap M \supset \text{int}_T(\overline{\text{int}_T V_r \cap X}^T) \cap M \supset \text{int}_T(\overline{\text{int}_T V_r}^T) \cap M \supset \text{int}_T V_r \cap M \supset V_r \cap M$; so $\text{int}_T(\overline{V_r \cap X}^T) \cap M = M \cap V_r$. Therefore, $\text{int}_T(\overline{V_r \cap X}^T) \cap M = \overline{V_r \cap X}^T \cap M$. Hence, $V_r \cap X \in L(h)$, $F \subset \overline{V_r \cap X}^T$ and $p \notin \overline{V_r \cap X}^T$.

Hence, L(h) is a base for the closed sets of T. This completes the proof of the theorem.

We now obtain the following corollary, which was noted previously by Steiner in [30] using other techniques.

5.8 COROLLARY. Let $T \in cX$. Suppose $T - X$ is countable. Then $T \in zX$.

Concerning the Wallman Compactification Problem

In [26] Steiner proved that βR (the Stone-Cech compactification of the reals with the usual topology) is regular Wallman. In the conclusion to his paper, Steiner said that although he proved that βR is regular Wallman, he could not show this for all Stone-Cech compactifications. Actually, this problem is at least as difficult as the original Wallman compactification problem (i.e., is every compactification Wallman?). In particular, we next show that if all Stone-Cech compactifications are regular Wallman, then all compact spaces are regular Wallman.

5.9 THEOREM. Every compact space is regular Wallman iff βX is regular Wallman for every locally compact space X.

Proof: Suppose βX is regular Wallman for every locally compact space X. Let T be a compact space. Let $t \in T$ which is not an isolated point in T. Let $X = T - \{t\}$. Then X is locally compact. Let $h: \beta X \to T$ be the canonical mapping. Since $M(h) \subset \{t\}$, then M(h) is finite. Hence, T is regular Wallman by corollary 5.4. This completes the proof of the theorem.

5.10 COROLLARY. If every Stone-Cech compactification is regular Wallman, then every compact space is regular Wallman.

CHAPTER VI

SUMMARY

Let T be a compactification of the topological space X. The matter of constructing T as a Wallman compactification of X can be looked at from two points of view. First, we may examine the Wallman bases on X for some Wallman base L on X such that T = wL, or secondly, we may examine the Wallman bases on T for some Wallman base L on T such that $T = wL_X$, where L_X is the restriction of L to X.

From the first viewpoint, the Wallman problem has two parts: to find Wallman bases on X and then to decide which Wallman bases (if any) "work" for some given compactification of X. This approach has been successful for the well-known compactifications T of X (for example, when T is the Stone-Cech compactification [corollary 4.6], the one-point compactification of a locally compact space [6, 4.3], the Freudenthal compactification of a rim-compact space [24], the Banaschewski compactification (maximal zero-dimensional compactification) of a zero-dimensional space [5] or the bounding-system compactifications of Gould [30], or when it is given that T - X is totally disconnected and X is locally compact [6] or when T has a countable number of multiple points with respect to the Stone-Cech compactification [theorem 5.7]). It is evident that, although this technique has shown that the familiar compactifications are Wallman, it is still rather restrictive. Specifically, the constructions depend on the compact space under consideration and hence, depend upon special properties of X, T or T - X. Consequently, one seeks to develop within this viewpoint a more general, but definitive, approach to the Wallman question. With this in mind we note that most of the above-mentioned compactifications are Z-compactifications. These questions then present themselves:

(i) Is every Wallman compactification a Z-compactification?

(ii) Is every Z-compactification of a space X determined by some subring of C(X); i.e., given a compactification T of X, does there exist a subring A of C(X) such that T = wZ[A]?

(iii) Is every Z-compactification determined by some subring of the ring of extendable functions; i.e., given a compactification T of X, does there exist a subring A of E(X,T) such that T = wZ[A]?

Although these questions remain unanswered, this seems a reasonable approach (i.e., via zero-set Wallman bases) for constructing any compactification as a Wallman compactification.

In this paper, however, we have emphasized the second viewpoint in the following context: given a compact space T and a Wallman base L on T, for what dense subspaces X of T will the restriction of L to X work; i.e., for what dense subspaces X of T do we have $T = wL_X$? It is precisely this question which motivates chapter III of this thesis where we present a simplified description for Wallman bases on compact spaces (theorem 3.2) as well as a necessary and sufficient condition in order that $T = wL_X$, where T is a compact space, L a Wallman base on T and X a dense subspace of T (theorem 3.3). In addition, we show that if T is a Wallman compactification of X, then we can always find a Wallman base L on T such that $T = wL_X$, where L_X is the restriction of L to X (theorem 3.7) (i.e., if the first approach "works", then so does the second).

The following question then becomes evident: given a compact space T and any dense subspace X of T, can one find a Wallman base L on T such that $T = wL_X$? The compact space T is dense Wallman simply means that T is a compact space for which the answer to this question is "yes".

In this context we have shown that every compact orderable space is dense Wallman (theorem 4.12) and that every compact F-space is dense Z-Wallman (theorem 4.11); i.e., every compact F-space is a Z-compactification of each of its dense subspaces. Steiner [29] has shown that every compact metric space is regular Wallman; hence, dense Z-Wallman. In general, if a compact space T has a Wallman base of regular closed sets, then T is dense Wallman (corollary 3.6). In this regard, the following questions remain unanswered:

(i) Is every compact space dense Wallman?

(ii) Is every compact space dense Z-Wallman?

(iii) Is every compact space regular Wallman?

Clearly an affirmative answer to any of the above questions would settle the Wallman problem. Furthermore, very little seems to be known concerning these questions. It is obvious that if a compact space T is regular Wallman, then T is dense Wallman, and, if T is dense Z-Wallman, then T is dense Wallman. But no other implications seem to be known.

As far as I know, Alo and Shapiro [2] and Steiner [28, 29, 30] have been the only ones to have worked on the Wallman problem in this context. Alo and Shapiro have shown that any zero-dimensional compact space is regular Wallman. From this, it is easy to show that every zero-dimensional compact space is dense Z-Wallman. In [29] Steiner shows that every compact metric space is regular Wallman; hence, every compact metric space is also dense Z-Wallman.

In [28] Steiner shows that the Stone-Cech compactification of the reals with the usual topology is regular Wallman. Steiner concludes [28] with the statement that although he could prove that the Stone-Cech compactification of the reals is regular Wallman, he could not show that all

Stone-Cech compactifications are regular Wallman. We conclude the main body of this thesis by showing that if the Stone-Cech compactification of every locally compact space is regular Wallman, then every compact space is regular Wallman. Hence, every compact space is regular Wallman if and only if every Stone-Cech compactification is regular Wallman.

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