Spring 2017

Music of the triangles: How students come to understand trigonometric identities and transformations

Neil Moshe Bornstein
University of New Hampshire, Durham

MUSIC OF THE TRIANGLES:
HOW STUDENTS COME TO UNDERSTAND TRIGONOMETRIC IDENTITIES AND TRANSFORMATIONS

BY

NEIL M. BORNSTEIN
B.A. Mathematics, Skidmore College, 2007
B.A. Classics, Skidmore College, 2007
M.S. Mathematics, University of New Hampshire, 2016

DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy
in
Mathematics Education

May, 2017

This dissertation has been examined and approved in partial fulfillment of the requirements for the degree of Doctorate in Philosophy in Mathematics Education by:

Dissertation Director, Sharon McCrone,
Professor of Mathematics Education

Karen Graham,
Professor of Mathematics Education

Marc Lessard,
Associate Professor of Space Physics

Sheree Sharpe,
Assistant Professor of Mathematics Education

Orly Buchbinder,
Assistant Professor of Mathematics Education

On April 12, 2017

Original approval signatures are on file with the University of New Hampshire Graduate School.

ACKNOWLEDGEMENTS

Thanks to Mike Yereniuk for coding the MATLAB program I used during my study. I told him what I wanted the program to do, and he made it happen. Even after moving on into his professional career, he still took time to help me make improvements.

Thank you to my friends and family who supported me during the roller coaster of graduate school. I wouldn’t have made it through without you.

Thank you to the students who participated in my studies by providing feedback on my MATLAB program, allowing me to observe them working on problems, and taking pre- and post-tests.

Finally, thank you to my advisors and committee for helping me shape my ideas into a coherent research project. They have influenced massive, positive changes since the first iteration of this project.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

MUSIC OF THE TRIANGLES: HOW STUDENTS COME TO UNDERSTAND TRIGONOMETRIC IDENTITIES AND TRANSFORMATIONS

by

Neil M. Bornstein

University of New Hampshire, May, 2017

Trigonometry is an essential part of mathematics education (NCTM, 2000; NGA, 2010). Trigonometry is prevalent in studies of pure mathematics as well as physical applications. Trigonometric identities and transformations are particularly important. However, students and even teachers have struggled to articulate and justify trigonometric concepts (Moore, 2013; Tuna, 2013). Students have also struggled with identities and transformations in non-trigonometric contexts (Borba & Confrey, 1996; Tsai & Chang, 2009). This paper will describe a research project which articulates the *critical stages* through which students must pass to understand trigonometric identities and transformations. These critical stages were first hypothesized based on a review of the literature. Then undergraduate precalculus students were recruited to participate in a series of task-based interviews in order to examine the process by which students come to understand and justify trigonometric identities and transformations. The critical stages were revised based on the results of these interviews. Following the interviews, hypothesized lesson plans for the subjects were revised and implemented. The implementation of the lesson plans did not collect enough information to draw any conclusions, but the critical stages underscore the importance of students being able to move fluidly among representations.

I. Introduction

Trigonometry is an essential part of mathematics education (NCTM, 2000; NGA, 2010). Pure mathematics frequently uses trigonometric concepts due to the complex relationship between trigonometric functions and the number *e* (Stein & Shakarachi, 2003). In particular, Euler's equation $e = cos(x) + (i)sin(x)$ leads to de Moivre's formula and the most beautiful equation in mathematics, $e^{i\pi} + 1 = 0$. Trigonometry is integral to the calculus sequence, and is also present in many science, technology, engineering, and mathematics (STEM) applications such as designing road reflectors (Popelska, 2011), digital image processing (Rosen, Usselman, & Llewellyn, 2005), and modeling periodic phenomena such as sound waves or temperature variations (Douglas, Christensen, & Orsak, 2008; Kuttruff, 1973; Lando & Lando, 1975).

Trigonometric *identities* and *transformations* are algebraic and graphical ways of representing the same idea: trigonometric functions can be manipulated in specific ways to produce predictable results, such as a different trigonometric function, or the negative of the original function. For example, $cos(-x) = cos(x)$, $sin(\frac{\pi}{2} - x) = cos(x)$, and $tan(x + \pi) = tan(x)$. In each of these cases, a trigonometric function, $T(x)$, has been *transformed* by additive and/or multiplicative operations. When these transformations result in equations that are generally true, they are called *identities*. Identities and transformations can be used in various fields of science and engineering to predict repetitions or changes in patterned behavior (Douglas et al., 2008; Kuttruff, 1973; Lando & Lando, 1975). For example, applying transformations to sound waves results in effects that are prevalent in audio industries: altering pitch, and amplifying or diminishing sound waves and their echoes (Rigden, 1977). Predicting and using echoes is integral to designing rooms and musical instruments with good acoustics. These echoes can also be manipulated digitally to create the illusion that the sound was produced in a different type of space, such as a narrow hallway. Transformed sinusoids can even be combined, potentially resulting in, among other changes, a different timbre – the quality of sound that describes the difference between, for instance, a voice and a violin, or between two different voices. Synthesizers can use these combinations to recreate the sounds of common musical instruments.

Unfortunately, many students and even teachers struggle with many aspects of trigonometry. Through a combination of questionnaires and interviews Akkoç (2008) and Tuna (2013) found that Turkish preservice teachers had poor understandings of radian measure. Few could correctly define radians (8% of 93 participants in Tuna's case), and even those who could were still likely to think of radians in terms of degrees and to assume that any trigonometric input that did not contain a π symbol was meant to be considered in degrees, even when explicitly told otherwise. These results are echoed by Moore (2013) in a study of American undergraduate students, and have also been noted in studies with inservice teachers (Topçu, Kertil, Yilmaz, & Öndar, 2006).

There are only a few studies on trigonometry learning, but they show students in trigonometry classrooms having difficulty examining situations and choosing appropriate *representations*. A *representation* is any thing that stands in for another thing (Pimm, 1995; Goldin & Kaput, 1996). For example, the symbol π stands for an irrational number approximately equal to 3.14; the word "addition" is a representation of the concept of combining multiple objects into a single object. Frequently used trigonometric representations include the *algebraic* representations – for example, $sin(x)$ – the *graphical* representations in the Cartesian plane, the *unit circle* representation, and the *right triangle* representation. Studies of high school students in England, Turkey, and Australia and undergraduates in the United States have shown that students had difficulty using any trigonometric representation except that which was most familiar to them; as a result, students have shown an inability to effectively approach many mathematical situations (Challenger, 2009; Delice & Roper, 2006; Gür, 2009; Kendal & Stacey, 1998; Weber, 2008). Challenger found that over the course of the trigonometry unit, students did not develop connections between the various representations and trigonometric concepts introduced throughout the course. This led students to develop isolated understandings of each individual topic rather than a cohesive understanding of the properties and applications of a few core ideas.

In light of these circumstances, the current study has been developed to investigate how students come to understand trigonometric identities –in particular, opposite angle identities, identities involving adding multiples of π to the input, and the sine-cosine cofunction identities – and transformations of trigonometric functions. A review of the literature has revealed a lack of studies of how students come to understand each of these concepts. The design of this study has been informed by previous studies on how students learn trigonometry as a unit (Challenger, 2009; Fi, 2003) or how students learn earlier trigonometric concepts such as angle measure (Moore, 2013) and the definition of sine (Demir & Heck, 2013), as well as identities and transformations in non-trigonometric contexts (Borba & Confrey, 1996; Hall & Giacin, 2013; Tsai & Chang, 2009). The content and organization of the current research study have been influenced by the previous studies' descriptions of the orders in which students learned these concepts, the methods used to help them learn, and the misconceptions that they faced. The study extends the previous research on identities and transformations to a trigonometric context using the results of prior research on students' learning of trigonometry. In particular, it examines how students view the relationships among the different representations of trigonometric functions.

Research Questions

This study was guided by the following research questions:

1) Through what critical stages do students pass as they come to understand trigonometric identities and transformations? That is, which actions, connections, or other ways of thinking are common to those students who go on to be able to justify their solutions of tasks involving these concepts?
2) How do students understand the relationship between the unit circle definitions of trigonometric functions and the identities and transformations of those functions? Is it critical that students be able to change from the algebraic representation to one with different affordances as they come to understand identities and transformations?
3) To what extent do students progress through the critical stages during a lesson plan developed with these stages as a framework?

In order to answer these questions, a three-part study was conducted. To answer the first research question, a set of critical stages of understanding was hypothesized for each learning goal. There are nine learning goals addressed by this study: (1) opposite angle identities, (2) identities of the form ($\theta + n\pi$) for some integer n, (3) cofunction identities, (4) correlating addition in the algebraic representation with shifting in the graphical representation, (5) correlating multiplication in the algebraic representation with stretching in the graphical transformations, (6) correlating transformations of the function input with horizontal transformations, (7) correlating transformations of the function output with vertical transformations, (8) noticing that horizontal transformations are counterintuitive, and (9) noticing that order of transformations can matter. The critical stages were generated from a review of the literature regarding how students learn trigonometry in general and how they learn identities and transformations in non-trigonometric contexts. To test these hypothesized stages, a two-stage study was conducted in which undergraduate precalculus students were asked to participate in task-based interviews (Goldin, 2000). These interviews occurred before the students had been presented with the relevant material in lecture. The tasks for these interviews were designed to guide students through the proposed critical stages, and the students' speech and written work were analyzed to determine whether any of the proposed critical stages were superfluous, inadequate, or otherwise in need of revision. During stage one of this study, students were not able to explore all of the relevant concepts in the allotted time. Because of this, the tasks were revised, and a second set of students, different from the first, was recruited to participate the following semester in the second stage. Results from both sets of interviews were used to answer the first two research questions.

While analyzing the interview data, particular attention was paid to students' uses of representations. It was noted how the students conceived of the relationships among the definitions and representations of the trigonometric functions. For instance, it was noted whether students connected the motion of a radius rotating around a unit circle with the generation of the graphs of the trigonometric functions, used changes in one representation to predict the changes in another representation, or chose representations that were appropriate for their goals. Previous research indicates that students' facility with different representations can greatly influence their success in developing an understanding of trigonometry (Challenger, 2009; Weber, 2005).

To answer the third research question, a final, confirmatory study was conducted. For this study, lesson plans were developed based on the hypothesized critical stages of understanding for each learning goal. The lesson plans were revised after analyzing the interviews and revising the critical stages. For example, two students showed a promising strategy for generalizing the cofunction identities through the unit circle representation that hadn't been considered previously; the lesson plan was revised to incorporate this strategy rather than one that relied on students viewing the identity as a pair of transformations.

Collectively, these studies have produced a set of critical stages of understanding for each learning goal under investigation and a corresponding lesson plan that utilizes the critical stages to guide students towards a good understanding of the learning goal. Gravemeijer and van Eerde (2009) refer to the set of critical stages and lesson plan for a topic as a Local Instruction Theory (LIT). In this paper, the literature underlying the hypothesized critical stages and lesson plans will be presented. Following this, the data collection and analysis processes will be detailed, after which results will be presented and discussed. The study will conclude with the revised critical stages and lesson plan, as well as study limitations and implications for future research.

II. Theoretical Framework

In this section, the theoretical framework underlying this study is discussed. This study used social constructivism as the theory of learning that influenced all of the decisions made in study design. Literature on LITs is presented to elaborate the goals of the study. To describe how the hypothesized critical stages and lesson plans were developed, relevant literature is presented regarding representation theory, teaching episodes, and students' understanding of trigonometry, identities, and transformations. Finally, the hypothesized critical stages for trigonometric identities and transformations will be detailed.

Social Constructivism

Social constructivism is the perspective that informs all of the decisions made during the design of this study. In this section, the four basic tenets of this perspective – learning is social; students learn by connecting pieces of information; learning is uniquely personal for each student; and learning is an active process – will be described, as well as the ways in which these tenets have informed different aspects of the study.

Social. Social constructivists believe that all personal meaning is inherently influenced by social experiences, that all learning is necessarily a social process, and that the roles of the teacher and student are inextricably linked as they construct meaning for a concept (Cobb et al., 1992; Ernest, 2006). According to social constructivists, a key element of the learning process is *negotiation* (Bauersfeld, 1995; Cobb et al., 1992; Ernest, 2006; Powell & Kalina, 2009). Negotiation can be seen as a continuous process of a teacher and student presenting their interpretations of the other's words and actions. The teacher uses the student's work to interpret the student's conception as one that needs to be corrected, built upon, or emphasized, and responds accordingly. The teacher's goal is to guide the student in building a conception of the topic that is consistent with the conceptions of the greater mathematical community.

Researchers taking a social constructivist perspective do not believe that learning can happen purely autonomously (Cobb, Jaworski, & Presmeg, 1996). They say that a student doing mathematics alone is still working in a social context. This lone student could build mathematical knowledge by interacting with an agent of the mathematical community such as a text or a real-world phenomenon that displayed mathematical properties. The student would attempt to construct knowledge by interacting with the object or phenomenon and interpreting the new information in light of previously held conceptions. Alternatively, without any external stimuli, the student could engage in an internal conversation. During this supposedly autonomous dialogue, the student would play both the role of teacher and student, reflecting upon the role of the new concept in terms of its place in the mathematical community of knowledge and in terms of how it is built out of the student's prior knowledge. This reflection is inseparable from the context of the knowledge accepted by the mathematics community.

Connective. According to constructivist theorists, understanding is gained as new ideas are connected to prior knowledge or as elements of prior knowledge are connected to each other in new ways (Confrey, 1990; Eli, Mohr-Schroeder, & Lee, 2013; Hiebert & Lefevre, 1986). Constructivist theorists believe that it is the strength, number, and organization of these connections that constitute understanding (Hiebert & Lefevre, 1986). Learners can achieve greater understanding by increasing the number of connections between their known concepts, strengthening existing connections, or by making observations about their connections (AAHE, ACPA, & NASPA, 1998; Engelkemeyer & Brown, 2008; Greeno & Hall, 1997; Hiebert &

Lefevre, 1986; NGA, 2010). These beliefs are reflected in current standards, which emphasize that making connections between mathematical ideas and reflecting on patterns of connections are integral to becoming successful mathematical students (NGA, 2010).

Skemp (1987) refers to instrumental and relational understanding: instrumental understanding is characterized by an ability to complete tasks and use algorithms but an inability to explain why the work is true, while students with relational understanding can explain why an algorithm is true and useful based on their other mathematical knowledge. Students with relational understandings have made connections between the algorithm and their prior knowledge, while students with instrumental understandings have typically only connected the algorithm to its associated problem type.

Sometimes when students construct their knowledge, they do so in a way that is not in line with conventional mathematical beliefs (e.g. believing that exponents distribute across binomials). This may be labeled as a *misconception*. Since misconceptions are not in line with mathematical beliefs, there must be a concrete reason why the misconception must not be true. Misconceptions may be *perturbed* by forcing the student to confront how the misconception does not fit with their more stable prior knowledge (Ely, 2010). For example, a student may believe that $sin(x + y) = sin(x) + sin(y)$. This is likely a function misconception. The student may treat all functions this way, or the student may be thrown off by this notation being slightly different than the standard, one-letter function notation (e.g. $f(x)$), leading them to treat the situation more like distribution than composition. If the student has a good conception of the sine function, then this misconception can be perturbed by pointing out that $sin(\frac{\pi}{2}) = 1$, but also $\frac{\pi}{2} = \frac{\pi}{6} + \frac{\pi}{3}$, and

$sin(\frac{\pi}{6}) + sin(\frac{\pi}{3}) = \frac{1}{2} + \frac{\sqrt{3}}{2}$, which is not equal to 1. Misconceptions such as these that arose during the interviews were used to inform a revision of the lesson plan. Material designed to perturb the misconceptions was inserted into the lecture.

It should be mentioned though that not all mistakes necessarily indicate a misconception. Students may commit clerical errors or they could make mistakes that they quickly correct upon reflection. For example, a student who claims that $sin(\frac{\pi}{3}) = \frac{1}{2}$ may have simply misremembered a special triangle and may be able to sketch out a triangle and correct themselves to $sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, rather than having a misconception such as confusing sine with cosine. This type of mistake would not indicate that the student must significantly modify or construct connections among their trigonometric conceptions.

Unique. Constructivist theorists believe that all understanding is individual because of unique personal interpretations of experiences (Cobb et al., 1992; Ely, 2010; Ernest, 2006; Olivier, 1989; von Glasersfeld, 1987). A person's conception of any idea– yellow, renaissance, trigonometry– will be influenced by their past experiences with that concept or related concepts. These conceptions may be incorrect or even impossible. Furthermore, constructivist theorists believe that students who have shared the same experience nevertheless interpret that experience differently from one another (Duit, 1995; Olivier, 1989). Since all understanding is individual, none can be declared objectively true; any concept could be awaiting perturbation (Ely, 2010). Social constructivists treat the concepts that the majority of the mathematical community believe to be true as the truth (Ernest, 2006). When a belief commonly held by the community is changed or built upon, then the social constructivist community changes its beliefs about what is "true", for example when Cantor explored the nuances of infinity (Barrow, 2006). As students make connections among their uniquely understood concepts, they must play an active role in the learning process.

Active. Since students interpret information in their own individual ways by connecting it to their own unique interpretations of prior knowledge, it is necessary for the student to be an active participant in the learning process (von Glasersfeld, 1987). According to constructivist theorists, learning is an active process, and just experiencing something without reflecting on it does not contribute significantly to learning. The learner must take an active role in order to develop, strengthen, or organize connections between previously understood concepts (Cobb et al., 1992; Confrey, 1990; Ernest, 2006; Greeno & Hall, 1997; von Glasersfeld). Since only the learner can know how the learner understands concepts, then only the learner can make connections between these understandings. The teacher can provide information for the student that enables the student to construct their own connections, but it is up to the student to take the initiative to actually connect the new information to the old knowledge and experiences. In addition to actively participating, the learner must also reflect on the experience. As the students construct a new concept, they must decide how it fits with their previously constructed knowledge (Confrey; von Glasersfeld). This reflection, decision, and the subsequent actions all require the student to be active. Every level of the learning process– receiving the information, processing it, reflecting on it, and presenting the interpretation– require an active learner.

Students' knowledge structures do not spontaneously come together once their constitutional materials have been assembled (Ernest, 2006; von Glasersfeld, 1987). For example, students who are aware that a full rotation of a circle is 2π radians and that trigonometric functions can be defined by different measures of this rotation must still reflect upon those ideas to realize that the trigonometric functions will be periodic. Even if a student acknowledges the existence of a connection between concepts without actively participating in the learning process, then the connection will be too weak to support deeper thought and future understandings. This view conflicts with one that is deeply rooted in tradition, that of a student as an empty vessel waiting to have knowledge poured into it by the teacher (Cohen, 2003). However, once a teacher has stated a concept, the students can only interpret that concept in terms of their prior knowledge, and if that prior knowledge has been formed by weak links between the foundational concepts, then the students will be unable to effectively make sense of the new concept.

Social constructivism has influenced the design of this research study. The critical stages were designed with the belief that students must actively connect their understandings of concepts such as the unit circle and algebraic representations of trigonometric functions. Additionally, the lesson plans were designed to include interactions between students, so that each learner would hear multiple perspectives and engage with the material in an explicitly social manner.

Local Instruction Theory

A Local Instruction Theory can be used as a tool to help guide students to actively engage with a new concept. A LIT is a framework for designing lesson plans for a particular topic, such as single digit multiplication or geometric series (Gravemeijer, 1994; Gravemeijer, 1999; Gravemeijer & van Eerde, 2009). A LIT consists of critical stages of understanding for the topic, an activity intended to guide students through the critical stages, and a theory as to why the activity will do so. In this section, the critical stages of a LIT are discussed in more detail, and examples are given of previous studies that developed LITs.

Critical stages are individual levels of understanding that the student must achieve to develop an understanding of the larger topic. These stages are fundamental to the learning process. That is, students who do not pass through all of the stages will not have developed a relational understanding of the material and perhaps not even an instrumental understanding. The critical stages may also have an optimized ordering. This could be the result of necessity: for instance, learners cannot notice that horizontal transformations behave non-intuitively before they have noticed how horizontal transformations behave. The ordering could also be intended to generalize some aspect of understanding: for example, moving from observations about the angles of a right triangle to observations about trigonometric operations on those angles. The critical stages are specific to the topic under investigation, but are generally applicable to students who are prepared to learn the topic. That is, the stages may be contingent upon some prerequisite knowledge, but this knowledge will not be classroom specific. Rather, it will be knowledge that is assumed to have been previously required in the curriculum.

To develop critical stages, a researcher familiar with the teaching and learning of the topic first conjectures a set of critical stages (Gravemeijer, 1994; Gravemeijer, 1999; Gravemeijer & van Eerde, 2009). These stages are based on the development and difficulties of the learning goals, which the researcher is familiar with through personal experience and a review of the literature. The researcher then tests and revises the critical stages by observing students developing their understandings of the learning goal. A lesson plan is developed based on the critical stages, and this lesson plan is in turn enacted, analyzed, and revised. This cycle can be continued indefinitely, similar to a lesson study (Lewis, Perry, & Murata, 2006). The LIT is formed by the “finished” product of one of the cycles: a set of critical stages, a lesson plan informed by those stages, and a theoretical justification of the lesson plan.

The LITs that I developed for trigonometric identities and transformations can be grouped with existing studies of how students learn other trigonometric topics in order to develop a trigonometry curriculum. The critical stages of each LIT can be used to determine an ordering of topics: if the critical stages of one topic contain knowledge developed while students learn a different topic, then the latter must be placed before the former in the curriculum. The collection of LITs would also identify each subtopic that must be covered during the course of the trigonometry unit. Students may require, for instance, knowledge of how multiple representations of the trigonometric functions are related to each other. In addition, the lesson plans could serve as examples of how the critical stages may be applied in the classroom. These lesson plans could also be modified to reflect differences in teaching philosophy or classroom logistics.

An example of an LIT can be found in Larsen’s (2013) study of two students in their guided reinvention of core advanced algebraic concepts. Larsen found that the critical stages relied on students creating and organizing a notation for their thoughts. He designed tasks and questions intended to provoke students to think about formalizing their thoughts, including the implications that occur from their conceptions of the material to their conceptions of their notation (e.g. how identities arising from reflections and rotations of a triangle are represented in their written records) and from their notation to the material (e.g. how notation of a group operation can be combined with conceptions of inverse and identity to simplify operations without physical manipulation).

An unsuccessful attempt to develop an LIT for commutative rings can be found in a study by Simpson and Stehlíková (2006). In this study, the teacher offered less guidance to the students, who were unable to connect the characteristics of an unfamiliar structure with known ring concepts and were therefore unable to progress through the proposed critical stages. The students were able to apply various concepts and techniques to the unfamiliar ring, but were not able to generalize or combine evidence into many productive deductions. It is possible that the students' knowledge was too disconnected to apply to any but the most familiar situations, similar to the previously cited trigonometric studies of understanding.

Representation Theory

In this section I will define representations and explain how I will use this construct in my LIT. *Representations* are the forms that we give to concepts; they are any things that stand in for any other things (Goldin & Kaput, 1996; Pimm, 1995; Wu & Puntambekar, 2012). In this way, representations can be thought of as symbols (Pimm). The representation is a signifier for some other signified concept. Epistemologically, symbol comes from two Greek words, $\beta\alpha\lambda\lambda\omega$ (BAH-loh), meaning "throw," and $\sigma\upsilon\mu$ (SOOM), meaning "together" (Liddell & Scott, 1889). Symbols "throw together" two concepts which become inextricably linked. One concept is the signifier, which is a written, vocalized, or otherwise expressed characterization of the signified concept. The signified concept is the idea that the signifier is intended to represent, including facets that are not made readily apparent by the signifier. For example, the word function signifies an extremely deep concept that is applicable to many fields, but this is not at all apparent from the word itself, or even its written definition.

In this study, representations were a useful tool for examining the connections that students made between concepts. These connections were used to create a model of how students connect concepts. These connections also served as the basis for the development of activities for the lesson plans that were designed to help guide students through critical stages. The constructivists argue that learners reason about new concepts in terms of other, known concepts; representations act as the media of that reasoning (Goldin & Kaput, 1996, p. 409). Social constructivism lends itself to the use of representations because of several similarities between how constructivists view learning and how representation theorists view the use of representations. For example, under representation theory: understanding is achieved by connecting representations; powerful representations are those that are connected to many other representations, and teachers must create models of how students think about and relate representations.

Goldin and Kaput (1996) divide representations into two types: external and internal. *External representations* are any observable phenomena that stand in for a represented idea, such as words, pictures, or diagrams. *Internal representations* are the ways that learners think about the underlying concept and its relationship to external representations. An internal representation of a concept includes the learner's beliefs, feelings, and attitudes about the concept, and the connections between that concept and other knowledge. The internal representations themselves are inherently unobservable, so teachers must model their interpretations of the students' internal representations in order to respond in a way that will help lead the students towards deeper understanding. Using this language, the external representation is the signifier aspect of the symbol, and the internal representation is the signified. The external representations are used to indicate the complete understanding that the students have of the concept: their internal representations.

Just as there are many facets of any concept, many different representations can be used for the same concept. For example, the sine function can be represented algebraically as $sin(x)$, graphically on the Cartesian plane, or dynamically as a y-value traversing the unit circle. Any individual representation will fail to capture the entirety of its signified concept, but each will have certain benefits. For example, the algebraic notation can precisely express ordered pairs of the function and concisely express the entire concept, while the graphical representation can simultaneously express every ordered pair of the sine function. This paper will use the language of Wu and Puntambekar (2012) and refer to the aspects of a concept that are emphasized by a particular representation as the *affordances* of that representation.

Teaching Episode

The critical stages of a LIT must be tested and refined in a real classroom setting (Gravemeijer & van Eerde, 2009). In addition, these stages are accompanied by a lesson plan that exemplifies the guidance for these critical stages. To test the lesson plan, Gravemeijer and van Eerde recommend a teaching experiment (Cobb et al., 2003; Confrey & Lachance, 2000; Lesh & Kelly, 2000; Steffe & Thompson, 2000). This study had originally intended to perform a more rigorous confirmatory study by following the development of the critical stages with a full-scale teaching experiment. However, this study was focused on developing the critical stages rather than testing the lesson plan, so a teaching experiment was deemed excessively rigorous.

Nevertheless, a teaching episode was conducted and analyzed using a similar framework. In this section the relevant aspects of a teaching experiment are discussed. That is, the recommended observer who would be present in a teaching experiment will not be detailed, nor will teaching macroexperiments, data collection during lecture, or relations among student learning, teacher learning, and administrative learning. These are all vital aspects of a teaching experiment that may only be touched upon in this chapter.

A single teaching experiment research study is termed a *teaching microexperiment* (Gravemeijer & Cobb, 2006; Simon & Tzur, 2004). During a teaching microexperiment, a researcher makes hypotheses about student learning, creates a lesson plan intended to test the hypotheses, enacts the lesson plan during a teaching episode, and collects and analyzes data from teaching the lesson. The analyzed data is used to refine the lesson plan and generate new hypotheses, creating a cycle of research. This cycle is a *teaching macroexperiment*, or a Mathematics Teaching Cycle (Simon & Tzur).

The first phase of a teaching experiment includes all of the preparation leading up to the teaching episode. This begins with developing a hypothesis to be tested. Through literature and classroom experience (each possibly in the form of previous studies from a teaching macroexperiment), the researcher will develop a hypothesis about student learning that is appropriate to be studied in a live classroom setting. For example, studies on students' classroom interactions, or an instructors' classroom management are best served with data from a teaching episode. The researcher then devises a learning goal and lesson plan intended to produce data on the hypothesis. This lesson plan will include a theory of learning and how it may be applied to the classroom context, activities to help students achieve the lesson goals that are justified by the learning theory, potential misconceptions that students may develop, and contingencies to contend with foreseeable difficulties.

The implementation of the lesson plan is the second phase of the teaching experiment. For reasons that are only sometimes predictable, lesson plans cannot be counted on to proceed as intended. For this reason, Steffe and Thompson (2000) recommend that the researcher be well-experienced in teaching the subject matter. The actions that the instructors take in altering their lesson plans are influenced by their preparation for the teaching episode, including their research hypotheses. These interactions provide data on the hypotheses, so they are recorded for analysis. The entire lesson plans are video recorded in an attempt to log as much of the classroom interaction as possible, and other researchers help by validating the primary researcher's observations.

The teaching microexperiment concludes with an analysis of the data. The classroom interactions, as well as any additional data collected such as interviews, questionnaires, or assessments, is analyzed in the context of the research hypotheses. The data analysis provides information to revise the activities, allowing them to be discarded, refined, or extended to generate new research hypotheses. These hypotheses may be tested in the live classroom setting of a teaching experiment to continue the teaching macroexperiment. In the context of a LIT, this involves testing how effectively the lesson plan facilitates students' paths through the critical stages and revising the lesson plan accordingly. The procedures for analysis of the teaching episode are discussed in detail in the Methods section.

Literature Review

In this section, the relevant literature regarding students' understanding of trigonometry, identities, and graphical transformations is reviewed. In particular, it will be noted how representations were used in these studies. The literature suggests that students' understanding of trigonometry is positively affected by teachers using a multitude of representations while consistently and explicitly describing the connections between these representations.

Learning trigonometry. There are several studies of how students learn the entire trigonometry unit (Challenger, 2009; Fi, 2003; Weber, 2005), as well as how they learn particular foundational pieces such as angle measure (Moore, 2013) or the definition of the sine function (Demir & Heck, 2013; Peterson, Averbeck, & Baker, 1998; Wood, 2011). However, a review of the literature has not revealed any studies of how students come to understand trigonometric identities or transformations. In preparation for developing the current study, literature was reviewed regarding how students learn trigonometry as a whole, and how students learn identities and transformations in non-trigonometric contexts. The trigonometry studies indicate that students are developing poor understandings of trigonometry. The studies related to identities and transformations showed techniques and activities that appear to be beneficial to students learning these topics. While discussing these studies, it will be noted how students' learning may have been affected by the instructors' and students' uses of representations.

Challenger (2009) used concept maps and interviews to examine British students' progress as they learned trigonometry in an advanced secondary school class. After the students had already successfully completed a lower-level trigonometry unit, Challenger found that the students entered the advanced course with an operational understanding of trigonometry characterized by identifying traits in the task description and applying algorithms to achieve a single, numerical answer. One of the instructors for the advanced course taught with a focus on representing the trigonometric functions in a multitude of ways and consistently noting how each representation denotes the same idea in different ways. Using Multiple External Representations (MERs) in this way, consistently presenting them and explicitly noting their connections, has been shown to be beneficial for students in past studies (Eitel, Scheiter, & Schüler, 2013; Lesh, Post, & Behr, 1987; Rau, Aleven, & Rummel, 2014; Wu & Puntambekar, 2012). The other instructor taught primarily with algebraic representations, only bringing in graphical representations briefly and sparingly to prove a particular point.

The students in Challenger's (2009) study who were taught using frequent, explicitly connected sets of multiple representations were each assessed to have developed stronger understandings of trigonometry than the students who were primarily taught using algebraic representations. The assessments included graded classroom assignments as well as interviews and concept maps implemented by Challenger. The former set of students demonstrated that they could choose representations appropriate to their tasks and could translate between representations as needed, which are important mathematical skills.

In contrast, Challenger (2009) noted that the students who were taught primarily with algebraic representations continued to understand the different trigonometric representations separately. These students could perform familiar tasks successfully when they were given a proper representation, but they did not demonstrate that they understood how the representations were connected to each other well enough to be able to comfortably change between different representations depending on the task requirements. They could produce the graphs for each of the trigonometric functions, but they could not identify how changes to one representation would affect another or solve problems for which it was necessary to glean information from multiple representations. The students in this study also generally did not show an ability to connect their knowledge of identities to any other piece of knowledge. With one exception, they were unable to express ideas that connected identities to any concept other than the trigonometric functions present in the identity, nor could they apply their knowledge of identities to any situation that was not similar to problems presented in lecture.

These results support previous comparative experiments that indicate that students who are provided with MERs outperform those who are only given single representations (Eitel et al., 2013; Rau et al., 2014). MERs have the benefit that different representations may have different affordances. For example, the graph of a sinusoid more prominently illustrates its zeros, while its algebraic representation more clearly shows its amplitude and phase shift. In comparative experiments, having access to representations with appropriate affordances has been shown to aid students' problem solving (Kendal & Stacey, 1998; Schnotz & Bannert, 2003). Also, students who were familiar with MERs were able to choose the one that had the most applicable affordances for a given situation (Lesh, Post, & Behr, 1987; Weber, 2008; Wu & Puntambekar, 2012).

In Australia, Kendal and Stacey (1998) compared the abilities of secondary school students receiving two different styles of trigonometry instruction. One group learned the basics of trigonometry through right triangles; the other learned through the unit circle. Clouding the issue is the fact that that students learning through the unit circle method were taught using only first quadrant angles and a particular algorithm for scaling the unit circle and orienting the triangles within it. These students were not taught to evaluate trigonometric functions using the unit circle, but rather were taught how to apply a triangle trigonometry algorithm to a unit circle context. Ultimately, these students were applying a similar algorithm to the tasks but with an added layer of interpretation.

Kendal and Stacey (1998) assessed each program's ability to instill in students a foundational understanding of the trigonometric functions by testing the students' abilities to find unknown lengths of right triangles. However, rather than assessing the students' understanding of trigonometric functions, their study seems to assess students' abilities to apply the functions to a particular problem context. This is more akin to a study by Schnotz and Bannert (2003) in which the data indicates that students who are presented with a representation that has affordances beneficial to the problem task were more likely to be able to solve the problem than students who were not provided with any representation, while students who were given a representation with inappropriate affordances were less likely to be able to solve it. In Kendal and Stacey's study, the students who had learned the algorithm with triangle representations were better able to solve these problems than students who had learned the circle scaling algorithm.

In a comparative experiment conducted during an undergraduate trigonometry unit in the United States, Weber (2005) compared students' understandings of the trigonometric functions after having been taught in one of two ways: one class was taught in a traditional definition-theorem-proof lecture format, while students in the other class were encouraged to view each representation both in its given context as well as describing a process. For example, $sin(x + \pi)$ as an algebraic representation provides the output of the sine function for an input π radians greater than x; it also describes the process of constructing a radius of angle $(x + \pi)$ radians on the unit circle and measuring the *y*-value of the endpoint of that radius. Students in this class were consistently asked to connect each trigonometric representation to a definition of the trigonometric functions. The students who were prompted to connect their work back to the definitions were assessed by Weber to have a better understanding of trigonometry than their counterparts on a test for conceptual understanding of trigonometry.

Even preservice mathematics teachers who have passed a course that covered trigonometry have been seen to struggle with relating different representations of trigonometric functions. Fi (2003) examined preservice teachers' pedagogical and subject content knowledge using concept maps, interviews, card sorting, and a set of trigonometric problems. He found that the preservice teachers had learned trigonometry in an instrumental manner and had difficulty finding connections among its various aspects. While his study was on students' understanding of a trigonometry unit as a whole, there was a section on identities and transformations that showed that the preservice teachers had misconceptions in these areas and had difficulties deriving and applying any identities except for the Pythagorean identity. The preservice teachers also could not demonstrate an understanding of the various effects of graphical transformations, including the counterintuitive properties of horizontal transformations. However, Fi's study did not examine the reasons for these particular difficulties.

There are several studies exhibiting positive techniques for teaching particular topics in trigonometry. Borba and Confrey (1996) describe a *rubber sheet* method of teaching function transformations, in which the student is asked to visualize graphical representations as being made of two transparent, malleable rubber sheets. One sheet contains the axes, while the other has the curve of function outputs. Horizontal transformations are considered to act on the sheet of axes, while vertical transformations act on the sheet with the curve. By considering transformations of the function input as acting on the inputs in the graphical representation (the *x*-axis), the non-intuitive properties of horizontal shifts and stretches are resolved. For example, shifting the axes to the right has the same effect as shifting the curve of outputs to the left.

Learning with representations. Two limitations with the rubber sheet method are that it is difficult to visually represent the acts of stretching axes and curves without a dynamic computer program, and it still does not offer any clues as to why the horizontal transformations should be applied in seeming defiance of the order of operations. For the latter, I will offer my own contribution in the methods section; for the former, if the technology is available, teachers may want to consider allowing students to use dynamic, interactive technologies. Dynamic and interactive representations have been shown to provide benefits to students' learning (Karadag & McDougall, 2009; Moreno-Armella, Hegedus, & Kaput, 2008 Özdemir & Ahvaz Reis, 2013; Zengin, Furkan, & Kutluca, 2012). Moreno-Armella, Hegedus, and Kaput examined the historical evolution of representations, from static, inert (non-interactive) representations such as textbooks through dynamic, continuously interactive representations. The authors argue that this evolution has generally resulted in better representations, as the new dynamic, interactive representations can be manipulated by the learner to show different situations as needed.

Dynamic representations can provide additional scaffolding for students as they attempt to visualize relationships between representations. For example, a graph of the cosine function presented alongside a representation of the unit circle with a radius whose endpoint is labeled with its value in terms of cosine makes a clear connection between the unit circle definition of cosine and a single point on the graph of cosine. Using these representations, it is up to the student to extrapolate how the rest of the graph results. However, if the unit circle representation dynamically shows the radius rotating while the graph of cosine is drawn simultaneously, then instead of a single explicit point of connection, there are an infinite number of connections. With the additional connections to reference, students could be less likely to develop certain misconceptions or begin work on unproductive paths (Salomon, 1993). For example, students working with the static representations may need to be reminded that we measure angles counterclockwise from the positive x-axis, while the dynamic representation models this automatically. In the latter scenario, the student may be led to wonder about the rotation direction and how it would be represented, while a student in the former may simply worry about which direction is correct.

Although the literature review has revealed studies where static representations such as triangles, equations, or circles are used to teach trigonometry, these representations seem to do a poor job of conveying the functional nature of trigonometric operations (Kendal & Stacey, 1998; Weber, 2008). For example, a representation of the wrapping function (where the real number line is drawn wrapped around a unit circle and each number corresponds to an arc length and subtended angle) implies that this is a function, but learners still don't necessarily see it as such after traditional instruction (Tuna, 2013). In contrast, the superiority of dynamic representations has been demonstrated numerous times (Özdemir & Ayvaz Reis, 2013; Zengin, Furkan, & Kutluca, 2012).

Representations are further improved by being made interactive. Interactive representations provide different responses based on how they are interacted with. For example, a scientific calculator provides the solutions to various computations given as inputs. Inert representations provide a single presentation of the concept for students to interpret. By contrast, interactive representations provide a feedback cycle (Goldin & Kaput, 1996; Martinovic & Karadag, 2012). The forms of the presentations are limited – the students' by how they can input information into the representation, and the representations' by how it has been designed to provide outputs – but the students may still continue to alter their presentations within the representation's parameters in order to refine their conceptions. For example, in the previously discussed representations linking the unit circle definition of the cosine function to its graph, consider if students were able to highlight points of the graph and be presented with the corresponding angle on the unit circle. Students using such a representation could explore the relationships between the representations and possibly discover some trigonometric identities. Ainsworth (2006) demonstrated that multiple representations that are linked in this way are helpful for students' learning. The extra scaffolding provided by these types of representations and the immediacy of the effects of altering representations aid students in noticing the relationships between them.

Dynamic, interactive representations have been successfully used to aid students' learning in trigonometry classrooms (Kessler, 2007; Rosen et al., 2005; Sokolowski & Rackley; 2011; Wilhelm & Confrey, 2005; Zengin et al., 2011). Studies by Kessler, by Rosen and colleagues, and by Wilhelm and Confrey examined the effects of providing students with such representations for trigonometric applications to the study of sound waves. It is possible that these lessons were effective because the applications allowed students to make additional connections among their experiences with the application, the external representations used in class, and their own understandings of the underlying concepts. Some researchers also theorized that students were more willing to actively engage with concepts that connect to their experiences out of school (Douglas et al., 2008; Kessler; Rosen et al.).

Although these studies show dynamic, interactive representations to be generally helpful, there is a danger that they can be used to automatically perform tasks that students should be reflecting on. While technology is helpful for students when it is used, among other ways, to perform tedious tasks at which students are proficient (Ellington, 2003), it should not be used to replace reasoning. In a study by Rosen and colleagues (2005), students were asked to use a computer program to solve trigonometry problems. These students were able to complete the tasks successfully, but when they were later asked to reflect on why their answers were true, they were unable to justify their responses other than to say that the computer confirmed that they were correct.

In contrast, Zengin and colleagues (2011) and Sokolowski and Rackley (2011) provided dynamic, interactive representations for students that showed waves on a string without an application for greater context. In these studies, as well as one by Wilhelm and Confrey (2005), students interacted with computer programs by exploring, hypothesizing, strategizing, testing, and generalizing their thoughts about the sinusoidal functions. Through these reflective interactions, the class was able to develop several properties of trigonometric functions on their own as opposed to accepting those properties as true without examination. In each of these cases, the authors describe students performing well on assessments after receiving these lessons and as having the ability to justify their work, indicating that these representations were helpful to the students' construction of knowledge for trigonometric topics.

Learning identities and transformations. Tsai and Chang (2009) studied students learning algebraic binomial identities through clothes-matching tasks supplemented with geometric representations. By finding the number of potential combinations of outfits, students were able to find patterns in the results that they had not predicted given just the algebraic representations. There was a common misconception with the algebraic representations that exponents would distribute across a binomial, and there were other distributive mistakes that were often made in these representations. When working with actual articles of clothing prior to the algebraic representations of this applied situation, the students were less likely to make these mistakes. This proficiency continued as the students were able to apply the properties that they had learned to subsequent algebraic manipulations as well as extend their ideas to trinomials.

On the topic of transformations, several studies have noted that students have difficulties justifying the effects of horizontal transformations (Barton, 2003; Borba & Confrey, 1996; Faulkenberry & Faulkenberry, 2010; Hall & Giacin, 2013). However, few have had much to contribute beyond asking students to remember that the horizontal transformations behave differently than the vertical. Two methods have been offered: an algebraic method from Hall and Giacin, and the rubber sheet method from Borba and Confrey.

Hall and Giacin (2013) describe an activity in which they guide a class of students through an examination of ordered pairs of transformed functions. The students are generally capable of grasping the vertical transformations, however there are the noted difficulties with horizontal ones. The authors guide the students through a *u*-substitution so that, for example, the transformed ordered pair $(2x + 1, y)$ could be rewritten as $(x', f(\frac{x' - 1}{2}))$. An analogous vertical transformation would be $(x, 2y + 1)$ rewritten as $(x, 2f(x) + 1)$ with no substitution necessary.

This method has the benefit of providing a representation where the order of horizontal representations is intuitive. However, it does have the drawback that it may not itself be intuitive. These substitutions are only made for transformations of the input, and students may not make the connection that $f(\frac{x' - 1}{2})$ represents the horizontal transformations that it is intended to. Students who have learned transformations instrumentally may misinterpret this new representation and graphically undo their algebraic work to arrive at the mistake that the authors hoped to avoid. Students viewing the subject for the first time may graph the function on the x'- and y'-axes and be unable to translate their work to the x- and y-axes, which would leave the students with an incorrect graph.

Borba and Confrey (1996) use what they refer to as the rubber sheet method to examine horizontal transformations. In this method, students are instructed to consider the axes and the curve of the function separately. Each can be considered as existing on a clear rubber sheet that can be stretched, reflected, and shifted without affecting the other rubber sheet. The vertical transformations affect the outputs of the function and therefore the curve of the function on the graph. These transformations are considered in the familiar manner in this method. Horizontal transformations, on the other hand, affect the axes, which move together since the y-axis is conventionally placed at $x = 0$. When the axes are transformed, they appear to have the opposite transformative effects upon the graph. For example, stretching the axes horizontally by a factor of two (then rescaling to be proportional to the units of *y*) appears to make the graph shrink, and shifting the axes to the left has the same effect as shifting the graph to the right.

Borba and Confrey's (1996) method may be more intuitive than Hall and Giacin's (2013) method. The latter requires the introduction of another representation, the ordered pair, while the former stays with the graphical representation. It is also an intuitive notion that the transformations of a function's input should affect the axes of the graph, since those represent the real number line of inputs. However, while the rubber sheet method does an adequate job explaining why the horizontal transformations have the effects that they do, it does not show why the order in which these transformations must be applied is counterintuitive. Stretching then shifting the axes still results in the wrong graph, despite that being the intuitive way to apply the transformations.

In this chapter, it has been argued that there are deficiencies in trigonometry education. Specifically, students leave trigonometry classes with poorly connected conceptions of trigonometric ideas, including trigonometric representations. It has also been argued here that social constructivism and representations will be useful tools in developing a LIT to help students learn trigonometric concepts well enough to justify them.

Critical Stages of Understanding for Trigonometric Identities

Based on the review of the literature and the researcher's experience teaching trigonometry, the following critical stages have been hypothesized for learning trigonometric identities:

0. *Prerequisite Knowledge*. Before learning trigonometric identities, students must possess an understanding of several concepts, notably algebraic and graphical representations of functions, and definitions of identity and the trigonometric functions.

1. *Notice a change in the algebraic representation*. Students must recognize that there has been a change to the algebraic representation of the "parent" function. In order to understand how the function has been affected, the student must first identify that there has been a change to the original function. At this stage, students only need to notice that there is a change.

2. *Change to a representation with better affordances*. The algebraic representations of trigonometric functions do not have good affordances for noticing the effects of these changes. Examining the effects in non-algebraic representations is a consistent theme in the literature (Barton, 2003; Borba & Confrey, 1996; Confrey, 1994; Fauleknerry & Faulkenberry, 2010).

3. *Notice that the changes to the algebraic representation correspond to changes in the other representation(s)*. Similar to stage one, the first step in classifying how these changes affect the trigonometric functions is noticing that there *is* some kind of correspondence between the representations. Tasks from the literature (Axler, 2013; Barton, 2003) ask students to identify a pattern that exists among a sequence of pairs of algebraic and graphical representations. This assumes that the students notice that there is a correspondence between the representations that could give rise to a meaningful pattern.

4a. *Notice that using the opposite input has predictable outcomes*. Using graphical representations, these transformations are viewed as horizontal reflections. In the case of cosine, this has no effect on the resulting graph, but for sine and tangent, the effects are the same as having undergone vertical reflection.

With the unit circle, a negative input is considered as an angle measured clockwise from the positive x-axis rather than counterclockwise. Cosine, given by the x-value of the endpoint of the radius, remains unchanged as the radius still oscillates between 1 and -1 in the same fashion. The y-values of the endpoint, however, take on the opposite values. Combining these two results with the tangent identity indicates that tangent will also take on opposite values under this transformation. These realizations, along with the rest of those in this stage, are learning goals for identities, and are thus necessarily critical stages.

4b. *Notice that adding multiples of π to the input results in predictable outcomes*. Students can use the graphical or circle representations to find how adding multiples of π to the input affect the values of the functions. In the graphical representations, these transformations correspond to horizontal shifts. Since tangent has period π, these transformations will result in identical graphs. Sine and cosine each have period 2π, so even multiples will also result in identical graphs. Furthermore, shifting by odd multiples of π will result in a graph that is perfectly out of phase with the original, or equivalently, a vertical reflection of the parent function.

If students use the unit circle to examine these transformations, they would find that the radius has been rotated either back to the starting position (for even multiples of π) or opposite the starting position (for odd multiples). For the former transformations, the trigonometric functions will produce the same values since the endpoint of the radius is in the same location; for the latter, since the beginning and ending radii are symmetric through the origin, their endpoints will have opposite x- and y-values, resulting in opposite cosine and sine values, but the same tangent values.

5. *Notice that* $cos(x - \frac{\pi}{2}) = sin(x)$ *and* $sin(x + \frac{\pi}{2}) = cos(x)$. It was hypothesized that in order to justify the cofunction identities, students would have to pass through an intermediate stage of understanding in which they recognized that the sine and cosine functions produce the same sets of outputs but for a set of inputs shifted by $\frac{\pi}{2}$. These proto-cofunction identities are not particularly useful on their own, but they were hypothesized to be useful for developing the cofunction identities. Students at this stage will have noticed connections between the sine and cosine functions and the angle $\frac{\pi}{2}$. Later, these connections will be synthesized into their more elegant, conventional forms.

6. *Reflect upon the relationships between cosine, sine, and* $\frac{\pi}{2}$ *in a right triangle representation*. In a right triangle, one of the angles is $\frac{\pi}{2}$ radians, meaning that the sum of the other two angles is $\frac{\pi}{2}$. So, if one of the angles is x radians, the other will be $(\frac{\pi}{2} - x)$ radians. Furthermore, the leg that is opposite one angle is adjacent to the other. Therefore, the sine of one angle will be equal to the cosine of the other and vice versa. Blackett and Tall (1991) used tabular data to give students "early insight into the complementary relationship between the increasing table of sines and the decreasing table of cosines" (p. 147). These tables were generated by examining right triangles in which the acute angles were multiples of ten degrees.

7. *Generalize cofunction identities using another representation*. The literature on trigonometric identities that was reviewed has stopped short of exploring students' conceptions of how this

identity can be extended beyond acute angles. It is reasonable to hypothesize that students must change to a representation other than right triangles in order to justify that the cofunction identities hold for non-acute angles. These stages are presented in Figure 1.

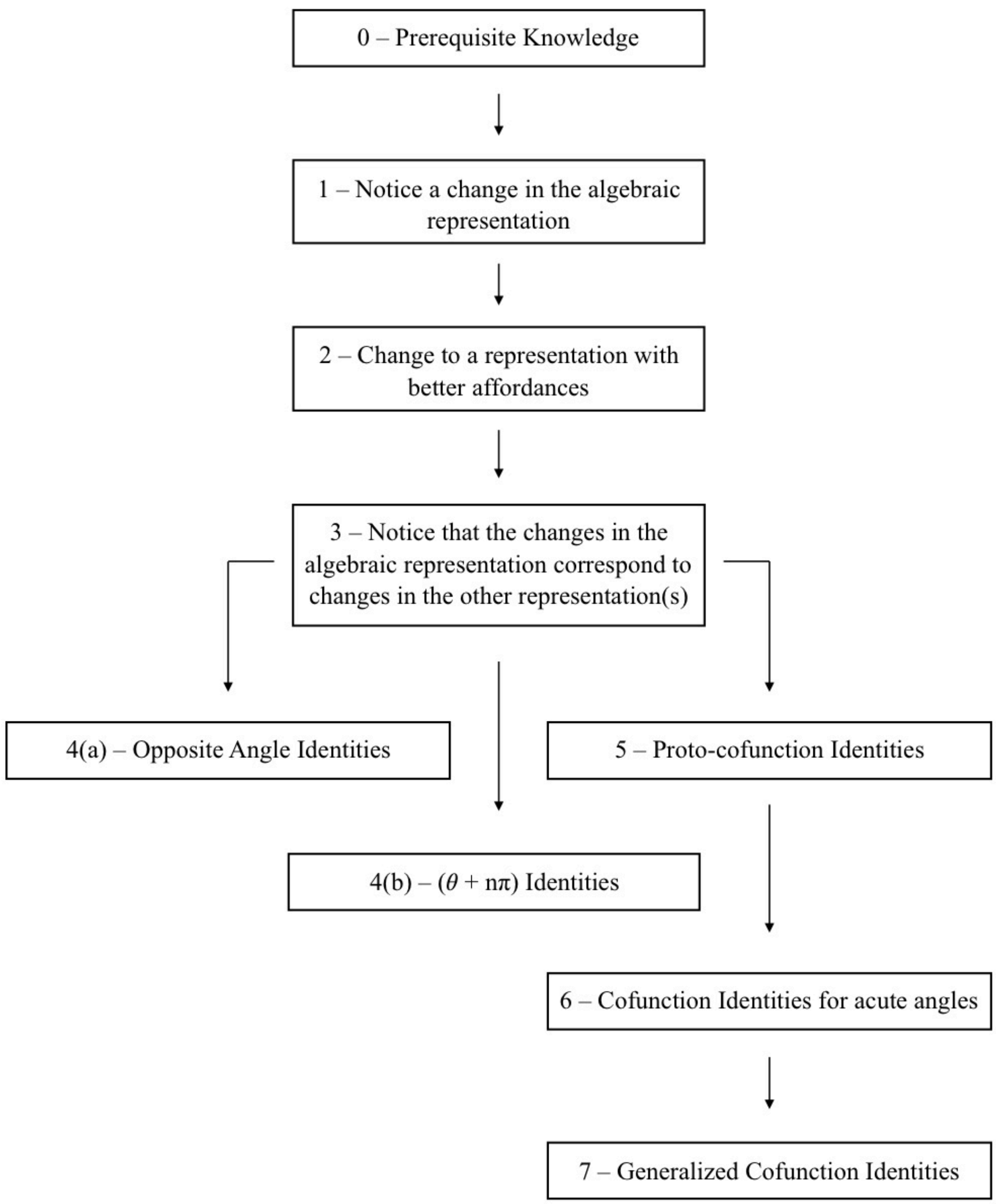


Figure 1. Identities critical stages

Critical Stages of Understanding for Trigonometric Transformations

Based on a review of the literature and the researcher's experience teaching trigonometry, the following critical stages have been hypothesized for learning transformations of trigonometric functions:

0. *Prerequisite Knowledge*. Before learning trigonometric transformations, students must possess an understanding of several concepts, notably algebraic and graphical representations of functions, and definitions the trigonometric functions.

1. *Notice a change in the algebraic representation*. Students must recognize that there has been a change, or transformation, to the algebraic representation of the "parent" function. In order to understand how the function has been affected, the student must first identify that there has been a change to the original function. At this stage, students only need to notice that there is a change.

In previous studies of transformations, this stage has sometimes been implicit, as in Barton's (2003) introductory task: "Sketch the following family of curves… $y = x^2$; $y = (x - 1)^2$; $y = (x - 2)^2$; $y = (x - 3)^2$." It is assumed that the students will notice that the representation has been changed. On the other hand, Borba and Confrey's (1996) example of a typical introductory task is "If $y = x^2 + 5$ is changed to $y = 2x^2 + 5$, how does the graph of the transformed function change" (p. 320)? In this case, the changes to the representations are explicitly remarked upon, however they are not the main focus of the problem. These examples indicate that the change must be noticed before further work can be accomplished.

2. *Change to a representation with better affordances*. The algebraic representations of trigonometric functions do not have good affordances for noticing the effects of transformations.

Examining the effects in non-algebraic representations is a consistent theme in the literature (Barton, 2003; Borba & Confrey, 1996; Confrey, 1994; Fauleknerry & Faulkenberry, 2010).

3. *Notice that changes to the algebraic representation correspond to changes in the other representation(s)*. Similar to stage one, the first step in classifying how these changes affect the trigonometric functions is noticing that there *is* some kind of correspondence between the representations. Tasks from the literature (Axler, 2013; Barton, 2003) ask students to identify a pattern that exists among a sequence of pairs of algebraic and graphical representations. This assumes that the students notice that a correspondence between the representations exists and that this correspondence could give rise to a meaningful pattern.

4a. *Classify changes to the algebraic representation as addition or multiplication*.

4b. *Classify changes to the algebraic representation as affecting the input or the output of the function.* Stage four involves the beginning of classification of transformations. Identifying the different ways in which the algebraic representation has been changed is a way to begin that process. There are other ways to classify algebraic transformations (such as those with positive or negative numbers), but I don't expect it to be critical for students to be that detailed in their classifications.

These classifications fit what Confrey (1994) refers to as the *template approach* to teaching transformations: classifying the effects of *a, b, c,* and d as a function $f(x)$ is transformed to $(a)f(bx + c) + d$. These classifications are one goal of tasks such as Barton's (2003) that asks students to sketch families of curves. They also serve as the basis for more sophisticated methods.

4c. *Classify changes to the graphical representation as shifting or stretching (or reflecting)*.

4d. *Classify changes to the graphical representation as horizontal or vertical*. Just as students notice that algebraic transformations can be classified as acting on the input or the output of the function, or as occurring through addition or multiplication, the graphical transformations can be classified as acting horizontally or vertically, or as shifting, stretching, or reflecting. Students may go so far as to classify the graphical transformations as rigid (shift) or proportional (stretch, shrink, or reflect) in order to have an equal number of types of algebraic and graphical classifications without singling out negative multiplication. However, it is not expected that students will need to make this distinction to develop a sufficient understanding.

Examining correspondences between algebraic and graphical representations may be used as a catalyst for classifications. This is another goal of a series of questions like Barton's (2003) that asks students to examine the correspondences between representations for a variety of transformations. These classifications may be interpreted in different ways, but they are present in all of the reviewed literature on graphical transformations. They are also present explicitly in the Common Core State Standards in the form of comparing "transformations that preserve distance and angle to those that do not" (p. 76).

(4e. *Classify changes to the unit circle as affecting the circle or the angle of the radius*.)

(4f. *Classify changes to the unit circle as positioning or scaling*.) If students choose to approach the transformations through the unit circle representation, they may classify transformations of sine and cosine as affecting either the circle (or equivalently the radius position or length) or the angle of the radius. The size of the circle could be changed to correspond with changes in amplitude, and the circle could be translated to account for what will be classified as vertical shifts in the sine or cosine graphs. Horizontal transformations correspond to changes to the radius' starting position and velocity.

The review of the literature has not revealed any studies including transformations of the unit circle, but if students choose to use this representation, then they should make the same sorts of classifications at this juncture as they would in other representations.

5a. C*lassify addition as shifting (or translating the circle and rotating the radius)*. Students must examine the relations among the representations of the trigonometric functions. Students may notice that addition and subtraction in the algebraic representation correspond to rigid shifts in the graph. If students chose to transform the unit circle to understand the effects of the algebraic transformations, they could find that addition to the output could be represented by translating the graph vertically (for sine) or horizontally (for cosine), while addition to the input could be represented by changing the starting angle for the radius as it rotates.

The classifications of stage five are present in each of the reviewed studies on transformations. The studies differ in their methods of using these classifications, but a representative sentiment (that also begins examining stage seven – counterintuitive horizontal transformations) is offered by Barton: "I want my students to make the crucial connection that the inclusion of the $(x - a)$ factor moves the curve a units in the positive direction" (p. 13). While the "crucial connection" may be that this shift is not in the intuitive direction, that assumes that the student has made a previous connection that subtracting a does in fact cause a shift.

5b. *Classify multiplication as stretching (or changing the radius of the circle and the speed of the radius).* Students may also notice that multiplication and division in the algebraic representation correspond to a stretch in the graphical representation. Because of the counterintuitive nature of the horizontal transformations, students cannot further distinguish between stretching and shrinking without parsing the types of transformations too finely for this stage. If students use the unit circle, they may find that they could represent multiplication by either scaling the size of the circle or the speed of the radius.

(5c.) *Classify negative multiplication as reflection (and the orientation of the circle or the direction of the radius' rotation*). Students may also feel it necessary to classify negative multiplication as reflection. It should not be critical that students make this distinction, but neither should it be critical that students view reflection as a type of stretch in order to develop a satisfactory understanding of transformations and identities. With time, students may come to view stretching, shrinking, and reflecting as part of the same continuum of transformations, but the important aspect of this stage is connecting multiplication with stretching.

5d. *Classify transformations of the input of the function as horizontal (and affecting the radius in the unit circle)*. As students transform the input of the function, the graph will be affected in the direction of the axis of inputs, the horizontal axis. In the unit circle, the inputs of the trigonometric functions are represented as the radii, so that is what these transformations affect.

5e. *Classify transformations of the output of the function as vertical (and affecting the circle of the unit circle representation)*. The transformations of a function's output will graphically affect that function along the axis of the outputs, the vertical axis. On the unit circle, the outputs of the cosine and sine function are given by the x- and y-values of the endpoints, so these transformations must transform the endpoints of the radii (i.e. the circle itself).

6. *Recognize that these graphical transformations affect the entire graph*. None of the transformations shift, stretch, or reflect only a piece of the graph. This means that

transformations have predictable effects on the entire domain or range. This stage is implicit in many studies of transformations, such as when Lando and Lando (1977) constructed a sinusoid modeling temperature annually and asked students to find the predicted temperatures for future dates. These predictions would not be possible to make if the transformations did not affect the entire function. Explicit evidence of this stage in student work can be seen in Borba and Confrey's (1996) case study, when the student, describing the effects of transforming $y = x$ to $y = 2x$, said to "take the whole graph paper and stretch it out…" (p. 330).

7. *Notice that the horizontal transformations act counterintuitively*. Students may approach the topic of transformations believing that any positive operation (or any operation greater than one) would move the graph upwards, rightwards, (or make it larger), while negative operations (or operations between zero and one) would move the graph downwards, leftwards, (or make it smaller). However, the horizontal transformations do not act in this way. This is likely what Barton (2003) is referring to in the quote mentioned earlier: "I want my students to make the crucial connection that the inclusion of the $(x - a)$ factor moves the curve a units in the positive direction" (p. 13). Borba and Confrey (1996), Faulkenberry and Faulkenberry (2010), and Hall and Giacin (2013) believed this connection to be crucial enough to devote their studies to examining ways to explain this counterintuitive behavior.

8. *Notice the effects that the transformations have on the period and phase of the functions*. Students should notice that horizontal stretches affect the periods of the functions. Mutiplying by a factor of absolute value greater than one results in a shorter period, while factors of absolute value less than one lengthen the period. Horizontal shifts, meanwhile, result in a change of phase unless they are an integer multiple of the period in length. Since the literature review did not

reveal any studies on transformations of periodic functions, I do not have evidence from prior work. However, it can readily be seen that transformations affecting the period and phase of a trigonometric function have important effects in engineering applications (Kuttruff, 1973; Rigden, 1977; Wilhelm & Confrey, 2005).

9. *Recognize that the order in which transformations are applied sometimes results in different graphs or outputs*. Students must notice that using the same set of transformations does not always result in the same graph or numerical value if the transformations are not applied in the same order. Students are familiar with the fact that the order in which operations are applied can have an effect on the result, and they must apply this reasoning to functions. Hall and Giacin (2013) recommend a combination of algebraic and graphical representations to familiarize students with this fact.

10. *Recognize that it matters when a rigid and proportional transformation are combined*. Since all of the proportional transformations have an axis line at their center, it affects the final output if the function is shifted in relation to the axes. Furthermore, two transformations of the same type may either be simplified into a single transformation (e.g. stretching vertically by a factor of two and reflecting vertically may be reduced to a vertical stretch by a factor of negative two), or may affect the function along different axes. In the latter case, the transformations will not affect each other, as addressed in the following critical stage.

11. *Recognize that the order only matters between transformations in the same direction*. Given a mix of horizontal and vertical transformations, the order of horizontal transformations matters, and the order of vertical transformations matters. However, the order between horizontal and vertical transformations does not matter. This can be justified graphically by noticing that

transforming a graph along one axis will not affect its position relative to the other axis. Therefore, transformations along the latter axis may be carried out before or after those on the former.

One of the strengths of Hall and Giacin's (2013) algebraic *horseshoe* method – in which the transformation process can be modeled with a diagram where transformations of the *x*- and *y*-variables are viewed vertically while the function effect bridges the two horizontally – is that the horizontal and vertical transformations are somewhat separated from each other in the algebraic notation. This may help students recognize that these transformations may take place in either order.

12. *The combination of horizontal transformations behaves counterintuitively*. Contrary to the order of operations, additive horizontal transformations are applied before multiplicative horizontal transformations. Confrey (1994) notes that students have been "totally perplexed by the result" (p. 222). The horseshoe method addresses this by unpacking the horizontal transformations with a substitution before applying the function. The unpacked version of the input shows that the order of horizontal transformations will be counterintuitive for the same reasons that other aspects of horizontal transformations were.

13. *Consider horizontal transformations individually in a representation with better affordances*. The algebraic and graphical representations do not emphasize why the horizontal transformations act the way they do. Teachers often simply tell students to remember that these transformations act non-intuitively (Borba & Confrey, 1996). By considering horizontal transformations with a substitution in the algebraic representation (Hall and Giacin, 2013), as transforming of the radius of the unit circle, or as transforming the coordinate axes (and not the curve of the graph) (Borba

& Confrey, 1996; Faulkenberry & Faulkenberry, 2010), students will have the opportunity to see the process by which intuitive transformations in these representations result in the counterintuitive effects in the graphical transformation.

In the algebraic representation, the new domain is found by unpacking the horizontal transformations, which results in the domain being transformed by the inverses of the additive or multiplicative transformations. In the graphical representation, since the transformations are affecting the inputs of the function, they should affect the inputs of the graph: the axes. For example, a horizontal stretch by a factor of two would double the distance of the unit measure. This transformation would affect the axes, and not the curve of the graph. Therefore, if the graph were rescaled, the effect would be the same as if the graph had been horizontally *shrunk* by a factor of two. On the unit circle, the starting position and speed of the radius can explain the starting position and frequency of the graph.

14. *Consider the order of horizontal transformations in a representation with better affordances*. To see the order of horizontal transformations behave intuitively, students may use either *u*-substitutions in the algebraic representation to consider the domain of the function (Hall & Giacin, 2013), or the unit circle representation. Transforming the axes of the graph in the order prescribed by the order of operations does not, however, result in the correct graph.

In the algebraic representation, the new domain is found by unpacking the horizontal transformations, which results in the domain being transformed first by the inverse of the additive transformation and second by the inverse of the multiplicative transformation. In the unit circle, where the additive horizontal transformations affect the starting position of the radius and the multiplicative transformations affect the speed, it is necessary to consider the former before the latter; one must place the radius in a starting position before setting it to move at a certain speed. These critical stages are presented in Figure 2.

Conclusion

This chapter has described the foundational theories and hypotheses upon which this study is established. Social constructivism is the theory of learning that was used to hypothesize the critical stages of understanding present in a LIT for trigonometric identities and transformations. These sets of critical stages have several similarities. This stands to reason since identities can be viewed as particular examples of transformations. Using social constructivism as a theory of learning, students learning these topics can be viewed as making connections among similar concepts. This will be reflected in the methods by the similarities in the processes of investigating each concept.

III. Methods

Research Approach

This research study is composed of a two-stage main study and a confirmatory study. The main study was intended to check the adequacy of the hypothesized critical stages and reveal any changes that needed to be made, as well as examine students' use of representations. The confirmatory study was designed to assess a lesson plan informed by those critical stages. The process of coming to understand a concept is complex. This study largely used qualitative methods to investigate this process. Students were interviewed as they came to understand a topic, and a grounded theory approach was used to analyze the interview data. This method of analysis was appropriate since this study intended to develop a new theory from raw data. In the confirmatory study, with a revised set of critical stages, a mix of qualitative and quantitative methods were used to assess the extents to which students progressed through the critical stages as a result of a teaching episode. The goal of this confirmatory study was to demonstrate one method by which the LITs could be implemented.

Setting and Participants

This study is targeted towards students learning trigonometry. As such, a population of undergraduate students were recruited who had not passed an undergraduate precalculus course. Ideally, the study participants would not have even taken a precalculus class at any level. However, from a recruitment perspective, this was not possible as it would not have yielded enough participants. This was not considered to be a significant issue both because of the length of time between the current research study and the participants' previous trigonometry experience – at least three months in all cases – and because the students generally did not

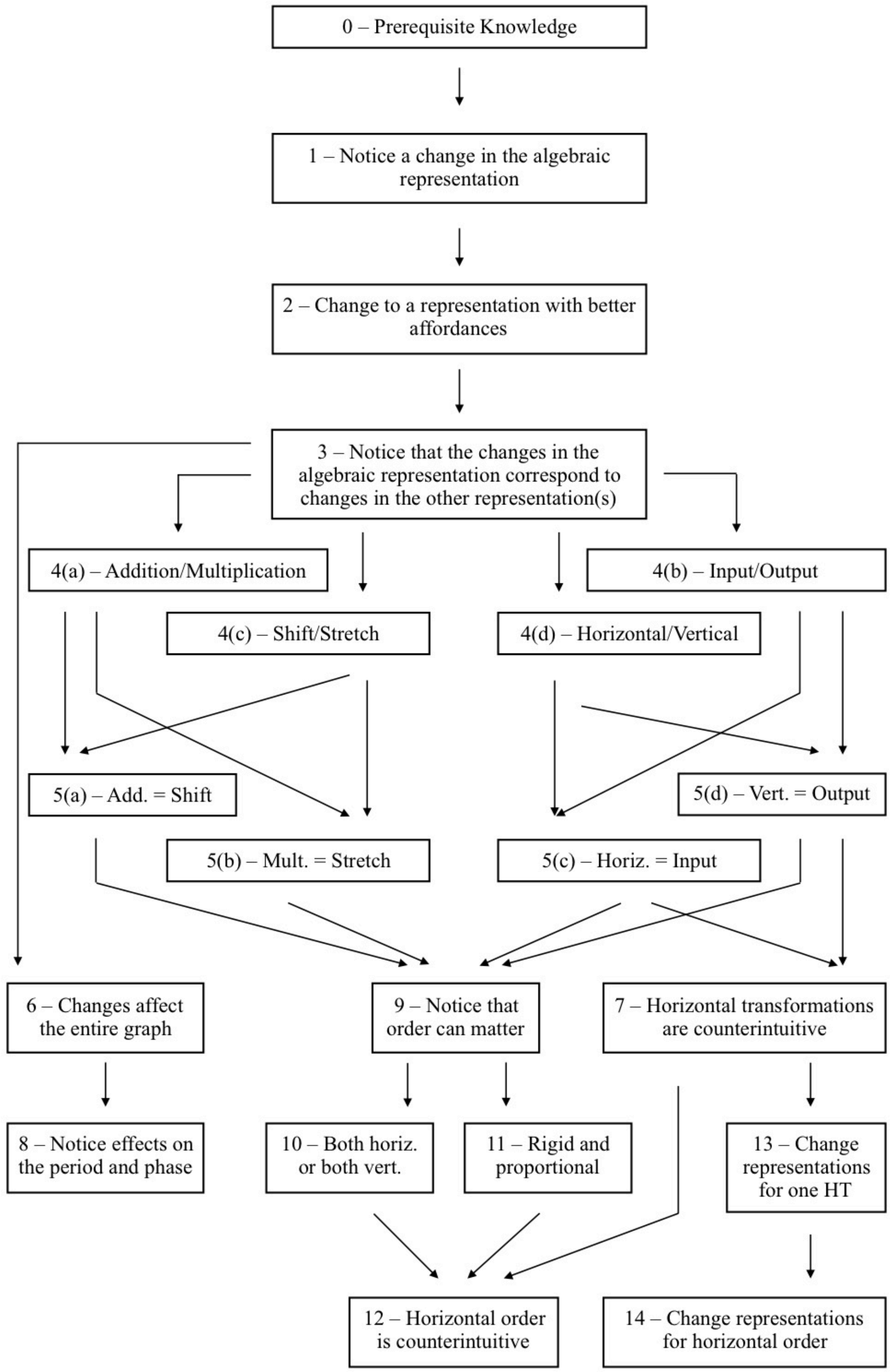


Figure 2. Transformations critical stages

understand the material well enough to test out of their undergraduate precalculus classes. By utilizing students who had not gained enough understanding to pass an undergraduate course or placement exam with trigonometry, this study was able to examine the processes by which the students developed understandings of trigonometric concepts and the difficulties that they encountered. It was hoped that a sample of students who displayed a range of levels of understanding could be recruited. This had the potential to provide more information than a homogeneous population regarding how critical stages should be expanded, introduced, deleted, and adapted to lesson plans. However, given the small number of participating students, every volunteer was included. Nevertheless, the participants did display a range of capabilities.

The participants were recruited from a medium-sized university in the northeast United States. Six students were recruited for the first stage, and an additional six were recruited for the second. The confirmatory study had sixteen participants. For the first stage, six students were recruited to participate in multiple interviews that covered identities and transformations of trigonometric functions. Interview protocol for this stage can be found in Appendix A. Morse (1994) and Ray (1994) recommend between six and twelve participants for interview-based studies provided that codes start repeating by the sixth interview. Repeating codes provide evidence that the thoughts and actions viewed among the students are general, not isolated incidents. Additionally, another mathematics education graduate student with qualitative research experience validated the coding to ensure that the same codes were in fact repeating. There were repeating codes, however these interviews were unable to cover all of the desired topics, so a second group of participants was recruited the following semester, and the interview tasks were revised. Stage two interview protocol can be found in Appendix B.

By having the same students participate in multiple interviews, the researcher was able to build a rapport with the students, which was helpful in subsequent interviews, since the interview did not need to begin anew each time (Seidman, 2013). This also allowed for a continuous construction of models of the students' understandings, which aided data analysis. The researcher was able to enter the second (and possibly third) interviews with more information with which to model the students' understandings. This allowed for a deeper investigation into students' understandings in the subsequent interviews. For example, in subsequent interviews, students were asked for clarification of any unclear answers from the first interview. In addition, having an idea of how students model their mathematics at the start of the interview made it easier to follow the students' reasoning and allowed for more material to be covered in the followup interview(s). This improved model of students' understanding provided better information with which to refine the proposed critical stages. Covering more material offered more data to work with, and more information about students' reasoning allowed more informed judgements to be made on what methods of reasoning are common to students at various stages.

For the confirmatory study, sixteen participants were recruited from an undergraduate precalculus course. These recruits were similar to the interview participants in that they were enrolled in an undergraduate precalculus course at the same university at the time of the study. These students took pre-tests to assess their level of attainment of the critical stages, which can be found in Appendix C. The lesson plan informed by the critical stages was enacted, after which the students took a post-test to determine how far they had progressed through the critical stages. The post-test was identical to the pre-test found in Appendix C. The participants in this study were students of the researcher. Because of this, the researcher was not directly involved in the recruitment process, and was unaware of who had chosen to participate in the confirmatory study until after final grades were submitted. The students were recruited by mathematics professors at the university who were not involved in teaching the precalculus class. The pre- and post-test data was also collected outside of class without direct involvement or knowledge of the researcher. Major differences between the studies are summarized in Table 3.1.

To recruit students to participate in the clinical interviews, permission was sought from the precalculus course coordinators to visit their classes and ask for participants. With their approval, each classroom was visited when it is was determined to be least intrusive for the instructors. The researcher was introduced and the goals and methods of the study were outlined. The potential benefits and risks that the students could encounter should they choose to participate were also explained. The instructor left the room while informed consent documents were distributed and questions from the students were fielded. Finally, the informed consent documents were collected. For the interviews, instructors were not informed of any students' choices regarding participation.

Table 3.1. Major Differences between the Studies

Study	Number of participants	Data Collection Methods	Identities and Transformations Combined?	Gender Information Collected?
Main Study (Stage One)	n = 6	Task-based Interviews	Yes	No
Main Study (Stage Two)	n = 6 (5M/1F)	Revised Task-based Interviews	No	Yes
Confirmatory Study	n = 16	Pre- and Post-tests, Audio recordings of group work during teaching episode	No	No

Data Collection

In this section, the data collection process will be detailed. For the main study, data was collected through task-based interviews. These interviews were audio- and video-recorded, and the students' work was kept. For the confirmatory study, one of the two lectures was audio- and video-recorded. During the group work that followed the lectures, each group was audio-recorded, and two video cameras captured the entire class. The work that students produced during the group work was also copied and kept. Additionally, the participating students took pre- and post-tests which were kept for analysis.

Task-based interviews. The main study included a series of task-based interviews (Goldin, 1997; Goldin, 2000; Zazkis & Hazzan, 1999) intended to find how students learn trigonometric identities and transformations.The interview questions and setting were designed to fit Goldin's model: (1) the questions increase in difficulty and abstractness; (2) they are appropriate for undergraduate students studying precalculus concepts; (3) the students were allowed to work freely and students were not prompted so long as their conceptions did not inhibit future work; (4) students were provided with pencils, colored pencils, paper, protractor, compass, ruler, and plastic $45-45-90$ and $30-60-90$ triangles (unlabeled) in order to enable them to approach the problems in a variety of ways; and (5) students were asked to elaborate and reflect on the reasoning that they presented. The interviews were video- and audio-recorded, and copies of all student work produced during these interviews was kept for analysis.

Students' work and reasoning from these interviews was used to inform and revise the critical stages for each learning goal. As students worked to justify each learning goal – the identities and transformation correlations under investigation – it was noted how successful students made connections among concepts and whether these connections were represented in the critical stages. It was anticipated that students may show reasoning that implies substages of understanding that must be added to the hypothesized critical stages, or they may show a variety of solution strategies that necessitate a reorganization of the stages. For example, in stage two of the main study, students showed the ability to justify the cofunction identities for acute angles without using the unit circle definitions of trigonometric functions. As a result, this learning goal may be achieved earlier than the original hypothesized critical stages imply.

Regarding the order of the critical stages, some orderings are believed to be necessary; others are believed to be likely. For example, it should be necessary for students to understand that there are horizontal transformations before noticing that they are counterintuitive in some ways. In comparison, it is likely that students will classify shifts and stretches before exploring why the horizontal ones behave the way they do. While the latter ordering may not be absolutely necessary, it is believed that the progression will generally take that order.

The data provided from these interviews was also used to inform the lesson plans used during the confirmatory study. In addition to informing the critical stages that have provided the framework for the lesson plan, the interview data contains examples of student reasoning and misconceptions. This makes it easier to predict and respond to student difficulties during the lesson plan. For example, a number of interviewees expressed the belief that horizontally shrinking a function is the same as vertically stretching it. As a result, the lecture was modified to emphasize the difference between these operations. The students were asked to notice that one of the transformations affected the range of the function while the other did not, and functions that do not have the same outputs cannot be equal to each other. The ways that students described using the affordances of various representations, how they connected the concepts under investigation with related concepts, the struggles that they faced and how they overcome them or not were all used to inform the resulting lesson plan.

The interview tasks were designed in accordance with previous work in the areas of students' understandings of trigonometry, identities and transformations (Axler, 2013; Barton, 2003; Blackett, 1990; Borba & Confrey, 1996; Challenger, 2009; Fi, 2003; Hall & Giacin, 2013; Sokolowski & Rackley, 2011; Weber, 2005). The current research is intended to build upon the results of these studies. Previous research has taken a more global view of trigonometry, or has addressed how well students learn rather than how they learn, or have studied these topics in non-trigonometric contexts. By using tasks derived from the prior research, my findings will be easier to situate within the literature. The interview begins with a question about the definition of identity in a mathematical context. This question is intended to ensure that the student and interviewer are in agreement about the types of mathematical constructs that are being examined through the rest of the interview. Non-trigonometric tasks were adjusted to use trigonometric functions. For example, instead of using the function $(x+5)^2+3(x+5)+5$ to explore a horizontal shift, the function $sin(x+\pi)$ was employed. Some problems that had originally used degrees to measure angles were adjusted to use radians. Also, the function inputs of some problems were changed from first quadrant angles in order to utilize the unit circle more often.

Following is a selection of interview tasks. This interview was intended to examine students' understandings of both trigonometric identities and transformations. It is noted what stages these tasks were intended to investigate. Also included are scripts of prompts for potential student responses. It is noted where interview tasks were influenced by tasks from previous research studies. The full list of interview tasks can be found in Appendix A.

0.

a. Have you ever passed a course with a trigonometry unit? If so, was it a high school or undergraduate course?
b. Have you ever taken a course with a trigonometry unit? If so, was it a high school or undergraduate course?
c. List the last three mathematics courses that you've taken.
d. What does identity mean (Challenger, 2009)?

- If the student is unable to answer:
  What does identity mean to you in a non-mathematical context?
- If the student describes or provides an example of an equality rather than an identity:
  What is the difference between identity and equality?
- If the student does not provide a trigonometric identity:
  What can you tell me about trigonometric identities?
- If the student can provide an example of identity (such as the Pythagorean or tangent) but not describe it further:
  Why might it be useful to know that those things are equal?

  This question is intended to examine students' understandings of the word "identity" and to inform the researcher of the students' potential familiarity with and readiness for the material.

1. Evaluate the following:

a. $cos(\frac{-\pi}{4})$, $cos(0)$, $cos(\frac{\pi}{4})$
b. $sin(\frac{-\pi}{2})$, $sin(0)$, $sin(\frac{\pi}{2})$
c. $cos(\frac{-\pi}{4}) + \pi$, $cos(0) + \pi$, $cos(\frac{\pi}{4}) + \pi$
d. $sin(\frac{-\pi}{2} + \pi)$, $sin(0 + \pi)$, $sin(\frac{\pi}{2} + \pi)$
e. $tan(\frac{-\pi}{3})$, $tan(0)$, $tan(\frac{\pi}{3})$
f. $\frac{-3}{2}tan(\frac{-\pi}{3})$, $\frac{-3}{2}tan(0)$, $\frac{-3}{2}tan(\frac{\pi}{3})$
g. $tan((\frac{-3}{2})(\frac{-\pi}{3}))$, $tan((0)(\frac{-\pi}{3}))$, $tan((\frac{-3}{2})(\frac{\pi}{3}))$

- If the student is uncomfortable or incapable of working with radians:
  Switch to degrees
- If the student believes that $f(-x) = -f(x)$ for all functions:
  Can you show me how you found $cos(x)$ and $cos(-x)$?
- If the student does not know how to perform the tasks:
  How would you define the trigonometric functions? or Are there any other ways you could represent the problem?

- If the student evaluates e.g. $sin(\frac{\pi}{2}) + \pi$ or $sin(\frac{\pi}{2}) + sin(\pi)$:

  What is $\frac{\pi}{2} + \pi$*?*

  This question is intended to prompt students to notice the differences between the parent algebraic representations and the transformed functions. This task also contains patterns in the questions and answers that could prompt the students to move to a representation with better affordances, interpret the situation in a new representation, find the values under consideration in the new representation, and compare those values This would mean the student had successfully achieved the first two critical stages.

2. Describe any relationships you've encountered regarding changes in the representations used during your work in the previous exercises (adapted from Barton, 2003; Fi, 2003).

   This question is intended to prompt students to reflect upon, hypothesize, and justify generalized relationships for trigonometric identities as is necessary for stage four.

3. Let $F(x) = sin(x)$; $g(x) = 2x$; $h(x) = x + \pi$. Write out and graph the following functions:

a. $F(h(x))$
b. $F(h(g(x)))$
c. $F(g(x))$
d. $F(g(h(x))$
e. $g(F(x))$
f. $g(F(h(x)))$
g. $F(h(x))$
h. $g(F(h(x)))$
i. $h(F(x))$
j. $g(h(F(x)))$
k. $g(F(x))$
l. $h(g(F(x)))$

- If the student is confused about compositions (e.g. order of application):

  Correct any misconceptions, noting previous compositions if applicable. This activity won't be productive with misunderstandings of composition, and it could affect future work.

  This series of tasks is intended to prompt students to notice that the order in which they apply transformations sometimes, but not always, affects the graphical transformation. Students may note in particular that the order of transformations matters when multiple transformations are applied horizontally and/or vertically. By writing out the algebraic representation, students may also begin to notice that the order of the horizontal transformations is counterintuitive in relation to the graphical representation. This addresses stages nine through twelve of the transformations critical stages.

4. How could you algebraically represent one or more transformations of sine, cosine, or tangent that results in the following functions:

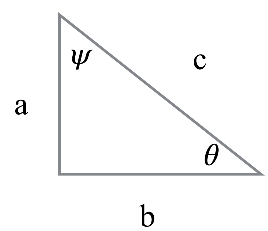
a. [Graph of $3cos(x) - 4$]
b. [Graph of $\frac{1}{4}sin(x) + 6$]

c. [Graph of $tan(2x + \frac{\pi}{4})$]

d. [Graph of $cos(\frac{\pi}{4}x - \frac{\pi}{3})$] (adapted from Borba & Confrey, 1996)

- If the student has mistakes in their graphs:
  What are some ordered pairs on your graph? and How do these ordered pairs relate to the algebraic function?
- If students do not note that there exist infinite ways of representing each function:
  Could you algebraically represent any of these graphs differently? Could you use the same or different parent functions to give different algebraic representations of these graphs?

This task is intended to provide students with further work to examine the counterintuitive nature of combinations of horizontal transformations. It also examines students abilities to move from the graphical to algebraic representations rather than from the algebraic to graphical representations. Students can demonstrate that they have achieved stages four through nine of the transformations critical stages with these tasks.



5. For the above right triangle, suppose $\theta = \frac{\pi}{8}$.

a. Evaluate ψ

b. Which leg is adjacent to θ?

c. Which leg is opposite ψ?

d. Find $cos(\theta)$

e. Find $sin(\psi)$ (adapted from Axler, 2013; Blackett, 1990).

- If the student is confused about adjacent/opposite or leg/hypotenuse:
  Define the term.

This task is intended to spur students to notice that, since all triangles have interior angles whose sum is π radians, then the acute angles of a right triangle must have a sum of $\frac{\pi}{2}$ radians. This exercise also implies that this identity should be true for all acute angles of right triangle trigonometry. This addresses Identities stages five and six.

Teaching episodes. Separate teaching episodes were conducted for trigonometric identities and for transformations. In this section I will describe how the three phases of each teaching microexperiment were enacted. First, the researcher prepared for the teaching episodes. The researcher then enacted the prepared lesson plans during the teaching episodes, and finally, the episodes were analyzed.

Preparation. To prepare for the teaching episode, Steffe and Thompson (2000) advise that it is critical to have teaching experience in the subject under investigation. The researcher should be familiar with how students think and reason about the subject. Students may have viable but unconventional approaches, and the researcher should not assume that the envisioned path to success will be the actual path to success for all (or possibly any) students. Furthermore, the researcher should be familiar with the types of conceptions that students may possess when they begin learning the topic. Students are likely to have learned the prerequisite material in a variety of ways, and they will bring these individual conceptions to their future learning. In this study, the researcher has been a teaching assistant for an undergraduate precalculus course five times, including twice as an instructor for a small section. The researcher has also taught two precalculus courses at the high school level. This has provided a firsthand experience of the types of reasoning that students may employ in a trigonometry unit.

Complementing this experience is the data collected through the main study. While the processes of creating lesson plans, evaluating students' work, and class discussions have been invaluable, the interview participants have gone into greater depth about their reasoning than is usually elicited from questions during class. This information has necessitated revisions to the proposed teaching episodes. For example, two interview participants proposed strategies to generalize the cofunction identities that included noticing that the relevant reference angles are complementary. This strategy was better suited for the lesson plans than the originally envisioned potential strategies. Approaching the concept through graphical transformations was impractical because the students had not been introduced to the graphs of the trigonometric functions during the previous lectures. Another potential strategy was to view $(\frac{\pi}{2} - \theta)$ as $(-\theta + \frac{\pi}{2})$ and interpret this as a pair of transformations of the radius of a unit circle. Since none of the participants in the interviews approached the task through either of these methods, they were discarded in favor of the students' promising strategy.

Hypothesized lesson plans. The second phase of the confirmatory study is the implementation of the prepared lesson plan through a teaching episode. The activities used in this study were originally designed to guide students through the hypothesized critical stages and have since been revised to reflect changes made to the critical stages after the main study. Each teaching episode covered two fifty-minute class periods. The first period was a lecture for approximately 125 students. The second period consisted of group work and a class discussion. This was conducted consecutively with three groups of approximately 25 students. The group work was made up of tasks intended to present students with situations that they would be able to approach with their prior knowledge of the definitions and representations of trigonometric functions. In this section, aspects of the intended lesson plans will be detailed, including motivations for specific tasks and a description of how the lecture and activities are intended to guide students through the critical stages.

Enacted lesson plans always vary from intended lesson plans (Usiskin, 1984). The lesson plan components described here include contingencies for anticipated student responses. The activities were designed to guide students towards thinking about the trigonometric functions in certain ways, but this did not guarantee that the students would in fact think about them in the intended manner. Because of this, it was necessary to anticipate potential student responses and outline a method by which the teacher-researcher could either use the students' alternate reasoning to guide them towards the required understanding, or perturb a misconception that the students may have indicated that they possessed. Even with these contingencies, it was likely that some students would provide unanticipated responses. In these situations it was necessary for the teacher-researcher to improvise a strategy to aid the students in developing the desired understanding. The preparation described earlier – experience teaching precalculus and knowledge obtained from the main study – provided the necessary skills to make these types of improvisations.

The hypothesized lesson plans were designed with the belief that both the lecture and group work class periods would be conducted with a small group, entirely consisting of study participants. The researcher intended to utilize the small class size to reduce the scope of the lecture and increase the scope of the group work. It was hypothesized that having the students engage in more independent investigation of the topics would provide more comprehensive data. However, as the design of the study developed, it became necessary to present the lecture to a much larger group of students that notably included students who would not be participating in the second day of group work. As a result, there are notable differences in some details of the hypothesized and revised lesson plans. These details will be noted, but the focus will be on revisions resulting from changes to the hypothesized critical stages.

Details for each of the two lesson plans will be presented in the following subsections. Generally though, the hypothesized lesson plans for each teaching episode consisted of an introductory lecture, in which the topic was motivated and students were guided through the critical stages. Following the lecture, students were asked to work in groups during a second class period. Group work has been chosen for these tasks in order to encourage the students to actively participate in the process of connecting their unit circle and trigonometric function conceptions in the hope of creating new identities and interpreting the effects of transformations. Students collaborating to solve mathematical problems are required to interpret the problem situation through their understandings of the material in order to make progress, whereas a student listening to a presentation of a solution, for example, may faithfully transcribe the material without considering it in any meaningful way.

Group work can also be more productive for all students involved than individual work. Students working together who understand different aspects of a concept will have opportunities to question their collaborators and build upon their deficiencies. Even if students working together have similar (mis)conceptions, they can benefit from working together (Doise & Mugny, 1979; Doise, Mugny, & Perret-Clermont, 1975). Students with similar conceptions still have their own personal experiences leading to personalized interpretations, and this means that they will have perspectives to share with each other to build more nuanced conceptions. In the confirmatory study, students worked in small groups both to echo the work of Doise and colleagues, as well as to provide more data for analysis.

During the group work, students interacted with each other, with the instructor, and, while studying transformations, with the TrigReps program. In each instance they provided their interpretations of the material– in forms such as answers, questions, or commands – to another student, to myself, or to the computer program. Once the students presented their interpretations, the listener interpreted that presentation, responded with a presentation of their own interpretation, and the cycle continued. During this cycle, each listener provided feedback with their interpretation, trying to come to an agreement on the concept with their interacting partner. The computer program is slightly different in that it has a comparatively limited number of interpretations and presentations that it can make. However, as the groups interacted with the computer, they had the opportunity to notice how the computer's feedback changed as the students' presentations changed. Through these interactions, students could notice how changes in various representations of the sine function corresponded to changes in the other representations. In this way, the computer and the student can be seen as jointly negotiating a construction. In the case of student-student or student-teacher interactions, the negotiation is likely to be a series of verbal and/or symbolic presentations of interpretations supplemented with gesticulations.

After each session of group work, students were asked to volunteer to present their work to the class. It was intended to have all of the students leave class having seen the general identities and transformations as well as examples of reasoning to support them. By having the students work through the material on their own and present their reasoning, they also heard multiple perspectives on the same topic. They heard their own groups' perspectives, as well as the presentations, and whatever comments were made on the presentations. These perspectives were each slightly different because of the personal experiences each person had. Some of these differences arose in the discussion, possibly providing insight into different nuances of the concepts. It was also useful to see how the presenting students stated the inferences that their groups made. The discussion was intended to spur reflection and further elaborations upon the students' written conclusions.

Data was gathered from each teaching episode in the form of pre- and post-tests, copies of student work produced during the episode, audio recordings of small group discussion, and video recordings of the whole class during group work. The pre- and post-tests were identical to each other for each teaching episode. Each test was made of a subset of questions from the stage two interview tasks. A subset was deemed sufficient since the tests were intended to assess how well students understood each stage, whereas during the interview, the tasks were intended to throughly investigate the processes by which students came to understand each stage. The latter required enough variety in tasks to leave students confident enough to generalize, while such tasks could be largely redundant for assessment. For example, while students build the knowledge to classify transformations, they likely need to see multiple examples in order to identify patterns and notice the differences in the representations of each transformation. On the other hand, to assess whether students have already made these connections requires only one or two such examples. In addition, the confirmatory study used a larger sample size, which helped generalize the data. The pre- and post-tests were used to gauge what critical stages each student had achieved at the beginning and end of each experiment. That is, the pre- and post-tests were designed to measure how well the teaching episodes helped individual students move through the critical stages.

Identities Lesson Plan. Originally, the majority of this lesson plan was intended to be examined by students during group work. Due to logistical changes, the material was presented to the students during a lecture, and they reinforced these concepts during group work in the following class period. The activities in this lesson plan were intended to utilize students' knowledge of the unit circle to guide them towards identifying operations that they can perform on the radius in order to produce predictable results. The class began with an introductory segment intended to focus students on the unit circle definitions of the trigonometric functions and to combat the misconception that $T(x + y) = T(x) + T(y)$.

To begin progressing through the trigonometric identities critical stages, it was hypothesized that students must first notice that there had been a change to the algebraic representation and that this representation did not have good affordances to examine this change. In order to guide students towards noticing these things a short introductory segment was planned. During this segment, students were presented with the situation that there is a number θ between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ that has the property that $sin(\theta) = \frac{3}{10}$ and asked what information could be gleaned about $sin(\theta + \frac{1}{10})$. Ideally, students would progress to the second and third hypothesized critical stages by changing to the unit circle representation in order to reason that $(\theta + \frac{1}{10})$ represents a rotation of the radius at angle θ by approximately 5.7 degrees. This would mean that $(\theta + \frac{1}{10})$ is still in the first quadrant and $sin(\theta + \frac{1}{10})$ is slightly greater than $\frac{3}{10}$ since the sine function increases in value as its input increases from 0 to $\frac{\pi}{2}$.

The primary misconceptions that the introduction aimed to address were that students believe that $sin(a + b) = sin(a) + b$ or that $sin(a + b) = sin(a) + sin(b)$. Without this stage zero knowledge, the students would be unable to construct meaningful identities. The latter is a misconception that has been noted before in other contexts where students treat functions as distributing across other operations without any supporting evidence (Tsai & Chang, 2009). If a student responded that $sin(\theta + \frac{1}{10}) = \frac{4}{10}$, or indicated that the value should be slightly greater than $\frac{3}{10}$ because $sin(\frac{1}{10})$ is a small number, the misconception was probed by re-presenting the task as "$sin(\frac{\pi}{2}) = 1$, so what can we tell about $sin(\frac{\pi}{2} + \frac{1}{10})$?" If the student was unperturbed and responded with "$\frac{11}{10}$" or "slightly above 1", the student was then asked to call to mind the range of sine. If the student could not recall the range, or if this still did not perturb the student, they were asked to call to mind the definitions of sine in order to move the discussion to the unit circle. Alternatively, if there were no responses forthcoming from the students, more leading questions were asked, such as: "What does it mean that $sin(\theta) = \frac{3}{10}$?", "What do θ and $(\theta + \frac{1}{10})$ represent?", or "How else could we represent the situation $sin(\theta) = \frac{3}{10}$?". If none of these prompted any of the students to respond productively, students would be presented with the definition of sine, and the situation would be examined in the unit circle representation.

One other difficulty that it was predicted students would face from stage zero is that the introductory problem is presented in radians rather than degrees. Since there are no units given, the assumed unit is radians, although some students persist in thinking of trigonometric functions in terms of degrees, especially when the input does not contain a multiple of π (Akkoç, 2008; Tuna, 2013). Should this misconception have arisen, students would have been reminded that the implied unit is in radians. If students were not able to work productively through the tasks in radians and switched to degree measure, that would still not have been an impediment to developing an understanding of the trigonometric identities. Once these identities were understood in degree measure, the remaining time could have been used to reinforce the students' understandings of radians and proportion with the goal of converting the identities and other trigonometric concepts from degrees to radians.

Following the introductory segment, students were led through a sequence of tasks intended to use the unit circle to guide them through the rest of the critical stages for opposite angle identities and identities involving adding multiples of π to the input. Students were led through the following tasks:

> *Draw a unit circle representation of* $cos(\theta) = a$ *and* $sin(\theta) = b$*. In the same picture, draw a radius with endpoint* $(cos(\theta + t), sin(\theta + t))$ *for some real value t.*
> *1) When is* $cos(\theta + t)$ *[*$sin(\theta + t)$*,* $tan(\theta + t)$*] less than, greater than, and equal to* $cos(\theta)$ *[*$sin(\theta)$*,* $tan(\theta)$*]?*
> *2) When is* $cos(\theta + t)$ *[*$sin(\theta + t)$*,* $tan(\theta + t)$*] equal to* $-cos(\theta)$ *[*$-sin(\theta)$*,* $-tan(\theta)$*]?*
> *3) Sketch the functions* $y = cos(\theta)$ *[*$sin(\theta)$*,* $tan(\theta)$*],* $y = a$ *[b,* $\frac{b}{a}$*], and* $y = -a$ *[−b,* $-\frac{b}{a}$*] on the same set of axes.*
> *4) Can you justify any general formulas for trigonometric (in)equalities based on your work?*

It was decided that students would work with the unit circle representation because it offers affordances for viewing the change to the input of the trigonometric functions. While the algebraic representation is unenlightening for approximating values of trigonometric functions, the unit circle can be used to approximate or exactly measure angles as well as *x*- and *y*-values. Students may have used their prior knowledge of the unit circle and radians to sketch a unit circle with radii at angles θ and $(\theta + t)$. They could apply their knowledge of trigonometric functions to label the endpoints of the radii $(cos(\theta),\ sin(\theta))$ and $(cos(\theta + t), sin(\theta + t))$, respectively. Students were shown the relationships using a first quadrant angle θ and asked to convince themselves that the same relationships would hold for angles in other quadrants. The tasks regarding when each trigonometric function acting on $(\theta + t)$ produces outputs that are greater than, less than, or equal to the respective function acting on θ was intended to help students develop dynamic conceptions of the trigonometric functions that change in value as the radius rotates about the unit circle.

The third task, in which trigonometric and constant functions were graphed, had two purposes. One was for students to notice where the ordered pairs of intersection occur so that they could affirm or improve their observations from the previous tasks. The other was for students to recall that the trigonometric functions extend beyond the interval $[0, 2\pi]$. Students should have noticed that there are an infinite number of intersections between the trigonometric functions and each of the constant functions. This was intended to lead students to recognize that the identities hold in general for multiples of π rather than a single instance.

A concern for this task was that students could only consider one period of the trigonometric functions, which could inhibit their abilities to generate general identities. After the identities were established, students were asked guiding questions such as "How many times does the trigonometric function intersect the constant functions?", "What are the domains of these functions?", or "How does your work represent $sin(-3\pi)$?" in an attempt to prompt the students to extend their identities.

If students felt that they were unable to approach any of the tasks, they were asked to call to mind the definitions of the functions in order to find the parts of the circle that are to the right of (or above) the endpoint of the radius, and hence have points that correspond to greater outputs for the trigonometric functions. This could be aided by having the student cover the lower (or left-most) portion of the circle, leaving only a relevant part of the circle uncovered. If students were able to graphically, but not algebraically, identify an angle that corresponds with an identity, they were asked to use their notion of symmetry (about an axis or the origin) to notice that the radii and endpoints are mirrored. This provided the students with a reference point from which they could draw triangles or measure distances to find the new input angles and function values.

The lecture ended with students being guided towards an understanding of the cofunction identities. Students were guided through introductory work to establish that the two acute angles of a right triangle are complementary. A right triangle was drawn on the board with the right angle labeled and one of the other angles labeled as $\frac{\pi}{8}$. The students were asked to find the missing angle. Students at this stage should be familiar with the fact that the sum of the angles of a triangle is π radians or 180˚. They were able to use this fact to find that the other acute angle can be found by subtracting the two given angles from π radians or 180˚, resulting in $\frac{3\pi}{8}$. Since one of the angles in a right triangle is always 90˚ or $\frac{\pi}{2}$ radians, this process can be further reduced to subtracting the known acute angle from the remaining 90˚ or $\frac{\pi}{2}$ radians. The process was then generalized for an angle θ.

During the lecture, a strategy suggested by students' work during the main study interviews was used. It was noted that if two radii are rotated in opposite directions from the positive x-axis, the angles formed by the radii sum to zero. Also, the reference triangles formed by the radii are congruent. Extending this idea, if the positively rotating radius starts from $\frac{\pi}{2}$ instead, then the angles formed by the radii sum to $\frac{\pi}{2}$. The reference triangles formed by these radii are also congruent, but oriented such that the height of one triangle is equal to the width of the other. As a result, the cosine of one angle is equal to the sine of the other, and the cofunction identity holds for all angles.

The preceding lesson plan was intended to guide students through the critical stages of understanding for opposite angle identities, identities involving adding a multiple of π to the input, and cofunction identities. Primarily through group work, it was hypothesized that these activities would help students to justify these identities using the unit circle representation rather than memorize them.

Transformations Lesson Plan. When the class was ready to investigate transformations of trigonometric functions, the following hypothesized lesson plan was intended to be utilized. This lesson plan was intended to provide students with the knowledge to justify the behaviors of transformations of trigonometric functions. In particular, it was hypothesized that this lesson plan would enable students to justify the counterintuitive behavior of horizontal transformations.

The transformations lecture began by using sound waves as motivation (Kessler, 2007; Kuttruff, 1973). One of the fundamental properties of a sound wave is periodicity, so students were asked to recall the periodic identities. Using these identities as examples, the definitions of period, frequency, and amplitude were introduced. Students were also introduced to Hertz (Hz), a unit of measurement for frequency, measured in number of periods per second. When the units for the x-values are defined as seconds, the measurement in Hz is equivalent to the frequency.

Students were told through the lecture that sound waves can be represented by sums of sine (or cosine) functions, where time is the input and pressure is the output. These changes in pressure are perceived as vibrations. Students were familiar with characteristics of sound waves such as volume and pitch, and through the activities that followed, they were offered opportunities to discover some mathematical properties of sound waves: higher amplitudes and frequencies correspond to higher volumes and pitches, respectively; vertical translations do not produce sound waves; reflections and horizontal translations produce sound waves indistinguishable from sinusoids lacking those transformations.

The set of tasks associated with this lecture were assigned for students to complete with the aid of a MATLAB computer program designed by the researcher, shown in Figure 3. This program is named TrigReps since its primary function is to display trigonometric representations. Given an algebraic representation of a sinusoid, TrigReps simultaneously produces (1) a static, graphical representation of the function on the Cartesian plane, (2) a dynamic representation of a radius rotating around a circle, and (3) an aural representation of a sounded tone which can be varied in volume and pitch corresponding to the function considered as a pressure wave. The students input values *a, b, c*, and *d* into $(a)sin(bx + c) + d$ and were shown a dynamic representation based on the unit circle definition of $sin(x)$. This dynamic representation showed a circle of radius *a* that had been shifted *d* units on the vertical axis, and the rotating radius of this circle started at *c* radians and moved counterclockwise at $\frac{b}{2\pi}$ revolutions per second, as seen in Figure 4. The screenshot in Figure 4 displays the program after inputs have been chosen but prior to the unit circle representation being animated and the aural representation being

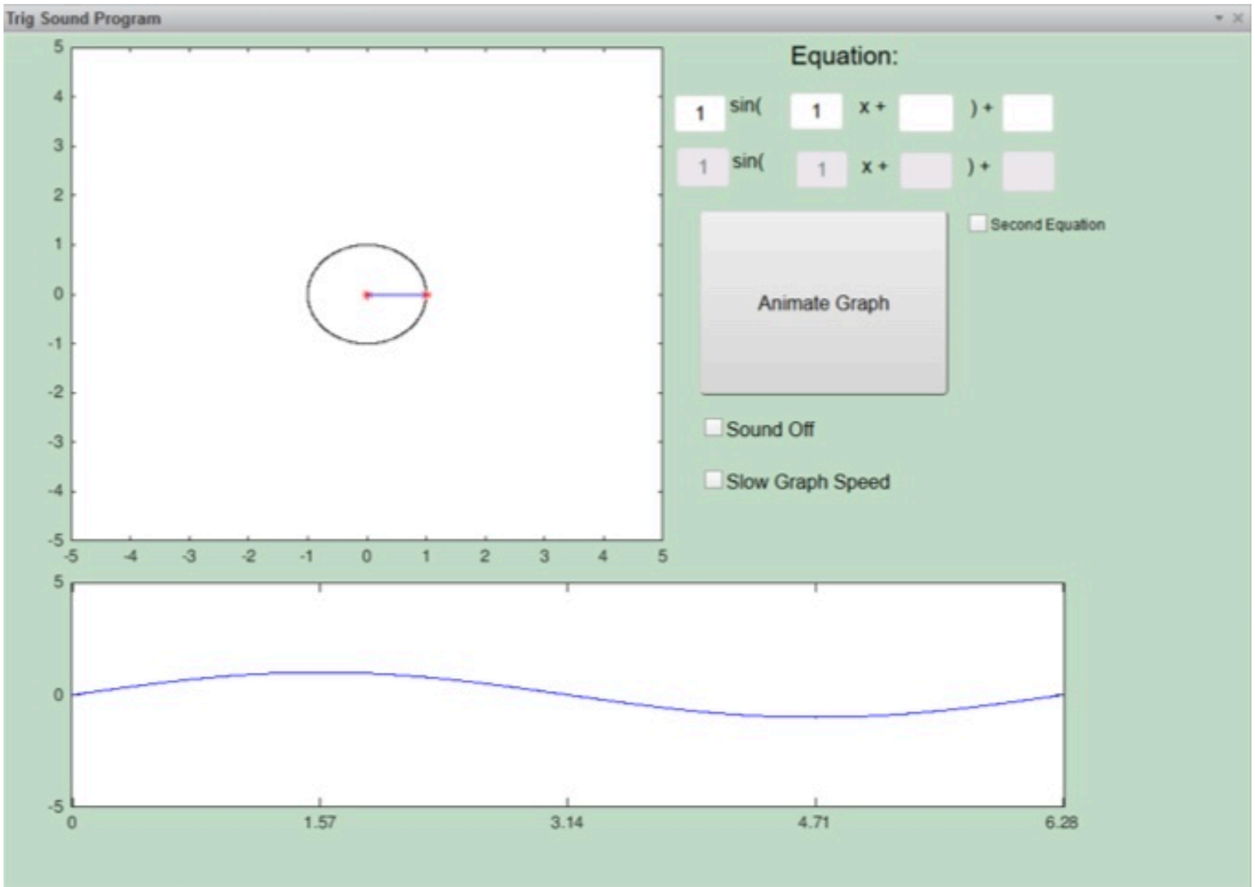


Figure 3. The TrigReps program with input *sin(x)*.

generated. There were also options to have the dynamic representation slowed down and to input two sine functions simultaneously. The former feature was available so that representations of audible sound waves were still understandable and relative differences could be noted despite the high speed of the rotating radius, while the latter was available so that students could see and hear the effects of multiple transformations simultaneously.

TrigReps was successfully piloted in an undergraduate precalculus class to determine its ease of use and clarity of message. During this pilot study, students were asked to Strongly Disagree, Disagree, Agree, or Strongly Agree with the statement “The computer program is simple to operate”. Three out of twelve students Strongly Agreed, six Agreed, one Disagreed,

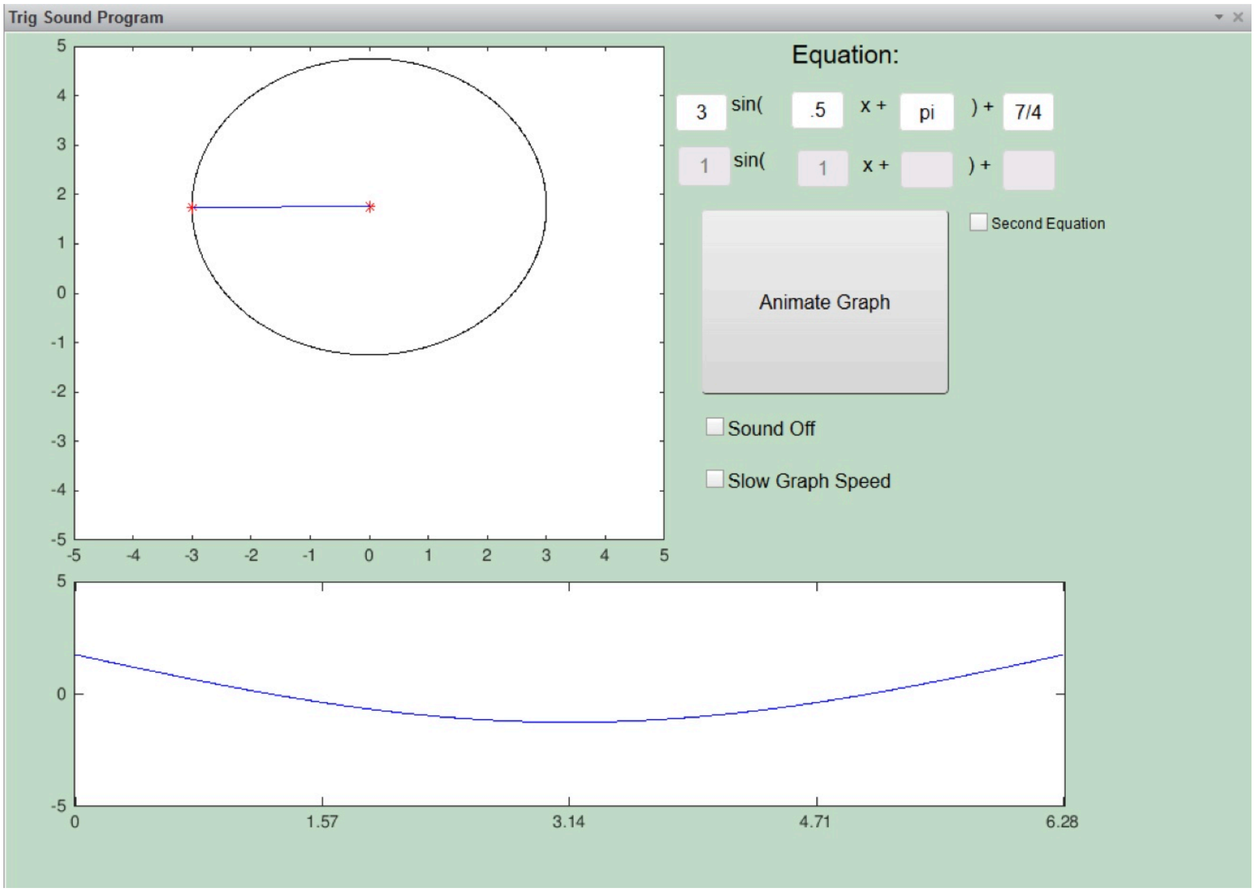


Figure 4. A transformed sinusoid represented in TrigReps

one wrote that “Once everything is set up it was simple to use,” and one wrote simply “No.” It seems reasonable to conservatively count the last as a Strongly Disagree and the penultimate as a Disagree. This raises the totals to two of twelve students who Disagreed, while one Strongly Disagreed. Some of the disagreement may be attributed to students experiencing technological difficulties getting the program to run. One of these students’ explanations for their disagreement with the program’s ease of operations included the statement “I thought initial guidance was necessary.” Instructions for opening the program were provided to each student, however in the confirmatory study the program was up and running before the students sat down to work. For students who did agree that the program was easy to operate, a typical response was that “the program is laid out in a very user-friendly setup." It seems that when the program was operational, it was intuitive to use.

From examining students' work during this pilot study, the questions asking students to reflect on the changes in the representations seem to have achieved their purpose. All twelve participants described how changes in the algebraic representations corresponded with changes in the graphical representations. Another eight students also included the aural representation, and three students wrote of connections between those three representations and the dynamic unit circle representation. Since it has been hypothesized that the unit circle representation will be necessary in order to understand combinations of horizontal transformations, this representation has been explicitly referenced in the tasks.

TrigReps utilized several representations in order to facilitate students' progressions through the critical stages for understanding transformations of trigonometric functions. Students should have been familiar with changes to the input of the algebraic representations of trigonometric functions after learning about trigonometric identities. Since the computer program required that the students changed the algebraic representation, it seemed that they would progress through the first hypothesized stage with minimal reflection on their actions. The inclusion of the graphical and unit circle representations similarly encouraged students to progress through the second hypothesized stage. The corresponding changes that students must notice to achieve the third critical stage should also have been fairly apparent using the computer program.

Students were asked to work on a set of tasks in groups of three or four at computer stations. A sample of the tasks is presented below, along with descriptions of how each task was intended to guide students through the hypothesized critical stages. The full set of tasks can be found in Appendix D.

Human hearing has range approximately 20 Hz - 20000 Hz. Not all of the functions that you input will produce sounds within your hearing range. Can you predict which of the functions will and will not produce sounds?

1. Input $f(x) = sin(x)$

a. Find and input a function with twice the amplitude.
b. Find and input a function with amplitude 0.2.
c. What do you notice about the four representations: algebraic, graphical, unit circle, and aural?

This task was intended to prompt students to find which value *a*, *b*, *c,* or *d* in the expression $(a)sin(bx + c) + d$ affects the function's amplitude. Ideally, the students would attempt to predict which value would produce those effects and the effects that the change in amplitude would have on the unit circle, graphical, and aural representations. In particular, multiplication of the parent function's output stretches the length of the radius, vertically stretches the graph, and alters the volume of the aural representation. However, at this frequency, the aural representation is outside the range of human hearing (stage four, five(b)).

3. Find and input a function with:

a. triple the frequency of $f(x) = sin(x)$.
b. frequency 1 Hz
c. What do you notice about the four representations?

This task was intended to guide students to notice that the *b*-value affects the function's frequency. Again, students would ideally make this prediction then test their hypothesis using TrigReps. By finding the function with frequency 1 Hz, it was intended that students would be guided towards finding the general formula for frequency $-sin(bx)$ has frequency $\frac{b}{2\pi}$ (stages four, five(b, d), six, eight).

6.

a. Input $f(x) = 2sin(x)$
b. Input $f(x) = sin(x) + 1$
c. Predict what will happen in each representation for the input $f(x) = 2sin(x) + 1$
d. Input $f(x) = 2sin(x) + 1$
e. Did the results match your prediction? If not, why not?

7.

a. Input $f(x) = sin(2x)$
b. Input $f(x) = sin(x + \frac{\pi}{4})$

c. Predict what will happen in each representation for the input $f(x) = sin(2x + \frac{\pi}{4})$
d. Input $f(x) = sin(x + \frac{\pi}{4})$
e. Did the results match your prediction? If not, why not?

It was intended that, after completing task six, students would find that the order of transformations can affect the resulting function. Ideally there would have been a discussion among members of the group to predict the results of the transformations in each representation. If students correctly predicted the results during this task, then it was hypothesized that students would make a similar prediction during task seven, which similarly examines the order of horizontal transformations. Following the same pattern would lead them to a false prediction for horizontal transformations, and it was hypothesized that this would prompt students to more thoroughly examine the relationships among the representations. This examination could lead to productive observations regarding the relationships between proportional and rigid transformations, or transformations of the input and output of the function (stages five through eleven)

The tasks in this activity were designed to help students progress through the hypothesized critical stages of function transformations. They were intended to help the students identify patterns in transformations of the sine function that help students to achieve critical stages. The activity also prompted students to reflect on the changes of each representation under the various transformations in various orders, in the hope that students would not mindlessly copy information from the program, as has happened in previous studies (Rosen et al., 2005).

These tasks asked students to examine the algebraic, graphical, and dynamic unit circle representations of families of sinusoids. This is similar to previously proposed exercises to examine graphical transformations (cf. Barton, 2003). However, where previous exercises gave students explicit algebraic representations of functions to examine (as in tasks six through nine), most of the tasks used in this study asked students to find the algebraic representations for functions that have certain graphical characteristics. This had the added benefit that students would make predictions about the effects of each type of transformation. If they chose to guess without reflecting on their previous work, then the exercise would become similar to (though possibly much longer than) previous exercises.

Each exercise also concluded by asking students what they noticed about the representations. These final tasks were intended to prompt students to reflect on their work and note how the changes in algebraic representations corresponded to changes in the graphical and unit circle representations. Students in the past have shown the ability to complete tasks successfully without reflecting on how their work is related to their prior knowledge (Rosen et al, 2005). Explicitly asking the students to reflect on the work that they have done can help consolidate their knowledge into generalizations of the transformations, as well as to note how the transformations can be represented in terms of the definitions of the trigonometric functions on the unit circle. The video recordings of group work would show if students had conversations about their predictions and reflections. Students' conversations and their responses to the questions about relationships among the representations that they had noticed provided information about whether the students were reflecting upon their work. Students could indicate that they were reflecting by using multiple representations to justify their predictions. Alternatively, students could guess and check with the computer and only note relatively unimportant details in their reflections, such as which inputs produced audible tones. It would also be revealing to see what entries students made in the computer program, which could reveal how they believe each of the values *a, b, c,* and *d* would affect the graph.

Some of the tasks also gave students opportunities to see some of the identities that they have previously worked on in other representations. Some transformations of $sin(x)$ result in $\pm sin(x)$ or $\pm cos(x)$, and students could predict how these transformations would look based on the task presentations and their prior knowledge. Students may not have predicted the cofunction identity since it is somewhat disguised, however I hoped that they would examine why those transformations result in a graph equivalent to that of the cosine function in their reflections. This topic was brought up in the class discussion afterwards as students classified the transformations and noted relationships between representations.

As students used the computer program to examine horizontal graphical transformations, they would have the opportunity to examine how these transformations could be viewed on a dynamic circle representation. Horizontal transformations could be seen as affecting the radius of the circle – the angle of which is the input of the trigonometric functions. Since the change in the angle of the radius varies in proportion with b, students could see that, for example, large values of b are represented by high frequency graphs, rather than graphs significantly stretched out from the y-axis. As for translations, changing the starting point of the radius can be viewed as changing the starting point ($x = 0$) of the graphical transformation. This is similar in effect to the rubber sheet method (Borba & Confrey, 1996) where horizontal transformations are seen to act upon the axes rather than the graph. By combining these two transformations, students would have the opportunity to note that the effect could be viewed most clearly in terms of first a rotation (translation) to establish the starting point, followed by a change in speed (stretch) to set the frequency. Finding exactly how much to translate after setting the frequency is less straightforward, more laborious, and requires an unconventional algebraic representation: $f(b(x+c))$. This combination could help students understand why the counterintuitive aspects of the horizontal transformations could be extended to the order of operations.

The activity concluded with a class discussion about the students' reflections on the representations. One goal of this discussion was for students to solidify their knowledge about the classifications of transformations and the relationships between the different representations. Students were welcome to use the computer program during the discussion to help answer any questions, such as those related to the horizontal transformations. For instance, it could be used to help explain why negative horizontal shifts actually move the graph to the right: if x is chosen to start at 0, then $sin(x)$ will start at 0, while $sin(x-\frac{\pi}{2})$ will start at -1, since $(x-\frac{\pi}{2})$ is $\frac{\pi}{2}$ radians clockwise of 0. It takes $\frac{\pi}{2}$ seconds for the radius to get to 0 radians, which helps explain why the graph shifts $\frac{\pi}{2}$ units to the left. The other goal of this discussion was to extend the new concepts of transformation to the cosine and tangent functions. Extending these concepts to cosine would be fairly straightforward, however extending to tangent could present difficulties because of the different domain, range, and period of tangent. It also could be more difficult for students to extend the idea of vertically stretching and shrinking a function that has an infinite range. The different period could cause difficulties for students who assume that the period of tangent can be calculated in the same way as cosine and sine. Ideally, there would have been sufficient student participation in the discussion for students to build these ideas on their own, but the researcher led rather than facilitated discussion as necessary by referencing the unit circle definitions.

The teaching episode lesson plans described in this section were intended to guide students through the hypothesized critical stages. The content of the lesson plans has been influenced both by the researcher's experience as a trigonometry instructor and by a review of the literature on students' learning of trigonometry, identities, and transformations. In particular, these lesson plans use MERs in an effort to prompt students to connect their various understandings of the trigonometric functions. In the following section, the methods of analysis will be described for both these teaching episodes and the preceding interviews.

Data Analysis

In this section, the analysis procedures will be explained for the three phases of the study: the two stages of main study interviews and the confirmatory teaching episodes. The theoretical framework detailed the two sets of critical stages that were the basis of the research: one for learning trigonometric identities, and one for learning graphical transformations of trigonometric functions. In this section, it will be explained how the collected data was analyzed and how the analysis could affect the ways in which the hypothesized critical stages and lesson plans were revised.

Before discussing the analysis methodology, it may be helpful to review the first two research questions:

> ***Research Question One:*** *Through what critical stages do students pass as they come to understand trigonometric identities and transformations?*
> *Which actions, connections, or other ways of thinking are common to those students who go on to be able to justify their solutions of tasks involving these concepts?*

> ***Research Question Two:*** *How do students understand the relationship between the unit circle definitions of trigonometric functions and the identities and transformations of those functions?*
> *Is it critical that students be able to change from the algebraic representation to one with different affordances as they come to understand identities and transformations?*

Coding. In order to begin identifying critical stages for a learning goal, it was first necessary to develop a system by which to analyze the students' thoughts and actions as they developed their understandings. The interview recordings were transcribed and coded in order to identify the methods by which students built major concepts from their prior knowledge (Burnard, 1991; Glaser, 1965). Before examining the transcriptions, a preliminary list of codes was hypothesized based on firsthand precalculus teaching experience and by cases documented in the research literature. It was predicted that certain codes would appear based on the hypothesized critical stages and common student errors seen in the literature review. For example, it was expected that a need would arise to have codes for students changing representations, using the Pythagorean identity, and making mistakes related to radian measure, among others.

Additionally, the notes taken during the data collection process were reviewed. These notes included unexpected mistakes and strategies, follow-up questions to ask, emerging patterns of student behavior, and potential modifications to the interview questions, and they influenced how additional codes were developed. As an example of how interview questions were modified, the first two participants interpreted the function $cos(x) + \pi$ as $cos(x + \pi)$. This may have been because the presence of the π was causing the students to assume that the value was added to the input. Despite how important it is for these students to address this misconception (Akkoç, 2008; Moore, 2013), this was not within the scope of the research plan. As a result, the future interview questions were altered to use $cos(x) + 1$. This distinction, although important, would not necessarily affect the students' abilities to learn trigonometric identities and transformations.

Each of the hypothesized codes did appear, and it will be discussed how and when they appeared in relation to the hypothesized critical stages in the Results and Discussion chapter. Many other codes were added as they were needed, such as one indicating that the student believed that the points on the graphs of the function should satisfy the unit circle equation. The final list of codes for each learning goal represented the students' thoughts and actions while examining the relevant concepts during the interview. Some subset of these codes should represent the final critical stages for each learning goal, since they captured the students' thoughts and actions as they progressed from incomplete understandings to justified understandings.

In order to determine the thoughts and actions that distinguished the successful students from the unsuccessful ones, it was necessary to determine which students displayed a justified understanding of each concept and which did not. If the students expressed that they understood a given concept coming into the interview, then they were asked to justify their understanding of that concept. On the other hand, if the students initially showed that they did not understand a concept, then specific prompts were used to encourage the students to make connections that were hypothesized to lead to the desired understanding. The students' reactions to the questions and prompts influenced the revision of the critical stages. For instance, when examining the cofunction identities, the interview prompts were designed around the hypothesis that students would need to consider $(\frac{\pi}{2} - \theta)$ as a pair of transformations – a leftward shift by $\frac{\pi}{2}$, followed by a horizontal reflection. In fact, interviewees were more likely to use directed, similar triangles in the unit circle. This discovery led to a major change in the hypothesized critical stages for this learning goal; the students demonstrated an alternative path to understanding through trigonometric definitions rather than transformations. This in turn led to a major change in the hypothesized lesson plan for this topic.

When students failed to demonstrate an understanding of a concept coming into the interview, or came to false conclusions during the interview, an attempt was made to find if there was a gap in the students' knowledge, a misconception of prerequisite knowledge, or if the students had only considered the concept through too narrow a lens and missed connections that would feature more prominently in a different representation. For instance, when examining the counterintuitive nature of horizontal transformations (as seen in Borba & Confrey, 1996), a student may have a gap in their knowledge such as having only learned the ratio definitions of the trigonometric functions. This would render transformations resulting in non-acute angles meaningless, as the student only has a conception for the trigonometric functions acting on acute angles in a right triangle. Alternatively, a student could have a misconception, such as believing that $sin(a + b) = sin(a) + sin(b)$. Errors were classified as misconceptions when the student did not correct themselves with a justified response upon having their attention called to the mistake. A student could also be stymied in developing further understanding by the limitations that the algebraic and graphical representations possess in illustrating the non-intuitive aspects of these transformations, as discussed in the literature review.

Transcriptions of the interviews were analyzed using a constant comparison method (Burnard, 1991; Glaser, 1965; Grbich, 2013; Hatch, 2002; Seidman, 2013). At the heart of this method is categorizing student utterances. In brief, every student utterance and action was placed into a category. At first, these categories were very specific, but similarities were sought in order to group categories together. For example, a group Changes to Representations could encapsulate the codes "changing the order of the transformations (can) change the graph" and "*student changes representation*". To ensure the validity of the coding process, another mathematics education doctoral candidate was asked to independently generate categories. The two sets were compared, and a discussion took place to resolve the few discrepancies that arose.

To develop the initial, specific codes, the open coding method was used, in which quotes or paraphrases of students' speech from the interview formed the codes. In some instances, the interview notes reflected that a particular action or phrase had been repeated by several students. For example, several students converted many of the radian values to degrees. Because of this, all of these instances were coded as R→D from the beginning rather than using quotations from students. Similarly, other hypothesized codes were used from the beginning. Students' written work produced during the interviews was also coded. It was noted which representations students used and whether each piece of work was justified by their speech or by other written pieces. In some cases, copies of images that the students drew have been included for clarity. These codes informed the revisions of the critical stages: the actions and thoughts that lead students from their prerequisite knowledge to the desired understanding.

The transcription notes and list of initial codes were used to begin developing a coding scheme. Many of the initial codes were similar to each other, and these were grouped together into single codes, which can be seen in Tables 3.2 and 3.3. Table 3.2 contains the codes that became revised critical stages. Table 3.3 contains the remainder of the codes used in the study. Based on the hypothesized critical stages, many of these codes had been predicted to be necessary, such as Ratio Definition or Noticing a Change to the Algebraic Representation. Other codes arose that had not been predicted, such as a code indicating that the student had noted the function values at $x = 0$ as an indicator of function behavior.

Table 3.2. Codes that became Critical Stages and Their Definitions

Code	Abbreviation	Explanation
Algebraic Manipulation	*AlgM*	Performing arithmetic or basic algebra.
Anchor Values	⚓	Creating or checking a hypothesis using a subset of the period. Generally positive multiples of π or $\pi/2$.
Change Angle	*C*∡	Recognizing that a change to one representation corresponds with a change to the angle in a unit circle.
Change Representation (to Unit Circle, Graph, Triangle, or Algebraic)	 *CR(UC)* *CR(Gr)* *CR(⊿)* *CR(Alg)*	Changing from one representation to another to continue working on the same task.
Determines Signs By Quadrant	*CAST*	Finding whether an answer would be positive or negative using either a mnemonic device or consideration of placement in relation to the axes.
Function Concept	*f(x)*	Examining an issue that is applicable to functions generally rather than periodic or trigonometric functions in particular.
Notice a Change to the Algebraic Representation	*NCAR*	Identifying that a transformation in some way has changed the algebraic representation.
Notice Correspondences between Representations	*NCorr*	Identifying how a change in one representation causes a change in another representation.
Ratio Definition	*RatDef*	Using SOHCAHTOA or the Right Triangle definitions of the trigonometric functions.
Reference Angle	*ref*∡	Using a reference angle to draw a right triangle in the unit circle.

Table 3.3. Codes that did not become Critical Stages

Code	Abbreviation	Explanation
Addition	+	Examining how addition is understood in a given representation.
Convert Radians to Degrees	*R*→ *D*	Converting from one angle measure unit to another.
Correct	√	Providing a correct answer to a task or to a step of a task.

Code	Abbreviation	Explanation
Degrees before Radians	*D > R*	Having views aligning with a degree-dominant mentality, such as believing that angles expressed without π cannot be measured in radians.
Horizontal	*H*	Examining horizontal transformations in any representation.
Incorrect	*X*	Providing an incorrect answer to a task or to a step of a task.
Input	*In*	Examining how function inputs are understood in a given representation.
Memorized	*Mem*	Justifying a statement by citing a memorized fact.
Multiples of π Identities	$\theta + n\pi$	Examining the effects of adding an integer multiple of π to the input of a trigonometric function.
Multiplication	*Mult*	Examining how multiplication is understood in a given representation.
Opposite Angle Identity	$(-\theta)$	Examining the effects of using the opposite input for a trigonometric function.
Origin	*Ori*	Inferring information about the function by examining whether it goes through the origin.
Output	*Out*	Examining how function outputs are understood in a given representation.
Prompt	*P*	Needing to be prompted by the researcher in order to productively move forward.
Pythagorean Identity	*PyID*	Using the Pythagorean Identity in reasoning.
References Previous Work	*PrevWk*	Noticing that a task can be solved using previous work.
Right Answer, Wrong Reason	*RAWR*	Ending with a correct answer despite incorrect reasoning.
Right Idea, Wrong Answer	*RIWA*	Using correct reasoning, but making a typo or other such mistake that leads to a wrong answer.
Shift	*Sh*	Examining shift transformations in any representation.
Special Triangles	*S⊿*	Using a 30-60-90 or 45-45-90 triangle.
Stretch	*St*	Examining stretch transformations in any representation.
Tangent Identity	*TanID*	Using the identity tan(x) = sin(x)/cos(x)
Unit Circle Definition	*UC Def*	Using the unit circle definitions of the trigonometric functions.
Vertical	*V*	Examining vertical transformations in any representation.

The lists of codes for each learning goal were collected and formed into critical stages. The critical stages were ultimately separated for each learning goal: opposite angle, $(\theta + n\pi)$, and cofunction identities; addition/shift, multiplication/stretch, input/horizontal, and output/vertical transformations; ordering of transformations; and horizontal transformations being counterintuitive.

This chapter has described the research process of the interviews and teaching episodes. Task-based interviews were conducted in order to examine the processes by which students came to understand trigonometric identities and transformations. The data collected from these interviews was used to revise the hypothesized critical stages and lesson plans. The following chapter will describe the data that was collected, and discuss how that data influenced the critical stages and lesson plans.

IV. Results and Discussion

In this chapter, the data collected through interviews and teaching episodes will be presented and analyzed. The analysis was used to revise the hypothesized critical stages, which were in turn used to revise the hypothesized lesson plans. The hypothesized stages were created based on a review of the literature, and they have been modified in light of the interview data. It will be noted where analysis of the data supported the hypothesized stages, showing that the stages successfully modeled students' learning sequences, as well as where the analysis has prompted changes to be made to the critical stages. Modifications include changes to the ordering of critical stages, as well as changes to the stages themselves. Some of these changes have been as small as noting a common misconception, or as large as accommodating successful student strategies that did not satisfy the critical stages as they had been hypothesized. This analysis allowed for the identification of any superfluous or omitted stages, as well as stages in need of modification or rearrangement.

The modifications of the critical stages have in turn necessitated modifications of the lesson plans. Changes to the lesson plans will be detailed and justified in this section. The lesson plans used during the teaching episodes were originally designed based on the hypothesized critical stages before data collection and were described previously in the methods chapter. After the main study, the hypothesized stages were revised, and the lesson plans were revised to reflect those revisions. The confirmatory study provided a data set to examine how the lesson plan helped guide students through the critical stages in a classroom setting.

After describing the changes to the lesson plans, data will be presented from the confirmatory study. This study was intended to demonstrate the feasibility of designing a lesson plan based on the critical stages. Analysis of this data will focus on how the enacted lesson plan did or did not guide students through the critical stages. Copies of students' group work, transcriptions of the students producing that work during the teaching episodes, and pre- and post-tests taken by the students the week before and up to 5 days after each teaching episode have been analyzed to determine how well the lesson plan was able to guide the students to the learning goals.

Results from the Main Study

The definition for *critical stage* used in this study involves the methods students use to justify the various identities and transformations. So, each instance where students used an identity or transformation during the interview was located, as well as situations in which the students failed to recognize an identity or transformation. For identities, each instance where a student worked with opposite angles, an angle that has an integer multiple of π added to it, or that has been subtracted from $\frac{\pi}{2}$ was identified. For transformations, it was noted where students worked with the following concepts: shift/addition; stretch/multiplication; horizontal/input; vertical/output; order of transformations; and counterintuitive aspects of horizontal transformations.

The interviews were separated into segments corresponding to each of these concepts. This made it possible to examine how each student built and applied their knowledge of each concept. Some of the *critical stages* involve simply noticing that these phenomena occur, but other stages involve investigating these concepts in particular ways. Students investigated multiple concepts simultaneously during many portions of the interviews, so there are many transcript sections that appear under multiple concepts (e.g. a passage in which a student pronounced that $f(x + k)$ shifted the graph vertically would be coded as horizontal/input, vertical/output, and shift/addition).

Each students' speech and work was examined to find instances of discussions of the learning goals. In each instance, it was determined whether the students used the learning goal correctly and whether the students were able to justify their answers. If the students were able to provide fully justified answers, then it was assumed that they must have taken a viable path to a good understanding. Therefore, even though it may not have been the most direct path to understanding, some of their work must be critical to developing a justified understanding. That is, any student who displayed a good understanding of a topic must have passed through the *critical stages*, and these stages would be represented in their interviews. On the other hand, at an individual level, if a student worked diligently, but did not use their work to make conceptual connections to establish a good understanding, then none of their work could be definitively said to be critical. At an individual level, if a student did not achieve competency, then it could not be inferred from the data that student provided what specifically it would take to achieve competency.

If a student had all correct, justified responses for a particular identity or transformation, then that student was coded as having developed a good understanding of that topic. Included in this group were students who self-corrected their mistakes. Students who initially could not justify correct responses, but expressed during the interview that they had developed a new connection and subsequently displayed correct reasoning were also included. For example, consider a student who was only familiar with the ratio definition of trigonometric functions, which caused him to fail to recognize the domains of the functions. Suppose further that during the interview, this student was prompted to consider the unit circle definition as an extension of the ratio definition, and subsequently could justify correct evaluations of non-acute angles. Then, despite perhaps having made repeated mistakes, this student would be coded as having understood that trigonometric functions could be applied to non-acute angles. Students not falling into any of these categories were coded as having not understood that particular topic.

The coding scheme described above was applied to the transcriptions, and in the sections that follow, the results will be given. The results have been organized by learning goal. In particular, it will be noted what codes were common to students who justified their understandings, as these codes will inform the revised critical stages.

Opposite angle identities. The work of the students who had successfully justified the *opposite angle identities* $sin(-\theta) = -sin(\theta)$, $cos(-\theta) = cos(\theta)$, and $tan(-\theta) = -tan(\theta)$ was examined in order to find the thoughts and actions that are critical to student success. This would provide data that could verify the hypothesized critical stages or imply that they should be modified. The critical stages must, by definition, have been achieved by all of the successful students, so their work was examined to determine common codes. It was determined that the second, third, and sixth participants of the first stage – F2, F3, and F6 – had developed good understandings of the topic. During the second stage, S1, S2, S4, S5, and S6 – all but the third participant – had also developed an understanding of the topic, as can be seen in Table 4.1. For each of these students, a list was made of every code that appeared in their examinations of opposite angle identities. Then, the intersection of those lists was found. This intersection was the set of thoughts and actions that were common to each of the students who developed a good understanding of the *opposite angle identities* (see Table 4.1). It was hypothesized that the critical stages for this learning goal would be present in this intersection.

Table 4.1. Participants' Understanding of Opposite Angle Identities.

Good Understanding	**Learning Goal**	**Poor Understanding**
F2, F3, F6; S1, S2, S4, S5, S6: *NCAR, CR(UC), C∡, ref∡, CAST, NCorr*	Opposite Angle Identities	F1, F4, F5; S3: *Mem*

While examining opposite angles in any of the tasks, the successful students first noted that the opposite input was a change from their defined functions (*NCAR*, see Table 3.2 for codes). They then moved to the unit circle (*CR(UC)*), used reference angles to determine the numerical answer (C∡, *ref∡*), used quadrants to determine the sign of that answer (*CAST*), and noticed a relationship between their original and transformed functions (*NCorr*), as can be seen in Table 4.2. The general reasoning can be characterized by an explanation from S2:

> "-*x* is just going in the opposite direction. So. because it's the *x*-value, it doesn't matter which direction I rotate. The *x*-value's going to be the same either way. Like this one. If I rotate 45° this way [gesticulates with his elbow as vertex, forearm as terminal ray – first as a positive angle, then negative], or 45° this way. As long as it's in the first or the fourth quadrant, it's going to be the same. Because [cosine] is an *x*-value."

In comparison, F1, F4, F5, and S3 did not show that they understood the opposite angle identities. These students commonly attempted to use memorized ordered pairs of the unit circle in order to solve the problems. These students either did not attempt to justify their answers, or they argued that negative function inputs will always lead to negative function outputs. F4 gave a typical argument, stating "that [$cos(\theta)$] is negative because θ is negative." This is more evidence that for students learning opposite angle identities, changing to a representation that allows consideration of $T(-x)$ in terms of $T(x)$, and using information about $-x$ in order to determine the sign of $T(-x)$ are both critical stages.

Codes and critical stages. Codes associated with each topic constitute the critical stages of learning for that topic. There may be multiple ways to justify a given learning goal, and therefore there may be multiple paths to understanding, which could imply different sets of critical stages. However, in some cases, there may be enough similarity between the approaches that the same set of critical stages may apply to either path. For example, if a student uses graphs to justify the opposite angle identity, he could still notice the opposite angle in the algebraic representation, change to the graphical representation, find a reference *x*-value and a transformed *x*-value, and compare their corresponding *y*-values.

In cases where there is not significant overlap between multiple sets of critical stages for a topic, each of the justifications could produce viable critical stages for learning that topic. In these cases, strategies have been listed separately. For example, some students approached the $(\theta + \pi)$ identities through reference angles on the unit circle. Other students used an algebraic approach to finding these identities. Since both approaches led to justified statements of identities, the set of codes from each approach are included as distinct, viable sets of critical stages for the same learning goal.

For the students who did not display a justified understanding of a learning goal, the common and unexpected codes that correspond with their work are listed in the left column of Table 4.3. These codes were not necessarily applied to every student who failed to achieve a good understanding, but they were prominent. These codes are included for two reasons: (1) there is a possibility that they are associated with a viable set of critical stages that is not

Table 4.2. Revised Critical Stages for Opposite Angle Identities with Supporting Quotes

Opposite Angle Identities **Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation	*NCAR*	“-X is…” (S2)
Move to a representation with better affordances	*CR(UC)*	“If I rotate 45º this way…” (S2)
Change angles	*C∡*	“…or 45º this way.” (S2)
Use a reference angle; Evaluate the function using similar triangles and the CAST diagram; Recognize that these changes to the algebraic representation correspond to changes in the other representation(s)	*ref∡; CAST; NCorr*	“As long as it’s in the first or the fourth quadrants, it’s going to be positive. Because it’s an x-value.” (S2); “The cosine of an opposite angle would still be the same as the cosine of the other angle.” (F3)

recognized here but which could be explored in future research, and (2) these codes could provide insight into students’ misconceptions.

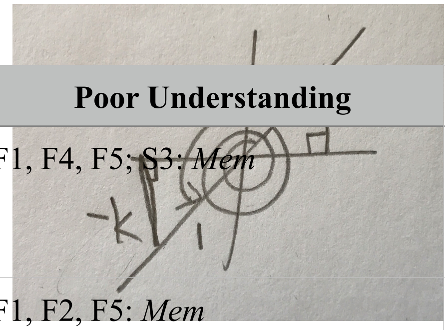
If there were students who displayed poor understanding of a concept, but had the same codes assigned to their work as the successful students, then that would indicate that there was a missing piece of information separating the students who had successfully justified their understanding with the students who had not. One possibility would be that there was some thought or action displayed by each of the successful students that had been overlooked during coding. Another possibility is that the unsuccessful students had a misconception that prevented them from developing a robust understanding. The situation where successful and unsuccessful students’ work resulted in the same codes did not arise. In cases where there was some overlap, there were always differences, such as a student who could not justify their work for opposite angle identities having an erroneous conception of negative angles on the unit circle.

The stage one interviews provided data that was used to revise the critical stages. Changes to the critical stages necessitated some revisions to the interview tasks for the second stage of the main study. The tasks were also revised to make the interviews shorter in order to cover all of the learning goals. In the subsections that follow, I will examine the responses of students from both sets of interviews. Each learning goal is discussed, and the intersection of codes for students who successfully justified their answers is noted. This set of codes constitutes the critical stages of understanding for that learning goal. For the Opposite Angle Identities, the critical stages are to *Notice a change to the algebraic representation, Move to a representation that has better affordances, Change angles, Use a reference angle, Evaluate the function using similar triangles and the CAST diagram,* and to *Notice a correspondence* (see Table 4.2). These stages overlap significantly with the critical stages for the rest of the learning goals related to identities that have been examined. It will also continue to be noted where students who did not achieve each learning goal faced significant or common difficulties.

***(θ+$n\pi$)* identities.** Identities of the form $T(\theta + \pi)$ were successfully expressed and justified by F3, F4, and F6 in the first stage of the main study, and by each participant in the second stage. Each of these students noticed that π had been added to the input of the function (*NCAR*). F6 and S1 simplified the input before evaluating the function (*AlgM*). These students compared these evaluations to the originals and generalized the relationship (*NCorr*).

The other successful students were more explicit in their use of the unit circle (*CR(UC)*), interpreting the addition of π as a change to the angle of the radius (*C*$\measuredangle$). More details can be found in Table 4.4. When these students used their new angles to create reference triangles

Table 4.3. Participants' Understanding of the Learning Goals



Good Understanding	Learning Goal	Poor Understanding
F2, F3, F6; S1, S2, S4, S5, S6: *NCAR, CR(UC), C∡, ref∡, CAST, NCorr*	*Opposite Angle Identities*	F1, F4, F5; S3: *Mem*
F3, F4, F6; S1, S2, S3, S4, S5, S6: *NCAR. CR(UC), ref∡, CAST; or NCAR, ⚓ AlgM*	$(\theta + \pi)$ *Identities*	F1, F2, F5: *Mem*
F2, F3, F4, F5, F6; S1, S2, S3, S4, S5, S6 *CR(UC), C∡, ref∡*	$(\theta + 2\pi)$ *Identities*	F1: *Mem, RAWR*
S2, S3: *NCAR, AlgM, RatDef, CR(UC), ref∡, CAST*	*Cofunction Identities (not addressed in stage one)*	S1, S4, S5, S6: *Idk, Xref∡*
F5; S2, S4, S5: *⚓, CR(UC), C∡, AlgM, CR(Gr)*	*Shift/Addition*	F1, F2, F3, F4, F6; S1, S3, S6: *Mem*
F5; S2, S5: *⚓, AlgM, Proportional/Rigid*	*Stretch/Multiplication*	F1, F2, F3, F4, F6; S1, S3, S4, S6: *Mem, V St = H Shrink, 3 • In => 3 • Out, Nyquist, Origin stretch*
F5; S3, S4, S5: *NCAR, ⚓, AlgM*	*H/In - V/Out*	F1, F2, F3, F4, F6 *Mem, V St = H Shrink*
F5; S2, S3: *NCAR, C∡, AlgM, ⚓, CR(UC), CR(Gr), NCorr*	*Horizontal Transformations are Counterintuitive*	F1, F2, F3, F4, F6; S1, S4, S5, S6: *Mem, Counter-Creep*
S1, S3, S5, S6: *Proportional/ Rigid, before/after applying the function*	*Order (not addressed in stage one)*	S2, S4: *The horizontal transformation order doesn't matter.*

similar to their original triangles, they were able to evaluate the trigonometric functions. F6 said that "the reference angle when you add [π] would be the same." An example of this reasoning can be seen in Figure 5. It was difficult for students to articulate why exactly the reference angles would be the same, but F3 said that the rotation would flip the triangle lengths over the x-axis.

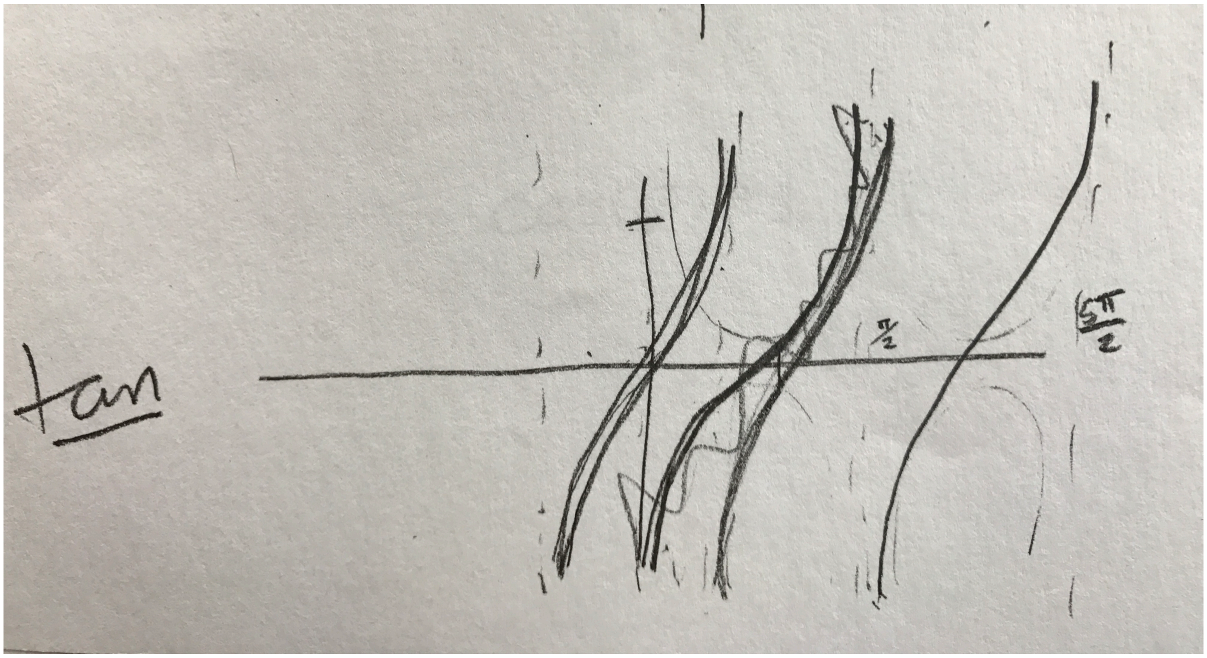


Figure 6. F5's drawing of tangent with period *2π.*

Similar to the opposite angle identities, each of these students explicitly referenced the quadrant of the angle to determine the sign of their answers (*CAST*). Additionally, when generalizing the identity, all of the students, including F6 and S1, used the unit circle to justify their arguments.

Two of the students who were not successful, F1 and F5, showed progress towards developing these identities, but did not generalize and justify their solutions. F1 approached some problems through the unit circle representation and reference angles, and in those instances he was able to justify answers consistent with the $(\theta + \pi)$ *identities*. He did not progress far enough through the interview to be explicitly prompted to think about these identities generally. Without this prompt, he did not make an effort to generalize the results of his work.

F5 showed mixed results for this identity. He examined $sin(\theta + \pi)$ graphically, evaluating the function at multiples of $\frac{\pi}{2}$. He correctly determined that $sin(\theta + \pi) = -sin(\theta)$.

However, he never justified an answer for cosine, and he indicated that he believed that tangent has period 2π, as seen in Figure 6. This would lead to incorrect answers that followed the pattern of sine, and because of this, F5 was coded as not having developed a good understanding.

F2 unsuccessfully attempted to recall the $(\theta + \pi)$ *identities* from memory. He provided some insight into his learning methods in the following exchange regarding an interview task in which he was asked to find $sin(\theta + \pi)$ given that $0 \leq \theta \leq \frac{\pi}{2}$, and $sin(\theta) = k$:

> **F2:** If that's an odd number [referring to the coefficient of π], you get $-k$.
> **R:** How'd you get that?
> **F2:** An identity.
> **R:** Can you explain that identity?
> **F2:** If that was an odd number, it would be negative sine, and if it was an even number, it would be positive sine.
> **R:** Do you know why that is?
> **F2:** No. It's one of the hardest identities I've had trouble memorizing. I try to memorize it instead of understanding it.

F2 claimed that $sin(\theta + n\pi) = sin(\theta)$ for even values of n, and $sin(\theta + n\pi) = -sin(\theta)$ for odd values of n. He never attempted to justify this identity and merely claimed it was true because he had memorized it. For the tangent function, he gave the right answer for the wrong reasons. He believed that tangent had a period of $\frac{\pi}{2}$, which would give the correct identity $tan(\theta + \pi) = tan(\theta)$, but would incorrectly state that generally $tan(\theta + \frac{k\pi}{2}) = tan(\theta)$. However, no matter what answers he provided, his lack of justification meant that he was coded as not having developed a good understanding.

All of the interview subjects except for F1 were able to justify the identities $T(\theta + 2\pi) = T(\theta)$. Each of the successful students used a unit circle representation (*CR(UC)*) and noted that 2π radians represents a full rotation around the unit circle (*ref∡*, *C∡*). The students noticed that this meant that the angles θ and $(\theta + 2\pi)$ are coterminal. For example, when explaining why $sin(\frac{-\pi}{2}) = sin(\frac{3\pi}{2})$, F6 stated "when it's just a normal circle, and you're finding measurements of angles, you go counterclockwise…. Then when you're searching for a negative measurement, you go clockwise. So then $\frac{-\pi}{2}$ and $\frac{3\pi}{2}$ fall in the same spot." These stages are reflected in Table 4.5. Several of these students incorrectly stated that these angles were the same and had to be prompted to recognize that they were different angles that intersected the unit circle in the same place. Despite this angle measure error, each of these students understood that adding 2π to any input of a trigonometric function would not change the output.

Despite the fact that the interview was explicitly about identities, the interviewer had defined identity, and the students comfortably used the $(\theta + 2\pi)$ identities, none of the students claimed that these were identities. For example, when examining $sin(x + 5\pi)$, F3 stated "I knew that 5π was the same as adding π, because it's two full rotations plus an extra one." The students' facility with this identity implied that this learning goal could be approached earlier than the other identities. However, students who successfully justified this identity were coded similarly to students who successfully justified the $(\theta + \pi)$ identities. Speculation regarding the reason for this disparity will be presented in the discussion section.

F1 performed relevant evaluations correctly when utilizing the unit circle, a reference angle, and the unit circle definitions of sine and cosine. However, when not using this strategy, he considered the functions to have periods of π "because that covers the range." Because of this

Table 4.4. Revised Critical Stages for ($\theta + \pi$) Identities with Supporting Quotes

***($\theta + \pi$) Identities* Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation; Simplify the algebraic representation	*NCAR; AlgM*	"$cos(\frac{\pi}{4})$… would be $\frac{5\pi}{4}$." (S1)
Evaluate the function at enough inputs to develop a pattern.	⚓	"So $cos(\frac{5\pi}{4})$ would be $\frac{\sqrt{2}}{2}$).... $cos(\pi + \pi)$ *is* 2π. So that would be 1. $2\pi + \pi$ is 3π. So -1." (S1)
Notice a correspondence	*NCorr*	"If the values are the same, the points should be the same." (S1)
Notice a change in the algebraic representation (did not simplify)	*NCAR*	"Then if I were to go 180…" (S2)
Move to a representation with better affordances; Change angles; Use a reference angle	*CR(UC); C∡; ref∡*	"It's on the other side…. It's the same angle." (S1)
Evaluate the function using similar triangles and the CAST diagram; Notice a correspondence	*CAST; NCorr*	"…it would also be $-b$." (S2)

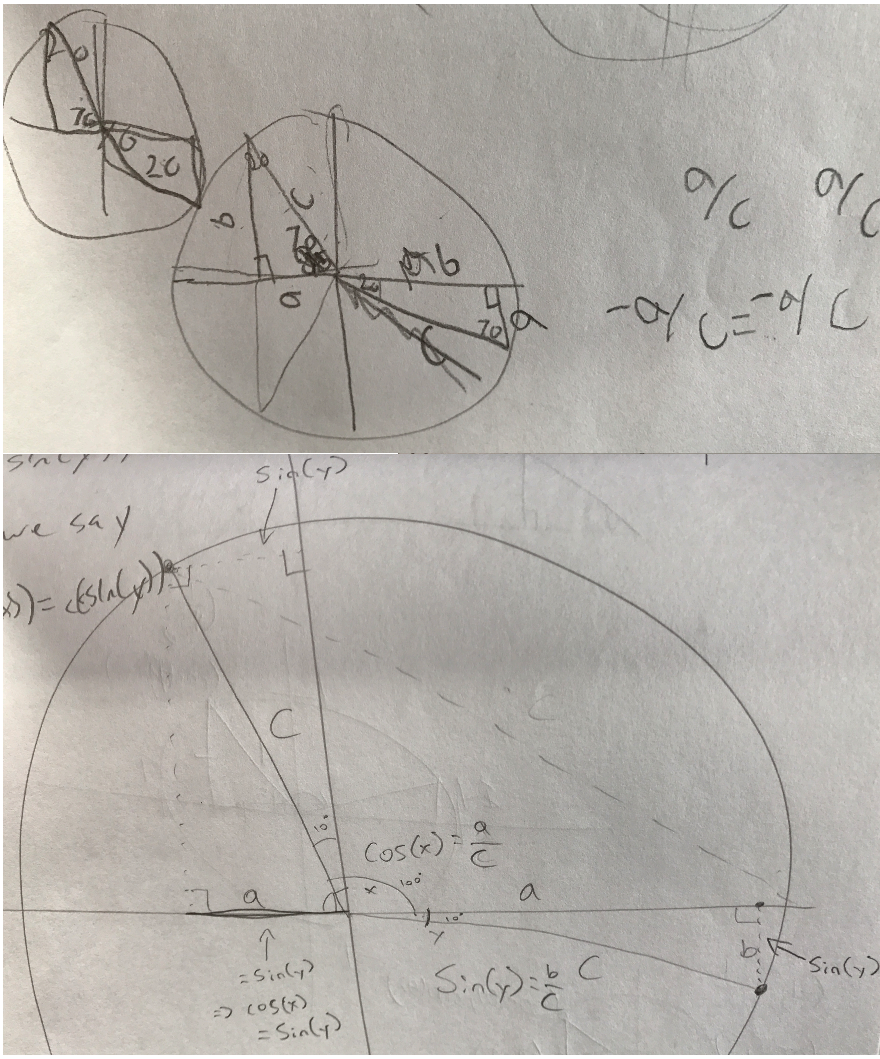
discrepancy, he was coded as not having a good understanding of the $(\theta + 2\pi)$ identities. F1's inconsistent conceptions prevented him from fully realizing these identities. The students who did not develop good understandings of the $(\theta + n\pi)$ generally did not reflect well enough upon their findings. Had F1, F2, or F5 reflected on their statements, they may have found them to be untrue, or they may have made some active, productive work towards convincing themselves of their truth.

Cofunction identities. Due to time restraints, the interviews in stage one did not cover the cofunction identities $cos(\frac{\pi}{2} - \theta) = sin(\theta)$ and $sin(\frac{\pi}{2} - \theta) = cos(\theta)$, so results are only included for the second stage interviewees in this area. All of the stage two participants noticed that the identities held when they were presented with a particular concrete example using a right triangle (*RatDef*). All of the participants were also able to generalize these identities for all complementary angles using the ratio definitions, recognizing that the leg opposite one of the acute angles is also the leg adjacent to the other acute angle (*NCAR*, *CR(Alg)*, *NCorr*). However, none of the students were able to rigorously justify the general identity. Some students claimed without justification that this identity must continue to hold for non-acute angles, such as when

Table 4.5. Revised Critical Stages for ($\theta + 2\pi$) Identities with Supporting Quotes

(θ + 2π) Identities **Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation	*NCAR*	"I knew that 5π was…" (F3)
Move to a representation with better affordances	*CR(UC)*	"When it's just a normal circle…" (F3)
Change angles; Use a reference angle	*C∡; ref∡*	"So then $\frac{-\pi}{2}$ and $\frac{3\pi}{2}$ fall in the same spot." (F6)
Notice a correspondence	*NCorr*	"…5π was the same as adding π because that's two full rotations…" (F3)

S1 said "I would assume that it would because it works for the acute ones." S5 attempted to use non-right triangles in a unit circle representation, as shown in Figure 7. She was looking to show that $cos(100°) = sin(-10°)$. After drawing the appropriate radii, she then connected the endpoints by what would be a chord in the unit circle. This is not a productive strategy, as the



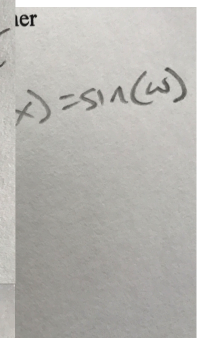


Figure 7.

(b)

Figure 8. Promising work from S2 (a) and S3 (b)

trigonometric functions are only applicable in right triangles. The angles labeled *w* and *x* in Figure 7 do not correspond with the ordered pairs at the ends of the radii.

Two students, S2 and S3, used strategies that were promising but ultimately unsuccessful, shown in Figures 8(a) and 8(b). These students used reference angles (*ref*∡) in the unit circle representation (*CR(UC)*), noticing that their reference angles were complementary (*AlgM*), and thus conformed to the established identity (*PrevWk*). These two students were unable to justify why pairs of angles that sum to $\frac{\pi}{2}$ would always produce complementary

(a)

reference angles, but their work was more productive than any of the other participants, as referenced in Table 4.6.

These promising approaches were incorporated into the identities lesson plan. Prior to the interviews, it was hypothesized that students would have to consider $(\frac{\pi}{2} - \theta)$ as a pair of transformations in order to generalize the cofunction identities for non-acute angles. Since the participants had not been introduced to the graphs of the trigonometric functions at the time of the study, it was hypothesized that students would have to consider the effects of these transformations on the radius of the unit circle – adding $\frac{\pi}{2}$ shifts the radius counterclockwise by $\frac{\pi}{2}$ radians and multiplying by (-1) reverses the rotation of the radius. However, the students' approach appeared more intuitive, as evidenced by the fact that two students took this approach, while none took the hypothesized approach through transformations. The lesson plan was revised to incorporate this strategy instead of the hypothesized strategy.

Another promising idea for the transformations lesson plan came from S1 while evaluating $-cos(x + \frac{\pi}{2})$ for various values of *x*. He said that "…adding $\frac{\pi}{2}$ flips the ones and zeros," referring to the *x*- and *y*-values in the ordered pairs where the unit circle intersects the axes. This idea was not pursued for two reasons: first, S1 did not refer back to his idea when examining the cofunction identities, whereas the only piece of information S2 and S3 failed to justify was that angles that sum to *π/2* will necessarily correspond with complementary reference angles; second, S1's observation may only be useful when viewing $T(\frac{\pi}{2} - x)$ as $T(-x + \frac{\pi}{2})$, which emphasizes that $\frac{\pi}{2}$ is being added to the input as part of two separate transformations, which none of the students explicitly did. Since the approach taken by S2 and S3 arose spontaneously and could be a promising method of showing that the cofunction identities generalize, it was adapted for use in the revised lesson plans.

Addition/shift transformations. The next learning goal addressed students understanding of the relationship between addition in the algebraic representation and shifts in the graphical representation. F5, S2, S4, and S5 were able to justify the relationship between addition and graphical shifts – that $T(x + c) + d$ will shift the function $T(x)$ leftwards by c units, and upwards by d units. In addition, F5 and S5 were successful in evaluating functions at regular intervals (*AlgM,* ⚓), plotting ordered pairs (*CR(Gr)*), and generalizing their results. In fact, while examining $sin(\frac{-\pi}{2} + \pi)$, F5 used the unit circle to perform his evaluations (*CR(UC)*). He stated, "$sin(\frac{-\pi}{2}$), so we're down here. Plus π, so that's going to take us all the way back here [to (0, 1)] …. So that's 1*"* ($C\measuredangle$, $ref\measuredangle$, *NCorr*)." S2 predicted that addition would cause rigid transformations of his ordered pairs, resulting in a shift of the graph. He said that a constant added to the function "shifts the whole thing up because you're not changing what the ratio is between input and output."

S4 used an approach similar to that espoused by Hall and Giacin (2013) and discussed in the literature review, in which the student takes a given ordered pair, then works backwards to find what original x-value would be transformed to the x-value in the given ordered pair, as seen in Table 4.7. S4 made up his own example to demonstrate his reasoning: "If $f(x) = 4$*,* and

Table 4.6. Revised Critical Stages for Cofunction Identities with Supporting Quotes

***Cofunction Identity* Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation; Algebraic manipulation	*NCAR, AlgM*	“Evaluate w? You would do $(180 - 75 - 90)°$, which is 15°.” (S1); “So, $x = 90 - w$, and $w = 90 - x$.” (S2)
Ratio Definition	*RatDef*	$Cos(q)$ – so adjacent over hypotenuse would be $\frac{j}{h}$…. $Sin(w)$ – opposite over hypotenuse would be $\frac{j}{h}$ again.” (S6)
Change representations to the unit circle; Change angles; Use a reference angle; Evaluate the function using similar triangles and the CAST diagram	*CR(UC); C∡; ref∡; CAST*	“…when you have a big angle like this… you follow it around the unit circle, and you make a reference angle based off what quadrant it’s in.
Notice a correspondence	*NCorr*	“It looks like I should be able to lift this triangle up and make it that same triangle…. $cos(x) = sin(y)$.” (S2)

$f(2x) = 4$, and the function is just this [$f(x)$], then this stays at 4 [*CR(Gr)*]. But with $f(2x)$ [*NCAR*], if x were to still be 4, it wouldn’t be true. It would be 8. So you have to divide by 2 to get 2 [*AlgM*, *NCorr*].” He also explained that additive transformations differ from multiplication in that “adding is only like a scale. It adds one to everything. Versus multiplying… stretches the slope….” Because of this, he was classified as having a good understanding of the addition/shift transformations learning goal.

Each of the students who could not justify the relationship between addition in the algebraic representation and shifting in the graphical representation relied upon memorization.

These students were confident that addition does result in graphical shifts, but were unable to justify that claim. They could not articulate the relationships between their work and their claims that addition causes shifts. The interviews also revealed that these students were unable to explain why multiplication and addition would result in different transformations. For example, when examining the difference between additive and multiplicative transformations, the following exchange took place between the interviewer and F3:

> **F3:** Because you're multiplying the function itself in this case by a different number, whereas normally you'd have the function and its input, but now you have three times the function, so it's not exactly the same as the original function.
> **R:** In [$cos(x) + 1$], you're adding something to it, and it seems like it's not the same as the original function.
> **F3:** I know. I honestly can't wrap my head around it. That's all I got.

Since these students made no effort to justify their claims, there was little insight into potential student errors. However, this example does reiterate that memorization provides a poor foundation for developing justified conceptions. F3 relied on memorization to such an extent that he was unable to articulate how addition produces different effects than multiplication.

Multiplication/stretch transformations. The multiplication/stretch transformation learning goal addresses the relationship between multiplication in the algebraic representation and stretching in the graphical representation. In order to justify that transformations of the form $(a)T(bx)$ will be stretched vertically by a factor of a and horizontally by a factor of $\frac{1}{b}$, S2 and S5 noticed [*NCAR*] that these are proportional effects – in contrast to the rigid effects of addition and shifting – while F5 inferred these relationships from his algebraic work [*AlgM,* ⚓]. As shown in Table 4.8, S2 described the effects of multiplicative transformations as changes to the slope of the function [*CR(Gr)*]. Describing the differences between additive and multiplicative transformations, he said “when you’re multiplying, you’ll end up changing the slope. For $y = mx + b$, you’re multiplying your slope by, say, two, so it gets twice as steep. With adding, you’re taking the already given slope and moving it up one, so it’s a parallel function one unit

Table 4.7. Revised Critical Stages for Addition/Shift Transformations with Supporting Quotes

***Addition/Shift Transformation* Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation	*NCAR*	“$Sin(\frac{-\pi}{2})$, so we’re down here. Plus π…” (F5)
Change to a representation with better affordances	*CR(UC)*	“...so we’re down here…” (F5)
Change angle; Use a reference angle	*C∡, ref∡,*	“Plus π, so that’s going to take us all the way back here [to (0, 1)].” (F5)
Evaluate using congruent triangles and the CAST diagram	*CAST*	“So that’s 1.” (F5)
Change to the graphical representation; Notice a correspondence	*CR(Gr); NCorr*	“[Plots transformed ordered pairs]… So it’ll shift it up all like that. They’re all the translated values of the base points, cosine of whatever.” (S2)
Notice a change in the algebraic representation	*NCAR*	“But with $f(2x)$...”
Change to a representation with better affordances	*CR(Gr)*	“…this stays at 4.”
Algebraic Manipulation; Notice a correspondence	*AlgM; NCorr*	“… if x were to still be 4, it wouldn’t be true. It would be 8. So you have to divide by 2 to get 2.”

higher" [*NCorr*]. In contrast, F5 compared transformed ordered pairs to the originals after multiplication, graphed each, and noted that multiplication correlates with a graphical stretch, similarly to how he justified that addition correlates with a graphical shift. Successful students referred either to a set of calculations showing that multiplication produces a proportional effect, or argued conceptually that multiplication is a proportional operation.

A number of additional mistakes occurred as students investigated stretches. For one, while S4 did notice that multiplication and stretching are both proportional, he failed to recognize that the graph is always stretched from an axis. He, along with S1 and S3, tracked the point given for $x = 0$, and stretched from the transformed point. When the interviewer pointed this out to S1, they had the following exchange:

R: So, we stretch away from the axis.
S1: You don't stretch away from, in this case, [$y =$]1?

That is, when examining $2(sin(x) + 1)$, S1 noted that the function originally passed through (0, 0), then was shifted to pass through (0, 1). He then stretched from the line $y = 1$. This mistake has the same effect as stretching before shifting, and provides students with an incorrect conception of the effects of transformation order.

Another mistake that arose during my interviews is related to the Nyquist frequency (Black, 1953, p. 7). The Nyquist frequency is the sinusoid with the largest frequency that passes through a given set of ordered pairs. Some interviewees graphed complex sinusoids in order to satisfy the given ordered pairs despite explicitly working with algebraic representations of the transformations that did not match their graphs. For example, S1 and S6 evaluated the function $cos(2x)$ for four inputs that all resulted in positive outputs. Because of this, they graphed

$cos(2x)$ entirely above the x-axis and with an inconsistent stretch, as seen in Figure 9. While the latter error may be solved or avoided by having students plot ordered pairs more frequently, the former error influenced how the students plotted those ordered pairs. Potential causes and solutions to this misconception will be elaborated upon in the discussion section.

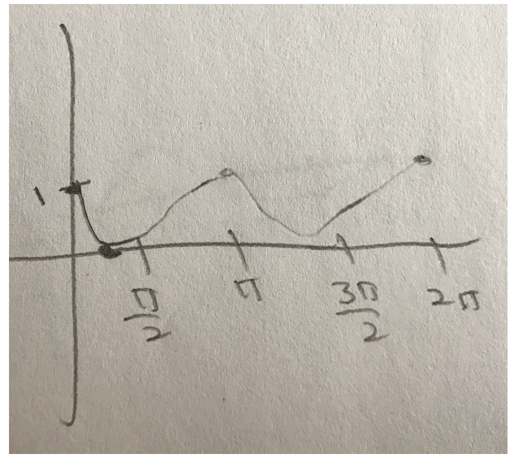


Figure 9. S1's graph of *cos(2x)*

Horizontal/input and vertical/output transformations. The hypothesized critical stages considered *horizontal/shift transformations* and *vertical/output transformations* as two separate

Table 4.8. Revised Critical Stages for Mult./Stretch Transformations with Supporting Quotes

***Multiplication/Stretch Transformation* Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation	*NCAR*	"Here I'm taking the output and multiplying it by whatever's in front of the cosine." (S2)
Evaluate the function at regular intervals	*AlgM,* ⚓	"$\frac{-3}{2}$ times $tan(\frac{\pi}{6})$. That equals $\frac{-3}{2}$ times $\frac{-1}{\sqrt{3}}$, which would give us $\frac{3}{2\sqrt{3}}$." (F5)
Change to the graphical representation	*CR(Gr)*	"So $(\frac{\pi}{6}, \frac{3}{2\sqrt{3}})$." (F5)
Notice a correspondence	*NCorr*	"For $y = mx + b$, you're multiplying your slope by, say, 2, so it gets twice as steep." (S2)

learning goals. It was theorized that these goals would be approached similarly, but that students would not achieve both simultaneously. However, the students who achieved these goals did not appear to consider them separately.

F5, S3, S4, and S5 justified that transformations of a function input affect the graph horizontally, and they were the same students who justified that transformations of a function output affect the graph vertically. In the algebraic representation, these students noticed whether the transformations occurred before or after applying the function (*NCAR*) and whether the graphical transformation was horizontal or vertical (*CR(GR)*). That is, transformations that occur before the function is applied result in horizontal transformations because they affect the function's input, whereas transformations that occur after the function is applied affect the output of the function, which is represented vertically on the graph. For example, S3 explained that $cos(x - \pi) + 2$ is shifted vertically by two by saying "Because once you find the cosine…. The cosine input is done [*AlgM*]. You find a separate number for cosine, then you add two to it for the *y*-value. It doesn't do anything to the cosine. It doesn't change that. It just adds two to what was there before." These students were also able to notice how the difference between *horizontal/input* and *vertical/output transformations* was reflected in other representations, as seen in Table 4.9. In terms of solution strategies, S3, S4, and S5 moved straight to ordered pairs in the graphical representation, while F5 used the unit circle to aid his calculations. F5 noted that horizontal transformations affected the input angle while the vertical transformations affected the value produced by the function. Since this observation did not lead or contribute to a justification, it was not coded as a critical stage.

Table 4.9. Revised Critical Stages for H/Input and V/Output Transformations with Supporting

***H/Input and V/Output Transformation* Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation; Evaluate the function at regular intervals	*NCAR, AlgM,* ⚓	"$cos(0)+1$ is 2 because $cos(0)$ is 1…." (S4)

The students who did not display a good understanding of *horizontal/input* and *vertical/output transformations* were largely able to answer questions correctly, but not justify their answers. These students had memorized what effects each algebraic transformation would cause. For example, when F6 was asked to explain the relationship between the algebraic and graphical representations of the function $-cos(x)+2$, he and the researcher had the following exchange:

> **F6:** So we need to move up two points. That's the plus two….
> **R:** Do you know why that's true?
> **F6:** No, it's just how I remember it.

Additionally, F6, S2, and S4 were convinced that horizontal stretches were identical to vertical shrinks and vice versa. While examining the graph of $tan(\frac{3}{2}x)$, F6 explained his reasoning to the researcher:

> **F6:** …That's a vertical shrink. You're multiplying the *x*-value by $\frac{3}{2}$, so if x is one, you're getting $\frac{3}{2}$. So there's going to be more distance between each *x*-value. The only thing I can picture is each *x*-value getting bigger, so each point having more distance between each. And the *y*-values are staying the same. So it's going to look like it's being pulled apart.
> **R:** You said vertical, and your gesticulations were horizontal.
> **F6:** Yeah, they kind of look the same, right? Because in a horizontal shrink, there's going to be less distance between each value. And it's going to look like a vertical stretch.

In comparison, F5 noticed that transformations in one direction did not affect the function in the other direction by noticing that horizontal transformations do not affect the range, and vertical transformations do not affect the *x*-intercepts. It was noted that F5's observation could be a useful prompt to help students notice that non-trivial horizontal transformations cannot be recreated with vertical transformations or vice versa. The error occurred often enough that the lesson plan was modified to incorporate F5's observation.

Horizontal transformations are counterintuitive. The participants were generally aware that there are counterintuitive aspects of horizontal transformations, but there were difficulties justifying the specifics. F5, S2, S3, and S4 successfully justified their answers by using a technique in which they worked backwards from their desired input to the original input. For example, S2 said that "in order to get outputs to stay the same you need to reduce the inputs by that amount." While explaining how he came to his conclusions, F5 said, "I was thinking of each of these as formulas, and taking different values of *x* and seeing where they would be on this graph. Drawing it was important to me." To underscore F5's use of drawing, he used the unit circle to aid in some of his function evaluations before moving to the graph, as shown in Table 4.10.

Table 4.10. Revised Critical Stages for H. Transformations being Counterintuitive with Supporting Quotes

***Horizontal Transformations are Counterintuitive* Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation	*NCAR*	"…you're adding π to each of your inputs." (S2)

***Horizontal Transformations are Counterintuitive* Revised Critical Stage(s)**	**Code(s)**	**Quote**
Evaluate the function at regular intervals	*AlgM,* ⚓	“So if θ was 0, you’d add π – $sin(\pi)$ – which is also 0. Then if I took the $sin(\frac{\pi}{4}+\pi)$ ...” (F5)
Change to the graphical representation	*CR(Gr)*	“So it gives you this point.” (S2)
Change to a representation with better affordances; Change angle; Use a reference angle	*CR(UC); C*∡*; ref*∡	[Examining $cos(-x)$] “…it’d still be down there. So these values are going to keep mirroring each other.” (F5)
Notice a correspondence	*NCorr*	“You’re changing the domain of the function. So your outputs are all staying the same, but you’re adding 1 to your inputs. So in order to get your outputs to stay the same, you need to reduce the inputs by that amount.” (S2)

The students who could not justify the counterintuitive nature of horizontal transformations uniformly attempted to repeat memorized information about what precisely is counterintuitive, but could not explain why some graphical transformations are intuitive and some are not. Most of these students could not consistently recall which aspects of graphical transformations are counterintuitive. This led to claims such as “multiplication in horizontal means division,” which led to algebraic claims such as $f(2\pi) = f(\frac{\pi}{2})$. For the prompt “Graph a cosine function horizontally stretched by a factor of two," S5 asked, "Do you mean a factor of two where you multiply everything by two? Or by a factor of two do you mean $\frac{1}{2}$?" Similarly, S4 asked, "So, it would condense then?" These students seemed to want to believe that everything about horizontal transformations is counterintuitive, even explicitly worded instructions.

Additionally, even the students who justified why horizontal transformations behaved counterintuitively individually struggled to explain why the order of horizontal transformations was counterintuitive. With prompting, all of the students noticed that their algebraic representations of pairs of horizontal transformations did not match their graphical representations, but none of them had explanations for why that was so. Some expressed displeasure that their burgeoning understanding was shown to be inadequate. It is likely beneficial to provide students with an explanation for the counterintuitive graphical behavior in addition to the fact that it is counterintuitive so as not to alienate the students from their work.

Order of transformations. The original task, discussed in the methods chapter, that was intended to help students notice that the order of transformations can have an effect on the resulting function did not serve its purpose. During the first stage, the interview questions regarding the order of transformations were presented using function composition. None of the stage one interviewees noticed that composition order corresponds with the order of transformations. The only observations any students made were that some orderings gave the same forms as others, and they believed this to be a typo. For example, when asking students to graph $F(x) = sin(x)$ first vertically stretched ($g(x)$) then horizontally shifted ($h(x)$) and vice versa, the students were asked to graph $g(F(x))$ followed by $g(F(h(x)))$, then $F(h(x))$ and

$g(F(h(x)))$. Unfortunately, the students focused on the presentation of the problem – the repetition of $g(F(h(x)))$ – rather than either the meaning of the algebraic or graphical representations.

For the second stage, the compositions were replaced with written commands, such as *Stretch* $cos(x)$ *vertically by a factor of two, then shift upwards by one*. S1, S3, S5, and S6 correctly applied these pairs of transformations by noticing that the order only mattered when horizontal transformations were combined or when vertical transformations were combined, but not for combinations of horizontal and vertical transformations, as seen in Table 4.11. These students noticed that the order of transformations could be indicated algebraically by inserting a set of parentheses around the first transformation. For example, the previously given example of a written command was written algebraically as $(2cos(x)) + 1$, while the reverse order was $2(cos(x) + 1)$. Figure 10 shows S3's algebraic representations using parentheses to emphasize the order of transformations. Furthermore, the successful students noticed that the parentheses only affected the transformations if both of the transformations occurred before the function was applied, or if both occurred after. When a horizontal transformation was combined with a vertical transformation, these students noticed that the parentheses that they inserted did not affect the function. S3 noted that "these two equations [*c* and *d*] are the same, and, shockingly enough, their graphs are the same. And these two expressions [*a* and *b*] are different, as are their graphs, which makes sense since they're graphs of the expressions."

A number of students had trouble noticing the different results of different orderings because they improperly stretched the functions. As mentioned earlier, some students tracked the changes to the ordered pair at $x = 0$, then stretched from the result. For example, after shifting

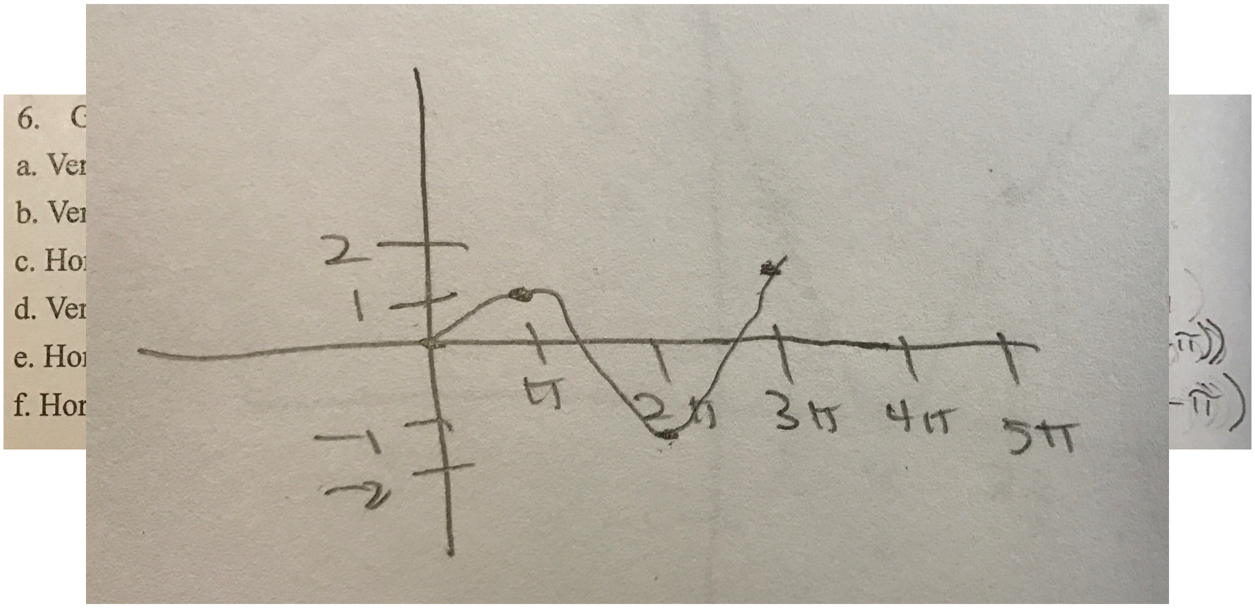


Figure 11. S1 shifted cosine by π then stretches from $x = \pi$ by two.

the cosine function π units to the right, S1 stretched from the line $x = \pi$, as seen in Figure 11. These students were prompted to evaluate function values based on their algebraic representations and compare these values to their graphs. In one case, a student was prompted that, based on their previous work, the graphs of $sin(x)$ and $sin(x + 2\pi)$ are identical. Since this was the case, based on his understanding of order, he would get different graphs if he stretched $sin(x)$, $sin(x + 2\pi)$, $sin(x + 4\pi)$, etc.; however, based on his understanding of $(\theta + 2\pi)$ identities, all of those graphs should be identical. This helped to convince the student that his method was incorrect, which was an improvement, but he did not infer that he should stretch from the y-axis. Therefore, another prompt must be found to fully combat this misconception.

The misconception that stretching should occur from a line other than an axis has resulted in a change to the hypothesized critical stages. In the hypothesized critical stages for transformations, there was a stage for recognizing that the transformations affected the entire graph. The critical stages have been revised to include this concept in the understanding of the individual stretch and shift transformations. The students who stretched their graphs from lines other than the axes were coded as not having achieved the learning goal of understanding stretches.

Table 4.11. Revised Critical Stages for Order of Transformations with Supporting Quotes

***Order of Transformations Can Matter* Revised Critical Stage(s)**	**Code(s)**	**Quote**
Notice a change in the algebraic representation; Notice that horizontal and vertical transformations are separated by the function operation	*NCAR; f(x)*	“Because of the parentheses. They don’t interact. One acts directly on the y-value…. Anything inside the parentheses is directly affecting the x-value.” (S3)
Notice that order between multiplication and addition matters	*AlgM*	"Because whatever the *x*-coordinate is on [6e], you’re multiplying the original *x*-coordinate by 2. Whereas in [6f], you’re multiplying (the original *x*-coordinate $+\pi$) times 2” (S5)

Parentheses appear to have been helpful for students to notice that order can matter. Students can even use their algebraic representations with parentheses to find that horizontal order is counterintuitive. However, if students are of the belief that shifting the graph also shifts the axis from which to stretch, then the parentheses will not serve their desired purpose. Having provided all of the results from the main study, this data will be analyzed and discussed in the following section.

Discussion of Research Question One

In this section the results of the main study will be interpreted. These studies were intended to provide information to revise the hypothesized critical stages – answering research question one – and inform how students' use of representations affected their abilities to justify their mathematical claims – answering research question two. Observations will be made about why certain codes or combinations of codes appeared in various stages. The hypothesized critical stages will be compared to the revised critical stages generated from the codes that appeared during the interviews. Finally, the implications that the revised critical stages have for the lesson plans will be discussed.

Considerations related to the order of stages. In this section, implications from the interviews related to the order of stages will be discussed. The codes collected during the interviews implied that the critical stages for justifying *$(\theta + 2\pi)$* identities and cofunction identities for acute angles can be achieved far earlier than had been hypothesized. Additionally, it was hypothesized that the critical stages for the cofunction identity would necessarily have to appear late in the sequence of critical stages. However, the students were able to make observations about the cofunction identity and even justify it for acute angles well before it was hypothesized that they would have the ability to. This section will present the evidence supporting the conclusion to move the associated critical stages earlier in the process.

All of the interview participants noticed that $T(x) = T(x + 2\pi)$ much earlier than the hypothesized stages imply that they should have been able to. Even though F1 noticed this identity, he went on to make statements about the functions' periods that were inconsistent with an understanding of this identity. The other participants were able to both notice that $T(x) = T(x + 2\pi)$ and justify that statement. Some of the students utilized and justified this identity before they had demonstrated that they had achieved any critical stages for other identities. For example, as mentioned in the $(\theta + n\pi)$ identities results section, F3 stated "I knew that 5π was the same as adding π, because it's 2 full rotations plus an extra one." They all seemed very comfortable with this identity based only on the unit circle definitions of the functions. However, most of the students also seemed to overgeneralize and conclude that each of the trigonometric functions has period 2π. This led to some confusion when the students graphed the tangent function and when they performed any horizontal transformations on the tangent function.

While the students justified the $(\theta + 2\pi)$ identities much earlier than other identities, they still progressed through the same set of critical stages. It is speculated that students may have been able to justify the $(\theta + 2\pi)$ identity sooner than the $(\theta + \pi)$ or opposite angle identities because they were investigating the relationships between trigonometric functions acting on the coterminal angles θ and $(\theta + 2\pi)$. Since the reference angles were identical, the comparisons were trivial. However, these students still proceeded through the same critical stages: noticing a difference in the algebraic representation, changing representations to one with better affordances, changing to a new reference angle, and noticing the correspondences between the values $T(\theta)$ and $T(\theta + 2\pi)$. Since these comparisons were trivial, the workload related to the *change angle, reference angle,* and *CAST diagram* codes was lessened. So, although the same codes are present, the trivial nature of their application has resulted in the critical stages for understanding the *$T(\theta + 2\pi)$* identities being achieved earlier than the corresponding stages for understanding *$T(\theta + \pi)$* or opposite angle identities.

Students were also able to approach the cofunction identities for acute angles much earlier than was implied by the hypothesized critical stages. The critical stages implied that students needed an understanding of how to justify opposite angle and $(\theta + n\pi)$ identities before they could understand the cofunction identities. However as was seen in the results section of this chapter, students were able to justify the cofunction identities for acute angles using only the ratio definitions of trigonometric functions and some algebraic manipulation. Since students did not need the unit circle definitions of the trigonometric functions to achieve this level of understanding, this critical stage – justifying the cofunction identities for acute angles – may be placed before many of the others. However, the difficulty that students faced attempting to generalize this identity to non-acute inputs and the promising strategies shown involving reference angles on the unit circle imply that this learning goal – justifying the cofunction identities for all real inputs – cannot be completed until much later. An argument implying that all pairs of angles that sum to $\frac{\pi}{2}$ should necessarily result in such reference angles may be too intricate for many students to understand while concepts such as unit circle angle measure and trigonometric functions are still new to them.

It is not clear how the critical stages should reflect the discrepancy between students' abilities to justify the cofunction identities for acute and non-acute angles. Based on the codes that appeared, students should be able to justify the cofunction identities for acute angles earlier than most other identities, since it only relies on the ratio definitions of the trigonometric functions. The facility with which students derived this identity corroborates this belief. But the codes did not identify a piece of knowledge that distinguished the students who made progress generalizing the identity from those who merely claimed that it should generalize. The latter students generally did not attempt to view the function in non-algebraic contexts; perhaps a larger sample size of successful students would have provided more codes and more separation between groups. A persistence code may have also separated the groups of students, since many who did not seriously attempt to justify the identity were satisfied with memorizing identities and the effects of transformations.

One example from this study that demonstrates the limitations of the implemented coding scheme was that there were no codes that differentiated S2 and S3 – the students who gave the most justified accounts of the cofunction identities – from other students who used the unit circle. S2 and S3 seemed to persist more and had more facility changing between representations, but these subjective observations have no supporting evidence. Perhaps future research on student affect or with a more refined coding scheme could differentiate between these students. S1 and S6 did not attempt to examine the relationships outside of acute angles, despite a prompt to examine a given example. S4 displayed an incomplete understanding of the relationship between the ratio and unit circle definitions of the trigonometric functions. He said "at least with triangles, I have a spot to measure from, versus the unit circle, where all I know is it goes from center to end." This indicates that S4 did not understand the relationship between reference angles on the unit circle and the ratio definition of right triangles. Similarly, when examining an angle of 100° on the unit circle, S5 drew a reference angle and lamented, "But then it's not 100° anymore." This indicates that she did not understand the relationship between the trigonometric functions evaluated at 100° and a reference angle in the second quadrant. Together, these examples provide some evidence that having the ability to fluidly change between unit circle and right triangle representations is a critical stage in justifying the cofunction identities. However, the current research has not been conceived with a framework for measuring the fluidity with which students move between representations. Rather, the second research question addresses the ways in which students use individual representations and how they make connections between representations, but not the ease with which they do so. This conclusion would agree with previous findings (Challenger, 2009; Weber, 2005) that mathematics students in general and trigonometry students in particular benefit from being able to easily change between multiple representations of the same concept.

Critical stage modifications. This section will describe events that occurred during the interviews which prompted the hypothesized stages themselves to be modified significantly. As has been mentioned in the methods chapter, one major modification is that the learning goals (and associated critical stages for) *horizontal/input* and *vertical/output transformations* have been combined. Another change is that credit has been given to an alternate algebraic approach towards recognizing that horizontal transformations are counterintuitive as described by Hall and Giacin (2013). Additional changes included elaborating the process by which students *notice a correspondence* between the unit circle and algebraic representations to include the codes for *changing angles*, using a *reference angle*, and using the *CAST* diagram. Also, a second method to generalize the cofunction identities has been integrated. Additionally, several student mistakes were prominent enough that it was determined that they should be mentioned with particular critical stages. The paragraphs that follow provide more details about these changes.

Originally, the learning goals for *Horizontal/Input* and *Vertical/Output* were hypothesized to be separate, similar to how *Addition/Shift* and *Multiplication/Stretch* are separated. However, the former pair seemed to be clearly dichotomous to the students, while the latter did not seem straightforward. That is, the students seemed to treat "not vertical" as synonymous with "horizontal," whereas it was not necessarily the case that "not shift" was the same as "stretch" or that "not addition" was "multiplication." This was evidenced by the fact that the group of students who could justify the *Horizontal/Input* learning goal was identical to the group of students who could justify the *Vertical/Output* learning goal. In comparison, S4 could justify the *Addition/Shift* learning goal, but not the *Multiplication/Stretch* one. This could be because the former pair are intuitively opposites, while multiplication is not considered to be the opposite of addition. Also, the students did not view all multiplication as stretching, but categorized it as stretches, shrinks, and flips. As a result of these differences between the learning goals, there were differences between the group of students who were coded as understanding *Addition/Shift* transformations and the group who were coded as understanding *Multiplication/Stretch*. In comparison, the group of students coded as having understood *Horizontal/Input* was identical to the group coded as having understood *Vertical/Output*. Therefore, joining these two learning goals was appropriate and supported by the data.

Some students approached the idea of counterintuitive transformations by using a method similar to that espoused by Hall and Giacin (2013). These students worked backwards to find what x-value would need to be transformed to produce a given y-value. The interviewees did not attempt to justify a general algorithm for any transformations as Hall and Giacin had demonstrated to a class, but some of the students were quite comfortable while examining the counterintuitive transformations in this way. The hypothesized critical stages, in contrast, were developed with the belief that this method is less beneficial than Borba and Confrey's (1996) rubber sheet method. The students who used Hall and Giacin's method applied it to individual x-values and generalized their results, rather than applying the method to a general point (x, y).

Further research is required to determine if this method is too difficult to generalize, if more time is required, or if this method is more beneficial for exploration while another method is more beneficial for generalization. In any case, this method has been accounted for and credit given to Hall and Giacin for recognizing its potential.

A third modification is associated with accepting less rigorous justification strategies. When investigating trigonometric identities, students tended to achieve the *Notice a Correspondence* critical stage through the unit circle representation. In particular, successful students interpreted the transformation as a change to the reference angle, and used the CAST diagram to determine the sign. This has led to a viable set of critical stages; there are other potential methods which could result in viable critical stages, notably using the graphical representation. However, the graph is in turn justified by the unit circle definitions. In the case of these particular students, they had not used the graphs to that point during their course, which may explain why they did not use the graphs to justify any identities. Additionally, students could generalize from a set of examples. Although this method could not be used to rigorously justify any of the learning goals, it is an acceptable justification for students at the precalculus level, as seen in these students' precalculus textbook (Axler, 2013). Since generalizing from a set of examples is acceptable in these students' precalculus course, it was accepted for this study, and a set of critical stages was created to accommodate this method.

For the interview question examining the order of transformations, the intention had been to examine the counterintuitive effects of horizontal transformations separately. However, numerous students began the exercises by writing out the task in algebraic notation rather than as in the written notation of the instructions. That is, students interpreted the written instructions to

"*Vertically stretch the function by a factor of 2*" as the algebraic expression $2f(x)$. The students' initiative was capitalized upon by asking them to note the impact of their parentheses, and whether the resulting expressions could be simplified to become identical. S3 noted, "these two equations are the same, and, shockingly enough, their graphs are the same. And these two expressions are different, as are their graphs, which makes sense since they're graphs of the expressions." This seemed to reinforce the idea that order of transformations can have an effect upon the result. Students saw that, after simplification, there were clear differences between the algebraic representations, which should result in differences between the graphical representations. While this algebraic method is promising, as it certainly perturbed some students, the interviews did not reveal any way to connect these observations to the graphical representations, which led to confusion for the students. As a result, there is no fundamental change made to the critical stages, but it is noted that if students find that their algebraic representations agree with their graphs but not their understanding of the effects of parentheses, those students should be directed to activities that bridge this gap in understanding before the student is negatively affected by their confusion. This was not a concern during the confirmatory studies because the algebraic, graphical, and dynamic unit circle representations were presented simultaneously, and the MATLAB program TrigReps did not allow students to insert their own parentheses.

Continuing examining the effect of parentheses on the algebraic expression, S1, S3, S4, and S6 noted how one transformation occurred before the function was applied, while the other occurred afterwards. This temporal dichotomy may be significant to the students in a way that the hypothesized critical stage of positional dichotomy – noticing whether the transformation is inside or outside the parentheses – is not. The students were not questioned regarding their word choice, but it could be examined in a future study.

The mistakes and methods noted above resulted in changes to the critical stages' order and content. Critical stages were modified and combined to produce the revised critical stages. In the following subsection, some student errors will be noted that did not result in fundamental changes to the critical stages but deserve some attention. These errors did not reveal levels of understanding through which students must progress, but they did reveal potential troubles that students could face while attempting to achieve those levels of understanding.

Notable student errors. Students made some errors that were notable but did not result in fundamental changes to the critical stages. In some cases, the errors were variations on stage zero errors – misunderstanding definitions. In other cases, the errors were prominent enough that they warranted inclusion in descriptions of the associated stages. These errors do not warrant their own stages because they aren't explicit levels of understanding that need to be achieved prior to fully understanding the learning goal, and it would be impossible to have a critical stage listing all of the mistakes that students shouldn't make. This section contains student mistakes which may hold some insight into ways in which students think and reason about trigonometric concepts. While none of these errors were important or prevalent enough to warrant explicit inclusion as a critical stage, it will be noted how these errors are connected to the revised critical stages.

The error that students made related to the Nyquist frequency – in which the students traced an incorrect sinusoid between a set of ordered pairs – may have been significantly influenced by the format of the task that was assigned to them. Since students were graphing the ordered pairs at $x = 0$, $x = \frac{\pi}{4}$, $x = \pi$, and $x = 2\pi$, the frequency change was not emphasized in the task or their work. A number of students expressed doubt about the graphs they drew from connecting the points in the simplest looking sinusoid. They gave two reasons for their doubt: (1) they were expecting a frequency change based on the algebraic representation of the transformation, and (2) all of the ordered pairs had non-negative y-values. For example, S1 and the researcher had the following exchange:

> **S1:** I was thinking *cos(π • 2).* I was thinking how to get- This is the only one that does a full rotation in *π*. And the way they did that was to multiply the inside by *2*. But that doesn't give you any- It doesn't drop below the *x*-axis….
> **R:** So what is the *2x* doing?
> **S1:** It's condensing it [H gesticulation].

S1 and S4 set aside their doubt and graphed the function as non-negative over the interval $[0, 2\pi]$. The design of this particular graphing task – task 3(g) of the revised transformations protocol – was not a significant limitation for this study, since there remained enough overlap of concepts across tasks that the students' understandings of stretches could still be observed in several instances. However, the answers provided by students in this study imply that tasks of this type can be successful if they are designed with these potential errors in mind. Function transformations could be used which result in both positive and negative outputs, and inputs can be chosen so that the Nyquist frequency is the desired frequency. This would potentially guide students away from this error.

Another common error that hadn't been hypothesized while developing the interview protocol was that students expected ordered pairs on the graphs of the function to also be on the unit circle. For example, when asked to plot $(\pi, cos(\pi))$, a number of students began at $x = -1$, since the input π on the unit circle corresponds to the point $(-1, 0)$. This error was possibly related to the fact that there was generally some confusion about the relationships among the x-value on the unit circle, the x-value on the graph, and cosine being defined as an x-value. The general ordered pair $(x, cos(x))$ confused some students since cosine is the x-value on the unit circle, but the y-value in that ordered pair. Some of these students were aided by being shown ordered pairs on the graph of $y = x^2$ as (x, x^2) and how the unit circle definitions for trigonometric functions correspond to the graphs. One student claimed that they had never been shown this connection before; the course he was taking at the time of the interview hadn't addressed it, and it is unknown whether or not it was addressed in his high school curriculum. These mistakes are related to stage zero knowledge – definitions of functions, ordered pairs, and the trigonometric functions – which implies that this error would be more properly placed with critical stages related to learning the unit circle definition of cosine and the connection between the graph and unit circle. These critical stages are not addressed in this study as they are assumed to be known by students examining trigonometric identities and transformations.

Since students were having difficulty with the repetition of the symbol x, it is reasonable to wonder if a different symbol should have been used. If the research tasks had used, for example, θ instead of x as the input variable, there would remain potential difficulties. Students may have only understood trigonometric functions as applicable to multiples of $\frac{\pi}{3}$ and $\frac{\pi}{4}$, or they may have only understood trigonometry as applicable in right triangles (Tuna, 2013). The former understanding defines the trigonometric functions as discrete rather than continuous, and the latter understanding ignores all negative inputs. Students may benefit from a repeated demonstration of the graph of the relation $x^2 + y^2 = 1$, the graphs of the trigonometric functions, and their relationship. A dynamic representation may help students make this connection and see sine and cosine as continuous functions, as discussed in Zengin, Furkan, and Kutluca (2012).

As has been mentioned, some students were coded as not having understood the relationship between multiplication and stretch transformations because they did not differentiate vertical stretches from horizontal shifts or vice versa. It could be helpful to ask students to track the range and x-intercepts of their functions as they learn about transformations. This could prompt the students to notice that the range remains unchanged under horizontal transformations, and the x-intercepts remain unchanged under vertical transformations. During the interview, there was no prompt about either the range or the x-intercepts. Instead, when left to reason on their own, the students frequently referred to the "slope" of the function. This was inadequate since, for example, $3sin(x)$ and $cos(3x)$ appear, without axes for context, to curve similarly. As a result, some students inferred that these functions must be the same. Without reference points such as x-intercepts or maximum and minimum values, these students had difficulty differentiating between these two types of transformations and were unable to achieve the *Horizontal/Input* and *Vertical/Output* learning goals.

A significant issue preventing students from achieving the learning goals related to counterintuitive transformations arose during the interviews, which will be referred to here as *counter-creep*. This name is derived from "Christmas-creep," the phenomenon by which Christmas decorations come out earlier every year. In this case, the students know that there are counterintuitive aspects of horizontal transformations, but they may be too eager to apply that knowledge. When students were asked to simplify $cos(2 \bullet \frac{\pi}{4})$, a number of students were confident that multiplication inside these parentheses is in actuality division. So, instead of $cos(2 \bullet \frac{\pi}{4}) = cos(\frac{\pi}{2})$, students would say that $cos(2 \bullet \frac{\pi}{4}) = cos(2 \div \frac{\pi}{4}) = cos(\frac{8}{\pi})$. This likely occurs because students memorize that there is something counterintuitive about horizontal transformations, but they don't remember what particular things are counterintuitive. As a result, they are liable to believe that any aspect might be a counterintuitive one. If that is the case, then this mistake will likely continue to occur until students can be shown why some aspects of horizontal transformations are counterintuitive and others are not. At the heart of this issue is the fact that students are attempting to memorize facts instead of learning how to justify their mathematical thinking. Students may need to be convinced that memorization is an inadequate strategy before they actively attempt to progress through the critical stages related to horizontal transformations being counterintuitive.

Students' errors are largely the result of attempting to supply answers before thinking through their work. These errors were occasionally exacerbated by students refusing to seek justification for their work. In the next section, it will be described how the lesson plans have been modified in order to guide students towards methods of justification and away from their flawed or unjustified arguments.

Implications for lesson plans. This section will examine how the changes to the critical stages have affected the corresponding lesson plans. Some changes have had more of an effect on the resulting lesson plans than others. The major changes are: (1) there are explicit warnings and scripted responses to newly recognized notable student errors; (2) the critical stages for horizontal and vertical transformations have been combined; (3) an algebraic approach to determining that horizontal transformations are counterintuitive has been validated; (4) the *notice a correspondence* critical stage has been refined to include *changing angles*, using a *reference angle*, and using the *CAST diagram* to determine the quantity and sign of the answer; and (5) the students' strategy for generalizing the cofunction identities has been employed.

Implications of student errors. When the lesson plans were being hypothesized, several common errors were not predicted. Some of these errors were trivial to account for, others involved substantial revision of the hypothesized lesson plans. For example, the error that students committed when they graphed a complex sinusoid through a set of ordered pairs with non-negative y-values was easily avoided by not providing the students with any tasks in which they had to extrapolate graph shapes from sets of ordered pairs.

The errors that ordered pairs on the graphs of trigonometric functions must satisfy the unit circle equation and that vertical stretches are equivalent to horizontal shrinks (and vice versa) were addressed briefly with counterexamples. Each was addressed during the lecture on trigonometric transformations. When the graphs were introduced, students were asked to check whether or not ordered pairs such as $(\frac{\pi}{2}, 1)$ or $(2\pi, 0)$ had x- and y-values that satisfied the unit circle equation. After some students responded in the negative, the researcher noted that the unit circle has an ordered pair (0, 1) and that $0^2 + 1^2 = 1$, so $(2\pi)^2 + 1^2$ must be greater than 1, and hence would not satisfy the unit circle equation. He then reminded the students of the relation among the unit circle and the definitions and graphs of the trigonometric functions.

During the segment reminding students of the relationship between the ordered pairs and the unit circle, students were also reminded of the multiple roles that the variable x would play during class. For instance, the ordered pairs $(x, cos(x))$ would appear on the graph of the cosine function. Students were instructed that the first x refers to the horizontal distance on the Cartesian plane, which is the input value x, while $cos(x)$ is the y-value of this ordered pair produced by cosine acting on the value x. Students had become confused because they associated sine with y-values and cosine with x-values in relation to their unit circle definitions. None of the students responded during lecture that they continued to be troubled by this distinction. However, as will be mentioned in study limitations, that does not necessarily mean the students were not actively troubled.

The lecture also contained a counterexample to the claim that horizontal stretches are equivalent to vertical shrinks and vice versa. The students were asked to compare the graphs of the functions $2sin(x)$ and $sin(2x)$. In particular, $2sin(x)$ has ordered pairs outside of the range of $sin(2x)$, and $sin(2x)$ has a different period and x-intercepts than $2sin(x)$. It was noted that, despite the similarities between the shapes of the graphs, and how similar their slopes looked when sketched, these graphs were in fact different. This meant that the functions themselves were different.

Finally, students were cautioned against attempting to memorize the effects of algebraic transformations. In addition to noting that memorization does not generally foster deep understandings, the students were reminded that it can be very difficult to memorize how exactly horizontal transformations differ from vertical ones and that believing that everything related to horizontal transformations is counterintuitive would often lead students astray during their work.

Implications of critical stage modifications. This section will detail how changes to the critical stages resulted in changes to the lesson plans. The critical stage modifications discussed here are that (1) horizontal/input and vertical/output stages were combined; (2) generalizing from a set of algebraic examples was recognized as a strategy; (3) the process by which students notice a correspondence between the unit circle and algebraic representations was refined; and (4) students' approach to generalizing the cofunction identity was incorporated.

The activities students participated in during the transformations teaching episode only gave options to multiply and add, which could imply that these operations are dichotomous. Additionally, the fact that the unit circle representation was also stretched and shifted may have prompted students to conclude that stretching and shifting were also dichotomous operations. To prevent a related potential misconception, the researcher framed shrinks as on a continuum of stretches in the hope that that students would not conclude that one of the operations referred to graphical shrinks. The researcher also emphasized that students could view stretches, shrinks, and reflections as a continuum of proportional transformations, where the effects were greater farther out from the appropriate axis. In contrast, shifts affected every ordered pair an equal amount. It was hoped that if students came to see these concepts as dichotomous in this context, then they would be more likely to classify them productively as types of transformations.

While the algebraic approach of generalizing from a sample of ordered pairs is sufficient for a precalculus class, it was hoped that the TrigReps program would help students justify that horizontal transformations act counterintuitively by helping the students make connections among the representations, in particular between the graphical, unit circle, and algebraic representations. For example, the effects of changing the speed and starting position of the radius can be seen to horizontally affect the graph, and are the result of transformations of the function's input. It was hypothesized that students would be able to use the changes to the algebraic representation to explain the changes to the unit circle representation, and could use these changes to justify the changes to the graphical representation. For example, multiplying the input by two doubles the speed that the function progresses through its inputs. This corresponds to the radius progressing through angles twice as fast and the endpoint progressing through its *x*- and *y*-values twice as fast, which corresponds to the graph horizontally shrinking by a factor of two (or stretching by a factor of one half). Furthermore, any shift of the starting position of the radius must necessarily occur before the change to its speed can be observed, which could help students to explain why the order of horizontal transformations is counterintuitive. Unfortunately, time limitations resulted in none of the students progressing through the tasks to the point where they were asked to investigate the counterintuitive effects of horizontal transformations.

For the hypothesized critical stages, it was assumed that students would notice a correspondence in some way between the algebraic representation and whatever representation that they had changed to. During the interviews, the unit circle was the most popular choice of alternate representations, and it was noted how students used this representation in particular to notice correspondences. Because of this, during the lecture on identities, the researcher emphasized the justification for the congruence of the reference triangles and for the signs given by the CAST diagram. Prior to the interviews, other methods had been considered to emphasize these justifications. Alternate methods for showing that unit circle triangles are congruent included using the fact that the diameters created sets of symmetric semi-circles or using the facts that vertical angles are congruent and all of the radii of a circle have equal length in order to craft an argument that reference angles 180° apart must create congruent reference triangles. These arguments were simplified to claim that, on the unit circle, triangles with equal reference angles are congruent, but students should check the orientations of the triangles. The orientation of the triangles is especially important in developing the cofunction identities since the height of one triangle will correspond to the width of another.

The hypothesized critical stages contained only one potential method for students to generalize the cofunction identities. It was hypothesized that it would be necessary for students to move to the unit circle representation and consider $(\frac{\pi}{2} - \theta)$ as a set of transformations on a radius. It was further hypothesized that students would have less difficulty if they viewed $(\frac{\pi}{2} - \theta)$ as $(-\theta + \frac{\pi}{2})$. This more explicitly shows the multiplication of θ by (-1) and the sum of this value with $\frac{\pi}{2}$; $(\frac{\pi}{2} - \theta)$ may be interpreted simply as one operation – the difference between two values. However, the two promising strategies seen during the interviews were very different from the hypothesized strategy. During the interviews, S2 and S3 noticed that angles that summed to $\frac{\pi}{2}$ consistently used complementary reference angles. They were unable to justify that this would necessarily be so, but their strategies provided evidence that this concept could be approached without transformations. During the second part of the lecture on identities, students were asked to notice that the angles created by two radii rotating in opposite directions from the positive *x*-axis will necessarily sum to zero, and if the positive rotating radius instead starts at $\frac{\pi}{2}$, then the angles will sum to $\frac{\pi}{2}$. Additionally, since one radius begins on the *x*-axis and one begins on the *y*-axis, the endpoints of the radii will have reversed *x*- and *y*-values. Since the endpoints have reversed *x*- and *y*-values, then the cosine of one angle is the sine of the other.

Thus, when the cosine and sine are respectively taken for two angles that sum to $\frac{\pi}{2}$, they will always produce equal values.

The interviews also prompted an additional logistical change. When students were investigating the task that explored the order of transformations during the interviews, many students attempted to perform both transformations at once. These students made significant mistakes that affected their abilities to achieve this critical stage. It was more difficult to convince some students than others to take multiple steps, but performing the transformations as two distinct operations was an important part of students' success. Without performing the actions separately, some students simply assumed that their results would be unsurprising and did not want to perform the work. The lesson plan was revised to separate combinations of transformations.

Also while attempting the task regarding order of transformations during the interview, the students spontaneously provided their interpretations of the algebraic representations of the pairs of transformations (see Figure 10). The algebraic representations helped students recognize that the order in which horizontal transformations are applied is also counterintuitive. Students were confident that they had placed the parentheses correctly reflecting the order of transformations. They were also confident that their graphs were correct, either because they were confident in their graphing skills, or because they were confident in the researcher's authority when he assured them that their graphs were correct. However, when the students were asked to use their algebraic representations to evaluate a few values in order to check that their algebraic representations aligned with their graphical representations, the students found that they did not. However, the algebraic representations did match up with the graph of the opposite ordering. Unfortunately, since this interview task showed students that there is something unexpected happening without helping them find why that is so, a number of students expressed displeasure. Many assumed that they must have made a mistake. Without any justification for why their representations did not align the way that they expected, students made comments such as “You blew my mind,” which is what S1 told the researcher when S1 considered applying addition before multiplication. Because of the students’ discomfort, the lesson plans were examined to determine whether this would cause an issue during the confirmatory study. It was not anticipated to cause difficulty during the transformations teaching episode because of the MERs provided to the students to investigate the effects of transformations. Students may be surprised that their graphical representations don’t match their predictions, but they could use the dynamic unit circle representation to connect their understandings of the definitions of the trigonometric functions and of graphical representations of transformations.

Conclusion. This section presented data and analysis for the first research question:

> *Through what critical stages do students pass as they come to understand trigonometric identities and transformations? That is, which actions, connections, or other ways of thinking are common to those students who go on to be able to justify their solutions of tasks involving these concepts?*

The thoughts and actions common to students who successfully justified trigonometric identities and transformations through task-based interviews were noted and compared to the hypothesized critical stages. Modifications were made to the critical stages, including combining some stages and elaborating on others. There was significant overlap between the critical stages for each of the learning goals. This is to be expected because the learning goals are all related to two subjects (identities and transformations). Furthermore, the identities can be viewed as a subset of particular transformations. Notably, each set of critical stages tended to include a need to change representations to one with better affordances. The only critical stages that did not include changing representations occurred when students were able to generalize from a set of algebraic examples such as $sin(x + \pi)$. With enough data points, students could reasonably conclude that $sin(x + \pi) = - sin(x)$. If, however, generalizing from a set of examples is not considered to be a justified understanding, then every set of critical stages requires a change in representations. Revised critical stages are presented in Figures 12 and 13 for a rigorous justification of the learning goals. Since students only justified why horizontal transformations are counterintuitive by pattern recognition, the rigorous justification is still hypothesized. This is signified by the dotted lines around these stages in Figure 12. This reinforces Weber's (2005) and Challenger's (2009) claims that successful trigonometry students must necessarily possess the ability to easily change between multiple representations.

Many students displayed incorrect understandings of stage zero concepts, such as angle measure, trigonometric function definitions, and function properties. These affected the students' abilities to progress through critical stages. For example, when a student found the angle $(\pi - \theta)$ on the unit circle instead of $(-\theta)$, his ability to notice correspondences between the associated trigonometric values was obviously affected. A better understanding of the stage zero concepts would allow students more opportunities to develop a conceptual understanding, rather than memorize facts. Interviewees repeatedly stated, especially when discussing transformations, that they had memorized the effects and had not attempted to justify the statements.

Discussion of Research Question Two

Students' use of representations will be discussed in this section. The hypothesized critical stages included several instances in which students must change representations in order to develop their understandings. For example, neither the algebraic nor right triangle representations offer appropriate affordances for generalizing the cofunction identities to non-acute angles. This led to a hypothesis that students would have to change to the graphical or unit circle representations in order to justify the generalized identity. It will be noted here whether it was possible for students to justify their answers using only the algebraic representations, and if not, what other representations they used to justify their work.

Identities. As mentioned in the previous section, some students were able to justify some identities simply through algebraic manipulation and pattern recognition. However, even these students were implicitly or briefly using the unit circle. When explaining the reasoning behind their evaluations, these students cited the unit circle definitions of the trigonometric functions. These students performed algebraic manipulation before and/or after using the unit circle to evaluate the trigonometric functions. Had the students known about the graphs of the trigonometric functions at this point in their course, then they could have potentially used transformations of the graphs to justify some identities. However, since the students had not worked with the graphs in class yet, it is not surprising that none of them used the graphs to make general arguments about identities.

Opposite angle identities. Despite not using the graphical representations, there were students who successfully justified several identities. The participants in this study who were able to justify the opposite angle identities did so using the unit circle representation. These students noticed that the endpoint of the radius at the opposite angle was reflected across the x-

axis. They could then apply their knowledge of the unit circle definitions to determine that this reflection would not produce a different output for the cosine function, but would give the opposite value for the sine function (see Table 4.2).

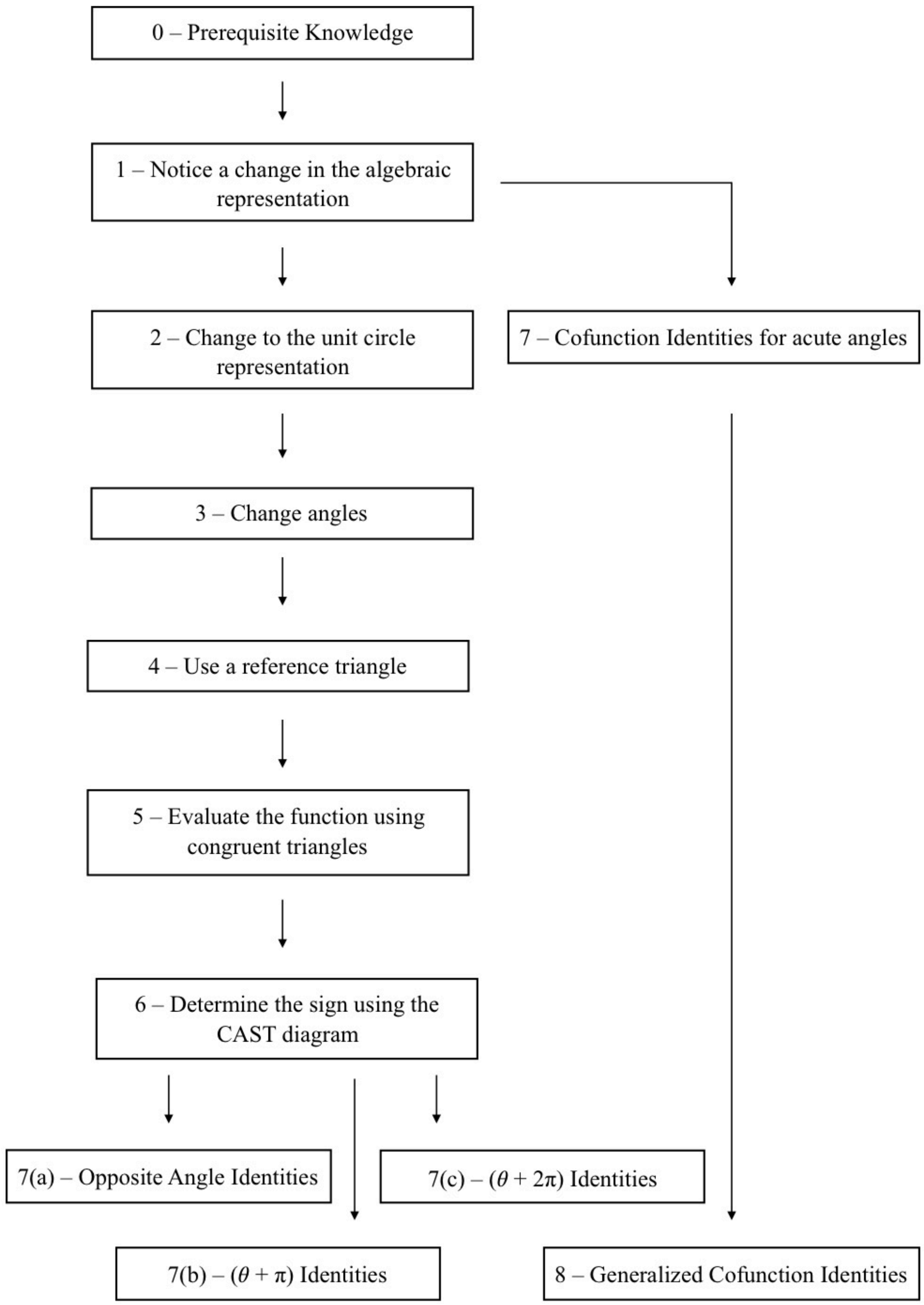


Figure 12. Revised critical stages for identities

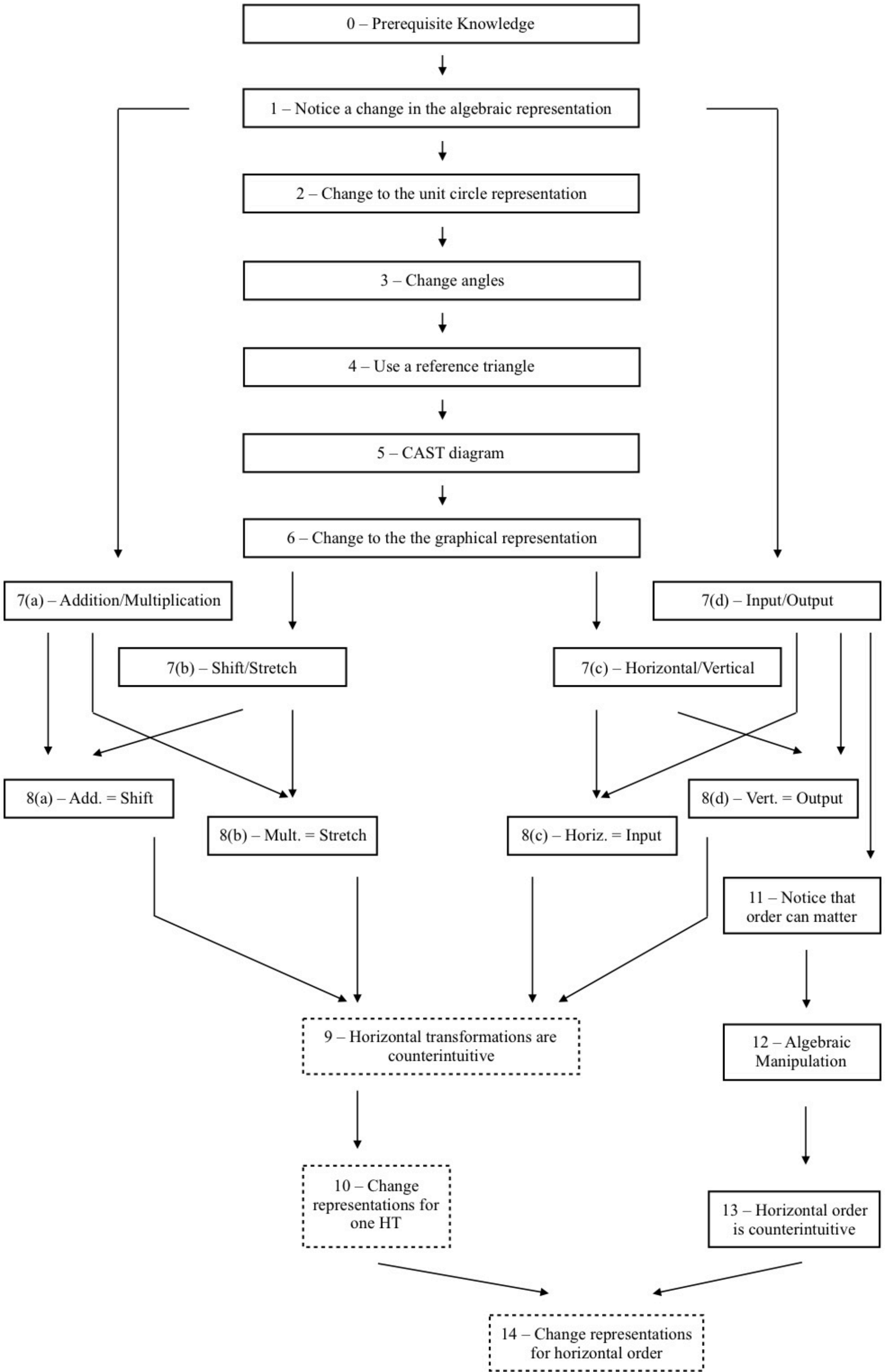


Figure 13. Revised critical stages for transformations

Notably, none of the students used the unit circle definition for the tangent function. They preferred to use the identity $tan(\theta) = \dfrac{sin(\theta)}{cos(\theta)}$ and their knowledge of sine and cosine. So, in the case of the opposite angle identities, the successful students concluded that $tan(-\theta) = \dfrac{sin(-\theta)}{cos(-\theta)} = \dfrac{-sin(\theta)}{cos(\theta)} = -tan(\theta)$. Had the students noticed that the slopes of the radii grow in equal and opposite proportions as θ and $-\theta$ grow, they may have found the opposite angle identity for tangent marginally faster. However, the unit circle definition for tangent may have had more significance for the identity $tan(\theta + \pi) = tan(\theta)$. In this case, the radii are 180° apart, and thus form a line segment. Since the slope is constant along this line, the tangent values will be equal for any outputs that differ by odd multiples of π.

Students were able to use reflection across the x-axis of the unit circle to reason about the effects of taking the opposite of an input angle. The graphical representation may have been another viable path had it been more prominent feature of their class. Students also may have generalized the identities after checking them for several values of θ, but none of them attempted this strategy.

(θ + nπ) identities. The algebraic representation was more prominent in justifying these identities than the opposite angle identities. Some students were able to justify identities of the form $T(\theta + n\pi)$ by evaluating the transformed functions algebraically and noticing how the results relate to $T(\theta)$. At the precalculus level, this pattern recognition is typically considered a justification for the general rule, as can be found in precalculus textbooks (see Axler, 2013). Other students used the unit circle to draw a general angle θ, and the angle $(\theta + \pi)$. As mentioned in the results section, the students had difficulty articulating how they knew that the reference angle at θ would produce a triangle congruent to the one created by the reference angle $(\theta + \pi)$. For example, given that $cos(x) = a$, S1 stated that $cos(x + \pi) = -a$, after which he and the researcher had the following exchange:

R: Why is it $-a$? Why isn't it $\frac{-2}{3}$?
S1: We're putting it in terms of a. If the adjacent side was $\frac{2}{3}$....
R: Why does it have to be the same x-value over there and over here?
S1: The hypotenuse is always going to be 1, so wherever you go out to would be the same.

It is a positive sign that, in this case, these students' intuition was reliable. Their intuition could perhaps be capitalized upon by prompting students to notice that the reference angles created by this line are vertical angles, and are thus congruent. Despite the difficulty of justifying that the reference triangles are congruent, the successful students continued with their work by applying the unit circle definitions of the trigonometric functions to determine the sign (positive or negative) of their answers.

Despite using the unit circle representations to justify the $(\theta + \pi)$ identities for sine and cosine, the students moved back to the algebraic representations to find the corresponding identity for tangent. Had they considered that tangent can be defined as the slope of the radius, they may have come to the same conclusion more quickly and without the potential difficulties that students face reducing terms and using identities and fractions. Through the unit circle representation, students may notice that the radii for angles θ and $(\theta + \pi)$ form a straight line, and precalculus students should be familiar with the fact that lines have constant slopes. Students' approaches to justifying the opposite angle and $(\theta + n\pi)$ identities typically used the unit circle and algebraic approaches. In the following section, the cofunction identities will be discussed, in which students needed to use the right triangle representations. Students then had difficulty changing from the right triangle representation to one that offered more affordances for justifying the identity for non-acute angles.

Cofunction identities. The stage one interview participants were not asked about the cofunction identities, but each of the stage two participants successfully justified the cofunction identities for acute angles using a right triangle representation. This representation, together with the ratio definitions of trigonometric functions and geometric knowledge about angles in a triangle, aided students in justifying the cofunction identities for acute angles.

Even so, only two students made significant progress generalizing these identities. These two students, S2 and S3, did so through unit circle representations rather than staying with right triangle representations. The unit circle helped these students examine the reference angles formed by pairs of angles that sum to $\pi/2$. They observed that these reference angles were complementary, and the reference triangles were arranged in such a way as to extend the identities to non-acute angles.

The unit circle appeared integral to S2 and S3's success in justifying identities. In fact, all of the successful students used the unit circle either explicitly or implicitly. Had the students been familiar with the graphs of the trigonometric functions through their class, it is possible that they could have used graphical transformations to justify their identities, but I did not see any evidence of this. None of the participants made any attempt to do so. Regardless, the hypothesis that students would need to change to a representation with better affordances than the algebraic one has been borne out by the collected data in relation to cofunction identities. That is, the unit circle and graphical representations can both be used to view an infinite number of data points for pairs of θ and $(\frac{\pi}{2} - \theta)$, whereas the algebraic representation only provides students with three values of θ – $\frac{\pi}{3}$, $\frac{\pi}{6}$, and $\frac{\pi}{4}$ – for which they can justify their responses.

Transformations. The learning goals for transformations largely involve relationships between algebraic and graphical representations – addition corresponds with shifts, transformations of the function's outputs correspond with vertical graphical transformations, the horizontal graphical transformations are counterintuitive based on students' understandings of algebraic properties, and so forth. So it is natural that both of these representations were frequently used as students sought to justify their answers. In comparison, the unit circle was only used to spontaneously explain transformations when F5 said that $cos(2x)$ would cause the radius to rotate twice around the circle for every one time that $cos(x)$ would. The fact that students chose algebraic and graphical representations over the unit circle and triangle representations – where students learned the definitions of the trigonometric functions – may be related to the students' reliance on memorization. While examining identities, students could use the unit circle definitions to reason through the effects of the changes. However, with transformations, the successful students largely performed algebraic evaluations before plotting points on the graph. It is hypothesized that it may be more difficult for students to generalize and recall how sets of ordered pairs were transformed on the Cartesian plane than how radii are transformed on the unit circle. A lesson plan reflecting this hypothesis was intended to be tested in the confirmatory study, but time restrictions did not allow for it.

To further investigate how students made sense of representing transformations, the students were asked during the interviews to describe how to portray trigonometric transformations using a circle and radius. While this was intuitive for some participants, such as F5, others struggled with the concept. For example, S3's description of how to show $2sin(x)$ using a circle and radius was to find $sin(x)$ the standard way, then to multiply the result by two. This reflects the students' algebraic process in generalizing the properties of transformations. The dynamic representation that students were provided during the confirmatory study was intended to help the students build stronger connections between the unit circle definitions and the effects of transformations by showing them ways to transform the unit circle itself to provide transformed outputs.

One goal of having students build stronger connections between the unit circle definitions and graphical transformations is to make it easier for students to justify the effects of transformations without performing several sets of evaluations. Students who develop a pattern through repeated algebraic work are not creating as rigorous a justification, may take longer to identify the pattern, may identify an incorrect pattern by evaluating at inputs that are spaced unevenly or too widely, and must endure the tedium of repeated algebraic work in order to justify each identity. Additionally, by using the unit circle, students may be able to justify why horizontal transformations are counterintuitive without memorization. By considering horizontal stretches and shifts as changes to the speed and starting position of the radius, the resulting graphs are intuitive: multiplying by large numbers increases the speed and hence the frequency; adding changes the starting value, which is akin to shifting the axes to the right or the graph to the left.

Finally, students successfully examined the order of transformations by using algebraic representations. The students inserted parentheses around the operation that they wished to be performed first, which resulted in different algebraic representations for combinations of transformations that signified different functions and identical algebraic representations for combinations of transformations that resulted in the same graph. For example, horizontally shifting left by one then vertically stretching by two produces an algebraic representation – $2(f(x+1))$ – that is identical to the one produced by performing the transformations in the opposite order – *(2f(x + 1))*. Or, as was earlier noted, S3 remarked, "these two equations are the same, and, shockingly enough, their graphs are the same. And these two expressions are different, as are their graphs, which makes sense since they're graphs of the expressions." While the task asked students to graph their functions, many students were making informed hypotheses about the effects of the order of transformations before they began graphing. However, the students who used the algebraic representation alone were uniformly unable to justify why the order of the horizontal transformations was counterintuitive. Relying only on the algebraic representation for this aspect could negatively affect students in the short-term as they struggle to understand why algebraic properties with which they are familiar seemingly do not hold.

Conclusion. This section presented data and analysis for the second research question:

> *How do students understand the relationship between the unit circle definitions of trigonometric functions and the identities and transformations of those functions? Is it critical that students be able to change from the algebraic representation to one with different affordances as they come to understand identities and transformations?*

The groups of students who successfully justified each of the trigonometric identities shared the ability to change between the algebraic and unit circle representations. When students failed to change between the algebraic and unit circle representations, they were generally unable to justify the identities. Changing to the unit circle representation helped them formulate productive hypotheses and justify their generalizations. In contrast, the groups of students who justified each learning goal for transformations typically made generalizations based off of a small set of ordered pairs. Generalizing from a small set of information is not as rigorous as justifications using the unit circle or graphical representations, and students struggled to justify several transformation concepts, notably when and why certain transformations are counterintuitive.

Out of the nine learning goals addressed in this study – (1) opposite angle identities, (2) $(\theta + n\pi)$ identities, (3) cofunction identities, (4) addition/shift transformations, (5) multiplication/stretch transformations, (6) input/horizontal transformations, (7) output/vertical transformations, (8) order of transformations, and (9) horizontal transformations being counterintuitive – there were four in which at least half the students could correctly justify their understandings. These were the learning goals for which students relied on the unit circle definition, or their understanding of order of operations – as when students generalized the results of transformations based on a small set of ordered pairs or correctly applied parentheses to their algebraic representations, imposing a new order for the given operations. Students struggled to bring their unit circle knowledge to bear on the other learning goals, and they struggled to justify those learning goals. This suggests that students who can interpret various trigonometric situations through their conceptions of the unit circle may have more success than students who cannot identify and change to a useful representation such as the unit circle or graphical representations, which agrees with previous literature (Challenger, 2009; Weber, 2005).

The results from this study suggest that students must connect their understandings of the algebraic and unit circle representations of transformations in order to justify trigonometric identities and graphical transformations. There were successful students who did not use the unit circle representation. However, when prompted, these students said that they based their reasoning in the algebraic representation on an implicit unit circle. In this way, these students could still be said to rely on the unit circle representation. Additionally, the method of justification through a set of examples is not mathematically rigorous, and those methods may not be accepted in other classrooms.

Results from the Confirmatory Study (Identities)

The confirmatory study was performed to demonstrate the utility of the critical stages as a framework for designing a lesson plan. None of the students who participated in the confirmatory study had participated in any previous portion of the study.

There were logistical difficulties collecting data from the confirmatory study. To be consistent with the other sections of precalculus being taught by other TAs, students worked in groups of three or four during recitation periods. For the purposes of this study, these groups had been temporarily arranged by the researcher’s advisors in order to have participating students working together. During the recitations, because of absences, some groups had to be combined. The researcher had no way of knowing whether the combined groups would be composed of participants, non-participants, or a mix. Additionally, there were technological shortcomings with the recording devices. Finally, despite repeated reminders, few participants completed the preand post-tests. Three students completed pre- and post-tests for both identities and transformations An additional two took only one test: the pre-test for identities. As a result of the low turnout, there were only three students, C2, C3, and C4, who produced pre- and post-tests and written group work, as seen in Table 4.12. There were another five students, C11, C12, C13, C14, and C16 who produced usable audio and written group work. Additionally, C11 and C12 took the pre-test for identities. There were five students, C1, C5, C6, C7, and C8 who produced only written group work. There was a single student, C15, who provided pre- and post-tests, audio recordings, and written work. Finally, students C9 and C10 were asked to work with non-participating students because of absences. As a result, their data was unable to be separated from the non-participating students' and was deemed unusable. This section will present the results based on data that were able to be collected. The results are presented by group in an attempt to be transparent about what conclusions could be drawn from the available data.

Group 1. The students C2, C3, C4, and C6 were grouped together for classwork during recitations. Although the audio recording device failed, students in this group did complete the activities and submit their work. Additionally, C2, C3, and C4 completed both sets of pre- and post-tests. The pre- and post-tests for C2, C3, and C4 will be presented individually, and their classwork will be presented collectively, since it is impossible to distinguish which members of the group provided which aspects of the submitted work.

Of the 12 tasks on the identical pre- and post-tests for identities, C2 improved from 6/12 correct to 7/12 correct. The improvement was due to a corrected special right triangle on the post-test. While C2's work on the final task of the pre-test used the tangent identity –

$tan(x) = \frac{sin(x)}{cos(x)}$ – and contained no other productive work, the work on the post-test shows an attempt to use the cofunction and $(\theta + \pi)$ identities in order to relate $cos(105°)$ and $tan(165°)$ to the given value of $sin(-15°)$, as seen in Figure 14. However, none of the identities cited are correct, nor is there evidence that he has changed representations in an effort to better understand the effects of these transformations. As a result, it is unclear from the pre- and post-tests that C2 made any progress through the critical stages for identities.

C3 scored 0/12 on the identities pre-test, only writing that he's sorry and can't do any of the tasks. On the post-test, his score improved to 4/12 with productive work on a fifth problem, but the other seven tasks were still blank. The correct responses show an understanding of some $(\theta + n\pi)$ identities, but none of the other identities. This indicates that C3 may have progressed through the critical stages of some $(\theta + n\pi)$ identities, but there was no other evidence that he had attempted to justify other identities. Similarly, C4 improved from 3/12 to 6/12 by correctly evaluating expressions of the form $cos(x + \pi)$ for various values of x in addition to the previous correct work using the opposite angle identity. Since C4's justification relied solely on the

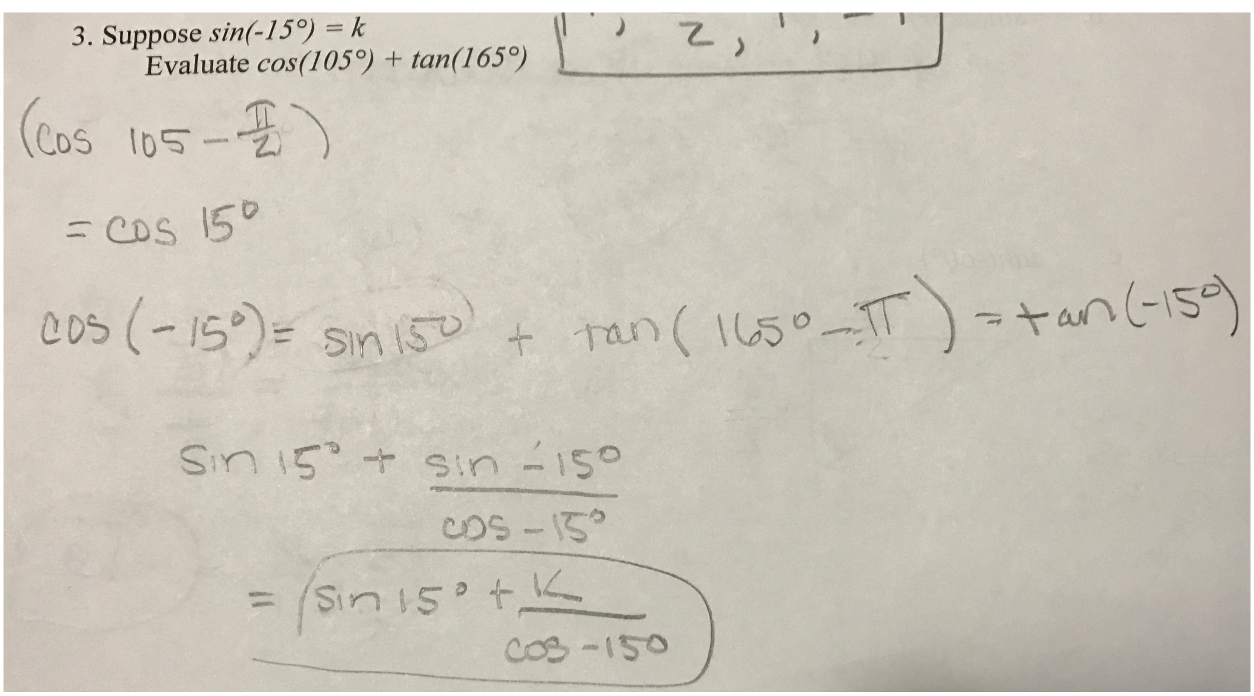


Figure 14. C2's attempts to use identities.

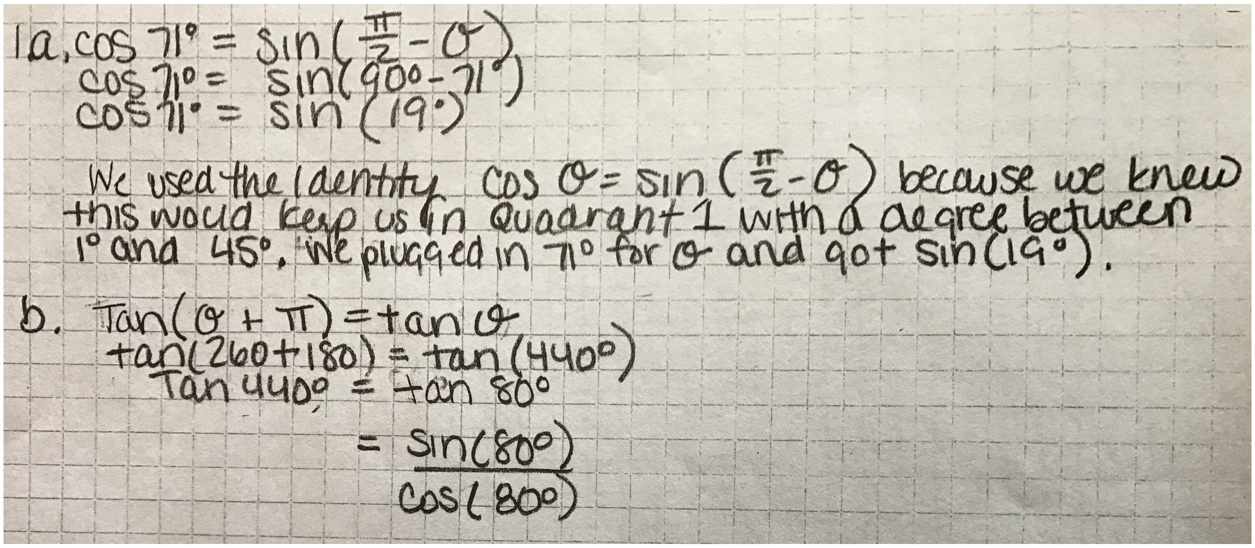
Table 4.12. Data collected during the confirmatory study

Student	Pre-Test (I)	Post-Test (I)	Pre-Test (II)	Post-Test (II)	Audio Recordings	Classwork
C1						X
C2, C3, C4	X	X	X	X		X
C5, C6, C7, C8						X
C9, C10						
C11, C12	X				X	X
C13, C14					X	X
C15	X	X	X	X	X	X
C16					X	X

algebraic representation, it is impossible to say whether he had progressed through any additional critical stages.

The collected class work from Group 1 reveals that they worked productively during the recitation period. However, it is impossible to determine which group member contributed which idea, or to understand how the students came to their conclusions without the audio recordings to accompany their written work. Nonetheless, the results of their group work are presented here. This group provided a well-constructed argument showing how they would evaluate $cos(71°)$ given the problem "Suppose that you have a table that gives you values for $sin(1°)$, $sin(2°)$, …, up to $sin(45°)$. Explain how you would find $cos(71°)$." They also made productive work towards evaluating $tan(260°)$ before running out of time, as seen in Figure 15.

With the data collected, it cannot be determined what effect, if any, the lesson plans had on these students. The justifications on the post-tests were not thorough enough to ensure that they had progressed through critical stages. One student used some $(\theta + n\pi)$ identities but did



1a. $\cos 71° = \sin(\frac{\pi}{2} - \theta)$
$\cos 71° = \sin(90° - 71°)$
$\cos 71° = \sin(19°)$

We used the identity $\cos \theta = \sin(\frac{\pi}{2} - \theta)$ because we knew this woud keep us in Quadrant 1 with a degree between 1° and 45°. We plugged in 71° for θ and got $\sin(19°)$.

b. $\tan(\theta + \pi) = \tan \theta$
$\tan(260 + 180) = \tan(440°)$
$\tan 440° = \tan 80°$
$= \frac{\sin(80°)}{\cos(80°)}$

Figure 15. Group 1’s identities classwork.

not justify the identities themselves on the post-test. The classwork indicates that at least one student in the group knows how to apply the cofunction identity to acute angles. However, it is not clear which student knows this or whether that student can justify it.

Group 2. Since group 2 provided audio data, it was easier to find who contributed which ideas to the group work. C11, C12, and C13 worked together and produced audio and written class work. Additionally, C11 and C12 took the pre-test for identities, scoring 4/12 and 6/12, respectively. As will be demonstrated in what follows, the transcriptions of their audio recordings show that, after having attended lecture, they were able to work together productively using trigonometric identities. C13 stated that he did not attend lecture, and he did not offer much of substance to the group’s conversation, but without any other information, it cannot be determined how he was affected by the teaching episode.

Working through the exercises, C11 and C12 continually attempted to find identities that connected their given information to their desired information. They checked whether the given information fit productively into any identities given during the lecture, and if they were unable to find any, they moved to the unit circle representation and created reference triangles. C11 used the cofunction identity to relate the desired information, $cos(71°)$, with the given information $sin(19°)$, saying "180 – 90 – 71 is 19. So $sin(19°) = cos(71°)$. C12 cited the identity "$sin(-\theta)$ is $-sin(\theta)$" to find $sin(-42°)$ given $sin(42°)$. When faced with a situation in which they could not find an applicable identity, these students referred to reference triangles on the unit circle. For example, they noticed that $\frac{\pi}{12}$, $\frac{-5\pi}{12}$, and $\frac{-13\pi}{12}$ create congruent reference triangles and therefore produced predictable *x*- and *y*-coordinates. Their written work for these inferences can be seen in Figure 16, and the following exchanges occurred in relation to tasks *3A* and *3B*, respectively:

> **C11:** We're looking for the *y*-coordinate of that.
> **C12:** Wouldn't it be the same thing, but negative?
> **C11:** It would be negative square root that value because it's down the same angle. If we're focusing on *x*- and *y*-coordinates, yeah. We're just taking this and flipping it over here.
>
> **C12:** It's cosine.
> **C11:** But it's the opposite angles, so it's the same thing, because we're just getting the *x*-coordinate off of that.
> **C12:** Negative, because it's in the second quadrant.

Comparing the audio recordings and work submitted by Group 2 with the pre-tests of C11 and C12 indicate that the identities lecture had a positive effect on these students. It appears that the students worked more correctly and productively after attending the identities lecture. However, without any data from the post-tests, it is impossible to say whether or not they would have improved individually from the pre-test without their classmates to help with the work.

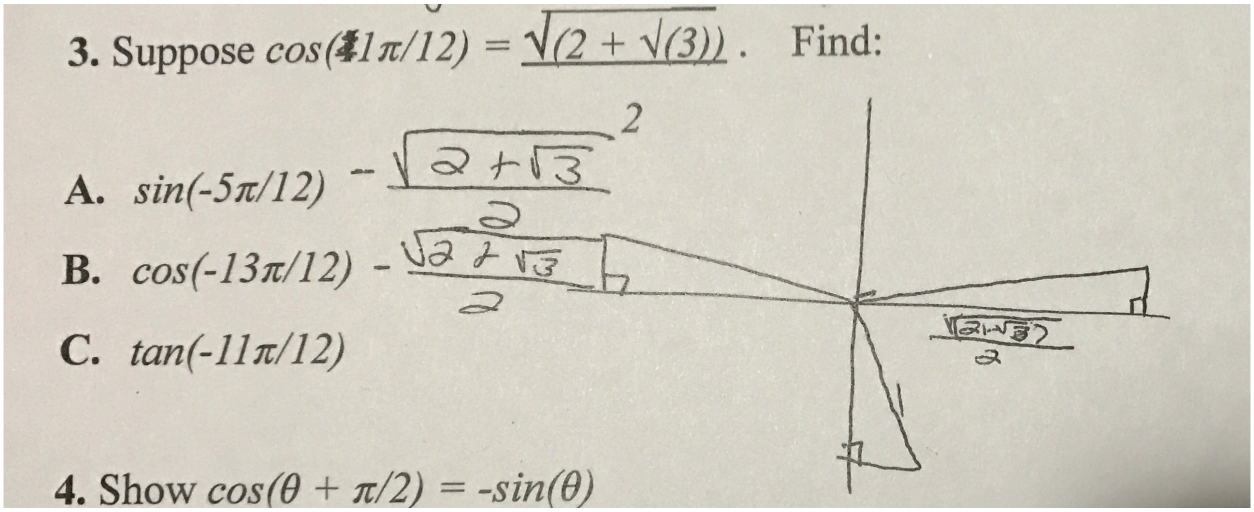


Figure 16. Classwork from C11, C12, and C13.

Group 3. C1, C5, C7, and C8 worked together as a group, but their audio equipment failed, and none of them took the pre- or post-tests. Their submitted work for the identities activity consists of a single, well-justified solution stating that $cos(71°) = sin(19°)$. Without any other information, it is impossible to draw conclusions from this data set.

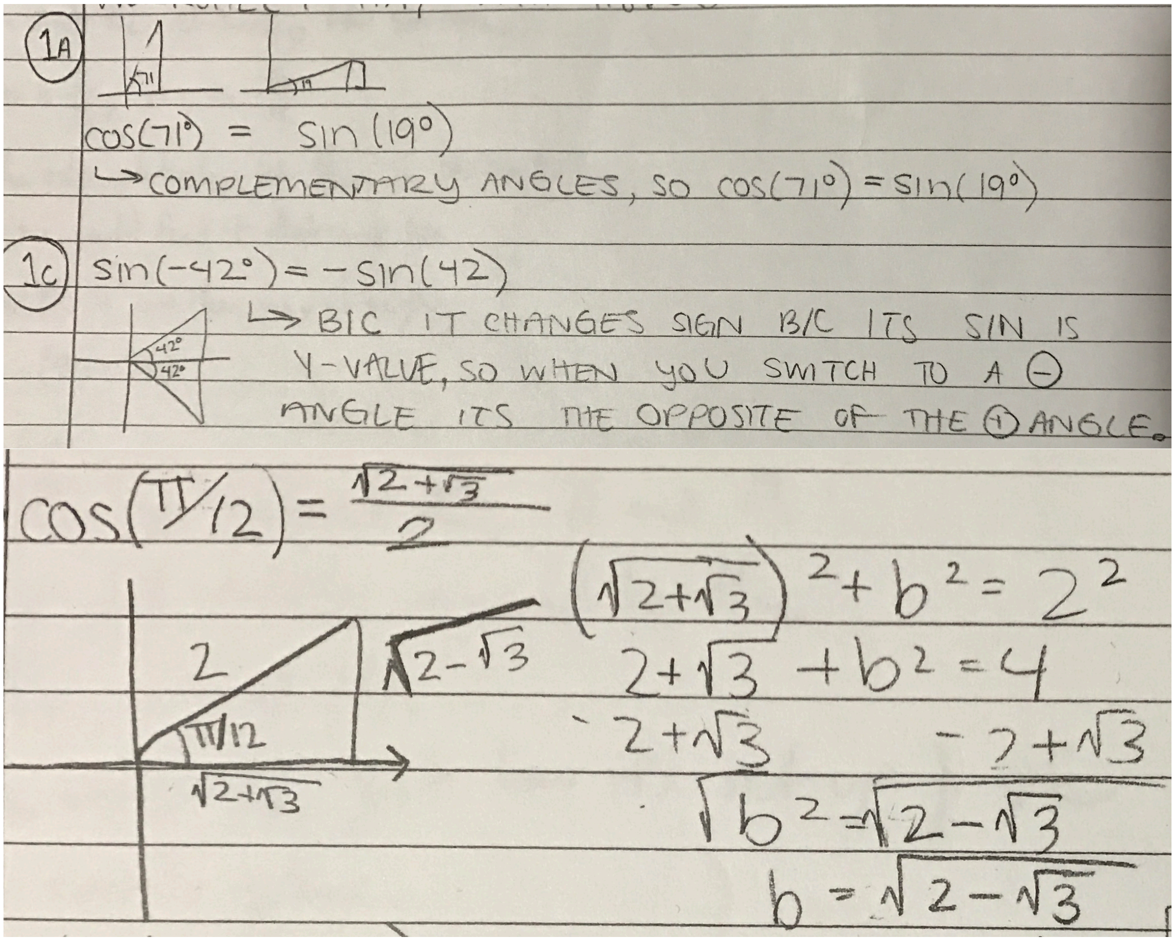
Group 4. C14, C15, and C16 produced audio and written classwork, and C15 took the pre- and post-tests. C15 improved from 3/12 to 10/12. Changes from the pre- to post-test for C15 include eliminating the mistake that $T(x + y) = T(x) + T(y)$, fixing trigonometric function definitions, and correcting the Pythagorean identity from $cos(x) + sin(x) = 1$ to $cos^2(x) + sin^2(x) = 1$.

Of the five tasks that these students were able to attempt during class, they correctly justified their answers for four. They used situated, congruent reference triangles on the unit circle to argue the cofunction and opposite angle identity tasks. For example, when attempting to find a relationship between $cos(71°)$ and $sin(19°)$, C14 noted that "if you draw a triangle, these two are complementary," to which C15 responded, "It's the same triangle. It's just how it's set up on the coordinate plane." They continued to use situated triangles to demonstrate the opposite angle identity for *sine*, drawing a congruent triangle in the fourth quadrant, as seen in Figure 17(a).

They combined this method with the Pythagorean Identity in order to solve the other two tasks. Given that $cos(\frac{\pi}{12}) = \frac{\sqrt{2+\sqrt{3}}}{2}$, these students drew a right triangle that agreed with the given information and had a hypotenuse of length two. They then found the length of the unknown leg. Having found the lengths of all sides of their triangle, they found how to situate the triangle in the coordinate plane in order to answer the questions, as seen in Figure 17(b).

Without pre- and post-tests from C14 or C16, it cannot be determined how well the lesson plan helped guide them through the critical stages. C15 appears to have made progress. The changes between pre- and post-tests cannot be directly attributed to being able to justify trigonometric identities. However, they do show an increased understanding of stage zero concepts that could have affected him as he attempted to progress through the stages. Additionally, his comment regarding the reference triangles for 71° and 19° being the same triangle reflects a well-developed understanding of the use of reference triangles in the unit circle representation. Although the results are not definitive since the tasks on the pre- and post-tests were able to be solved without explicitly using identities, these results are promising.

The results from the identities section of the confirmatory study show that some students benefited from attending lecture by learning applicable identities and strategies. In particular, creating reference triangles on the unit circle was a productive strategy for two groups who could not find how to apply identities directly through the algebraic representation.



(a)

(b)

Figure 17. Classwork from Group 4 using identities through the algebraic representation (a) and the unit circle representation (b).

Results from the Confirmatory Study (Transformations)

This portion of the confirmatory study was intended to demonstrate that the critical stages for understanding function transformations could be used as a framework to design a lesson plan for teaching transformations of trigonometric functions. Although some audio devices failed and many students did not take either the pre- or post-test, the data has been reviewed in an endeavor to draw some conclusions and offer potential leads for future research.

The group work consisted primarily of working on tasks using TrigReps. While its capabilities were not fully utilized, the data collected do indicate some positive results. Students were able to work with the various representations to confirm or correct their predictions about the effects of transformations.

Group 1. Recall that Group 1 consisted of C2, C3, C4, and C6. Of these students, all but C6 took the pre- and post-tests related to transformations. C2 improved his score from 2/10 on the pre-test to 5/10 on the post-test. These assessments revealed that after the lecture he was able to correctly answer problems related to algebraically interpreting graphical representations of vertical transformations, and he was able to determine the period of the tangent function given a graphical representation.

On the pre-test, C3 drew an incorrect cosine graph*,* then left the rest of the test blank. On the post-test, the cosine graph was the correct shape, but there were no labels on the x-axis. After this, about half the exam was completed, but none of it was correct. C3 scored 0/10 on both tests, but there was more effort given on the post-test.

C4 showed noticeable improvement between pre- and post-tests for transformations, increasing his score from 0/10 to 5/10. On the post-test, C4 showed the ability to graph the cosine function and determine the periods of various trigonometric functions when given either algebraic or graphical representations. Two significant errors that appeared in C4's responses on the pre-test were that (1) the period could be determined graphically by finding the length over which a function mapped to its entire domain, and that (2) if a function had a period of π, then it must be the tangent function. This led to the conclusion that the graph of a horizontally transformed sinusoid represented a tangent function, as seen in Figure 18.

These students' group work consisted of correct answers to the first two sets of tasks with imprecisely worded reasoning, such as "adding and subtracting at the end of the graph would cause the graph to move up and down. When the adding and subtracting is in between the parentheses, then the graph will be moved left and right." A generous interpretation would be that these students understand the material but not all of the vocabulary; a conservative interpretation would be that the students have algorithmic but not conceptual understandings. Without the audio recordings, not much more can be said. More data than that would be required to make any significant inferences.

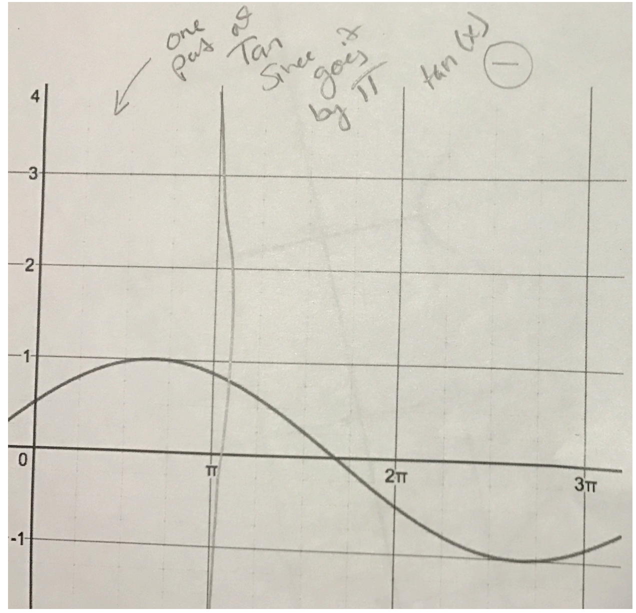


Figure 18. C4's Transformations pre-test.

The pre- and post-assessments imply that the teaching episodes had a significant effect on the students of Group 1. C2 showed that he had made correct classifications of graphical transformations. Without more work on the post-test or audio recordings from the recitation, it cannot be determined if C2 is applying algorithmic knowledge, or if he has justified his understandings of the classifications that he made. C4 improved his stage zero knowledge to a level where he could productively approach some of the material. In this sense, the lesson plan was helpful, but did not achieve its goals of guiding these students through all of the critical stages.

Group 2. Recall that Group 2 provided audio recordings and group work from the recitation period. None of the students in this group took pre- or post-assessments for transformations. C13 stated that he had not attended the identities lecture. While he did not make a similar admission for the transformations lesson, after analyzing the audio recordings it was determined that he still did not contribute anything productive to the group work.

C11, C12, and C13 were able to use TrigReps to successfully complete each of the tasks that they attempted from the classwork. The audio recordings reveal that these students were confused about some transformations, but were able to use the program to help them understand, as evidenced by the following exchanges in which students were attempting to find functions that had double the amplitude of $sin(x)$, and triple the frequency of $sin(x)$, respectively:

C11: Twice the amplitude would be $2sin(x)$.
C12: That or $sin(2x)$.
C11: I'd say that has twice the amplitude, right?
C12: $sin(2x)$ doubles the frequency. $2sin(x)$ would double the amplitude.

C11: Triple frequency, that's $sin(3x)$.
C12: Three or $\frac{1}{3}$?
C11: You may be correct, sir…. No, definitely three.

In these instances, the students were able to use the computer program to perform the tedious, repetitive action of plotting points in order to check their hypotheses about the effects of transformations. Using a computer also avoided the potential difficulty related to the Nyquist frequency – fitting a lower frequency sinusoid through the plotted points.

Although C11 and C12 briefly discussed the changes that they noticed in each representation during the exercises, it cannot be determined whether they were making meaningful connections between the effects of the transformations on the different representations. When examining horizontal shifts, they remarked:

C12: Clearly the equations are changing and the graphs are moving.

> **C11:** Algebraically we're changing it in- and outside of the parentheses. Graphically we see it shifting. On the unit circle, we see θ changing.
> **C12:** And we still haven't heard anything [referring to the aural representation].
> **C11:** True story.

C11 and C12 correctly note how the algebraic and graphical representations are affected, but it cannot be determined whether they note that the starting position of the radius – what C11 refers to as "θ changing" – is directly related to the graphical ordered pair at $x = 0$. While it remains to be seen if students are actively making connections among their concepts of the various representations, the students' answers indicate that TrigReps can adequately demonstrate trigonometric functions in different representations.

Since none of the members of Group 2 completed pre- or post-tests, and their audio recordings do not reveal whether the students were actively attempting to create connections among concepts or whether they were passively observing the effects of various transformations, few determinations can be made about how the students have progressed through the critical stages. From the exchange regarding doubling the amplitude, it appears that C11 and C12 had classified multiplication as stretching, but were not confident in their classifications of horizontal and vertical transformations. Since the students did not give any indication that they were relating the changes in each representation to one other as opposed to simply listing them, it cannot be determined whether these students could justify their reasoning.

Group 3. There were not many inferences to make using the data provided by Group 3. C1, C5, and C7 submitted written work, but did not take pre- or post-tests, and their audio recorder failed. This group submitted correct work after the recitation on transformations, and for these activities there is some evidence of their thought processes. For instance, they noted that in

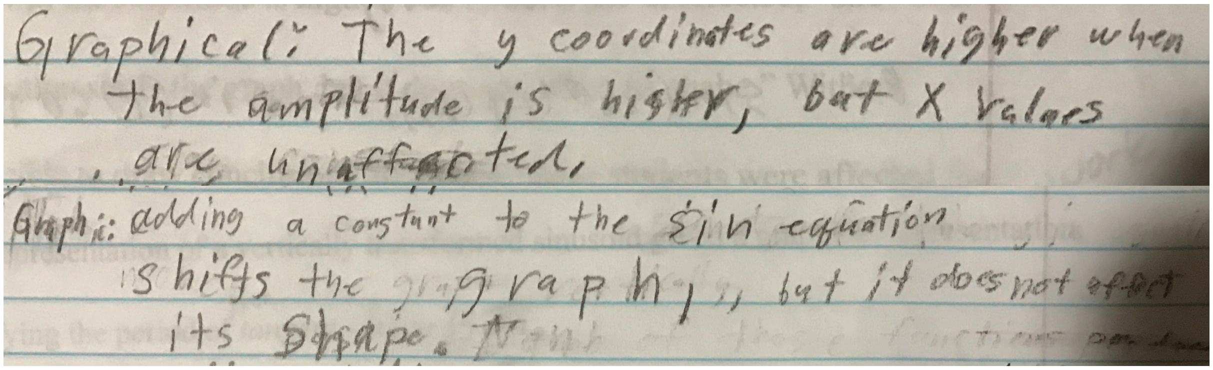


Figure 19. Written work from Group 3.

the graphical representation "the *y*-coordinates are higher when the amplitude is higher, but *x*-values are unaffected," and "adding a constant to the sin[e] equation shifts the graph, but it does not affect its shape," as seen in Figure 19. The work that this group performed to arrive at their correct answers and these conclusions is not present in their written work. Without additional data, it is impossible to draw conclusions about how these students were affected by the lesson plan. Without audio recordings, pre-tests, or post-tests, this group has not contributed meaningful data to determine the efficacy of the lesson plan.

Group 4. Recall that Group 4 provided audio recordings and C15 took the pre- and post-tests. C15 increased his score on the transformations tests from 4/10 to 6/10. The questions that he improved upon were related to finding the algebraic representation of a vertically transformed sinusoid given a graphical representation, and identifying the period of the tangent function.

During the recitation period examining transformations of trigonometric functions, Group 4 completed the first four sets of tasks with all correct answers. Furthermore, the audio recording reveals that these students made correct predictions regarding function transformations. While finding a function with twice the amplitude of $f(x) = sin(x)$, the following remarks were made:

C14: For twice the amplitude, do we just do two sine [meaning $2sin(x)$]?

C16: That made the amplitude greater. The peaks are taller.

While looking for a function that shifted $f(x) = sin(x)$ down by π, C15 said to "put $(-\pi)$ in the d slot," meaning the fourth input for $\_sin(\_x + \_) + \_$ in the computer program.

Even so, this group failed to predict some counterintuitive aspects of horizontal transformations. When they were attempting to find a way to shift the function $sin(x)$ to the left by $\frac{\pi}{2}$ units, the following exchange occurred:

> **C15:** To the left by $\frac{\pi}{2}$. Did that go to the right?
> **C16:** So it'd be $sin(x + \frac{\pi}{2})$.

They also noticed some connections among representations. In the first example below, a connection was made between the unit circle representation and the graphical representation during a horizontal shift, as seen in Figure 20. In the second, C15 correctly predicted the relationship between the algebraic, aural, and potentially graphical representations. It is unclear whether the first use of the word frequency was in relation to the graph or aural representations, but the remainder of that quote can be inferred to be referring to the aural representation.

> **C15:** To the right by seven. The radius went to a different spot.
>
> **C15:** This will be a lower frequency. Oh, that was so low! It's a little hum, like a little submarine.

When they were unable to predict the behavior of the different representations, this group was able to use TrigReps to explore the effects of the various representations. C16 asked the group, "What's the difference between inside and outside the parentheses?" Their submitted work demonstrated that they had discovered the difference at least among shift transformations:

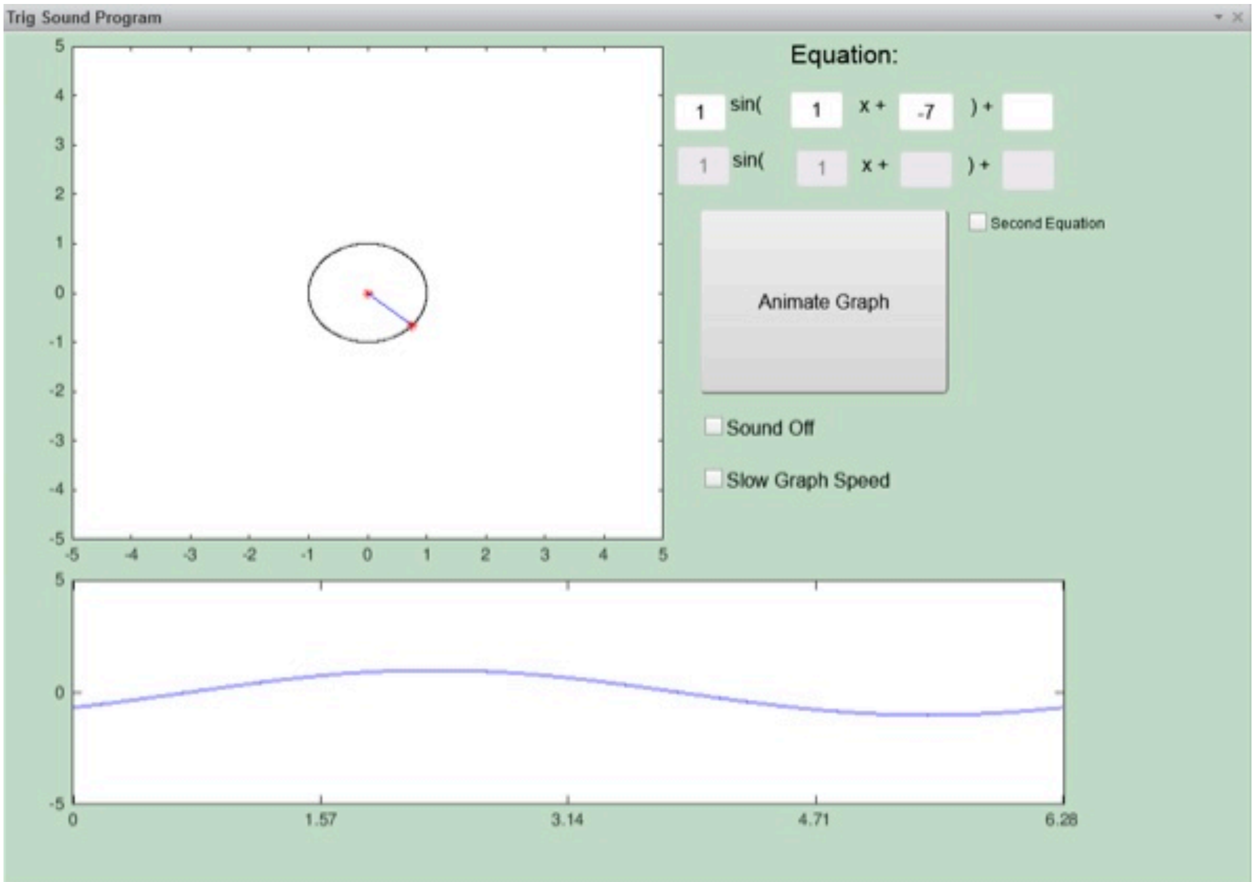


Figure 20. Group 4's work with TrigReps

"Changing values within the sine parentheses shifts the graph left or right. Changing values outside the parentheses moves the graph and unit circle up or down."

Group 4's written work also indicated that they noticed the corresponding changes among representations. They wrote that "increasing the frequency increases the pitch, the number of cycles per second on the graph, [and] the speed of the radius on the unit circle." It is not clear that these students have developed conceptual understandings of the trigonometric concepts, but the collective work does indicate that they were seeking to understand the relationships at more than an algorithmic level. The MATLAB program TrigReps provided avenues for students to explore these relationships. Had the students had more time with the program, they would have had the ability to examine the order of transformations using the program. In particular, the program is hypothesized to be helpful for students seeking to explain why the order of horizontal transformations is applied counterintuitively. This is especially useful because the main study demonstrated that students had difficulty justifying why the order is counterintuitive, and some students' remarks indicate that they may have found this to be disconcerting.

Discussion of Research Question Three (Identities)

The confirmatory study was intended to examine the effectiveness of lesson plans designed to help guide students through the critical stages of understanding. In this section, the results of the confirmatory evidence will be interpreted with respect to trigonometric identities. Particular emphasis will be placed on identifying evidence that students have progressed through critical stages. Students for whom there was no meaningful data have been omitted.

C2. On the pre-test, C2 displayed the ability to correctly evaluate trigonometric functions. The only correct application of trigonometric identities was the use of the fact that $sin(x + 2\pi) = sin(x)$, as seen in Figure 21. He attempted to use other identities, but each of his uses of other identities was flawed in some way. On the post-test, C2 attempted to use several identities, as was shown in Figure 14, but there were mistakes in each case. He wrote that $sin(-15^{\circ}) = sin(15^{\circ})$, that $tan(165^{\circ} - \pi = tan(-15^{\circ})$, and that $cos(-15^{\circ}) = sin(15^{\circ})$. The latter two mistakes are difficult to interpret, but it is clear that he was not applying conventional, correct identities. There is no evidence that he had done more than notice a change to the algebraic representations. Figure 14 shows that he noticed a change to algebraic representation $cos(-15^{\circ})$, however it is not clear that he has done any productive reasoning to justify what the effects of this change might be. Beyond noticing the change to the algebraic representation, it is

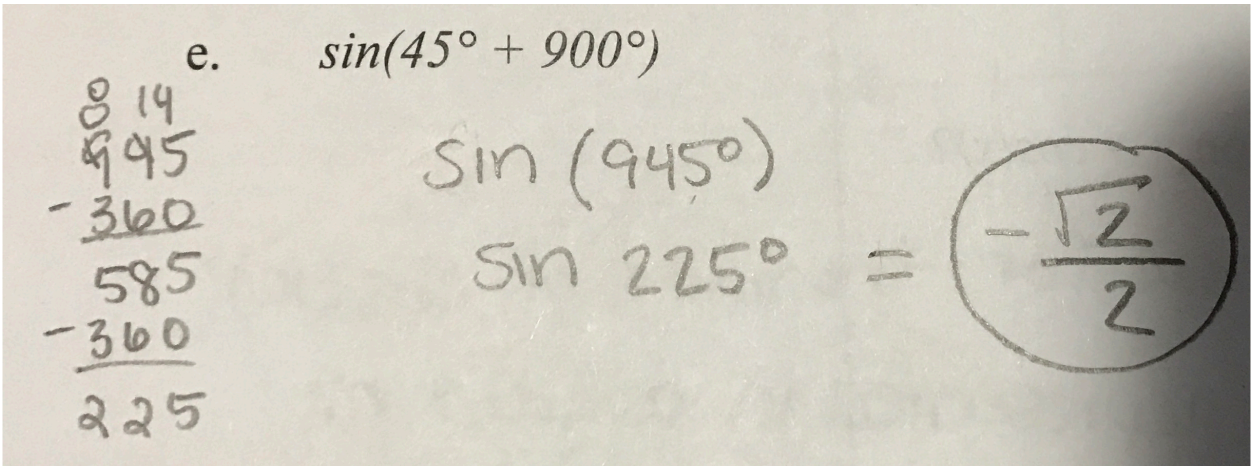


Figure 21. C2's use of the *sin(x + 2π)* identity.

not able to be determined whether this student has used any other representations or attempted to develop a pattern in the algebraic representation that would lead to a justified identity.

C3. As mentioned previously, C3 scored 0/12 on the pre-test for identities, and improved to a 4/12 on the post-test. There is evidence that he achieved some critical stages of understanding necessary for several learning goals, and there is evidence that he understands the $T(x + 2\pi) = T(x)$ identities. Since the pre-test was left blank, it is reasonable to conclude that he had not made meaningful progression through the critical stages at that time. The pre-test does not provide any evidence that he had achieved the critical stage of considering changes to the algebraic representation in other representations, and it is unclear whether he noticed that the algebraic representations had been changed. On the post-test, there is evidence of knowledge of trigonometric identities, as seen in Figure 22. In part b, it is possible that there was a clerical error – dropping the negative sign from $sin(45°)$ – and the problem was completed correctly. However, a conservative interpretation would suggest that C3 does not understand that $sin(\theta + \pi) = -sin(\theta)$. In part c, although there is no answer given, the student suggests that he understands that $tan(x + 2\pi) = tan(x)$. This is corroborated by stating elsewhere on the post-test that $cos(405°) = cos(45°)$ and $sin(45° + 900°) = sin(225°)$. However, he did not demonstrate that he understands that $tan(x + \pi) = tan(x)$ as well.

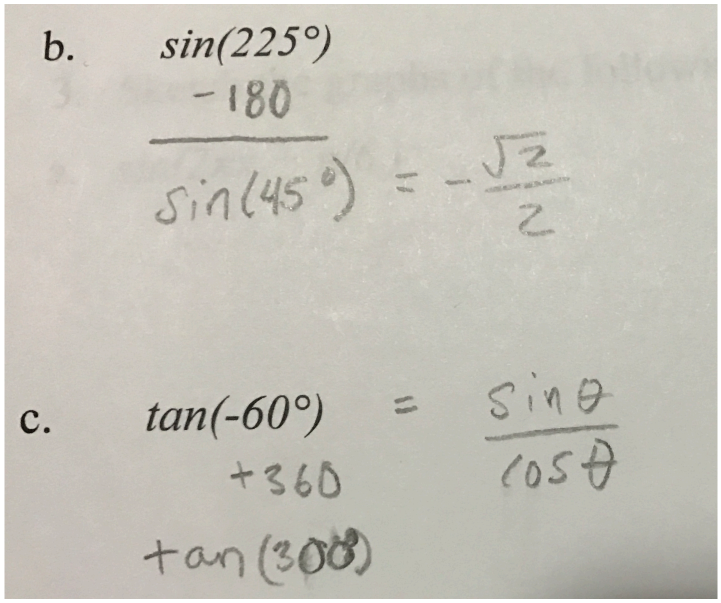


Figure 22. C3's post-test use of identities.

C11. C11 took the pre-test, but not the post-test for identities. His pre-test indicates that he progressed through the critical stages for the $(\theta + \pi)$ identities, as seen in Figure 21. In part *d*, it appears that he has noticed that $cos(405°)$ is a change to the algebraic representation of $cos(45°)$, and in part *e*, it appears that he noticed that the algebraic representation of $sin(45°)$ has been altered by a 900° transformation. He noticed a change to the algebraic representation and moved to the unit circle representation in order to relate the desired value to known values. He did not state the use of quadrants to determine the signs of his answers, so it cannot be determined whether he used this strategy, but his work is consistent with having used it, as seen in Figure 23. Based on his success, it is reasonable to hypothesize that he made a vocabulary error when he used the word "cotangent" instead of "coterminal."

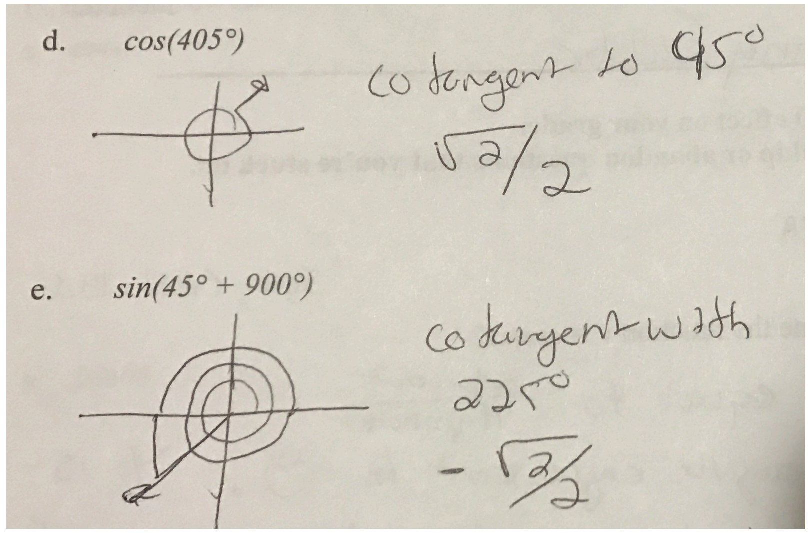


Figure 23. C11's use of the *(θ + 2π)* identities in his pre-test.

During the group work, C11 cited the cofunction identity and correctly applied it to the relevant task. He also collaborated with his classmates to use the unit circle and congruent reference triangles to apply opposite angle and $(\theta + \pi)$ identities. This work suggests that he has a conceptual understanding of these identities and that he could potentially derive the identities that he hasn't demonstrated.

C12. C12's pre-test contains one sign mistake, where he has labeled a Quadrant 3 angle with positive x- and y-values, leading him to state that $sin(225°) = \frac{\sqrt{2}}{2}$. His other work indicates that this is an error unlikely to remain upon reflection. Figure 24 shows that C12 could use the unit circle to justify $(\theta + 2\pi)$ identities, even though there is an arithmetic error. These are the only instances in the pre-test in which he used identities.

While C11 contributed the most to Group 2's conversation, C12 demonstrated significant trigonometric knowledge and helped his group to work productively. He cited the opposite angle identity $sin(-\theta) = -sin(\theta)$ and was aware of the signs of the trigonometric functions based on the quadrant of the angle. C12 did not use any opposite angle identity during the pre-test. It is possible that he knew this identity during the pre-test but did not apply it. A post-test may have given insight into this, since opposite angle identities can simplify some of the work and thus

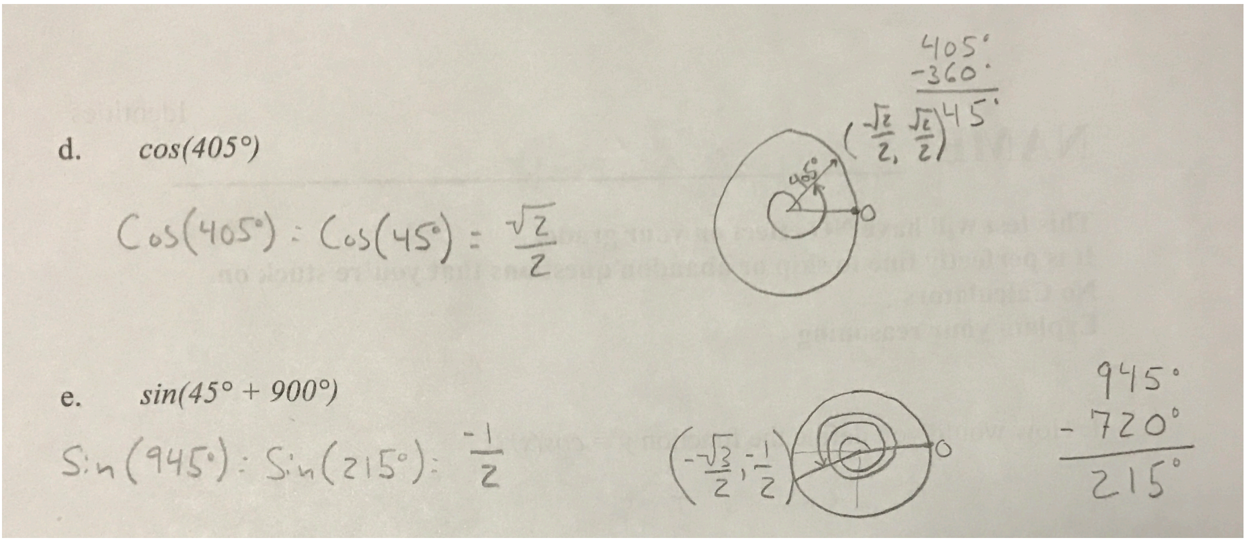


Figure 24. C12's identities pre-test.

have the potential to be used if known.

Since C11 led the conversation during the recitation period and C12 did not complete a post-test, it is difficult to determine what effect the lesson plan had on him. His work and speech show that he was comfortable moving between representations and noticing correspondences between reference triangles. This suggests that he finished the lesson plan having at least achieved the critical stages through applying the CAST diagram. However, it cannot be definitively determined if the lesson plan impacted achievement of any learning goals.

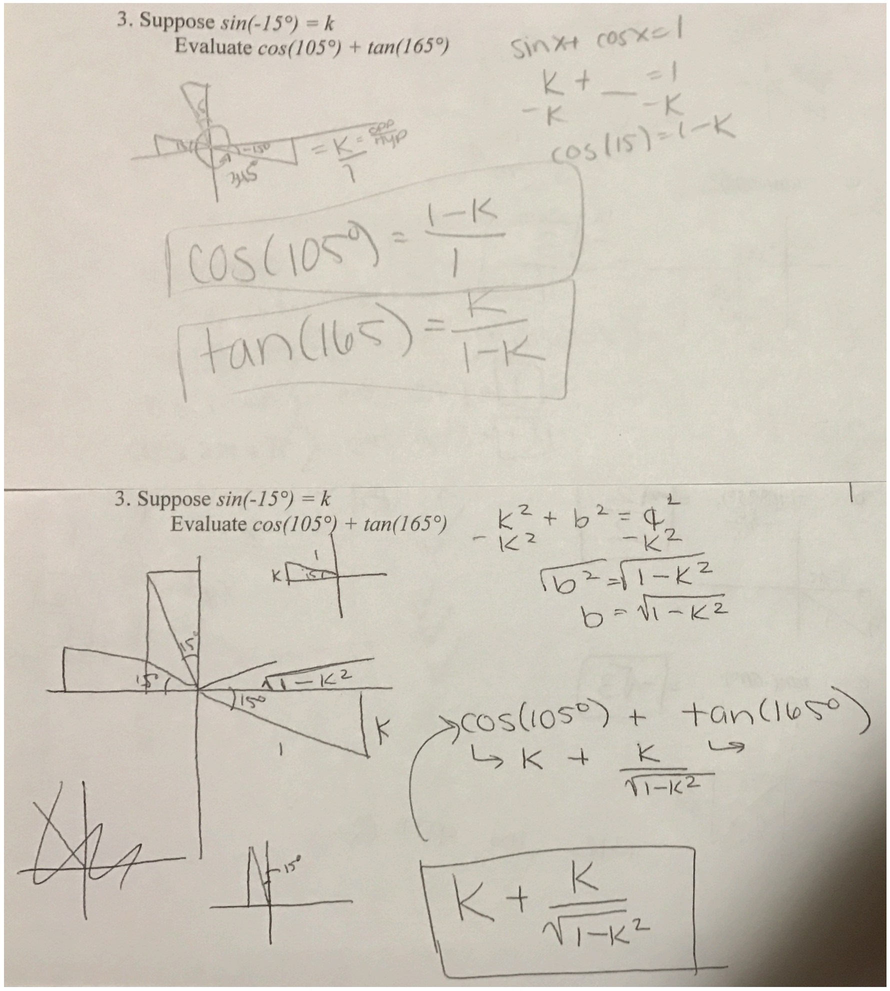
C15. C15 showed significant improvement between pre- and post-test scores. Figure 25 shows that, during the pre-test, he noticed a change in the algebraic representation of the sine function and changed to the unit circle representation to utilize its better affordances. He also changed angles, but was not able to notice correct, productive correspondences among those angles. During the post-test, in addition to correcting the Pythagorean identity, C15 corrected which legs of the reference triangles have length *k*. Although he cites no identities, it appears that he has achieved all of the requisite critical stages.

Overall, the results from the identities section of the confirmatory study show that students benefited from having the ability to utilize different representations. However, the plethora of strategies through which students can approach trigonometric problems had a detrimental impact on the usefulness of the pre- and post-tests: many students gave correct, justified answers that did not require the use of identities. An additional difficulty was that the scarcity of audio recordings during the recitation periods made it difficult to draw conclusions. Even so, Figures 16, 17(b), and 25 show arguments similar to that made by the researcher during the lecture regarding congruent triangles on the unit circle. However, without more data, it cannot be determined what effect the lesson plan in particular had on these students. Potential improvements to the lessons and assessments will be discussed in the future research section.

Discussion of Research Question Three (Transformations)

In this section, the results of the confirmatory study will be interpreted with regards to students progressing through the critical stages of understanding for function transformations. These students used the previously described TrigReps program to examine the effects of transformations on graphical, unit circle, and aural representations. Students for whom there was no meaningful data have been omitted.

C2. C2 applied transformations inconsistently during both the pre- and post-tests. For example, he sketched an accurate graph of $3cos(x) - 4$ on the post-test, which he was unable to do during the pre-test. However, given a graph of $sin(\frac{1}{2}x - \frac{\pi}{6})$, he stated that the algebraic



(a)

(b)

Figure 25. C15's pre-test (a) and post-test (b).

representation would be $sin(\frac{1}{2}x) + \frac{\pi}{4}$. If the audio recording device assigned to this group had worked correctly, it could have provided some insight as to whether this was simply a misplaced parentheses or whether he does not understand the relationships between the algebraic and graphical representations of horizontal and vertical transformations. When discussing similar situations with the group, it could be seen if C2 advocated that addition to the function output would result in horizontal transformations. If not, it would lend credence to the idea that he merely misplaced a parentheses. C2's reasoning on the post-test would also be helpful in determining how he conceived of horizontal transformations. However, the post-test contained no reasoning or justification for C2's observations regarding transformations. In conclusion, all that can be determined is that C2 had marginally improved his ability to perform tasks related to transformations of trigonometric functions.

C3. As with identities, C3's pre-test was almost entirely blank, and his post-test was barely an improvement. While the answers are not correct, C3's post-test does have appropriately transformed functions. For example, C3 identified a horizontally stretched and shifted graph as $cos(2x)$, which is a horizontal transformation. However, there is no justification or reasoning for any of the answers, so, without the audio recording, it cannot be determined whether these connections were memorized, meaningfully understood, or guessed.

C4. C4 showed an improved understanding of transformations on the post-test. Even for incomplete answers, the work on the post-test is more productive than the pre-test, as seen in Figure 26. However, it cannot be determined how C4 arrived at the graph from his pre-test. Of note is that the *x*-values in Figure 26(a) descend in magnitude the farther away they are from the origin. It could be that he believes that multiplying by 2π flips the graph vertically in some way while adding $\frac{\pi}{6}$ shifts it to the right by $\frac{\pi}{6}$. Although, because of the labels on the *x*-axis, it cannot be determined how C4 arrived at his solution. His answer on the post-test implies that he correlates adding $\frac{\pi}{6}$ to the input as a shift to the left by that amount. However, because the x-axis is unlabeled, it cannot be determined if he understands how the multiplication or order of transformations affect the graph. He did provide a number of correct answers, but without any explicit reasoning from the audio recordings or the pre- or post-tests, it cannot be determined

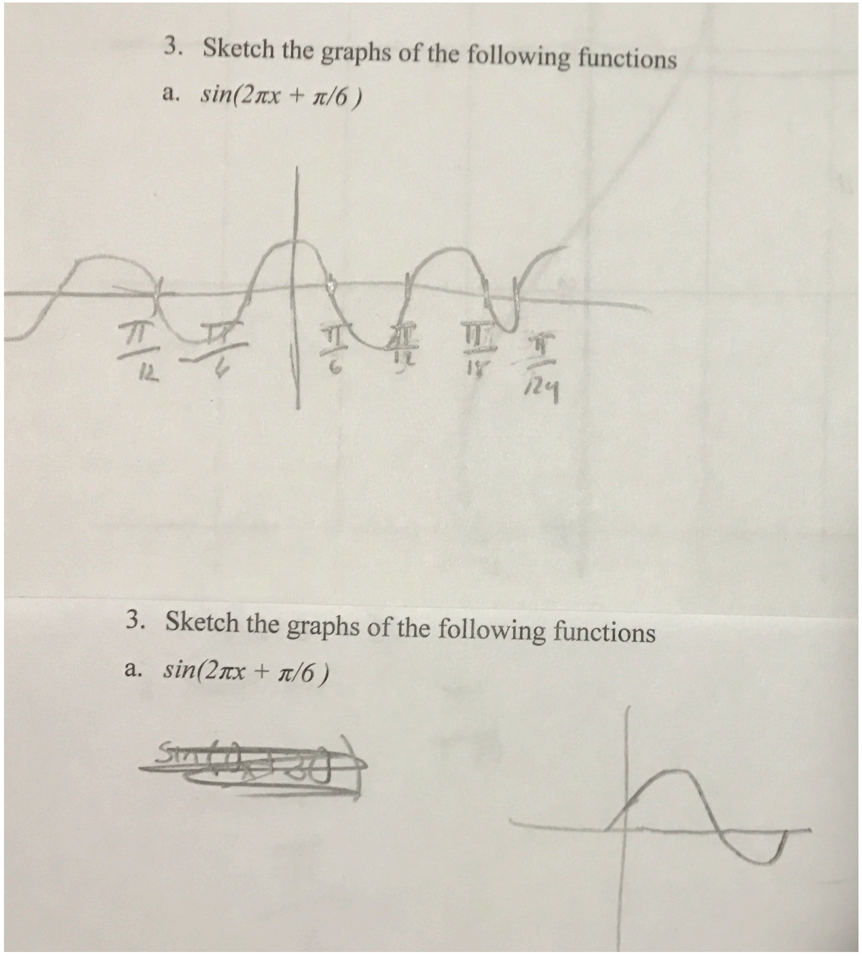


Figure 26. C4's pre-test (a) and post-test (b) for transformations.

whether he had an algorithmic or conceptual understanding of any concepts, or whether he understood any of the concepts under investigation at all.

C11 and C12. Without pre- or post-tests, it cannot be determined how the lesson plan affected how these students progressed through the critical stages. The audio recordings show that they were uncertain about how transformations affect the various representations but that they were able to use TrigReps to arrive at the correct answer, such as in the previously cited exchange:

C11: Triple frequency, that's $sin(3x)$.

C12: Three or $\frac{1}{3}$?

C11: You may be correct, sir…. No, definitely three.

TrigReps allowed these students to quickly check their hypotheses about the effects of each transformation. However, since they did not take pre- or post-tests, it cannot be determined whether they came into the teaching episode at that level of understanding, or whether they understood these observations following the teaching episode.

C15. There are slight differences between C15's pre- and post-tests for transformations. On the pre-test, C15 identified the graphs of two sinusoids as "cosine graph shifted up by two because the zeros are on $y = 2$. Something like $y = cos(x) + 2$," and "sine graph. Shifted to the right, so something like $y = sin(x - \#)$." On the post-test, he correctly identified the former as $y = - cos(x) + 2$. The other post-test response, though incorrect, shows a greater understanding than was displayed on the pre-test. That graph is identified as $y = sin(2x + \frac{3\pi}{4})$. This indicates that C15 had identified the change in frequency for this graph and the fact that this corresponds with multiplication of the input. However, he did not indicate that he understands that multiplication of the input by numbers larger than one results in horizontal shrinks rather than stretches, nor did he indicate that he understands that the horizontal shift will be applied before the horizontal stretch.

There is evidence from the audio recordings and collected work that C15 finished the lesson with some understandings of these concepts. The group worked productively on the assigned tasks, and C15 contributed significantly to the group's discussion. One of the transformation tasks explicitly referenced that a transformation would affect the volume of the sound produced. While attempting to produce a louder sound, C15 said "What if we do amplitude as well? I heard that a lot clearer…. I guess that's why they call it an amp [referring to an amplifier, as for a musical instrument]." This indicates that he was making connections between the algebraic and aural representations of vertical stretches, and also that TrigReps helped him make connections between the mathematical material and his personal experiences. For a horizontal stretch, he noted "we decided to do $3x$, and we can see three humps" for x-values between zero and 2π. However, on the post-test, he associated a horizontal stretch by a factor of two with multiplication by two, indicating that he did not fully understand the counterintuitive aspects of these stretches.

In summary, C15's pre-test provides evidence that he had achieved the learning goals related to classifying horizontal and vertical transformations, as well as shifts. The audio recordings and post-test indicate that he had made connections among the representations well enough to justify the classification of stretches, but not well enough to correctly identify counterintuitive aspects of horizontal transformations.

There was evidence that the lesson plan helped guide students through some of the critical stages of understanding function transformations. However, the lack of students who took both the pre- and post-tests severely limits the strength of this evidence. Audio recordings reveal some classifications that students have made. Without pre- and post-tests however, it cannot be determined whether the lesson plan helped guide students towards those classifications. Furthermore, the students were unable to complete all of the tasks assigned to them, which means that the audio recordings and class work do not provide information on whether the students understood the effects of the order of transformations.

Conclusion. There is some evidence that the lesson plans helped guide students through the critical stages for understanding trigonometric identities and transformations. Based on pre- and post-test results for C2, C3, C4, and C15, there is evidence of modest improvements by these students. They improved their average scores on the identities tests by 3.75 points out of 12, and on the transformations tests by 2.5 points out of 10. There is evidence that, between pre- and post-tests, C2 achieved the learning goals related to classifying horizontal and vertical transformations, as well as shifts and stretches; C3 achieved learning goals related to the $(\theta + n\pi)$ identities; and C15 achieved the learning goal of classifying stretches and the critical stages necessary to achieve the identities learning goals. The strategies employed by C15 do not demonstrate a knowledge of identities, however there are also no misuses of identities. It is not clear whether C15 was able to infer the trigonometric identities from his work since he never stated them explicitly.

None of the students were able to finish the assigned tasks during the recitation periods. It is possible that, with different tasks, the students would have had sufficient time to demonstrate their knowledge of each concept. However, it may be the case that there were too many learning goals to be assessed in a total of 100 minutes. Regardless, the lesson plan must be modified to allow students the opportunities to examine more subject matter in the time allotted.

The tasks for the transformations activity should also be modified in order to prompt students to reflect more upon correspondences among representations. Rosen and colleagues (2008) did not ask students to reflect upon these correspondences and were concerned that students were not appreciating how changes to one representation would affect other representations. The tasks for this confirmatory study explicitly asked students to notice correspondences among the representations, however, it is not clear that students reflected upon how the representations are related to each other. That is, while students may have noticed that multiplication of the input related to the frequency in the graphical representation and the speed of rotation for the radius of the unit circle representation, they did not show that they understood how the speed of rotation affects the frequency with which the endpoint of the radius passes through each value of the function range. This task should be revised to ask students how the changes in representations affect each other instead of asking what the correspondences are.

Because of the technological difficulties and the poor participation rate, the confirmatory study is inconclusive. Some of the students' work indicates that, after the lecture, they were able to use the congruent triangles in the unit circle representation to work productively towards solving tasks, but the lack of pre- and post-test data makes it impossible to quantify the effectiveness of the lesson plan. Similarly, the audio recordings and written work indicate that TrigReps is a useful tool to examine the effects of transformations, but it is not clear how effective the program and tasks were at helping guide students through the critical stages of understanding for transformations.

V. Conclusion, Study Limitations, and Implications for Future Research

The main study was intended to examine the critical stages of understanding through which students need to pass as they come to understand trigonometric identities and transformations. The confirmatory study was intended to demonstrate that these critical stages could be used as a framework to design a lesson plan. The studies began with hypothesized critical stages and lesson plans based on these stages that were influenced by a review of the literature and the researcher's personal experience as a precalculus instructor. Through approximately forty hours of task-based interviews with precalculus students, data was collected to inform and revise these critical stages. The revisions to the critical stages necessitated revisions to the lesson plans, which were then enacted. Data was collected before, during, and after these teaching episodes. However, not enough data was collected to draw conclusions about the effectiveness of these particular lesson plans. The sample sizes of students who took the pre- and post-tests, who submitted group work, or who submitted audio recordings of their group work were too small to draw any conclusions about how effective this lesson plan was in guiding the class through critical stages. In this chapter, general observations will be made regarding: (1)

the revised critical stages of understanding, (2) students' misconceptions, (3), students' use of representations, and (4) similarities and differences to previous studies on students' understanding of trigonometry, identities, and transformations. The chapter will conclude with study limitations and implications for future research.

Critical Stages

The most significant contributions that this study has made to the field of mathematics education are the sets of critical stages of understanding for each learning goal. The critical stages provide a framework for creating lesson plans by explicitly listing the thoughts and actions common to successful students. They separate each learning goal into smaller, more easily managed concepts. These stages can be used to examine how a lesson plan may help guide a student towards a justified understanding of the concepts, to identify an obstacle to a student's understanding, or to sequence topics in a curriculum. The critical stages also contain significant errors and misconceptions that students may face as they examine each topic. It is important for instructors to be cognizant of common errors as they design and implement lesson plans.

Additionally, this study supports and extends previous literature that has noted the importance of understanding different trigonometric representations. Weber (2005) and Challenger (2009) have noted that students must have the ability to fluidly change between representations in order to develop well-justified understandings of trigonometric concepts. The revised critical stages note in particular what representations students must have familiarity with and how they must be utilized. While students who confined themselves to the algebraic representation were able to notice patterns and convince themselves of the content, only the students who used multiple representations developed rigorous justifications.

The structure of the critical stages was revised during the study to make it easier to understand the specific topics under investigation. Most significantly, rather than considering every desired piece of knowledge as a critical stage, each of the identities – opposite angle, $(\theta + n\pi)$, and cofunction – and each transformation classification – addition/shift, multiplication/stretch, input/horizontal, output/vertical, order of transformations, and the counterintuitive aspects of horizontal transformations – were considered to be learning goals, and the critical stages of understanding were defined as the thoughts and actions found to be necessary to achieve those learning goals. That is, instead of viewing it is an additional identities critical stage for students to notice that using the opposite angle has predictable effects, only the thoughts and actions that led students to notice this fact were considered to be the critical stages. Noticing that using opposite angles has predictable effects was termed a learning goal instead of a critical stage. Originally, all of the critical stages and learning goals were combined in one list in an attempt to emphasize how the concepts of identities and transformations are related to each other and how the learning processes for each of these concepts would be similar. This list has been separated into the two lists of hypothesized critical stages in chapter two. It is believed that separating the learning goals makes each one clearer and that the repetition of critical stages will emphasize how closely related the learning goals are. The following paragraphs will note specific modifications to the critical stages, and the justifications for these modifications.

In the hypothesized critical stages, it was proposed that, after moving from the algebraic representation to a representation with better affordances such as the unit circle or graphical, students would recognize that changes to the algebraic representation correspond to changes in the other representation(s). After the main study, this was revised to more explicitly describe how students would use other representations to notice these correspondences. Students who used the unit circle representation used reference angles to draw triangles congruent to ones for which they had been given information. The students then used the CAST diagram to determine the signs of the trigonometric functions applied to these triangles. Students who used algebraic or graphical representations evaluated the functions at regular intervals in order to establish a pattern relating the original trigonometric function to the one under examination. For example, after several evaluations, these students noticed that $sin(x + \pi)$ produced the opposite outputs of $sin(x)$.

The hypothesized critical stage that students must notice that transformations affect the entire graph was revised to be an aspect of understanding the individual transformations. For example, during the stage two interviews, several students stretched their graphs from lines other than the axes. Rather than saying that these students had not achieved a distinct critical stage of understanding, it was determined that these students had not fully understood the relationship between multiplication in the algebraic representation and stretching in the graphical representation. They had not understood that, since the multiplication was applied to all real numbers x, it affected the entire graph.

Similarly, the critical stage necessitating that students notice the effects of transformations on period and phase were revised to be an aspect of understanding how multiplication and addition in the algebraic representation are related to stretching and shifting transformations in the graphical representation. Making observations about the period and phase were considered to be applications of these understandings rather than necessary stages of understanding.

The resulting lists of critical stages for each learning goal shared a great deal of overlap. For example, in each case, students had to notice a change to the given algebraic representation and ultimately notice a correspondence between the original algebraic representation and the changed, or transformed, algebraic representation. This demonstrates the similarity between the processes for learning each of the identities and how the identities are related to transformations. Understanding each of the identities involves noticing correspondences between a trigonometric function and a specific transformed trigonometric function. The processes of finding correspondences are generally similar, and these correspondences are particular instances of transformations of trigonometric functions, so there are numerous similarities between the processes of coming to understand trigonometric identities and transformations.

Several critical stages have also been modified to make reference to notable student misconceptions. For example, it was not predicted that students would stretch their graphs from lines other than the *x*- and *y*-axes. However, this misconception was prominent enough that it should be explicitly noted, since it seems likely to be helpful in designing a lesson plan. Since the purpose of the critical stages is to provide a framework for creating a lesson plan, it would be helpful to note common errors and misconceptions that may occur as students attempt to achieve each critical stage. Other notable misconceptions included believing that ordered pairs on the graphs of trigonometric functions would satisfy the unit circle equation; believing that a single sinusoid – the Nyquist frequency – corresponded with a given set of ordered pairs; and believing that everything related to horizontal transformations was counterintuitive, including the words "stretch" and "shrink."

The critical stages for trigonometric identities and transformations described in this study share similarities with previous studies on students' understanding of trigonometry. It was hypothesized that students would need to change representations in order to understand the effects of transformations. Weber (2005) and Challenger (2009) each emphasized that the students who could move fluidly between trigonometric representations tended to be successful in their trigonometry classes. Previous studies had found that using inappropriate representations can inhibit a student from coming to understand the topic under investigation (Schnotz & Bannert, 2003), which agrees with data collected during this study. This study found that the algebraic representation was not generally helpful for students. When students used representations that were aligned with the goals of their investigations – such as using the unit circle to justify symmetries of reference triangles – they tended to be successful. On the other hand, using inappropriate representations – such as using right triangles to investigate non-acute angles – did not lead to justified responses from the students. This is reflected by the prevalence of critical stages advocating for a particular representation.

Although it was not a focus of this study, the data collected during this study supports previous research that found that students have difficulty with radian measure (Akkoç, 2008; Moore, 2013; Tuna, 2013). For example, some students were confused about how many radians are in a circle. Some students believed that any term containing a multiple of π necessarily denoted an angle. Because of these difficulties, students were encouraged during interviews to use whichever units they were comfortable with. Since students' conception of angle measure was beyond the scope of this study, it was not noted what effects this had upon students' justifications of identities or transformations.

Previous studies on students' understanding of identity have found that students had difficulty understanding and applying identities using only the algebraic representation (Fi, 2003; Tsai & Chang, 2009). It was hypothesized during this study that students would need to supplement their algebraic representations with other representations that have better affordances. While some students were able to achieve several of the learning goals using only the algebraic representations, the majority of students who successfully justified identities or transformations utilized the affordances of the unit circle or graphical representations to justify their understandings.

In conclusion, the hypothesized critical stages were largely supported by the collected data. The critical stages for learning each of the trigonometric identities and transformations are similar, which reflects the similarities between the concepts. A major revision to the critical stages is that the methods by which students notice a correspondence between the algebraic and unit circle representations has been elaborated upon. The revised critical stages also reinforce the idea that students must have the ability to move between representations as they learn trigonometry. Finally, some critical stages have been revised to include significant misconceptions or errors encountered during the interviews.

Study Limitations

Although this study has collected data supporting critical stages of understanding for trigonometric identities and transformations, there are some factors that limit the generalizability of these stages. This study was limited by its sample size. More interview participants could have led to more refined critical stages. Additional student perspectives could have offered more details regarding how students came to understand each concept, or alternative paths to understanding. Some critical stages could be found to be superfluous if additional students achieved learning goals without passing through all of the revised critical stages. During this study, students used the unit circle representation more often than any other to justify their understandings of trigonometric identities. Additional information regarding how students use algebraic, graphical, or other representations to justify trigonometric identities would be helpful for supporting the conclusions drawn during this study.

As well as being small, the population in these studies were not diverse. Stage two of the main study only had productive data collected from one female student. A second female student participated, but she struggled with the stage zero material to the point that her interview data did not contribute to the development of the critical stages of understanding. Demographic information was not collected for stage one of the main study or the confirmatory study. Without a diverse data set, it is more difficult to make an argument that the results of this study should generalize to other classrooms.

During the interviews and group activities, students may have been reluctant to share all of their thoughts despite repeated prompts. Students may have refrained from giving answers that they believed were obvious. For example, during the interview, students who were coded as not having understood the differences between graphical shifts and stretches may have recognized that multiplication of real numbers behaves proportionally while addition does not, but they may not have said so. The students may have believed that this distinction between the operations was not remarkable enough to mention, or they may have believed that it was not closely related to trigonometry and was thus not relevant to the study. The students who were coded as unsuccessful at differentiating between the effects of addition and multiplication may have understood this concept but not mentioned it because they were afraid of being embarrassed for stating something obvious.

The presence of audio- and video-recorders may have suppressed students' actions during nearly every phase of this research. Students may have preferred to be recorded giving no answer rather than an incorrect or obvious one. The lecture on trigonometric identities was not recorded, and some students responded to questions that were addressed to the class at large. The lecture on transformations was recorded, but the camera was focused on the researcher without any students in frame. Additionally, the students were informed multiple times that none of the video recordings used during lecture or recitation of the confirmatory study would be transcribed, used in the study, or otherwise shown to anyone; the recording was made strictly for the researcher to observe himself as he delivered the lecture. None of the students answered or asked any questions during the video-recorded lecture.

The confirmatory study was originally envisioned as two lectures and two recitations for each topic, conducted with 20-30 students, all of whom would be participating in the study. This would have allowed more flexibility in the methods for guiding students through the critical stages. Logistics necessitated a final version of the confirmatory study that constrained data collection by reducing the length by half and having the lectures be delivered to a group of students that contained both participants and non-participants that was almost twice as large as originally anticipated. This meant that audio recordings and work produced during recitation could only be collected during a single fifty-minute period for each topic.

Implications for Future Research

Future research could include re-implementing the confirmatory study. The confirmatory study was not able to collect enough data to draw conclusions. A teaching experiment using either this study's revised lesson plan or a different lesson plan created using the critical stages as a framework could be conducted, and it could be noted how the lesson plan facilitates or inhibits students' abilities to advance through the critical stages. Alternatively, the study protocol could be revised to include task-based interviews instead of written assessments for the pre- and post-tests. This would allow the researcher to assess whether the students could justify their understandings of the identities and transformations instead of relying on the students to provide justified reasoning of their own volition on a written assessment.

A trigonometry curriculum could be developed using results from this study combined with results from similar studies on other areas of trigonometry. Previous studies have examined how students come to understand the sine function (Demir & Heck, 2013; Peterson et al., 1998; Wood, 2011) and angle measure (Moore, 2013). Research still must be done on how students come to understand inverse trigonometric functions. After this, research could be done connecting the results from these studies in a coherent way to form a trigonometry curriculum through lessons proving the laws of sines and cosines. Similar to how there is overlap between the critical stages of understanding for identities and transformations, there is likely to be overlap among critical stages for other topics. The collection of critical stages could be investigated to find optimal orderings among all viable orderings. For example, this study has found that the cofunction identities can be justified for acute angles earlier than many of the other identities can be justified. By examining the critical stages of understanding for other topics, a reason could be found for having students justify this identity early in their studies, and a curriculum could be designed to reflect that. However, if no reason is found to separate justifications of the cofunction identities for acute and non-acute angles, then it would make sense to keep those critical stages near each other in the curriculum.

TrigReps could be further refined and researched. It is designed to simultaneously provide four representations of transformations of the sine function: (1) the algebraic representation, (2) a graphical representation on the Cartesian plane, (3) a dynamic unit circle representation, and (4) an aural representation of the sinusoid as a pressure wave. A previous study using the program asked students to discuss its ease of use; future research could focus on how effective it is in helping students progress through critical stages. In particular, it was believed that the dynamic unit circle representation would be helpful for students justifying why the order of horizontal transformations has counterintuitive characteristics. It was hypothesized that students would notice that the change to the radius's starting position must occur before the effects of changing its speed could be seen. However, the students in the confirmatory study did not progress far enough through the classwork to provide evidence that they could notice the effects of multiple horizontal transformations on the unit circle representation. A study could be conducted to test the effectiveness of a dynamic unit circle representation at helping students progress through the critical stages related to combinations of horizontal transformations behaving counterintuitively.

TrigReps could also be helpful in the effort to motivate students. During the teaching episode for transformations, C15 reacted to an aural representation by exclaiming "Oh, that was so low! It's a little hum, like a little submarine." The enthusiasm with which this was said is promising in regards to the ability of the MATLAB program to help motivate students. The pure tones that are provided by this program may inspire students to question whether sinusoids can be used to represent more commonly experienced sounds, and how to transform the sine function to do so. Additionally, students may wish to explore the methods by which sinusoids can be transformed to produce effects such as wah-wah or auto-tune. While TrigReps cannot presently be used to examine those concepts, it could potentially be modified to examine distortion or echoing effects. Even without these modifications, the program could be used to motivate explorations into these concepts, and it can be used to explain that noise-cancelling effects work by producing a sound wave that is identical to the "noise" in frequency, amplitude, and timbre, but is perfectly out of phase.

Replicating this study with other populations would lend credence to the theory that the critical stages developed in this study are general for all students learning trigonometry. A replication could also collect more demographic information with which to inform generalizations about the critical stages for different groups or further refine alternate paths to understanding.

References

AAHE, ACPA, & NASPA (1998). *Powerful partnerships: A shared responsibility for learning*. Washington, DC: ACPA.

Akkoç, H. (2008). Pre-service mathematics teachers' concept images of radian. International Journal of Mathematical Education in Science and Technology, 39(7), 857-878.

Axler, S. (2013). *Precalculus: A prelude to calculus* (6th ed.). Hoboken, NJ: Wiley.

Barrow, J.D. (2006). *The infinite book: A short guide to the boundless, timeless, and endless*. New York: Vintage.

Barton, D. (2003). Using technology to teach the 'Transformation of graphs'. *Micromath, 19*(2), 12-15.

Bauersfeld, H. (1995). The structuring of the structures: Development and function of mathematizing as a social practice. In L.P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 137-158). Hillsdale, NJ: Lawrence Erlbaum.

Black, H.S. (1953). *Modulation theory.* Van Nostrand: New York.

Blackett, N. (1990). *Developing understanding of trigonometry in boys and girls using a computer to link numerical and visual representations*. (Doctoral dissertation, University of Warwick, 1990).

Blackett, N. & Tall, D. (1991). Gender and the versatile learning of trigonometry using computer software. *Proceedings of the International Group for the Psychology of Mathematics Education, 15*(1), 144-151.

Borba, M.C. & Confrey, J. (1996). A student's construction of transformations of functions in a multiple representational environment. *Educational Studies in Mathematics, 31*(3), 319-337.

Burnard, P. (1991). A method of analyzing interview transcripts. *Nurse Education Today, 11*, 461-466.

Challenger, M. (2009). *From triangles to a concept: A phenomenographic study of A-level students' development of the concept of trigonometry*. (Unpublished doctoral dissertation). University of Warwick, Coventry, England.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Cobb, P., Jaworski, B., & Presmeg, N. (1996). Emergent and sociocultural views of mathematical activity. In L.P. Steffe, P. Nesher, P. Cobb, G.A. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 3-19). New York: Routledge.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education 23*(1), 2-33.

Confrey, J. (1990). What Constructivism Implies for Teaching. In R.B. Davis, C.A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 107-122). *Journal for Research in Mathematics Education Monograph No. 4*.

Confrey, J. (1994). Six approaches to transformation of functions using multi-representational software. *Proceedings of the XVIII Psychology of Mathematics Education* (pp. 217-224). Lisbon: Lisbon University.

Confrey, J. & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design. In A.E. Kelly & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 231-265).

Delice, A. & Roper, T. (2006). Implication of a comparative study for mathematics education in the English education system. *Teaching Mathematics and its Applications, 25*(2), 64-72.

Demir, Ö. & Heck, A. (2013). A new learning trajectory for trigonometric functions. In E. Faggiano & A. Montone (Eds.), *Proceedings of ICTMT, 11* (pp. 119-124). Bari, Italy: University of Bari.

Doise, W. & Mugny, G. (1979). Individual and collective conflicts of contraptions in cognitive development. *European Journal of Psychology, 9*, 105-108.

Doise, W., Mugny, G., & Perret-Clermont, A. (1975). Social interaction and the development of cognitive operations. *European Journal of Social Psychology, 5*(3), 367-383.

Douglas, S.C., Christensen, M.P., & Orsak, G.C. (2008). Designing pre-college curricula and technology: Lessons learned from the infinity project. *Proceedings of the IEEE, 96*(6), 1035-1048.

Duit, R. (1995). The constructivist view: A fashionable and fruitful paradigm for science education research and practice. In L.P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 271-285). Hillsdale, NJ: Lawrence Erlbaum.

Eitel, A., Scheiter, K., & Schüler, A. (2013). How inspecting a picture affects processing of text in multimedia learning. *Applied Cognitive Psychology, 27*, 451-461.

Eli, J.A., Mohr-Schroeder, M.J., & Lee, C.W. (2013). Mathematical connections and their relationship to mathematics knowledge for teaching geometry. *School Science and Mathematics Journal, 113*(3), 120-134.

Ellington, A. J. (2003). A meta-analysis of the effects of calculators on students' achievement and attitude levels in precollege mathematics classes. *Journal for Research in Mathematics Education, 34*(5), 433-463.

Ely, R. (2010). Nonstandard Student Conceptions about Infinitesimals. *Journal for Research in*

Mathematics Education, 42(2), 117-146.

Even, R. (1998). Factors involved in linking representations of functions. *Journal for Mathematical Behavior, 17*(1), 105-121.

Engelkemeyer, S.W. & Brown, S.C. (1999). Powerful partnerships: A shared responsibility for learning. In T.J. Marchese (Ed.), *AAHE Bulletin 51*(1-10) 1998-1999 (pp. 10-12). Washington, D.C.: American Association of Higher Education.

Ernest, P. (2006). Reflections on theories of learning. *Zentralblatt für Didaktik der Mathematik, 36*(1), 3-7.

Faulkenberry, E.D. & Faulkenberry, T.J. (2010). Transforming the way we teach function transformations. *The Mathematics Teacher, 104*(1), 29-33.

Fi, C.D. (2003). *Preservice secondary school mathematics teachers knowledge of trigonometry: Subject matter knowledge, pedagogical content knowledge, and envisioned pedagogy*. (Doctoral dissertation). University of Iowa. Ames, IA.

Glaser, B.G. (1965). The constant comparative method of qualitative analysis. *Social Problems, 12*(4), 436-445.

Goldin, G.A. (1997). Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education Monograph, 9*, 40-62, 164-177.

Goldin, G.A. (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In R. Lesh and A.E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 517-545). Hillsdale, NJ: Lawrence Erlbaum.

Goldin, G.A. & Kaput, J.J (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L.P. Steffe, P. Nesher, P. Cobb, G.A. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397-430. New York: Routledge.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25*(5), 443-471.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155-177.

Gravemeijer, K. & Cobb, P. (2006). Design research from the learning design perspective. In J. Akker, K. Gravemeijer, S. McKenney & N. Nieveen (Eds.), *Educational design research* (pp. 17-51). London: Routledge.

Gravemeijer, K. & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *The Elementary School Journal, 109*(5), 510-524.

Grbich, C. (2013). *Qualitative data analysis: An introduction* (2nd ed.). Thousand Oaks, CA: SAGE.

Greeno, J.G. & Hall, R.P. (1997). Practicing representation: Learning with and about representational forms. *The Phi Delta Kappan, 78*(5), 361-367.

Guest, G., Bunce, A., & Johnson, L. (2006). How many interviews are enough? An experiment with data saturation and variability. *Field Methods, 18*(1), 59-82.

Gür, H. (2009). Trigonometry learning. *New Horizons in Education, 57*(1), 67-80.

Hall, B. & Giacin, R. (2013). Exploring function transformations using the Common Core. *The Mathematics Teacher, 107*(2), 132-137.

Hatch, J.A. (2002). *Doing qualitative research in education settings*. Albany, NY: State

University of New York Press.
Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*, (pp. 1-27). Hillsdale, NJ: Larence Erlbaum.

Karadag, Z. & McDougall, D. (2009). Dynamic worksheets: Visual learning with the guidance of Polya. *MSOR Connections, 9*(2), 13-16.

Kendal, M. & Stacey, K. (1998). *Teaching trigonometry. The Australian Mathematics Teacher, 54*(1), 34-39.

Kessler, B. (2007). A "sound" approach to Fourier transforms using music to teach trigonometry. In R. Sarhangi & J. Barralo (Eds.), *2007 Bridges Donostia Conference Proceedings,* 135-142. Tarquin Publications.

Kuttruff, H. (1973). *Room acoustics.* Halsted Press: New York.

Lando, B.M. & Lando, C.A. (1977). Is the graph of temperature variation a sine curve? An application for trigonometry classes. *The Mathematics Teacher, 70*(6), 534-537.

Larsen, S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *Journal of Mathematical Behavior, 32*(4), 726-742.

Lesh, R. & Kelly, A. (2000). Multitiered teaching experiments. In A.E. Kelly & R.A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 197-230). Mahwah, NJ: Lawrence Erlbaum.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum.

Martinovic, D. & Karadag, Z. (2012). Dynamic and interactive mathematics learning environments: The case of teaching the limit concept. *Teaching Mathematics and Its Applications, 31*, 41-48.

Moore, K.C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics, 83*, 225-245.

Moreno-Armella, L., Hegedus, S.J., & Kaput, J.J. (2008). From static to dynamic mathematics: Historical and representational perspectives. *Educationl Studies in Mathematics, 68*, 99-111.

Morse, J. (1994). Designing funded qualitative research. In N.K. Denzil & Y.S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 220-235). Thousand Oaks, CA: Sage.

NCTM, (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

NGA (2010). *Common Core State Standards for Mathematics*. Washington, D.C.: Author.

Olivier, A. (1989). Handling pupils' misconceptions. *Pythagoras, 48*, 17-25.

Özdemir, S. & Ayvaz Reis, Z. (2013). The effect of Dynamic and Interactive Mathematics Learning Environmentss (DIMLE), supporting multiple representations on perceptions of elementary mathematics pre-service teachers in problem solving process. *Mevlana International Journal of Education, 3*(3), 85-94.

Peterson, B.E., Averbeck, P., & Baker, L. (1998). Sine curves and spaghetti. *The Mathematics Teacher, 91*(7), 564-567.

Pimm, D. (1995). *Symbols and meaning in school mathematics*. New York: Routledge.

Popelska. S.R. (2011). Corner reflector mathematics. *The Mathematics Teacher, 105*(2), 112-119.

Powell, K.C. & Kalina, C.J. (2009). Cognitive and social constructivism: Developing tools for an effective classroom. *Education, 130*(2), 241-250.

Rau, M.A., Aleven, V., & Rummel, N. (2014). Successful learning with multiple graphical representations and self-explanation prompts. *Journal of Educational Psychology, 107*(1), 30-46.

Ray, M.A. (1994). The richness of phenomenology: Philosophic, theoretic, and methodologic concerns. In J.M. Morse (Ed.), *Critical issues in qualitative research methods* (pp. 117-133). Thousand Oaks, CA: Sage.

Rigden, J.S. (1977). *Physics and the sound of music*. John Wiley and Sons: New York.

Rosen, G., Usselman, M., & Llewellyn, D. (2005). *Relating high school mathematics through sound and images.* Paper presented at 2005 American Society for Engineering Education Southeast Section Conference.

Salomon, G. (1993). No distribution without individuals' cognition: A dynamic interactional view. In G. Salomon's (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 111-138). New York: Cambridge University Press.

Schnotz, W. & Bannert, M. (2003). Constructivism and interference in learning from multiple representation. *Learning and Instruction, 13*, 141-156.

Seidman, I. (2013). *Interviewing as qualitative research: A guide for researchers in education and the social sciences* (4th ed). New York: Teachers College Press.

Simon, M.A. & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the Hypothetical Learning Trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Simpson, A., & Stehlíková, N. (2006). Apprehending mathematical structure: A case study of coming to understand a commutative ring. *Educational Studies in Mathematics, 61*, 347-371.

Skemp, R.R. (1987). Relational understanding and instrumental understanding. In The psychology of learning mathematics (pp. 152-163). New Jersey: Lawerence Erlbaum.

Stein, E.M. & Shakarchi, R. (2003). *Complex Analysis*. Princeton, NJ: Princeton University Press.

Stein, M.K. & Smith, M.S. (2010). The influence of curriculum on students' learning. In B.J. Reys & R.E. Reys (Eds.), Mathematics Curriculum: Issues, trends, and future directions. NCTM 72nd yearbook.Reston, VA: NCTM

Sokolowski, A. & Rackley, R. (2011). Teaching harmonic motion in trigonometry: Inductive inquiry supported by physics simulations. *Australian Senior Mathematics Journal, 25*(1), 45-53.

Steffe, L.P., & Thompson, P.W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A.E. Kelly & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum.

Topçu, T., Kertil, M., Yilmaz, K., & Öndar, O. (2006). Pre-service and in-service mathematics teachers' concept image of radian. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the*

Psychology of Mathematics Education, 5, pp. 281-288. Prague: PME.

Tsai, Y. & Chang, C. (2009). Using combinatorial approach to improve students' learning of the distributive law and multiplicative identities. *International Journal of Science and Mathematics Education, 7*, 501-531.

Tuna, A. (2013). A conceptual analysis of the knowledge of prospective mathematics teachers about degree and radian. *World Journal of Education, 3*(4), 1-9.

Usiskin, Z. (1984). Needed changes in mathematics curricula. In T.A. Romberg & D.M. Stewart (Eds.), School Mathematics: Options for the 1990s: Proceedings of the Conference (pp. 53-65). Washington DC: NCTM.

von Glasersfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 3-17). Hillsdale, NJ: Lawrence Erlbaum.

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal, 17*(3), 91-112.

Weber, K. (2008). Teaching trigonometric functions: Lessons learned from research. *The Mathematics Teacher, 102*(2), 144-150.

Wilhelm, J. & Confrey, J. (2005). Designing project-enhanced environments. *The Science Teacher, 72*(9), 42-45.

Wood, W.E. (2011). Squigonometry. *Mathematics Magazine, 84*(4), 257-265.

Wu, H. & Puntambekar, S. (2012). Pedagogical affordances of multiple external representations in scientific processes. *Journal of Science Education and Technology, 21*, 754-767.

Zazkis, R. & Hazzan, O. (1999). Interviewing in mathematics education research: Choosing the questions. *Journal of Mathematical Behavior, 17*(4), 429-439.

Zengin, Y., Furkan, H., & Kutluca, T. (2012). The effect of dynamic mathematics software Geogebra on student achievement in teaching of trigonometry. *Procedia - Social and Behavioral Sciences, 31*, 183-187.

Appendix A

Main Study Stage One Protocol

0.

a. Have you ever passed a course with a trigonometry unit? If so, was it a high school or undergraduate course?
b. Have you ever taken a course with a trigonometry unit? If so, was it a high school or undergraduate course?
c. List the last three mathematics courses that you've taken.
d. What does identity mean (Challenger, 2009)?
   Intended to: assess prior knowledge of identities in preparation for learning trigonometric identities (stage zero)

If the student is unable to answer:

> What does identity mean to you in a non-mathematical context?

If the student describes or provides an example of an equality rather than an identity:

> What is the difference between identity and equality?

If the student does not provide a trigonometric identity:

> What can you tell me about trigonometric identities?

If the student can provide an example of identity (such as the Pythagorean or tangent) but not describe it further:

> Why might it be useful to know that those things are equal?

Question zero is intended to examine students' understandings of the word "identity" and to inform the researcher of the students' potential familiarity with the material.

1. Sketch graphs of the following functions:

a. *sin(x)*
b. *cos(x)*

c. *tan(x)* (adapted from Barton, 2003)

Intended to: identify misconceptions that students may have about the graphs of the trigonometric functions and establish the "parent" graphs which will be compared to the transformations (stage zero).

If the student has the wrong period, amplitude, phase, or otherwise incorrectly graphs a function:

How is your graph related to the definitions of the functions?

If the students' mistakes persist:

How is your graph related to right triangles or the unit circle?

2. Evaluate the following:

a. $cos(-\pi/4)$, $cos(0)$, $cos(\pi/4)$
b. $sin(-\pi/2)$, $sin(0)$, $sin(\pi/2)$
c. $cos(-\pi/4) + \pi$, $cos(0) + \pi$, $cos(\pi/4) + \pi$
d. $sin(-\pi/2 + \pi)$, $sin(0 + \pi)$, $sin(\pi/2 + \pi)$
e. $tan(-\pi/3)$, $tan(0)$, $tan(\pi/3)$
f. $-(3/2)tan(-\pi/3)$, $-(3/2)tan(0)$, $-(3/2)tan(\pi/3)$
g. $tan(-(3/2)(-\pi/3))$, $tan(-(3/2)(0))$, $tan(-(3/2)(\pi/3))$

Intended to: prompt students to notice the differences in the algebraic representations of transformed trigonometric functions, notably through addition and multiplication on the inputs and outputs. Also intended to imply patterns that could prompt the student to move a representation with better affordances, interpret the situation in the new representation, find the values under consideration in the new representation, and compare those values (stages one and two).

If the student is uncomfortable or incapable of working with radians:

Switch to degrees

If the student believes that $f(-x) = -f(x)$ for all functions:

Can you show me how you found $cos(x)$ and $cos(-x)$?

If the student does not know how to perform the tasks:

How would you define the trigonometric functions? or Are there any other ways you could represent the problem?

If the student evaluates e.g. $sin(\pi/2) + \pi$ or $sin(\pi/2)+ sin(\pi)$:

What is $\pi/2 + \pi$?

3. Plot the points:

a. $(-\pi/4, cos(-\pi/4) + \pi)$, $(0, cos(0) + \pi)$, $(\pi/4, cos(\pi/4) + \pi)$
b. $(-\pi/2, sin(-\pi/2 + \pi))$, $(0, sin(0 + \pi))$, $(\pi/2, sin(\pi/2 + \pi))$
c. $(-\pi/3, -(3/2)tan(-\pi/3))$, $(0, -(3/2)tan(0))$, $(\pi/3, -(3/2)tan(\pi/3))$
d. $(-\pi/3, tan(-(3/2)(-\pi/3)))$, $(0, tan(-(3/2)(0)))$, $(\pi/3, tan(-(3/2)(\pi/3)))$
e. Do you notice any relationships between the sets of ordered pairs that you have drawn and the graphs of the functions *sin(x)*, *cos(x)*, and *tan(x)*? (adapted from Barton, 2003)

Intended to: prompt students to notice how their pairs of inputs and outputs are changed graphically with each algebraic transformation (stages three and four).

4. Predict how the graphs of the following functions will differ from those in question two:

a. *sin(x + π)*
b. *cos(x) + 1*
c. *(3/2)tan(x)*
d. *tan((3/2)x)*
e. *cos((1/2)x)*
f. *sin(-x)*
g. *-sin(x)*
h. *cos(2πx)* (adapted from Borba & Confrey, 1996).

Intended to: prompt students to move from a pointwise consideration of transformations to a global one (Even, 1998), and to reflect upon snd extend the results of task two (stages four through seven).

5. Graph the functions from question five. Do you notice any relationships or properties? Why do you think that is (adapted from Barton, 2003; Hall & Giacin, 2013)?

Intended to: prompt students to notice how the transformations affect the graphical representations of the functions (stage three) as well as begin to classify the effects of the transformations (stages four through seven).

If the student has mistakes in their graph(s), such as from assuming that horizontal transformations will behave similarly to vertical ones:

What are some ordered pairs on your graph? and How do these ordered pairs relate to the algebraic representation (e.g. *tan((3/2)x)*)?

6. Find all *x* such that *2cos(x) = 1* (adapted from Challenger, 2009).

Intended to: assess students' conceptions of the transformations as globally affecting the function. That is, the *entire* graph of cosine is stretched vertically in the above example (stage six).

If the student gives a single answer:

Could you sketch graphs of *y = 2cos(x)* and *y = 1*?

7. How could you algebraically represent one or more transformations of sine, cosine or tangent that results in the following functions:

a. [Graph of *3sin(x)*]
b. [Graph of *-cos(x) + 2*]
c. [Graph of *tan(-2x)*]
d. [Graph of *cos(x)*] (adapted from Borba & Confrey, 1996; Hall & Giacin, 2013)

Intended to: examine how students think about the classification of transformations and how to use them to achieve specific results. It will also be interesting to see how students approach the fact that three of the graphs can be given as transformations of either the cosine or sine functions. In particular, *d* appears to be a parent function, cosine. (stages five through nine).

If the student has mistakes in their graphs:

What are some ordered pairs on your graph? and How do these ordered pairs relate to the algebraic function?

If students do not note that there exist infinite ways of representing each function:

Could you algebraically represent any of these graphs differently? Could you use the same or different parent functions to give different algebraic representations of these graphs?

8. A function is defined as *periodic* if there exists a number p such that $f(x + p) = f(x)$ for every x in the domain of f. The number p is said to be the *period*. Find the period of the following functions:

a. $sin(x)$

b. $tan(x)$

c. [Graph of $(1/2)sin(6x)$]

d. $2cos((1/10)x)$

e. $asin(bx + c) + d$ (adapted from Sokolowski & Rackley, 2011).

Intended to: examine students' understandings of transformations on periodic functions (stage eight).

If the student gives too large a period, such as 2π for tangent:

Can you draw a line with slope 1/2 through the unit circle (or at $y = 1/2$ and $x = 1/2$ for sine and cosine, respectively)? What angles θ make $tan(\theta) = 1/2$?

9. Suppose $0 < \theta < \pi/2$ and $sin(\theta) = k$. Evaluate (in terms of k):

a. $sin(\theta + 5\pi)$

b. $cos(-\theta)$

c. $tan(\theta - \pi)$ (adapted from Axler, 2013).

Intended to: prompt the student to use the identities that they have identified in the previous exercise. If the students have not shown an understanding of generalized relationships, then this example could prompt them by showing them a problem between the previous two exercises in terms of abstractness (stage nine).

10. Describe any relationships you've encountered regarding changes in the representations used during your work in the previous exercises (adapted from Barton, 2003; Fi, 2003).

Intended to: prompt the student to reflect, hypothesize, and justify generalized relationships for trigonometric identities of the form $f(x + k\pi)$ and $f(-x)$ for integer values of k (stages four, five, eight, nine).

If they state incorrect relationships:

Only ask the standard followup questions. If the mistakes persist through questions five and six, the students will be questioned more thoroughly.

11. Let $F(x) = sin(x)$; $g(x) = 2x$; $h(x) = x + \pi$. Write out and graph the following functions:

a. $F(h(x))$

b. $F(h(g(x)))$

c. $F(g(x))$

d. *F(g(h(x))*
e. *g(F(x))*
f. *g(F(h(x)))*
g. *F(h(x))*
h. *g(F(h(x)))*
i. *h(F(x))*
j. *g(h(F(x)))*
k. *g(F(x))*
l. *h(g(F(x)))*

Intended to: prompt students to notice that the order in which they apply the transformations sometimes, but not always, affects the graphical transformation (stage ten). Students may note in particular that the order of transformations matters when multiple transformations are applied horizontally and/or vertically (stages eleven and twelve). By writing out the algebraic representation, students may also begin to notice that the order of the horizontal transformations is non-intuitive in relation to the graphical representation (stage thirteen).

If the student is confused about compositions (e.g. order of application):

Correct any misconceptions, noting previous compositions if applicable. This activity won't be productive with misunderstandings of composition, and it could affect future work.

12. Describe how could you represent the following functions using a circle and radius:

a. *sin(x)*
b. *2sin(x)*
c. *sin(2x)*
d. *sin(x - π/4)*

Intended to: prompt students to think about transforming the representation of the input of the function. Students have viewed graphical transformations as acting upon the output representation (the curve) rather than the input representation (the axes) (Borba & Confrey, 1996; Hall & Giacin, 2013). These transformations acting upon a circle and radius can potentially be more clearly seen as acting separately upon the input (the speed and starting rotational position of the radius) or the output (the size and vertical placement of the circle and radius) (stage fourteen).

If the student does not know how to use the unit circle for part *b*:

Could you alter the unit circle in some way to make it more helpful?

If the student adjusts the unit circle horizontally instead of adjusting the input for *c*, *d*:

Could you draw an angle at *π/4* radians? What is the resulting sine value?

If the result is *sin(π/4)*:

How is *sin(2x)* different than *sin(x)*?

If the result is *sin(2(π/4))*:

How does your circle represent this?

If the student can make no progress on *c*, *d*:

What does the *x* represent in *cos(x)* in the circle representation?

13. How could you algebraically represent one or more transformations of sine, cosine, or tangent that results in the following functions:

a. [Graph of $3\cos(x) - 4$]
b. [Graph of $(1/4)\sin(x) + 6$]
c. [Graph of $\tan(2x + \pi/4)$]
d. [Graph of $\cos((\pi/4)x - (\pi/3))$] (adapted from Borba & Confrey, 1996)

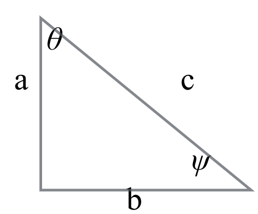
Intended to: provide students with further work to examine the non-intuitive nature of combining horizontal transformations (stages fourteen and fifteen).

If the student has mistakes in their graphs:

What are some ordered pairs on your graph? and How do these ordered pairs relate to the algebraic function?

If students do not note that there exist infinite ways of representing each function:

Could you algebraically represent any of these graphs differently? Could you use the same or different parent functions to give different algebraic representations of these graphs?



14. For the above right triangle, suppose $\theta = \pi/8$.

a. Evaluate ψ
b. Which leg is adjacent to θ?
c. Which leg is opposite ψ?
d. Find $\cos(\theta)$
e. Find $\sin(\psi)$ (adapted from Axler, 2013; Blackett, 1990).

Intended to: spur students to notice that, since all triangles have interior angles whose sum is π radians, then the acute angles of a right triangle must have a sum of $\pi/2$ radians (stages seven and eight). This exercise also implies that this identity should be true for the acute angles of right triangle trigonometry (stage sixteen).

If the student is confused about adjacent/opposite or leg/hypotenuse:

Define the term.

15. Will this always be true for complementary angles?

Intended to: generalize the results of the previous exercise. In order to do so, the student will have to move to a different representation since right triangles can only represent acute angles. The student may choose the unit circle or the graphs of these functions in order to generalize beyond acute angles (stage seventeen).

If the student does not know how to explore generalization:

How else could you represent $\cos(\theta)$ and $\sin(\pi/2 - \theta)$?

16. Which of the following are equivalent? Put a circle around them and join them with a line.

sin(x) *cos(x)* *tan(x)* *sin(x - π)* *-sin(x)* *cos(x - π)*

sin(π/2 - x) *tan(x + π)* *sin(x + π)* *sin(x) + 1* *1 + sin(x)* *sin(x)/cos(x)*

(Challenger, 2009).

Intended to: assess how the students have come to understand the trigonometric identities encountered thus far, including whether they are reflecting on their generalizations. For example, students should note that adding multiples of *π* to the tangent function has a different effect than adding to cosine or sine. Students may be hesitant to connect *tan(x + π)* to *tan(x)* unless they have considered the effects of the angle change on the tangent function in particular (stages nine and seventeen).

17. Draw a concept map for trigonometry, including trigonometric identity and transformation concepts. Write and circle "Trigonometry" in the center of the page. Write and circle other concepts that are related to trigonometry. Draw uni- or bi-directional arrows between related concepts, and write on those arrows a word or short phrase describing the connection. Write in as many concepts and arrows as are necessary to show how you believe all of these trigonometric concepts are related to each other. An example of a concept map for multiplication has been provided (adapted from Challenger, 2009; Fi, 2003).

Intended to: examine how students think about trigonometric identities and transformations in relation to their prior knowledge, notably the unit circle definitions of the trigonometric functions.

Appendix B

Main Study Stage Two Protocol

Identities

0.

a. Have you ever taken a course with a trigonometry unit? How long ago did you take it? Was that in high school or college? What grade did you get?
b. List the last three mathematics courses that you've taken.
c. What's your major?
    If the student is undecided or undeclared:
        Have you been thinking of any major? What fields of study or employment interest you?

1. What does mathematical identity mean (Challenger, 2009)?
    Intended to: assess prior knowledge of identities in preparation for learning trigonometric identities (stage zero)
    If the student is unable to answer:
        Would you consider 2 = 2, (2/4 = 1/2, 5x = 7, tan(x) = sin(x)/cos(x), x + a - a = x) to be an identity? Why or why not?
    If the student does not provide a trigonometric identity:
        What can you tell me about trigonometric identities?
    If the student can provide an example of identity (such as the Pythagorean or tangent) but not describe it further:
        Why might it be useful to know that values on the left side of that equation are equal to the values on the right side of the equation?

2. How would you define the functions $y = sin(x)$, $y = cos(x)$, and $y = tan(x)$?

Intended to: establish prerequisite knowledge of trigonometric functions (stage zero).

If they don't use the unit circle:

prompt for alternate definitions.

3. Evaluate the following:

a. $cos(0), cos(\pi/4), cos(\pi), cos(2\pi),$

b. $sin(0), sin(\pi/4), sin(\pi), sin(2\pi)$

c. $cos(-0), cos(-\pi/4), cos(-\pi), cos(-2\pi),$

d. $cos(0 + \pi), cos(\pi/4 + \pi), cos(\pi + \pi), cos(2\pi + \pi)$

Intended to: prompt students to notice the differences in the algebraic representations of transformed trigonometric functions, specifically transformations adding multiples of π to the input or taking the opposite input. Also intended to imply patterns that could prompt the student to move a representation with better affordances, interpret the situation in the new representation, find the values under consideration in the new representation, and compare those values (stages one through three).

If the student is uncomfortable or has difficulty working with radians:

Switch to degrees

If the student believes that $f(-x) = -f(x)$ for all functions:

Can you show me how you found $cos(\pi/4)$ and $cos(-\pi/4)$?

If the student does not know how to perform the tasks:

How would you define the trigonometric functions? or Are there any other ways you could represent the problem?

If the student evaluates e.g. $cos(0) + \pi$ or $cos(0) + cos(\pi)$:

What is $0 + \pi$?

4. Suppose $cos(x) = a; sin(x) = b$. Evaluate (in terms of a and/or b):

a. $cos(-x)$

b. $cos(x + \pi)$

c. $cos(x + 2\pi)$

d. $sin(-x)$

e. $sin(x + \pi)$

f. $sin(x + 2\pi)$

g. $tan(x)$

h. $tan(-x)$

i. $tan(x + \pi)$

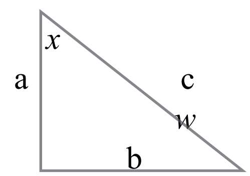
j. $tan(x + 2\pi)$

Intended to: prompt the student to use the identities that they have started to develop in the previous exercise. If they have not yet changed to an alternate representation, this exercise should prompt them to do so. (stages two through four).

If students believe that the starting value or quadrant matters for x, ask them to check one of the relationships for multiple values or quadrants.

5. What do you notice about tasks *a*, *b*, and *c* in relation to the given information that $cos(x) = a$? What do you notice about *d*, *e*, and *f* in relation to the given information that $sin(x) = b$? What do you notice about *g*, *h, i*, and *j*?

Intended to: prompt the student to reflect, hypothesize, and justify generalized relationships for trigonometric identities of the form $f(x + k\pi)$ and $f(-x)$ for integer values of k (stage four).



6. For the above right triangle, suppose $x = \pi/8$.

a. Evaluate *w*
b. Which leg is adjacent to *x*?
c. Which leg is opposite *w*?
d. Find $cos(x)$
e. Find $sin(w)$ (adapted from Axler, 2013; Blackett, 1990).
f. How are those values related?

Intended to: spur students to notice that, since all triangles have interior angles whose sum is π radians, then the acute angles of a right triangle must have a sum of $\pi/2$ radians (stages seven and eight). This exercise also implies that this identity should be true for the acute angles of right triangle trigonometry (stage five).

If the student is confused about adjacent/opposite or leg/hypotenuse:

Define the term.

7. Will the relationship that you found in the last problem be true in general for two angles whose sum is $\pi/2$? That is, would the relationship hold for $cos(591°)$ and $sin(-501°)$, as well as all other such pairs?

Intended to: generalize the results of the previous exercise. In order to do so, the student will have to move to a different representation since right triangles can only represent acute angles. The student may choose the unit circle or the graphs of these functions in order to generalize beyond acute angles (stage six).

If the student does not know how to explore generalization:

How else could you represent $\cos(x)$ and $sin(\pi/2 - x)$?

Transformations:

1. Sketch a graph of the function $cos(x)$ (adapted from Barton, 2003).

Intended to: identify misconceptions that students may have about the graph of the cosine function and establish the “parent” graph which will be compared to the transformations (stage zero).

If the student has the wrong period, amplitude, phase, or otherwise incorrectly graphs a function:

How is your graph related to the definitions of the functions?

If the students' mistakes persist:

How is your graph related to right triangles or the unit circle?

2. Evaluate the following: [present previous work for a, b, e]
a. *cos(0), cos(π/4), cos(π), cos(2π),*
b. *cos(-0), cos(-π/4), cos(-π), cos(-2π)*
c. *-cos(0), -cos(π/4), -cos(π), -cos(2π)*
d. *cos(0) + 1, cos(π/4) + 1, cos(π) + 1, cos(2π) + 1*
e. *cos(0 + π), cos(π/4 + π), cos(π + π), cos(2π + π)*
f. *2cos(0), 2cos(π/4), 2cos(π), 2cos(2π)*
g. *cos(2·0), cos(2·(π/4)), cos(2·π), cos(2·2π)*

Intended to: prompt students to notice the differences in the algebraic representations of transformed trigonometric functions, notably through addition and multiplication on the inputs and outputs. Also intended to imply patterns that could prompt the student to move a representation with better affordances, interpret the situation in the new representation, find the values under consideration in the new representation, and compare those values (stages one through three).

If the student is uncomfortable or incapable of working with radians:

Switch to degrees

If the student believes that *f(-x) = -f(x)* for all functions:

Can you show me how you found *cos(x)* and *cos(-x)*?

If the student does not know how to perform the tasks:

Recall how you defined cosine in the previous interview [present work]

If the student evaluates e.g. *cos(π/2 + 1)* as *cos(π/2) + 1* or *cos(π/2)+ cos(1)*:

What is *π/2 + 1*?

3. Plot points on a Cartesian graph with *0, π/4, π, 2π* as x-values and the answers from the previous exercise as y-values. (E.g. 3a would be plotting the points *(0, cos(0)) = (0, 1),* then *(π/4, cos(π/4)), (π, cos(π)),* and *(2π, cos(2π))*.) These points are not necessarily on the unit circle.

Intended to: prompt students to notice how their pairs of inputs and outputs are changed graphically with each algebraic transformation (stages three through eight).

If the student believes that these ordered pairs should all be on the unit circle:

cos(2π) = 1. So the last ordered pair is (2π, 1). Is that on the unit circle?

If the student has mistakes in their graphs, such as incorrectly labeled points:

What are some ordered pairs on your graph? and How do these ordered pairs relate to the algebraic function?

4. Can you describe any relationships between the values in 2a and the values calculated in the other parts of question 2? How do these values relate to the points that you plotted in problem 3?

Intended to: prompt the student to reflect, hypothesize, and justify generalized relationships for trigonometric transformations (stages four through seven).

5. How could you algebraically represent one or more transformations of the cosine function that results in the following functions:
a. [Graph of cos$(x - \pi/2)$]
b. [Graph of $-cos(x) + 2$]
c. [Graph of $cos(-2x)$]
d. [Graph of $cos(x)$]

Intended to: examine how students think about the classification of transformations and how to use them to achieve specific results. It will also be interesting to see how students approach the fact that *d* appears to be a parent function, cosine. (stages five through seven).

If students do not note that there exist infinite ways of representing each function:

Could you algebraically represent any of these graphs differently? Could you use the same or different parent functions to give different algebraic representations of these graphs?

6. Graph a cosine function:
a. Vertically stretched by a factor of 2, then vertically shifted by 1.
b. Vertically shifted by 1, then vertically stretched by a factor of 2.
c. Horizontally stretched by a factor of 2, then vertically shifted by 1.
d. Vertically shifted by 1, then horizontally stretched by a factor of 2.
e. Horizontally stretched by a factor of 2, then horizontally shifted by π.
f. Horizontally shifted by π, then horizontally stretched by a factor of 2.

Intended to: prompt students to notice that the order in which they apply the transformations sometimes, but not always, affects the graphical transformation (stage ten). Students may note in particular that the order of transformations matters when multiple transformations are applied horizontally and/or vertically (stage eleven). Students may also begin to notice that the order of the horizontal transformations is non-intuitive in relation to the algebraic representations (stage eleven).

If the student treats, for example, vertical stretches as horizontal shrinks:

Refer back to the plotted points. Note the zeros, range.

7. What did you notice about the graphs that resulted from problem 6?

8. How could you algebraically represent one or more transformations of sine, cosine, or tangent that results in the following functions:
a. [Graph of $3cos(x) - 4$]
b. [Graph of $(1/4)cos(x) + 6$]
c. [Graph of $cos(2x + \pi/4)$]
d. [Graph of $cos((\pi/4)x - (\pi/3))$] (adapted from Borba & Confrey, 1996)

Intended to: provide students with further work to examine the non-intuitive nature of combining horizontal transformations (stage twelve).

If students do not note that there exist infinite ways of representing each function:

Could you algebraically represent any of these graphs differently? Could you use the same or different parent functions to give different algebraic representations of these graphs?

9. Describe how could you represent the following functions using a circle and radius:

a. *cos(x)*
b. *2cos(x)*
c. *cos(2x)*
d. *cos(x - π/4)*

Intended to: prompt students to think about transforming the representation of the input of the function. Students have viewed graphical transformations as acting upon the output representation (the curve) rather than the input representation (the axes) (Borba & Confrey, 1996; Hall & Giacin, 2013). These transformations acting upon a circle and radius can potentially be more clearly seen as acting separately upon the input (the speed and starting rotational position of the radius) or the output (the size and vertical placement of the circle and radius) (stages nine, thirteen).

If the student does not know how to use the unit circle for part *b*:

Could you alter the unit circle in some way to make it more helpful?

If the student adjusts the unit circle horizontally instead of adjusting the input for *c*, *d*:

Could you draw an angle at *π/4* radians? What is the resulting sine value?

If the result is *cos(π/4)*:

How is *cos(2x)* different than *cos(x)*?

If the result is *cos(2(π/4))*:

How does your circle represent this?

If the student can make no progress on *c*, *d*:

What does the *x* represent in *cos(x)* in the circle representation?

10. A function is defined as *periodic* if there exists a number *p* such that *f(x + p) = f(x)* for every *x* in the domain of *f*. The number *p* is said to be the *period*. Find the period of the following functions:

a. *sin(x)*
b. *tan(x)*
c. [Graph of *(1/2)sin(6x)*]
d. *2cos((1/10)x)*
e. *asin(bx + c) + d* (adapted from Sokolowski & Rackley, 2011).

Intended to: examine students' understandings of transformations on periodic functions (stage eight).

If the student gives too large a period, such as 2*π* for tangent:

Can you draw a line with slope 1/2 through the unit circle (or at *y = 1/2* and *x = 1/2* for sine and cosine, respectively)? What angles *θ* make *tan(θ) = 1/2*?

Appendix C

Pre-Post Tests

Identities:

NAME: ______________________________

This test will have NO effect on your grade.
It is perfectly fine to skip or abandon questions that you're stuck on.
No Calculators
Explain your reasoning

1. How would you define the function $y = cos(x)$

2. Evaluate the following:
a. $cos(0)$

b. $sin(5\pi/4)$

c. $tan(-\pi/3)$

d. $cos(9\pi/4)$

e. $sin(\pi/4 + 5\pi)$

3. Suppose $sin(-15°) = k$

Evaluate $cos(105°) + tan(165°)$

4. $\cos(0 + \pi)$, $\cos(\pi/4 + \pi)$, $\cos(\pi + \pi)$, $\cos(2\pi + \pi)$

Transformations:

NAME: ____________________________

This test will have NO effect on your grade.
It is perfectly fine to skip or abandon questions that you're stuck on.
No Calculators
Explain your reasoning

1. Sketch a graph of the function *cos(x)*.

2. How could you algebraically represent one or more transformations of the sine, cosine, or tangent function that results in the following functions:

a. See graph

b. See graph

3. Sketch the graphs of the following functions

a. $\sin(2\pi x + \pi/6)$

(Sketch the graph of the function)

b. $3\cos(x) - 4$

4. A function is defined as *periodic* if there exists a number p such that $f(x + p) = f(x)$ for every x in the domain of f. The number p is said to be the *period*. Find the period of the following functions:

a. $\sin(x)$

b. $\tan(x)$

c. See graph

d. $2cos((1/10)x)$

e. $asin(bx + c) + d$

Appendix D

Group Work Tasks

Identities

Draw a unit circle representation of $cos(x) = m$ and $sin(x) = n$.
Choose a real number t.
Draw a radius with endpoint $(cos(x + t), sin(x + t))$.

1.
   a. When is $cos(x + t)$ greater than $cos(x)$?
   b. When is $cos(x + t)$ less than $cos(x)$?
   c. When is $cos(x + t)$ equal to $cos(x)$?
   d. When is $cos(x + t)$ equal to $-cos(x)$?

2.
   a. When is $sin(x + t)$ greater than $sin(x)$?
   b. When is $sin(x + t)$ less than $sin(x)$?
   c. When is $sin(x + t)$ equal to $sin(x)$?
   d. When is $sin(x + t)$ equal to $-sin(x)$?

3.
   a. When is $tan(x + t)$ greater than $tan(x)$?
   b. When is $tan(x + t)$ less than $tan(x)$?
   c. When is $tan(x + t)$ equal to $tan(x)$?
   d. When is $tan(x + t)$ equal to $-tan(x)$?

4.

a. On one set of axes, sketch the graphs of $y = cos(x); y = m; and\ y = -m$.
b. On a second set of axes, sketch the graphs of $y = sin(x); y = n; and\ y = -n$.
c. On a third set of axes, sketch the graphs of $y = tan(x); y = m/n; and\ y = -m/n$.

5.

Can you justify any general formulas for the trigonometric (in)equalities based on your work?

6.

Find all angles x such that $cos(x) = m$

7.

Suppose $\theta + \psi = 90°$. We have justified that $cos(\theta) = sin(\psi)$ for acute angles, and we have seen one way to extend this property to all real numbers θ and ψ. Can you find another way to justify this property?

Transformations

Human hearing has range approximately 20 Hz - 20000 Hz. Not all of the functions that you input will produce sounds within your hearing range. Can you predict which of the functions will and will not produce sounds?

1. Input $f(x) = sin(x)$
a. Find and input a function with twice the amplitude.
b. Find and input a function with amplitude 0.2.
c. What do you notice about the four representations: algebraic, graphical, unit circle, and aural?

2. Find and input a function that shifts the graph:
a. down by 2π.
b. up by 3/2
c. to the left by π/2
d. to the right by 7
e. What do you notice about the four representations?

3. Find and input a function with:
a. triple the frequency of $f(x) = sin(x)$.
b. frequency 1 Hz
c. What do you notice about the four representations?

For tasks 4 and 5, check the box that allows for a second set of inputs. Also, please check the box on the dynamic representation to slow it down. These tasks use numbers that are too large for the representation to effectively display.

4.

a. Find and input a function with frequency 440 Hz.
b. Find and input a second function with twice the frequency.
c. Find and input a second function with 3/2 the frequency.
d. Find and input a second function with 17/19 the frequency.
e. What do you notice about the four representations?

5.
a. Find and input a function with frequency 220 Hz.
b. Find and input a second function with a different frequency and 1/3 the amplitude.
c. Find and input a second function with a different frequency and twice the amplitude.
d. What do you notice about the four representations?

6.
a. Input $f(x) = 2sin(x)$
b. Input $f(x) = sin(x) +1$
c. Predict what will happen in each representation for the input $f(x) = 2sin(x) + 1$
d. Input $f(x) = 2sin(x) + 1$
e. Did the results match your prediction? If not, why not?

7.
a. Input $f(x) = sin(2x)$
b. Input $f(x) = sin(x + \pi/4)$
c. Predict what will happen in each representation for the input $f(x) = sin(2x + \pi/4)$
d. Input $f(x) = sin(2x + \pi/4)$
e. Did the results match your prediction? If not, why not?

8.
a. Input $f(x) = sin(2x)$
b. Input $f(x) = sin(x) + 1$
c. Predict what will happen in each representation for the input $f(x) = sin(2x) + 1$
d. Input $f(x) = sin(2x) + 1$
e. Did the results match your prediction? If not, why not?

9.
a. Input $f(x) = 2sin(x)$
b. Input $f(x) = sin(x + \pi/4)$
c. Predict what will happen in each representation for the input $f(x) = 2sin(x + \pi/4)$
d. Input $f(x) = 2sin(x + \pi/4)$
e. Did the results match your prediction? If not, why not?

10.
a. Find and input a function with frequency 220 Hz.

b. Find and input a second function that is completely out of phase with your function from part *a*.
c. Predict what will happen in each representation as one of these functions changes in amplitude.
d. Input three different amplitudes for one of these functions one at a time. Record your inputs and note the effects that they have on the four representations.