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On decompositions and Connes's embedding problem of finite von Neumann algebras

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ON DECOMPOSITIONS AND CONNES'S EMBEDDING PROBLEM OF FINITE VON NEUMANN ALGEBRAS

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DEDICATION

I dedicate to my parents and my wife, without whose caring support, this thesis would not have been possible to be done.

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ABSTRACT

ON DECOMPOSITIONS AND CONNES'S EMBEDDING PROBLEM OF FINITE VON NEUMANN ALGEBRAS

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A longstanding open question of Connes asks whether every finite von Neumann algebra embeds into an ultraproduct of finite-dimensional matrix algebras. As of yet, algebras verified to satisfy Connes's embedding property belong to just a few special classes (e.g. amenable algebras and free group factors). In this dissertation we establish Connes's embedding property for von Neumann algebras satisfying Popa's co-amenability condition. Some decomposition properties of finite von Neumann algebras are also investigated.

Chapter 1 reviews von Neumann algebras, completely bounded mappings, conditional expectations, tensor products, crossed products, direct integrals, and Jones basic construction.

Chapter 2 introduces new decompositions of finite von Neumann algebras which we call Γ-thin, strongly Γ-thin, and weakly Γ-thin, etc. We also consider the singly-generated problem, and compute the cohomology in such decompositions of finite von Neumann algebras.

In Chapter 3 we show by estimation of free entropy that free group factors lack the type of decompositions discussed in Chapter 2.

In Chapter 4 we investigate co-amenability and Connes's embedding problem.

CHAPTER 1

INTRODUCTION

1.1 Background

F.J. Murray and J. von Neumann [Von30, MV36, MV37, Von40, MV43] introduced and studied "rings of operators," which were later renamed "von Neumann algebras" by J. Dixmier in 1957. *Von Neumann algebras* are strong-operator closed self-adjoint subalgebras of the algebra of all bounded linear transformations on a Hilbert space. One calls a von Neumann algebra whose center consists of scalar multiplies of the identity a *factor*. Every von Neumann algebra has structure equal to a direct integral of factors. This makes factors the building blocks for all von Neumann algebras.

Murray and von Neumann [MV36] classified factors by means of their relative dimension functions. *Finite factors* have dimension functions with finite range. (More generally, one calls a von Neumann algebra *finite* if it admits a faithful normal trace.) The dimension function of a finite factor gives rise to a (unique, when normalized) tracial state.

Finite-dimensional finite factors are full matrix algebras $M_n(\mathbb{C})$, $n = 1, 2, \ldots$.

Infinite-dimensional finite factors are called factors of type II_1, sometimes described as continuous matrix algebras. A factor is *hyperfinite* if it can be weakly approximated by finite-dimensional matrix algebras. In [MV37], Murray and von Neumann provided the first two examples of non-isomorphic factors of type II_1, the two-generator free group factor and the permutation group factor. They also established the uniqueness of the hyperfinite factor $\mathcal{R}$ of type II_1. The permutation group factor is the hyperfinite factor $\mathcal{R}$ of type II_1. The hyperfinite factor of type II_1 occurs as a subfactor in every factor of type II_1. A. Connes [Con76] famously showed that every subfactor of $\mathcal{R}$ is hyperfinite. Embeddings into an ultrapower of $\mathcal{R}$ plays a key role in his proof. Accordingly, Connes asks whether every factor of type II_1 with a separable predual embeds into some ultrapower of $\mathcal{R}$; this is known as Connes's embedding problem.

In this thesis, we will study Connes's embedding problem for finite von Neumann algebras satisfying Popa's co-amenability [PM03] and show that a new class of finite von Neumann algebras can be embedded into an ultrapower of $\mathcal{R}$. F. Rădulescu [Ra02] calls a discrete group *hyperlinear* if it faithfully embeds into the unitary group of an ultrapower of $\mathcal{R}$. For group von Neumann algebras, Connes's embedding problem reduces to whether any discrete countable group is hyperlinear. We will show that any group with a hyperlinear co-amenable subgroup is itself hyperlinear.

Gromov [Gro99] introduced sofic groups, easily seen to be hyperlinear. In fact, many groups [ElSz05, Pe08] are known to be sofic, but whether every group is sofic, or even just whether every hyperlinear group is sofic, remains open.

The other factor of type II_1 introduced in [MV37] is the free group factor. Much about free group factors remains unknown. Despite much attention, the question of isomorphism between the two-generator free group factor and the three-generator free group factor remains open. Attacking on this problem, D. Voiculescu [VDN92] introduced free probability theory which included many tools such as free entropy. In the framework of free probability theory, Connes's embedding problem is equivalent to the emptiness of a certain set connected with the definition of free entropy. In [GePo98], L. Ge and S. Popa introduced a new type of decomposition for factors of type II_1. They expressed a factor of type II_1 as the weak-operator closure of the linear span of a product of abelian von Neumann subalgebras and the hyperfinite subfactors of type II_1. This decomposition provides a tool to study free group factors. Ge and Popa showed that many factors of type II_1 are *thin*; i.e. equal to the weak-operator closure of the linear span of a product of two hyperfinite von Neumann subalgebras. In contrast, by estimating the free entropy of a finite generating set in a thin factor, Ge and Popa showed that free group factors are not thin. Hyperfinite von Neumann algebras and abelian von Neumann algebras (i.e. type I_1 von Neumann algebras) are building blocks for the decomposition of von Neumann algebras. More building blocks such as property Γ factors could be used.

We extend the decomposition defined in [GePo98] and introduce new decompositions that we call Γ-thin, strongly Γ-thin, and weakly-thin etc. We show that the free group factors do not have this type of decompositions either.

1.2 Preliminaries

Throughout this thesis, we always denote by $\mathbb{C}$ ($\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively) the complex number field (the real number field, the group of all integers, the set of all positive integers respectively).

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ with an inner product $\langle \cdot, \cdot \rangle$: $\mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$ satisfying:
(i) $\langle a\xi_1 + b\xi_2, \eta \rangle = a\langle \xi_1, \eta \rangle + b\langle \xi_2, \eta \rangle$,
(ii) $\langle \xi, \eta \rangle = \overline{\langle \eta, \xi \rangle}$,
(iii) $\langle \xi, \xi \rangle \geq 0$,
(iv) $\langle \xi, \xi \rangle = 0$ only when $\xi = 0$,
whenever ξ_1, ξ_2, ξ, η are in $\mathcal{H}$, and a, b are in $\mathbb{C}$. The norm $\|\cdot\|$ on the Hilbert space $\mathcal{H}$ induced by the inner product $\langle \cdot, \cdot \rangle$ is then defined by $\|\xi\| = \langle \xi, \xi \rangle^{1/2}$, whenever $\xi \in \mathcal{H}$.

Now let $T : \mathcal{H} \mapsto \mathcal{H}$ be a linear operator acting on the space $\mathcal{H}$ as above, whose operator norm is given by

$$\|T\| = \sup\{\|T\xi\| : \xi \in \mathcal{H}, \|\xi\| \leq 1\}.$$

We say T is a *bounded operator* if $\|T\| < \infty$. The adjoint of T on the Hilbert space $\mathcal{H}$, denoted by T^*, can be defined as follows:

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle,$$

whenever ξ, η are in $\mathcal{H}$. From now on, we always consider T as a bounded operator on $\mathcal{H}$, unless otherwise stated.

Below are several properties that the bounded operators enjoy.

Lemma 1 *For all bounded operators T, S on a Hilbert space $\mathcal{H}$ and $a, b \in \mathbb{C}$, we have that*

1. $(aT + bS)^* = \bar{a}T^* + \bar{b}S^*$,
2. $(TS)^* = S^*T^*$,
3. $(T^*)^* = T$,
4. $\|T^*T\| = \|T\|^2$.

We say T is *normal* if $TT^* = T^*T$; is *self-adjoint* if $T^* = T$; is *unitary* if $TT^* = T^*T = I$, where I is the identity on $\mathcal{H}$. Actually, self-adjoint operators and unitary operators are normal operators, while it is not true vice versa.

Let us recall more types of bounded operators. We say T is *positive* if $\langle T\xi, \xi\rangle \geq 0$ for any ξ in $\mathcal{H}$; T is a(n) (orthogonal) *projection* if $T^* = T = T^2$. Projections are positive and positive operators are self-adjoint.

Now let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators acting on the Hilbert space $\mathcal{H}$. Although there are many topologies on $\mathcal{B}(\mathcal{H})$, we will focus on the following three topologies: norm topology, strong-operator topology, and weak-operator topology. Suppose $\{T_\alpha\}_\alpha$ is a net of operators on $\mathcal{H}$. We say T_α is convergent to T in *norm topology* if $\|T_\alpha - T\|$ is convergent to 0; in *strong-operator topology* if $\|(T_\alpha - T)\xi\|$ is convergent to 0 for all ξ in $\mathcal{H}$; in *weak-operator topology* if $\langle T_\alpha\xi, \eta\rangle$ is convergent to $\langle T\xi, \eta\rangle$ for all ξ, η in $\mathcal{H}$.

Finally, we can successfully give the definition of C* algebra, which is important to von Neumann algebras introduced in the following section. An algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ over $\mathbb{C}$ is called a *$*$-algebra* if $T \in \mathfrak{A}$ implies $T^* \in \mathfrak{A}$. We say $\mathfrak{A}$ is a *C^* algebra* if the $*$-algebra $\mathfrak{A}$ is closed in norm topology.

There is also an alternative way to define a C* algebra. Suppose $\mathfrak{A}$ is a Banach algebra over $\mathbb{C}$. Let $* : A \mapsto A^*$ be *an involution* from $\mathfrak{A}$ onto $\mathfrak{A}$ for all $A \in \mathfrak{A}$ satisfying that, for all T, S in $\mathfrak{A}$ and a, b in $\mathbb{C}$,

1. $(aT + bS)^* = \bar{a}T^* + \bar{b}S^*$,

2. $(TS)^* = S^*T^*$,

3. $(T^*)^* = T$.

Then, a Banach algebra $\mathfrak{A}$ with an involution $*$ is a C* algebra if the additional equation $||T^*T|| = ||T||^2$ holds for any T in $\mathfrak{A}$.

Von Neumann Algebras

A *-algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a *von Neumann algebra* if $\mathcal{M}$ is closed in weak-operator topology. Denote the commutant of $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ by $\mathcal{M}'$, and the center of $\mathcal{M}$ by $\mathscr{C}(\mathcal{M})$. Any projection in the center of $\mathcal{M}$ is called a *central projection* in $\mathcal{M}$. According to the double commutant theorem for von Neumann algebras, a *-algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is von Neumann algebra if $\mathcal{M} = (\mathcal{M}')'(= \mathcal{M}'')$. All von Neumann algebras are C* algebras. A von Neumann algebra $\mathcal{M}$ is a *factor* if the center of $\mathcal{M}$ consists of only scalar multiplies of the identity; i.e. $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$. In particular, $\mathcal{B}(\mathcal{H})$ is a factor. Each von Neumann algebra is a direct integral of factors.

Let $\mathcal{M}$ be a von Neumann algebra described as above, and let E, F be projections in $\mathcal{M}$. We say that E is *equivalent* to F in $\mathcal{M}$, denoted by $E \sim F(\mathcal{M})$, if there exists an element V in $\mathcal{M}$ such that $V^*V = E$ and $VV^* = F$. Here V is called a *partial isometry* from the range $E(\mathcal{H})$ of E onto the range $F(\mathcal{H})$ of F. The *central carrier* P of an element A in $\mathcal{M}$ is the central projection P satisfying $P = I - \vee_\alpha P_\alpha$ for any central projection P_α in $\mathcal{M}$ with $P_\alpha A = 0$.

A projection E in a von Neumann algebra $\mathcal{M}$ is said to be *infinite* relative to $\mathcal{M}$ whenever $E \sim E_0 < E$ for some projection E_0 in $\mathcal{M}$. Otherwise E is called *finite* relative to $\mathcal{M}$. A projection E is a *minimal projection* (or an atom) in a von Neumann algebra $\mathcal{M}$ if E is non zero and contains no non zero proper subprojections in $\mathcal{M}$. A von Neumann algebra $\mathcal{M}$ is *finite* if the identity I is finite; $\mathcal{M}$ is *semi-finite* if there is a finite projection $E \in \mathcal{M}$ whose central carrier is the identity I.

Now let us focus on the case when $\mathcal{M}$ is a factor. We say $\mathcal{M}$ is a *factor of type* I if $\mathcal{M}$ contains a minimal projection — of *type* I_n if the identity I is the sum of n equivalent minimal projections. All $n \times n$ full matrix algebras are factors of type I_n for $n \in \mathbb{N}$. An example of a factor of type I_∞ is $\mathcal{B}(\mathcal{H})$. A factor $\mathcal{M}$ is of *type* II if $\mathcal{M}$ has no minimal projections but has a finite projection — of *type* II_1 if I is finite — of *type* II_∞ if I is infinite. Each factor of type II_∞ is a tensor product of a factor of type II_1 and a factor of type I_∞. A factor $\mathcal{M}$ is of *type* III if $\mathcal{M}$ contains no finite projections. According to [Tak73], every factor of type III is a continuous crossed product of a factor of type II_∞ by the real line $\mathbb{R}$.

As an example, see the following:

Example 2 *Let G be a discrete group with a unit e, and $l^2(G)$ be the Hilbert space spanned by the elements in G with inner product $\langle\cdot,\cdot\rangle$ given by*

$$\left\langle \sum_{g\in G}\lambda_g g, \sum_{g\in G}\mu_g g\right\rangle = \sum_{g\in G}\lambda_g\bar{\mu}_g.$$

Denote by $\mathcal{L}_G$ the von Neumann algebra generated by L_g for all g in G, i.e. $\mathcal{L}_G = \{L_g : g \in G\}'' \subset \mathcal{B}(l^2(G))$, where L_g is the shift operator on $l^2(G)$ satisfying $L_g h = gh$, for any h in G. A discrete group G is infinite-conjugacy-class (I.C.C.) if the conjugacy class of g is infinite for all $g \in G$ but unit e. One result showed in [KR] claims that G is I.C.C. if and only if $\mathcal{L}_G$ is a factor of type II_1.

More precisely, consider the case when G is the non-abelian free group $\mathcal{F}_2$ on two generators, which is I.C.C.. The result above tells us that the corresponding group von Neumann algebra $\mathcal{L}_{\mathcal{F}_2}$ is a factor of type II_1. Another example is the permutation group Π. Suppose Π_n, $n \in \mathbb{N}$, is the group of all permutations on the set $\{-n, \ldots, -1, 0, 1, \ldots, n\}$, Π_n embeds into Π_{n+1} naturally and the permutation group $\Pi = \cup_n \Pi_n$. Then the permutation group Π is an I.C.C. group and the permutation group von Neumann algebra $\mathcal{L}_\Pi$ is a factor of type II_1. Moreover, Murray and von Neumann proved that $\mathcal{L}_{\mathcal{F}_2}$ and $\mathcal{L}_\Pi$ are not isomorphic (see [KR], Chapter 6).

To proceed with our arguments, we need to recall a few basic facts about GNS construction.

Let $\mathcal{M}$ be a von Neumann algebra and $\rho : \mathcal{M} \mapsto \mathbb{C}$ be a linear functional on $\mathcal{M}$. The norm of the linear functional ρ on $\mathcal{M}$ is defined by

$$\|\rho\| = \sup\{|\rho(T)| : T \in \mathcal{M}, \|T\| \leq 1\},$$

and ρ is *bounded* if $\|\rho\| < \infty$. A bounded linear functional ρ is *normal* if it is weak-operator continuous on the closed unit ball $(\mathcal{M})_1$ of $\mathcal{M}$; is *faithful* if $\rho(A^*A) = 0$ implies $A = 0$, for all A in $\mathcal{M}$; is *positive* if $\rho(I) = \|\rho\|$; is a *state* if $\rho(I) = 1 = \|\rho\|$; is a *tracial state* if ρ is a state and $\rho(TS) = \rho(ST), \forall T, S \in \mathcal{M}$. In [MV36], Murray and von Neumann proved that only factors of type I_n and II_1 have tracial states, where $n \in \mathbb{N}$.

The linear space of all bounded linear functionals on $\mathcal{M}$ forms the dual of $\mathcal{M}$, denoted by $\mathcal{M}^\#$. The linear space of all normal linear functionals on $\mathcal{M}$, denoted by $\mathcal{M}_\#$, is a Banach space. The space $\mathcal{M}_\#$ is a *predual* of $\mathcal{M}$; i.e. $(\mathcal{M}_\#)^\# = \mathcal{M}$. It is well-known that the predual $\mathcal{M}_\#$ of $\mathcal{M}$ is weak* dense in $\mathcal{M}^\#$.

In order to establish the GNS construction, we still need to introduce two notations.

A *representation* φ of a C^* algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ is a *-homomorphism from $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$. For each unit vector ξ in $\mathcal{H}$, i.e. $\|\xi\| = 1$, a linear functional $\omega_\xi = \langle \cdot \xi, \xi \rangle$ on $\mathcal{B}(\mathcal{H})$ is called a *vector state*.

Theorem 3 (GNS Construction, see [KR], Theorem 4.5.2) *If ρ is a state on a C^* algebra $\mathfrak{A}$, then there exists a representation π_ρ of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_\rho$ and a vector $\xi_\rho \in \mathcal{H}_\rho$ such that $\rho = \omega_{\xi_\rho} \circ \pi_\rho$, i.e.*

$$\rho(A) = \langle \pi_\rho(A)\xi_\rho, \xi_\rho \rangle,$$

whenever $A \in \mathfrak{A}$.

Proof. Let

$$\mathcal{L}_\rho = \{A \in \mathfrak{A} : \rho(A^*A) = 0\}.$$

Since $\rho(B^*A) = 0$ for all $A \in \mathcal{L}_\rho, B \in \mathfrak{A}$, $\mathcal{L}_\rho$ is a closed left ideal of $\mathfrak{A}$. The equation

$$\langle A + \mathcal{L}_\rho, B + \mathcal{L}_\rho \rangle = \rho(B^*A)$$

gives an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{A}/\mathscr{L}_\rho$. Denote by $\mathcal{H}_\rho$ the completion of $\mathfrak{A}/\mathscr{L}_\rho$ relative to the inner product $\langle \cdot, \cdot \rangle$. Therefore $\mathcal{H}_\rho$ is a Hilbert space.

For all A, B in $\mathfrak{A}$, we define

$$\pi(A)(B + \mathscr{L}_\rho) = AB + \mathscr{L}_\rho,$$

and so $\pi(A)$ is a linear operator on $\mathfrak{A}/\mathscr{L}_\rho$. For all A, B in $\mathfrak{A}$,

$$\begin{aligned}
&\|A\|^2\|B + \mathscr{L}_\rho\|^2 - \|\pi(A)(B + \mathscr{L}_\rho)\|^2 \\
&= \|A\|^2\|B + \mathscr{L}_\rho\|^2 - \|AB + \mathscr{L}_\rho\|^2 \\
&= \|A\|^2\langle B + \mathscr{L}_\rho, B + \mathscr{L}_\rho\rangle - \langle AB + \mathscr{L}_\rho, AB + \mathscr{L}_\rho\rangle \\
&= \|A\|^2\rho(B^*B) - \rho(B^*A^*AB) \\
&= \rho(B^*(\|A\|^2 I - A^*A)B) \geq 0,
\end{aligned}$$

hence $\|\pi(A)\| \leq \|A\|$ and $\pi(A)$ is bounded. Consequently it can be extended to a bounded operator on $\mathcal{H}_\rho$, denoted by $\pi_\rho(A)$. We now show that $\pi_\rho(A)$ is a representation of $\mathfrak{A}$When $A = I$, $\pi_\rho(I)$ is the identity on $\mathcal{H}_\rho$. Clearly, for all A, B, C in $\mathfrak{A}$, a, b in $\mathbb{C}$,

$$\begin{aligned}
\pi_\rho(aA + bB)(C + \mathscr{L}_\rho) &= (a\pi_\rho(A) + b\pi_\rho(B))(C + \mathscr{L}_\rho), \\
\pi_\rho(AB)(C + \mathscr{L}_\rho) &= \pi_\rho(A)\pi_\rho(B)(C + \mathscr{L}_\rho), \\
\langle \pi_\rho(A)(B + \mathscr{L}_\rho), C + \mathscr{L}_\rho\rangle &= \langle B + \mathscr{L}_\rho, \pi_\rho(A^*)(C + \mathscr{L}_\rho)\rangle.
\end{aligned}$$

Moreover, since $\mathfrak{A}/\mathscr{L}_\rho$ is dense in $\mathcal{H}_\rho$, we have

$$\begin{aligned}
\pi_\rho(aA + bB) &= a\pi_\rho(A) + b\pi_\rho(B), \\
\pi_\rho(AB) &= \pi_\rho(A)\pi_\rho(B), \\
\pi_\rho(A)^* &= \pi_\rho(A^*).
\end{aligned}$$

This proves that π_ρ is a representation of $\mathfrak{A}$ on $\mathcal{H}_\rho$. Let $\xi_\rho = I + \mathscr{L}_\rho \in \mathfrak{A}/\mathscr{L}_\rho$. Then

$$\pi_\rho(A)\xi_\rho = A + \mathscr{L}_\rho, \forall A \in \mathfrak{A}.$$

Therefore, $\pi_\rho(\mathfrak{A})\xi_\rho(= \mathfrak{A}/\mathscr{L}_\rho)$ is dense in $\mathcal{H}_\rho$, and hence, for all A in $\mathfrak{A}$

$$\langle \pi_\rho(A)\xi_\rho, \xi_\rho\rangle = \langle A + \mathscr{L}_\rho, I + \mathscr{L}_\rho\rangle = \rho(A).$$

■

Remark 1 *If ρ is faithful, then $\mathscr{L}_\rho = 0$, and thus the Hilbert space $\mathcal{H}_\rho$ is the completion of $\mathfrak{A}/\mathscr{L}_\rho (= \mathfrak{A})$ relative to the inner product given by $\langle A, B \rangle = \rho(B^* A)$, for all A, B in $\mathfrak{A}$. The space $\mathcal{H}_\rho$ is also denoted by $L^2(\mathfrak{A}, \rho)$, which will be used frequently in the following sections.*

Theorem 3 focuses on the case when ρ is bounded. Actually, it can also be extended to the case when ρ is a weight, which is an unbounded linear functional.

Now let us recall the definition of a weight. For a von Neumann algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$, let $\mathfrak{A}^+$ be the set of all positive elements in $\mathfrak{A}$. A linear mapping $\rho : \mathfrak{A}^+ \mapsto [0, \infty]$ is called a *weight* on $\mathfrak{A}$ if

$$\rho(H + K) = \rho(H) + \rho(K), \rho(aH) = a\rho(H), \forall H, K \in \mathfrak{A}^+, 0 \leq a \in \mathbb{R}$$

Let

$$\begin{aligned} \mathscr{N}_\rho &= \{A \in \mathfrak{A} : \rho(A^* A) < \infty\}, \\ N_\rho &= \{A \in \mathfrak{A} : \rho(A^* A) = 0\}, \\ F_\rho &= \{A \in \mathfrak{A}^+ : \rho(A) < \infty\}, \\ \mathscr{M}_\rho &= span\{A : A \in F_\rho\}. \end{aligned}$$

A weight ρ is *faithful* if $N_\rho = \{0\}$; ρ is *semi-finite* if $\mathscr{M}_\rho$ is weak-operator dense in $\mathcal{M}$; ρ is *normal* if there is a family of positive normal linear functionals $\{\rho_\alpha\}_\alpha$ such that $\rho(H) = \sum_\alpha \rho_\alpha(H)$ for any H in F_ρ; ρ is a tracial weight on $\mathfrak{A}$ if $\rho(AA^*) = \rho(A^* A)$, for all A in $\mathfrak{A}$. Since $\mathscr{M}_\rho$ is the linear span of F_ρ, the weight ρ can be extended to a linear functional on $\mathscr{M}_\rho$, denoted by ρ again.

To see the GNS construction induced from a weight, we refer to the textbooks such as [KR] for a much more complete analysis.

In ending this section, we will show that for a fixed tracial weight on a semi-finite von Neumann algebra, there exists a simple relation between any normal state on the von

Neumann algebra and a positive unbounded operator. Before this, we would like to recall some notations about unbounded operators.

An (unbounded) operator T on a Hilbert space $\mathcal{H}$ is closed if the graph $\{(\xi, T\xi) : \xi \in \mathcal{H}\}$ of T is closed under the norm given by $\|(\xi, T\xi)\| = \|\xi\| + \|T\xi\|$ for any $\xi \in \mathcal{H}$. We say T is densely defined if its domain is dense in $\mathcal{H}$. In particular, every bounded operator is closed and densely defined. A closed, densely defined operator T is affiliated with a von Neumann algebra $\mathcal{M}$ on $\mathcal{H}$, denoted by $T\eta\mathcal{M}$, if $UTU^* = T$ for any unitary operator U in $\mathcal{M}'$. For more about unbounded operators, we refer to [KR]. To state an important result about unbounded operators, we denote by $|T|$ the absolute value of T for any operator T; i.e. $|T| = (T^*T)^{1/2}$. Then the result [KR] is that a closed, densely defined T has a polar decomposition $T = V|T|$, where V is a partial isometry from the range of T^* onto the range of T. Moreover, if $T\eta\mathcal{M}$, then $|T|\eta\mathcal{M}$.

Lemma 4 *Suppose $\mathcal{M}$ is a semi-finite von Neumann algebra with a separable predual and a faithful normal tracial weight Tr. Then for any normal state ϕ on $\mathcal{M}$, there is a(n) (unbounded) positive operator H affiliated with $\mathcal{M}$ such that $\phi(X) = Tr(HX)$ for any $X \in \mathcal{M}$.*

Proof. Let $\{E_{1,\alpha}\}_\alpha$ be an orthogonal family of projections in $\mathcal{M}$ maximal with respect to the property $\phi(E_{1,\alpha}) > Tr(E_{1,\alpha})$ and $E_1 = I - \sum_\alpha E_{1,\alpha}$. By induction, for $n \in \mathbb{N}$, let $\{E_{n,\beta}\}_\beta$ be an orthogonal family of projections in $(I - E_{n-1})\mathcal{M}(I - E_{n-1})$ maximal with respect to the property $\phi(E_{n,\beta}) > nTr(E_{n,\beta})$. Let $E_n = I - \sum_\beta E_{n,\beta}$. Then $E_n \leq E_{n+1}$ and E_n must converges to I in the strong-operator topology. Otherwise, we take $E = I - \lim_n E_n$. Then

$$\phi(E) = \lim_n \phi(I - E_n) \geq \lim_n nTr(I - E_n) \geq \lim_n nTr(E) > 0,$$

and $\phi(E)$ goes to ∞ as n goes to ∞ which leads a contradiction. Since $\mathcal{M}$ is semi-finite and separable, there exists a sequence $\{F_n\}_n$ of projections such that $\lim_n F_n = I$, F_n are finite and $F_n \leq E_n$. Then for $\phi|_{F_n\mathcal{M}F_n} \leq nTr|_{F_n\mathcal{M}F_n}$, there exists a positive element K_n' in the unit ball $(F_n\mathcal{M}F_n)_1$ of $F_n\mathcal{M}F_n$ such that $\phi(F_nXF_n) = nTr(K_n'X)$ for all $X \in \mathcal{M}$. Let $K_n = nK_n'$.

Since

$$Tr(K_{n+1}F_nXF_n) = \phi(F_{n+1}F_nXF_nF_{n+1}) = Tr(K_nX)$$

for all $X \in \mathcal{M}$, we have $F_nK_{n+1}F_n = K_n$. Let K be the least upper bound of $\{K_n\}$. By [KR] Chapter 5, K is positive, $K\eta\mathcal{M}$ and $Tr(K) = 1$. We pick H as K. ■

Special Mappings

In this section, I will mainly introduce two mappings: norm one projection and conditional expectation. The relationships between these two mappings is also discussed.

Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra of a von Neumann algebra $\mathcal{M}$. A linear mapping $\Psi : \mathcal{M} \mapsto \mathcal{N}$ is a *norm one projection* if $||\Psi(X)|| \leq ||X||, \forall X \in \mathcal{M}$, and $\Psi(Y) = Y, \forall Y \in \mathcal{N}$.

A linear mapping $\Phi : \mathcal{M} \mapsto \mathcal{N}$ is a *conditional expectation* if, for any X in $\mathcal{M}$, Y_1, Y_2 in $\mathcal{N}$, we have

1. $\Phi(X) \geq 0$ when $X \geq 0$,

2. $\Phi(I) = I$,

3. $\Phi(Y_1XY_1) = Y_1\Phi(X)Y_2$.

There is a well-known result showing that the two mappings described above are actually equivalent (see [Tak] for reference). More precisely,

Proposition 5 *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras, Φ a linear mapping from $\mathcal{M}$ onto $\mathcal{N}$. Then Φ is a norm one projection if and only if Φ is a conditional expectation.*

Proof. First, we assume that Φ is a norm one projection from $\mathcal{M}$ onto $\mathcal{N}$ and X is a positive element in $\mathcal{M}$. For any state ρ on $\mathcal{N}$, we have that $\rho \circ \Phi$ is a state on $\mathcal{M}$ since $(\rho \circ \Phi)(I) = 1 = ||\rho \circ \Phi||$. If $\Phi(X)$ is not positive, then there exists a state ρ such that

$(\rho \circ \Phi)(X)$ is negative or an imaginary number, this contradicts the positivity of $\rho \circ \Phi$. Thus Φ is positive. For any projection E in $\mathcal{N}$, X in $(\mathcal{M})_1$, we shall prove

$$\Phi(XE) = \Phi(XE)E, E\Phi(EX) = \Phi(EX).$$

We assume that $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$ and consider operator $\Phi(XE)(I - E)$ in $\mathcal{N}$ with λ in $\mathbb{R}$. Then, we have that

$$\begin{aligned}
||\lambda\Phi(XE)(I-E) + XE||^2 &= ||\bar{\lambda}(I-E)\Phi(XE)^* + EX^*||^2 \\
&= \sup_{||\xi||\leq 1} ||\bar{\lambda}(I-E)\Phi(XE)^*\xi + EX^*\xi||^2 \\
&= \sup_{||\xi||\leq 1} ||\bar{\lambda}(I-E)\Phi(XE)^*\xi||^2 + ||EX^*\xi||^2 \\
&\leq \lambda^2||(I-E)\Phi(XE)^*||^2 + ||EX^*||^2 \\
&\leq \lambda^2||\Phi(XE)(I-E)||^2 + 1.
\end{aligned}$$

On the other side, we obtain that

$$\begin{aligned}
||\lambda\Phi(XE)(I-E) + XE|| &\geq ||\Phi(\lambda\Phi(XE)(I-E) + XE)|| \\
&= ||\Phi(\lambda\Phi(XE)(I-E) + \Phi(XE)|| \\
&= ||\Phi((1+\lambda)\Phi(XE)(I-E) + \Phi(XE)E|| \\
&\geq ||((1+\lambda)\Phi(XE)(I-E)||.
\end{aligned}$$

Combining the above two equations, we get for $\lambda \in \mathbb{R}$,

$$\lambda^2||\Phi(XE)(I-E)||^2 + 1 \geq (1+\lambda)^2||\Phi(XE)(I-E)||^2.$$

Then we have

$$2\lambda||\Phi(XE)(I-E)|| \leq 1 - ||\Phi(XE)(I-E)||.$$

If λ is large enough, the left-hand side of the equation go to ∞ and the right-hand side is a constant, then the contradiction yields $\Phi(XE)(I-E) = 0$. Symmetrically, $(I-E)\Phi(EX) = 0$, and thus

$$\Phi(XE) = \Phi(XE)E + \Phi(XE)(I-E) = \Phi(XE)E = \Phi(XE)E + \Phi(X(I-E))E = \Phi(X)E.$$

Similarly, $\Phi(EX) = E\Phi(X)$. According to the spectral theorem (for example, [KR], Theorem 5.2.2), and the fact that any element can be written as a linear combination of self-adjoint elements, we have

$$\Phi(Y_1XY_2) = Y_1\Phi(X)Y_2,$$

whenever Y_1, Y_2 are in $\mathcal{N}$. Hence Φ is a conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$.

In the other direction, we assume that Φ is a conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. By the definition of conditional expectations, we have $\Phi(Y) = Y$ for any Y in $\mathcal{N}$. Since Φ is positive, we have

$$0 \leq \Phi((X - \Phi(X))^*(X - \Phi(X))) = \Phi(X^*X) - \Phi(X^*)\Phi(X),$$

and then

$$\|\Phi(X)\|^2 = \|\Phi(X^*)\Phi(X)\| \leq \|\Phi(X^*X)\| \leq \|X^*X\| = \|X\|^2.$$

Thus Φ is a norm one projection from $\mathcal{M}$ onto $\mathcal{N}$. ■

There are still some more mappings we would like to mention here as they will be discussed later. Suppose $\mathcal{M}, \mathcal{N}$ are von Neumann (or C*) algebras. A linear mapping $\Psi : \mathcal{M} \mapsto \mathcal{N}$ is *positive* if $\Psi(X) \geq 0$ for all $X \geq 0$. A linear mapping $\Psi : \mathcal{M} \mapsto \mathcal{N}$ is *completely positive* if for any $n \in \mathbb{N}$, the linear mapping $\Psi_n : M_n(\mathcal{M}) \mapsto M_n(\mathcal{N})$ is *positive*, where Ψ_n is given by $\Psi_n([X_{ij}]_{i,j=1}^n) = [\Psi(X_{ij})]_{i,j=1}^n, X_{ij} \in \mathcal{M}, [X_{ij}]_{i,j=1}^n \in M_n(\mathcal{M})$. A linear mapping $\Psi : \mathcal{M} \mapsto \mathcal{N}$ is *completely bounded* if

$$\|\Psi\|_{cb} = \sup_{n\geq 1} \|\Psi_n\| < \infty,$$

where

$$\|\Psi_n\| = \sup\{\|\Psi_n(X)\| : X \in M_n(\mathcal{M}), \|X\| \leq 1\}.$$

Finally, a linear mapping $\Psi : \mathcal{M} \mapsto \mathcal{N}$ is *completely contractive* if $\|\Psi\|_{cb} \leq 1$.

Two Products

In our main work, we shall frequently use two products for von Neumann algebras: the tensor product and the crossed product.

First, let us recall the tensor product. Suppose $\mathcal{M}, \mathcal{N}$ are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Let $\mathcal{H} \otimes \mathcal{K}$ be the Hilbert space tensor product of $\mathcal{H}$ and $\mathcal{K}$. For any $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, a simple tensor product $A \otimes B$ is the bounded linear operator on $\mathcal{H} \otimes \mathcal{K}$ given by $A \otimes B(\xi \otimes \eta) = A\xi \otimes B\eta$ for all $\xi \in \mathcal{H}, \eta \in \mathcal{K}$. Then the *von Neumann algebra tensor product* $\mathcal{M}\bar{\otimes}\mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ acting on the Hilbert space $\mathcal{H} \otimes \mathcal{K}$ is the von Neumann algebra $\{A \otimes B : A \in \mathcal{M}, B \in \mathcal{N}\}'' \subset \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

The following theorem is very important, and will be used in the sections below from time to time. For the proof and more details, we refer to [KR, Tak].

Theorem 6 *Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. The commutant $(\mathcal{M}\bar{\otimes}\mathcal{N})'$ of $\mathcal{M}\bar{\otimes}\mathcal{N}$ on $\mathcal{H} \otimes \mathcal{K}$ is isomorphic to $\mathcal{M}'\bar{\otimes}\mathcal{N}'$.*

Crossed products are used mainly for studying properties of von Neumann algebras that are invariant under *-isomorphisms. Suppose $\mathcal{M}$ is a von Neumann algebra acting on $L^2(\mathcal{M}, \tau)$ with a faithful normal tracial state τ. Let G be a discrete group with a unit e and $\sigma : G \mapsto Aut(\mathcal{M})$ be a trace-preserving group homomorphism. That is $\tau \circ \sigma_g = \tau$, for any g in G. The *crossed product* of a von Neumann algebra $\mathcal{M}$ by the discrete group G, denoted by $\mathcal{M} \rtimes_\sigma G$, can be described as below.

Denote by $\|\cdot\|_2$ the tracial norm of $\mathcal{M}$ given by $\|X\|_2 = \tau(X^*X)^{1/2}, \forall X \in \mathcal{M}$. Since σ_g is an automorphism of $\mathcal{M}$ and $\tau = \tau \circ \sigma_g$ for any g in G, we have $\|X\|_2 = \|\sigma_g(X)\|_2$ for any X in $\mathcal{M}$ and g in G. Then we can define a unitary operator V_g on $L^2(\mathcal{M}, \tau)$ such that $V_g\hat{X} = \widehat{\sigma_g(X)}$ for any g in G, X in $\mathcal{M}$, where $\hat{X}$, is a vector in $L^2(\mathcal{M}, \tau)$ corresponding to X.

Let $\mathcal{K} = \oplus_{g\in G}\mathcal{H}_g$, where $\mathcal{H}_g$ is a copy of $L^2(\mathcal{M}, \tau)$. For any T in $\mathcal{B}(\mathcal{K})$, its corresponding matrix form is $[T_{p,q}]_{p,q\in G}$ satisfying $T_{p,q} \in \mathcal{B}(L^2(\mathcal{M}, \tau))$. We embed $\mathcal{M}$ into $\mathcal{B}(\mathcal{K})$ such that X has matrix form $[X\delta_{p,q}]_{p,q\in G}$ in $\mathcal{B}(\mathcal{K})$ for any $X \in \mathcal{M}$, where $\delta_{p,q} = 0$ if $p \neq q$; $\delta_{p,q} = 1$ if $p = q$. Let U_g be the element in $\mathcal{B}(\mathcal{K})$ whose corresponding matrix form is $[\delta_{p,gq}V_g]$, where V_g is the unitary operator described above.

Finally, the *crossed product* $\mathcal{M} \rtimes_\sigma G$ of $\mathcal{M}$ by G is the von Neumann algebra

$$\mathcal{M} \rtimes_\sigma G = \{X, U_g : X \in \mathcal{M}, g \in G\}'' \subset \mathcal{B}(\mathcal{K}).$$

All elements in the crossed product have form $\sum_{g\in G} X_g U_g$, where $X_g \in \mathcal{M}$. The trace τ_1 on $\mathcal{M} \rtimes_\sigma G$ is given by

$$\tau_1\left(\sum_{g\in G} X_g U_g\right) = \tau(X_e).$$

Direct Integrals

Let $\mathcal{X}$ be a σ-compact, locally compact (Borel measure) space, μ be the completion of a Borel measure on $\mathcal{X}$, and let $\{\mathcal{H}_p\}_p$ be a family of separable Hilbert spaces indexed by the points p of $\mathcal{X}$. We say that a separable Hilbert space $\mathcal{H}$ is the *direct integral* of $\{\mathcal{H}_p\}_p$ over $(\mathcal{X}, \mu)$ (we write $\mathcal{H} = \int_{\mathcal{X}} \oplus \mathcal{H}_p d\mu(p)$) when, for each $\xi \in \mathcal{H}$, there exists a corresponding function $p \mapsto \xi(p)$ such that $\xi(p) \in \mathcal{H}_p$ for each p and

(i) $p \mapsto \langle \xi(p), \eta(p) \rangle$, for all $\xi, \eta \in \mathcal{H}$ is μ-integrable,

$$\langle \xi, \eta \rangle = \int_{\mathcal{X}} \langle \xi(p), \eta(p) \rangle d\mu(p).$$

(ii) if $u_p \in \mathcal{H}$ for all $p \in \mathcal{X}$ and $p \mapsto \langle u_p, \xi(p) \rangle$ is integrable for all $\xi \in \mathcal{H}$, then there is a u in $\mathcal{H}$ such that $u(p) = u_p$ for almost every $p \in \mathcal{X}$.

We say that $\int_{\mathcal{X}} \oplus \mathcal{H}_p d\mu(p)$ and $p \mapsto \xi(p)$ are the (direct integral) decompositions of $\mathcal{H}$ and $\xi \in \mathcal{H}$ respectively.

If $\mathcal{H}$ is the direct integral of $\{\mathcal{H}_p\}_p$ over $(\mathcal{X}, \mu)$, an operator T in $\mathcal{B}(\mathcal{H})$ is said to be *decomposable* when there is a function $p \mapsto T(p)$ on $\mathcal{X}$ such that $T(p) \in \mathcal{B}(\mathcal{H}_p)$ and, for each $\xi \in \mathcal{H}$, $T(p)\xi(p) = (T\xi)(p)$ for almost every p. If, in addition, $T(p) = f(p)I_p$, where I_p is the identity operator on $\mathcal{H}_p$, we say T is *diagonalizable*. In general, a (separable) Hilbert space $\mathcal{H}$ has direct integral decomposition relative to an abelian von Neumann algebra $\mathcal{A}$ on $\mathcal{H}$. We state some related theorem as follows.

Theorem 7 (See [KR]) *If $\mathcal{A}$ is an abelian von Neumann algebra on the separable Hilbert space $\mathcal{H}$ there is a (locally compact complete separable metric) measure space $(\mathcal{X}, \mu)$ such that $\mathcal{H}$ is (unitarily equivalent to) the direct integral of Hilbert spaces $\{\mathcal{H}_p\}_p$ over $(\mathcal{X}, \mu)$*

and $\mathcal{A}$ is (unitarily equivalent to) the algebra of diagonalizable operators relative to this decomposition.

If $\mathcal{H}$ is the direct integral of Hilbert spaces $\{\mathcal{H}_p\}$ over (X, μ), a representation ϕ of a C* algebra $\mathfrak{A}$ on the Hilbert space $\mathcal{H}$ is said to be decomposable over (X, μ), when there exists a representation ϕ_p of $\mathfrak{A}$ on $\mathcal{H}_p$ such that for any $A \in \mathfrak{A}$, $\phi(A)$ is decomposable and $\phi(A)(p) = \phi_p(A), a.e.$ A von Neumann algebra $\mathcal{M}$ is decomposable on $\mathcal{H}$ with $p \mapsto \mathcal{M}_p$, if $\mathcal{M}$ contains a norm separable C* subalgebra $\mathfrak{A}$ strong-operator dense in $\mathcal{M}$ such that the identity representation l of $\mathfrak{A}$ is decomposable and $l_p(\mathfrak{A})$ is strong-operator dense in $\mathcal{M}$. We state the following theorem to indicate that every von Neumann algebra has the direct integral decomposition relative to its center.

Theorem 8 (See [KR]) *If $\mathscr{A}$ is an abelian von Neumann subalgebra of the center C of a von Neumann algebra $\mathcal{M}$ on a separable Hilbert space $\mathcal{H}$ and $\{\mathcal{H}_p\}$ is the direct integral decomposition of $\mathcal{H}$ relative to $\mathscr{A}$, then C_p is the center of $\mathcal{M}_p$ almost everywhere. In particular, $\mathcal{M}_p$ is a factor a.e. if and only if $\mathscr{A} = C$.*

A state ϕ of a von Neumann algebra $\mathcal{M}$ could be decomposable according to Theorem 9 below.

Theorem 9 (See [KR]) *If $\mathcal{H}$ is a direct integral of Hilbert spaces $\{\mathcal{H}_p\}$ over (X, μ), $\mathcal{M}$ is a decomposable von Neumann algebra on $\mathcal{H}$, ϕ is a normal state on $\mathcal{M}$. Then there is a mapping $p \mapsto \phi_p$, where ϕ_p is positive normal linear functional on $\mathcal{M}_p$ and $\phi(A) = \int_X \phi_p(A(p))d\mu(p), \forall A \in \mathcal{M}$.*

Jones Basic Construction

In 1983, V. R. Jones introduced a new construction for von Neumann algebras, which is known as Jones basic construction. It has many applications, especially in the index theory of subfactors, some of whose basic definitions will be introduced at the end of this section.

Suppose $\mathcal{B} \subset \mathcal{N}$ is an inclusion of von Neumann algebras with a faithful normal tracial state τ. Let $\mathbb{E}_{\mathcal{B}}$ be the trace-preserving conditional expectation from $\mathcal{N}$ onto $\mathcal{B}$. Let $\mathcal{N}$

act on $L^2(\mathcal{N}, \tau)$ which is the Hilbert space from the GNS construction induced by τ (refer to section 1.2.2). We identify $L^2(\mathcal{B}, \tau)$ as a Hilbert subspace of $L^2(\mathcal{N}, \tau)$. For any X in $\mathcal{N}$, denote by $\hat{X}$ the vector of $L^2(\mathcal{N}, \tau)$ corresponding to X. Let $E_{\mathcal{B}}$ be the projection from $L^2(\mathcal{N}, \tau)$ onto $L^2(\mathcal{B}, \tau)$ with $E_{\mathcal{B}}\hat{X} = \widehat{\mathbb{E}_{\mathcal{B}}(X)}$ for any X in $\mathcal{N}$ and J the conjugation on $L^2(\mathcal{N}, \tau)$ given by $J\hat{X} = \hat{X}^*$ for any X in $\mathcal{N}$. Denote by $\langle \mathcal{N}, \mathcal{B} \rangle$ the von Neumann algebra $\{\mathcal{N}, E_{\mathcal{B}}\}'' \subset \mathcal{B}(L^2(\mathcal{N}, \tau))$ generated by $\mathcal{N}$, $E_{\mathcal{B}}$ and one has that $\langle \mathcal{N}, \mathcal{B} \rangle = J\mathcal{B}'J$.

The (Jones) basic construction for $\mathcal{B} \subset \mathcal{N}$ is then defined to be the inclusions $\mathcal{B} \subset \mathcal{N} \subset \langle \mathcal{N}, \mathcal{B} \rangle$ (see [Jon83]). The following property of Jones basic construction is very important to our work, see [SM08] for its complete proof and analysis.

Theorem 10 *Let $\mathcal{B}$ be a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{N}$ with a faithful normal tracial state τ. There exists a unique normal semi-finite faithful tracial weight Tr on $\langle \mathcal{N}, \mathcal{B} \rangle$ satisfying $Tr(XE_{\mathcal{B}}Y) = \tau(XY)$, for X, Y in $\mathcal{N}$.*

Now let us recall some basic concepts from the index theory of subfactors, which will be required later. Let $\mathcal{M}$ be a finite factor with the trace τ acting on a Hilbert space $\mathcal{H}$. Suppose the commutant $\mathcal{M}'$ of $\mathcal{M}$ is finite and its trace is denoted by τ'. Then the *coupling constant* $dim_{\mathcal{M}}(\mathcal{H})$ of $\mathcal{M}$ is defined as $\tau(E_{\mathcal{M}'\xi})/\tau'(E_{\mathcal{M}\xi})$, where ξ is a non zero vector in $\mathcal{H}$ and $E_{\mathcal{A}\xi}$ is the projection onto the closure of the subspace $\mathcal{A}\xi$. This definition, due to Murray and von Neumann [MV37], is independent of ξ. If $\mathcal{N}$ is a subfactor of $\mathcal{M}$, the index of $\mathcal{N}$ in $\mathcal{M}$, denoted by $[\mathcal{M} : \mathcal{N}]$, is defined as $dim_{\mathcal{N}}(\mathcal{H})/dim_{\mathcal{M}}(\mathcal{H})$. This definition, due to Jones [Jon83], is independent of $\mathcal{H}$. If $\mathcal{H} = L^2(\mathcal{M}, \tau)$, then $[\mathcal{M} : \mathcal{N}] = dim_{\mathcal{N}}(L^2(\mathcal{M}, \tau))$. The remarkable result in [Jon83] is that, the set of all possible values of index is given by

$$\{4\cos^2 \pi/n | n = 3, 4, \ldots\} \cup \{r \in \mathbb{R} | r \geq 4\} \cup \{\infty\}.$$

CHAPTER 2

DECOMPOSITIONS OF FINITE VON NEUMANN ALGEBRAS

In this chapter, we begin with some definitions of building blocks for decompositions of finite von Neumann algebras. A factor is hyperfinite if it contains an ascending sequence of full matrix algebras weak-operator dense in itself. For instance, $\mathcal{B}(\mathcal{H})$ is a hyperfinite factor of type I_n, where $\mathcal{H}$ is a Hilbert space with dimension $n \in \mathbb{N} \cup \{\infty\}$, while the permutation group factor (See Chapter 1, section 1.2.1) is a hyperfinite factor of type II_1. The hyperfinite factor of type II_1 is known to be unique (see [KR], chapter 12).

Let $\mathcal{M}$ be a factor of type II_1 with the trace τ. The type II_1 factor $\mathcal{M}$ is said to have property Γ if for any finitely many elements $X_1, \ldots, X_n$ in $\mathcal{M}$ and $\epsilon > 0$, there exists a unitary element U in $\mathcal{M}$ with $\tau(U) = 0$ such that

$$\|X_iU - UX_i\|_2 < \epsilon, i = 1, 2, \ldots, n.$$

An alternative formulation is that for any finitely many elements $X_1, \ldots, X_n$ in $\mathcal{M}$, there exists a sequence $\{U_k\}_{k=1}^{\infty}$ of trace zero unitary elements in $\mathcal{M}$ satisfying

$$\lim_{k\to\infty} \|X_iU_k - U_kX_i\|_2 = 0, i = 1, 2, \ldots, n.$$

For a free ultrafilter ω on $\mathbb{N}$, a sequence $\{X_n\}_n$ of elements in $\mathcal{M}$ is an ω-central sequence of $\mathcal{M}$ if $\lim_{n\to\omega} \|X_nX - XX_n\|_2 = 0$ for any X in $\mathcal{M}$ and $\sup_n\{\|X_n\|\} < \infty$ (for more details see Chapter 4). All ω-central sequences of $\mathcal{M}$ form a finite von Neumann algebra, denoted by $\mathcal{M}_\omega$, which is also called a ω-central sequence algebra of $\mathcal{M}$. The hyperfinite factor of type II_1 has property Γ (for example, see [KR]). Moreover, D. McDuff [Mc70] proved that if the ω-central sequence algebra of a separable factor $\mathcal{M}$ of type II_1 is not abelian, then $\mathcal{M}$ is (isomorphic to) the tensor product of the hyperfinite factors of type II_1 and itself. In this case, $\mathcal{M}$ is called a McDuff factor.

A von Neumann algebra $\mathcal{M}$ is said to have property T if there exists $\epsilon > 0$, $X_1, \ldots, X_n$ in $\mathcal{M}$ such that for any $\mathcal{M} - \mathcal{M}$ bimodule $\mathcal{H}$ and any vector ξ in $\mathcal{H}$, with $\|\xi\| = 1$ and $\|X_i\xi - \xi X_i\| < \epsilon$ for $i = 1, \ldots, n$, there exists a vector η in $\mathcal{H}$, $\eta \neq 0$ which is central: $X\eta = \eta X$ for all $X \in \mathcal{M}$. Recall the definition of Kazhdan's property T for group: a countable discrete group G has property T of Kazhdan if there exists an $\epsilon > 0$ and a compact subset K of G such that every unitary representation $\pi : G \mapsto \mathcal{B}(\mathcal{H})$ of G on a Hilbert space $\mathcal{H}$ having a non zero vector ξ in $\mathcal{H}$ with $\|\pi(g)\xi - \xi\| < \epsilon$ for all g in K also has a non zero invariant vector. In [CoJ85], Connes and Jones proved that a countable discrete group has property T of Kazhdan if and only if its corresponding group von Neumann algebra has property T. For example, the linear group $PSL_n(\mathbb{Z})$ of all $n \times n$ matrices with entries in $\mathbb{Z}$ with determinant one module $\{\pm I\}$ when $n \geq 4$ is even and $SL_n(\mathbb{Z})$ of all $n \times n$ matrices with entries in $\mathbb{Z}$ with determinant one when $n \geq 3$ is odd have property T and then group von Neumann algebras $\mathcal{L}_{PSL_n(\mathbb{Z})}$, $n \geq 4$ even and $\mathcal{L}_{SL_n(\mathbb{Z})}$, $n \geq 3$ odd, have property T.

Definition 11 *A factor $\mathcal{M}$ of type II_1 with the trace τ acting on the Hilbert space $L^2(\mathcal{M}, \tau)$ is Γ-thin if there are two subfactors $\mathcal{N}_1$, $\mathcal{N}_2$ with property Γ in $\mathcal{M}$ such that*

$$\mathcal{M} = \overline{sp}\mathcal{N}_1\mathcal{N}_2,$$

in the sense of weak-operator topology on $\mathcal{B}(L^2(\mathcal{M}, \tau))$. Similarly, one can define a series of "thin" factors. If $\mathcal{N}_1, \mathcal{N}_2$ are subfactors with property T, $\mathcal{M}$ then is called T-thin; if $\mathcal{N}_1$ is property Γ subfactor and $\mathcal{N}_2$ is property T subfactor, $\mathcal{M}$ is called Γ-T-thin.

If $\mathcal{N}_1, \mathcal{N}_2$ are replaced by hyperfinite von Neumann subalgebras of $\mathcal{M}$ in the definition above, the factor $\mathcal{M}$ is called thin factor as defined in [GePo98]. If one of $\mathcal{N}_1, \mathcal{N}_2$ is an abelian von Neumann algebra or a hyperfinite von Neumann subalgebras of $\mathcal{M}$, we have a.Γ-thin factors, h.Γ-thin factors etc.

Definition 12 *A type II_1 factor $\mathcal{M}$ with the trace τ acting on the Hilbert space $L^2(\mathcal{M}, \tau)$ is strongly Γ-thin if there are property Γ subfactors $\mathcal{N}_1, \mathcal{N}_2$ of $\mathcal{M}$ such that*

$$\overline{sp}\mathcal{N}_1\xi\mathcal{N}_2 = L^2(\mathcal{M}, \tau)$$

for every non zero vector ξ in $L^2(\mathcal{M}, \tau)$. If $\mathcal{N}_1, \mathcal{N}_2$ are property T subfactors, $\mathcal{M}$ then is called strongly T-thin; if $\mathcal{N}_1$ is property Γ subfactor and $\mathcal{N}_2$ is property T subfactor, $\mathcal{M}$ is called strongly Γ-T-thin factor.

If $\mathcal{N}_1, \mathcal{N}_2$ are replaced by hyperfinite von Neumann subalgebras of $\mathcal{M}$ in the definition above, the factor $\mathcal{M}$ is called strongly thin factor as defined in [GePo98]. If one of $\mathcal{N}_1, \mathcal{N}_2$ is an abelian von Neumann subalgebra or a hyperfinite von Neumann subalgebra, we have strongly a.Γ-thin factors, strongly h.Γ-thin factors.

Definition 13 *A factor $\mathcal{M}$ of type II_1 with the trace τ acting on the Hilbert space $L^2(\mathcal{M}, \tau)$ is m-weakly Γ-thin if there are two property Γ subfactors $\mathcal{N}_1$, $\mathcal{N}_2$ of $\mathcal{M}$ and vectors $\xi_1, \ldots, \xi_m$ in $L^2(\mathcal{M}, \tau)$ such that*

$$L^2(\mathcal{M}, \tau) = \overline{sp}\mathcal{N}_1\{\xi_1, \ldots, \xi_m\}\mathcal{N}_2.$$

If $\mathcal{N}_1, \mathcal{N}_2$ are property T subfactors, we say $\mathcal{M}$ is m-weakly T-thin; if $\mathcal{N}_1$ is property Γ subfactor and $\mathcal{N}_2$ is property T subfactor, we say $\mathcal{M}$ is m-weakly Γ-T-thin.

If $\mathcal{N}_1, \mathcal{N}_2$ are replaced by hyperfinite von Neumann subalgebras of $\mathcal{M}$ in the definition above, the factor $\mathcal{M}$ is called weakly thin factor as defined in [GePo98]. If one of $\mathcal{N}_1, \mathcal{N}_2$ are an abelian von Neumann subalgebra or a hyperfinite von Neumann subalgebra, we have n-weakly a.Γ-thin factors, n-weakly h.Γ-thin factors.

In the other words,

$$\text{"strongly } \Gamma\text{-thin} \Rightarrow \Gamma\text{-thin} \Rightarrow \text{weakly } \Gamma\text{-thin".}$$

Lemma 14 *Let $\mathcal{M}$ be a property Γ factor of type II_1 with the trace τ and P a non zero projection of $\mathcal{M}$. Then $P\mathcal{M}P$ has property Γ.*

Proof. By a result of Connes ([Con76], Theorem 2.1), $\mathcal{M}$ has property Γ if and only if the C* algebra $C^*(\mathcal{M}, \mathcal{M}')$ generated by $\mathcal{M}$ and $\mathcal{M}'$ in $L^2(\mathcal{M}, \tau)(= \mathcal{H})$ contains no nonzero compact operator; i.e. $C^*(\mathcal{M}, \mathcal{M}') \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Since P is a non zero projection in $\mathcal{M}$, we have

$$C^*(P\mathcal{M}P, \mathcal{M}'P) \cap \mathcal{K}(P\mathcal{H}) = \{0\}$$

and hence $P\mathcal{M}P$ has property Γ. ∎

Lemma 15 *a) Let $\mathcal{M}$ be a type II_1 factor and P a non zero projection in $\mathcal{M}$ with $\frac{1}{k} \leq \tau(P) \leq \frac{1}{k-1}$ for some positive integer k. If $\mathcal{M}$ is n-weakly Γ-thin, then $P\mathcal{M}P$ is nk^2-weakly Γ-thin; if $P\mathcal{M}P$ is n-weakly Γ-thin, then $\mathcal{M}$ is $4n$-weakly Γ-thin.*
a') Let $\mathcal{M}$ be a type II_1 factor and P a non zero projection in $\mathcal{M}$. Then $\mathcal{M}$ is strongly Γ-thin if and only if $P\mathcal{M}P$ is.
b) Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of type II_1 factors with $k-1 < [\mathcal{M}, \mathcal{N}] \leq k$ for some k. If $\mathcal{M}$ is n-weakly Γ-thin, then $\mathcal{N}$ is nk^2-weakly Γ-thin; if $\mathcal{N}$ is n-weakly Γ-thin, then $\mathcal{M}$ is $4n$-weakly Γ-thin.
c) $\mathcal{M} \otimes M_n(\mathbb{C})$ is (n-weakly, strongly) Γ-thin if $\mathcal{M}$ is.

Proof. a)We assume that $\mathcal{M}$ is n-weakly Γ-thin. Then there are vectors $\xi_1, \ldots, \xi_n$ in $L^2(\mathcal{M}, \tau)$ and property Γ subfactors $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{M}$ such that $L^2(\mathcal{M}, \tau) = \overline{sp}\mathcal{N}_1\{\xi_1, \ldots, \xi_n\}\mathcal{N}_2$. Up to unitary conjugations, we may assume that $P \in \mathcal{N}_1 \cap \mathcal{N}_2$. Because $\mathcal{M}$ is a factor of type II_1, there are unitary elements U, V in $\mathcal{M}$ such that UPU^* in $\mathcal{N}_1$ and VPV^* in $\mathcal{N}_2$. Then we may replace $\mathcal{N}_1$ by $U\mathcal{N}_1U^*$, $\mathcal{N}_2$ by $V\mathcal{N}_2V^*$ and ξ_j by $U\xi_jV^*$. Since $1/k \leq \tau(P)$, we can choose a matrix unit system $\{E_{jl}, j, l = 1, \ldots, k\}$ for some matrix subalgebra of $\mathcal{N}_1$

such that $E_{11} \leq P$. Similarly, we have $\{F_{jl}\}_{j,l=1}^{k}$ for $\mathcal{N}_2$ and $F_{11} \leq P$. Thus we have that

$$\begin{aligned} L^2(P\mathcal{M}P, \tau_P) &= \overline{sp} P\mathcal{N}_1\{\xi_1, \ldots, \xi_n\}\mathcal{N}_2 P \\ &= \overline{sp} P\mathcal{N}_1(\sum_j E_{jj})\{\xi_1, \ldots, \xi_n\}(\sum_j F_{jj})\mathcal{N}_2 P \\ &= \overline{sp} P\mathcal{N}_1(\sum_j E_{j1}E_{11}E_{1j})\{\xi_1, \ldots, \xi_n\}(\sum_j F_{j1}F_{11}F_{1j})\mathcal{N}_2 P \\ &= \overline{sp} P\mathcal{N}_1 E_{11}\{E_{1j}\xi_1 F_{l1}, \ldots, E_{1j}\xi_n F_{l1}, j, l = 1, \ldots, k\}F_{11}\mathcal{N}_2 P \\ &= \overline{sp} P\mathcal{N}_1 P\{E_{1j}\xi_1 F_{l1}, \ldots, E_{1j}\xi_n F_{l1}, j, l = 1, \ldots, k\}P\mathcal{N}_2 P, \end{aligned}$$

where $\tau_P = \tau/\tau(P)$. Since $P\mathcal{N}_1P$ and $P\mathcal{N}_2P$ are type II_1 factors with property Γ, we have that $P\mathcal{M}P$ is nk^2-weakly Γ-thin. If $P\mathcal{M}P$ is n-weakly Γ-thin, then we pick a subprojection E of P with trace $1/k$. Since $\tau(E)/\tau(P) \geq \frac{k-1}{k} > 1/2$ and the argument above can be applied to subfactor $E\mathcal{M}E$ of $P\mathcal{M}P$, $E\mathcal{M}E$ is $4n$-weakly Γ-thin. Let $E\mathcal{M}E = \overline{sp}\mathcal{N}_3\{\eta_1, \ldots, \eta_{4n}\}\mathcal{N}_4$ where $\mathcal{N}_3, \mathcal{N}_4$ are subfactors of $E\mathcal{M}E$ and $\eta_1, \ldots, \eta_{4n}$ are in $L^2(E\mathcal{M}E, \tau_E)$. Since E is a projection with trace $1/k$ in $\mathcal{M}$, we know that $\mathcal{M} \simeq M_k(\mathbb{C}) \otimes E\mathcal{M}E$. Then

$$\begin{aligned} L^2(\mathcal{M}, \tau) &= L^2(M_k(\mathbb{C}) \otimes E\mathcal{M}E, \tau) \\ &= \overline{sp} M_k(\mathbb{C}) \otimes \mathcal{N}_3\{1 \otimes \eta_1, \ldots, 1 \otimes \eta_{4n}\}M_k(\mathbb{C}) \otimes \mathcal{N}_4 \end{aligned}$$

where 1 is the identity of $M_k(\mathbb{C})$. By [SM08], Theorem 13.4.5, we know that $M_k(\mathbb{C}) \otimes \mathcal{N}_3$ and $M_k(\mathbb{C}) \otimes \mathcal{N}_4$ have property Γ. Hence $\mathcal{M}$ is $4n$-weakly Γ-thin.

a') follows from a).

b) We assume that $\mathcal{M}$ is n-weakly Γ-thin. Then there are vectors $\xi_1, \ldots, \xi_n$ in $L^2(\mathcal{M}, \tau)$ and property Γ subfactors $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{M}$ such that $L^2(\mathcal{M}, \tau) = \overline{sp}\mathcal{N}_1\{\xi_1, \ldots, \xi_n\}\mathcal{N}_2$. Suppose $E_{\mathcal{N}}$ is the projection from $L^2(\mathcal{M}, \tau)$ onto $L^2(\mathcal{N}, \tau)$. Let P be a projection in $\mathcal{M}$ such that there exists unitary element W in $\langle \mathcal{M}, \mathcal{N} \rangle$ on $L^2(\mathcal{M}, \tau)$ with $WPW^* = E_{\mathcal{N}}$ and $\tau(P) = [\mathcal{M} : \mathcal{N}]^{-1} = \tau(E_{\mathcal{N}})$, where τ is the normalized trace on $\langle \mathcal{M}, \mathcal{N} \rangle$ extending the trace τ on $\mathcal{M}$. Up to unitary conjugations, we may assume that $P \in \mathcal{N}_1 \cap \mathcal{N}_2$. Because $\mathcal{M}$ is a factor of type II_1, there are unitary elements U, V in $\mathcal{M}$ such that UPU^* in $\mathcal{N}_1$ and VPV^* in $\mathcal{N}_2$. Then we may replace $\mathcal{N}_1$ by $U\mathcal{N}_1U^*$, $\mathcal{N}_2$ by $V\mathcal{N}_2V^*$ and ξ_j by $U\xi_jV^*$. Since $1/k \leq \tau(P)$, we can choose a matrix unit system $\{E_{jl}, j, l = 1, \ldots, k\}$ for some matrix subalgebra of $\mathcal{N}_1$ such that $E_{11} \leq P$. Similarly, we have $\{F_{jl}\}_{j,l=1}^k$ for $\mathcal{N}_2$ and $F_{11} \leq P$. Thus we have that

$$\begin{aligned}
L^2(P\mathcal{M}P, \tau_P) &= \overline{sp} P\mathcal{N}_1\{\xi_1, \ldots, \xi_n\}\mathcal{N}_2 P \\
&= \overline{sp} P\mathcal{N}_1(\sum_j E_{jj})\{\xi_1, \ldots, \xi_n\}(\sum_j F_{jj})\mathcal{N}_2 P \\
&= \overline{sp} P\mathcal{N}_1(\sum_j E_{j1}E_{11}E_{1j})\{\xi_1, \ldots, \xi_n\}(\sum_j F_{j1}F_{11}F_{1j})\mathcal{N}_2 P \\
&= \overline{sp} P\mathcal{N}_1 E_{11}\{E_{1j}\xi_1 F_{l1}, \ldots, E_{1j}\xi_n F_{l1}, j, l = 1, \ldots, k\}F_{11}\mathcal{N}_2 P \\
&= \overline{sp} P\mathcal{N}_1 P\{E_{1j}\xi_1 F_{l1}, \ldots, E_{1j}\xi_n F_{l1}, j, l = 1, \ldots, k\}P\mathcal{N}_2 P.
\end{aligned}$$

Since $P\mathcal{N}_1P$ and $P\mathcal{N}_2P$ are type II_1 factors with property Γ, we have that $P\mathcal{M}P$ is nk^2-weakly Γ-thin. Then $WP\mathcal{M}PW^*$ is nk^2-weakly Γ-thin. Since

$$WP\mathcal{M}PW^* = WPW^*W\mathcal{M}W^*WPW^* = E_\mathcal{N}W\mathcal{M}W^*E_\mathcal{N} \subset \mathcal{N}E_\mathcal{N},$$

$WP\mathcal{M}PW^*$ is a subfactor of $\mathcal{N}E_\mathcal{N}$. But $L^2(WP\mathcal{M}PW^*) = WL^2(P\mathcal{M}P) = E_\mathcal{N}L^2(\mathcal{M}) = L^2(E_\mathcal{N}\mathcal{M}E_\mathcal{N}) = L^2(\mathcal{N}E_\mathcal{N})$, and we obtain that $WP\mathcal{M}PW^* = \mathcal{N}E_\mathcal{N}$. Therefore $\mathcal{N}E_\mathcal{N}$ is also nk^2-weakly Γ-thin. Since $\mathcal{N}E_\mathcal{N}$ acting on $L^2(\mathcal{N}E_\mathcal{N})$ is unitarily equivalent to $\mathcal{N}$ acting on $L^2(\mathcal{N})$, $\mathcal{N}$ is nk^2-weakly Γ-thin. If $\mathcal{N}$ is n-weakly Γ-thin, $\mathcal{N}E_\mathcal{N}$ is n-weakly Γ-thin and $P\mathcal{M}P$ is n-weakly Γ-thin, then we pick a subprojection E of P with trace $1/k$. Since $\tau(E)/\tau(P) \geq \frac{k-1}{k} > 1/2$ and the argument above can be applied to subfactor $E\mathcal{M}E$ of $P\mathcal{M}P$, $E\mathcal{M}E$ is $4n$-weakly Γ-thin. Since E is a projection with trace $1/k$ in $\mathcal{M}$, we know that $\mathcal{M} \simeq M_k(\mathbb{C}) \otimes E\mathcal{M}E$. Hence $\mathcal{M}$ is $4n$-weakly Γ-thin.

c) We assume that $\mathcal{M}$ is n-weakly Γ-thin. Then there are vectors $\eta_1, \ldots, \eta_n$ in $L^2(\mathcal{M}, \tau)$ and property Γ subfactors $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{M}$ such that $L^2(\mathcal{M}, \tau) = \overline{sp}\mathcal{N}_1\{\eta_1, \ldots, \eta_n\}\mathcal{N}_2$.

$$L^2(\mathcal{M} \otimes M_k(\mathbb{C}), \tau) = \overline{sp}\mathcal{N}_1 \otimes M_k(\mathbb{C})\{\eta_1 \otimes 1, \ldots, \eta_n \otimes 1\}\mathcal{N}_2 \otimes M_k(\mathbb{C})$$

where 1 is the identity of $M_k(\mathbb{C})$. By [SM08], Theorem 13.4.5, we know that $\mathcal{N}_1 \otimes M_k(\mathbb{C})$ and $\mathcal{N}_2 \otimes M_k(\mathbb{C})$ have property Γ. Hence $\mathcal{M} \otimes M_k(\mathbb{C})$ is n-weakly Γ-thin. ■

2.1 Γ-Thin

We begin with the simplest decomposition "a.a.-thin". The hyperfinite factor $\mathcal{R}$ of type II_1 is a.a.-thin. To see this, given an irrational number θ, suppose $\mathcal{A}_\theta$ is the reduced C* algebra generated by two unitary operators, U and V, satisfying the twisted commutation relation $UV = \exp(2\pi i\theta)VU$ with the trace τ given by $\tau(\sum_{\iota,j}\lambda_{\iota,j}U^\iota V^j) = \lambda_{0,0}$, where $\lambda_{\iota,j} \in \mathbb{C}$, $\sum_{\iota,j}\lambda_{\iota,j}U^\iota V^j$ is in $\mathcal{A}_\theta$. Let $(\mathcal{H}_\tau, \pi_\tau, \xi_\tau)$ be the triple from the GNS construction induced by τ. Then the weak-operator closure of the representation π_τ of $\mathcal{A}_\theta$ induced by the trace τ is the hyperfinite factor $\mathcal{R}$ of type II_1. Let $\mathcal{A}_U$ be the abelian von Neumann subalgebra generated by $\pi_\tau(U)$, $\mathcal{A}_V$ the abelian von Neumann subalgebra generated by $\pi_\tau(V)$. We obtain that $\mathcal{R} = \overline{sp}\mathcal{A}_U\mathcal{A}_V$.

If factors of type I are considered in this decomposition, we have that all (weakly) separable factors of type I are a.a.-"thin".

Suppose $\mathcal{H}$ is a n-dimensional Hilbert space with an orthogonal normal basis $\xi_1, \xi_2, \ldots, \xi_n$, where $n \in \mathbb{N}$. Define unitary operators U and V on $\mathcal{H}$ such that $U\xi_j = e^{2\pi\iota j/n}\xi_j$ for $j = 1, \ldots, n$ and $V\xi_k = \xi_{k+1}$ for $k = 1, \ldots, n-1$, $V\xi_n = \xi_1$. Let $\{E_{jk}\}_{j,k=1}^n$ be a system of matrix units for $\mathcal{B}(\mathcal{H})$ such that $E_{jk}\xi_k = \xi_j$ for $j, k = 1, \ldots, n$. Since $\frac{1}{n}\sum_{k=0}^{n-1}(e^{-2\pi\iota d/n}U)^{k-1} = E_{d,d}$ for $d = 1, \ldots, n$, and $E_dV^{d-l} = E_{d,l}$ for $d, l = 1, \ldots, n$, then the algebra generated by U, U^*, V, V^* contains all matrix units $\{E_{jk}\}_{j,k}$ of $\mathcal{B}(\mathcal{H})$, and hence it is $\mathcal{B}(\mathcal{H})$ which is isomorphic to $M_n(\mathbb{C})$. Moreover, $UV = e^{2\pi\iota/n}VU$. Let $\mathcal{A}_U$ be the abelian von Neumann subalgebra generated by U, $\mathcal{A}_V$ the abelian von Neumann subalgebra generated by V. We obtain that $M_n(\mathbb{C}) = sp\mathcal{A}_U\mathcal{A}_V$.

Suppose $\mathcal{H}$ is a countably infinite dimensional Hilbert space with an orthogonal normal basis $\{\xi_j\}_{j\in\mathbb{Z}}$. Define unitary operators U and V on $\mathcal{H}$ such that $U\xi_j = e^{2\pi\iota j\theta}\xi_j$ for $j \in \mathbb{Z}$ and $V\xi_k = \xi_{k+1}$ for $k \in \mathbb{Z}$, where θ is an irrational number. Let $\{E_{jk}\}_{j,k\in\mathbb{Z}}$ be a system of matrix units of $\mathcal{B}(\mathcal{H})$. Since θ is an irrational number, $\{m\theta + n : m, n \in \mathbb{Z}\}$ is dense in the real line $\mathbb{R}$. Let $p \geq 2$ be a natural number. Then there exist sequences $\{m_k\}_k$ and $\{n_k\}_k$ of integers such that $\lim_k m_k\theta + n_k = \frac{1}{p}$. Therefore, $\lim_k^{SOT} U^{m_k} = U_p$, where U_p is a unitary operator on $\mathcal{H}$ such that $U_p\xi_j = e^{2\pi\iota j/p}\xi_j$ for j in $\mathbb{Z}$. Since $\frac{1}{p}\sum_{j=0}^{p-1} U_p^j = \sum_{j\in\mathbb{Z}} E_{pj,pj} (= E_p)$, we have that $E_{00} = \lim_p \prod_{j=2}^{p} E_j$ in strong-operator topology. Let $\mathcal{A}_U$ be the abelian von Neumann algebra generated by U, U^*, and $\mathcal{A}_V$ be the abelian von Neumann algebra generated by V, V^*. Thus E_{00} is in the von Neumann algebra $\mathcal{A}_U$ generated by U, U^*. If we replace U by $e^{-2\pi k\theta}U$ for $k \in \mathbb{Z}$, then we get that E_{kk} is in $\mathcal{A}_U$. Moreover, $E_{ll}V^{l-d} = E_{ld}$ for $l, d \in \mathbb{Z}$. Thus we have that U, V generate $\mathcal{B}(\mathcal{H})$ as a von Neumann algebra and $UV = e^{2\pi i\theta}VU$. Finally, we obtain that $\mathcal{B}(\mathcal{H}) = \overline{sp}\mathcal{A}_U\mathcal{A}_V$.

Now we state a theorem in [GePo98] proved by L. Ge and S. Popa to give an example of an a.Γ-thin factor. Let G be a discrete group with unit e and $\sigma : G \mapsto Aut(\mathcal{B})$ a group action of G on a von Neumann algebra $\mathcal{B}$. We say that σ acts ergodically on $\mathcal{M}$ if the following condition is satisfied: if $X \in \mathcal{M}$ and $U_gXU_g^* = X$ for each $g \in G$, then X is a scalar multiple of I; and that σ is properly outer when $\sigma_g(X)X_0 = X_0X$ for all X in $\mathcal{B}$ implies that $g = e$ or $X_0 = 0$. It is known that the properly outerness of σ is equivalent to the condition $\mathcal{B}' \cap (\mathcal{B} \rtimes_\sigma G) = \mathscr{C}(\mathcal{B})$, where $\mathscr{C}(\mathcal{B})$ is the center of $\mathcal{B}$.

Theorem 16 (See [GePo98]) *Let $\mathcal{B}$ be a finite von Neumann algebra with no atoms and with a faithful normal trace τ. Let G be a countable discrete group and σ a τ-preserving, properly outer action of G on $\mathcal{B}$. Denote by $\mathcal{M} = \mathcal{B} \rtimes_\sigma G$ the crossed product of $\mathcal{B}$ by σ. Then there exist an abelian subalgebra $\mathcal{A}$ of $\mathcal{B}$ and a unitary element $U \in \mathcal{M}$ such that $\mathcal{M} = \overline{sp}\mathcal{B}U\mathcal{A} = \overline{sp}\mathcal{B}U\mathcal{A}U^*$.*

Corollary 17 *Let $\mathcal{B}, \sigma, G$ be given as in Theorem 16. Assume that σ acts ergodically on the center of $\mathcal{B}$ and $\mathcal{B}$ is a property Γ or T factor. Then $\mathcal{M}$ is a.Γ-thin or a.T-thin.*

In theorem 16, if $\mathcal{B}$ is an abelian von Neumann algebra, then we have that $\mathcal{M} = \mathcal{B} \rtimes_\sigma G$ which is a.a.-thin. Let $\mathbb{Z}^2$ be the group $\{(m, n) : m, n \in \mathbb{Z}\}$ with addition $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. For any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} (= g)$ in $SL_2(\mathbb{Z})$, the action α of g on $\mathbb{Z}^2$ is given by $(m, n)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (am + cn, bm + dn)$, for any $(m, n) \in \mathbb{Z}^2$. The group action α acts ergodically on $\mathbb{Z}^2$. In fact, if $(m, n)g = (m, n)$ for any $g \in SL_2(\mathbb{Z})$, we see that $(m, n) = (0, 0)$. Then the crossed product $\mathcal{L}_{\mathbb{Z}^2} \rtimes_\alpha SL_2(\mathbb{Z})$ is a factor of type II_1

(See [KR] Chapter 8 for more details). But it is not the hyperfinite factor of type II_1 since it contains a free group subfactor $\mathcal{L}_{SL_2(\mathbb{Z})}$. The crossed product $\mathcal{L}_{\mathbb{Z}^2} \rtimes_\alpha SL_2(\mathbb{Z})$ is a.a.-thin by the corollary above.

In [CoJ85], Connes and Jones showed that a type II_1 factor with property T is not a subfactor of the free group factor $\mathcal{L}_{\mathcal{F}_n}$, where $n \geq 2$. This indicates that the free group factor is not a.T-thin, Γ-T-thin, T-thin.

If the conditions on the group action are removed, i.e. $\mathcal{M} \rtimes_\alpha G$ for any group action $\alpha : G \mapsto Aut(\mathcal{M})$, we have $\mathcal{M} \rtimes_\alpha G = \overline{sp}\mathcal{M}\mathcal{L}_G$. Therefore if $\mathcal{M}$ has property Γ and group G has property T, $\mathcal{M} \rtimes_\alpha G$ is Γ-T thin.

Any tensor product of two type II_1 factors is Γ-thin or McDuff-thin provided that we use McDuff factors as building blocks in the corresponding decompositions. That is, if $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$, where $\mathcal{M}_1$, $\mathcal{M}_2$ are factors of type II_1 and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be hyperfinite subfactors in $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, then

$$\mathcal{M} = \overline{sp}(\mathcal{M}_1 \overline{\otimes} \mathcal{R}_2)(\mathcal{R}_1 \overline{\otimes} \mathcal{M}_2).$$

Hyperfinite length $\ell_h(\mathcal{M}) = \min\{n|\text{there are hyperfinite subalgebras } \mathcal{R}_1, \ldots, \mathcal{R}_n \text{ of } \mathcal{M}$ such that $\overline{sp}\mathcal{R}_1 \cdots \mathcal{R}_n = \mathcal{M}\}$ for a given type II_1 factor $\mathcal{M}$ was defined in [GePo98] and they proved that property Γ factors have hyperfinite length ≤ 2 and any tensor product of two type II_1 factors has hyperfinite length ≤ 3. It has been proved in [GePo98] that a factor of type II_1 with property Γ is thin factor. We see that Γ-thin factors have hyperfinite length ≤ 4. Similarly, length $\ell_a(\mathcal{M}) = \min\{n|\text{there are abelian *-subalgebras } \mathcal{A}_1, \ldots, \mathcal{A}_n \text{ of } \mathcal{M}$ such that $\overline{sp}\mathcal{A}_1 \cdots \mathcal{A}_n = \mathcal{M}\}$ for a given type II_1 factor $\mathcal{M}$ could be defined. If factor $\mathcal{M}$ of type II_1 is Γ-thin, $\ell_a(\mathcal{M}) \leq 8$.

2.2 Strongly Γ-Thin

Proposition 18 *There is no strongly a.a.-thin factor of type II_1.*

Proof. Suppose $\mathcal{M}$ is a strongly a.a.-thin factor of type II_1 with the trace τ and $L^2(\mathcal{M}, \tau) = \overline{sp}\mathcal{A}_1\xi\mathcal{A}_2$ for all nonzero vector $\xi \in L^2(\mathcal{M}, \tau)$, where $\mathcal{A}_1$, $\mathcal{A}_2$ are MASAs in $\mathcal{M}$. Let P be a projection in $\mathcal{A}_1$ such that $P \neq 0, I$ and Q a projection in $\mathcal{A}_2$ such that $Q \sim P(\mathcal{M})$. Then there is a unitary operator U in $\mathcal{M}$ such that $Q = UPU^*$. Since $L^2(\mathcal{M}, \tau) = \overline{sp}\mathcal{A}_1\xi\mathcal{A}_2$ for all nonzero vector $\xi \in L^2(\mathcal{M}, \tau)$, we have

$$L^2(\mathcal{M}, \tau) = \overline{sp}\mathcal{A}_1(\xi U)U^*\mathcal{A}_2 U$$

for all non zero vector $\xi \in L^2(\mathcal{M}, \tau)$. We note that $P \in \mathcal{A} \cap U^*\mathcal{A}_2 U$. Let ξ be $\hat{U}^*$ the vector in $L^2(\mathcal{M}, \tau)$ corresponding to a unitary operator U^*. Then $\overline{sp}\mathcal{A}_1\hat{I}U^*\mathcal{A}_2 U = L^2(\mathcal{M}, \tau)$, i.e. $\mathcal{A}_1 \vee U^*\mathcal{A}_2 U = \mathcal{M}$ and $\mathcal{A}_1' \cap U^*\mathcal{A}_2' U = \mathcal{M}'$, where $\mathcal{A} \vee \mathcal{B}$ means the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{B}$. Since $P \in \mathcal{A} \cap U^*\mathcal{A}_2 U$, we have $P \in \mathcal{A}' \cap U^*\mathcal{A}_2' U = \mathcal{M}'$ and P is in the center of $\mathcal{M}$. But $\mathcal{M}$ is a factor, so P must be 0 or I. This is a contradiction. Therefore there is no strongly a.a.-thin factor of type II_1. ■

All non prime factors of type II_1 are strongly Γ-thin. Suppose $\mathcal{M} = \mathcal{M}_1\overline{\otimes}\mathcal{M}_2$ is a non prime factor, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are factors of type II_1, and $\mathcal{R}_i$ is an irreducible hyperfinite subfactor in $\mathcal{M}_i$ for $i = 1, 2$ (See [SM08], Theorem 13.2.3). Then from $\mathcal{M}_1\overline{\otimes}\mathcal{R}_2 \cap \mathcal{R}_1\overline{\otimes}\mathcal{M}_2 = \mathcal{R}_1\overline{\otimes}\mathcal{R}_2$ and $(\mathcal{R}_1\overline{\otimes}\mathcal{R}_2)' \cap \mathcal{M}_1\overline{\otimes}\mathcal{M}_2 = \mathbb{C}I$, by [GePo98] Proposition 2.2, we get that $\mathcal{M}$ is strongly Γ-thin. For convenience, we quote the proposition as follows:

Proposition 19 (See [GePo98],Proposition 2.2) *Assume that $\mathcal{N}_0$ and $\mathcal{N}_1$ are subfactors of a type II_1 factor $\mathcal{M}$ such that $\overline{sp}\mathcal{N}_0\mathcal{N}_1 = \mathcal{M}$ and $(\mathcal{N}_0 \cap \mathcal{N}_1)' \cap \mathcal{M} = \mathbb{C}$. Then $\overline{sp}\mathcal{N}_0\xi\mathcal{N}_1 = L^2(\mathcal{M}, \tau)$, for any non zero ξ in $L^2(\mathcal{M}, \tau)$. Equivalently, $\mathcal{N}_0 \vee J\mathcal{N}_1 J = \mathcal{B}(L^2(\mathcal{M}, \tau))$, or also, $\mathcal{N}_0' \cap \langle \mathcal{M}, \mathcal{N}_1 \rangle = \mathbb{C}I$.*

In [GePo98], S. Popa and L. Ge formed a strongly thin factor by using symmetric enveloping type II_1 factor. Now we shall use a similar process to form a strongly Γ-thin factor.

Let $\mathcal{Q} \subset \mathcal{P}$ be an inclusion of factors of type II_1 with Jones index $[\mathcal{P} : \mathcal{Q}] < \infty$. Let τ be the trace on $\mathcal{M}$. Assume that the inclusion $\mathcal{Q} \subset \mathcal{P}$ is extremal, i.e. $[P\mathcal{P}P, \mathcal{Q}P] = \tau(P)^2[\mathcal{P} : \mathcal{Q}]$ for any projection $P \in \mathcal{Q}' \cap \mathcal{P}$. Let $e_{\mathcal{Q}}$ denote the Jones's projection for $\mathcal{Q} \subset \mathcal{P}$ and $\mathcal{P} \boxtimes_{e_{\mathcal{Q}}} \mathcal{P}^{op}$ be the symmetric enveloping type II_1 factor associated with $\mathcal{Q} \subset \mathcal{P}$ (See [GePo98, Po99] for more details). We describe $\mathcal{P} \boxtimes_{e_{\mathcal{Q}}} \mathcal{P}^{op}$ as follows.

If $\mathcal{S}_0 = C^*(\mathcal{P}, e_Q, J\mathcal{P}J)$ is the C* algebra generated by $\mathcal{P}$, e_Q and $J\mathcal{P}J$ on $L^2(\mathcal{P}, \tau)$, then $\mathcal{S}_0$ has a unique positive normalized trace, denoted by τ once again. $\mathcal{S}_0$ can be generated by its * algebra $\bigcup_n (J\mathcal{P}J)\mathcal{P}_n(J\mathcal{P}J)$, where $\{\mathcal{P}_n\}_{n\geq 1}$ is the Jones tower for $Q \subset \mathcal{P}$ in the representation on $L^2(\mathcal{P}, \tau)$ given by some choice of the tunnel $\mathcal{P} \supset Q \supset Q_1 \supset \cdots$, i.e. $\mathcal{P}_n$ is, by definition, equal to $(JQ_{n-1}J)'$, $n \geq 1$. One then defines $\mathcal{P} \boxtimes_{e_Q} \mathcal{P}^{op}$ to be the type II_1 factor $\{\pi_\tau(\mathcal{S}_0)\}''(= \mathcal{S})$, where π_τ is the GNS representation for $(\mathcal{S}_0, \tau)$. We identify $\mathcal{P}, \mathcal{P}_n$, and e_Q with their images via π_τ and denote by ^{op} the anti-automorphism, implemented by $X \mapsto JX^*J$ on $L^2(\mathcal{P}, \tau)$. Then $\mathcal{P}' \cap \mathcal{S} = \mathcal{P}^{op}$, $(\mathcal{P}^{op})' \cap \mathcal{S} = \mathcal{P}$ and more generally $\mathcal{P}'_n \cap \mathcal{S} = Q^{op}_{n-1}$, $(Q^{op}_{n-1})' \cap \mathcal{S} = \mathcal{P}_n$. Moreover, denote $\mathcal{R}^{st}_m = (\bigcup_n (Q'_n \cap \mathcal{P}_m))^-$, the weak-operator closure of $\bigcup_n (Q'_n \cap \mathcal{P}_m)$ for $m = 0, 1, 2, \ldots$ where $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{R}^{st}_0 = \mathcal{R}^{st}$, and denote $\mathcal{P}_\infty = (\bigcup_n \mathcal{P}_n)^-(\subset \mathcal{S})$. Then we have $(\mathcal{R}^{st})^{op} \subset \mathcal{P}_\infty$, $\mathcal{P} = sp\mathcal{R}^{st}Q_n$ and $sp\mathcal{P}_n(\mathcal{R}^{st})^{op} \subset \mathcal{P}_\infty$, for each n. So we have $\bigcup_n sp(\mathcal{P}^{op}\mathcal{P}_n\mathcal{P}^{op}) \subset sp\mathcal{P}^{op}\mathcal{P}_\infty$. Thus, $\mathcal{S} = \overline{sp}\mathcal{P}^{op}\mathcal{P}_\infty$. If Q has property Γ and $[\mathcal{P} : Q] < \infty$, $\mathcal{P}$ has property Γ by [PoPi], and $\mathcal{P}_\infty$ has property Γ from the definition of property Γ von Neumann algebra, then $\mathcal{P} \boxtimes_{e_Q} \mathcal{P}^{op}$ is Γ-thin. Finally, by [GePo98], Proposition 2.2 and [Po99], one obtains that $\mathcal{P} \boxtimes_{e_Q} \mathcal{P}^{op}$ is strongly Γ-thin.

2.3 Weakly Γ-Thin

Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion of type II_1 factors. Denote by

$$qN_{\mathcal{M}}(\mathcal{N}) = \{X \in \mathcal{M} | \exists X_1, \ldots, X_n \in \mathcal{M} \text{ such that } X\mathcal{N} \subset \sum_{i=1}^{n} \mathcal{N}X_i \text{ and } \mathcal{N}X \subset \sum_{i=1}^{n} X_i\mathcal{N}\}.$$

We call $qN_{\mathcal{M}}(\mathcal{N})$ the quasi-normalizer of $\mathcal{N}$ in $\mathcal{M}$. $\mathcal{N}$ is said to be quasi-regular in $\mathcal{M}$, if $qN_{\mathcal{M}}(\mathcal{N})'' = \mathcal{M}$.

Now we state a proposition from [GePo98] to show an example of a weakly a.Γ-thin factor.

Proposition 20 (See [GePo98]) *Assume that $\mathcal{N} \subset \mathcal{M}$ is an irreducible inclusion of type II_1 factors with $\mathcal{N}$ quasi-regular in $\mathcal{M}$. Then there are an abelian subalgebra $\mathcal{A}$ in $\mathcal{N}$ and a vector ξ in $L^2(\mathcal{M}, \tau)$ such that $\overline{sp}\mathcal{A}\xi\mathcal{N} = L^2(\mathcal{M}, \tau)$.*

Corollary 21 *Let $\mathcal{N}$ be given as in Proposition 20. Assume that $\mathcal{N}$ has property Γ or T etc. Then $\mathcal{M}$ is weakly a.Γ-thin or weakly a.T-thin etc.*

In [Po99], S. Popa showed that if $\mathcal{N} \subset \mathcal{M}$ is an extremal inclusion of type II_1 factors, then $\mathcal{M} \vee \mathcal{M}^{op}$ is quasi-regular in symmetric enveloping type II_1 factor $\mathcal{M} \boxtimes_{e_{\mathcal{N}}} \mathcal{M}^{op}$. This is to say $\mathcal{M} \boxtimes_{e_{\mathcal{N}}} \mathcal{M}^{op}$ is weakly a.Γ-thin if $\mathcal{N}$ has property Γ.

2.4 Singly Generated

In [GePo98], L. Ge and S. Popa pointed out that many factors of type II_1 are singly generated such as property Γ factors, strongly thin factors, non prime factors, and n-weakly thin factors etc. With new definitions given in the chapter, we could add some more singly generated factors as follows:

Theorem 22 *Suppose $\mathcal{M}$ is a factor of type II_1 satisfying one of the following properties:*
a) $\mathcal{M}$ has a quasi-regular subalgebra $\mathcal{B} \subset \mathcal{M}$ with property Γ with $\mathcal{B}' \cap \mathcal{M} \subset \mathcal{B}$;
b) $\mathcal{M}$ is strongly Γ-thin.
Then $\mathcal{M}$ is singly generated.

Proof. a) If $\mathcal{B}$ is quasi-regular in $\mathcal{M}$ then $P\mathcal{B}P$ is quasi-regular in $P\mathcal{M}P$ for any projection $P \in \mathcal{B}$. Also, $(P\mathcal{B}P)' \cap P\mathcal{M}P \subset P\mathcal{B}P$. $P\mathcal{M}P$ is a.Γ weakly thin by Corollary 21, in particular it is generated by 5 self-adjoint elements. Taking P of trace $\frac{1}{5}$, it follows that $\mathcal{M}$ can be generated by two self-adjoint elements by [GePo98], Lemma 6.3.
b) By Lemma 15, if $\mathcal{M}$ is strongly Γ-thin then $P\mathcal{M}P$ is strongly Γ-thin for any non zero projection $P \in \mathcal{M}$ and then [GePo98] Lemma 6.3 applies. ■

2.5 Cohomology

In [GePo98], S. Popa and L. Ge claimed that if a type II_1 factor $\mathcal{M}$ is n-weakly a.h.-thin for some $n \in \mathbb{N}$, then $H^2(\mathcal{M}, \mathcal{M}) = 0$. Here we fill the details of the proof.

Let $\mathfrak{A}$ be a C* algebra acting on a Hilbert space $\mathcal{H}$ and $\mathcal{V}$ a two-sided $\mathfrak{A}$-bimodule $\mathfrak{A}$ or $\mathcal{B}(\mathcal{H})$. For any $n \geq 1$, $\mathfrak{A}^n$ will denote the n-fold Cartesian product of copies of $\mathfrak{A}$. The space of bounded n-linear maps $\phi : \mathfrak{A}^n \mapsto \mathcal{V}$ will be denoted by $\mathcal{L}^n(\mathfrak{A}, \mathcal{V})$. For $n = 0$, we let $\mathcal{L}^0$ be $\mathcal{V}$. The coboundary map $\partial : \mathcal{L}^n(\mathfrak{A}, \mathcal{V}) \mapsto \mathcal{L}^{n+1}(\mathfrak{A}, \mathcal{V})$ is defined as follows. For $n = 0$, ∂V is the derivation $X \mapsto XV - VX$, $X \in \mathfrak{A}$. When $n \geq 1$ and $\phi \in \mathcal{L}^n(\mathfrak{A}, \mathcal{V})$, $\partial\phi \in \mathcal{L}^{n+1}(\mathfrak{A}, \mathcal{V})$ is defined by

$$\begin{aligned}\partial\phi(X_1, \ldots, X_{n+1}) &= X_1\phi(X_2, \ldots, X_n) \\ &+\sum_{i=1}^{n-1}(-1)^i\phi(X_1, \ldots, X_{i-1}, X_iX_{i+1}, X_{i+2}, \ldots, X_n) \\ &+(-1)^n\phi(X_1, \ldots, X_n)X_{n+1}\end{aligned}$$

for $X_i \in \mathfrak{A}, 1 \leq i \leq n+1$. It is known that $\partial\partial = 0$. Thus the image of $\partial : \mathcal{L}^{n-1}(\mathfrak{A}, \mathcal{V}) \mapsto \mathcal{L}^n(\mathfrak{A}, \mathcal{V})$, denoted by $Im\partial$, is contained in the kernel of $\partial : \mathcal{L}^n(\mathfrak{A}, \mathcal{V}) \mapsto \mathcal{L}^{n+1}(\mathfrak{A}, \mathcal{V})$, denoted by $Ker\partial$. Then the n-th Hochschild cohomology group $H^n(\mathfrak{A}, \mathcal{V})$ is the quotient of the two vector spaces, i.e. $H^n(\mathfrak{A}, \mathcal{V}) = Ker\partial/Im\partial$.

Theorem 23 (See also [CPSS97]) *Suppose $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a factor of type II_1, $\mathcal{A}$ is a subalgebra of $\mathcal{M}$. $\mathcal{B}$ is a fixed abelian C* subalgebra of $\mathcal{M}'$. Let $\phi : \mathcal{M} \mapsto \mathcal{B}'$ be a bounded $\mathcal{A}$-bimodule map. Then ϕ has a norm preserving extension to $C^*(\mathcal{A}, \mathcal{B})$-bimodule map $\psi : C^*(\mathcal{M}, \mathcal{B}) \mapsto C^*(\mathcal{B}', \mathcal{B})$.*

Proof. Since $\mathcal{M}$ is a factor, the multiplication map $m \otimes m' \mapsto mm'$ on the algebraic tensor product $\mathcal{M} \odot \mathcal{M}'$ is a monomorphism. This allows us to define a C* norm on $\mathcal{M} \odot \mathcal{M}'$ by

$$\|\sum_i m_i \otimes m_i'\|_1 = \|\sum_i m_i m_i'\|.$$

Denote by $\mathcal{M} \otimes_1 \mathcal{M}'$ the completion of $\mathcal{M} \odot \mathcal{M}'$ with respect to norm $\|\cdot\|_1$. There is a unique C* norm on the tensor product of two C* algebras whenever one is abelian and so the restriction of $\|\cdot\|_1$ to $\mathcal{M} \otimes_1 \mathcal{B}$ must equal to the spatial C* norm $\|\cdot\|_{\min}$. Therefore the multiplication map $\rho : \mathcal{M} \odot \mathcal{B} \mapsto C^*(\mathcal{M}, \mathcal{B})$ given by $\rho(m \otimes b) = mb$ extends to an isometric isomorphism between $\mathcal{M} \otimes_{\min} \mathcal{B}$ and $C^*(\mathcal{M}, \mathcal{B})$.

Let Ω be the maximal ideal space of $\mathcal{B}$. Then $\mathcal{B}$ and $C(\Omega)$ are isomorphic, and we regard an element $b \in \mathcal{B}$ as a continuous function $b(\omega)$ on Ω. Then, for any C* algebra $\mathcal{D}$, $\mathcal{D} \otimes_{\min} \mathcal{B}$ may be identified with the algebra of $\mathcal{D}$-valued continuous functions on Ω. Replacing $\mathcal{D}$ by $\mathcal{M}$ and $\mathcal{B}'$, we obtain

$$\begin{aligned} \|\sum_i \phi(m_i) \otimes b_i\|_{min} &= \sup_{\omega\in\Omega} \|\sum_i \phi(m_i) b_i\| \\ &= \sup_{\omega\in\Omega} \|\phi(\sum_i m_i b_i(\omega))\| \leq \|\phi\| \sup_{\omega\in\Omega} \|\sum_i m_i b_i\| \\ &= \|\phi\| \|\sum_i m_i \otimes b_i\|_{min}, \end{aligned}$$

for $m_i \in \mathcal{M}, b_i \in \mathcal{B}$. Thus there is a bounded map $\phi \otimes I : \mathcal{M} \otimes_{\min} \mathcal{B} \mapsto \mathcal{B}' \otimes_{\min} \mathcal{B}$ defined on elementary tensors by

$$(\phi \otimes I)(m \otimes b) = \phi(m) \otimes b, \quad m \in \mathcal{M}, b \in \mathcal{B},$$

and $\|\phi \otimes I\| \leq \|\phi\|$. Since $\mathcal{B}$ is an abelian C* subalgebra, we can define an isometric, $\pi : \mathcal{B}' \otimes \mathcal{B} \mapsto C^*(\mathcal{B}', \mathcal{B})$, by $\pi(b' \otimes b) = b'b$. Then we obtain

$$C^*(\mathcal{M}, \mathcal{B}) \xrightarrow{\rho^{-1}} \mathcal{M} \otimes_{min} \mathcal{B} \xrightarrow{\phi\otimes I} \mathcal{B}' \otimes_{min} \mathcal{B} \xrightarrow{\pi} \mathcal{B}'$$

Define $\psi = \rho^{-1} \circ (\phi \otimes I) \circ \pi$. Then $\psi(m) = \phi(m)$ for all $m \in \mathcal{M}$ and $\psi(mb) = \pi(\phi(m) \otimes b) = \phi(m)b = \psi(m)b$ for all $m \in \mathcal{M}$ and $b \in \mathcal{B}$. Furthermore, for $a_1, a_2 \in \mathcal{A}$, $b, b_1, b_2 \in \mathcal{B}$, and $m \in \mathcal{M}$,

$$\begin{aligned} \psi(a_1 b_1 (mb) a_2 b_2) &= \psi(a_1 m a_2 b_1 b b_2) = \phi(a_1 m a_2) b_1 b b_2 \\ &= a_1 \phi(m) a_2 b_1 b b_2 = a_1 b_1 \phi(m) b a_2 b_2 \\ &= a_1 b_1 \psi(mb) a_2 b_2. \end{aligned}$$

Thus ψ is a C*$(\mathcal{A}, \mathcal{B})$-bimodule map. ■

Theorem 24 (See [SiSm98]) *Suppose $\mathcal{E} \subset \mathcal{B}(\mathcal{H})$ is a C* algebra, $\mathcal{A} \subset \mathcal{E}$ is C* subalgebra with cyclic vector ξ, and $\mathcal{A}$-module map $\phi : \mathcal{E} \mapsto \mathcal{B}(\mathcal{H})$ is bounded. Then ϕ is completely bounded and $\|\phi\|_{cb} = \|\phi\|$.*

Proof. Without loss of generality, we assume that $\|\phi\| = 1$ and assume that for some $n \in \mathbb{N}$, the norm of $\phi_n : M_n(\mathcal{E}) \mapsto M_n(\mathcal{B}(\mathcal{H}))$ exceeds one. Then there exists an element $(E_{ij}) \in M_n(\mathcal{E})$ of unit norm such that $\|\phi(E_{ij})\| > 1$. Then vectors $\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$ maybe chosen from the unit ball of $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ such that $|\langle \phi(E_{ij}) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \rangle| > 1$. Since $\mathcal{A}$ has cyclic vector, we may choose elements $a_i, b_i \in \mathcal{A}$ such that $\|a_i\xi - \xi_i\|$ and $\|b_i\xi - \eta_i\|$ are so small that $\|\begin{pmatrix} a_1\xi \\ \vdots \\ a_n\xi \end{pmatrix}\|, \|\begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{pmatrix}\| < 1$ and $|\langle \phi(E_{ij}) \begin{pmatrix} a_1\xi \\ \vdots \\ a_n\xi \end{pmatrix}, \begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{pmatrix} \rangle| > 1$. We shall assume temporarily that $a = \sum_i a_i^* a_i$ and $b = \sum_i b_i^* b_i$ are invertible elements, and remove this restriction at the end of the proof.

Let $\tilde{\eta} = a^{1/2}\xi$, $\tilde{\xi} = b^{1/2}\xi$, $c_i = a_i a^{-1/2}$ and $d_i = b_i b^{-1/2}$. Then $c_i\tilde{\eta} = a_i\xi$ and $d_i\tilde{\xi} = b_i\xi$ and $|\langle \sum_{ij} \phi(c_i^* E_{ij} d_j)\tilde{\xi}, \tilde{\eta}\rangle| > 1$ by using the module properties of ϕ. Now, $\|\tilde{\xi}\|^2 = \langle b^{1/2}\xi, b^{1/2}\xi\rangle = \|\begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{pmatrix}\| < 1$ and $\sum_{ij} c_i^* E_{ij} d_j$ may be expressed as

$$(c_1^* \cdots c_n^*)\,(E_{ij}) \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

which has norm at most one. It follows that $\|\phi\| > 1$ and the desired contradiction is reached.

A modification is necessary if either $\sum_i a_i^* a_i$ or $\sum_i b_i^* b_i$ fails to be invertible. We replace $(E_{ij}) \in M_n(\mathcal{E})$ by $(E_{ij}) \oplus 0 \in M_{n+1}(\mathcal{E})$ and vectors $\begin{pmatrix} a_1\xi \\ \vdots \\ a_n\xi \end{pmatrix}, \begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \end{pmatrix}$ by $\begin{pmatrix} a_1\xi \\ \vdots \\ a_n\xi \\ \epsilon\xi \end{pmatrix}, \begin{pmatrix} b_1\xi \\ \vdots \\ b_n\xi \\ \epsilon\xi \end{pmatrix}$

respectively for some sufficiently small $\epsilon > 0$. Note that the new vector will still have norms less than 1. The argument above can be applied again to complete the proof. ■

Corollary 25 *Suppose $\mathcal{M}$ is an n-weakly a.h.-thin factor of type II_1 with the trace τ and*

$$\overline{sp}\mathcal{A}_1\{\xi_1,\ldots,\xi_n\}\mathcal{R}_2 = L^2(\mathcal{M},\tau),$$

where $\xi_1,\ldots,\xi_n \in L^2(\mathcal{M},\tau)$, $\mathcal{A}_1$ is an abelian von Neumann subalgebra of $\mathcal{M}$ and $\mathcal{R}_2$ is a hyperfinite von Neumann subalgebras= of $\mathcal{M}$. Let J be the canonical conjugation of $\mathcal{M}$ on $L^2(\mathcal{M},\tau)$ and $\mathcal{B} = J\mathcal{A}_1 J$. Then every bounded $\mathcal{R}_2$-bimodule map $\phi : \mathcal{M} \mapsto \mathcal{B}'$ is completely bounded.

Proof. Let $\mathcal{D} = \mathcal{B} \otimes \mathbb{C}I_n$ where I_n is the identity of $M_n(\mathbb{C})$. Then $\phi_n : \mathcal{M} \otimes M_n(\mathbb{C}) \mapsto \mathcal{B}' \otimes M_n(\mathbb{C}) = \mathcal{D}'$ is a $\mathcal{R}_2 \otimes M_n(\mathbb{C})$-bimodule map. By Theorem 23, there is a bounded $C^*(\mathcal{R}_1 \otimes M_n(\mathbb{C}),\mathcal{D})$-bimodule map $\psi : C^*(\mathcal{M} \otimes M_n(\mathbb{C}),\mathcal{D}) \mapsto C^*(\mathcal{D}',\mathcal{D})$ and $\|\psi\| = \|\phi_n\|$. Since $\overline{\text{sp}}\mathcal{A}_1\{\xi_1,\ldots,\xi_n\}\mathcal{R}_2 = L^2(\mathcal{M},\tau)$, $C^*(\mathcal{R}_1 \otimes M_n(\mathbb{C}),\mathcal{D})$ has a cyclic vector $\begin{pmatrix}\xi_1\\ \vdots \\ \xi_n\end{pmatrix}$. By Theorem 24, ψ is completely bounded, therefore ϕ_n is completely bounded and hence ϕ is completely bounded. ■

Theorem 26 *Suppose $\mathcal{M}$ is an n-weakly a.h.-thin factor of type II_1 and $\overline{sp}\mathcal{A}\{\xi_1,\ldots,\xi_n\}\mathcal{R} = L^2(\mathcal{M},\tau)$ with $\xi_1,\ldots,\xi_n \in L^2(\mathcal{M},\tau)$. Then $H^2(\mathcal{M},\mathcal{M}) = 0$.*

Proof. Suppose $\theta : \mathcal{M} \times \mathcal{M} \mapsto \mathcal{M}$ is a 2-cocycle on $\mathcal{M}$, i.e. $\partial\theta = 0$. We shall construct a bounded map $\alpha : \mathcal{M} \mapsto \mathcal{M}$ such that $\theta = \partial\alpha$, showing that all such 2-cocycles are coboundaries. We may restrict attention to 2-cocycles which are $\mathcal{R}$-multimodular. Let J be the canonical conjugation of $\mathcal{M}$ and $\mathcal{B} = J\mathcal{A}J$. By [KR71], there is a bounded map $\phi : \mathcal{M} \mapsto \mathcal{B}'$ such that $\theta = \partial\phi$. By Corollary 25, ϕ is completely bounded, and since θ is a completely bounded 2-cocycle, there exists a completely bounded map $\alpha : \mathcal{M} \mapsto \mathcal{M}$ such that $\theta = \partial\alpha$. ■

We would like to point out that $H^3(\mathcal{M},\mathcal{M}) = 0$ holds for n-weakly a.h.-thin factor, $\mathcal{M}$ (More details see [CPSS97]).

CHAPTER 3

FREE ENTROPY

Free entropy was introduced by Voiculescu[Vo94] in the free probability theory in 1994. Due to its discovery, several longstanding problems in finite von Neumann algebras were answered. The free entropy is also a powerful tool for studying factors of type II_1. The purpose of this chapter is to borrow the idea of the free entropy to propose that there are factors of type II_1 which are not weakly Γ-thin, strongly Γ-thin, or Γ-thin etc.

3.1 Basic Notation

In this section, we shall recall some basic notations in the free probability theory.

Let $(\mathfrak{A}, \phi)$ be a C* algebra with a state ϕ. This pair $(\mathfrak{A}, \phi)$ is a so-called C* probability space. A family $\{\mathfrak{A}_\iota\}_{\iota\in I}$ of unital (*-)subalgebras of $\mathfrak{A}$ is called (*-)*free* if $\phi(a_1 a_2 \cdots a_n) = 0$ whenever $a_j \in \mathfrak{A}_{l_j}$, $l_1 \neq l_2 \neq \cdots, \neq l_n$ and $\phi(a_{l_j}) = 0$, $\forall j$. A family $\{S_\iota\}_{\iota\in I}$ of subsets of $(\mathfrak{A}, \phi)$ is free if the family $\{\mathfrak{A}_\iota\}$ of (*-)subalgebra is (*-)free, where $\mathfrak{A}_\iota$ is the (*-) algebra generated by S_ι.

Let $\mathbb{C}\langle X_i | i \in I\rangle$ be the noncommutative polynomial ring with an identity 1 and $(\mathfrak{A}, \phi)$ be as above. If $(A_\iota)_{\iota\in I}$ is a family of elements in $\mathfrak{A}$, then the *joint distribution* of $(A_\iota)_{\iota\in I}$ is $\mu : \mathbb{C}\langle X_i | i \in I\rangle \mapsto \mathbb{C}$ given by $\mu(P) = \phi(h(P))$, where $h : \mathbb{C}\langle X_i | i \in I\rangle \mapsto \mathfrak{A}$ is an algebraic unital homomorphism with $h(X_\iota) = A_\iota, \forall i \in I$, $P \in \mathbb{C}\langle X_i | i \in I\rangle$. In particular, when the cardinality of index set I is 1, the distribution of A in $\mathfrak{A}$ is $\mu : \mathbb{C}\langle X\rangle \mapsto \mathbb{C}$ given by

$\mu(P) = \phi(P(A))$, for any $P \in \mathbb{C}\langle X\rangle$.

As is well known, the Gaussian law plays a key role in the probability theory. In the free probability theory, the Gaussian law is replaced by the semicircle law. It can be described as the distribution $\gamma_{a,r} : \mathbb{C}\langle X\rangle \mapsto \mathbb{C}$ given by

$$\gamma_{a,r}(t^k) = \frac{2}{\pi r^2}\int_{a-r}^{a+r} t^k \sqrt{r^2 - (t-a)^2}dt.$$

A self-adjoint element $A \in \mathfrak{A}$ having semicircle law is called *semicircular element*. A unitary element U in $\mathfrak{A}$ is Haar unitary if $\phi(U^k) = 0, k \in \mathbb{Z}, k \neq 0$.

In order to discuss our work in chapter 4 better, here we would like to recall some concepts such as limit distribution, asymptotically free, and von Neumann algebra free product.

For each $n \in \mathbb{N}$, let $(T_i^{(n)})_{i\in I}$ be a family of noncommutative random variables in C* algebra $\mathfrak{A}_n$ with a state φ_n. Then the sequence of joint distributions μ_n of $(T_i^{(n)})_{i\in I}$ converges as $n \to \infty$ if there exists a distribution μ such that

$$\mu_n(P) \mapsto \mu(P), n \to \infty$$

for every $P \in \mathbb{C}\langle X_i | i \in I\rangle$. We call μ the limit distribution of the sequence and write $\mu_n \to \mu$.

Now, let $I = \cup_{j\in J} I_j$ be a partition of I. A sequence of families $(\{T_i^{(n)} | i \in I_j\})_{j\in J}$ of sets of noncommutative random variables is said to be asymptotically free as $n \to \infty$ if it has a limit distribution μ and if $\{X_i | i \in I_j\}_{j\in J}$ is a free family of sets of random variables in $(\mathbb{C}\langle X_i | i \in I\rangle, \mu)$.

Suppose $\mathcal{M}_1, \mathcal{M}_2$ are finite von Neumann algebras with faithful normal tracial states τ_1, τ_2 acting on the Hilbert spaces $L^2(\mathcal{M}_i, \tau_i)$ respectively . Let $\mathcal{H}_i = L^2(\mathcal{M}_i, \tau_i)$ and let ξ_i be a distinguished unit vector $\hat{I}$ in $\mathcal{H}_i$ corresponding to the identity I in $\mathcal{M}_i$ for $i = 1, 2$. Then their Hilbert space free product $(\mathcal{H}, 1)(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ is given by

$$\mathcal{H} = \mathbb{C}1 \oplus \bigoplus_{n\geq 1}(\bigoplus_{i_1\neq i_2 \quad \neq i_n} \overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n}),$$

where $\overset{\circ}{\mathcal{H}}_i = \mathcal{H}_i \ominus \mathbb{C}\xi_i$, is the orthocomplement of $\mathbb{C}\xi_i$ in $\mathcal{H}_i$, for $i = 1, 2$.

Denote

$$\mathcal{H}(i) = \mathbb{C}1 \oplus \bigoplus_{n\geq 1} \Big(\bigoplus_{\substack{i_1\neq i_2\cdots\neq i_n \\ i_1\neq i}} \overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n} \Big)$$

and define unitary operator $V_i : \mathcal{H}_i \otimes \mathcal{H}(i) \mapsto \mathcal{H}$, for $i = 1, 2$, as follows:

$$\begin{aligned} \xi_i \otimes 1 &\mapsto 1, \\ \overset{\circ}{\mathcal{H}}_i \otimes 1 &\mapsto \overset{\circ}{\mathcal{H}}_{i_n}, \\ \xi_i \otimes (\overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n}) &\mapsto \overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n}, \\ \overset{\circ}{\mathcal{H}}_i \otimes (\overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n}) &\mapsto \overset{\circ}{\mathcal{H}}_i \otimes \overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n} \end{aligned}$$

Let λ_i be the representation of $\mathcal{M}_i$ on $\mathcal{H}$ given by

$$\lambda_i(A) = V_i(A \otimes I_{\mathcal{H}(i)})V_i^*,$$

whenever $A \in \mathcal{M}_i$ for $i = 1, 2$. Then *the von Neumann algebra free product* $\mathcal{M}_1 * \mathcal{M}_2$ is

$$\{\lambda_1(A_1), \lambda_2(A_2) : A_i \in \mathcal{M}_i, i = 1, 2\}'' \subset \mathcal{B}(\mathcal{H})$$

whose trace $\tau = \tau_1 * \tau_2$ given by $\tau(A) = \langle A1, 1\rangle, \forall A \in \mathcal{M}_1 * \mathcal{M}_2$.

At the end of this section, we will state some lemmas which will be used to prove one of my work in the following section (Theorem 30). We omit its proofs and refer to [Ge97, Ge98] for complete analysis. To state lemmas, we need some more notations.

Let $\mathbb{C}\langle X_1, \ldots, X_t, X_1^*, \ldots, X_t^*\rangle$ be the noncommutative polynomial ring with involution $*$ satisfying $(X_{j_1} \cdots X_{j_g})^* = X_{j_g}^* \cdots X_{j_1}^*$. In the chapter, we will use $\mathbb{C}\langle X_1, \ldots, X_t\rangle$ to denote the $*$-ring $\mathbb{C}\langle X_1, \ldots, X_t, X_1^*, \ldots, X_t^*\rangle$ and write $\varphi(X_1, \ldots, X_t)$ instead of $\varphi(X_1, \ldots, X_t, X_1^*, \ldots, X_t^*)$ for $\varphi \in \mathbb{C}\langle X_1, \ldots, X_t\rangle$. Let $M_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and τ_k be the normalized trace on $M_k(\mathbb{C})$; i.e. $\tau_k = \frac{1}{k}Tr_k$, where Tr_k is the usual trace on $M_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $M_k(\mathbb{C})$. Let $M_k(\mathbb{C})^n$ be the direct sum of n copies of $M_k(\mathbb{C})$ and let $(M_k)_R$ be the closed ball of the $k\times k$ matrix algebra $M_k(\mathbb{C})$ with radius R under its operator norm and M_k^{sa} the set of all self-adjoint $k \times k$ matrices. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $M_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \ldots, A_n)\|_2^2 = \tau_k(A_1^*A_1) + \cdots + \tau_k(A_n^*A_n)$$

for all $(A_1, \ldots, A_n)$ in $M_k(\mathbb{C})^n$. Finally, let $\|\cdot\|_e$ denote the euclidean norm on $M_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \ldots, A_n)\|_e^2 = Tr_k(A_1^* A_1) + \cdots + Tr_k(A_n^* A_n)$$

for all $(A_1, \ldots, A_n)$ in $M_k(\mathbb{C})^n$.

Lemma 27 *Define the mapping*

$$\Phi : (W_1, W_2, \ldots, W_t) \mapsto (\varphi_1(W_1, \ldots, W_t), \ldots, \varphi_r(W_1, \ldots, W_t))$$

from $((M_k)_1)^t$ into $M_k(\mathbb{C})^r$, where $\varphi_1, \ldots, \varphi_r \in \mathbb{C}\langle X_1, \ldots, X_t \rangle$. Then there is a constant $D(\Phi)$ (independent of k) such that

$$\|\Phi(W_1, \ldots, W_t) - \Phi(W_1', \ldots, W_t')\|_e \le D(\Phi)\|(W_1, \ldots, W_t) - (W_1', \ldots, W_t')\|_e$$

for any $(W_1, \ldots, W_t)$ and $(W_1', \ldots, W_t')$ in $((M_k)_1)^t$. Note that the constant $D(\Phi)$ may depend on t. All the above is true when $\|\cdot\|_e$ is replaced by $\|\cdot\|_2$.

Lemma 28 *For every $\delta > 0$, there is an $0 < \epsilon < \delta$, such that for every finite factor $\mathcal{M}$ with trace τ, if A is an element in the unit ball of $\mathcal{M}$ such that*

$$\|A^*A - AA^*\|_2 \le \epsilon, \quad \|I - AA^*\|_2 \le \epsilon$$

then there is a unitary U in $\mathcal{M}$ such that $\|A - U\|_2 \le \delta$.

Lemma 29 *Let $\boldsymbol{B}(r)$ be a ball of radius r in $\mathbb{R}^n$. For any δ in $(0, r)$, if $\{\boldsymbol{B}_s(\delta)\}_{s \in \mathbb{S}}$ is a δ-net for $\boldsymbol{B}(r)$ with the minimal cardinality, then*

$$(\frac{r}{\delta})^n \le |\mathbb{S}| \le (\frac{3r}{\delta})^n,$$

where $|\mathbb{S}|$ is the cardinality of $\mathbb{S}$. Similar upper bound holds for any convex bodies euclidean spaces where the radius r is replaced by the diameter of the convex body.

3.2 Free Orbit-Dimension

In [HadSh], Shen and Hadwin introduced the concept of a free orbit-dimension. It simplified the computation of Voiculescu's free entropy dimension. In this section, we shall discuss, briefly, the concepts of free entropy, free entropy dimension and free orbit-dimension.

For every $\omega > 0$, the ω-orbit-ball $\mathcal{U}(B_1, \ldots, B_n; \omega)$ centered at $(B_1, \ldots, B_n)$ in $M_k(\mathbb{C})^n$ is the subset of $M_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $M_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1, \ldots, A_n) - (WB_1W^*, \ldots, WB_nW^*)\|_2 < \omega.$$

For every $R > 0$, $(M_k(\mathbb{C})^n)_R$ is the subset of $M_k(\mathbb{C})^n$ consisting of all these $(A_1, \ldots, A_n)$ in $M_k(\mathbb{C})^n$ such that $\max_{1 \le j \le n} \|A_j\| \le R$. Note that $(M_k(\mathbb{C})^n)_R = ((M_k)_R)^n$.

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state τ, and $X_1, \ldots, X_n$ be self-adjoint elements in $\mathcal{M}$. For any positive R and ϵ, and any m, k in $\mathbb{N}$, let $\Gamma_R(X_1, \ldots, X_n; m, k, \epsilon)$ be the subset of $(M_k^{sa})^n$ consisting of all $(A_1, \ldots, A_n)$ in $(M_k^{sa})^n$ such that $(A_1, \ldots, A_n)$ is contained in $(M_k(\mathbb{C})^n)_R$, and

$$|\tau_k(A_{i_1} \cdots A_{i_q}) - \tau(X_{i_1} \cdots X_{i_q})| < \epsilon,$$

for all $1 \le i_1, \ldots, i_q \le n$, and all q with $1 \le q \le m$. Let Λ be Lebesgue measure on $(M_k^{sa})^n$ corresponding to the euclidean norm $\|\cdot\|_e$.

Now we define, successively,

$$\begin{aligned}
\chi_R(X_1, \ldots, X_n; m, k, \epsilon) &= \log \Lambda(\Gamma_R(X_1, \ldots, X_n; m, k, \epsilon)), \\
\chi_R(X_1, \ldots, X_n; m, \epsilon) &= \limsup_{k \to \infty}(k^{-2}\chi_R(X_1, \ldots, X_n; m, k, \epsilon) + \frac{n}{2}\log k), \\
\chi_R(X_1, \ldots, X_n) &= \inf\{\chi_R(X_1, \ldots, X_n; m, \epsilon) : m \in \mathbb{N}, \epsilon > 0\}, \\
\chi(X_1, \ldots, X_n) &= \sup_{R>0} \chi_R(X_1, \ldots, X_n).
\end{aligned}$$

We call $\chi(X_1, \ldots, X_n)$ the *free entropy* of $(X_1, \ldots, X_n)$.

For technical reasons, Voiculescu introduced a "modified" free entropy in [Vo96]. Let $X_1, \ldots X_n, Y_1, \ldots, Y_p, n \ge 1, p \ge 0$ be self-adjoint random variables in a finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ, and $\Gamma_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, k, \epsilon)$ be the image of the projection of $\Gamma_R(X_1, \ldots, X_n, Y_1, \ldots, Y_p; m, k, \epsilon)$ onto the first n components, in another words, $(A_1, \ldots, A_n)$ is in $\Gamma_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, k, \epsilon)$ if there are elements $B_1, \ldots, B_p$ in M_k^{sa} such that $(A_1, \ldots, A_n, B_1, \ldots, B_p)$ is in $\Gamma_R(X_1, \ldots, X_n, Y_1, \ldots, Y_p; m, k, \epsilon)$.

We can define similarly,

$$
\begin{aligned}
&\chi_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, k, \epsilon) \\
&\quad = \log \Lambda(\Gamma_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, k, \epsilon)), \\
&\chi_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, \epsilon) \\
&\quad = \limsup_{k \to \infty} (k^{-2} \chi_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, k, \epsilon) + \frac{n}{2} \log k), \\
&\chi_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p) \\
&\quad = \inf\{\chi_R(X_1, \ldots, X_n : Y_1, \ldots, Y_p; m, \epsilon) : m \in \mathbb{N}, \epsilon > 0\}, \\
&\chi(X_1, \ldots, X_n : Y_1, \ldots, Y_p) \\
&\quad = \sup_{R>0} (X_1, \ldots, X_n : Y_1, \ldots, Y_p).
\end{aligned}
$$

We call $\chi(X_1, \ldots, X_n : Y_1, \ldots, Y_p)$ the modified free entropy of $X_1, \ldots, X_n$ in presence of $Y_1, \ldots, Y_p$.

Although the free entropy is defined for self-adjoint elements, for modified free entropy $\chi(X_1, \ldots, X_n : Y_1, \ldots, Y_p)$, we need not assume that $Y_1, \ldots, Y_p$ are self-adjoint elements. Instead we may write $\chi(X_1, \ldots, X_n : Y_1, \ldots, Y_p)$ as $\chi(X_1, \ldots, X_n : A_1, \ldots, A_p, B_1, \ldots, B_p)$ where $A_j = Y_j + Y_j^*$ and $B_j = -i(Y_j - Y_j^*)$ for each j.

The (modified) free entropy dimension $\delta(X_1, \ldots, X_n : Y_1, \ldots, Y_p)$ is defined by

$$
\begin{aligned}
&\delta(X_1, \ldots, X_n : Y_1, \ldots, Y_p) \\
&\quad = n + \limsup_{\epsilon \to 0} \frac{\chi(X_1 + \epsilon S_1, \ldots, X_n + \epsilon S_n : S_1, \ldots, S_n, Y_1, \ldots, Y_p)}{|\log \epsilon|}
\end{aligned}
$$

where $\{S_1, \ldots, S_n\}$ is a semicircular family and $\{X_1, \ldots, X_n, Y_1, \ldots, Y_p\}$ and $\{S_1, \ldots, S_n\}$ are free.

For $\omega > 0$, the ω-orbit covering number $\nu(\Gamma_R(X_1, \ldots, X_n; m, k, \epsilon), \omega)$ is the minimal number of ω-orbit-balls that cover $\Gamma_R(X_1, \ldots, X_n; m, k, \epsilon)$ with the centers of these ω-orbit-balls in $(M_k(\mathbb{C})^n)_R$.

Now we define

$$\begin{aligned}
\mathfrak{K}(X_1,\ldots,X_n;\omega,R) &= \inf_{m\in\mathbb{N},\epsilon>0}\limsup_{k\to\infty}\frac{\log(\nu(\Gamma_R(X_1,\ldots,X_n;m,k,\epsilon),\omega))}{-k^2\log\omega},\\
\mathfrak{K}(X_1,\ldots,X_n;\omega) &= \sup_{R>0}\mathfrak{K}(X_1,\ldots,X_n;\omega,R),\\
\mathfrak{K}_1(X_1,\ldots,X_n) &= \limsup_{\omega\to 0}\mathfrak{K}(X_1,\ldots,X_n;\omega),\\
\mathfrak{K}_2(X_1,\ldots,X_n) &= \sup_{0<\omega<1}\mathfrak{K}(X_1,\ldots,X_n;\omega),
\end{aligned}$$

where $\mathfrak{K}_1(X_1,\ldots,X_n)$ is the free orbit-dimension of $X_1,\ldots,X_n$ and $\mathfrak{K}_2(X_1,\ldots,X_n)$ is the upper free orbit-dimension of $X_1,\ldots,X_n$.

The relation between free entropy dimension and free orbit dimension was derived in [HadSh] as:

$$\delta(X_1,\ldots,X_n)\leq\mathfrak{K}_1(X_1,\ldots,X_n)+1\leq\mathfrak{K}_2(X_1,\ldots,X_n)+1.$$

Suppose $\mathcal{M}$ is a finitely generated von Neumann algebra with a faithful normal tracial state τ. Then the free orbit-dimension $\mathfrak{K}_1(\mathcal{M})$ of $\mathcal{M}$ is

$$\mathfrak{K}_1(\mathcal{M})=\sup\{\mathfrak{K}_1(X_1,\ldots,X_n):X_1,\ldots,X_n\text{ generate }\mathcal{M}\},$$

while the upper free orbit-dimension $\mathfrak{K}_2(\mathcal{M})$ of $\mathcal{M}$ is defined as

$$\mathfrak{K}_2(\mathcal{M})=\sup\{\mathfrak{K}_2(X_1,\ldots,X_n):X_1,\ldots,X_n\text{ generate }\mathcal{M}\}.$$

If $\mathcal{M}$ is a von Neumann algebra with a faithful normal tracial state τ and $\mathfrak{K}_2(\mathcal{M})=0$, then $\mathfrak{K}_2(\mathcal{M}\otimes M_n(\mathbb{C}))=0$.

In [HadSh], Hadwin and Shen showed that the class of finite von Neumann algebra $\mathcal{M}$ with upper free orbit dimension $\mathfrak{K}_2(\mathcal{M})=0$ is closed under the following three operations:

(1) Suppose $\mathfrak{K}_2(\mathcal{N}_1)=\mathfrak{R}_2(\mathcal{N}_2)=0$ and $\mathcal{N}_1\cap\mathcal{N}_2$ is diffused. Then $\mathfrak{K}_2(\{\mathcal{N}_1\cup\mathcal{N}_2\}'')=0$.

(2) Suppose $\mathcal{M}=\{\mathcal{N},U\}''$, where $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ with $\mathfrak{K}_2(\mathcal{N})=0$ and U is a unitary element in $\mathcal{M}$ satisfying, for a sequence $\{V_n\}$ of Haar unitary elements in $\mathcal{N}$, $dist_{\|\cdot\|_2}(UV_nU^*,\mathcal{N})\to 0$. Then $\mathfrak{K}_2(\mathcal{M})=0$.

(3) Suppose $\{\mathcal{N}_j\}_{i=1}^{\infty}$ is an ascending sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\Re_2(\mathcal{N}_i) = 0$ for all $i \geq 1$, and $\mathcal{M} = \overline{\cup_i \mathcal{N}_i}^{SOT}$. Then $\Re_2(\mathcal{M}) = 0$.

Many factors of type II_1, such as property Γ factors, have upper free orbit dimension zero.

3.3 The Estimate of Free Entropies

One of my main results in this thesis is to estimate the free entropy of any generating subset of m-weakly Γ-thin factor. More precisely,

Theorem 30 *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a faithful normal tracial state τ, $X_1, \ldots, X_n$ self-adjoint elements in $\mathcal{M}$ such that $X_1, \ldots, X_n$ generate $\mathcal{M}$ as a von Neumann algebra. Suppose there are subfactors $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{M}$ with property Γ, operators $Y_1, \ldots, Y_q$ in $\mathcal{M}$ such that the trace-norm distance from each X_j to the linear span of $\{WY_iW' : W \in \mathcal{U}(\mathcal{N}_1), W' \in \mathcal{U}(\mathcal{N}_2), i = 1, \ldots, q\}$ is less than $\omega (< 1)$. Let a be the constant $\max_{1\leq j\leq n}\{\|X_j\|_2 + 1\}$. Then we have that*

$$\chi(X_1, \ldots, X_n) \leq C(n, q, a) + (n - 2q - 2 - \omega) \log \omega,$$

where $C(n, q, a)$ is a constant depending on n, q and a.

Proof. From our assumptions in the theorem, there are unitary operators $U_1, \ldots, U_p$ in $\mathcal{N}_1$, $V_1, \ldots, V_{p'}$ in $\mathcal{N}_2$ and constants $\lambda(j, s_{ji}, q_{ji}, s'_{ji})$ where $s_{ji} \in \{1, \ldots, p\}$, $s'_{ji} \in \{1, \ldots, p'\}$, $q_{ji} \in \{1, \ldots, q\}$, $i = 1, \ldots, i_j$ for some integer i_j dependent on j, $j = 1, \ldots, n$, such that

$$\|X_j - \sum_{i=1}^{i_j} \lambda(j, s_{ji}, q_{ji}, s'_{ji}) U_{s_{ji}} Y_{q_{ji}} V_{s'_{ji}}\|_2 < \omega$$

Let

$$\begin{aligned} &\varphi_j(U_1, \ldots, U_p, V_1, \ldots, V_{p'}, Y_1, \ldots, Y_q) \\ = &\sum_{i=1}^{i_j} \lambda(j, s_{ji}, q_{ji}, s'_{ji}) U_{s_{ji}} Y_{q_{ji}} V_{s'_{ji}}, \quad j = 1, \ldots, n \end{aligned}$$

Here φ_j will be viewed as a noncommutative polynomial with variables in $U_1, \ldots, U_p$, $V_1, \ldots, V_{p'}$

Suppose $\Phi : (\mathcal{U}(k))^{p+p'} \mapsto (M_k)^n$ is the mapping given by

$$(W_1, \ldots, W_{p+p'}) \mapsto (\varphi_1(W_1, \ldots, W_{p+p'}), \ldots, \varphi_n(W_1, \ldots, W_{p+p'})),$$

for each $(W_1, \ldots, W_{p+p'})$ in $(\mathcal{U}(k))^{p+p'}$. There is a positive constant D such that

$$\begin{aligned} &\|\Phi(W_1, \ldots, W_{p+p'}) - \Phi(W_1', \ldots, W_{p+p'}')\|_e \\ \leq \ & D\|(W_1, \ldots, W_{p+p'}) - (W_1', \ldots, W_{p+p'}')\|_e \end{aligned}$$

for $(W_1, \ldots, W_{p+p'})$ and $(W_1', \ldots, W_{p+p'}')$ in $(\mathcal{U}(k))^{p+p'}$. Here $(\mathcal{U}(k))^{p+p'}$ is naturally imbedded in $(M_k)^{p+p'}$.

Since D is a constant and p, p' are given, there is a n_0 in $\mathbb{N}$ such that $(D\sqrt{p+p'})^{(p+p')/n_0} \leq 2$. We may assume that $n_0 \geq \frac{p+p'}{\omega}$. In Lemma 28, take $\delta = \frac{\omega}{D\sqrt{(p+p')n_0}}$. Then there is $\epsilon_1 < \delta$ such that if the condition in Lemma 28 is satisfied, the results will follow. Since $\mathcal{N}_1$, $\mathcal{N}_2$ have property Γ [Dix69], there are mutually orthogonal family of projections $\{P_i\}_{i=1}^{n_0}$ with equal trace $\tau(P_i) = \frac{1}{n_0}$ in $\mathcal{N}_1$ and $\{P_i'\}_{i=1}^{n_0}$ with equal trace $\tau(P_i')\frac{1}{n_0}$ in $\mathcal{N}_2$ such that

$$\|\sum_{i=1}^{n_0} P_i U_t P_i - U_t\|_2 < \frac{\epsilon_1}{4} \quad t = 1, \ldots, p,$$

and

$$\|\sum_{i=1}^{n_0} P_i' V_t P_i' - V_t\|_2 < \frac{\epsilon_1}{4} \quad t = 1, \ldots, p'.$$

In the following, we shall estimate

$$\chi(X_1, \ldots, X_n : U_1, \ldots, U_p, V_1, \ldots, V_{p'}, Y_1, \ldots, Y_q, \{P_i\}_{i=1}^{n_0}, \{P_i'\}_{i=1}^{n_0}).$$

We begin by describing elements in

$$\Gamma_R(X_1, \ldots, X_n, U_1, \ldots, U_p, V_1, \ldots, V_{p'}, Y_1, \ldots, Y_q, \{P_i\}_{i=1}^{n_0}, \{P_i'\}_{i=1}^{n_0}; m, k, \epsilon)$$

for some large R in $\mathbb{R}$, large m, k in $\mathbb{N}$ and small ϵ. To simplify our estimates, we assume that $\frac{k}{n_0}$ is an integer. By a standard argument, one obtains that there are a positive ϵ_0 and

m_0, k_0 in $\mathbb{N}$ such that, if $0 < \epsilon < \epsilon_0$, $m \geq m_0$, $k \geq k_0$ and

$$(A_1, \ldots, A_n, \ldots)$$
$$\in \Gamma_R(X_1, \ldots, X_n, U_1, \ldots, V_1, \ldots, V_{p'}, U_p, T_1, \ldots, T_q, \{P_i\}_{i=1}^{n_0}, \{P'_i\}_{i=1}^{n_0}; m, k, \epsilon),$$

then there exists a mutually orthogonal family of projections $\{Q_i\}_{i=1}^{n_0}$ with equal trace (corresponding to $\{P_i\}_{i=1}^{n_0}$), $\{Q'_i\}_{i=1}^{n_0}$ with equal trace (corresponding to $\{P'_i\}_{i=1}^{n_0}$), unitary elements $G_1, \ldots, G_p$ (corresponding to $U_1, \ldots, U_p$), unitary elements $H_1, \ldots, H_{p'}$(corresponding to $V_1, \ldots, V_{p'}$), and elements $T_1, \ldots T_q$ (corresponding to $Y_1, \ldots, Y_q$) such that

$$\|A_j - \varphi_j(G_1, \ldots, G_p, H_1, \ldots, H_{p'} : T_1, \ldots, T_q)\|_2 < \omega, \quad j = 1, \ldots, n$$

$$\|\sum_{i=1}^{n_0} Q_i G_t Q_i - G_t\|_2 < \frac{\epsilon_1}{4}, \quad t = 1, \ldots, p,$$

$$\|\sum_{i=1}^{n_0} Q'_i H_t Q'_i - H_t\|_2 < \frac{\epsilon_1}{4}, \quad t = 1, \ldots, p'.$$

For each large k (with assumption that $\frac{k}{n_0}$ is an integer), decompose M_k into a tensor product $M_{n_0} \otimes M_{\frac{k}{n_0}}$ and let $\{E_{st} : s, t = 1, \ldots, n_0\}$ be a given matrix unit system for $M_{n_0} \otimes \mathbb{C}I$. Then there are unitary matrices W and W' in $\mathcal{U}(k)$ such that $WQ_iW^* = E_{ii}$, and $W'Q'_iW'^* = E_{ii}$ for $i = 1, \ldots, n_0$. Thus for each WG_tW^*, let $D_t = \sum_{i=1}^{n_0} E_{ii}WG_tW^*E_{ii}, t = 1, \ldots, p$ and $D_{ti} = E_{ii}WG_tW^*E_{ii}, i = 1, \ldots, n_0$. we thus have

$$\|D_t^* D_t - D_t D_t^*\|_2 < \epsilon_1, \|I - D_t D_t^*\|_2 < \epsilon_1, \quad t = 1, \ldots, p$$

and

$$\|D_{ti}^* D_{ti} - D_{ti} D_{ti}^*\|_2 < \epsilon_1, \|I - D_{ti} D_{ti}^*\|_2 < \epsilon_1, \quad i = 1, \ldots, n_0$$

and therefore there are $\frac{k}{n_0} \times \frac{k}{n_0}$ unitary matrices $G_t^{(1)}, \ldots, G_t^{(n_0)}$ in $M_{\frac{k}{n_0}}$ such that

$$\|G_t^{(i)} - D_{ti}\|_2 \leq \frac{\omega}{D\sqrt{pn_0}}, t = 1, \ldots, p, i = 1, \ldots, n_0$$

$$\|D_t - G'_t\|_2 \leq \frac{\omega}{D\sqrt{p}}, t = 1, \ldots p$$

$$\|WG_tW^* - G'_t\|_2 \leq \frac{2\omega}{D\sqrt{p}}, \quad t = 1, \ldots, p,$$

where $G'_t = \sum_{i=1}^{n_0} E_{ii} \otimes G_t^{(i)}$. Similarly, for $W'H_tW'^*$, we obtain $\frac{k}{n_0} \times \frac{k}{n_0}$ unitary matrices $H_t^{(1)}, \ldots, H_t^{(n_0)}$ such that

$$\|W'H_tW'^* - H'_t\|_2 \le \frac{2\omega}{D\sqrt{p}}, \quad t = 1, \ldots, p,$$

where $H'_t = \sum_{i=1}^{n_0} E_{ii} \otimes H_t^{(i)}$.

We also know that there is a σ-net $(U'_t)_{t\in\mathbb{S}(k)}$ in $\mathcal{U}(k)$ with respect to the uniform norm such that $|\mathbb{S}(k)| < (C/\sigma)^{k^2}$ for each k in $\mathbb{N}$, where C is a universal constant. We choose σ to be $\omega/2a$. Hence there is a $U'_r, U'_{r'}$ in $\mathcal{U}(k)$ such that $\|W - U'_r\| \le \sigma$, $\|W'W^* - U'_{r'}\| \le \sigma$. It follows that

$$\|U'_r A_j U'^*_r - WA_jW^*\|_2 \le \omega$$

and

$$\|U'_r A_j U'^*_r - \varphi_j(WG_1W^*, \ldots, WG_pW^*,$$
$$W'H_1W'^*, \ldots, W'H_{p'}W'^*, WT_1W'^*, \ldots, WT_qW'^*)W'W^*\|_2 \le 2\omega$$

for $j = 1, \ldots, n$. Since

$$\|\varphi_j(WG_1W^*, \ldots, WG_pW^*, W'H_1W'^*, \ldots, W'H_{p'}W'^*$$
$$WT_1W'^*, \ldots, WT_qW'^*)\|_2 \le \|A_j\|_2 + \omega < a$$

we have

$$\|U'_r A_j U'^*_r - \varphi_j(WG_1W^*, \ldots, WG_pW^*,$$
$$W'H_1W'^*, \ldots, W'H_{p'}W'^*, WT_1W'^*, \ldots, WT_qW'^*)U'_{r'}\|_2 \le 3\omega$$

for $j = 1, \ldots, n$.

We also know that there is a θ-net $(W_s)_{s\in\mathbb{T}(k/n_0)}$ with respect to the Euclidean metric such that $|\mathbb{T}(k/n_0)| < (C\sqrt{k/n_0}/\theta)^{k^2/n_0^2}$, where C is a universal constant and θ is an arbitrary constant in $(0, \sqrt{k/n_0}]$.

Thus there are $W_{s_1}, \ldots, W_{s_{pn_0}}, s_1, \ldots, s_{pn_0} \in \mathbb{T}(k/n_0)$ and $W'_{s'_1}, \ldots, W'_{s'_{pn_0}}, s'_1, \ldots, s'_{p'n_0} \in \mathbb{T}(k/n_0)$ such that

$$\|W_{s_j} - G^{(j \mod n_0)}_{[\frac{j}{n_0}]+1}\|_e \le \theta, \quad j = 1, \ldots, pn_0$$

$$\|W'_{s'_j} - H^{(j \bmod n_0)}_{[\frac{j}{n_0}]+1}\|_e \le \theta, \quad j = 1, \ldots, p'n_0$$

Let $W^{(j)} = \sum_{i=1}^{n_0} E_{ii} \otimes W_{s_{(j-1)n_0+i}}$ for $j = 1, \ldots, p$ and $W'^{(j)} = \sum_{i=1}^{n_0} E_{ii} \otimes W'_{s'_{(j-1)n_0+i}}$ for $j = 1, \ldots, p'$.

Let $B_j(s, s', r')$ be $\varphi_j(W^{(1)}, \ldots, W^{(p)}, W'^{(1)}, \ldots, W'^{(p')}, WT_1W'^*, \ldots, WT_qW'^*)U'_{r'}$ for $j = 1, \ldots, n$. Now, we have

$$\begin{aligned}
&\|U'_rA_jU'^*_r - B_j(s, s', r')\|_e \\
\le\ & \|U'_rA_jU'^*_r - \varphi_j(WG_1W^*, \ldots, WG_pW^*, W'H_1W'^*, \ldots, W'H_{p'}W'^*, \\
& WT_1W'^*, \ldots, WT_qW'^*)U'_{r'}\|_e + \\
& \|\varphi_j(WG_1W^*, \ldots, WG_pW^*, W'H_1W'^*, \ldots, W'H_{p'}W'^*, WT_1W'^*, \ldots, WT_qW'^*) - \\
& -\varphi_j(G'_1, \ldots, G'_p, H'_1, \ldots, H'_{p'}, WT_1W'^*, \ldots, WT_qW'^*)\|_e + \\
& +\|\varphi_j(G'_1, \ldots, G'_p, H'_1, \ldots, H'_{p'}, WT_1W'^*, \ldots, WT_qW'^*) \\
& -\varphi_j(W^{(1)}, \ldots, W^{(p)}, W'^{(1)}, \ldots, W'^{(p')}, WT_1W'^*, \ldots, WT_qW'^*)\|_e \\
\le\ & 3k^{1/2}\omega + D\sqrt{p+p'}\frac{2k^{1/2}\omega}{D\sqrt{p+p'}} + D\sqrt{p+p'}\theta.
\end{aligned}$$

Let θ be $\frac{k^{1/2}\omega}{D\sqrt{p+p'}}$. Then

$$\|U'_rA_jU'^*_r - B_j(s, s', r')\|_e < 6\omega\sqrt{k},$$

Define a linear mapping $\phi : (M_k)^q \mapsto (M_k^{sa})^n$ as follows:

$$\begin{aligned}
&\phi(S_1, \ldots, S_q) \\
=\ & \Big(\frac{1}{2}\varphi_j(W^{(1)}, \ldots, W^{(p)}, W'^{(1)}, \ldots, W'^{(p)}, S_1, \ldots, S_q)U'_{r'} \\
& + \frac{1}{2}U'^*_{r'}\varphi^*_j(W^{(1)}, \ldots, W^{(p)}, W'^{(1)}, \ldots, W'^{(p)}, S_1, \ldots, S_q)\Big)_{j=1,\ldots,n}.
\end{aligned}$$

Let $\mathcal{T}$ be the range of ϕ in $(M_k^{sa})^n$. It is easy to see that $\mathcal{T}$ is a real linear subspace of $(M_k^{sa})^n$ whose real dimension is not greater than $2qk^2$. By adjoining linearly independent elements of $(M_k^{sa})^n$, if necessary, we may assume that the real dimension of $\mathcal{T}$ is precisely $2qk^2$. Let $\mathcal{T}'$ be the orthogonal complement of $\mathcal{T}$ in $(M_k^{sa})^n$. Then $\mathcal{T}'$ has real dimension $(n-2q)k^2$.

Now let $\mathbb{B}(s, s', r')$ be the ball of radius $(nk)^{1/2}a$ in $\mathcal{T}$ and $\mathbb{B}'(s, s', r')$ be the ball of radius $6(nk)^{1/2}\omega$ in $\mathcal{T}'$ with respect to Euclidean norms. The volumes of the two balls are

$$\pi^{\frac{1}{2}2qk^2}\Gamma(1 + \frac{1}{2}2qk^2)^{-1}(nka^2)^{\frac{1}{2}2qk^2}$$

and

$$\pi^{\frac{1}{2}(n-2q)k^2}\Gamma(1 + \frac{1}{2}(n-2q)k^2)^{-1}(36nk\omega^2)^{\frac{1}{2}(n-2q)k^2}$$

Let $(B_1, \ldots, B_n)$ in $\mathcal{T}$ be the image of $(U_r'A_1U_r'^*, \ldots, U_r'A_nU_r'^*)$ under the orthogonal projection from $(M_k^{sa})^n$ onto $\mathcal{T}$. Since

$$\|(U_r'A_1U_r'^*, \ldots, U_r'A_nU_r'^*)\|_e \le (nk)^{1/2}(a-1),$$

we have $\|(B_1, \ldots, B_n)\|_e \le (nk)^{1/2}(a-1)$ and $(B_1, \ldots, B_n) \in \mathbb{B}'(s, s', r')$. Since

$$(B_1(s, s', r'), \ldots, B_n(s, s', r')) \in \mathcal{T}$$

and

$$\|(U_r'A_1U_r'^*, \ldots, U_r'A_nU_r'^*) - (B_1(s, s', r'), \ldots, B_n(s, s', r'))\|_e < 6(nk)^{1/2}\omega,$$

we know that $(U_r'A_1U_r'^*, \ldots, U_r'A_nU_r'^*) - (B_1, \ldots, B_n)$ is both orthogonal to $\mathcal{T}$ and lies in $\mathbb{B}'(s, s', r')$. Thus

$$(U_r'A_1U_r'^*, \ldots, U_r'A_nU_r'^*) \in \mathbb{B}(s, s', r') \oplus \mathbb{B}'(s, s', r').$$

We have proved that, if $m > m_0$, $k > k_0$ and $0 < \epsilon < \epsilon_0$, then

$$\Gamma_R(X_1, \ldots, X_n : \ldots ; m, k, \epsilon) \subset \bigcup_{\substack{s_1 \quad s_{pn_0} \in \mathbb{T}(k/n_0) \\ s_1' \quad s'_{pn_0} \in \mathbb{T}(k/n_0)}} \bigcup_{r, r' \in \mathbb{S}(k)} (U_r'^*)^{(n)} \mathbb{B}(s, s', r') \oplus \mathbb{B}'(s, s', r')(U_r')^{(n)},$$

where $(U_r')^{(n)}$ is $(U_r', \ldots, U_r')$. Thus

$$\begin{aligned}
&\Lambda(\Gamma_R(X_1, \ldots, X_n : \ldots; m, k, \epsilon)) \\
&\leq |\mathbb{T}(k/n_0)|^{(p+p')n_0} |\mathbb{S}_k|^2 \Lambda(\mathbb{B}(s, s', r'))\Lambda(\mathbb{B}'(s, s', r')) \\
&\leq (C\sqrt{k}/\theta)^{\frac{k^2}{n_0^2}((p+p')n_0)} (C/\sigma)^{2k^2} \pi^{\frac{1}{2}nk^2} \Gamma(1 + \frac{1}{2}2qk^2)^{-1} \Gamma(1 + \frac{1}{2}(n-2q)k^2)^{-1} \\
&\quad \cdot (nk)^{\frac{1}{2}nk^2} a^{2qk^2} (6\omega)^{(n-2q)k^2} \\
&= (\frac{CD\sqrt{p+p'}}{\omega})^{\frac{(p+p')k^2}{n_0}} (\frac{2aC}{\omega})^{2k^2} \pi^{\frac{1}{2}nk^2} \Gamma(1 + \frac{1}{2}2qk^2)^{-1} \Gamma(1 + \frac{1}{2}(n-2q)k^2)^{-1} \\
&\quad \cdot (nk)^{\frac{1}{2}nk^2} a^{2qk^2} (6\omega)^{(n-2q)k^2}
\end{aligned}$$

As before, D is a constant and it follows that $(D\sqrt{p+p'})^{(p+p')/n_0} < 2$, $n_0 > \frac{p+p'}{\omega}$ and the fact that $\Gamma(1+x) \geq x^x e^{-x}$ (Stirling's formula), we have

$$\begin{aligned}
&\Lambda(\Gamma_R(X_1, \ldots, X_n : \ldots; m, k, \epsilon)) \\
&\leq (\frac{C}{\omega})^{k^2\omega} 2^{k^2} (\frac{2aC}{\omega})^{2k^2} \pi^{\frac{1}{2}nk^2} (\frac{1}{2}2qk^2)^{-\frac{1}{2}2qk^2} \\
&\quad \cdot (\frac{1}{2}(n-2q)k^2)^{-\frac{1}{2}(n-2q)k^2} e^{\frac{1}{2}nk^2} (nk)^{\frac{1}{2}nk^2} a^{2qk^2} (6\omega)^{(n-2q)k^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\chi(X_1, \ldots, X_n) = \chi(X_1, \ldots, X_n : \ldots) \\
&\leq \limsup_{k\to\infty}(k^{-2} \log \Lambda(\Gamma_R(X_1, \ldots, X_n : \ldots, m, k, \epsilon)) + \frac{n}{2}\log k) \\
&= \limsup_{k\to\infty}(\omega \log \frac{C}{\omega} + \log 2 + 2\log\frac{2aC}{\omega} - q\log qk^2 - \frac{1}{2}(n-2q)\log\frac{1}{2}(n-2q)k^2 \\
&\quad + \frac{1}{2}n + \frac{1}{2}n\log nk + 2q\log a + (n-2q)\log 6\omega + \frac{n}{2}\log k) \\
&\leq \log 2C + 2\log 2aC - q\log q - \frac{n-2q}{2}\log\frac{n-2q}{2} + \frac{n}{2} + \\
&\quad + \frac{n}{2}\log n + 2q\log a + (n-2q)\log 6 + (n-2q-2-\omega)\log\omega \\
&= C(n, q, a) + (n-2q-2-\omega)\log\omega.
\end{aligned}$$

■

Corollary 31 *The free group factor $\mathcal{L}_{F_n}$ when $n > 2q+2$ is not q-weakly Γ-thin.*

In Theorem 30, the subfactors with property Γ in $\mathcal{M}$ can be replaced by subfactors having Cartan subalgebras. In [HadSh], D. Hadwin and J. Shen prove a more general case by using the idea of free orbit-dimension. We state the theorem below:

Theorem 32 (See [HadSh]) *Suppose $\mathcal{M}$ is a type II_1 factor with the trace τ and there exist von Neumann subalgebras $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{M}$ with $\mathfrak{K}_2(\mathcal{N}_1) = \mathfrak{K}_2(\mathcal{N}_2) = 0$ and vectors $\xi_1, \ldots, \xi_n$ in $L^2(\mathcal{M}, \tau)$ such that*

$$\overline{sp}^{\|\ \|_2} \mathcal{N}_1\{\xi_1, \ldots, \xi_n\}\mathcal{N}_2 = L^2(\mathcal{M}, \tau)$$

*Then $\mathfrak{K}_1(\mathcal{M}) \leq 1 + 2n$ and $\delta(\mathcal{M}) \leq 2 + 2n$. Thus $\mathcal{M}$ is not *-isomorphic to $\mathcal{L}_{\mathcal{F}_m}$ for $m > 2 + 2n$.*

In the theorem above, when $\xi_n = \hat{X}_n$ and X_n is self-adjoint in $\mathcal{M}$, we have that $\mathfrak{K}_1(\mathcal{M}) \leq 1 + n$ and $\delta(\mathcal{M}) \leq 2 + n$ from the proof of the theorem above, where $\mathcal{M}$ is given as in the theorem. Therefore the free group factor $\mathcal{L}_{\mathcal{F}_m}$, for $m > 3$, is not Γ-thin, all free group factors are not strongly Γ-thin, $\mathcal{L}_{\mathcal{F}_m}$ for $m > 4$ is not 1-weakly Γ-thin.

In [HadSh], they also applied the theorem above to the case when a factor of type II_1 contains a subfactor with a finite index and the subfactor has upper free orbit-dimension zero. Suppose $\mathcal{N} \subset \mathcal{M}$ is an inclusion of factors of type II_1 and $[\mathcal{M} : \mathcal{N}] = r < \infty$. If $\mathfrak{K}_2(\mathcal{N}) = 0$, then $\mathfrak{K}_1(\mathcal{M}) \leq 2[r] + 3$ and $\delta(\mathcal{M}) \leq 2[r] + 4$, where $[r]$ is the integer part of r. The result is rough in some sense, as you can see that the estimation depends on index r. Actually, we can improve the result as follows.

Corollary 33 *Suppose $\mathcal{N} \subset \mathcal{M}$ is an inclusion of factors of type II_1 and $[\mathcal{M} : \mathcal{N}] = r < \infty$. If $\mathfrak{K}_2(\mathcal{N}) = 0$, then $\delta(\mathcal{M}) \leq 3$.*

Proof. By [Po86], there exists a MASA $\mathcal{A}$ in $\mathcal{M}$ that is also a MASA in $\langle \mathcal{M}, \mathcal{N} \rangle$; i.e. $\mathcal{A}' \cap \langle \mathcal{M}, \mathcal{N} \rangle = \mathcal{A}$. Consequently $\mathcal{A} \vee J\mathcal{N}J = \mathcal{A}'$ and $\mathcal{M} = \overline{sp}\mathcal{A}\mathcal{N}$. Thus, we have $\delta(\mathcal{M}) \leq 3$ in view of the theorem above. ∎

CHAPTER 4

CONNES'S EMBEDDING PROBLEM

In Quantum Physics, observed quantities are described by operators while researchers use large matrices to replace operators for the sake of convenience of computation. In general, this method is not correct in mathematics, but it is reasonable to ask when operators can be approximated by matrices. Similarly, for von Neumann algebras, researchers could ask whether any separable factor of type II_1 can be asymptotically embedded into matrix algebras. In the language of ultrapower of von Neumann algebras, this problem can be rephrased as whether any separable factor of type II_1 can be embedded into the ultrapower $\mathcal{R}^\omega$ of the hyperfinite factor $\mathcal{R}$ of type II_1. This is the Connes's embedding problem. It was first proposed by A. Connes [Con76] in 1976.

4.1 Ultrapower of von Neumann Algebras

We begin with the definition of an ultrafilter. An ultrafilter ω on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ such that

1. the empty set $\emptyset \notin \omega$,
2. for any $A, B \in \omega$, $A \cap B \in \omega$,
3. for any $A \subset \mathbb{N}$, $A \in \omega$ or $\mathbb{N} \backslash A \in \omega$.

An example of an ultrafilter is obtained by choosing an element $a \in \mathbb{N}$ and letting ω be the collection of all subsets of $\mathbb{N}$ that contain a. Such ultrafilters are called principal ultrafilters; Ultrafilters not of this form are called free. Free ultrafilters on $\mathbb{N}$ can be identified as points in $\beta(\mathbb{N})\backslash\mathbb{N}$, where $\beta(\mathbb{N})$ is the Stone-Cêch compactification of $\mathbb{N}$. In addition, $\mathbb{N}$ can be replaced by any infinite set.

Suppose $\mathbb{S}$ is another set, $f : \mathbb{N} \mapsto \mathbb{S}$ is a mapping and $E \subset \mathbb{S}$. Then $f(n)$ is eventually in E along ω if $f^{-1}(E) = \{n \in \mathbb{N} : f(n) \in E\} \in \omega$. If $\mathbb{S}$ is a topological space, then $f(n)$ converges to $s \in \mathbb{S}$ along ω, denoted by $\lim_{n\to\omega} f(n) = s$, if $f(n)$ is eventually in each neighborhood of s. It is known that if $\mathbb{S}$ is a compact Hausdorff space, then $\lim_{n\to\omega} f(n)$ always exists in $\mathbb{S}$ for every $f : \mathbb{N} \mapsto \mathbb{S}$ and every ultrafilter ω on $\mathbb{N}$.

Regarding an ultrafilter as a topological space, one can define a product of ultrafilter. Let α, α' be two ultrafilters on infinite sets I and J respectively. The tensor product $\alpha \otimes \alpha'$ is the ultrafilter defined by setting

$$S \in \alpha \otimes \alpha' \Leftrightarrow \{i \in I : \{j \in J : (i, j) \in S\} \in \alpha'\} \in \alpha.$$

Lemma 34 *Let $\{x_i^j\}_{(i,j)\in I\times J}$ be a bounded subset of $\mathbb{C}$. Then*

$$\lim_{i\to\alpha} \lim_{j\to\alpha'} x_i^j = \lim_{(i,j)\to\alpha\otimes\alpha'} x_i^j.$$

Proof. Let $x = \lim_{i\to\alpha} \lim_{j\to\alpha'} x_i^j$. Fixing $\epsilon > 0$, we obtain $A = \{i \in I : |\lim_{j\to\alpha'} x_i^j - x| < \epsilon/2\} \in \alpha$ and $A_i = \{j \in J : |x_i^j - \lim_{j\to\alpha'} x_i^j| < \epsilon/2\}$. Then

$$X = \{(i, j) \in I \times J : i \in A, j \in A_i\} \subseteq \{(i, j) \in I \times J : |x_i^j - x| < \epsilon\}.$$

Since $X \in \alpha \otimes \alpha'$ and ϵ is arbitrary, the equation follows. ■

Suppose $\mathcal{M}$ is a factor of type II_1 with a separable predual and the trace τ. Let ω be any free ultrafilter on $\mathbb{N}$. Let $\oplus_\infty\mathcal{M}$ be the direct sum of a countable number of copies of $\mathcal{M}$ i.e.

$$\oplus_\infty\mathcal{M} = \left\{\{X^{(n)}\}_n : X^{(n)} \in \mathcal{M}, \sup_n \|X^{(n)}\| < \infty\right\},$$

and

$$\mathcal{I}_\omega = \left\{\{X^{(n)}\}_n \in \oplus_\infty\mathcal{M} : \lim_{n\to\omega} \tau(X^{(n)*}X^{(n)}) = 0\right\}.$$

It is well known that $\mathcal{I}_\omega$ is a maximal ideal in $\oplus_\infty\mathcal{M}$. The quotient $\oplus_\infty\mathcal{M}/\mathcal{I}_\omega$, which is called an ultrapower of $\mathcal{M}$, is a C* algebra, denoted by $\mathcal{M}^\omega$. The linear functional τ_ω on $\mathcal{M}^\omega$ defined by $\tau_\omega(X) = \lim_{n\to\omega}\tau(X^{(n)}), \forall X = \{X^{(n)}\}_n + \mathcal{I}_\omega \in \oplus_\infty\mathcal{M}/\mathcal{I}_\omega$ is a trace on $\mathcal{M}^\omega$. The center-valued function $\mathbb{T}$ defined by $\mathbb{T}(X) = \{\tau(X^{(n)})\}, \forall X = \{X^{(n)}\}_n \in \oplus_\infty\mathcal{M}$ is a center-valued trace from $\oplus_\infty\mathcal{M}$ to $\ell^\infty = \{\{a_n\} : a_n \in \mathbb{C}, \sup_n |a_n| < \infty\}$. The center-valued norm $\|\cdot\|_\mathbb{T}$ is given by $\|X\|_\mathbb{T} = \mathbb{T}(X^*X)^{1/2}, \forall X \in \oplus_\infty\mathcal{M}$. By the theory of abelian C* algebras, we identify ℓ^∞ as $C(\beta\mathbb{N})$. Let $X = \{X^{(n)}\}_n + \mathcal{I}_\omega \in \mathcal{M}^\omega$. $\{X^{(n)}\}_n$ represents X and without confusion, we write $X = \{X^{(n)}\}_n$.

Denote by $\mathcal{M}_\omega$ the relative commutant of $\mathcal{M}$ in $\mathcal{M}^\omega$; i.e. $\mathcal{M}_\omega = \mathcal{M}' \cap \mathcal{M}^\omega$. Now we will give some basic properties of an ultrapower of factor $\mathcal{M}$ of type II$_1$.

Lemma 35 *Suppose $\mathcal{M}$ is a factor of type II_1. Then $\mathcal{M}^\omega$ is a non-separable factor of type II_1.*

Proof. We shall split the proof into three steps. First, we shall prove $\mathcal{M}^\omega$ is a von Neumann algebra. Second, that $\mathcal{M}^\omega$ is a II$_1$ factor. And last, we shall show that it is not separable under trace norm.

Step I. To show $\mathcal{M}^\omega$ is a von Neumann algebra, it is suffice to show that the close unit ball of $\mathcal{M}^\omega$ is complete in the $\|\cdot\|_2$-norm induced by τ_ω, denoted by $\|\cdot\|_\omega$. Let $\{A_k\}_k$ be a sequence in the unit ball of $\mathcal{M}^\omega$ with $\|A_{k+1} - A_k\|_\omega \le 2^{-k}$ for all $k \ge 1$. By [KR], Lemma 10.1.6, for each A_k, there exist B_k in $\oplus_\infty\mathcal{M}$ such that $\|A_k\|_{\mathcal{M}^\omega} = \|B_k\|$ and $A_k = B_k + \mathcal{I}_\omega$.

By induction on k, we shall choose a sequence C_k in $\oplus_\infty\mathcal{M}$ with property that $C_1 = B_1$, $A_k = C_k + \mathcal{I}_\omega$ and

$$\|C_{k+1} - C_k\|_\mathbb{T} < 2^{-k+1}I, k \ge 1.$$

Suppose that $C_1, \ldots, C_k$ have been chosen for some $k \ge 1$.

$$\begin{aligned}\|B_{k+1} - C_k\|_\mathbb{T}(\omega) &= \mathbb{T}((B_{k+1} - C_k)^*(B_{k+1} - C_k))^{1/2}(\omega)\\ &= \mathbb{T}((B_{k+1} - C_k)^*(B_{k+1} - C_k)(\omega))^{1/2}\\ &= \|A_{k+1} - A_k\|_\omega < 2^{-k}\end{aligned}$$

Let $\mathscr{V} = \{s \in \beta\mathbb{N} : \|B_{k+1} - C_k\|_{\mathbb{T}}(s) < 2^{-k+1}\}$. Then $\mathscr{V}$ is a open neighborhood of ω in the compact Hausdorff space $\beta\mathbb{N}$. So by the Urysohn's lemma, there is a $Z \in C(\beta\mathbb{N})$ such that $0 \leq Z \leq 1$, $Z(\omega) = 1$ and $Z(s) = 0$ for all $s \in \beta\mathbb{N}/\mathscr{V}$. Let $C_{k+1} = ZB_{k+1} + (I - Z)C_k$. Therefore

$$\|B_{k+1} - C_{k+1}\|_\omega = \|B_{k+1} - C_{k+1}\|_{\mathbb{T}}(\omega) = (I - Z)\|(B_{k+1} - C_k)\|_{\mathbb{T}}(\omega) = 0$$

and

$$\|C_{k+1} - C_k\|_{\mathbb{T}} = \|Z(B_{k+1} - C_k)\|_{\mathbb{T}} = Z\|(B_{k+1} - C_k)\|_{\mathbb{T}} < 2^{-k+1}I,$$

which completes the induction. $\{C_k\}$ is a $\|\cdot\|_2$-Cauchy sequence in the unit ball of $\oplus_\infty\mathcal{M}$ and converges to C in this unit ball. Let $A = C + \mathcal{I}_\omega$.

$$\begin{aligned}
\|A - A_k\|_\omega &= \|C - C_k\|_{\mathbb{T}}(\omega) \\
&\leq \limsup_{j\to\infty} |\max \|C_j - C_k\|_{\mathbb{T}}| \\
&\leq \limsup_{j\to\infty} |\max \sum_{i=k}^{j-1} \|C_{i+1} - C_i\|_{\mathbb{T}}|^2| \\
&\leq \sum_{i=k}^{j-1} 2^{-i} \leq 2^{-k+1}
\end{aligned}$$

Therefore $\mathcal{M}^\omega$ is a von Neumann algebra.

Step II. Suppose the center of $\mathcal{M}^\omega$ does not consist of scalars multiplies of the identity. Let $P = \{P^{(n)}\}_n$ be a center projection in $\mathcal{M}^\omega$ with trace λ, where $P \notin \{0, I\}$ and suppose $P^{(n)}$ are projections in $\mathcal{M}$ with the same trace as P in $\mathcal{M}^\omega$. For each $P^{(n)}$, there is a unitary element $U^{(n)}$ in $\mathcal{M}$ such that $\|P^{(n)} - U^{(n)}P^{(n)}U^{(n)*}\|_2 > \sqrt{\lambda - \lambda^2} - 1/n$, otherwise by the Dixmier approximation thereom ([KR] Thereom 8.3.5) we would have

$$\sqrt{\lambda - \lambda^2} = \|P^{(n)} - \tau(P^{(n)})\|_2 \leq \sqrt{\lambda - \lambda^2} - 1/n.$$

Let $U = \{U^{(n)}\}_n$. Then $\|UP - PU\|_\omega \geq \sqrt{\lambda - \lambda^2}$, U does not commute with P and hence $\mathcal{M}^\omega$ is a factor. Since $\mathcal{M} \subset \mathcal{M}^\omega$ and τ_ω is a trace on $\mathcal{M}^\omega$, $\mathcal{M}^\omega$ is a factor of type II_1. Alternatively, observing that any two projections with the same trace in $\mathcal{M}^\omega$ are equivalent in $\mathcal{M}^\omega$, $\mathcal{M}^\omega$ is a factor.

Step III. Embed $\otimes_1^\infty M_2(\mathbb{C}) \simeq \mathcal{R}$ into $\mathcal{M}$ as a subfactor. Define $U_{t_j} \in M_2(\mathbb{C})$ as I if $t_j = 0$; $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if $t_j = 1$. For any sequence $t = (t_j) \in \{0,1\}^\infty$, define $U_t = \{\otimes_{j=1}^n U_{t_j}\}_n \in \mathcal{M}^\omega$. If $s, t \in \{0,1\}^\infty$ are not equal at some j_0, then $\tau(\otimes_{j=1}^n U_{s_j})(\otimes_{j=1}^n U_{t_j}) = 0, \forall n \geq j_0$. Therefore $\tau_\omega(U_s U_t) = 0$ and $\{U_t : t \in \{0,1\}^\infty\}$ is an orthogonal set in $L^2(\mathcal{M}^\omega)$. Thus $\mathcal{M}^\omega$ is not separable under the trace norm. ■

In particular, for the hyperfinite factor $\mathcal{R}$ of type II_1, the ultrapower $\mathcal{R}^\omega$ of $\mathcal{R}$ is a non-separable factor of type II_1.

What we would like to mention here is that if we replace each summand of $\oplus_\infty \mathcal{M}$ by a finite factor with its trace, one can get a finite factor again. For example, for any free ultrafilter ω on $\mathbb{N}$, $M_n(\mathbb{C})^\omega = M_n(\mathbb{C})$. Suppose $\{n_k\}_k$ is a increasing sequence of natural numbers and $n_k \to \infty$ as $k \to \infty$. The ultraproduct $M_{n_k}(\mathbb{C})^\omega$ of matrix algebras given by

$$M_{n_k}(\mathbb{C})^\omega = \oplus_{k=1}^\infty M_{n_k}(\mathbb{C})/\mathcal{I}_\omega,$$

where

$$I_\omega = \left\{\{X^{(k)}\}_k \in \oplus_{k=1}^\infty M_{n_k}(\mathbb{C}) : \lim_{k\to\omega} tr_{n_k}(X^{(k)*}X^{(k)}) = 0\right\},$$

and tr_{n_k} is the normalized trace on $M_{n_k}(\mathbb{C})$. Moreover $M_{n_k}(\mathbb{C})^\omega$ is a factor of type II_1.

Without specification, throughout this section, ω will denote a free ultrafilter on $\mathbb{N}$.

Theorem 36 ([GeH01, Con75, SM08]) *Let $\mathcal{M}$ be a separable factor of type II_1, $\mathcal{M}^\omega$ an ultrapower of $\mathcal{M}$, $\mathcal{M}_\omega$ the relative commutant of $\mathcal{M}$ in $\mathcal{M}^\omega$.*

1) Any self-adjoint, positive, unitary element or projection $A \in \mathcal{M}^\omega$ or $\mathcal{M}_\omega$ can be represented by a sequence $\{A^{(n)}\}$ of self-adjoint, positive, unitary elements or projections in $\mathcal{M}$.

2) Let E, F be equivalent projections in $\mathcal{M}^\omega$ or $\mathcal{M}_\omega$; i.e. $E \overset{V}{\sim} F$. V has a representing sequence of partial isometries in $\mathcal{M}$.

3) Any $p \times p$ matrix units in $\mathcal{M}^\omega$ or $\mathcal{M}_\omega$ can be represented by a sequence of $p \times p$ matrix units in $\mathcal{M}$.

Proof. The details of the proof can be found in [GeH01, Con75, SM08]. ■

With the continuum hypothesis, Ge and Hadwin [GeH01] proved the following amazing theorem:

Theorem 37 ([GeH01]) *Assume the continuum hypothesis. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal trace τ. If $\mathcal{M}$ is trace-norm separable, then $\mathcal{M}^{\omega}$ and $\mathcal{M}^{\omega'}$ are *-isomorphic von Neumann algebras for any free ultrafilters ω and ω' on $\mathbb{N}$. Moreover, the relative commutant of $\mathcal{M}$ in $\mathcal{M}^{\omega}$ is *-isomorphic to that of $\mathcal{M}$ in $\mathcal{M}^{\omega'}$.*

It is known that property Γ of factors of type II_1 can distinct free group factors and the hyperfinite factor. Recall that a factor $\mathcal{M}$ of type II_1 has property Γ if for any given $n \in \mathbb{N}$, finitely many elements $X_1, \ldots, X_n$ in $\mathcal{M}$ and $\epsilon > 0$, there exists trace-zero unitary element $U \in \mathcal{M}$ such that $\|UX_i - X_iU\|_2 < \epsilon$ for $i = 1, \ldots, n$. In 1943, Murray and von Neumann [MV43] proved that $\mathcal{R}$ has property Γ and so does $\mathcal{R}^{\omega}$. In general, we have

Proposition 38 *Suppose $\mathcal{M}$ is a factor of type II_1. $\mathcal{M}$ has property Γ if and only if $\mathcal{M}^{\omega}$ has property Γ.*

Proof. Suppose $\mathcal{M}$ has property Γ. For m in $\mathbb{N}$, $A_1, \ldots, A_m$ in $\mathcal{M}^{\omega}$, write $A_k = \{A_k^{(n)}\}_n$, $k = 1, \ldots, m$. For $A_k^{(j)}$, $1 \le k \le m, 1 \le j \le n$, there exists unitary element $U^{(n)} \in \mathcal{M}$ with trace zero such that $\|U^{(n)}A_k^{(j)} - A_k^{(j)}U^{(n)}\| < 1/n$. Let $U = \{U^{(n)}\}$. Then $\|UA_k - A_kU\|_{\omega} = 0$ and $UA_k = A_kU$.

Suppose $\mathcal{M}^{\omega}$ has property Γ. For any $A_1, \ldots, A_m \in \mathcal{M}$, $m \ge 1$, $m \in \mathbb{N}$, $\epsilon > 0$, and since $A_1, \ldots, A_m$ can be viewed as elements in $\mathcal{M}^{\omega}$, there is a unitary element U in $\mathcal{M}^{\omega}$ such that $\|UA_k - A_kU\|_{\omega} < \epsilon/2$. Writing $U = \{U^{(n)}\}_n$, we see there is a $U^{(n_0)}$ in $\{U^{(n)}\}$ such that $\|U^{(n_0)}A_k - A_kU^{(n_0)}\|_2 < \epsilon$. ■

A factor $\mathcal{M}$ of type II_1 is a prime factor if $\mathcal{M}$ is not (isomorphic to) a tensor product of two factors of type II_1. S. Popa and L. Ge etc show that for any factor $\mathcal{M}$ of type II_1, $\mathcal{M}^{\omega}$ is a prime factor and has no Cartan subalgebras [FGL06]. Let $\mathfrak{M}_k^{sa}$ be the set of all self-adjoint elements in $M_k(\mathbb{C})$, and $\mathcal{U}(\mathfrak{M}_k^{sa})$ be the set of all unitary elements in $\mathfrak{M}_k^{sa}$.

Lemma 39 (See [Vo94], lemma 4.3) *Given $\epsilon > 0$ there is $N \in \mathbb{N}$ and $\delta > 0$, so that for all $k \in \mathbb{N}$, $A, B \in \mathfrak{M}_k^{sa}$, $\|A\| \leq 1$ if $|\tau_k(A^p) - \tau_k(B^p)| < \delta$ for $1 \leq p \leq N$, then there is $U \in \mathscr{U}(\mathfrak{M}_k^{sa})$, so that $\tau_k((B - UAU^*)^2) < \epsilon$.*

Lemma 40 (See [Po83]) *Suppose $\mathcal{M}$ is a factor of type II_1 and ω a free ultrafilter on $\mathbb{N}$. Let $\mathcal{M}^\omega$ be the ultrapower of $\mathcal{M}$ and $\mathcal{A}_1$, $\mathcal{A}_2$ be two non-atomic abelian von Neumann subalgebras of $\mathcal{M}^\omega$ with separable preduals. Then there is a unitary element U in $\mathcal{M}^\omega$ such that $U^*\mathcal{A}_1 U = \mathcal{A}_2$.*

Proof. Since $\mathcal{A}_1$, $\mathcal{A}_2$ are non-atomic abelian von Neumann algebras with separable preduals, they are isomorphic to $L^\infty[0, 1]$. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are generated by Haar unitary elements U_1 and U_2 respectively. Write $U_1 = \{U_1^{(n)}\}_n$ and $U_2 = \{U_2^{(n)}\}_n$ for $U_1^{(n)}$ and $U_2^{(n)}$ in $\mathcal{M}$. We may assume that $U_j^{(n)}$ lies in a finite dimensional abelian subalgebra of $\mathcal{M}$ (otherwise, replace $U_j^{(n)}$ by such an element close to it in trace norm). Since U_1 and U_2 are Haar unitary elements, we may assume that $U_1^{(n)}$ and $U_2^{(n)}$ have the same distribution by Lemma 39 and $U_1^{(n)} = \sum_{j=1}^{s_n} \lambda_j E_j^{(n)}$, $U_2^{(n)} = \sum_{j=1}^{s_n} \lambda_j F_j^{(n)}$ for $E_1^{(n)}, \ldots, E_{s_n}^{(n)}$ and $F_1^{(n)}, \ldots, F_{s_n}^{(n)}$ in $\mathcal{M}$ such that $\tau(E_i^{(n)}) = \tau(F_j^{(n)})$, $\sum_{j=1}^{s_n} E_j^{(n)} = \sum_{j=1}^{s_n} F_j^{(n)} = I$. From [KR],Lemma 12.2.5, there is a unitary element $U^{(n)}$ in $\mathcal{M}$ such that $(U^{(n)})^* E_j^{(n)} U^{(n)} = F_j^{(n)}$ for all $j = 1, \ldots, s_n$. Then $(U^{(n)})^* U_1^{(n)} U^{(n)} = U_2^{(n)}$. Let $U = \{U^{(n)}\}_n$ in $\mathcal{M}^\omega$. Then $U^* U_1 U = U_2$ and $U^*\mathcal{A}_1 U = \mathcal{A}_2$. ■

Lemma 41 (See [Po83]) *Suppose ω is a free ultrafilter on $\mathbb{N}$. Let $\mathcal{A}_1$, $\mathcal{A}_2$ be two non-atomic abelian von Neumann subalgebras of $\mathcal{R}_\omega$ with separable preduals. Then there is a unitary element U in $\mathcal{R}_\omega$ such that $U^*\mathcal{A}_1 U = \mathcal{A}_2$.*

Proof. The proof of this lemma is similar to Lemma 40. The only difference is that the resulting unitary element U lies in $\mathcal{R}_\omega$. Since $\mathcal{R}$ is hyperfinite, we may choose full matrix subalgebras $M_{2^k}(\mathbb{C}) \subseteq M_{2^{k+1}}(\mathbb{C})$ such that $\cup_{k=1}^\infty M_{2^k}(\mathbb{C})$ is weak-operator dense in $\mathcal{R}$. once more $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic to $L^\infty[0, 1]$. Suppose $\mathcal{A}_1$, $\mathcal{A}_2$ are generated by Haar unitary elements U_1 and U_2 respectively. Write $U_1 = \{U_1^{(n)}\}$ and $U_2 = \{U_2^{(n)}\}$ for $U_1^{(n)}$ and $U_2^{(n)}$ in $\mathcal{R}$. Since U_1, U_2 commute with $\mathcal{R}$, we may assume that $U_1^{(n)}$ and $U_2^{(n)}$ commute with $M_{2^n}(\mathbb{C})(\subset \mathcal{R})$. We may also assume that $U_j^{(n)}$ lies in a finite dimensional abelian subalgebra of $M_{2^n}(\mathbb{C})' \cap \mathcal{R}$ for $j = 1, 2$. Since U_1 and U_2 are Haar unitary elements, we may assume that $U_1^{(n)}$ and $U_2^{(n)}$ have the same distribution by Lemma 39 and $U_1^{(n)} = \sum_{j=1}^{s_n} \lambda_j E_j^{(n)}$, $U_2^{(n)} = \sum_{j=1}^{s_n} \lambda_j F_j^{(n)}$ for $E_1^{(n)}, \ldots, E_{s_n}^{(n)}$ and $F_1^{(n)}, \ldots, F_{s_n}^{(n)}$ in $M_{2^n}(\mathbb{C})' \cap \mathcal{R}$ such that $\tau(E_i^{(n)}) = \tau(F_j^{(n)})$, $\sum_{j=1}^{s_n} E_j^{(n)} = \sum_{j=1}^{s_n} F_j^{(n)} = I$. From [KR], Lemma 12.2.5, there is a unitary element $U^{(n)}$ in $M_{2^n}(\mathbb{C})' \cap \mathcal{R}$ such that $(U^{(n)})^* E_j^{(n)} U^{(n)} = F_j^{(n)}$ for all $j = 1, \ldots, s_n$. Then $(U^{(n)})^* U_1^{(n)} U^{(n)} = U_2^{(n)}$. Let $U = \{U^{(n)}\}_n$ in $\mathcal{R}^\omega$. Then $U \in \mathcal{R}_\omega$, $U^* U_1 U = U_2$ and $U^* \mathcal{A}_1 U = \mathcal{A}_2$. ■

Lemma 42 (See [Po83]) *Suppose $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{M}$, where $\mathcal{M}$ is a type II_1 von Neumann algebra with a trace τ. Let U be a unitary operator in $\mathcal{M}$ such that for any $\epsilon > 0$, there is a finite dimensional abelian von Neumann algebra $\mathcal{A}_\epsilon$ of $\mathcal{B}$ such that $\tau(E) < \epsilon$ for all minimal projections E in $\mathcal{A}_\epsilon$, and $U\mathcal{A}_\epsilon U^*$ and $\mathcal{B}$ are orthogonal with respect to τ, then U is orthogonal to the set of normalizers $\{V \in \mathcal{M} : V\mathcal{B}V^* = \mathcal{B}, V \text{ unitary}\}$ of $\mathcal{B}$ in $\mathcal{M}$, denoted by $\mathcal{N}(\mathcal{B})$. In particular, U is orthogonal to $\mathcal{B}$ and $\mathcal{B}' \cap \mathcal{M}$.*

Proof. Let $E_1, \ldots, E_n$ be minimal projections in $\mathcal{A}_\epsilon$ and $\sum_i E_i = I$. Then for any $V \in \mathcal{N}(\mathcal{B})$ and $\epsilon > 0$, we have

$$\tau(UE_iU^*V^*E_iV) = \tau(UE_iU^*)\tau(V^*E_iV) = \tau(E_i)^2, \quad \forall i.$$

This implies:

$$\begin{aligned}
|\tau(VU)|^2 &\leq \|VU\|_2^2 = \|E_{\mathcal{A}_\epsilon' \cap \mathcal{M}}(VU)\|_2^2 \\
&= \|\sum_i E_i VUE_i\|_2^2 = \sum_i \|E_i VUE_i\|_2^2 \\
&= \sum_i \tau(VUE_iU^*V^*E_i) = \sum_i \tau(E_i)^2 \leq \epsilon.
\end{aligned}$$

Therefore $\tau(VU) = 0$. Since $\mathcal{B}$ is the span of $\mathcal{N}(\mathcal{B})$ and for any $T \in \mathcal{B}' \cap \mathcal{M}$,

$$\begin{aligned}
|\tau(TU)|^2 &\leq \|TU\|_2^2 = \|E_{\mathcal{A}_\epsilon' \cap \mathcal{M}}(TU)\|_2^2 \\
&= \|\sum_i E_i TUE_i\|_2^2 = \sum_i \|E_i TUE_i\|_2^2 \\
&= \sum_i \tau(TUE_iU^*T^*E_i) \leq \|T\|^2 \sum_i \tau(E_i)^2 \leq \|T\|^2\epsilon.
\end{aligned}$$

Then $\tau(TU) = 0, \forall T \in \mathcal{B}' \cap \mathcal{M}$ and U is orthogonal to $\mathcal{B}$ and $\mathcal{B}' \cap \mathcal{M}$. ■

A subalgebra $\mathcal{B}$ of a von Neumann algebra $\mathcal{M}$ is a Cartan subalgebra if span$\mathcal{N}(\mathcal{B}) = \mathcal{M}$.

Theorem 43 (See [FGL06, Po83]) *$\mathcal{M}^\omega$ is prime and has no Cartan subalgebras. Moreover, $\mathcal{R}_\omega$ is also a prime factor of type II_1 and has no Cartan subalgebras.*

Proof. Suppose $\mathcal{M}^\omega = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ for some factors $\mathcal{M}_1$ and $\mathcal{M}_2$ of type II_1. Choose non-atomic abelian subalgebras $\mathcal{A}_1$ of $\mathcal{M}_1$ and $\mathcal{A}_2$ of $\mathcal{M}_2$ such that $\mathcal{A}_1, \mathcal{A}_2$ are weak-operator separable. From Lemma 40, there is a unitary element U in $\mathcal{M}^\omega$ such that $U^*\mathcal{A}_1 U = \mathcal{A}_2$ which is orthogonal to $\mathcal{M}_1 \otimes \mathbb{C}I$. From Lemma 42, U is orthogonal to the normalizers of $\mathcal{M}_1$ in $\mathcal{M}^\omega$. But the normalizers of $\mathcal{M}_1$ generate $\mathcal{M}^\omega$ as a von Neumann algebra. This contradicts the assumption that U lies in $\mathcal{M}^\omega$. Therefore $\mathcal{M}^\omega$ is prime. Similarly, using Lemma 41, we can show that $\mathcal{R}_\omega$ is also prime.

Suppose $\mathcal{A}$ is a MASA in $\mathcal{M}^\omega$. Let $\mathcal{B}$ be a separable diffuse abelian von Neumann subalgebra of $\mathcal{A}$. Then $\mathcal{B}$ is isomorphic to $L^\infty[0,1]$ and suppose $\mathcal{B}$ is generated by a Haar unitary U. Write $U = \{U^{(n)}\}_n$, we may assume that $U^{(n)}$ lies in a finite dimensional algebra and $U^{(n)} = \sum_{i=1}^{s_n} \lambda_i E_{ii}^{(n)}$, where $\{E_{ij}^{(n)}\}_{i,j=1}^{s_n}$ is a self-adjoint system of matrix units. Let $V^{(n)} = \sum_{i=1}^{s_n-1} E_{i,i+1}^{(n)} + E_{s_n,1}$ and $V = \{V^{(n)}\}$. Then V is a Haar unitary and $C = \{V\}''$ is orthogonal to $\mathcal{B}$ and $\mathcal{B}' \cap \mathcal{M}^\omega$. By Lemma 40, there exists W such that $W\mathcal{B}W^* = C$. Then by Lemma 42 W is orthogonal to $\mathcal{A}$ and $\mathcal{N}(\mathcal{A})''$. Therefore $\mathcal{A}$ is not a Cartan subalgebra. Similarly, by Lemma 41, $\mathcal{R}_\omega$ has no Cartan subalgebras. ■

Lemma 44 (See [FGL06]) *Suppose $\mathcal{M}$ is a subfactor of $\mathcal{R}^\omega$ with a separable predual. Then $\mathcal{M}' \cap \mathcal{R}^\omega$ contains a 2×2 full matrix algebra.*

Proof. Suppose $A_1, A_2, \ldots$ are in the unit ball of $\mathcal{M}$ so that they are ultraweakly dense in the ball. Write $A_j = \{A_j^{(n)}\}_n$ with $A_j^{(n)}$ in $\mathcal{R}$. For any given n and $\{A_l^{(k)} : 1 \le k, l \le n\}$, there is a 2×2 matrix unit system $\{E_{st}^{(n)}\}_{s,t=1}^2$ in $\mathcal{R}$ such that $\|A_l^{(k)}E_{st}^{(n)} - E_{st}^{(n)}A_l^{(k)}\|_2 \le \frac{1}{n}$, for $1 \le k, l \le n$ and $1 \le s, t \le 2$. Let $E_{st} = \{E_{st}^{(n)}\}_n$ in $\mathcal{R}^\omega$. Then $\{E_{st}\}_{s,t=1}^2$ commutes with $A_1, A_2, \ldots$, and is a 2×2 matrix unit system in $\mathcal{R}^\omega$. ■

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Since

$$(\oplus_\infty \mathcal{N} + \mathcal{I}_\omega)/\mathcal{I}_\omega \simeq (\oplus_\infty \mathcal{N})/(\oplus_\infty \mathcal{N}) \cap \mathcal{I}_\omega$$

$\mathcal{N}^\omega$ can be embedded into $\mathcal{M}^\omega$ as a von Neumann subalgebra.

Lemma 45 *For any $\epsilon > 0$ and $A \in \mathcal{M}$, there is unitary element U such that*

$$\|UA - AU\|_2 \geq \|A - \mathbb{E}_{\mathcal{N}' \cap \mathcal{M}}(A)\|_2 - \epsilon,$$

where $\mathbb{E}_{\mathcal{N}'\cap\mathcal{M}}$ is the trace preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}' \cap \mathcal{M}$.

Proof. Suppose that

$$\|UA - AU\|_2 = \|UAU^* - A\|_2 < \|A - \mathbb{E}_{\mathcal{N}'\cap\mathcal{M}}(A)\|_2 - \epsilon (= \alpha)$$

for all unitary elements U in $\mathcal{N}$. Let $Co\{UAU^* : U \in \mathcal{N}\}$ be the minimal convex set containing all UAU^* with U a unitary in $\mathcal{N}$. For any $X \in Co\{UAU^* : U \in \mathcal{N}\}$, we have $\|X - A\|_2 < \alpha$. But $\mathbb{E}_{\mathcal{N}'\cap\mathcal{M}}(A)$ lies in the weak-operator closure of $Co\{UAU^* : U \in \mathcal{N}\}$ and we shall have contradiction $\|\mathbb{E}_{\mathcal{N}'\cap\mathcal{M}}(A) - A\|_2 \leq \alpha$. The lemma follows. ∎

Lemma 46 $(\mathcal{N}^\omega)' \cap \mathcal{M}^\omega = (\mathcal{N}' \cap \mathcal{M})^\omega$.

Proof. From $\oplus_\infty(\mathcal{N}' \cap \mathcal{M}) = (\oplus_\infty \mathcal{N}') \cap (\oplus_\infty \mathcal{M})$, we obtain $(\mathcal{N}' \cap \mathcal{M})^\omega \subseteq (\mathcal{N}^\omega)' \cap \mathcal{M}^\omega$. For any $X = \{X^{(n)}\} \in (\mathcal{N}^\omega)' \cap \mathcal{M}^\omega$, we see $\{\mathbb{E}_{\mathcal{N}'\cap\mathcal{M}}(X^{(n)})\}$ is in $(\mathcal{N}' \cap \mathcal{M})^\omega$. For $X^{(n)} \in \mathcal{M}$, there exists unitary element $U^{(n)} \in \mathcal{N}$ such that

$$\|U^{(n)}X^{(n)} - X^{(n)}U^{(n)}\|_2 \geq \|X^{(n)} - \mathbb{E}_{\mathcal{N}'\cap\mathcal{M}}(X^{(n)})\|_2 - 1/n.$$

Let $U = \{U^{(n)}\} \in \mathcal{N}^\omega$. Then

$$\|UX - XU\|_\omega \geq \|X - \{\mathbb{E}_{\mathcal{N}'\cap\mathcal{M}}(X^{(n)})\}\|_\omega,$$

but $UX = XU$, therefore $X = \{\mathbb{E}_{\mathcal{N}'\cap\mathcal{M}}(X^{(n)})\} \in \mathcal{N}' \cap \mathcal{M}$. ∎

Let $\{\mathcal{N}_n\}$ be a sequence of von Neumann subalgebras of $\mathcal{M}$. Let $\mathcal{N}_n^\omega = \oplus_\infty \mathcal{N}_n + \mathcal{I}_\omega/\mathcal{I}_\omega$. By the proof of the lemma above, we actually have

$$(\mathcal{N}_n^\omega)' \cap \mathcal{M}^\omega = (\mathcal{N}_n' \cap \mathcal{M})^\omega.$$

Proposition 47 *Suppose $\{\mathcal{A}_n\}$ is a sequence of MASAs of $\mathcal{M}$. Then $\mathcal{A}_n^\omega$ is a MASA in $\mathcal{M}^\omega$.*

Proof. From $\mathcal{A}_n = \mathcal{A}_n' \cap \mathcal{M}$, we get

$$\mathcal{A}_n^\omega = (\mathcal{A}_n' \cap \mathcal{M})^\omega = (\mathcal{A}_n^\omega)' \cap \mathcal{M}^\omega.$$

■

Problem 48 *Are these all the MASAs of $\mathcal{M}^\omega$? If not, what is a counterexample?*

Proposition 49 (See [Po81]) *No MASA of $\mathcal{M}^\omega$ is separable.*

Proof. Suppose $\mathcal{A}$ is a separable MASA of $\mathcal{M}^\omega$. Then $\mathcal{A}$ can be generated by a positive element $A = \{A^{(n)}\}$, where $A^{(n)}$ are positive elements. Let $\mathcal{A}_n \subset \mathcal{M}$ be a MASA in $\mathcal{M}$ such that $A^{(n)} \in \mathcal{A}_n$. Then $\oplus_\infty \mathcal{A}_n + \mathcal{I}_\omega / \mathcal{I}_\omega$ is abelian and contains A, so it is $\mathcal{A}$. Since $\mathcal{M}^\omega$ is continuous and $\mathcal{A}$ is separable and maximal abelian, one can find projections $\{E_{k,n}\}_{\substack{1 \le k \le 2^n \\ n \ge 0}} \subset \mathcal{A}$ such that

1) $\overline{\text{span}}^w \{E_{k,n}\} = \mathcal{A}$;

2) $\tau_\omega(E_{k,n}) = 2^{-n}, 1 \le k \le 2^n, n \ge 0$;

3) $E_{2k-1,n} + E_{2k,n} = E_{k,n-1}$.

One can choose by induction over n, k, sequence $(E_{k,n}^{(m)})_m$ in $\oplus_\infty \mathcal{A}$ such that

1) $\overline{\text{span}}^w \{E_{k,n}^{(m)}\} = \mathcal{A}$;

2) $\tau_\omega(E_{k,n}^{(m)}) = 2^{-n}, 1 \le k \le 2^n, n \ge 0$;

3) $E_{2k-1,n}^{(m)} + E_{2k,n}^{(m)} = E_{k,n-1}^{(m)}$.

Take $E^{(m)} = \sum_{k=1}^{2^m} E_{2k-1,m}^{(m)}$ and let $E = \{E^{(m)}\}$. Then $E \in \mathcal{A}$ and $\tau_\omega(E) = 1/2$. Moreover $\tau_\omega(E E_{k,n}) = 1/2 \tau_\omega(E_{k,n})$ for all k, n so that $\tau_\omega(EX) = 1/2 \tau_\omega(X)$ for all $X \in \mathcal{A}$. In particular $\tau_\omega(E) = \tau_\omega(E \cdot E) = 1/4$ which is a contradiction. ■

Lemma 50 *Suppose ω, ω' are free ultrafilters on $\mathbb{N}$. Then*

$$(\mathcal{M}^{\omega})^{\omega'} = \mathcal{M}^{\omega\otimes\omega'}.$$

Proof. Any $X = (\mathcal{M}^{\omega})^{\omega'}$ may be represented by a with representing sequence $\{X_n\}_n \subset \mathcal{M}^{\omega}$. Similarly, write $X_n = \{X_n^{(k)}\}_k$, where $X_n^{(k)} \in \mathcal{M}$. Therefore $X = \{X_n^{(k)}\}_{k,n}$ and $\{X_n^{(k)}\}_{k,n}$ could be viewed as elements in $\mathcal{M}^{\omega\otimes\omega'}$. By Lemma 34, the lemma holds. ■

Let $\mathcal{M}$ be a factor of type II_1 with the trace τ acting on the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, \tau)$. Let $\mathcal{H}^{\omega}$ be the ultraproduct of copies of $\mathcal{H}$, which is the Hilbert space of all the equivalence classes of elements in $\oplus_{\infty}\mathcal{H}$ with respect to equivalence relation that $(\xi^{(n)}) \sim (\eta^{(n)})$ if and only if $\lim_{\omega} \|\xi^{(n)} - \eta^{(n)}\| = 0$. $\mathcal{H}^{\omega}$ is a Hilbert space with inner product $\langle\{\xi^{(n)}\}, \{\eta^{(n)}\}\rangle = \lim_{\omega}\langle\xi^{(n)}, \eta^{(n)}\rangle$. In general, $\mathcal{M}^{\omega}$ does not act on $\mathcal{H}^{\omega}$. However $\mathcal{M}^{\omega}$ acts on a subspace of $\mathcal{H}^{\omega}$.

Proposition 51 (See [Con76]) *Let $\mathcal{H}_{\omega}$ be the set of $\xi = \{\xi^{(n)}\} \in \mathcal{H}^{\omega}$ which satisfy that for any $\epsilon > 0$, there exists $a > 0$ such that*

$$\lim_{n\to\omega} \|E_{(a,\infty)}(|\xi^{(n)}|)|\xi^{(n)}|\|_2 < \epsilon.$$

where $E_{(a,\infty)}(|\xi^{(n)}|)$ is the spectral projection of $|\xi^{(n)}|$ corresponding to (a, ∞). Then $\mathcal{H}_{\omega}$ is a closed subspace of $\mathcal{H}^{\omega}$ and $\mathcal{M}^{\omega}$ acts on $\mathcal{H}_{\omega}$ in a standard way with the vector $I = \{I\}$ as cyclic and separating trace vector and the map $\{\xi^{(n)}\} \mapsto \{J\xi^{(n)}\}$ as canonical involution, where J is involution of $\mathcal{M}$.

Proof. We have to check that $\mathcal{H}_{\omega}$ is the closure in $\mathcal{H}^{\omega}$ of the set of vectors $\{x^{(n)}\}$, $\|x^{(n)}\|_{\infty}$ bounded. Assume that $\xi = \{\xi^{(n)}\} \in \mathcal{H}_{\omega}$ and let $\epsilon > 0$. Then for some $a > 0$ one has $\lim_{k\to\infty} \|\xi_k E_{(a,\infty)}|\xi_k|\| < \epsilon$ so that the vector $\eta = \{\eta_k\}_k$, $\eta_k = \xi_k(I - E_{(a,\infty)})|\xi_k|$ is at less than ϵ of ξ and satisfies $\|\eta_k\|_{\infty} \leq a$ for all $k \in \mathbb{N}$. Conversely, let $\epsilon \in (0, 1)$ and $a > 0$ and assume that $\|\xi_k\|_2 \leq 1$, $\|x^{(n)} - \xi^{(n)}\|_2 \leq \epsilon$ for all k, where $\|x^{(n)}\|_{\infty} \leq a$ for all $k \in \mathbb{N}$. By inequality (see[Con76],Proposition 1.2.1)

$$\||A| - |B|\|_2^2 \leq \||A|^2 - |B|^2\|_1 \leq \|A - B\|_2(\|A\|_2 + \|B\|_2), \forall A, B \in \mathcal{M},$$

$\||x^{(n)}| - |\xi^{(n)}|\|_2 \leq (3\epsilon)^{1/2}$ and then $\||\xi^{(n)}|E_{(a,\infty)}(|\xi^{(n)}|)\|_2 < 2(3\epsilon)^{1/2}$. ■

4.2 Embeddings into Ultrapower

Let $\mathcal{M}$ be a finite von Neumann algebra with a separable predual and a faithful normal tracial state τ, and let $\mathcal{M}_{s.a.}$ be the set of all self-adjoint elements in $\mathcal{M}$. For all n in $\mathbb{N}$ and $X_1, \ldots, X_n$ in $\mathcal{M}$ with $X_j = X_j^*$ for $j = 1, \ldots, n$, finite set $\mathcal{S} = \{X_1, X_2, \ldots, X_n\} \subset \mathcal{M}_{s.a.}$ has matricial microstates if for every m in $\mathbb{N}$ and every $\epsilon > 0$ there are $k \in \mathbb{N}$ and $k \times k$ matrices $A_1, \ldots, A_n$ such that whenever $1 \leq p \leq m$ and $i_1, \ldots, i_p \in \{1, \ldots, n\}$, we have

$$|tr_k(A_{i_1}A_{i_2}\cdots A_{i_p}) - \tau(X_{i_1}X_{i_2}\cdots X_{i_p})| < \epsilon,$$

where tr_k is the normalized trace on $M_k(\mathbb{C})$.

A von Neumann algebra $\mathcal{M}$ with a separable predual and a faithful normal tracial state τ is *embeddable* into $\mathcal{R}^\omega$ if there is a *-isomorphism Φ of $\mathcal{M}$ into an ultrapower $\mathcal{R}^\omega$ of $\mathcal{R}$ with $\tau_\omega \circ \Phi = \tau$.

Proposition 52 *Suppose $\mathcal{M}$ is a von Neumann algebra with a separable predual and a faithful normal tracial state τ. Then the following are equivalent:*
1) $(\mathcal{M}, \tau)$ is embeddable into $\mathcal{R}^\omega$
2) Any finite subset $\mathcal{S} \subset \mathcal{M}_{s.a.}$ has matricial microstates.
3) If $\mathcal{S}_0 \subset \mathcal{M}_{s.a.}$ is a generating set for $\mathcal{M}$ (i.e. the von Neumann algebra generated by $\mathcal{S}_0$ is $\mathcal{M}$), then any finite subset $\mathcal{S}$ of $\mathcal{S}_0$ has matricial microstates.

Proof. 1) $\Rightarrow$ 2): $\forall m \in \mathbb{N}$, let $X_1, \ldots, X_m$ be any self-adjoint elements in $\mathcal{M}$. Since $\mathcal{M}$ can be embedded into $\mathcal{R}^\omega$, we identify $\mathcal{M}$ as a von Neumann subalgebra of $\mathcal{R}^\omega$. Then $\tau = \tau_\omega|_\mathcal{M}$, $X_j = \{X_j^{(n)}\}_n$ and $X_j^{(n)} \in \mathcal{R}$ for $j = 1, \ldots, n$. By [KR], Theorem 12.2.2, for $\epsilon > 0$ and $X_j^{(n)}, j = 1, \ldots, n$ there is a finite type I subfactor $\mathcal{N}$ of $\mathcal{R}$ isomorphic to $M_k(\mathbb{C})$ for some $k \in \mathbb{N}$ and $A_j^{(n)} \in \mathcal{N}, j = 1, \ldots, m$ such that $\|X_j^{(n)} - A_j^{(n)}\|_2 < \epsilon$. Assume that $X_j^{(n)}, j = 1, \ldots, m$ lies in the same finite type I subfactor. Since for any $l \in \mathbb{N}$, $1 \leq p \leq l$,$i_1, \ldots, i_p \in \{1, \ldots, m\}$, $X_{i_1} \cdots X_{i_p} = \{X_{i_1}^{(n)} \cdots X_{i_p}^{(n)}\}_n$, for any $\epsilon > 0$, there is an integer $N > 0$ such that $|\tau_\omega(X_{i_1} \cdots X_{i_p}) - \tau_\mathcal{R}(X_{i_1}^{(n)} \cdots X_{i_p}^{(n)})| < \epsilon$ when $n > N$. Since $\mathcal{M}$ is a subfactor of $\mathcal{R}^\omega$ and $\mathcal{N}$ is a subfactor of $\mathcal{R}$, we have that $\tau = \tau_\omega|_\mathcal{M}$ and the trace $\tau_\mathcal{N}$ is $\tau_\mathcal{R}|_\mathcal{N}$. If we identify

$\mathcal{N}$ as $M_k(\mathbb{C})$, then $\tau_{\mathcal{N}} = tr_k$. Therefore $|\tau(X_{i_1} \cdots X_{i_p}) - tr_k(X_{i_1}^{(n)} \cdots X_{i_p}^{(n)})| < \epsilon$ and $\{X_1, \ldots, X_m\}$ has matricial microstates.

2) $\Rightarrow$ 3): 3) follows directly from 2).

3) $\Rightarrow$ 1): Suppose $X_1, X_2, \ldots$ is a generating set for $\mathcal{M}$ whose elements are self-adjoint. For any integer $m \geq 1$, $\{X_1, \ldots, X_m\}$ is a finite subset of the generating set. Then there are $k_m \in \mathbb{N}$ and $k_m \times k_m$ matrices $A_j^{(m)}, j = 1, \ldots, m$ such that whenever $1 \leq p \leq m$ and $i_1, \ldots, i_p \in \{1, \ldots, m\}$, we have

$$|\tau_{k_m}(A_{i_1}^{(m)} A_{i_2}^{(m)} \cdots A_{i_p}^{(m)}) - \tau(X_{i_1} X_{i_2} \cdots X_{i_p})| < 1/m,$$

where τ_{k_m} is the normalized trace on $M_{k_m}(\mathbb{C})$. Let $A_j = \{A_j^{(m)}\}_m \in \mathcal{R}^\omega, j = 1, \ldots$. Then

$$\tau_\omega(A_{i_1} A_{i_2} \cdots A_{i_p}) = \tau(X_{i_1} X_{i_2} \cdots X_{i_p})$$

$i_1, \ldots, i_p \in \{1, 2, \ldots\}$. We define a homomorphism Ψ from the the algebra generated by $X_1, \ldots, X_n, \ldots$ to the algebra generated by $A_1, \ldots, A_n, \ldots$ such that $\Psi(X_j) = A_j, j = 1, \ldots, n, \ldots$. By the equation above, we can obtain $\tau_\omega \circ \Psi = \tau$, Ψ is well-defined and moreover Ψ can be extended to be a *-isomorphism of $\mathcal{M}$. Therefore $(\mathcal{M}, \tau)$ can be embedded into $\mathcal{R}^\omega$. ■

In the proof of the above proposition, we have that $(\mathcal{M}, \tau)$ is embeddable into $\mathcal{R}^\omega$ if and only if $(\mathcal{M}, \tau)$ is embeddable into $M_{n_k}(\mathbb{C})_k^\omega$, for some increasing sequence $\{n_k\}$ of natural numbers.

For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the free group on n generators $g_1, \ldots, g_n$. For $m \in \mathcal{F}_n$ let the length of m be the sum of the absolute values of the exponents of the g_i in the reduced form of m. For operators $X_1, \ldots X_n$ in von Neumann algebra $\mathcal{M}$, let $\tilde{X}(m)$ be the operator obtained in replacing each g_i by the corresponding X_i, g_i^{-1} by X_i^* and finding the product in $\mathcal{M}$. So $m \mapsto \tilde{X}(m)$ is the map of $\mathcal{F}_n$ in $\mathcal{M}$ such that $X^{g_i} = X_i$, $X_i^{g_i^{-1}} = X_i^*$, $\tilde{X} = (X_1, \ldots, X_n)$.

Let $\mathcal{F}(k)$ be the set of all words $m \in \mathcal{F}_n$ whose length is less than or equal to k. In general, a finite set $\mathcal{S} = \{X_1, X_2, \ldots, X_n\} \subset \mathcal{M}$ has matricial microstates if for every $k \in \mathbb{N}$, $m \in \mathcal{F}(k)$, and $\epsilon > 0$ there are $k' \in \mathbb{N}$ and $k' \times k'$ matrices $A_1, \ldots, A_n$ such that

$$|tr_k(\tilde{A}(m)) - \tau(\tilde{X}(m))| < \epsilon,$$

where $tr_{k'}$ is the normalized trace on the $k' \times k'$ matrix algebra $M_{k'}(\mathbb{C})$.

Throughout this section, $\mathcal{M}$, $\mathcal{N}$ will be considered the von Neumann algebras with separable preduals and faithful normal tracial states $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$ respectively and suppose $(\mathcal{M}, \tau_{\mathcal{M}})$ and $(\mathcal{N}, \tau_{\mathcal{N}})$ are embeddable into $\mathcal{R}^{\omega}$.

Lemma 53 *Suppose $(\mathcal{M}, \tau_{\mathcal{M}})$ is embeddable into $\mathcal{R}^{\omega}$, P is a nonzero projection in $\mathcal{M}$ and $\tau_P = \tau_{\mathcal{M}}/\tau_{\mathcal{M}}(P)$ is a faithful normal tracial state on $P\mathcal{M}P$. Then $(P\mathcal{M}P, \tau_P)$ is embeddable into $\mathcal{R}^{\omega}$.*

Proof. Since $(\mathcal{M}, \tau)$ is embeddable into $\mathcal{R}^{\omega}$, view $\mathcal{M}$ as a subfactor of $M_{n_k}(\mathbb{C})_k^{\omega}$ for some increasing sequence $\{n_k\}_k \subset \mathbb{N}$, then P has a representing sequence $\{P^{(n)}\}_n$ where $P^{(n)}, n \geq 1$ are projections in $M_{n_k}(\mathbb{C})$. For $m \in \mathbb{N}$, $X_1, \ldots, X_m \in P\mathcal{M}P$, since $PX_iP = X_i$ and $X_i = \{X_i^{(n)}\}_n$, $i = 1, \ldots, m$, $\{P^{(n)}X_i^{(n)}P^{(n)}\}_n$ represents X_i too. Therefore $(P\mathcal{M}P, \tau_P)$ is embeddable into $(P^{(k)}M_{n_k}(\mathbb{C})P^{(k)})_k^{\omega}$ and then $\mathcal{R}^{\omega}$. ∎

In [FGL06], Fang, Ge and Li proved an interesting result on embedding. We state it below and include its proof.

Proposition 54 (See [FGL06]) *Let $\mathcal{R}$ be the hyperfinite factor of type II_1 and ω a free ultrafilter on $\mathbb{N}$. Then ultrapower $\mathcal{R}^{\omega}$ can be embedded into $\mathcal{R}_{\omega}$.*

Proof. Since $\mathcal{R} \simeq \otimes_1^{\infty}\mathcal{R}$, we shall show that $\mathcal{R}^{\omega}$ can be embedded into $(\otimes_1^{\infty}\mathcal{R})_{\omega}$. For any $A = \{A_n\}_n$ in $\mathcal{R}^{\omega}$ with A_n in $\mathcal{R}$, define $\phi(A)$ to be an element in $(\otimes_1^{\infty}\mathcal{R})^{\omega}$ corresponding to the sequence $A_1 \otimes I \otimes I \otimes \cdots, I \otimes A_2 \otimes I \otimes \cdots, \cdots$ in $(\otimes_1^{\infty}\mathcal{R})^{\omega}$. $\phi(A)$ is a central sequence and thus ϕ induces an embedding from $\mathcal{R}^{\omega}$ into $(\otimes_1^{\infty}\mathcal{R})_{\omega}$. ∎

Proposition 55 *Suppose that $(\mathcal{M}, \tau_{\mathcal{M}})$ and $(\mathcal{N}, \tau_{\mathcal{N}})$ are von Neumann algebras with faithful normal traces $\tau_{\mathcal{M}}, \tau_{\mathcal{N}}$ and separable preduals embeddable into $\mathcal{R}^{\omega}$. The von Neumann algebra tensor product $(\mathcal{M}\bar{\otimes}\mathcal{N}, \tau_{\mathcal{M}} \otimes \tau_{\mathcal{N}})$ is embeddable into $\mathcal{R}^{\omega}$.*

Proof. We shall show that for any $p, n \in \mathbb{N}$, unitary elements $X_1, \ldots, X_n \in \mathcal{M}\bar{\otimes}\mathcal{N}$, $\epsilon > 0$, there exist $k \in \mathbb{N}$ and $k \times k$ matrices $C_1, \ldots, C_n$ such that for $m \in \mathcal{F}(p)$

$$|\tau_{\mathcal{M}} \otimes \tau_{\mathcal{N}}(\tilde{X}(m)) - tr_k(\tilde{C}(m))| < \epsilon.$$

The algebraic tensor product of $\mathcal{M}$ and $\mathcal{N}$ is trace-norm dense in $\mathcal{M}\bar{\otimes}\mathcal{N}$ and so by the Kaplansky density theorem, there exist positive integers $l_1, \ldots, l_n$ and $Y_i^{(j)} \in \mathcal{M}, Z_i^{(j)} \in \mathcal{N}$, $j = 1, \ldots, l_i$, $i = 1, \ldots, n$ such that $\|X_i - \sum_{j=1}^{l_i} Y_i^{(j)} \otimes Z_i^{(j)}\|_2 < \epsilon/p$ and $\|\sum_{j=1}^{l_i} Y_i^{(j)} \otimes Z_i^{(j)}\| \leq 1$. Since $(\mathcal{M}, \tau_{\mathcal{M}})$ and $(\mathcal{N}, \tau_{\mathcal{N}})$ are embeddable into $\mathcal{R}^\omega$, for p and $Y_i^{(j)} \in \mathcal{M}, Z_i^{(j)} \in \mathcal{N}$, $j = 1, \ldots, l_i$, $i = 1, \ldots, n$, there exist $k_1, k_2 \in \mathbb{N}$, $k_1 \times k_1$ matrices $A_i^{(j)}$, and $k_2 \times k_2$ matrices $B_i^{(j)} \in \mathcal{N}$, $j = 1, \ldots, l_i$, $i = 1, \ldots, n$ such that

$$\|\tau_{\mathcal{M}}(\tilde{Y}(m)) - tr_{k_1}(\tilde{A}(m))\|_2 < \frac{\epsilon}{pl_1 \cdots l_n},$$

$$\|\tau_{\mathcal{N}}(\tilde{Z}(m)) - tr_{k_2}(\tilde{B}(m))\|_2 < \frac{\epsilon}{pl_1 \cdots l_n}.$$

Combining the two inequalities above, we obtain,

$$\|\tau_{\mathcal{M}} \otimes \tau_{\mathcal{N}}(\tilde{X}(m)) - tr_k(\tilde{C}(m))\|_2 < \epsilon,$$

where $C_i = \sum_{j=1}^{l_i} A_i^{(j)} \otimes B_i^{(j)}$, and $k = k_1 k_2$. This proves the proposition. ■

In particular, for any $k \in \mathbb{N}$, $(\mathcal{M} \otimes M_k(\mathbb{C}), \tau_{\mathcal{M}} \otimes tr_k)$ is embeddable into $\mathcal{R}^\omega$.

Lemma 56 *If any von Neumann algebra with a separable predual and a faithful normal tracial state generated by two self-adjoint elements is embeddable into $\mathcal{R}^\omega$, then any finite von Neumann algebra $\mathcal{M}$ with a separable predual and a faithful normal tracial state is embeddable into $\mathcal{R}^\omega$.*

Proof. Suppose $\mathcal{M}$ is generated by countably many self-adjoint elements $A_1, A_2, \ldots$ in its unit ball. Let $\widetilde{A_j} = \alpha_j A_j + \beta_j I$, $\alpha_j, \beta_j \in \mathbb{R}$, and choose proper α_j and β_j such that $\frac{1}{j} \leq \|\widetilde{A_j}\| \leq \frac{1}{2j(j+1)}$. Replace A_j by $\widetilde{A_j}$. Suppose $\mathcal{R} = \bigotimes_\infty M_2^{(n)}(\mathbb{C})$ is the hyperfinite II_1 factor. Let $\{E_{ts}^{(n)}\}_{t,s=1}^2$ be the 2×2 system of matrix units of the nth copy of matrix algebra. We shall show that $\mathcal{M} \otimes \mathcal{R}$ can be generated by two self-adjoint elements. Let

$$S_1 = A_1 \otimes E_{11}^{(1)} + A_2 \otimes E_{22}^{(1)} E_{11}^{(2)} + \cdots = \sum_{j=1}^{\infty} (A_j \otimes (\prod_{i=1}^{j-1} E_{22}^{(i)}) E_{11}^{(j)})$$

$$S_2 = (E_{12}^{(1)} + E_{21}^{(1)}) + \frac{1}{2^2} E_{22}^{(1)} (E_{12}^{(2)} + E_{21}^{(2)}) + \cdots = \sum_{j=1}^{\infty} \frac{1}{j^2} \prod_{i=1}^{j-1} E_{22}^{(i)} (E_{12}^{(j)} + E_{21}^{(j)}).$$

By the function calculus for C* algebras to S_1, we have $\prod_{i=1}^{j-1} E_{22}^{(i)} E_{11}^{(j)}$ are in $\{S_1\}''$. From the construction of S_2, $\mathcal{R} \subset \{S_1, S_2\}''$ and so $\mathcal{M}\overline{\otimes}\mathcal{R} \subset \{S_1, S_2\}''$. But $S_1, S_2 \in \mathcal{M} \otimes \mathcal{R}$, and thus $\mathcal{M} \otimes \mathcal{R}$ can be generated by two self-adjoint elements. By assumption, $(\mathcal{M}, \tau)$ is embeddable into $\mathcal{R}^\omega$. ■

In 1987, D. Voiculescu introduced the free probability theory and found that the free independence in noncommutative probability space can be approximated by the independence of Gaussian random matrices. More details can be find in [VDN92] but here we shall show that the von Neumann algebra free product of two embeddable von Neumann algebras is embeddable into $\mathcal{R}^\omega$.

Lemma 57 *Let $\tau_{\mathbb{Z}}$ be the vector tracial state on $\mathcal{L}_{\mathbb{Z}}$. Suppose $(\mathcal{M}, \tau_{\mathcal{M}})$ is embeddable into $\mathcal{R}^\omega$. The von Neumann algebra free product $(\mathcal{M} * \mathcal{L}_{\mathbb{Z}}, \tau_{\mathcal{M}} * \tau_{\mathbb{Z}})$ is embeddable into $\mathcal{R}^\omega$.*

Proof. Suppose $\mathcal{M}$ can be generated by two self-adjoint elements X_1, X_2 in its unit ball, (otherwise consider $\mathcal{M}\overline{\otimes}\mathcal{R}$) and $\mathcal{M} \subset \mathcal{R}^\omega$. Let $X_j = \{X_j^{(n)}\}_n, X_j^{(n)} \in \mathcal{R}$, and assume that $X_j^{(n)}, j = 1, 2$ lies in the same type I subfactor $M_{N(n)}(\mathbb{C})$ of $\mathcal{R}$ for some positive integer $N(n)$ dependent on n. Then by [VDN92], Theorem 4.2.2, and the fact that $(\otimes_\infty L^\infty[0,1]) \otimes M_{nN}(\mathbb{C})$ is a von Neumann subalgebra of $\mathcal{R}$, there exists Guassian random matrices $Y(m, N(n)) \in (\otimes_\infty L^\infty[0,1]) \otimes M_{mN(n)}(\mathbb{C}), m \geq 1$ such that $(Y(m, N(n)), I \otimes M_{N(n)}(\mathbb{C}) \otimes I)$ is asymptotically free as $m \to \infty$, where $Y(m, N(n))$ is given as in [VDN92] theorem 4.1.2. Let $X'_j = \{I \otimes X_j^{(n)} \otimes I\}, j = 1, 2$ and $Y = \{Y(n, N(n))\}_n$. Then $X'_j, j = 1, 2$ is free from Y in $\mathcal{R}^\omega$ and Y is a semicircle element. Therefore $(\mathcal{M} * \mathcal{L}_{\mathbb{Z}}, \tau_{\mathcal{M}} * \tau_{\mathbb{Z}})$ is embeddable into $\mathcal{R}^\omega$. ■

Proposition 58 *Let $(\mathcal{M}, \tau_{\mathcal{M}})$ and $(\mathcal{N}, \tau_{\mathcal{N}})$ be von Neumann algebras with separable preduals and faithful normal tracial states $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$ respectively. Suppose $(\mathcal{M}, \tau_{\mathcal{M}})$ and $(\mathcal{N}, \tau_{\mathcal{N}})$ are embeddable into $\mathcal{R}^\omega$. Let τ be the trace $\tau_{\mathcal{M}} * \tau_{\mathcal{N}}$ on the von Neumann algebra free product $\mathcal{M} * \mathcal{N}$. Then $(\mathcal{M} * \mathcal{N}, \tau)$ is embeddable into $\mathcal{R}^\omega$.*

Proof. We only have to show $\mathcal{M} * \mathcal{N}$ can be embedded into $(\mathcal{M} \otimes \mathcal{N}) * \mathcal{L}_{\mathbb{Z}}$. Let U be the Haar unitary that generates $\mathcal{L}_{\mathbb{Z}}$. Since $\mathcal{M}$ is free from $U\mathcal{N}U^*$, we have $\mathcal{M} * \mathcal{N}$ is a subfactor of $(\mathcal{M} \otimes \mathcal{N}) * \mathcal{L}_{\mathbb{Z}}$ and is thus embeddable into $\mathcal{R}^\omega$ with its tracial state. ■

It is known that any separable finite von Neumann algebra of type I is embeddable into the hyperfinite factor $\mathcal{R}$ and also into an ultrapower $\mathcal{R}^{\omega}$. By the proposition 58 and [VDN92], Theorem 2.6.2, we have that free group factors is embeddable into $\mathcal{R}^{\omega}$.

In 1993, D. Voiculescu [Vo93, Vo94, Vo96] developed the free probability theory and introduced the free entropy for factors of type II_1. From the definition of free entry, we see that the Connes's embedding problem is equivalent to whether free entropy is well-defined on a separable factor of type II_1.

4.3 Hyperlinear Groups

One important example of von Neumann algebras introduced by Murray and von Neumann is the group von Neumann algebra arising from the left (or right) regular representation of an infinite countable (discrete) group. F. Rădulescu found that whether a group von Neumann algebras is embeddable into $\mathcal{R}^{\omega}$ only depends on the property of the group itself. Hence he [Ra02] introduced the hyperlinearity of group in 2002.

A group G is *hyperlinear* ([Ra02, CP09]) if G embeds faithfully into $\mathcal{U}(\mathcal{R}^{\omega})$. By [Ra02], Proposition 2.5, a countably discrete group G is hyperlinear if and only if the group von Neumann algebra $(\mathcal{L}_G, \tau_e)$ is embeddable into $\mathcal{R}^{\omega}$, where τ_e is the tracial vector state on $\mathcal{L}_G$ given by $\tau_e(X) = \langle Xe, e\rangle$ for all $X \in \mathcal{L}_G$. Moreover, F. Rădulescu showed that any non-residually finite Baumslag group $\langle a, b | ab^3a^{-1} = b^2\rangle$ is hyperlinear.

A group G (with unit e) is residually finite if for every nontrivial element $g \in G$, there is a homomorphism π from G to a finite group such that $\pi(g) \neq e$.

Lemma 59 *A residually finite discrete countable group G with unit e is hyperlinear.*

Proof. Let $\{e, g_1, g_2, \ldots\}$ be an enumeration of G and ρ_n be a group homomorphism of G into a finite group F_n such that $\rho_n(g_n) \neq e$. For any integer $n \geq 1$, since $\prod_{k=1}^{n} \rho_k(g_l) \neq e$ for $l = 1, \ldots, n$, let $U_l^{(n)} = L_{\prod_{k=1}^{n} \rho_k(g_l)} \in \mathscr{B}(l^2(\prod_{k=1}^{n} F_k))$. Define $U_l = \{U_l^{(n)}\}_n$, $l = 1, 2, \ldots,$ where $U_l^{(n)} = I$ if $l \leq n$. By the definition of $U_l^{(n)}$, we have that the group generated by

$I, U_l, l = 1, 2, \ldots$ is isomorphic to G. Then G can be faithfully embedded into $\mathcal{U}(\mathcal{R}^\omega)$, and therefore $\mathcal{L}_G$ can be embedded into $\mathcal{R}^\omega$. ■

For any integer $n \geq 2$, $SL_n(\mathbb{Z})$ is a linear group with matrix multiplication given by $n \times n$ matrices with entries in $\mathbb{Z}$ and determinant equal to 1. For any element g in $SL_n(\mathbb{Z})$, suppose p is a prime number larger than any entry of g and π is a group homomorphism from $SL_n(\mathbb{Z})$ to $SL_n(\mathbb{Z}_p)$ such that it maps each entry a to $a + p\mathbb{Z}$ in $\mathbb{Z}_p$. Since $SL_n(\mathbb{Z}_p)$ is a finite group, $SL_n(\mathbb{Z})$ is residually finite, and $\mathcal{L}_{SL_n(\mathbb{Z})}$ is embeddable into $\mathcal{R}^\omega$ by the lemma above.

Lemma 60 *Any non-abelian free group $\mathcal{F}_m$ on m generators, $2 \leq m \in \mathbb{N}$ or $m = \aleph_0$, is residually finite.*

Proof. Suppose $\mathcal{F}_m$ is a free group on m generators $g_1, \ldots, g_m$ and $g_{i_1}^{\epsilon_1} \cdots g_{i_k}^{\epsilon_k}$ is a reduced word in $\mathcal{F}_m$, where $i_1 \neq i_2 \neq \ldots \neq i_k \in \{1, \ldots, m\}$ and $\epsilon_1, \ldots \epsilon_k \in \mathbb{Z}\backslash\{0\}$, $n = \sum_{j=1}^k |\epsilon_j|$. We shall construct a homomorphism π from $\mathcal{F}_m$ into $\prod_{n+1}$, the permutation group on $\{1, \ldots, n+1\}$, such that $\pi(g_{i_1}^{\epsilon_1} \cdots g_{i_k}^{\epsilon_k}) \neq 1$. Let $f_i = \pi(g_i)$, for $i = 1, \ldots, m$. If the generator g_i is not in the reduced word $g_{i_1}^{\epsilon_1} \cdots g_{i_k}^{\epsilon_n}$, let $f_i = e$. Let $\eta_j = \sum_{l=1}^j |\epsilon_l|$ for $j = 1, \ldots, k$. Define $h_{i_j} = \begin{pmatrix} \eta_{j-1}+1 & \cdots & \eta_j \\ \eta_{j-1} & \cdots & \eta_j - 1 \end{pmatrix}$ for $j = 2, \ldots, k$ and $h_{i_1} = \begin{pmatrix} 1 & 2 & \cdots & \eta_1 \\ n+1 & 1 & \cdots & \eta_1 - 1 \end{pmatrix}$. Let $f_i = \prod\{h_{i_j}^{s_j}, i_j = i\}$, where $s_j = 1$ if $\epsilon_j > 0$; $s_j = -1$ if $\epsilon_j < 0$. Since $i_1 \neq i_2 \neq \ldots \neq i_k$, f_i is well-defined when g_i is in the reduced word. Moreover $h_{i_1} \cdots h_{i_k}(n+1) = 1$, and hence $h_{i_1} \cdots h_{i_k} \neq e$ and $f_{i_1}^{\epsilon_1} \cdots f_{i_k}^{\epsilon_k} \neq e$. This proves that $\mathcal{F}_m$ is residually finite. ■

As a corollary of the above lemma, we see that a free group factor $\mathcal{L}_{\mathcal{F}_m}$, $2 \leq m \in \mathbb{N}$ is embeddable into $\mathcal{R}^\omega$. K. Dykema [Dyk94] and F. Radulescu [Ra94] introduced, independently, the interpolated free group factor $\mathcal{L}_{\mathcal{F}_t}$, $t > 1$. These factors can be obtained from the free group factors by suitable compression with projections. Note that the embeddable property is preserved by the compression with a projection in a factor. Thus, we have that $\mathcal{L}_{\mathcal{F}_t}$, $t > 1$ is embeddable into $\mathcal{R}^\omega$.

A group is locally embeddable into finite groups (an LEF group, for short) if for every finite subset $F \subset G$, there is a partially defined monomorphism i of F into a finite group,i.e

$i(xy) = i(x)i(y)$ for any $x, y \in F$. By the definition, a residually finite group is LEF from the definition.

A notion similar to a hyperlinear group in group theory is a sofic group. The sofic group was first defined by Gromov [Gro99]. A group G is *sofic* if for every finite $F \subset G$ and every $\epsilon > 0$, there exists $n \in \mathbb{N}$ and an (F, ϵ)-almost homomorphism $j : F \mapsto \prod_n$. An (F, ϵ)-almost homomorphism j is a map with the property: if $g, h \in F$ and $gh \in F$, then $d_{\text{hamm}}(j(g)j(h), j(gh)) < \epsilon$ and if $e \in F$, then $d_{\text{hamm}}(j(e), Id) < \epsilon$, which is uniformly injective: $d_{\text{hamm}}(j(g), j(h)) \geq 1/4$ whenever $g, h \in F, g \neq h$. From this definition, a sofic group is hyperlinear. Unfortunately it is unknown whether a hyperlinear group is sofic.

A discrete group G with unit e is amenable if G admits a left invariant mean. A positive linear functional $\phi : l^\infty(G) \mapsto \mathbb{C}$ is an invariant mean if $\phi(e) = 1$ and ϕ is invariant under left translations. Alternatively, a discrete group G is amenable if for every finite subset $F \subset G$ and $\epsilon > 0$, there is a finite subset $A \subset G$ (A is called a Følner set for F and ϵ) such that for each $g \in F$, $|gA \triangle A| < \epsilon |A|$. This is known as Følner condition.

A group G (with unit e) is residually amenable if for every nontrivial element $g \in G$, there is a homomorphism π from G to an amenable group such that $\pi(g) \neq e$. By this definition, every amenable group is residually amenable.

A group G is initially subamenable if every finite subset $F \subset G$ admits an $(F, 0)$-almost monomorphism into an amenable group Γ. It is clear that every residually amenable group is initially subamenable and every initially subamenable group is sofic from the definition. In particular, every amenable group is sofic. The following diagram summarizes these properties.

$$\begin{array}{ccccc}
\text{Factorization Property} & & \overset{Propert\ T}{\Longrightarrow} & & \text{Residually Finite} \\
 & \Nwarrow & & \Nearrow & \\
\Downarrow & & \text{Residually Amenable} & & \Downarrow\Uparrow (f.p.) \\
 & & \Downarrow\Uparrow (f.p.) & & \\
\Downarrow & & \text{Initially Subamenable} & \Leftarrow & LEF \\
 & & \Downarrow & & \\
\text{Hyperlinear} & \Leftarrow & \text{Sofic} & &
\end{array}$$

4.4 Co-amenability of von Neumann subalgebras

Co-amenability was first raised in the group theory, but Co-amenability of von Neumann subalgebras was introduced by S. Popa in [Po86, Po99, PM03].

Let $\mathcal{N}$ be a finite von Neumann algebra with a separable predual and a faithful normal tracial state τ and $\mathcal{B} \subset \mathcal{N}$ a von Neumann subalgebra. The subalgebra $\mathcal{B}$ is co-amenable in $\mathcal{N}$ if there exists a norm one projection Ψ of $\langle \mathcal{N}, \mathcal{B} \rangle$ onto $\mathcal{N}$. One also says that $\mathcal{N}$ is amenable relative to $\mathcal{B}$.

We present an important property of co-amenability of von Neumann subalgebras as follows and omit its proof.

Proposition 61 (See [PM03], Proposition 5) *With the notation as above. $\mathcal{B}$ is co-amenable in $\mathcal{N}$ if and only if there exists a state ψ on $\langle \mathcal{N}, \mathcal{B} \rangle$ extending the tracial state τ on $\mathcal{N}$ with $\psi(UXU^*) = \psi(X)$ for all $X \in \langle \mathcal{N}, \mathcal{B} \rangle$, $U \in \mathcal{U}(\mathcal{N})$.*

In [PM03], N. Monoid and S. Popa pointed out that co-amenability of a von Neumann subalgebra is equivalent to a kind of Følner type condition. Inspired by this, we show the following main theorem of this section.

Theorem 62 (Main Theorem) *Let $\mathcal{N}$ be a von Neumann algebra with a separable predual and a faithful normal trace τ. Suppose $\mathcal{B}$ is a von Neumann subalgebra co-amenable in $\mathcal{N}$*

and $(\mathcal{B},\tau)$ is embeddable into an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1 factor $\mathcal{R}$. Then $(\mathcal{N},\tau)$ is embeddable into $\mathcal{R}^\omega$.

To prepare the proof of the main theorem, we review some notations and results of direct integrals.

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state τ acting on a separable Hilbert space $\mathscr{H}$, and $\mathcal{M}'$ be the commutant of $\mathcal{M}$ on $\mathscr{H}$, and $C = \mathcal{M} \cap \mathcal{M}'$ their center. By [KR], Charper 14, there is a (locally compact complete separable metric) measure space $(\mathcal{X},\mu)$ such that $\mathscr{H}$ is (unitarily equivalent to) the direct integral of Hilbert spaces $\{\mathscr{H}_p\}$ over $(\mathcal{X},\mu)$; i.e $\mathscr{H} = \int_X \mathscr{H}_p d\mu(p)$. Moreover, $\mathcal{M}, \mathcal{M}', \tau$ have direct integral decomposition relative to C; i.e. $\mathcal{M} = \int_X \mathcal{M}(p)d\mu(p)$, $\mathcal{M}' = \int_X \mathcal{M}'(p)d\mu(p)$, $\tau = \int_X \tau_p d\mu(p)$ where $\mathcal{M}(p), \mathcal{M}'(p)$ are factors acting on $\mathscr{H}_p$ a.e., τ_p is the trace on $\mathcal{M}(p)$ a.e. and $\mathcal{M}(p)' = \mathcal{M}'(p)$ on $\mathscr{H}_p$ a.e. In addition, if $\mathcal{M}'$ has faithful normal tracial state τ', then τ' has direct integral decomposition relative to C too, i.e. $\tau' = \int_X \tau'_p d\mu(p)$, where τ'_p is the trace on $\mathcal{M}'(p)$ a.e.

Recall the Lance's weak expectation property (WEP) for C* algebra and quotient C* algebra of a C* algebra with WEP:

A C* algebra $\mathfrak{A}$ has the weak expectation property (WEP) (or is "WEP") if there exist a Hilbert space $\mathscr{H}$ and completely positive and complete contractions $T_1 : \mathscr{B}(\mathscr{H}) \mapsto \mathfrak{A}^{\#\#}$ and $T_2 : \mathfrak{A} \mapsto \mathscr{B}(\mathscr{H})$ such that the inclusion map $i_{\mathfrak{A}} : \mathfrak{A} \mapsto \mathfrak{A}^{\#\#}$ satisfies $T_1T_2 = i_{\mathfrak{A}}$. A C* algebra $\mathfrak{B}$ is a quotient C* algebra of a C* algebra with WEP (i.e. QWEP) if there exist WEP C* algebra $\mathfrak{A}$ and *-homomorphism π from $\mathfrak{A}$ onto $\mathfrak{B}$.

To complete the proof the main theorem, we need the following four lemmas.

Lemma 63 *Suppose $\mathcal{M}$ is a von Neumann algebra with a separable predual and a faithful normal tracial state τ. Let C be the center of $\mathcal{M}$. Then $\forall n \in \mathbb{N}$, given $X_1, \ldots, X_n \in \mathcal{M}$ and $\epsilon > 0$, there exist finite subset $F \subset \mathcal{U}(\mathcal{M})$ and $0 < \delta < \epsilon$ such that for any normal state $\psi \in \mathcal{M}_\#$ with $\|U\psi U^* - \psi\| \le \delta$ for all $U \in F$ and $\tau|_C = \psi|_C$, we have $|\psi(X_j) - \tau(X_j)| \le \epsilon$, $j = 1, \ldots, n$.*

Proof. Assume that for any finite subset $F \subset \mathcal{U}(\mathcal{M})$ and $0 < \delta < \epsilon$, there exists a normal state $\psi_{F,\delta}$ with $\|U\psi_{F,\delta}U^* - \psi_{F,\delta}\| \leq \delta$ for all $U \in F$ and $\psi_{F,\delta}|_C = \tau|_C$, while $|\psi_{F,\delta}(X_j) - \tau(X_j)| > \epsilon$ for some $j \in \{1, \ldots, n\}$. Let $S = \{\psi_{F,\delta} : F \subset \mathcal{U}(\mathcal{M}) \text{ is finite}, 0 < \delta < \epsilon\}$. Then S is a net with order $(F, \delta) \leq (F', \delta')$ given by $F \subset F'$, $\delta \geq \delta'$. By weak* compactness of the state space of $\mathcal{M}$, there is an accumulation point ψ of S in $\mathcal{M}^\#$ which commutes with U for all $U \in \mathcal{U}(\mathcal{M})$, $\psi|_C = \tau|_C$, and $|\psi(X_j) - \tau(X_j)| > \epsilon$ for some j. Therefore ψ is a different tracial state on $\mathcal{M}$ with $\psi|_C = \tau|_C$ which is not possible. ■

Lemma 64 *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful normal tracial state τ acting on a separable Hilbert space $\mathcal{H}$ and the commutant $\mathcal{M}'$ of $\mathcal{M}$ on $\mathcal{H}$ is finite. Let τ' be a faithful normal tracial state on $\mathcal{M}'$. Then $(\mathcal{M}, \tau)$ is embeddable into $\mathcal{R}^\omega$ if and only if $(\mathcal{M}', \tau')$ is embeddable into $\mathcal{R}^\omega$.*

Proof. By [Kir93], Corollary 3.8, $\mathcal{M}$ is QWEP if and only if $\mathcal{M}'$ is QWEP. Then by [Kir93], Theorem 4.1, $(\mathcal{M}, \tau)$ is embeddable into $\mathcal{R}^\omega$ if and only if $\mathcal{M}$ is QWEP and $(\mathcal{M}', \tau')$ is embeddable into $\mathcal{R}^\omega$ if and only if $\mathcal{M}'$ is QWEP. Therefore, $(\mathcal{M}, \tau)$ is embeddable into $\mathcal{R}^\omega$ if and only if $(\mathcal{M}', \tau')$ is embeddable into $\mathcal{R}^\omega$. ■

Lemma 65 *With the notations in the theorem. Let C be the center of $\mathcal{N}$. Then $\mathcal{N}$, $\mathcal{B}$ and τ have unique direct integral decomposition relative to C over some (locally compact complete separable) measure space (X, μ) i.e.*

$$\mathcal{N} = \int_X \mathcal{N}(p)d\mu(p), \ \mathcal{B} = \int_X \mathcal{B}(p)d\mu(p), \ \tau = \int_X \tau_p d\mu(p).$$

$(\mathcal{N}(p), \tau_p)$ is embeddable into $\mathcal{R}^\omega$ a.e. if and only if $(\mathcal{N}, \tau)$ is embeddable into $\mathcal{R}^\omega$.

Proof. By [KR], Charpter 14, we have that $\mathcal{N}$, $\mathcal{B}$ and τ have unique direct integral decomposition relative to C over some (locally compact complete separable) measure space (X, μ) and $\mathcal{N}(p)$ is factor a.e. Suppose $L^2(\mathcal{N}, \tau)$ is the direct integral of Hilbert spaces $\{L^2(\mathcal{N}, \tau)_p\}_p$ over (X, μ). Let $J(p)$ be an operator on $L^2(\mathcal{N}, \tau)_p$ such that $J(p)T(p)\hat{I}(p) = T^*(p)\hat{I}(p)$ a.e. where $T = \int_X T(p)d\mu(p) \in \mathcal{N}$, I is the identity on $L^2(\mathcal{N}, \tau)$, and $\hat{I} = \int_X \hat{I}(p)d\mu(p)$. Let J be the canonical conjugation on the Hilbert space $L^2(\mathcal{N}, \tau)$ such that $JT\hat{I} = T^*\hat{I}$ for any

$T \in \mathcal{N}$. Since $J(p)$ is the canonical conjugation on the Hilbert space $L^2(\mathcal{N}, \tau)_p$, we have $J = \int_X J(p) d\mu(p)$ and

$$\langle \mathcal{N}(p), \mathcal{B}(p) \rangle = J(p)\mathcal{B}(p)' J(p) \text{ a.e.}$$

Now we shall show

$$\int_X \langle \mathcal{N}(p), \mathcal{B}(p) \rangle d\mu(p) = C' \cap \langle \mathcal{N}, \mathcal{B} \rangle.$$

For any $T \in C' \cap \langle \mathcal{N}, \mathcal{B} \rangle$, we have $T = \int_X T(p) d\mu(p)$. Since $\langle \mathcal{N}(p), \mathcal{B}(p) \rangle = J(p)\mathcal{B}(p)' J(p)$, to show $T \in \int_X \langle \mathcal{N}(p), \mathcal{B}(p) \rangle d\mu(p)$, we only have to show $T(p)$ commutes with $J(p)\mathcal{B}(p)J(p)$; i.e.

$$T(p)J(p)B(p)J(p) = J(p)B(p)J(p)T(p), a.e. \forall B \in \mathcal{B}$$

This implies $TJBJ = JBJT, \forall B \in \mathcal{B}$. But T is in $JB'J = \langle \mathcal{N}, \mathcal{B} \rangle$, the commutant of $J\mathcal{B}J$. Therefore

$$\int_X \langle \mathcal{N}(p), \mathcal{B}(p) \rangle d\mu(p) \supset C' \cap \langle \mathcal{N}, \mathcal{B} \rangle.$$

On the other hand, if $T \in \int_X \langle \mathcal{N}(p), \mathcal{B}(p) \rangle d\mu(p)$, then T commutes with C and $T \in \langle \mathcal{N}, \mathcal{B} \rangle$ and hence

$$\int_X \langle \mathcal{N}(p), \mathcal{B}(p) \rangle d\mu(p) \subset C' \cap \langle \mathcal{N}, \mathcal{B} \rangle.$$

Therefore $\int_X \langle \mathcal{N}(p), \mathcal{B}(p) \rangle d\mu(p) = C' \cap \langle \mathcal{N}, \mathcal{B} \rangle$.

By [Kir93], Corollary 3.7, we have $\mathcal{N}(p)$ is QWEP a.e. if and only if $\mathcal{N}$ is QWEP. By [Kir93], we have that $(\mathcal{N}, \tau)$ is embeddable into $\mathcal{R}^\omega$ if and only $\mathcal{N}$ is QWEP; $(\mathcal{N}(p), \tau_p)$ is embeddable into $\mathcal{R}^\omega$ a.e. if and only if $\mathcal{N}(p)$ is QWEP a.e. Therefore $(\mathcal{N}, \tau)$ is embeddable into $\mathcal{R}^\omega$ if and only if $(\mathcal{N}(p), \tau_p)$ is embeddable into $\mathcal{R}^\omega$. ■

Lemma 66 *With the notations in the theorem. If $(\mathcal{B}, \tau)$ is embeddable into $\mathcal{R}^\omega$ and E is non-zero projection in $\langle \mathcal{N}, \mathcal{B} \rangle$ with $Tr(E) < \infty$. Then $(E\langle \mathcal{N}, \mathcal{B} \rangle E, Tr/Tr(E))$ is embeddable into $\mathcal{R}^\omega$.*

Proof. Since $(\mathcal{B}, \tau)$ is embeddable into $\mathcal{R}^\omega$, $(J\mathcal{B}J, J\tau J)$ is embeddable into $\mathcal{R}^\omega$, where $J\tau J$ is the tracial state on $J\mathcal{B}J$ given by $J\tau J(Y) = \tau(JYJ)$, for all $Y \in J\mathcal{B}J$. Let C_E be the central support of E in $\langle \mathcal{N}, \mathcal{B} \rangle$. Then $C_E \in J\mathcal{B}J$ and by Lemma 53, $(J\mathcal{B}JC_E, J\tau J/\tau(JC_EJ))$

is embeddable into $\mathcal{R}^\omega$. Since $J\mathcal{B}JE$ is *-isomorphic to $J\mathcal{B}JC_E$ and the tracial state τ_1 on $J\mathcal{B}JE$ induced by $J\tau J/\tau(JC_EJ)$ is given by $\tau_1(YE) = \tau(JYJ)/\tau(JC_EJ)$ for all $Y \in J\mathcal{B}JC_E$, $(J\mathcal{B}J, \tau_1)$ is embeddable into $\mathcal{R}^\omega$. By Lemma 64, $(EJ\mathcal{B}'JE, Tr/Tr(E))$ is embeddable into $\mathcal{R}^\omega$. ∎

Now we begin the proof of the Main Theorem:

Proof. For $k \in \mathbb{N}$, let $\mathcal{F}(k)$ be the set of all words $m \in \mathcal{F}_n$ whose length is less than k. Let C be the center of $\mathcal{N}$. By results of Kirchberg [Kir93], whether $\mathcal{N}$ is embeddable into $\mathcal{R}^\omega$ is independent of the choice of the faithful normal tracial state τ. We assume that $\tau|_C$ is multiplicative. To prove $\mathcal{N}$ can be embedded into $\mathcal{R}^\omega$, we shall show that for unitary operators $U_1, \ldots, U_n \in \mathcal{N}$, $\epsilon > 0$, $k \in \mathbb{N}$, there exists $k' \in \mathbb{N}$ and $k' \times k'$ matrices $V_1, \ldots, V_n$ such that

$$|\tau(\tilde{U}(m)) - tr_{k'}(\tilde{V}(m))| < \epsilon, \forall m \in \mathcal{F}(k). \tag{4.1}$$

Let $S = \{U(m) : m \in \mathcal{F}(k)\}$. By Lemma 63, there exists fintie subset $F_0 \subset \mathcal{U}(\mathcal{N})$ such that for any normal state $\psi \in \mathcal{N}_\#$ with $\|U\psi U^* - \psi\| < \delta$ for all $U \in F_0$, we have $|\psi(X) - \tau(X)| < \epsilon$ for all $X \in S$. Let $F = F_0 \cup S = \{X_1, \ldots, X_p\}$, where p is the cardinality of F.

Next, we shall use Day's convexity trick. Let $\langle \mathcal{N}, \mathcal{B}\rangle_\#$ be the predual of $\langle \mathcal{N}, \mathcal{B}\rangle$ and $\langle \mathcal{N}, \mathcal{B}\rangle_\#^p$ be the Banach space $\langle \mathcal{N}, \mathcal{B}\rangle_\#^p$ with norm $\|(\phi_1, \ldots, \phi_p)\| = \sum \|\phi_j\|$. Then

$$\sum \phi_j(Y_j) = \langle(\phi_1, \ldots, \phi_p), (Y_1, \ldots, Y_p)\rangle$$

identifies the product von Neumann algebra $\langle \mathcal{N}, \mathcal{B}\rangle^p$ with the dual of $\langle \mathcal{N}, \mathcal{B}\rangle_\#^p$.

Let

$$\mathcal{G} = \{(\psi - X_1\psi X_1^*, \ldots, \psi - X_p\psi X_p^*) | \psi \text{ is a normal state on } \langle \mathcal{N}, \mathcal{B}\rangle\}.$$

Then $\mathcal{G}$ is a convex subset of $\langle \mathcal{N}, \mathcal{B}\rangle_\#^p$ and its weak and norm closure coincide. Since $\mathcal{B}$ is co-amenable in $\mathcal{N}$, by Proposition 61, there is a state ϕ on $\langle \mathcal{N}, \mathcal{B}\rangle$ invariant under $Ad(U)$ for all $U \in F$. Since the set of normal states is weakly dense in the state space of $\langle \mathcal{N}, \mathcal{B}\rangle$, there is a net of normal states converging weakly to the state ϕ. So the weak, and hence norm, closure of $\mathcal{G}$ contains $(0, \ldots, 0)$. Then let ψ be a normal state on $\langle \mathcal{N}, \mathcal{B}\rangle$ with

$$\|\psi - U\psi U^*\| \leq (\delta/24pk)^{16}, \forall U \in F.$$

For the normal state ψ, there exists a positive element $H\eta\langle\mathcal{N},\mathcal{B}\rangle$ with $Tr(H^2) = 1$ such that $\psi(X) = Tr(HXH)$, then

$$||UH^2U^* - H^2||_{1,Tr} \leq (\delta/24pk)^{16}||H^2||_{1,Tr}$$

for all $U \in F$. By adjusting ψ, we could assume that H is a bounded positive operator in $\langle\mathcal{N},\mathcal{B}\rangle$.

By Powers-Størmer inequality (See [PS70, Haa75]),

$$||UHU^* - H||_{2,Tr} \leq (\delta/24pk)^8||H||_{2,Tr}, \forall U \in F$$

By [Con76], Theorem 1.2.2, for set $\{H, UHU^*|U \in F\}$, there exists a projection $E \in \langle\mathcal{N},\mathcal{B}\rangle$ with $Tr(E) < \infty$ such that

$$||UEU^* - E||_{2,Tr} = ||E - U^*EU||_{2,Tr} \leq \delta/4k||E||_{2,Tr},$$

for all $U \in F$ and $||H - EH||_{2,Tr} \leq \delta/4k||H||_{2,Tr}$.

Let ψ_0 be the normal state on $\mathcal{N}$ defined by $\psi_0(X) = Tr(EXE)/Tr(E)$ for all $X \in \mathcal{N}$. Since

$$\psi_0(UYU^*) = Tr(EUYU^*E)/Tr(E) = Tr(U^*EUY)/Tr(E),$$

for any Y in $(\mathcal{N})_1$,

$$\begin{aligned} &|\psi_0(Y) - \psi_0(UYU^*)|Tr(E) \\ = \;& |Tr(EY) - Tr(U^*EUY)|, \\ = \;& |Tr((E - U^*EU)Y)| = |Tr(|E - U^*EU|V^*Y)|, \\ \leq \;& Tr(|E - U^*EU|)^{1/2}Tr(|E - U^*EU|^{1/2}V^*YY^*V|E - U^*EU|^{1/2})^{1/2}, \\ \leq \;& Tr(|E - U^*EU|) \leq ||E - U^*EU||_{2,Tr}(||E||_{2,Tr} + ||U^*EU||_{2,Tr}), \\ \leq \;& \delta/2k||E||_{2,Tr}^2, \end{aligned}$$

where $|E - U^*EU|V^*$ is the polar decomposition of $E - U^*EU$. Then we obtain $||\psi_0 - U\psi_0U^*|| \leq \delta/2k \leq \delta/4$ for all $U \in F_0 \subset F$ and $k \geq 2$. Since $Tr(CE) = \tau(C)Tr(E)$ for all $C \in C$, $\psi_0|_C = \tau|_C$. By Lemma 63, we have for all $m \in \mathcal{F}(k)$,

$$|Tr(E\tilde{U}(m)E)/Tr(E) - \tau(\tilde{U}(m))| = |\psi_0(\tilde{U}(m)) - \tau(\tilde{U}(m))| \leq \epsilon/4. \tag{4.2}$$

Now for unitary operators $U_1, \ldots, U_n \in \mathcal{N}$, $\epsilon > 0$, $k \in \mathbb{N}$, let

$$W_1 = EU_1E, \ldots, W_n = EU_nE \in E\langle \mathcal{N}, \mathcal{B} \rangle E.$$

Then

$$\begin{aligned} |Tr(\tilde{U}(m))/Tr(E) - Tr(\tilde{W}(m))/Tr(E)| &\leq length(m)\delta/4k \\ &\leq \delta/4 \leq \epsilon/4, \forall m \in \mathcal{F}(k). \end{aligned}$$

Hence, $\forall m \in \mathcal{F}(k)$

$$|Tr(\tilde{U}(m))/Tr(E) - Tr(\tilde{W}(m))/Tr(E)| \leq \epsilon/4. \tag{4.3}$$

Since $(\mathcal{B}, \tau)$ is embeddable into $\mathcal{R}^\omega$, by Lemma 66, $(E\langle \mathcal{N}, \mathcal{B} \rangle E, Tr/Tr(E))$ is also embeddable into $\mathcal{R}^\omega$. Then by Proposition 52, there exist $k' \in \mathbb{N}$ and $k' \times k'$ matrices $V_1, \ldots, V_n$ such that

$$|Tr(\tilde{W}(m))/Tr(E) - tr_{k'}(\tilde{V}(m))| \leq \epsilon/2, \forall m \in \mathcal{F}(k), \tag{4.4}$$

where $tr_{k'}$ is the normalized trace on $M_{k'}(\mathbb{C})$. Then combining equations (4.2), (4.3), and (4.4), we reach our goal (see equation 4.1) and have

$$|\tau(\tilde{U}(m)) - tr_{k'}(\tilde{V}(m))| \leq \epsilon$$

and $(\mathcal{N}, \tau)$ is embeddable into $\mathcal{R}^\omega$. ■

A subgroup H of a group G is called co-amenable in G if there exists a G-invariant mean on the space $l^\infty(G/H)$.

Corollary 67 *Suppose $\mathcal{B}_0$ is a finite von Neumann algebra with a separable predual and a faithful normal tracial state τ_0 and G is a countably discrete group with unit e. Let $\sigma : G \mapsto Aut(\mathcal{B}_0)$ be a trace-preserving cocycle action on $(\mathcal{B}_0, \tau_0)$. Let $\mathcal{N} = \mathcal{B}_0 \rtimes_\sigma G$ be the corresponding crossed product von Neumann algebra with faithful normal tracial state τ given by $\tau(\sum_{g \in G} B_g U_g) = \tau_0(B_e)$, where $B_g \in \mathcal{B}_0, g \in G$. Suppose H is a subgroup of G co-amenable in G and $(\mathcal{B}(= \mathcal{B}_0 \rtimes_\sigma H), \tau)$ is embeddable into $\mathcal{R}^\omega$. Then $(\mathcal{N}, \tau)$ is embeddable into $\mathcal{R}^\omega$.*

Proof. By [PM03], Proposition 6, $\mathcal{B}$ is co-amenable in $\mathcal{N}$ if and only if the group H is co-amenable in G. Since $(\mathcal{B}, \tau)$ is embeddable into $\mathcal{R}^\omega$, by Theorem 62, $(\mathcal{N}, \tau)$ is embeddable into $\mathcal{R}^\omega$. ■

In the above corollary, let $\mathcal{B}_0$ be $\mathbb{C}I$, we have following corollary.

Corollary 68 *Let G be a countable (discrete) group. Suppose H is a hyperlinear subgroup co-amenable in G. Then G is hyperlinear.*

Proof. Since H is hyperlinear, $(\mathcal{L}_H, \tau_e)$ is embeddable into $\mathcal{R}^\omega$. By [PM03], Corollary 7, $\mathcal{L}_H$ is co-amenable in $\mathcal{L}_G$, since H is co-amenable in G. By Theorem 62, $(\mathcal{L}_G, \tau_e)$ is embeddable into $\mathcal{R}^\omega$. Therefore G is hyperlinear. ■

Let H be any group and $\theta : H \mapsto H$ an injective homomorphism. Denote by $G = H*_\theta$ the corresponding HNN-extension, i.e.

$$G = \langle H, t | tht^{-1} = \theta(h), \forall h \in H \rangle.$$

By [PM03], Proposition 2, H is co-amenable in G. Then the HNN-extension of a hyperlinear group is a hyperlinear group again.

In Corollary 67, if H is $\{e\} \subset G$, then G is an amenable group and we have:

Corollary 69 *Suppose $\mathcal{B}$ is finite von Neumann algebra with a separable predual and a faithful normal tracial state τ and G is an amenable countably discrete group. Let $\sigma : G \mapsto Aut(\mathcal{B})$ be a trace-preserving cocycle action on $(\mathcal{B}, \tau)$. Let $\mathcal{B}_0 \rtimes_\sigma G$ be the corresponding crossed product von Neumann algebra with faithful normal tracial state τ given by $\tau(\sum_{g \in G} B_g U_g) = \tau_0(B_e)$, where $B_g \in \mathcal{B}_0, g \in G$. Then $(\mathcal{B} \rtimes_\sigma G, \tau)$ is embeddable into $\mathcal{R}^\omega$.*

4.5 Similarity Property

Let us recall Kadison's similarity problem [Ka55]. Let $\mathfrak{A}$ be a unital C* algebra and $\phi : \mathfrak{A} \mapsto \mathcal{B}(\mathcal{H})$ a unital homomorphism. Kadison's similarity problem is whether the condition that

ϕ is bounded implies that ϕ is similar to a *-homomorphism, i.e. $\exists S : \mathcal{H} \mapsto \mathcal{H}$ is invertible such that $\phi_S : X \mapsto S^{-1}\phi(X)S$ is a *-homomorphism. In [Haa75], Haagerup proved that ϕ is similar to a *-homomorphism if and only if ϕ is completely bounded and

$$||\phi||_{cb} = \inf\{||S^{-1}||||S|| : \phi_S \text{ is *-homomorphism.}\}$$

An operator algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is said to be of length $\leq d$ if there is a constant K such that, for any n and any X in $M_n(\mathfrak{A})$, there is an integer $N = N(n, X)$ and scalar matrices $\alpha_0 \in M_{n,N}(\mathbb{C})$, $\alpha_1 \in M_N(\mathbb{C})$, ..., $\alpha_{d-1} \in M_N(\mathbb{C})$, $\alpha_d \in M_{N,n}(\mathbb{C})$ together with diagonal matrices $D_1, \ldots, D_d$ in $M_N(\mathfrak{A})$ satisfying

$$\begin{cases} X = \alpha_0 D_1 \alpha_1 D_2 \cdots D_d \alpha_d \\ \prod_0^d ||\alpha_i|| \prod_1^d ||D_i|| \leq K||X||. \end{cases}$$

Denote by $\ell(\mathfrak{A})$ the length of $\mathfrak{A}$; i.e the smallest d for which the two equations above holds. Let

$$d(\mathfrak{A}) = \inf\{\alpha \geq 0 | \exists K, \forall \phi, ||\phi||_{cb} \leq K||\phi||^\alpha\},$$

where ϕ denotes an arbitrary unital homomorphism from $\mathfrak{A}$ to $\mathcal{B}(\mathcal{H})$.

G. Pisier [Pi99, Pi00, Pi] showed that $\ell(\mathfrak{A}) = d(\mathfrak{A})$ for any unital operator algebra $\mathfrak{A}$ which is the similarity degree of $\mathfrak{A}$.

Proposition 70 *Let G be a discrete group, $(\mathcal{B}_0, \tau_0)$ a finite von Neumann algebra with a normal faithful tracial state and $\sigma : G \mapsto Aut(\mathcal{B}_0, \tau_0)$ a trace preserving cocycle action of G on $(\mathcal{B}_0, \tau_0)$. Let $\mathcal{N} = \mathcal{B}_0 \rtimes_\sigma G$ be the corresponding crossed product von Neumann algebra with its normal faithful tracial state given by $\tau(\sum_{g \in G} b_g u_g) = \tau_0(b_e)$. Let $H < G$ be a subgroup co-amenable in G and $\mathcal{B} = \mathcal{B}_0 \rtimes_\sigma H$. If $\mathcal{N}$ is a factor and $\mathcal{B}$ has similarity degree d, then $\mathcal{N}$ has similarity degree of at most $9d + 8$.*

Proof. Suppose ϕ is a unital bounded representation of $\mathcal{N}$ on a Hilbert space $\mathcal{H}$ such that $\overline{sp}\phi(\mathcal{N})\mathcal{H} = \mathcal{H}$. Then $\phi|_{\mathcal{B}}$ is a bounded representation of $\mathcal{B}$, and so there is an invertible operator S_0 on $\mathcal{H}$ such that $S_0\phi|_{\mathcal{B}}S_0^{-1}$ is a *-representation of $\mathcal{B}$ and $||S_0^{-1}||||S_0|| \leq K||\phi|_{\mathcal{B}}||^d$. Let $\phi_0 = S_0\phi S_0^{-1}$. Then ϕ_0 is a bounded representation of $\mathcal{N}$. We have to estimate the complete bounded norm of ϕ_0. To do this, we may and will assume that the representation has an at most countable cyclic set. In this case [Ch81] there is a *-representation π of $\mathcal{N}$ on $\mathcal{H}$ such that for any vector ξ in $\mathcal{H}$, there exists a bounded injective operator X with dense range and a vector η satisfying

$$\forall Y \in \mathcal{N} : \phi_0(Y)X = X\pi(Y); \|X\| \leq 2\|\phi_0\|^2; X\eta = \xi; \|\eta\| \leq \|\xi\|.$$

The first property admits a homomorphism ψ of $\pi(\mathcal{N})$ into $\mathcal{B}(\mathcal{H})$ by $A \mapsto \overline{XAX^{-1}}$ and $\|\psi\| = \|\phi_0\|$, whereas the second shows that ψ is ultrastrongly continuous since $\psi(A)\xi = XA\eta$. We will denote by ψ again the extension of ψ to the von Neumann algebra generated by $\pi(\mathcal{N})$. In this algebra we will let F denote the maximal finite central projection and let $\mathcal{D}$ be a copy of the compact operators placed inside $(I - F)\pi(\mathcal{N})$, such that $I - F$ belongs to the weak closure of $\mathcal{D}$. Then $\mathcal{D} + \mathbb{C}F$ is a nuclear C* algebra, by [Ch81], we can perturb ψ with a Z in $GL(\mathcal{H})$ such that $Ad(Z) \circ \psi$ is trivial on $\mathcal{D} \oplus \mathbb{C}F$ and $\|Z^{-1}\|\|Z\| \leq \|\phi_0\|^2$. The new homomorphism $Ad(Z) \circ \psi$ decomposes naturally into an orthogonal direct sum. The restriction to the properly infinite part is by construction completely bounded with complete bounded norm less than $\|\phi_0\|^3$. The restriction to the finite part yields homomorphisms π_F and Δ of the finite von Neumann algebra $\mathcal{N}$ into $\mathcal{B}(F\mathcal{H})$ given by

$$\pi_F(Y) = \pi(Y)|_{F\mathcal{H}} \text{ and } \Delta(Y) = (ZX)F\pi_F(Y)(ZX)^{-1}|_{F\mathcal{H}}.$$

Since a finite representation of a finite representation of a finite factor is ultrastrongly continuous because of the uniqueness of the trace, we see that Δ is ultrastrongly continuous.

Let $F_n \nearrow G/H$ be a net of finite Følner sets, which we identify with some sets of representatives $F_n \subset G$. Since Δ is unital bounded, the set $|F_n|^{-1} \sum_{s \in F_n} \Delta(U_s)^* \Delta(U_s)$ in the von Neumann algebra generated by $\Delta(\mathcal{N})$ has a strong-operator accumulation point. The accumulation point is positive. So let S be the square root of it. We have

$$\|S\xi\|^2 = \lim_n \frac{1}{|F_n|} \sum_{s \in F_n} \|\Delta(U_s)\xi\|^2$$

and hence, $\|\Delta\|^{-1} \leq S \leq \|\Delta\|$. For any unitary element U in $\mathcal{B}_0$, let $V_s = U_s U U_s^*$ in $\mathcal{B}_0$.

Then

$$\begin{aligned}
S^2\Delta(U)\xi &= \lim_n \frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U_s)^*\Delta(U_s)\Delta(U)\xi \\
&= \lim_n \frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U_s)^*\Delta(V_sU_s)\xi \\
&= \lim_n \frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U)\Delta(U_s)^*\Delta(U_s)\xi \\
&= \Delta(U)S^2\xi.
\end{aligned}$$

For any unitary element U_g, $g \in G$ in $\mathcal{N}$, let $h_s s' = sg$ if sg is in F_n. Since F_n is a Følner set and $\Delta(U_{h_s})$ is a unitary, we have that

$$\begin{aligned}
S^2\Delta(U_g)\xi &= \lim_n \frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U_s)^*\Delta(U_s)\Delta(U_g)\xi \\
&= \lim_n \frac{1}{|F_n|}\sum_{s\in F_n}\Delta(U_s)^*\Delta(U_{sg})\xi \\
&= \lim_n \frac{1}{|F_n|}\sum_{s'\in F_ng}\Delta(U_g)\Delta(U_{s'})^*\Delta(U_{s'})\xi \\
&= \Delta(U_g)S^2\xi.
\end{aligned}$$

Let $\mathcal{N}_0$ be the *-subalgebra in $\mathcal{N}$ generated by $\mathcal{B}_0$ and U_g, $g \in G$. For any element A_0 in $\mathcal{N}_0$, we have $S^2\Delta(A_0)\xi = \Delta(A_0)S^2\xi$, for all $\xi \in \mathcal{H}$. By the Kaplansky density theorem, for any A in the unit ball of $\mathcal{N}$, there is a net of $\{A_\alpha\}$ in the unit ball of $\mathcal{N}_0$ convergent to A in the strong-operator topology.

Since Δ is strong-operator continuous, $\Delta(A_\alpha)$ converges to $\Delta(A)$, then $\|Ad(S)\circ\Delta\| \leq 1$ and Δ is completely bounded with completely bounded norm $\|\Delta\|_{cb} \leq \|\Delta\|^2$. Thus

$$\begin{aligned}
\|\phi\|_{cb} &\leq \|S_0^{-1}\|\|S_0\|\|\phi_0\|_{cb} \\
&\leq K\|\phi\|^d\|Z\|\|Z^{-1}\|\|\Delta\|_{cb} \\
&\leq K\|\phi\|^d\|\phi_0\|^2\|\phi_0\|^6 \\
&\leq K^9\|\phi\|^{9d+8},
\end{aligned}$$

since $\|S_0^{-1}\|\|S_0\| \leq K\|\phi|_{\mathcal{B}}\| \leq K\|\phi\|\,\|Z\|\|Z^{-1}\| \leq \|\phi_0\|^2 \leq (K\|\phi\|^{d+1})^2$ and $\|\Delta\| \leq \|Z\|\|Z^{-1}\|\|\phi_0\| \leq \|\phi_0\|^3$. ■

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