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Homotopy mapping spaces

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HOMOTOPY MAPPING SPACES

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To Clarice

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TABLE OF CONTENTS

ABSTRACT

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In algebraic topology, one studies the group structure of sets of homotopy classes of maps (such as the homotopy groups $\pi_n(X)$) to obtain information about the spaces in question. It is also possible to place natural topologies on these groups that remember local properties ignored by the algebraic structure. Upon choosing a topology, one is left to wonder how well the added topological structure interacts with the group structure and which results in homotopy theory admit topological analogues. A natural place to begin is to view the n-th homotopy group $\pi_n(X)$ as the quotient space of the iterated loop space $\Omega^n(X)$ with the compact-open topology. This dissertation contains a systematic study of these quotient topologies, giving special attention to the fundamental group.

The quotient topology is shown to be a complicated and somewhat naive approach to topologizing sets of homotopy classes of maps. The resulting groups with topology capture a great deal of information about the space in question but unfortunately fail to be a topological group quite often. Examples of this failure oc-

curs in the context of a computation, namely, the topological fundamental group of a generalized wedge of circles. This computation introduces a surprising connection to the well-studied free Markov topological groups and indicates that similar failures are likely to appear in higher dimensions.

The complications arising with the quotient topology motivate the introduction of well-behaved, alternative topologies on the homotopy groups. Some alternatives are presented, in particular, free topological groups are used to construct the finest group topology on $\pi_n(X)$ such that the map $\Omega^n(X) \to \pi_n(X)$ identifying homotopy classes is continuous. This new topology agrees with the quotient topology precisely when the quotient topology does result in a topological group and admits a much nicer theory.

INTRODUCTION

The fact that classical homotopy theory is insufficient for studying spaces with homotopy type other than that of a CW-complex has motivated the introduction of a number of invariants useful for studying spaces with complicated local structure. For instance, in Čech theory, one typically approximates complicated spaces with "nice" spaces and takes the limit or colimit of an algebraic invariant evaluated on the approximating spaces. The approach taken in this dissertation is to directly transfer topological data to algebraic invariants by endowing them with natural topologies that behave nicely with respect to the algebraic structure. While this second approach does not yield purely algebraic objects, it does have the advantage of allowing direct application of the rich theory of topological algebra. The notion of "topologized" homotopy invariant seems to have been introduced by Hurewicz in [Hur35] and studied subsequently by Dugundji in [Dug50]. Whereas these early methods focused on "finite step homotopies" through open covers of spaces, we are primarily interested in the properties of spaces of homotopy classes of maps $[X, Y]_*$.

The topological fundamental group $\pi_1^{top}(X)$ of a based space (X, x), as first specified by Biss [Bis02], is the fundamental group $\pi_1(X, x)$ endowed with the natural topology that arises from viewing it as a quotient space of the space of loops based at x. This choice of topological structure makes π_1^{top} particularly useful for studying the homotopy of spaces that lack 1-connected covers. The brevity of this construction is rather deceiving since the topology of $\pi_1^{top}(X, x)$ is typically very complicated. In fact, for over 10 years [May90, Bis02], it was thought that this construction always results in a topological group. The initial intention of this research was to determine the validity of this assertion. We actually produce a plethora of counterexamples and shed light on a number of "defects" of the functor π_1^{top}. Recently, Fabel [Fab09] has shown that the Hawaiian earring group $\pi_1^{top}(\mathbb{HE})$ fails to be a topological group independently of this work.

In a first algebraic topology course, one learns early on that the fundamental group of a wedge of circles is the free group on the set indexing the wedge. One might similarly expect a generalized or "non-discrete" wedge of circles (constructed here as a suspension space $\Sigma(X_+)$) to have topological fundamental group with some similar notion of "freeness." This computation is one of the main contributions of this dissertation to the theory of topological fundamental groups. A surprising consequence is that $\pi_1^{top}(\Sigma(X_+))$ either fails to be a topological group or is one of the well-studied but notoriously complicated free (Markov) topological groups. Since realizing free groups as fundamental group is an important tool in many fields it is hoped that this geometric interpretation of many quasitopological and free topological groups will provide useful in topological algebra.

The complications that arise with the quotient topology motivate the introduction of new topologies, however, there are many natural choices. Each is likely to have its own benefits and uses. In many situations, a topology on a homotopy group may be defined to remember a specific local properties of a space. For this reason, the author does not argue that one topology is "right" where others are "wrong" or that one is "better" than another. For instance, the main power of the quotient topology is the universal property of quotient spaces and the enormous amount of data that it remembers about loops representing homotopy classes. The topology of $\pi_n^\tau(X)$ introduced in this dissertation is constructed to give a group topology from the quotient topology by removing as few open sets in the quotient topology as possible. As with the quotient topology, its primary attribute is its universal property and connection to the topology of loop spaces.

0.1 **Notation**

The following notation will be used:

Topological spaces:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, I = [0,1]$ - Non-negative integers, integers (both discrete), rational, real numbers with the standard topology, standard unit interval.

- For each integer $n \geq 1$ and $j \in \{1, ..., n\}$, let K_n^j be the closed interval $\left[\frac{j-1}{n}, \frac{j}{n}\right] \subset I$.

- For $\epsilon > 0$, let $B^n(\epsilon) = \{x \in \mathbb{R}^n | ||x|| \leq \epsilon\}$. In the case $\epsilon = 1$, we write $B^n = B^n(1)$ or sometimes e^n. Let $S^n = \partial B^{n+1} = \{x \in \mathbb{R}^{n+1} | |x| = 1\}$. When considered as based spaces, the basepoint of B^n and S^n will be $(1, 0, 0, ..., 0)$ unless otherwise stated. Let $E^n(\epsilon) = int(B^n(\epsilon))$ (interior in $\mathbb{R}^n$) and $E^n = int(B^n)$. for $n \geq 0$.

Categories: In general, if a, b are objects of a category C, $C(a, b)$ denotes the set of morphisms $a \to b$ in C. C^{op} will denote the opposite category with the direction of arrows reversed.

- **Set**, **Set$_*$**, **Top**, **Top$_*$**, **hTop**, and **hTop$_*$** - category of sets, based sets, topological spaces, based spaces, homotopy category of spaces, and homotopy category of based spaces. If C is a full subcategory of **Top** or **Top$_*$**, **h**C denotes the corresponding full subcategory of **hTop** or **hTop$_*$**.

- **Haus** and **Haus$_*$** denote the full subcategories of **Top** and **Top$_*$** consisting of Hausdorff spaces.

- $\mathbf{Top^{(n)}}$ - the category consisting of n-tuples $(X, A_1, ..., A_{n-1})$ where $A_i \subseteq X$. A morphism $(X, A_1, ..., A_{n-1}) \to (Y, B_1, ..., B_{n-1})$ is a map $f : X \to Y$ such that $f(A_i) \subseteq B_i$ for each i.

- $\mathbf{Top_*^{(n)}}$ - the category whose objects are pairs $(\mathbf{X}, x)$ where $\mathbf{X} = (X, A_1, ..., A_{n-1}) \in \mathbf{Top^{(n)}}$ and $x \in A_i$ for each i. Morphisms are basepoint preserving morphisms in $\mathbf{Top^{(n)}}$.

- **Mon, cMon, Grp, Ab** - the category of monoids, commutative monoids, groups, abelian groups.

- **MonwTop, GrpwTop** - the category of monoids (resp. groups) with topology. Objects are monoids (groups) with topology with no restriction on the continuity of operations. Morphisms are continuous monoid (group) homomorphisms.

- **TopMon, TopcMon** - the category of topological monoids and topological commutative monoids viewed as full subcategories of **MonwTop**.

- **TopGrp, TopAb** - the category of topological groups and topological abelian groups viewed as full subcategories of **GrpwTop**.

Functors:

- $(-)_+ : \mathbf{Top} \to \mathbf{Top_*}$ is the functor adding disjoint basepoint to unbased space. It is left adjoint to the functor $U : \mathbf{Top_*} \to \mathbf{Top}$ forgetting basepoint.

- For based spaces X, Y, $X \wedge Y = X \times Y / X \vee Y$ is the smash product.

- S : **Top** $\to$ **Top** - unreduced suspension given by $SX = X \times I/\sim$ where $(x,t) \sim (y,s)$ if either $s = t = 0$ or $s = t = 1$.

- Σ : **Top**$_*$ $\to$ **Top**$_*$ - reduced suspension given on a based space (X,x) by

$$\Sigma X = \Sigma(X,x) = \frac{X \times I}{X \times \{0,1\} \cup \{x\} \times I} \cong X \wedge S^1$$

  with canonical choice of basepoint. We typically denote a point in ΣX as $x \wedge t$, the image of $(x,t) \in X \times I$ in the quotient. Σ is left adjoint to the loop functor Ω : **Top**$_*$ $\to$ **Top**$_*$.

- C : **Top** $\to$ **Top**$_*$ denotes the unbased cone functor which is $CX = \frac{X \times I}{X \times \{1\}} \cong I \wedge X_+$ on an unbased space X. The image of $X \times \{1\}$ is chosen as the basepoint of CX. Sometimes C will denote the reduced cone $CX = X \wedge (I,1)$ but this distinction will be clear from context.

0.2 Outline

Chapter I includes preliminaries on function spaces with the compact-open topology and quotient spaces. In particular, a convenient basis is constructed for spaces of paths and the concept of restricted paths and neighborhoods is introduced.

Chapter II contains general theory regarding homotopy mapping spaces $[X,Y]_*$ (quotient topology). In particular, we study the interaction of the topology of $[X,Y]_*$ with algebraic structure arising naturally when X is a co-H-space space or

Y an H-space. Subsequently, we conclude that many homomorphisms in exact sequences of homotopy mapping sets are continuous in this topological setting. A characterization of discreteness of homotopy mapping spaces is included using local connectivity results of Wada [Wad54]. This allows us to know when the added topological structure does not provide any new information. Lastly, we introduce three alternative topologies on homotopy mapping sets (particularly the homotopy groups). Individually, these topologies require the use of an adjunction from topological algebra, the theory of k-spaces, and the inverse system approach to shape theory.

Chapter III deals with topology of path component spaces. These space arise naturally in the study of homotopy mapping spaces and receive a detailed treatment. Of particular interest is the preservation of limits and colimits and the path component spaces of monoids and groups.

Chapter IV is a study of the topological fundamental group and contains the main results of this dissertation. We provide some basic theory of these groups and go on to make a very general computation, namely, $\pi_1^{top}(\Sigma(X_+))$ for an arbitrary space X. This computation is akin to computing the fundamental group of a wedge of circles. Applying many results from the Appendix, we study the topology of $\pi_1^{top}(\Sigma(X_+))$ in detail and characterize when $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group. The last section of this chapter contains some theory of the alternative topology of $\pi_1^{\tau}(X)$ introduced in Chapter II.

Chapter V includes some comments on potential applications and extensions of these results, in particular a conjecture about the higher topological homotopy groups.

APPENDIX contains some theory of monoids and groups with topology where multiplication is not necessarily continuous. Much of this Appendix cannot be found in the literature, in particular, the large portion on quotient topologies on free groups. Results from the appendix are used throughout Chapters II,III, and IV.

CHAPTER I

PRELIMINARIES: TOPOLOGY

1.1 Function spaces

For unbased spaces X and Y we let $M(X, Y)$ denote the set of maps $\mathbf{Top}(X, Y)$ with the compact-open topology. A subbase for this topology consists of sets of the following form:

$$\langle K, U\rangle = \{f : X \to Y | f(K) \subseteq U \text{ where } K \subseteq X \text{ compact and } U \text{ open in } Y\}.$$

If X and Y have basepoint, let $M_*(X, Y)$ denote the set $\mathbf{Top}_*(X, Y)$ of basepoint preserving maps with the subspace topology of $M(X, Y)$. Additionally, for n-tuples $\mathbf{X} = (X, A_1, ..., A_{n-1}), \mathbf{Y} = (Y, B_1, ..., B_{n-1}) \in \mathbf{Top^{(n)}}$ and based n-tuples $(\mathbf{X}, x), (\mathbf{Y}, y) \in \mathbf{Top_*^{(n)}}$, let $M(\mathbf{X}, \mathbf{Y})$ and $M_*((\mathbf{X}, x), (\mathbf{Y}, x))$ be the hom-sets $\mathbf{Top^{(n)}}(\mathbf{X}, \mathbf{Y})$ and $\mathbf{Top_*^{(n)}}((\mathbf{X}, x), (\mathbf{Y}, y))$ respectively with the subspace topology of $M(X, Y)$. Notationally, we will not distinguish $\langle K, U\rangle$ as being a subbasis element for the topology of $M(X, Y)$ or any of these subspaces. We may mean $\langle K, U\rangle \cap M_*(X, Y)$ when we write $\langle K, U\rangle$, however the distinction should be clear from context. In the based case, the basepoint of a set of maps will be the constant map unless otherwise stated. We will often make use of the following functorial nature of these mapping spaces.

Functorality 1.1 For $C = \mathbf{Top}, \mathbf{Top}^{(n)}$, we have that $M(-,-) : C^{op} \times C \to \mathbf{Top}$ is a bifunctor. Similarly, in the based case $C = \mathbf{Top}_*, \mathbf{Top}_*^{(n)}$, $M_*(-,-) : C^{op} \times C \to \mathbf{Top}_*$ is a bifunctor.

Proof. Certainly $C(-,-) : C^{op} \times C \to \mathbf{Set}$ is a bifunctor. If therefore suffices to show that $M(-,-)$ is well-defined on morphisms. We prove the case $C = \mathbf{Top}$ and the others follow from the fact that restrictions of continuous functions are continuous. Let $f : X \to Y$ be a morphism in $\mathbf{Top}$. We show that $f^{\#} = M(id, f) : M(Z, X) \to M(Z, Y)$ given by $f^{\#}(g) = f \circ g$ and $f_{\#} = M(f, id) : M(Y, Z) \to M(X, Z)$ given by $f_{\#}(g) = g \circ f$ are continuous for any $Z \in \mathbf{Top}^{(n)}$. For $\langle K, U\rangle \subset M(Z, Y)$ and $g \in (f^{\#})^{-1}(\langle K, U\rangle)$, we have

$$g \in \langle K, f^{-1}(U)\rangle \cap M(Z, X) \subseteq (f^{\#})^{-1}(\langle K, U\rangle)$$

and so $f^{\#}$ is continuous. On the other hand, if $\langle K, U\rangle \subset M(X, Z)$ and $g \in (f_{\#})^{-1}(\langle K, U\rangle)$, then

$$g \in \langle f(K), U\rangle \cap M(Y, Z) \subseteq (f_{\#})^{-1}(\langle K, U\rangle)$$

and $f_{\#}$ is continuous. ∎

In a few cases, it will be convenient to shorten notation. For integer $n \geq 1$ and based space $(X, x) \in \mathbf{Top}_*$ let $\Omega^n(X, x) = M_*(S^n, X)$. When the basepoint is clear from context, we simply write $\Omega^n(X)$. It is well-known that $\Omega^n(X)$ is naturally homeomorphic to both $\Omega(\Omega^{n-1}(X))$ and the relative mapping space $M_*((I^{n+1}, \partial I^n), (X, \{x\}))$. The relative loop space of a pair $((X, A), x) \in \mathbf{Top}_*^{(2)}$ is $\Omega^n(X, A) = M_*((B^n, S^{n-1}), (X, A))$

and is naturally homeomorphic to both $M_*((I^n, \partial I^n), (X, A))$ and the space of maps of triples

$$M\Big(\big(I^n, I^{n-1}\times\{0\}, I^{n-1}\times\{1\}\cup\partial(I^{n-1})\times I\big), (X, A, \{x\})\Big).$$

in $\mathbf{Top}^{(3)}$. We now recall some basic facts regarding the compact-open topology.

Proposition 1.2 *An inclusion $B \subseteq Y$ induces an inclusion $i : M(X, B) \hookrightarrow M(X, Y)$.*

Proof. Let $C \subset X$ be compact and U be open in Y. That i is an inclusion follows from the equation $i(\langle C, U \cap B\rangle) = \langle C, U\rangle \cap M(X, B)$. ■

Theorem 1.3 *[Eng89, 3.4] If Y is T_0 (resp. T_1, Hausdorff, Regular and T_1, Tychonoff), then so is $M(X, Y)$.*

Remark 1.4 If X is an unbased space, there is a homeomorphism $M(*, X) \cong X$, $f \mapsto f(*)$ and for any based space Y, there is a homeomorphism $M_*(X_+, Y) \cong M(X, Y)$, $f \mapsto f|_X$.

Remark 1.5 Since $S^0 \wedge X \cong X$ for every $X \in \mathbf{Top}_*$, the previous remark implies that for arbitrary $X, Y \in \mathbf{Top}_*$ there are canonical homeomorphisms

$$M_*(S^0, M_*(X, Y)) = M_*(*_+, M_*(X, Y)) \cong M(*, M_*(X, Y)) \cong M_*(X, Y).$$

Lemma 1.6 *[Eng89, Proposition 3.4.5] Let X be any space, Y_λ be a family of spaces with projections $p_\lambda : Y = \prod_\lambda Y_\lambda \to Y_\lambda$. The canonical map $M(X, Y) \to \prod_\lambda M(X, Y_\lambda)$, $f \mapsto (p_\lambda \circ f)$ is a homeomorphism. It restricts to a homeomorphism $M_*(X, Y) \to \prod_\lambda M_*(X, Y_\lambda)$ in the based case.*

Of particular interest to us are the homeomorphisms $\Omega(\prod_\lambda X_\lambda) \cong \prod_\lambda \Omega(X_\lambda)$.

Lemma 1.7 *[Eng89, 3.4.B] Let X_λ and Y_λ be a family of spaces indexed by the same set where each X_λ is Hausdorff. The product operation $\prod_\lambda M(X_\lambda, Y_\lambda) \to M(\prod_\lambda X_\lambda, \prod_\lambda Y_\lambda)$, $(f_\lambda) \mapsto \prod_\lambda f_\lambda$ is an embedding. This restricts to an embedding of based mapping spaces as well.*

Lemma 1.8 *Let X_λ a family of based Hausdorff spaces, Y be a based space, and $j_\lambda : X_\lambda \hookrightarrow \bigvee_\lambda X_\lambda$ be the canonical inclusions. The canonical map $M_*(\bigvee_\lambda X_\lambda, Y) \to \prod_\lambda M_*(X_\lambda, Y)$, $f \mapsto (f \circ j_\lambda)$ is a continuous bijection and is a homeomorphism of based spaces when X_λ is Hausdorff for each λ.*

Proof. Since $\mathbf{Top}_*(-, Y) : \mathbf{Top}_*^{\mathrm{op}} \to \mathbf{Set}_*$ preserves colimits and is induced by the inclusion j_α, this map is clearly a continuous bijection. Let X_λ be Hausdorff and x_0 be the basepoint of $\bigvee_\lambda X_\lambda$. Let $\langle C, U\rangle \subset M_*(\bigvee_\lambda X_\lambda, Y)$. Since C is compact and X_λ is closed in $\bigvee_\lambda X_\lambda$, $C \cap X_\lambda$ is compact for each λ. But since each X_λ is Hausdorff, we have $(X_\lambda - \{x_0\}) \cap C = \emptyset$ for all but finitely many λ. Suppose $\lambda_1, \dots, \lambda_m$ are the indices for which $(X_\lambda - \{x_0\}) \cap C \neq \emptyset$. Since X_{λ_i} is Hausdorff $X_{\lambda_i} \cap C$ is a compact subset of X_{λ_i}. Let $V_{\lambda_i} = \langle X_{\lambda_i} \cap C, U\rangle$ and $V_\lambda = M_*(X_\lambda, Y)$ when $\lambda \neq \lambda_i$. This makes $V = \prod_\lambda U_\lambda$ an open neighborhood in $\prod_\lambda M_*(X_\lambda, Y)$. Clearly if $f \in \langle C, U\rangle$, then $(f \circ j_\lambda) \in V$. Conversely, if $(f \circ j_\lambda) \in V$, then $f(C) = f(\bigcup_i X_{\lambda_i} \cap C) = \bigcup_i (f \circ j_\lambda)(X_{\lambda_i} \cap C) \subset U$ and we have $f \in \langle C, U\rangle$. Therefore $f \mapsto (f \circ j_\lambda)$ is a homeomorphism. ∎

Lemma 1.9 *Let X_λ and Y_λ be a family of based spaces indexed by the same set where each X_λ*

is Hausdorff. The wedge operation $\prod_\lambda M_*(X_\lambda, Y_\lambda) \to M_* (\bigvee_\lambda X_\lambda, \bigvee_\lambda Y_\lambda)$, $(f_\lambda) \mapsto \bigvee_\lambda f_\lambda$ *is an embedding.*

Proof. The inclusions $j_{Y_\lambda} : Y_\lambda \to \bigvee_\lambda Y_\lambda$ induce inclusions (1.2) $(j_{Y_\lambda})_\# : M_*(X_\lambda, Y_\lambda) \to M_*(X_\lambda, \bigvee_\lambda Y_\lambda)$. Together, these give the embedding

$$\prod_\lambda (j_{Y_\lambda})_\# : \prod_\lambda M_*(X_\lambda, Y_\lambda) \to \prod_\lambda M_*\left(X_\lambda, \bigvee_\lambda Y_\lambda\right) \cong M_*\left(\bigvee_\lambda X_\lambda, \bigvee_\lambda Y_\lambda\right)$$

where the homeomorphism is from the previous lemma. This map is the desired embedding. ∎

The Continuity of Evaluation 1.10 If $X, Y \in$ **Top** with X locally compact Hausdorff, then the evaluation map $ev : X \times M(X, Y) \to Y$, $ev(x, f) = f(x)$ is continuous [Mun00, Theorem 46.10]. If $X, Y \in$ **Top$_*$** with X locally compact Hausdorff, then $ev : X \wedge M_*(X, Y) \to Y$ is continuous [Mau70, Theorem 6.2.31].

Exponential Law 1.11 [Dug66, Theorem 5.3] If X is Hausdorff and Y is locally compact Hausdorff, then for every space Z, the natural map $\eta : M(X, M(Y, Z)) \to M(X \times Y, Z)$, $\eta(f)(x, y) = f(x)(y)$ is a homeomorphism.

Based Exponential Law 1.12 [Mau70, Theorem 6.2.38] If X, Y are compact Hausdorff based spaces, then for every based space Z, the natural map $\eta_* : M_*(X, M_*(Y, Z)) \to M_*(X \wedge Y, Z)$, $\eta_*(f)(x \wedge y) = f(x)(y)$ is a homeomorphism of based spaces.

Metrizable function spaces are also of interest.

Theorem 1.13 *[Eng89, 4.2.17 & 4.2.18] If X is compact Hausdorff and Y is metrizable, then $M_*(X,Y)$ is metrizable. If, in addition, X is metrizable and Y is separable, then $M_*(X,Y)$ is separable.*

1.1.1 Restricted paths and neighborhoods

In order to study quotients of spaces of paths and loops, it will be necessary to study operations on paths and to find a convenient basis for the topology of the *free path space* $M(I,X)$. Sometimes we will write $P(X)$ for $M(I,X)$, $P(X,x) = \{p \in P(X) | p(0) = x\}$, and $P(X,x,y) = \{p \in P(X) | p(0) = x, p(1) = y\}$. We first consider concatenation of paths. For any fixed, closed subinterval $A = [a,b] \subseteq I$, we make use of the following notation. Let $H_A : I \to A$ be the unique, increasing, linear homeomorphism. For a path $p : I \to X$, the *restricted path of p to A* is the composite $p_A = p|_A \circ H_A : I \to A \to X$. As a convention, if $A = \{t\} \subseteq I$ is a singleton, $p_A : I \to X$ will denote the constant path at $p(t)$. Note that if $0 = t_0 \leq t_1 \leq ... \leq t_n = 1$, knowing the paths $p_{[t_{i-1},t_i]}$ for $i = 1, ..., n$ uniquely determines p.

This definition allows us to easily define the concatenation of paths. If $p_1, ..., p_n : I \to X$ are paths such that $p_i(1) = p_{i+1}(0)$ for each $i = 1, ..., n-1$, the *n-fold concatenation* of these paths is the unique path

$$q = *_{i=1}^{n} p_i = p_1 * p_2 * \cdots * p_n$$

such that $q_{K_n^i} = p_i$ for each i (recall $K_n^i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$). The *reverse* of a path p is $p^{-1}(t) = p(1-t)$.

The ability to restrict open neighborhoods to smaller and smaller paths will be convenient Chapter IV. The following construction is introduced to this end. If $\mathscr{U} = \bigcap_{j=1}^{n}\langle C_j, U_j\rangle$ is a basic open neighborhood of a path $p \in P(X)$, then $\mathscr{U}_A = \bigcap_{A\cap C_j\neq\emptyset}\langle H_A^{-1}(A \cap C_j), U_j\rangle$ is a basic open neighborhood of p_A called the *restricted neighborhood* of $\mathscr{U}$ to A. If $A = \{t\}$ is a singleton, then $\mathscr{U}_A = \bigcap_{t\in C_j}\langle I, U_j\rangle = \langle I, \bigcap_{t\in C_j} U_j\rangle$. On the other hand, if $\mathscr{U} = \bigcap_{j=1}^{n}\langle C_j, U_j\rangle$ is a basic open neighborhood of the restricted path p_A, then $\mathscr{U}^A = \bigcap_{j=1}^{n}\langle H_A(C_j), U_j\rangle$ is a basic open neighborhood of p called the *induced neighborhood* of $\mathscr{U}$ on A. If $A = \{t\}$ is a singleton so that p_A is a constant map, we let $\mathscr{U}^A = \bigcap_{j=1}^{n}\langle \{t\}, U_j\rangle = \langle \{t\}, \bigcap_{j=1}^{n} U_j\rangle$.

Lemma 1.14 *For any basic open neighborhood $\mathscr{U} = \bigcap_{j=1}^{n}\langle C_j, U_j\rangle$ in $P(X)$ and closed interval $A \subseteq I$, we have $(\mathscr{U}^A)_A = \mathscr{U} \subseteq (\mathscr{U}_A)^A$. The second inclusion also holds when A is a singleton.*

Proof. It is easy to see that

$$(\mathscr{U}^A)_A = \left(\bigcap_{j=1}^{n}\langle H_A(C_j), U_j\rangle\right)_A = \bigcap_{A\cap H_A(C_j)\neq\emptyset}\langle H_A^{-1}(A \cap H_A(C_j)), U_j\rangle$$

Since $H_A(C_j) \subseteq A$, this is clearly $\mathscr{U}$. This gives the first inclusion. To prove the second, we note that

$$(\mathscr{U}_A)^A = \bigcap_{A\cap C_j\neq\emptyset}\langle H_A\left(H_A^{-1}(A \cap C_j)\right), U_j\rangle = \bigcap_{A\cap C_j\neq\emptyset}\langle A \cap C_j, U_j\rangle$$

Clearly, if $f(C_j) \subset U_j$, then $f(A \cap C_j) \subseteq U_j$ whenever $A \cap C_j \neq \emptyset$. If $A = \{t\}$, then $(\mathscr{U}_A)^A = \left(\langle I, \bigcap_{t\in C_j} U_j\rangle\right)^A = \langle \{t\}, \bigcap_{t\in C_j} U_j\rangle$. Clearly, if $f(C_j) \subseteq U_j$, then $f(t) \in U_j$

whenever $t \in C_j$. ■

A slightly more useful variant of the previous lemma is:

Lemma 1.15 *If* $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = 1$ *and* $\mathcal{U} = \bigcap_{j=1}^{n} \langle C_j, U_j \rangle$ *such that* $\bigcup_{j=1}^{n} C_j = I$, *then* $\mathcal{U} = \bigcap_{i=1}^{n} (\mathcal{U}_{[t_{i-1}, t_i]})^{[t_{i-1}, t_i]}$.

Proof. For convenience of notation, let $A_i = [t_{i-1}, t_i]$. 2. of Lemma 1.14 gives the first inclusion $\mathcal{U} \subseteq \bigcap_{i=1}^{n} (\mathcal{U}_{A_i})^{A_i}$. Now suppose

$$f \in \bigcap_{i=1}^{n} (\mathcal{U}_{A_i})^{A_i} = \bigcap_{i=1}^{n} \bigcap_{A_i \cap C_j \neq \emptyset} \langle A_i \cap C_j, U_j \rangle.$$

Clearly, if $t \in C_j = \bigcup_{A_i \cap C_j \neq \emptyset} A_i \cap C_j$. then $f(t) \in U_j$. Therefore, $f(C_j) \subseteq U_j$ for each j, that is $f \in \mathcal{U}$. ■

The condition that the union of the C_j be the whole interval is to avoid taking the intersection with a potentially empty set.

1.1.2 A convenient basis for free path spaces

Let $B_{P(X)}$ denote the basis for the topology of $P(X)$ generated by the subbasis of elements $\langle K, U \rangle$ and let $\mathscr{B}_X$ be a basis for the topology of X which is closed under finite intersection (for instance, the topology of X). We now find a new basis $\mathscr{B}_{P(X)} \subset B_{P(X)}$. To do so we must know how to subdivide the unit interval in an orderly manner. We use the following convention: For each integer $m \geq 1$, let $[m] = \{1, \ldots, m\}$ and for each $j \in [m]$, let $K_m^j = \left[\frac{j-1}{m}, \frac{j}{m}\right]$ be the corresponding closed subinterval of I. For any integer $p \geq 1$ and subset $S \subseteq [m]$ we let $pS = \{j \in [pm] | j = pq + 1 - k \text{ for } q \in S \text{ and } k \in [p]\}$. For example, if $S = \{1, 3\} \subset [4]$, then

$3S = \{1,2,3,7,8,9\} \subset [12]$. Now an interval K_m^j may be subdivided evenly into p smaller intervals $K_m^j = \bigcup_{l \in p\{j\}} K_{mp}^l$. Moreover, note that for any set $S \subseteq [m]$ and any $p \geq 1$ we have $pS \subset [mp]$ and $\bigcup_{j \in S} K_m^j = \bigcup_{i \in pS} K_{mp}^i$.

Lemma 1.16 *If $C \subseteq I$ is a compact set and $U \subseteq I$ is an open set containing C, then there is an integer $M > 1$ and a set $S \subseteq [M]$ such that $C \subseteq \bigcup_{j \in S} K_M^j \subseteq U$.*

Proof. Write $U = \coprod_k U_k$ where each U_k is an open interval in I. Since C is compact, take $U_1, ..., U_m$ to be the U_k such that $U_k \cap C \neq \emptyset$. Take a finite cover $\{(s_l, t_l)\}_l$ of C where $s_l, t_l \in \mathbb{Q}$ and $(s_l, t_l) \subseteq U_i$ for some i. For each i, let $\frac{a_i}{b_i} = \min\{s_l | (s_l, t_l) \subset U_i\}$ and $\frac{c_i}{d_i} = \max\{t_l | (s_l, t_l) \subset U_i\}$. Now for each $i = 1, ..., m$ we have $\left[\frac{a_i}{b_i}, \frac{c_i}{d_i}\right] \subseteq U_{k_i}$ and

$$C \subseteq \coprod_{i=1}^{m} \left[\frac{a_i}{b_i}, \frac{c_i}{d_i}\right] \subset \coprod_{i=1}^{m} U_{k_i} \subseteq U.$$

Now let $M = lcm(b_1, ..., b_m, d_1, ..., d_m)$ so that

$$\left[\frac{a_i}{b_i}, \frac{c_i}{d_i}\right] = \bigcup_{j=\frac{Ma_i}{b_i}}^{\frac{Mc_i}{d_i}} K_M^j \text{ and thus } C \subseteq \coprod_{i=1}^{m} \bigcup_{j=\frac{Ma_i}{b_i}}^{\frac{Mc_i}{d_i}} K_M^j \subset \coprod_{i=1}^{m} U_{k_i} \subseteq U.$$

So the desired set $S \subseteq [M]$ is

$$S = \bigcup_{i=1}^{m} \left\{\frac{Ma_i}{b_i}, \frac{Ma_i}{b_i} + 1, ..., \frac{Mc_i}{d_i}\right\}.$$

■

Using previous notation we may state a special case of the Lebesgue lemma in terms of the compact-open topology of $P(X)$.

Lebesgue Lemma 1.17 [Mun00, Lemma 27.5] *If* $p \in P(X)$ *and* $\{U_i\}_{i=1}^{M}$ *is a finite open cover of the compact image* $\alpha(I) \subset X$*, then there is an integer* $N > 1$ *such that* $p \in \bigcap_{j=1}^{N} \langle K_N^j, U_{i_j} \rangle$ *for not necessarily distinct* $i_j \in [M]$.

Lemma 1.18 *For each open neighborhood* $W = \bigcap_{i=1}^{n} \langle C_i, U_i \rangle$ *of path* $p : I \to X$ *in* $P(X)$*, there is an open neighborhood* V *of* p *of the form* $V = \bigcap_{j=1}^{M} \langle D_j, V_j \rangle \subseteq W$ *where* $\bigcup_{i=1}^{M} D_i = I$ *and* $V_j \in \mathscr{B}_X$ *for each* $j = 1, 2, ..., M$.

Proof. For each i, choose a finite open cover $\mathscr{V}^i = \left\{V_j^i\right\}_{j \in [m_i]}$ of compact space $p(C_i) \subseteq X$ such that $\bigcup_{j \in [n_i]} V_j^i \subseteq U_i$ and $V_j^i \in \mathscr{B}_X$ for each ordered pair (i, j). Now $\left\{\alpha^{-1}\left(V_j^i\right)\right\}_{j \in [n_i]}$ is a finite cover of compact metric subspace $C_i \subseteq I$. So for each i we may use the Lebesgue lemma to find and integer $m_i > 1$ such that for each $l \in [m_i]$ we have:

$$K_{m_i}^l \cap C_i \subseteq p^{-1}\left(V_{j_l}^i\right) \text{ for some } j_l \in [m_i]$$

Now

$$p \in V_1 = \bigcap_{i \in [n]} \bigcap_{l \in [m_i]} \left\langle K_{m_i}^l \cap C_i, V_{j_l}^i \right\rangle \subseteq W$$

The set V_1 satisfies the desired inclusion into W however the union of the compact sets $K_{m_i}^l$ is not necessarily the entire unit interval. To fix this we cover $p(I)$ with finitely many basic open neighborhoods $\{Y_k\}_{k \in [N]} \subset \mathscr{B}_X$. Now $\left\{p^{-1}(Y_k)\right\}_{k \in [N]}$ is a cover of compact metric space I and we may apply the Lebesgue lemma again to find an integer $Q > 1$ such that for each $q \in [Q]$, $K_Q^q \subset p^{-1}\left(Y_{k_q}\right)$ for some $k_q \in [N]$. Therefore $p \in V_2 = \bigcap_{q \in [Q]} \left\langle K_Q^q, Y_{k_q} \right\rangle$. Now we let $V = V_1 \cap V_2 \subseteq W$ which is of the form desired. ■

Lemma 1.19 *For each open neighborhood* $W = \bigcap_{i=1}^{n} \langle C_i, U_i \rangle$ *of path* $p \in P(X)$ *where* $\bigcup_{i=1}^{n} C_i = I$, *there is an integer* $N \geq 1$ *and an open neighborhood* $p \in \bigcap_{j=1}^{N} \langle K_N^j, V_j \rangle \subseteq W$ *where each* V_j *is some intersection of the* U_i, *i.e.* $V_j = \bigcap_{i \in T_j} U_i$ *for some* $T_j \subseteq [n]$.

Proof. We have $C_i \subset \alpha^{-1}(U_i)$ for each i and so by Lemma 1.16 there is an integer $M_i \geq 1$ and set $S_i \subseteq [M_i]$ such that

$$C_i \subseteq \bigcup_{l \in S_i \subset [M_i]} K_{M_i}^l \subseteq p^{-1}(U_i)$$

Let $N = lcm(M_i)$ and $P_i = \frac{N}{M_i}$. We now re-index and find

$$p \in \bigcap_{i=1}^{n} \bigcap_{l \in S_i \subset [M_i]} \langle K_{M_i}^l, U_i \rangle = \bigcap_{i=1}^{n} \bigcap_{j \in P_i S_i \subset [N]} \langle K_N^j, U_i \rangle.$$

Since $\bigcup_{i=1}^{n} C_i = I$, the compact set K_N^j appears at least once in the intersection for each $j \in [N]$. Therefore the open set

$$V_j = \bigcap_{\alpha(K_N^j) \subset U_i} U_i$$

is non-empty. We certainly have $p \in \bigcap_{j=1}^{N} \langle K_N^j, V_j \rangle$ and we claim the inclusion $\bigcap_{j=1}^{N} \langle K_N^j, V_j \rangle \subseteq \bigcap_{i=1}^{n} \langle C_i, U_i \rangle$. If $\beta(K_N^j) \subseteq V_j$ for each j, then

$$\beta(C_i) \subseteq \beta\left(\bigcup_{l \in S_i \subset [M_i]} K_{M_i}^l \right) = \beta\left(\bigcup_{j \in P_i S_i \subset [N]} K_N^j \right)$$

But for each $j \in P_iS_i$, we have $K_N^j \subset \alpha^{-1}(U_i)$ implying that $V_j \subseteq U_i$ by the definition of V_j. Therefore $\beta(C_i) \subset \bigcup_{j \in P_iS_i} V_j \subseteq U_i$. ■

Since we assumed basis $\mathscr{B}_X$ is closed under finite intersections Lemmas 1.18 and 1.19 imply the next result.

Theorem 1.20 *If W is any open neighborhood of path $p : I \to X$, there is an integer $N \geq 1$ and an open neighborhood $\bigcap_{j=1}^{N} \langle K_N^j, V_j \rangle$ of p contained in W such that $V_j \in \mathscr{B}_X$ for each $j \in [N]$. Moreover, if $\mathscr{B}_{P(X)}$ is the collection of neighborhoods of the form $\bigcap_{j \in [N]} \langle K_N^j, V_j \rangle$ with $V_j \in \mathscr{B}_X$, $\mathscr{B}_{P(X)}$ is a basis for the topology of $P(X)$ which is closed under finite intersections.*

Proof. The first statement follows directly from Lemmas 1.18 and 1.19. To see that $\mathscr{B}_{P(X)}$ is closed under finite intersection we suppose $\mathscr{U} = \bigcap_{i \in [M]} \langle K_M^i, U_i \rangle$ and $\mathscr{V} = \bigcap_{j \in [N]} \langle K_N^j, V_j \rangle$ are neighborhoods in $\mathscr{B}_{P(X)}$. We find a common partition of I by letting $P = lcm(M, N)$ and for each $k \in [P]$, we let $W_k = U_{i_k} \cap V_{j_k} \in \mathscr{B}_X$ whenever $K_P^k \subseteq K_N^{i_k} \cap K_M^{j_k}$. It is easy to see that $\mathscr{U} \cap \mathscr{V} = \bigcap_{k \in [P]} \langle K_P^k, W_k \rangle \in \mathscr{B}_{P(X)}$. ■

Intuitively, Theorem 1.20 allows us to restrict our use of basic neighborhoods in $P(X)$ to neighborhoods that resemble "finite sets of ordered instructions."

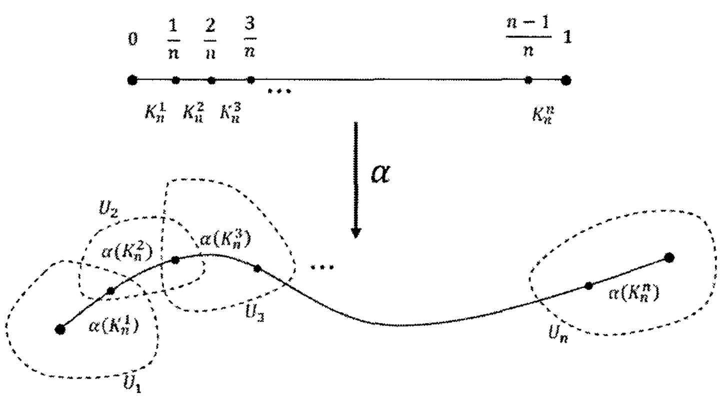


Figure 1: *An illustration of a basic open neighborhood as an element of $\mathscr{B}_{P(X)}$*

Using this basis, we may easily prove some basic facts about path spaces.

Lemma 1.21 *Let $C_n = \{(p_1, ..., p_n) \in P(X)^n | p_i(1) = p_{i+1}(0)\}$. For each $n \geq 1$, the n-fold concatenation map $c_n : C_n \to P(X)$ $(p_1, ..., p_n) \mapsto *_{i=1}^{n} p_i$ is continuous. The operation $r : P(X) \to P(X)$, $\alpha \mapsto \alpha^{-1}$ of taking a loop to its reverse is also continuous*

Proof. First, we note that $r^{-1}(\langle C, U\rangle) = \langle\{1 - t | t \in C\}, U\rangle$ for each subbasis set $\langle C, U\rangle \subseteq P(X)$. Therefore r is continuous. Let $*_{i=1}^{n} p_i \in \bigcap_{j=1}^{m} \langle K_m^j, U_j\rangle$. We may suppose that n divides m, in particular that $kn = m$. Now $p_i \in U_{K_n^i} = \bigcap_{j\in k\{i\}\subset[m]} \langle K_m^j, U_j\rangle$ for each $i \in [n]$ and

$$(p_1, ..., p_n) \in \left(U_{K_n^1} \times \cdots \times U_{K_n^n}\right) \cap C_n \subset c_n^{-1}\left(\bigcap_{j=1}^{m} \langle K_m^j, U_j\rangle\right).$$

Therefore c_n is continuous. ■

1.2 Quotient spaces

The following lemma is a basic fact that we will refer to repeatedly. It is a direct consequence of the universal property of quotient spaces.

Quotient-Square Lemma 1.22 *Suppose A, B, C, D are spaces and the diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow[g]{} & D \end{array}$$

commutes in **Set**. *If p is quotient and f, q are continuous, then g is continuous. If f, p, q are quotient, then so is g.*

It is well known that the product of two quotient maps may fail be a quotient map. This failure naturally leads to the introduction of so-called convenient categories of spaces. Since many of the quotient spaces considered in this dissertation fail to be Hausdorff we take Brown's approach [Bro06, §5.9].

Definition 1.23 A space X is a *k-space* if it has the final topology with respect to all maps $C \to X$ for all compact Hausdorff spaces C. Equivalently [Bro06, 5.9.1], X is a k-space if and only if it is the quotient of a sum of compact Hausdorff spaces.

Let **kTop** denote the full subcategory of **Top** consisting of k-spaces. The inclusion functor **kTop** $\to$ **Top** has a left adjoint k : **Top** $\to$ **kTop** which is the identity on the underlying sets and functions. The identity $k(X) \to X$ is always continuous and $k(X) = X$ if and only if X is a k-space. For spaces X, Y, let $X \times_k Y = k(X \times Y)$. This satisfies $k(X) \times_k k(Y) \cong X \times_k Y$ and gives a well-defined categorical product in **kTop**.

Fact 1.24 [Bro06] The following are well-known facts regarding k-spaces.

1. If X is a k-space and Y is locally compact Hausdorff, then $X \times Y$ is a k-space.
2. Every quotient space of a k-space is a k-space.
3. First countable spaces and locally compact Hausdorff spaces are k-spaces.
4. If $f_i : X_i \to Y_i$, $i = 1, 2$ are quotient maps of k-spaces, then $f_1 \times_k f_2$ is quotient.
5. The previous two facts and the fact that finite products of first countable spaces are first countable imply that if X, Y are first countable and $q : X \to Y$ is quotient, then $q^n : X^n \to Y^n$ is quotient for every $n \geq 1$.

6. Parts 1.,3.,4. imply that if X, Y are locally compact Hausdorff and $q : X \to Y$ is quotient, then $q^n : X^n \to Y^n$ is quotient for every $n \geq 1$.

We also consider another class of spaces whose quotients are well behaved.

Definition 1.25 A space X is a *k_ω-space* if X is the inductive limit of a sequence of compact subsets, i.e. $X = \bigcup_{n\geq 1} K_n$ and A is closed in X if and only if $A \cap K_n$ is closed in K_n for each $n \geq 1$.

Theorem 1.26 [Mic68, 7.5] *If X is a Hausdorff k_ω-space and $q : X \to Y$ is quotient, then $q^n : X^n \to Y^n$ is quotient for every $n \geq 1$.*

As mentioned in the introduction, we are interested in quotients of mapping spaces. Simple descriptions of such objects are often hard to come by. While computationally challenging, there is an intuitive method of constructing a basis of open neighborhoods for any quotient space. We take this approach so that if $q : Y \to Z$ is a quotient map, a basis for Z may be described in terms of open coverings of Y.

Definition 1.27 For any space Y, a *pointwise open cover* of Y is an open cover $\mathscr{U} = \{U^y\}_{y\in Y}$ where each point $y \in Y$ has a distinguished open neighborhood U^y containing it. Let $Cov(Y)$, be the directed set of pointed open covers of Y where the direction is given by pointwise refinement: If $\mathscr{U} = \{U^y\}_{y\in Y}, \mathscr{V} = \{V^y\}_{y\in Y} \in Cov(Y)$, then we say $\mathscr{U} \preceq \mathscr{V}$ when $V_y \subseteq U_y$ for each $y \in Y$.

We also make use of the following notation: If $\mathscr{U} = \{U^y\}_{y\in Y} \in Cov(Y)$ is a pointwise open covering of Y and $A \subseteq Y$, let $\mathscr{U}(A) = \bigcup_{a\in A} U^a$.

Construction 1.28 Suppose $q : Y \to Z$ is quotient map, $z \in Z$, and $\mathscr{U} \in Cov(Y)$ is a fixed point-wise open cover. We construct open neighborhoods of z in Z in the most unabashed way, that is, by recursively "collecting" the elements of Y so that our collection is both open and saturated. We begin by letting $O_q^0(z, \mathscr{U}) = \{z\}$. For integer $n \geq 1$, we define $O_q^n(z, \mathscr{U}) \subseteq Z$ as

$$O_q^n(z, \mathscr{U}) = q\left(\mathscr{U}\left(q^{-1}\left(O_q^{n-1}(z, \mathscr{U})\right)\right)\right)$$

It is clear that $O_q^{n-1}(z, \mathscr{U}) \subseteq O_q^n(z, \mathscr{U})$ for all $n \geq 1$. We then may take the union

$$O_q(z, \mathscr{U}) = \bigcup_n O_q^n(z, \mathscr{U})$$

Note that if $y \in q^{-1}\left(O_q(z, \mathscr{U})\right)$, then $U^y \subset q^{-1}\left(O_q(z, \mathscr{U})\right)$ so that $O_q(z, \mathscr{U})$ is open in Z. Also, if $\mathscr{W} = \{W^y\}_{y \in Y}$ is another point-wise open cover of Y such that $q(W^y) \subseteq q(U^y)$ for each $y \in Y$, then $O_q(z, \mathscr{W}) \subseteq O_q(z, \mathscr{U})$. The neighborhood $O_q(z, \mathscr{U})$ is said to be the *open neighborhood of z in Z generated by* $\mathscr{U}$. It is easy to see that for each open neighborhood V of z in Z, there is a pointwise open covering $\mathscr{V} \in Cov(Y)$ such that $z \in O_q(z, \mathscr{V}) \subseteq V$. In particular, let $V^y = q^{-1}(V)$ when $q(y) \in V$ and $V^y = Y$ otherwise.

Theorem 1.29 *The neighborhoods* $O_q(z, \mathscr{V})$ *for* $\mathscr{V} \in Cov(Y)$ *form a neighborhood base at* z *in* Z.

The points in $q^{-1}\left(O_q(z, \mathscr{U})\right)$ can be described as follows: For each $y \in q^{-1}\left(O_q(z, \mathscr{U})\right)$, there is an integer $n \geq 1$ and a sequence of loops $y_0, y_1, \dots, y_{2n+1}$ such that

- $y_0 = y$ and $y_{2n+1} = \alpha$
- $q(y_{2i}) = q(y_{2i+1})$ for $i = 0, 1, \ldots, n$
- $y_{2i+1} \in U^{y_{2i+2}}$ for $i = 0, 1, \ldots, n-1$

In this sense, the neighborhood $O_q(z, \mathscr{U})$ is an alternating "collection" of fibers and nearby points (the "nearby" being determined by the elements of $\mathscr{U}$).

CHAPTER II

HOMOTOPY MAPPING SPACES

2.1 Homotopy mapping sets as quotient spaces

Definition 2.1 Two maps $f, g : X \to Y$ are *homotopic* (we write $f \simeq g$) if there is a map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. For each $t \in I$, we let H_t be the restriction $H|_{X \times \{t\}} : X \to Y$. If $f, g \in M_*(X, Y)$ are based maps, H is said to *preserve basepoint* if $H_t \in M_*(X, Y)$ for all $t \in I$. If $\mathbf{X}, \mathbf{Y} \in \mathbf{Top}^{(n)}$, and $f, g \in M(\mathbf{X}, \mathbf{Y})$, H *preserves relative structure* if $H_t \in M(\mathbf{X}, \mathbf{Y})$ for all $t \in I$.

Homotopy defines an equivalence relation on $M(X, Y)$. Similarly, homotopy preserving basepoint, relative structure, or both give equivalence relations on the appropriate mapping spaces.

Homotopy Mapping Spaces 2.2 Let

$$[X, Y], [X, Y]_*, [\mathbf{X}, \mathbf{Y}] \text{ and } [\mathbf{X}, \mathbf{Y}]_*$$

be the quotient space of mapping space

$$M(X, Y), M_*(X, Y), M(\mathbf{X}, \mathbf{Y}) \text{ and } M_*(\mathbf{X}, \mathbf{Y})$$

respectively where homotopy classes of maps are identified. These spaces will be referred to as *homotopy mapping spaces*. In the based case, the homotopy class of the constant map will be chosen as the basepoint of the homotopy mapping space.

The underlying sets of homotopy mapping spaces are, of course, the homotopy mapping sets found throughout classical algebraic topology.

Universal Property 2.3 The quotient topology on $[X, Y]_*$ is the finest topology on the underlying set of $[X, Y]_*$ such that the canonical surjection $\pi : M_*(X, Y) \to [X, Y]_*$ is continuous. The same statement holds for the unbased and relative cases as well.

Now we observe the functorality of homotopy mapping spaces. Recall that $[-,-]_* : \mathbf{Top}_*^{op} \times \mathbf{Top}_* \to \mathbf{Set}_*$ is a functor where for maps $f : W \to X, g : Y \to Z$, we have $f^* = [f, id_Y]_* : [X, Y]_* \to [W, Y]_*, f^*([h]) = [h \circ f]$ and $g_* = [id_W, g]_* : [W, Y]_* \to [W, Z]_*, g_*([k]) = [g \circ k]$. Together, we have $[g, f] = g_* \circ f^*$.

Functorality 2.4 $[-,-]_* : \mathbf{Top}_*^{op} \times \mathbf{Top}_* \to \mathbf{Top}_*$ *is a bifunctor. The analogous statement holds for the unbased and relative cases.*

Proof. We already have defined $[X, Y]_*$ as a space and it is basic homotopy theory that the non-topological functor $[-,-]_* : \mathbf{Top}_*^{op} \times \mathbf{Top}_* \to \mathbf{Set}_*$ is a functor. Therefore it suffices to check the continuity of functions $[g, f]_* = g_* \circ f^*$ for each pair of maps $f : W \to X, g : Y \to Z$. Recall that $M_*(-,-) : \mathbf{Top}_*^{op} \times \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor by 1.1. In particular, the induced maps $f^\#(k) = k \circ f$ and $g_\#(k) = g \circ k$ in the following diagram are continuous.

$$\begin{array}{ccccc} M_*(X,Y) & \xrightarrow{f^\#} & M_*(W,Y) & \xrightarrow{g_\#} & M_*(W,Z) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ [X,Y]_* & \xrightarrow[f^*]{} & [W,Y]_* & \xrightarrow[g_*]{} & [W,Z]_* \end{array}$$

Here the vertical maps are the canonical quotient maps. It follows from the Quotient Square Lemma that $g_* = [id_W, g]_*$ and $f^* = [f, id_Y]_*$ are continuous. Therefore the composite $g_* \circ f^*$ is continuous. ■

Remark 2.5 Since homotopic maps $f, g : X \to Y$ induce the same continuous maps $f_* = g_*$ and $f^* = g^*$ on homotopy mapping spaces we actually have a functor $[-,-]_* : \mathbf{hTop}_*^{\mathrm{op}} \times \mathbf{hTop}_* \to \mathbf{Top}_*$ (and similarly for the unbased and relative cases).

Proposition 2.6 *If X, Y, Z are spaces with X Hausdorff and Y locally compact Hausdorff, then there is a natural homeomorphism* $[X, M(Y,Z)] \cong [X \times Y, Z]$. *If X, Y, Z are based spaces where X and Y are compact Hausdorff, there is a natural homeomorphism* $[X, M_*(Y,Z)]_* \cong [X \wedge Y, Z]_*$.

Proof. By the exponential laws 1.11 and 1.12, there are natural homeomorphisms $\eta : M(X, M(Y,Z)) \to M(X \times Y, Z)$ and $\eta_* : M_*(X, M_*(Y,Z)) \to M_*(X \wedge Y, Z)$. Since homeomorphisms of mapping spaces preserve homotopy classes, there are commuting diagrams of spaces:

$$\begin{array}{ccc} M(X,M(Y,Z)) & \xrightarrow[\cong]{\eta} & M(X\times Y,Z) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ [X,M(Y,Z)] & \xrightarrow{\cong} & [X\times Y,Z] \end{array} \qquad \begin{array}{ccc} M_*(X,M_*(Y,Z)) & \xrightarrow[\cong]{\eta_*} & M_*(X\wedge Y,Z) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ [X,M_*(Y,Z)]_* & \xrightarrow{\cong} & [X\wedge Y,Z]_* \end{array}$$

The vertical maps are the canonical quotient maps. By the Quotient Square Lemma, the bottom maps are homeomorphisms. ■

Example 2.7 For an unbased space X, $\pi_0^{top}(X) = [*, X]$ is the *path component space* of X. If X is a based space, then $\pi_0^{top}(X) = [S^0, X]_*$ is the path component space of X. It is easy to see that $[S^0, X]_* \cong [*, X]$ as unbased spaces for any choice of basepoint of X. A detailed study of the topology of path component spaces is appears in Chapter 3.

Proposition 2.8 *For unbased (resp. based) spaces X, Y, there is a canonical homeomorphism $[X, Y] \cong \pi_0^{top}(M(X, Y))$, $[X_+, Y]_* \cong [X, Y]$, and $[X, Y]_* \cong \pi_0^{top}(M_*(X, Y))$*

Proof. These homeomorphisms are induced by those the natural homeomorphisms of function spaces in Remarks 1.4 and 1.5. ■

Example 2.9 For a based space $(X, x) \in \mathbf{Top}_*$ and integer $n \geq 1$, $\pi_n^{top}(X, x) = [S^n, X]_* \cong \pi_0^{top}(\Omega^n(X, x))$ is the *n-th topological homotopy group* of (X, x). When the basepoint is clear from context, we simply write $\pi_n^{top}(X)$. For $n = 1$, this is often referred to in the literature [Bis02] as the *topological fundamental group* of (X, x). The higher topological homotopy groups first appear in [GHMM08]. For a based pair $((X, A), x) \in \mathbf{Top}_*^{(2)}$ and integer $n \geq 1$, $\pi_n^{top}(X, A) = [(B^n, S^{n-1}), (X, A)]_* \cong \pi_0^{top}(\Omega^n(X, A))$ is the *n-th relative topological homotopy group* of $((X, A), x)$. Note that $\pi_1^{top}(X, A)$ is a space but may not have group structure.

Remark 2.10 There are canonical homeomorphisms $\pi_m^{top}(\Omega^n(X)) \cong \pi_{m+n}^{top}(X)$ for integers $n, m \geq 0$. When $n + m \geq 1$ this is also an isomorphism of groups. There are canonical homeomorphisms $\pi_m^{top}(\Omega^n(X, A)) \cong \pi_{m+n}^{top}(X, A)$ for integers $m \geq 0$ and $n \geq 1$. When $m + n \geq 2$ this is also an isomorphism of groups.

2.2 Multiplication in homotopy mapping spaces

The characterization of the spaces admitting natural group structures on homotopy mapping sets is classical [Whi78, III, 4. & 5.]. We use this to study the interaction of topological and natural algebraic structure in homotopy mapping spaces. Since our interest lies mainly in the homotopy groups, we will typically assume the presence of basepoint though similar results hold for the unbased and relative cases.

A map $f : X \to Y_1 \times Y_2$ will sometimes be denoted $(f_1, f_2) = (p_1 \circ f, p_2 \circ f)$ where $p_i : Y_1 \times Y_2 \to Y_i$ are the projections. Additionally, a function $f : X_1 \vee X_2 \to Y$ will sometimes be written as $\langle f_1, f_2 \rangle = \langle f \circ j_1, f \circ j_2 \rangle$ where $j_i : X_i \hookrightarrow X_1 \vee X_2$ are the obvious inclusions.

2.2.1 H-spaces

We first recall the conditions for inducing monoid and group structure in $[X, Y]_*$ when Y is fixed. Suppose (Y, y_0) is a based space and $i_1, i_2 : Y \to Y \times Y$ are the inclusions $i_1(y) = (y, y_0)$, $i_2(y) = (y_0, y)$ for $y \in Y$. Let $p_1, p_2 : Y \times Y \to Y$ be the projections such that $p_\alpha \circ i_\alpha = id_Y$ and $\Delta : Y \to Y \times Y$ be the diagonal map.

Definition 2.11 (Y, y_0) is a *homotopy associative H-space* (or just *H-space*) if there is a map $\mu : Y \times Y \to Y$ such that $i_1 \circ \mu = i_2 \circ \mu = id_Y$ and $\mu \circ (\mu \times id_Y) \simeq \mu \circ (id_Y \times \mu)$ as based maps $Y^3 \to Y$. A homotopy associative H-space Y is *group-like* if there is also a map $j : Y \to Y$ (called a homotopy inverse) such that the diagram

$$\begin{array}{ccc} Y \times Y & \xrightarrow{j \times id_Y} & Y \times Y \\ \uparrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mu} \\ Y & \xrightarrow{c_{y_0}} & Y \\ \downarrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle \mu} \\ Y \times Y & \xrightarrow{id_Y \times j} & Y \times Y \end{array}$$

commutes up to homotopy. A map $f : Y \to Y'$ is a *map of H-spaces (group-like spaces)* if and only if the diagram

$$\begin{array}{ccc} Y \times Y & \xrightarrow{f \times f} & Y' \times Y' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ Y & \xrightarrow[f]{} & Y' \end{array}$$

commutes up to homotopy.

If Y is an H-space, $[-, Y]_* : \mathbf{Top}_*^{\mathrm{op}} \to \mathbf{Mon}$ is functor to the category of monoids and if Y is group-like, $[-, Y]_* : \mathbf{Top}_*^{\mathrm{op}} \to \mathbf{Grp}$ is a functor to the category of groups. For $f, g \in M_*(X, Y)$, let $f \cdot g = \mu \circ (f \times g) \circ \Delta$. The multiplication of homotopy classes $[f], [g] \in [X, Y]_*$ is given by $[f] * [g] = [f \cdot g]$ and the identity of is the homotopy class of the constant map. The fact that $[-, Y]_*$ is a well-defined functor $\mathbf{Top_*}^{op} \to \mathbf{Top_*}$ implies the following.

Functorality 2.12 *If Y is an H-space (resp. is group-like), then $[-, Y]_* : \mathbf{Top_*}^{op} \to$ **MonwTop** (resp. $[-, Y]_* : \mathbf{Top_*}^{op} \to$ **GrpwTop**) is a well-defined functor. Since homotopic maps $f, g : X \to W$ induce the same homomorphisms $[W, Y]_* \to [X, Y]_*$ for H-space Y, these functors factor through the homotopy category $\mathbf{hTop_*}^{op}$.*

Lemma 2.13 *Let $f_0, f_1 \in M_*(W, Y)$ and $g_0, g_1 \in M_*(X, Z)$ such that $f_0 \simeq f_1$ and $g_0 \simeq g_2$. Then $(f_0 \times g_0) \simeq (f_1 \times g_1)$. Consequently, the map in Theorem 1.7 induces a well-defined function $[\times] : [W, Y]_* \times [X, Z]_* \to [W \times X, Y \times Z]_*$, $([f], [g]) \mapsto [f \times g]$.*

Proof. Suppose our based homotopies are $H : W \times I \to Y$, $H_0 = f_0, H_1 = f_1$ and $G : X \times I \to Z$, $G_0 = g_0, G_1 = g_1$. Let $K : W \times X \times I \to Y \times Z$ such that $K(w, x, t) = (H(w, t), G(x, t))$. This is clearly a continuous homotopy $f_0 \times g_0 \simeq f_1 \times g_1$. ■

We now show that in some cases $[-, Y]_*$ takes values in a category where the algebraic and topological structures interact nicely. To do this we need a basic fact about the compact-open topology proved in Chapter 1.1. We also use the semitopological monoids and quasitopological groups defined in A.1 and A.11 of the Appendix.

Theorem 2.14 *Let Y be an H-space. Then $[-, Y]_* : \mathbf{Haus_*^{op}} \to$ **sTopMon** is a functor to the category of semitopological monoids. If, in addition, Y is group-like, then $[-, Y]_* : \mathbf{Haus_*^{op}} \to$ **qTopGrp** is a functor to the category of quasitopological groups.*

Proof. By 2.12 it suffices to show that $[X,Y]_*$ is a semitopological monoid (with continuous inversion when X is Hausdorff Y is group-like) for arbitrary X. We first show that for each $f \in M_*(X,Y)$, right multiplication and left multiplication by $[f] \in [X,Y]_*$ are continuous operations. Recall that $f \cdot g = \mu \circ (f \times g) \circ \Delta$ and $[f] * [g] = [f \cdot g]$ for $f, g \in M_*(X,Y)$. The bottom row of the commuting diagram

$$\begin{array}{ccccccc} M_*(X,Y) \times M_*(X,Y) & \xrightarrow{\times} & M_*(X \times X, Y \times Y) & \xrightarrow{\Delta^\#} & M_*(X, Y \times Y) & \xrightarrow{\mu_\#} & M_*(X,Y) \\ \downarrow{\scriptstyle \pi \times \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ [X,Y]_* \times [X,Y]_* & \xrightarrow[{[\times]}]{} & [X \times X, Y \times Y]_* & \xrightarrow[\Delta^*]{} & [X, Y \times Y]_* & \xrightarrow[\mu_*]{} & [X,Y]_* \end{array}$$

is precisely the multiplication of homotopy classes. The product operation $\times$ is continuous (Lemma 1.7) and $[\times]$ is well-defined (Lemma 2.13). It follows that the composition of the top three maps (which is the dot operation) is continuous. Now we restrict this diagram to two diagrams

$$\begin{array}{ccc} \{f\} \times M_*(X,Y) & \xrightarrow{f \cdot (-)} & M_*(X,Y) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \{[f]\} \times [X,Y]_* & \xrightarrow{[f]*(-)} & [X,Y]_* \end{array} \qquad \begin{array}{ccc} M_*(X,Y) \times \{f\} & \xrightarrow{(-) \cdot f} & M_*(X,Y) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ [X,Y]_* \times \{[f]\} & \xrightarrow{(-)*[f]} & [X,Y]_* \end{array}$$

In both diagrams the top map is continuous as the restriction of the continuous dot operation. The Quotient Square Lemma implies both left and right multiplication by $[f]$ are continuous. Therefore $[X,Y]_*$ is a semitopological monoid. In the case that Y is group-like with homotopy inverse $j : Y \to Y$, we have $(f \circ j) \cdot f \simeq f \cdot (f \circ j) \simeq c_{y_0}$ in $M_*(X,Y)$. So if $j_\# : M_*(X,Y) \to M_*(X,Y)$ is post-composition with j, then $j_* = [id_X, j]_*$ is inversion in $[X,Y]_*$. Since $[-,-]_*$ is well-defined as a functor to $\mathbf{Top}_*$, j_* is continuous. ■

Remark 2.15 We have similar functors in the unbased and relative cases. Let Y be an H-space (group-like) and $B \subseteq Y$ be an H-space (group-like) such that the inclusion $B \hookrightarrow Y$ is a map of H-spaces (group-like spaces). We can show that $[(X, A), (Y, B)]_*$ is a semitopological monoid (quasitopological group) using the arguments from the proof of the previous theorem.

Example 2.16 For all Hausdorff X and arbitrary Y, the homotopy mapping space $[X, \Omega(Y)]_*$ is a quasitopological group since $\Omega(Y)$ is group-like via concatenation $\Omega(Y) \times \Omega(Y) \to \Omega(Y)$. For $n \geq 2$ $[X, \Omega^n(Y)]_* \cong [X, \Omega(\Omega^{n-1}(Y))]_*$ is a quasitopological abelian group.

Proposition 2.17 *If $f : X \to X'$ and $g : Y \to Y'$ are based maps, then the maps*

$$\Omega(g)_* : [X, \Omega(Y)]_* \to [X, \Omega(Y')]_* \text{ and } (f)^* : [X', \Omega(Y)]_* \to [X, \Omega(Y)]_*$$

given by $\Omega(g)_*([k]) = [\Omega(g) \circ k]$ and $f^*([k]) = [k \circ f]$ are continuous group homomorphisms.

Proof. The functorality of Ω gives the continuity of $\Omega(g) : \Omega(Y) \to \Omega(Y')$. The functorality of homotopy mapping spaces then guarantees the functorality of $\Omega(g)_*$ and f^*. See [AGP02, 2.8.6] for a proof of the fact that these are actually group homomorphisms. ■

Explicit examples of homotopy mapping spaces (with group structure) failing to be topological groups appear in Chapter 4. This failure is a serious complication arising from our choice of the quotient topology. We are, however, interested in conditions that do imply the continuity of multiplication.

Corollary 2.18 *Let X be Hausdorff and Y be an H-space (resp. group-like) so that $[X,Y]_*$ is a semitopological monoid (resp. quasitopological group). If the product map $\pi \times \pi$: $M_*(X,Y) \times M_*(X,Y) \to [X,Y]_* \times [X,Y]_*$ is quotient, then $[X,Y]_*$ is a topological monoid (resp. group).*

Proof. If $\pi\times\pi$ is quotient, applying the Quotient Square Lemma to the first diagram in the proof of Theorem 2.14 implies that multiplication $[X,Y]_* \times [X,Y]_* \to [X,Y]_*$, $([f],[g]) \mapsto [\mu \circ (f \times g) \circ \Delta]$ is continuous. ■

A nice application here is that:

Theorem 2.19 *If X is compact Hausdorff and Y is a metrizable group-like space, then*

1. *$[X,Y]_*$ is first countable $\Leftrightarrow$ $[X,Y]_*$ is a pseudometrizable topological group.*
2. *$[X,Y]_*$ is first countable and T_1 $\Leftrightarrow$ $[X,Y]_*$ is a metrizable topological group*

If, in addition, X is metrizable and Y is separable, then

1. *$[X,Y]_*$ is second countable $\Leftrightarrow$ $[X,Y]_*$ is a separable, pseudometrizable topological group.*
2. *$[X,Y]_*$ is second countable and T_1 $\Leftrightarrow$ $[X,Y]_*$ is a separable, metrizable topological group*

Proof. In the first set of conditions, $M_*(X,Y)$ is metrizable (see 1.13) and therefore first countable. If $[X,Y]_*$ is first countable, then $\pi \times \pi : M_*(X,Y) \times M_*(X,Y) \to$

$[X, Y]_* \times [X, Y]_*$ is quotient by 1.24. The previous corollary then implies that $[X, Y]_*$ is a topological group. Every first countable topological group is pseudometrizable (Appendix A.26). Since every T_1 pseudometric space is a metric space and pseudometric spaces are first countable, the rest of the statements are immediate.

If we add the conditions that X is metrizable and Y is separable, then $M_*(X, Y)$ is a separable metric space (see 1.13). Since the continuous image of a separable space is separable, $[X, Y]_*$ is separable. The statements follow from those in the first set of conditions and the basic fact that separable pseudometric spaces are second countable. ■

Theorem 2.20 *Let X be a Hausdorff space and Y be group-like. If $[X, Y]_*$ is locally compact Hausdorff, then it is a topological group.*

Proof. It is a celebrated theorem of R. Ellis [AT08, Theorem 2.3.12] that every locally compact Hausdorff semitopological group is a topological group. Since $[X, Y]_*$ is a quasitopological group and therefore a semitopological group, the theorem follows. ■

Lemma 2.21 *If X_1, X_2 are spaces and Y is group-like, the inclusions $j_i : X_i \to X_1 \vee X_2$ induce a continuous group isomorphism $\phi : [X_1 \vee X_2, Y]_* \to [X_1, Y]_* \times [X_2, Y]_*$.*

Proof. The inclusions induce continuous homomorphisms $(j_i)^* : [X_1 \vee X_2, Y]_* \to [X_i, Y]_*$, $(j_i)^*([f]) = [f \circ j_i]$. Together these induce a continuous group isomorphism

$\phi : [X_1 \vee X_2, Y]_* \to [X_1, Y]_* \times [X_2, Y]_*$. The inverse homomorphism (see [Whi78, Theorem 5.20]), which is not necessarily continuous, is given by $\phi^{-1}([f],[g]) = [\langle f, g\rangle]$. ■

Theorem 2.22 *The following are equivalent for any Hausdorff space X and group-like space Y:*

1. $\phi : [X \vee X, Y]_* \to [X, Y]_* \times [X, Y]_*$ *is a homeomorphism.*

2. *The product map $\pi \times \pi : M_*(X, Y) \times M_*(X, Y) \to [X, Y]_* \times [X, Y]_*$ is quotient.*

3. $[X \vee X, Y]_*$ *is a topological group.*

Proof. 1. $\Leftrightarrow$ 2. follows from applying the Quotient Square Lemma to the commuting diagram

$$\begin{array}{ccc} M_*(X \vee X, Y) & \xrightarrow{\cong} & M_*(X, Y) \times M_*(X, Y) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi \times \pi} \\ [X \vee X, Y]_* & \xrightarrow[\phi]{} & [X, Y]_* \times [X, Y]_* \end{array}$$

where the top map is the natural homeomorphism of Lemma 1.8. For 2. $\Rightarrow$ 3., if $\pi \times \pi$ is quotient, then multiplication in $[X, Y]_*$ is continuous by Corollary 2.18. But 2. also implies $\phi : [X \vee X, Y]_* \cong [X, Y]_* \times [X, Y]_*$ is a homeomorphism and isomorphism of groups. Thus $[X \vee X, Y]_*$ is homeomorphic to the product of topological groups and also must be a topological group. To prove 3. $\Rightarrow$ 1. we suppose $[X \vee X, Y]_*$ is a topological group and that x_0, y_0 are the basepoints of X, Y respectively. Let $k_1, k_2 : X \vee X \to X$ be the maps collapsing the second and first summands respectively and $m : [X \vee X, Y]_* \times [X \vee X, Y]_* \to [X \vee X, Y]_*$ be continuous multiplication. If $j_i : X \hookrightarrow X \vee X$ are the two inclusions, then $k_1 \circ j_1 = id_X = k_2 \circ j_2$

and $k_2 \circ j_1 = c_{x_0} = k_1 \circ j_2$. Therefore $f \circ k_1 = \langle f, c_{y_0} \rangle$ and $f \circ k_2 = \langle c_{y_0}, g \rangle$. Now we have the continuous composite

$$\psi : [X, Y]_* \times [X, Y]_* \xrightarrow{(k_1)^* \times (k_2)^*} [X \vee X, Y]_* \times [X \vee X, Y]_* \xrightarrow{\quad m \quad} [X \vee X, Y]_*$$

where $\psi([f], [g]) = (k_1)^*([f])(k_2)^*([g]) = [f \circ k_1][g \circ k_2] = [\langle f, c_{y_0} \rangle][\langle c_{y_0}, g \rangle]$. It suffices to check that $\psi = \phi^{-1}$. Indeed, we have $[\langle f, c_{y_0} \rangle][\langle c_{y_0}, g \rangle] = \phi^{-1}(([f], [c_{y_0}]))\phi^{-1}(([c_{y_0}], [g])) = \phi^{-1}(([f], [c_{y_0}])([c_{y_0}], [g])) = \phi^{-1}([f], [g])$. ■

Proposition 2.23 *Let Y be a group-like space and $r : X \to A$ be a retraction of based spaces. The inclusion $i : A \hookrightarrow X$ induces a retraction $i^* : [X, Y]_* \to [A, Y]_*$ in* **GrpwTop**. *If $[X, Y]_*$ is a topological group, so is $[A, Y]_*$.*

Proof. Since $r \circ i = id_A$ by definition, functorality gives $i^* \circ r^* = (r \circ i)^* = id_{[X,A]_*}$ where i^* and r^* are continuous homomorphisms. Therefore i^* is a retraction of groups with topology. By Corollary A.18 of the Appendix, $[A, Y]_*$ is a topological group whenever $[X, Y]_*$ is. ■

Similarly, we have:

Proposition 2.24 *A retract $r : Y \to Y'$ of group-like spaces induces a retraction $r_* : [X, Y]_* \to [X, Y']_*$ of groups with topology for every space X. If $[X, Y]_*$ is a topological group, so is $[X, Y']_*$.*

Proof. Certainly r_* is a continuous homeomorphism [Whi78, III, 4.20]. It is a retraction since if $i : Y \hookrightarrow Y'$ is the inclusion of group-like spaces such that $r \circ i = id_{Y'}$, then $r_* \circ i_*$ is the identity of $[X, Y']_*$. ■

In addition to the continuity of multiplication, we are confronted with the following complication.

Proposition 2.25 *Let Y be an H-space. For every $n \geq 1$, the power function $p_n : [X, Y]_* \to [X, Y]_*$, $p_n([f]) = [f]^n$ is continuous.*

Proof. Fix an $n \geq 1$, let $\Delta_n : M_*(X, Y) \to M_*(X, Y)^n$ be the diagonal map and let $m_n : M_*(X, Y)^n \to M_*(X, Y)$, $(f_1, ..., f_n) \mapsto f_1 \cdot (f_2 \cdot ... \cdot (f_{n-1} \cdot f_n))$ be the continuous dot operation (as in Theorem 2.14) iterated on mapping spaces. Since the operation is associative up to homotopy, the square

$$\begin{array}{ccc} M_*(X,Y) & \xrightarrow{m_n \circ \Delta_n} & M_*(X,Y) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ [X,Y]_* & \xrightarrow[p_n]{} & [X,Y]_* \end{array}$$

commutes in **Top**$_*$. By the Quotient Square Lemma, p_n is continuous. ■

Proposition 2.25 illustrates a property of homotopy mapping spaces which is not present in all quasitopological groups.

Corollary 2.26 *There is a quasitopological group G which is not isomorphic (in* **qTopGrp***) to $[X, Y]_*$ for any Hausdorff space X and group-like space Y. Consequently, $[-, Y]_* :$* **Haus**$_*^{op} \to$ **qTopGrp** *is not essentially surjective for any group-like Y.*

Proof. We construct a quasitopological abelian group G such that the square function $G \to G$, $g \mapsto g^2$ is discontinuous. Consider subset $K = \left\{\frac{\epsilon}{3^n} | n \geq 1, \epsilon = \pm 1\right\}$ of the additive group of reals $\mathbb{R}$. Take a subbase for $\mathbb{R}$ consisting of sets $U(r, \delta) =$

$\{r + t | t \in (-\epsilon, \epsilon) - K\}$ for $r \in \mathbb{R}$ and $\delta > 0$. Since $U(r_1, \delta) + r_2 = U(r_1 + r_2, \delta)$ for any $r_1, r_2 \in \mathbb{R}$ and $-U(0,\delta) = U(0,\delta)$, the topology generated makes $\mathbb{R}$ a Hausdorff quasitopological group. The map $s : \mathbb{R} \to \mathbb{R}$, $s(t) = 2t$ is discontinuous since the sequence $\frac{1}{2(3^n)}$ converges to 0 but $s\left(\frac{1}{2(3^n)}\right) = \frac{1}{3^n}$ does not. ∎

2.2.2 co-H-spaces

Now we study the dual notions of the previous section. Let (X, x_0) be a based space, $j_1, j_2 : X \to X \vee X$ be the inclusions into the first and second copies of X, $i_1, i_2 : X \hookrightarrow X \times X$ be the obvious inclusions, and $k : X \vee X \to X \times X$ be the map $k = \langle i_1, i_2 \rangle$. Let $q_1, q_2 : X \vee X \to X$ be the unique maps such that $q_\alpha = p_\alpha \circ i_\alpha$ for $\alpha = 1, 2$. Let $\nabla : X \vee X \to X$ be the folding map so that $j_\alpha \circ \nabla = id_X$ for $\alpha = 1, 2$.

Definition 2.27 X is a *homotopy coassociative co-H-space* (or just *co-H-space*) if there is a map $\theta : X \to X \vee X$ such that $q_1 \circ \theta \simeq id_X \simeq q_2 \circ \theta$ in $M_*(X, X)$ and $(\theta \vee id_X) \circ \theta \simeq (id_X \vee \theta) \circ \theta$ as based maps $X \to X \vee X \vee X$. A co-H-space X is *cogroup-like* if there is a map $j : X \to X$ (called a homotopy coinverse) such that diagram

$$\begin{array}{ccc} X \vee X & \xrightarrow{j \vee id_X} & X \vee X \\ \theta \uparrow & & \downarrow \nabla \\ X & \xrightarrow{c_x} & X \\ \theta \downarrow & & \uparrow \nabla \\ X \vee X & \xrightarrow{id_X \vee j} & X \vee X \end{array}$$

commutes up to homotopy. A map $f : X \to X'$ is a *map of co-H-spaces (cogroup-like* *spaces)* if and only if the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \theta \downarrow & & \downarrow \theta' \\ X \vee X & \xrightarrow[f \vee f]{} & X' \vee X' \end{array}$$

commutes up to homotopy.

If X is a co-H-space, $[X,-]_* : \mathbf{Top}_* \to \mathbf{Mon}$ is functor to the category of monoids and if X is a cogroup-like, $[X,-]_* : \mathbf{Top}_* \to \mathbf{Grp}$ is a functor to the category of groups. For $f, g \in M_*(X, Y)$, let $f \cdot g = \nabla \circ (f \vee g) \circ \theta$. The multiplication of homotopy classes $[f], [g] \in [X, Y]_*$ is given by $[f] * [g] = [f \cdot g]$ and the identity of is the homotopy class of the constant map. For details regarding this algebraic structure we again refer to [Whi78, III, 4. & 5.]. The fact that $[X,-]_*$ is well-defined as a functor $\mathbf{Top}_* \to \mathbf{Top}_*$ implies the following.

Functorality 2.28 If X is an co-H-space (resp. is cogroup-like), then $[X,-]_* : \mathbf{Top}_* \to \mathbf{MonwTop}$ (resp. $[X,-]_* : \mathbf{Top}_* \to \mathbf{GrpwTop}$) is a well-defined functor. Since homotopic maps $f, g : Y \to Z$ induce the same homomorphisms $[X, Y]_* \to [X, Z]_*$ for co-H-space X, these functors factor through the homotopy category $\mathbf{hTop}_*$.

Lemma 2.29 *If $f_0, f_1 \in M_*(W, Y)$ and $g_0, g_1 \in M_*(X, Z)$ such that $f_0 \simeq f_1$ and $g_0 \simeq g_2$, then $(f_0 \vee g_0) \simeq (f_1 \vee g_1)$. Consequently, the map in Theorem 1.9 induces a well-defined function* $[\vee] : [W, Y] \times [X, Z] \to [W \vee X, Y \vee Z]$, $([f], [g]) \mapsto [f \vee g]$.

Proof. Suppose our based homotopies are $H : W \wedge I_+ \to Y$, $H_0 = f_0, H_1 = f_1$ and $G : X \wedge I_+ \to Z$, $G_0 = g_0, G_1 = g_1$. There is a natural homeomorphism

$h : (W \vee X) \wedge I_+ \cong (W \wedge I_+) \vee (X \wedge I_+)$. Therefore $(H \vee G) \circ h : (W \vee X) \wedge I_+ \to Y \vee Z$ is a homotopy $f_0 \vee g_0 \simeq f_1 \vee g_1$. ■

Now for the dual of Theorem 2.14.

Theorem 2.30 *Let X be a Hausdorff co-H-space. Then $[X, -]_* : \mathbf{Top}_* \to \mathbf{sTopMon}$ is a functor from the category of spaces to the category of semitopological monoids. If, in addition, X is cogroup-like, then $[X, -]_* : \mathbf{Top}_* \to \mathbf{qTopGrp}$ is a functor to the category of quasitopological groups.*

Proof. By 2.28, it suffices to show that $[X, Y]_*$ is a semitopological monoid (with continuous inversion when X is cogroup-like) for any space Y. We first show that for each $f \in M_*(X, Y)$, right multiplication and left multiplication by $[f] \in [X, Y]_*$ are continuous operations. Recall that $f \cdot g = \nabla \circ (f \vee g) \circ \theta$ and $[f] * [g] = [f \cdot g]$ for $f, g \in M_*(X, Y)$. The bottom row of the commuting diagram

$$\begin{array}{ccccccc}
M_*(X,Y) \times M_*(X,Y) & \xrightarrow{\vee} & M_*(X \vee X, Y \vee Y) & \xrightarrow{\theta^{\#}} & M_*(X, Y \vee Y) & \xrightarrow{\nabla_{\#}} & M_*(X,Y) \\
\downarrow{\scriptstyle \pi \times \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
[X,Y]_* \times [X,Y]_* & \xrightarrow[{[\vee]}]{} & [X \vee X, Y \vee Y]_* & \xrightarrow[\theta^*]{} & [X, Y \vee Y]_* & \xrightarrow[\nabla_*]{} & [X,Y]_*
\end{array}$$

is precisely the multiplication of homotopy classes. The wedge operation $\vee$ is continuous (Lemma 1.9) and $[\vee]$ is well-defined (Lemma 2.29). It follows that the composition of the top three maps (which is the dot operation) is continuous. Now we restrict this diagram to two diagrams

$$\begin{array}{ccc} \{f\}\times M_*(X,Y) & \xrightarrow{f\cdot(-)} & M_*(X,Y) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \{[f]\}\times[X,Y]_* & \xrightarrow{[f]*(-)} & [X,Y]_* \end{array} \qquad \begin{array}{ccc} M_*(X,Y)\times\{f\} & \xrightarrow{(-)\cdot f} & M_*(X,Y) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ [X,Y]^{top}_*\times\{[f]\} & \xrightarrow{(-)*[f]} & [X,Y]_* \end{array}$$

In both diagrams the top map is continuous as the restriction of the continuous dot operation. The Quotient Square Lemma implies left and right multiplication by $[f]$ are continuous. Therefore $[X,Y]_*$ is a semitopological monoid. In the case that X is cogroup-like with homotopy coinverse $j : X \to X$, we have $(f\circ j)\cdot f \simeq f\cdot(f\circ j) \simeq c_{y_0}$ in $M_*(X,Y)$. So if $j^{\#} : M_*(X,Y) \to M_*(X,Y)$ is pre-composition with j, then $j^* = [j, id_Y]_*$ is inversion in $[X,Y]_*$. Since $[-,-]_*$ is well-defined as a functor to $\mathbf{Top}_*$, j^* is continuous. ■

Remark 2.31 We have similar functors in the unbased and relative cases. Let X be a based Hausdorff co-H-space (cogroup-like) and $A \subseteq X$ be a co-H-space (cogroup-like) such that the inclusion $A \hookrightarrow X$ is a map of co-H-spaces (cogroup-like spaces). We can show that $[(X,A),(Y,B)]_*$ is a semitopological monoid (quasitopological group) using the arguments from the proof of the previous theorem. This, for instance gives that the relative homotopy groups $\pi_n(Y,B)$ are quasitopological groups for $n \geq 2$.

The proofs of the following results are dual to those in the previous section.

Example 2.32 For Hausdorff X and arbitrary Y, the homotopy mapping space

$[\Sigma X, Y]_*$ is a quasitopological group since ΣX is co-group-like via the usual comultiplication $\theta : \Sigma X \to \Sigma X \vee \Sigma X$ (See [AGP02, 2.10.2]). By Prop. 2.6, $[\Sigma X, Y]_*$ is naturally isomorphic to $[X, \Omega(Y)]_*$ (from Example 2.16) as a quasitopological group. Of course $[\Sigma^n X, Y]_*$ for $n \geq 2$ is a quasitopological abelian group.

Proposition 2.33 *If $f : X \to X'$ and $g : Y \to Y'$ are based maps, then the maps*

$$g_* : [\Sigma X, Y]_* \to [\Sigma X, Y']_* \text{ and } (\Sigma f)^* : [\Sigma X', Y]_* \to [\Sigma X, Y]_*$$

given by $g_*([k]) = [g \circ k]$ and $(\Sigma f)_*([k]) = [k \circ \Sigma f]$ are continuous group homomorphisms.

Corollary 2.34 *Let X be a Hausdorff co-H-space (resp. cogroup-like space) and Y be arbitrary so that $[X, Y]_*$ is a semitopological monoid (resp. quasitopological group). If the product map $\pi \times \pi : M_*(X, Y) \times M_*(X, Y) \to [X, Y]_* \times [X, Y]_*$ is quotient, then $[X, Y]_*$ is a topological monoid (resp. group).*

Theorem 2.35 *If X is a compact Hausdorff cogroup-like space and Y is metrizable, then*

1. *$[X, Y]_*$ is first countable $\Leftrightarrow$ $[X, Y]_*$ is a pseudometrizable topological group.*

2. *$[X, Y]_*$ is first countable and T_1 $\Leftrightarrow$ $[X, Y]_*$ is a metrizable topological group*

If, in addition, X is metrizable and Y is separable, then

1. *$[X, Y]_*$ is second countable $\Leftrightarrow$ $[X, Y]_*$ is a separable, pseudometrizable topological group.*

2. *$[X, Y]_*$ is second countable and T_1 $\Leftrightarrow$ $[X, Y]_*$ is a separable, metrizable topological group*

Theorem 2.36 *Let X be a Hausdorff cogroup-like space and Y be a space. If $[X,Y]_*$ is locally compact Hausdorff, then it is a topological group.*

Remark 2.37 Note that the previous two theorems apply to the topological homotopy groups $\pi_n^{top}(X) = [S^n, X]_*$, $n \geq 1$ and can easily be modified to apply to the relative topological homotopy groups

$$\pi_n(X,A) = [(B^n, S^{n-1}), (X,A)]_* = [(\Sigma B^{n-1}, \Sigma S^{n-1}), (X,A)]_*, n \geq 2.$$

Lemma 2.38 *If X is a cogroup-like and Y_1, Y_2 are spaces, then the projections $p_i : Y_1 \times Y_2 \to Y_i$ induce a continuous group isomorphism $\phi : [X, Y_1 \times Y_2]_* \to [X, Y_1]_* \times [X, Y_2]_*$ given by $\phi([(f,g)]) = ([f],[g])$.*

Theorem 2.39 *For any Hausdorff cogroup like space X and space Y, the following are equivalent:*

1. *$\phi : [X, Y \times Y]_* \to [X,Y]_* \times [X,Y]_*$ is a homeomorphism.*

2. *The product map $\pi \times \pi : M_*(X,Y) \times M_*(X,Y) \to [X,Y]_* \times [X,Y]_*$ is quotient.*

3. *$[X, Y \times Y]_*$ is a topological group.*

Proposition 2.40 *Let X be a cogroup-like space and $r : Y \to B$ be a retraction of based spaces. The induced homomorphism $r^* : [X,Y]_* \to [X,B]_*$ is a retraction of quasitopological groups. If $[X,Y]_*$ is a topological group, so is $[X,B]_*$.*

Proposition 2.41 *Let $r : X \to X'$ be a retraction of cogroup-like spaces. The inclusion $i : X' \hookrightarrow X$ induces a retraction $i^* : [X,Y]_* \to [X',Y]_*$ of quasitopological groups for every* *space Y. If $[X, Y]_*$ is a topological group, so is $[X', Y]_*$.*

Example 2.42 For any non-connected space $X = A \sqcup B$ (A, B are disjoint and open), the retraction $X \to S^0$ collapsing A and B to points induces a retraction $\Sigma X \to S^1$ of cogroup-like spaces. It is a result of Chapter 4 that $[S^1, Y]_*$ is not always a topological group. The previous proposition then implies that whenever $[S^1, Y]_*$ fails to be a topological group, $[\Sigma X, Y]_*$ also fails to be a topological group. Consequently, for any non-connected space X, $[\Sigma X, -]_*$ does not take values in **TopGrp**. This is particularly interesting for the spaces $\Sigma(X_+)$ studied in Chapter 4. Another interesting example is when $X = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., 0\right\} \subset \mathbb{R}$ which is non-connected and ΣX is homeomorphic to the Hawaiian earring $\mathbb{HE}$ described further in Example 4.24. The algebraic sturcture of the groups $[\mathbb{HE}, X]_*$ are used in [KR06, KR10].

Lemma 2.43 *Let X be a co-H-space. For every $n \geq 1$, the power function $p_n : [X, Y]_* \to [X, Y]_*$, $p_n([f]) = [f]^n$ is continuous.*

Corollary 2.44 *There is an abelian quasitopological group G which is not isomorphic to $[X, Y]_*$ for any Hausdorff cogroup-like space X. Consequently, $[X, -]_* :$ **Top$_*$** $\to$ **qTopGrp** is not essentially surjective for any Hausdorff, cogroup-like X.*

2.3 Homotopy sequences

Exact sequences involving homotopy mapping sets arise on many occasions in homotopy theory. In this section, we observe that these exact sequences very often are realized as exact sequences in the category of quasitopological (abelian) groups.

Cofiber and fiber sequences 2.45 Notation in this section is borrowed from [AGP02], however, since we are interested in a purely based setting we refer to the content of Whitehead [Whi78, III §6]. Though Whitehead works in the convenient category of k-spaces, the arguments for cofiber and fiber sequences do not require this assumption.

Fix a based map $f : (X, x_0) \to (Y, y_0)$. Let $CX = X \wedge (I, 1)$ be the reduced cone of X and $C_f = Y \cup_f CX$ the reduced mapping cone of f. Specifically, C_f is the quotient of $Y \sqcup CX$ by the relation $f(x) \sim x \wedge 0$ for each $x \in X$. Let $i_1 : Y \hookrightarrow C_f$ be the inclusion. The cofiber sequence of f is

$$X \xrightarrow{f} Y \xrightarrow{i_1} C_f \xrightarrow{i_2} C_{i_1} \xrightarrow{i_3} C_{i_2} \xrightarrow{i_4} \cdots$$

where C_{i_k} is the reduced mapping cone of i_k and $i_{k+1} : C_{i_{k-1}} \hookrightarrow C_{i_k}$ is the canonical inclusion. It is well known that there is a homotopy commutative diagram

$$\begin{array}{ccccccccccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i_1} & C_f & \xrightarrow{i_2} & C_{i_1} & \xrightarrow{i_3} & C_{i_2} & \xrightarrow{i_4} & C_{i_3} & \xrightarrow{i_5} & C_{i_4} & \xrightarrow{i_4} & C_{i_5} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\
X & \xrightarrow[f]{} & Y & \xrightarrow[i_1]{} & C_f & \xrightarrow[i_1]{} & \Sigma X & \xrightarrow[\Sigma f]{} & \Sigma Y & \longrightarrow & C_{\Sigma f} & \longrightarrow & \Sigma^2 X & \xrightarrow[\Sigma^2 f]{} & \Sigma^2 Y & \longrightarrow & \cdots
\end{array}$$

where vertical maps are homotopy equivalences.

The two horizontal sequences are useful since applying $[-, Z]_*$ for any based space Z yields two isomorphic exact sequences of groups (and sets when group structure is not present). The functorality of $[-, Z]_*$ as a homotopy mapping space then gives

Theorem 2.46 *For any based map $f : X \to Y$ and space Z, there is a sequence of continuous homomorphisms (and functions)*

$$[X,Z]_* \xleftarrow{(f)^*} [Y,Z]_* \xleftarrow{(i_1)^*} [C_f,Z]_* \longleftarrow [\Sigma X,Z]_* \xleftarrow{(\Sigma f)^*} [\Sigma Y,Z]_* \longleftarrow$$

$$\longleftarrow [C_{\Sigma f},Z]_* \longleftarrow [\Sigma^2 X,Z]_* \xleftarrow{(\Sigma^2 f)^*} [\Sigma^2 Y,Z]_* \longleftarrow \cdots$$

given by applying $[-,Z]_*$ to the cofiber sequence. In particular, if we truncate the sequence at $[\Sigma X,Z]_*$ (resp. at $[\Sigma^2 X,Z]_*$), then we get an exact sequence in **qTopGrp** *(resp.* **qTopAb***).*

We may also consider the dual fiber sequence of $f : (X,x_0) \to (Y,y_0)$. The homotopy fiber of f is $P_f = \{(x,p) \in X \times P(X) | p(0) = y_0 \text{ and } p(1) = f(x)\}$ with basepoint (x_0, c_{x_0}). Let $q_1 : P_f \to X$ be the projection which has fiber $q_f^{-1}(x_0) \cong \Omega(X)$. The cofiber sequence of f is

$$\cdots \longrightarrow P_{q_2} \xrightarrow{q_3} P_{q_1} \xrightarrow{q_2} P_f \xrightarrow{q_1} X \xrightarrow{f} Y$$

where P_{q_k} is the mapping path space of q_k and $q_{k+1} : P_{q_k} \hookrightarrow P_{q_{k-1}}$ is the canonical projection. It is well known that there is a homotopy commutative diagram

$$\begin{array}{ccccccccccccccccc}
\cdots \longrightarrow & P_{q_5} & \longrightarrow & P_{q_4} & \xrightarrow{q_5} & P_{q_3} & \xrightarrow{q_4} & P_{q_2} & \xrightarrow{q_3} & P_{q_1} & \xrightarrow{q_2} & P_f & \xrightarrow{q_1} & X & \xrightarrow{f} & Y \\
& \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow id & & \downarrow id & & \downarrow id \\
\cdots \longrightarrow & \Omega^2(X) & \xrightarrow[\Omega^2(f)]{} & \Omega^2(Y) & \longrightarrow & P_{\Omega(f)} & \longrightarrow & \Omega(X) & \xrightarrow[\Omega(f)]{} & \Omega(Y) & \xrightarrow[q_2]{} & P_f & \xrightarrow[q_1]{} & X & \xrightarrow[f]{} & Y
\end{array}$$

where all vertical maps are homotopy equivalences.

We may apply the functor $[W,-]_*$ for any space W gives rise to two isomorphic exact sequences of groups (and sets when group structure is not present). As before the fact that the functions in the sequence are induced by maps implies continuity of the morphisms in the exact sequence.

Theorem 2.47 *For any based map $f : X \to Y$ and space W, there is a sequence of continuous homomorphisms (and functions)*

$$\cdots \longrightarrow [W, \Omega^2(X)] \xrightarrow{(\Omega^2(f))_*} [W, \Omega^2(Y)] \longrightarrow [W, P_{\Omega(X)}] \longrightarrow [W, \Omega(X)] \longrightarrow$$

$$\xrightarrow{(\Omega(f))_*} [W, \Omega(Y)] \longrightarrow [W, P_f] \longrightarrow [W, X] \xrightarrow{(f)_*} [W, Y]$$

given by applying $[W, -]_*$ to the fiber sequence. In particular, if we truncate the sequence at $[W, \Omega(Y)]_*$ (resp. at $[\Sigma X, Z]_*$) and suppose W is locally compact Hausdorff, then we get an exact sequence in **qTopGrp** *(resp.* **qTopAb***).*

Similarly, there is a cofiber and fiber sequence for a based pair of maps $f : (X, A) \to (Y, B)$ which each give rise to an exact sequence after applying the appropriate functor $[(W_1, W_2), -]_*$ or $[-, (Z_1, Z_2)]_*$. In each case, the fact that the functions in the exact sequences are induced by continuous maps of pairs implies continuity of morphisms in the sequence.

Homotopy sequence of a pair 2.48 The particularly useful example of fiber and cofiber sequences is the homotopy sequence of a pair of a based pair $(X, A) \in \mathbf{Top}_*^{(2)}$. Recall that $\Omega^n(X, A)$ may be viewed as the space of maps of triples

$$\left(I^n, I^{n-1} \times \{0\}, I^{n-1} \times \{1\} \cup \partial(I^{n-1}) \times I\right) \to (X, A, \{x\}).$$

and the relative homotopy groups $\pi_n^{top}(X,A) \cong \pi_0^{top}(\Omega^n(X,A))$. Pre-composition with the inclusion $I^{n-1} \times \{0\} \hookrightarrow I^n$ gives a map $\partial : \Omega^n(X,A) \to \Omega^{n-1}(A)$. Applying the path component functor π_0^{top}, we see that the connecting homomorphism $\partial_* : \pi_n^{top}(X,A) \to \pi_{n-1}^{top}(A)$ in the long exact homotopy sequence of the pair (X,A) is continuous. The inclusion of mapping spaces $j : \Omega^n(X) \hookrightarrow \Omega^n(X,A)$ induces the continuous homomorphism $j_* : \pi_n^{top}(X) \to \pi_n^{top}(X,A)$ on path components which also appears in the homotopy sequence of (X,A). Together, these observations imply the following proposition.

Theorem 2.49 *For every based pair $(X,A) \in \mathbf{Top}_*^{(2)}$ with inclusion $i : A \hookrightarrow X$, there is a long exact sequence*

$$\cdots \longrightarrow \pi_n^{top}(A) \xrightarrow{i_*} \pi_n^{top}(X) \xrightarrow{j_*} \pi_n^{top}(X,A) \xrightarrow{\partial_*} \pi_{n-1}^{top}(A) \longrightarrow \cdots \longrightarrow \pi_1^{top}(X)$$

in the category of quasitopological groups.

Proposition 2.50 *Let $p : E \to B$ be a Hurewicz fibration of path connected spaces with fiber F. There is a long exact sequence*

$$\cdots \longrightarrow \pi_n^{top}(F) \xrightarrow{i_*} \pi_n^{top}(E) \xrightarrow{p_*} \pi_n^{top}(B) \xrightarrow{\partial_*} \pi_{n-1}^{top}(F) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1^{top}(B) \longrightarrow \pi_0^{top}(F) \longrightarrow *$$

in the category of quasitopological groups.

Proof. It is clear that the inclusion $i : F \hookrightarrow E$ and fibration $p : E \to B$ induce continuous homomorphisms i_* and p_*. Let $\Omega(B) \to B$ be the constant map which is the restriction of the path fibration $M_*(I,B) \to B$. This lifts to a map $\Omega(B) \to F$ which induces the connecting homomorphisms $\partial_* = \pi_{n-1}^{top}(\partial) : \pi_n(B) \to \pi_{n-1}^{top}(F)$ on homotopy groups. ■

2.4 Discreteness of homotopy mapping spaces

It is also worthwhile to note when quotient topology on $[X, Y]_*$ fails to provide any new information, that is, when it has the discrete topology. In this section, we assume that all spaces are path connected and Hausdorff.

Proposition 2.51 *Suppose X, Y are based spaces such that either X is Hausdorff and cogroup-like or that Y is group-like. The following are equivalent:*

1. *$[X, Y]_*$ is a discrete group.*

2. *The singleton containing the identity is open in $[X, Y]_*$.*

3. *For every null-homotopic, based map $f : X \to Y$, there is a basic open neighborhood $\bigcap_{i=1}^{n} \langle K_i, U_i \rangle$ of f in $M_*(X, Y)$ containing only null-homotopic maps.*

Proof. 1. ⇔ 2. follows from the fact that all translations in a quasitopological group are homeomorphisms. 2. ⇔ 3. follows directly from the definition of the quotient topology. ■

Remark 2.52 Recall from 2.7 that the path component space $\pi_0^{top}(X)$ of a space X is the quotient space of X obtained by identifying path components. In Proposition

2.8, we noted that for arbitrary X and Y, there is a natural homeomorphism $[X, Y]_* \cong \pi_0^{top}(M_*(X, Y))$ taking the homotopy class of f to the path component of f. Moreover, this is a group isomorphism when X is cogroup-like or Y is group-like. Since a quotient space Z of X is discrete if and only if the fibers of the quotient map $X \to Z$ are open in X, these observations allow us to characterize the discreteness of homotopy mapping spaces in terms of local connectedness properties of mapping spaces. We make use of the following notions of connectedness.

Definition 2.53 Let Y be a space and $k \geq 0$ be an integer.

- Y is *k-connected* if $\pi_n(Y) = 0$ for $n = 0, 1, ..., k$.
- Y is *locally k-connected* at $y \in Y$ if for every neighborhood U of y there is a k-connected open neighborhood V of y contained in U. Y is *locally k-connected* if it is locally k-connected at all of its points. A space is locally 0-connected precisely when it is locally path connected.
- Y is *semilocally k-connected* at $y \in Y$ if there is an open neighborhood U of y such that the inclusion $U \hookrightarrow Y$ induces the trivial homomorphism $\pi_k(U, y) \to \pi_k(Y, y)$. Y is *semilocally k-connected* if it is semilocally k-connected at all of its points.
- Y is *well k-connected* (for $k \geq 1$) at $y \in Y$ if Y is semilocally k-connected at y and locally (k-1)-connected at y. Y is *well k-connected* if it is well k-connected at all of its points. Being well 0-connected is the same as being semilocally 0-connected.

Proposition 2.54 *A space X is semilocally 0-connected if and only if $\pi_0^{top}(X)$ is discrete. Consequently, if X is locally path connected, then $\pi_0^{top}(X)$ is discrete.*

Proof. Let $\pi_X : X \to \pi_0^{top}(X)$ denote the quotient map identifying path components. Note that for each $x \in X$, $\pi_X^{-1}(\pi_X(x)) \subseteq X$ is the path component of x. If X is semilocally 0-connected and $x \in X$, then there as an open neighborhood U of x such that $U \hookrightarrow X$ induces the constant function $\pi_0(U) \to \pi_0(X)$. This means precisely that $U \subseteq \pi_X^{-1}(\pi_X(x))$. Therefore $\pi_X^{-1}(\pi_X(x))$ is open in X and since π_X is quotient the singleton $\{\pi_X(x)\}$ is open in $\pi_0^{top}(X)$. Conversely, if $\pi_0^{top}(X)$ is discrete, $x \in X$, and $U = \pi_X^{-1}(\pi_X(x))$ is open in X and the inclusion $U \hookrightarrow X$ induces the constant function $\pi_0(U) \to \pi_0(X)$. It is obvious that every locally path connected space is semilocally 0-connected. ■

Corollary 2.55 *The homotopy mapping space $[X, Y]_*$ is discrete if and only if $M_*(X, Y)$ is semilocally 0-connected.*

The characterizations in 2.51 and 2.55 are general but are not particularly illuminating. We refine our focus to the case when X is a finite polyhedron. Recall that an *m-dimensional finite polyhedron* X is a space homeomorphic to the geometric realization $|K|$ of an m-dimensional finite simplicial complex K. Any such space may be embedded in $\mathbb{R}^{N(X)}$ for some $N(X) \geq 1$. A subpolyhedron $S \subseteq X$ is a subspace which is homeomorphic to the geometric realization of a subcomplex of K and so is a polyhedron itself. In [Wad54], H. Wada proves the following theorem.

Theorem 2.56 (Wada) *Let X be an m-dimensional finite polyhedron and Y a Hausdorff space.*

1. *If Y is locally k-connected, then $M(X, Y)$ is locally (l-m) connected.*
2. *If Y is well k-connected, then $M(X, Y)$ is well (l-m) connected.*

In particular, we are interested in the case when $l = m$.

Corollary 2.57 *If X is an m-dimensional finite polyhedron and Y is Hausdorff and well m-connected, then the homotopy mapping space $[X, Y]$ of unbased maps has the discrete topology.*

Proof. By Theorem 2.56, $M(X, Y)$ is well 0-connected and so $\pi_0^{top}(M(X, Y)) \cong [X, Y]$ is discrete by Proposition 2.54. ■

Wada also proved a relative version of Theorem 2.56. We use this to prove the based and relative versions of Corollary 2.57. For integer $p \geq 1$ let $[p]$ be the finite set $\{0, 1, ..., p\}$. Fix subpolyhedra $X_1, ..., X_p$ of m-dimensional finite polyhedron X and closed subspaces $Y_1, ..., Y_p$ of Hausdorff space Y. Let $Q_0 = X$, $Y_0 = Y$, and $\mathbf{X} = (X_0, X_1, ..., X_p), \mathbf{Y} = (Y_0, Y_1, ..., Y_p) \in \mathbf{Top}^{(\mathbf{p+1})}$. Recall that $M(\mathbf{X}, \mathbf{Y})$ is the subspace of $M(X, Y)$ consisting of maps $f : X \to Y$ such that $f(X_i) \subset Y_i$ for each $i = 1, ..., p$. For each subset $S \subseteq [p]$, let $m_S = dim\,(\bigcap_{s \in S} X_s)$ and $Y_S = \bigcap_{s \in S} Y_s$. Wada's relative theorem is:

Theorem 2.58 (Wada) *If Y_S is locally l_S-connected (resp. well l_S-connected) for each $S \subseteq [p]$ (where $l_S \geq m_S$), then $M(\mathbf{X}, \mathbf{Y})$ is locally N_p-connected (resp. well N_p-connected),* *where*

$$N_p = \min_{S\subseteq[p]}(l_S - m_S)$$

Corollary 2.59 *Suppose* $X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_p$ *and* $Y = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_p$. *If* X_i *is* m_i*-dimensional and* Y_i *is well* l_i*-connected (where* $l_i \geq m_i$*) for each* $i = 0, \ldots, p$, *then* $[\mathbf{X}, \mathbf{Y}]$ *is discrete.*

Proof. Since $\pi_0^{top}(M(\mathbf{X}, \mathbf{Y})) \cong [\mathbf{X}, \mathbf{Y}]$, it suffices to show that $M(\mathbf{X}, \mathbf{Y})$ is well N-connected for some $N \geq 0$. Note that for each $S \subseteq [p]$, we have $m_S = \dim(X_{\max(S)}) = m_{\max(S)}$ and $Y_S = Y_{\max(S)}$. Since Y_S is well $l_{\max(S)}$-connected for each $S \subseteq [p]$ (where $l_S = l_{\max(S)} \geq m_{\max(S)} = m_S$), Theorem 2.58 tells us that $M(\mathbf{X}, \mathbf{Y})$ is well N-connected for

$$N = \min_{S\subseteq[p]}(l_S - m_S) = \min_{S\subseteq[p]}(l_{\max(S)} - m_{\min(S)}) = \min_{0\leq i\leq p}(l_i - m_i) \geq 0.$$

■

Let x, y be basepoints for X, Y respectively. and suppose $X_p = \{x\}$ and $Y_p = \{y\}$ are singletons. The next corollary follows directly from applying Corollary 2.59 to $M(\mathbf{X}, \mathbf{Y}) \cong M_*(((X_0, .., X_{p-1}), x), ((Y_0, .., Y_{p-1}), y))$.

Corollary 2.60 *If* $X = X_0 \supseteq X_1 \supseteq \ldots\ldots \supseteq X_p = \{x\}$ *and* $Y = Y_0 \supseteq Y_1 \supseteq \ldots\ldots \supseteq Y_p = \{y\}$ *such that* X_i *is* m_i*-dimensional and* Y_i *is well* l_i*-connected (where* $l_i \geq m_i$*), then* $[((X_0, .., X_{p-1}), x), ((Y_0, .., Y_{p-1}), y)]_*$ *is discrete.*

In particular, if $p = 1$, then we have:

Corollary 2.61 *If* (X, x) *is an* m*-dimensional polyhedron and* (Y, y) *is well* m*-connected, then* $[X, Y]_*$ *is discrete.*

Example 2.62 For instance, if Y is a locally contractible space such as a manifold or CW-complex then $[X,Y]_*$ is discrete for any based finite polyhedron X.

The next few statements and examples are direct applications of this section to the topological homotopy groups. Since S^n is an n-dimensional polyhedron we have:

Theorem 2.63 *For $n \geq 0$, $\pi_n^{top}(Y) = [S^n, Y]_*$ is a discrete space (group when $n \geq 1$) whenever Y is well n-connected.*

Example 2.64 For a locally contractible space Y (such as a manifold, CW-complex, or polyhedron), $\pi_n^{top}(Y)$ is discrete for all $n \geq 0$.

Theorem 2.65 *For $n \geq 1$, a closed subspace A of X containing the basepoint x of X, the relative homotopy group $\pi_n^{top}(X,A) \cong [(B^n, S^{n-1}), (X,A)]_*$ of the pair $((X,A),x) \in \mathbf{Top}_*^{(2)}$ is discrete (group when $n \geq 2$) whenever X is well n-connected and A is well (n-1)-connected.*

Example 2.66 For a based CW-pair (X,A) or a manifold X and submanifold $A \subseteq X$, $\pi_n^{top}(X,A)$ is discrete for all $n \geq 0$.

As noted in [GHMM08], discreteness is also connected to the cardinality of homotopy mapping spaces. The following is an obvious extension of [GHMM08, Theorem 3.8].

Proposition 2.67 *If X is compact metric space and Y is a separable metric space, and $[X,Y]_*$ is discrete, then $[X,Y]_*$ is countable.*

Proof. The function space $M_*(X,Y)$ is a separable metric space (1.13) and $[X,Y]_*$ is separable as the continuous image of a separable space. Every discrete separable space is countable. ∎

2.5 Alternative topologies for homotopy mapping sets

The complications arising in the study of homotopy mapping spaces motivate the introduction of alternative topologies. There are certainly many candidates, especially when restricted to the fundamental group. In this section, three alternative topologies are introduced. The first approach makes use of the reflection functor τ studied in the Appendix (A.3.1). The second approach essentially translates the entire conversation into the convenient category $\mathbf{kTop_*}$ of based k-spaces discussed in Chapter 1.2. The last approach makes direct use of shape theory (the inverse system approach).

A direct consequence of the failure of $[X,Y]_*$ to be a topological group (for cogroup-like X or group-like Y) is that the functors $[X,-]_* : \mathbf{Top_*} \to \mathbf{qTopGrp}$ and $[-,Y]_* : \mathbf{Top_*} \to \mathbf{qTopGrp}$ fail to preserve products (See 2.22 and 2.39). Prior to our three constructions, we observe that this is part of a more general phenomenon. Let $U : \mathbf{TopGrp} \to \mathbf{Grp}$ be the functor forgetting topological structure.

Proposition 2.68 *Let X be a cogroup-like space. Suppose $[X,-]_*^{TG} : \mathbf{Top_*} \to \mathbf{TopGrp}$ is a functor such that $U \circ [X,-]_*^{TG}$ is the homotopy mapping set $[X,-]_* : \mathbf{Top_*} \to \mathbf{Grp}$. Then $[X,-]_*^{TG}$ preserves finite products.*

Proof. Let (Y_1, y_1) and (Y_2, y_2) be based spaces. The $Y_1 \times Y_2 \to Y_i$ induce continuous homomorphisms $[X, Y_1 \times Y_2]_*^{TG} \to [X, Y_i]_*^{TG}$, $i = 1, 2$ which induce a continuous group homomorphism $\phi : [X, Y_1 \times Y_2]_*^{TG} \to [X, Y_1]_*^{TG} \times [X, Y_1]_*^{TG}$ which is known to be a group isomorphism (2.38). Let $j_i : Y_i \hookrightarrow Y_1 \times Y_2$ be the inclusions $j_1(y) = (y, y_2)$, $j_2(z) = (y_1, z)$. These induce continuous homomorphisms $J_i : [X, Y_i]_*^{TG} \to [X, Y_1 \times Y_2]_*^{TG}$. Let μ be the continuous multiplication of $[X, Y_1 \times Y_2]_*^{TG}$. The composite $\mu \circ (J_1 \times J_2)$ is a continuous map $[X, Y_1]_*^{TG} \times [X, Y_1]_*^{TG} \to [X, Y_1 \times Y_2]_*^{TG}$. It is clear that this map is the inverse of ϕ. ■

The dual statement follows similarly.

Proposition 2.69 *Let Y be a group-like space. Suppose $[-, Y]_*^{TG} :$ **Top**$_*$$^{\mathbf{op}} \to$ **TopGrp** is a functor such that $U \circ [-, Y]_*^{TG}$ is the homotopy mapping set $[-, Y]_* :$ **Top**$_*$$^{\mathbf{op}} \to$ **Grp**. Then $[-, Y]_*^{TG}$ preserves finite products (recall that a product in **Top**$_*$$^{\mathbf{op}}$ is just a wedge in **Top**$_*$).*

2.5.1 The functor τ and $[X, Y]_*^{\tau}$

Fix a group-like space Z and a cogroup-like space W so that $[-, Z]_* :$ **Top**$_*$$^{\mathbf{op}} \to$ **GrpwTop** and $[W, -] :$ **Top**$_* \to$ **GrpwTop** are functors. We apply the reflection functor τ from the Appendix A.3.1. Recall that for any group with topology G, $\tau(G)$ is the unique topological group with continuous homomorphism $G \to \tau(G)$ universal with respect to continuous homomorphisms from G to topological groups. It is a great convenience that the underlying group of $\tau(G)$ is G and the universal arrow $G \to \tau(G)$ is the continuous identity homomorphism (which is open if and only if G is already a topological group). For based spaces X, Y let

$[W,X]_*^\tau = \tau([W,X]_*)$ and $[Y,Z]_*^\tau = \tau([Y,Z]_*)$. Since $[-,Z]_*^\tau$ and $[W,-]_*^\tau$ are defined as composites of functors, they are themselves functors taking values in **TopGrp**.

Functorality 2.70 *$[W,-]_*^\tau$: $\mathbf{Top}_*$ → **TopGrp** and $[-,Z]_*^\tau$: $\mathbf{Top}_*^{\mathbf{op}}$ → **TopGrp** are product preserving (2.68,2.69) functors. The identity maps give natural transformations $[W,-]_* \to [W,-]_*^\tau$ and $[-,Z]_* \to [-,Z]_*^\tau$ with components in* **GrpwTop**.

These new topologies on homotopy mapping sets are characterized by the following property.

Universal Property 2.71 The topology of $[W,X]_*^\tau$ (resp. $[Y,Z]_*^\tau$) is the finest group topology on the set $[W,X]_*$ such that $\pi : M_*(W,X) \to [W,X]_*$ (resp. $\pi : M_*(Y,Z) \to [Y,Z]_*$) is continuous.

Remark 2.72 Since F_M preserves quotient maps and the quotient maps

$$m : F_M([W,X]_*) \to [W,X]_* \text{ and } m : F_M([Y,Z]_*) \to [Y,Z]_*$$

are quotient by definition, the composites $m \circ F_M(\pi) : F_M(M_*(W,X)) \to [W,X]_*$ and $m \circ F_M(\pi) : F_M(M_*(Y,Z)) \to [Y,Z]_*$ are also quotient.

Example 2.73 For $n \geq 1$, let $\pi_n^\tau(X) = \tau(\pi_n^{top}(X))$ and for $n \geq 2$, let $\pi_n^\tau(X,A) = \tau(\pi_n^{top}(X,A))$. This means we have functors π_1^τ : $\mathbf{Top}_*$ → **TopGrp**, π_n^τ : $\mathbf{Top}_*$ → **TopAb**, $n \geq 2$, π_2^τ : $\mathbf{Top}_*^{(2)}$ → **TopGrp**, and π_n^τ : $\mathbf{Top}_*^{(2)}$ → **TopAb**, $n \geq 3$ such that the underlying functors to **Grp** and **Ab** are the usual homotopy and relative homotopy group functors.

The following are basic facts that result from the properties of τ (see Chapter A.3.1).

Proposition 2.74 *For spaces X and Y,*

1. *$id : [W, X]_* \cong [W, X]_*^\tau$ if and only if $[W, X]_*$ is a topological group.*

2. *$[W, X]_*^\tau$ is discrete if and only if $[W, X]_*$ is discrete.*

3. *If $[W, X_1]_* \cong [W, X_2]_*$ in* **GrpwTop**, *then $[W, X_1]_*^\tau \cong [W, X_2]_*^\tau$ in* **TopGrp**. *In other words, $[W, -]_*^\tau$ is a weaker invariant than $[W, -]_*$.*

4. *If $[W, X]_*^\tau$ is a Hausdorff topological group, then the homotopy mapping space $[W, X]_*$ with the quotient topology is functionally Hausdorff.*

The analogous results hold for $[Y, Z]_*^\tau$.

The case of the fundamental group π_1^τ is considered in more detail in Chapter 4.

2.5.2 k-spaces, the functor k, and $[X, Y]_*^k$

Here we move the conversation into a convenient category of spaces (that includes non-Hausdorff spaces) where products of quotient maps are quotients. We use the k-space from Chapter 1.2, however, other convenient categories offer similar approaches. All facts of k-spaces that we do not prove here appear in [Bro06]. For k-spaces X, Y, let $T_*(X, Y)$ be the set $\mathbf{Top}_*(X, Y)$ with the test-open topology of [Bro06, 5.9]. A subbasis is given by sets $\langle t, U\rangle = \{f : X \to Y | f \circ t(C) \subseteq U\}$ for some compact Hausdorff space C and map $t : C \to X$. If X is Hausdorff, the test-open topology is the same as the compact-open topology. Certainly, $T_*(-,-)$: $\mathbf{kTop_*}^{\mathrm{op}} \times \mathbf{kTop_*} \to \mathbf{Top_*}$ is a functor, however, it does not take values in $\mathbf{kTop_*}$. The category $\mathbf{kTop_*}$ becomes enriched over itself when we give $\mathbf{Top_*}(X,Y)$ the topology of $K_*(X,Y) = k(T_*(X,Y))$ for $X, Y \in \mathbf{kTop_*}$.

For $X, Y \in \mathbf{kTop_*}$, let $[X,Y]^k_*$ be the set $[X,Y]_*$ with the quotient topology with respect to $\pi : K_*(X,Y) \to [X,Y]_*$. Since every quotient of a k-spaces is a k-space [Bro06, 5.9.1], $[X,Y]^k_*$ is a k-space. Clearly this gives a functor $[-,-]_* : \mathbf{kTop_*}^{\mathrm{op}} \times \mathbf{kTop_*} \to \mathbf{kTop_*}$. The same construction may be made in the unbased and relative cases as well.

By Fact 1.24.4, $\pi \times_k \pi : K_*(X,Y) \times_k K_*(X,Y) \to [X,Y]^k_* \times_k [X,Y]^k_*$ is quotient. This fact is precisely the step which hindered us from asserting that multiplication in the group $[X,Y]_*$ was continuous. Additionally, for $X, Y \in \mathbf{kTop_*}$, the product operation

$$K_*(X,Y) \times_k K_*(X,Y) \to K_*(X \times_k X, Y \times_k Y)\,,\ (f,g) \mapsto f \times_k g$$

is continuous.

Let **kMon** and **kGrp** be the categories of k-monoids and k-groups. These are precisely monoid and group objects in $\mathbf{kTop_*}$.

Functorality 2.75 *If Y is an H-space (resp. group-like) and a k-space, then $[-,Y]^k_*$ is a functor* $\mathbf{kTop_*}^{\mathrm{op}} \to \mathbf{kMon}$ *(resp.* $\mathbf{kTop_*}^{\mathrm{op}} \to \mathbf{kGrp}$*)*

Proof. Let Y be an H-space with multiplication $\mu : Y \times_k Y \to Y$ and X be a k-space with diagonal $\Delta : X \to X \times_k X$. For any space X, Consider the diagram

$$\begin{array}{ccc} K_*(X,Y) \times_k K_*(X,Y) & \longrightarrow & K_*(X,Y) \\ \downarrow{\scriptstyle \pi \times_k \pi} & & \downarrow{\scriptstyle \pi} \\ [X,Y]^k_* \times_k [X,Y]^k_* & \longrightarrow & [X,Y]^k_* \end{array}$$

in **kTop**$_*$. This is precisely the large rectangular diagram in the proof of Theorem 2.14. Specifically the top map is the operation taking (f,g) to the map

$$\mu \circ (f \times_k g) \circ \Delta : X \xrightarrow{\Delta} X \times_k X \xrightarrow{f \times_k g} Y \times_k Y \xrightarrow{\mu} Y$$

and the bottom map is the monoid multiplication $([f],[g]) \mapsto [\mu \circ (f \times_k g) \circ \Delta]$. But $(f,g) \mapsto f \times_k g$ is continuous and $\mu_\# : K_*(X, Y \times_k Y) \to K_*(X,Y)$ and $\Delta^\# : K_*(X \times_k X, Y \times_k Y) \to K_*(X, Y \times_k Y)$ are continuous by functorality. The top map is the composite of these three maps and is therefore continuous. We have already noted that the left vertical map is quotient. By the Quotient Square Lemma, multiplication in $[X,Y]^k_*$ is continuous. Additionally, if $f : X_2 \to X_1$ is a map the functorality of $k \circ T_*(-,Y)$ and the Quotient Square Lemma imply the continuity of the monoid homomorphism $f^* : [X_1,Y]^k_* \to [X_2,Y]^k_*$. If, in addition Y is group-like with homotopy inverse $j : Y \to Y$, then $j_\# : T_*(X,Y) \to T_*(X,Y)$, $f \mapsto j \circ f$ is continuous by the functorality of $T_*(X,-)$. Applying k illustrates the continuity of $j_\# : K_*(X,Y) \to K_*(X,Y)$ and the Quotient Square Lemma again gives the continuity of the inversion map j_*. ∎

The dual of this statement with analogous proof is:

Functorality 2.76 *If X is a co-H-space (resp. cogroup-like) and a k-space, then $[X,-]^k_*$ is* *a functor* **kTop**$_*$ $\to$ **kMon** *(resp.* **kTop**$_*$ $\to$ **kGrp***)*

Example 2.77 For $n \geq 0$ the sphere S^n is a k-space. For a k-space X, we let $\pi_n^k(X)$ be the topological group$[S^n, X]_*^k$ be the topological group. If X is not a k-space, then one could define $\pi_n^k(X) = \pi_n^k(k(X))$.

Proposition 2.78 *Let $X, Y \in$ **kTop**$_*$ where X is Hausdorff. The identity $[X, Y]_*^k \to [X, Y]_*$ is continuous. Moreover, if $M_*(X, Y)$ is a k-space, then $[X, Y]_*^k \cong [X, Y]_*$ as spaces.*

Proof. Since X is Hausdorff, $M_*(X, Y) \cong T_*(X, Y)$. Since the identity $K_*(X, Y) = k(T_*(X, Y)) \to T_*(X, Y) \cong M_*(X, Y)$ is continuous, the identity $[X, Y]_*^k \to [X, Y]_*$ is continuous. If $M_*(X, Y)$ is a k-space, then $K_*(X, Y) = k(T_*(X, Y)) = T_*(X, Y) \cong M_*(X, Y)$ and therefore $[X, Y]_*^k \cong [X, Y]_*$. ■

Though very often the topologies of $[X, Y]_*^k$ and $[X, Y]_*$ agree, the main difference of these two approaches lies in the difference of products in **kGrp** and **Grp**.

2.5.3 The topological shape groups and $\pi_n^{Sh}(X)$

In the previous two sections, we topologized general homotopy mapping sets $[X, Y]_*$. In this section, we use shape theory to topologize the homotopy groups. The author thanks Paul Fabel for suggesting the following application of shape theory which seems to be fairly well-known (for instance [Mel09]). Application to the quotient topology of $\pi_n^{top}(X)$ is also of interest. The reader is referred to [MS82] for all preliminaries of the inverse system approach to shape theory. The homotopy category of polyhedra **hPol**$_*$ is the full-subcategory of **hTop**$_*$ consisting of spaces with the homotopy type of a polyhedron. It is well known [MS82, §4.3, Theorem 7] that **hPol**$_*$ is a dense subcategory of **hTop**$_*$. This means for each based space X, there is an **hPol**$_*$-expansion $X \to (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ universal with respect to other morphisms $X \to (Y_\mu, q_{\mu\mu'}, M)$ in **pro – hPol**$_*$ (here X is treated as a rudimentary system indexed by a singleton). Specifically, the expansion consists of maps $p_\lambda : X \to X_\lambda$ such that $p_\lambda = p_{\lambda\lambda'}p_{\lambda'}$ whenever $\lambda' \geq \lambda$ in directed set Λ. These maps induce continuous homomorphisms $(p_\lambda)_* : \pi_n^{top}(X) \to \pi_n^{top}(X_\lambda)$ for each $n \geq 1$.

Remark 2.79 The results in Chapter 2.4 indicate that $\pi_n^{top}(X_\lambda)$ is a discrete group since X_λ has the homotopy type of a polyhedron.

Definition 2.80 The *n-th topological homotopy pro-group* of a based space is the inverse system $pro{-}\pi_n^{top}(X) = \left(\pi_n^{top}(X_\lambda), (p_{\lambda\lambda'})_*, \Lambda\right)$ of discrete groups in **pro–TopGrp** where the bonding maps are the induced, continuous homomorphisms $(p_{\lambda\lambda'})_* : \pi_n^{top}(X_{\lambda'}) \to \pi_n^{top}(X_\lambda)$. The *n-th topological shape homotopy group* of X is the limit $\check{\pi}_n^{top}(X) = \varprojlim pro{-}\pi_n(X) = \varprojlim \pi_n^{top}(X_\lambda)$ which, as an inverse limit of discrete groups, is a Hausdorff topological group. The isomorphism class of $\check{\pi}_n^{top}(X)$ does not depend on the choice of **hPol**$_*$-expansion. There are analogous constructions in the unbased and relative cases as well.

Remark 2.81 It is a well-known fact of shape theory that a compact metric space X, has an **hPol**$_*$-expansion $X \to (X_n, p_{n,n+1}, \mathbb{N})$ indexed by the integers. Since $\check{\pi}_n^{top}(X)$ is a subspace of a countable product of discrete spaces, it is a metrizable topological group.

The continuous homomorphisms $(p_\lambda) : \pi_n^{top}(X) \to \pi_n^{top}(X_\lambda)$ satisfy $(p_\lambda)_* = (p_{\lambda\lambda'})_*(p_{\lambda'})_*$ and therefore induce a canonical, continuous homomorphism $\Phi_X : \pi_n^{top}(X) \to$

$\check{\pi}_n^{top}(X)$ to the limit. Let **HTopGrp** (resp. **HTopAb**) be the full subcategory of **TopGrp** (resp. **TopAb**) consisting of Hausdorff topological (resp. abelian) groups.

Functorality 2.82 *$\check{\pi}_1^{top}$: **hTop$_*$** $\to$ **HTopGrp** is a functor and $\Phi : \pi_1^{top} \to \check{\pi}_1^{top}$ is a natural transformation with components in **qTopGrp**. For $n \geq 2$, $\check{\pi}_n^{top}$: **hTop$_*$** $\to$ **HTopAb** is a functor and $\Phi : \pi_n^{top} \to \check{\pi}_n^{top}$ is a natural transformation with components in **qTopAb**.*

Proof. Forgetting the topological structure gives the usual shape group functors $\check{\pi}_1$: **hTop$_*$** $\to$ **Grp** and $\check{\pi}_n$: **hTop$_*$** $\to$ **Ab** for $n \geq 2$ [MS82, Ch II, §3.3, Corollary 2]. We have already seen that $\check{\pi}_n^{top}$ is well-defined on objects and so it suffices to show that a based map $f : X \to Y$ induces a continuous homomorphism $\check{f}_* : \check{\pi}_n^{top}(X) \to \check{\pi}_n^{top}(Y)$. If $X \to (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $Y \to (Y_\mu, q_{\mu\mu'}, M)$ are **hPol$_*$**-expansion for X and Y respectively, then a based map $f : X \to Y$ induces a map

$$(f_\mu, \phi) : (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (Y_\mu, q_{\mu\mu'}, M)$$

of inverse systems in **hPol$_*$** [MS82, Ch I, §2.1, Theorem 1]. Here (f_μ, ϕ) consists of a function $\phi : M \to \Lambda$ and continuous maps $f_\mu : X_{\phi(\mu)} \to Y_\mu$, $\mu \in M$ such that whenever $\mu \leq \mu'$ there is a $\lambda \in \Lambda$ such that $\lambda \geq \phi(\mu), \phi(\mu')$ and $f_\mu p_{\phi(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{\phi(\mu')\lambda}$. The f_μ induce continuous homomorphisms $(f_\mu)_* : \pi_n^{top}(X_{\phi(\mu)}) \to \pi_n^{top}(Y_\mu)$ such that whenever $\mu \leq \mu'$ there is a $\lambda \in \Lambda$ such that $\lambda \geq \phi(\mu), \phi(\mu')$ and $(f_\mu)_*(p_{\phi(\mu)\lambda})_* = (q_{\mu\mu'})_*(f_{\mu'})_*(p_{\phi(\mu')\lambda})_*$. Therefore,

$$((f_\mu)_*, \phi) : \left(\pi_n^{top}(X_\lambda), (p_{\lambda\lambda'})_*, \Lambda\right) \to \left(\pi_n^{top}(Y_\mu), (q_{\mu\mu'})_*, M\right)$$

is a morphism of inverse systems in **HTopGrp** and induces a continuous homomorphism

$$\check{f}_* : \check{\pi}_n^{top}(X) = \varprojlim\left(\pi_n^{top}(X_\lambda), (p_{\lambda\lambda'})_*, \Lambda\right) \to \varprojlim\left(\pi_n^{top}(Y_\mu), (q_{\mu\mu'})_*, M\right) = \check{\pi}_n^{top}(Y)$$

on the limits. To see the naturality of Φ, we regard $\pi_n^{top}(X)$ and $\pi_n^{top}(Y)$ as rudimentary inverse systems in **qTopGrp**, let $f : X \to Y$ be a based map, and consider the commuting square of inverse systems

$$\begin{array}{ccc} \pi_n^{top}(X) & \xrightarrow{\quad f_* \quad} & \pi_n^{top}(Y) \\ \downarrow & & \downarrow \\ \left(\pi_n^{top}(X_\lambda), (p_{\lambda\lambda'})_*, \Lambda\right) & \xrightarrow[((f_\mu)_*, \phi)]{} & \left(\pi_n^{top}(Y_\mu), (q_{\mu\mu'})_*, M\right) \end{array}$$

Applying $\varprojlim$, we obtain the square

$$\begin{array}{ccc} \pi_n^{Sh}(X) & \xrightarrow{\Phi_X} & \check{\pi}_n^{top}(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \check{f}_*} \\ \pi_n^{Sh}(Y) & \xrightarrow[\Phi_Y]{} & \check{\pi}_n^{top}(Y) \end{array}$$

for the naturality of Φ. ■

The primary application we make of shape groups is the following.

Proposition 2.83 *If $n \geq 1$ and the canonical map $\Phi : \pi_n^{top}(X) \to \check{\pi}_n^{top}(X)$ is an injection, then $\pi_n^{top}(X)$ is a functionally Hausdorff quasitopological group.*

Proof. Since $\check{\pi}_n^{top}(X)$ is a Hausdorff topological group, it is functionally Hausdorff. Any space continuously injecting into a functionally Hausdorff space is functionally

Hausdorff. ■

The existence of the natural homomorphism $\Phi_X : \pi_n(X) \to \check{\pi}_n^{top}(X)$ hints at an alternative topology for $\pi_n(X)$. Let $\pi_n^{Sh}(X)$ be the n-th homotopy group of X with the initial topology with respect to Φ. This topology is generated by the sets $\Phi_X^{-1}(U)$ where U is open in $\check{\pi}_n^{top}(X)$. By A.35 of the Appendix, $\pi_n^{Sh}(X)$ is a topological group.

Functorality 2.84 π_1^{Sh} : **Top**$_*$ $\to$ **TopGrp** *is a functor and for* $n \geq 2$, π_n^{Sh} : **Top**$_*$ $\to$ **TopAb** *is a functor.*

Proof. It suffices to prove that the homomorphism $f_* : \pi_n^{Sh}(X) \to \pi_n^{Sh}(Y)$ induced by a based map $f : X \to Y$ is continuous. But this follows directly from applying A.36 of the Appendix to the naturality diagram of Φ:

$$\begin{array}{ccc} \pi_n^{Sh}(X) & \xrightarrow{\Phi_X} & \check{\pi}_n^{top}(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \check{f}_*} \\ \pi_n^{Sh}(Y) & \xrightarrow[\Phi_Y]{} & \check{\pi}_n^{top}(Y) \end{array}$$

■

The next proposition follows directly from the definition of the initial topology:

Proposition 2.85 $\pi_n^{Sh}(X)$ *is Hausdorff if and only if the homomorphism* $\Phi_X : \pi_n(X) \to \check{\pi}_n(X)$ *is injective.*

The injectivity of Φ_X has received a significant amount of attention in the case $n = 1$. In the effort to characterize the spaces X for which Φ_X is injective (for fixed n), a simpler description of the topology of $\pi_n^{Sh}(X)$ is desirable.

Proposition 2.86 *If X is a compact metric space, then $\pi_n^{Sh}(X)$ is a pseudometrizable topological group.*

Proof. By Remark 2.81, $\check{\pi}_n(X)$ is metrizable. In general, if Y is a metric space with metric $d : Y^2 \to \mathbb{R}$ and a space X has the initial topology with respect to a function $f : X \to Y$, then $\rho = d \circ (f \times f) : X^2 \to \mathbb{R}$ is a pseudometric for X. It is clear that ρ satisfies the axioms of a pseuometric. Also, if $B_r^d(y_0) = \{y \in Y | d(y_0, y) < r\}$ and $B_r^\rho(x_0) = \{x \in X | \rho(x_0, x) = d(f(x_0), f(x)) < r\}$, the equation $B_r^\rho(x_0) = f^{-1}(B_r^d(f(x_0)))$ indicates that the initial topology on X agrees with the topology induced by the pseudometric ρ. ■

Theorem 2.87 *If a map $f : X \to Y$ induces an isomorphism $\check{f}_* : \check{\pi}_n(X) \to \check{\pi}_n(Y)$ of topological groups, then the continuous homomorphism $f_* : \pi_n^{Sh}(X) \to \pi_n^{Sh}(Y)$ is quotient onto its image.*

Proof. We use the naturality diagram

$$\begin{array}{ccc} \pi_n^{Sh}(X) & \xrightarrow{\Phi_X} & \check{\pi}_n^{top}(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \check{f}_*} \\ \pi_n^{Sh}(Y) & \xrightarrow[\Phi_Y]{} & \check{\pi}_n^{top}(Y) \end{array}$$

Let $U \subseteq Im(f_*)$ such that $f_*^{-1}(U)$ is open in $\pi_n^{Sh}(X)$. Since $\pi_n^{Sh}(X)$ has the initial topology with respect to Φ_X, we have $f_*^{-1}(U) = \Phi_X^{-1}(V)$ for open $V \subseteq \pi_n^{Sh}(X)$. Since $\check{f}_*$ is a homeomorphism and Φ_Y is continuous, it suffices to check the equality $\Phi_Y^{-1}(\check{f}_*(V)) \cap Im(f_*) = U$. If $u \in U \subseteq Im(f_*)$, and $\alpha \in f_*^{-1}(U) = \Phi_X^{-1}(V)$ such that

$f_*(\alpha) = u$, then $\Phi_Y(u) = \check{f}_* \circ \Phi_X(\alpha) \in \check{f}_*(V)$ and therefore $u \in \Phi_Y^{-1}(\check{f}_*(V))$. For the other inclusion, if $\Phi_Y(u) \in \check{f}_*(V)$ and $f_*(\alpha) = u$, then $\check{f}_* \circ \Phi_X(\alpha) = \Phi_Y(u) \in \check{f}_*(V)$. Since $\check{f}_*$ is bijective, we have $\Phi_X(\alpha) \in V$ and therefore $\alpha \in \Phi_X^{-1}(V) = f_*^{-1}(U)$. This implies $u = f_*(\alpha) \in U$. ∎

From this theorem, we begin to get a feel for the strength of π_n^{Sh} as an invariant.

Corollary 2.88 *If a map $f : X \to Y$ induces an isomorphism $\check{f}_* : \check{\pi}_n(X) \to \check{\pi}_n(Y)$ of topological groups, and an isomorphism $f_* : \pi_n(X) \to \pi_n(Y)$ of groups, then $f_* : \pi_n^{Sh}(X) \to \pi_n^{Sh}(Y)$ is an isomorphism of topological groups.*

The following is a comparison of the four topologies defined on the homotopy groups:

Corollary 2.89 *For any space X, the identity maps $\pi_n^k(k(X)) \to \pi_n^{top}(X) \to \pi_n^{\tau}(X) \to \pi_n^{Sh}(X)$ are continuous.*

Proof. The identity $k(X) \to X$ is continuous and so the identities $\pi_n^k(k(X)) \to \pi_n^{top}(k(X)) \to \pi_n^{top}(X)$ are continuous by Proposition 2.78 and the functorality of π_n^{top}. The identity $\pi_n^{top}(X) \to \pi_1^{\tau}(X)$ is continuous by the construction of τ. The identity $\pi_n^{top}(X) \to \pi_n^{Sh}(X)$ is continuous by the universal property of spaces with initial topologies. Since $\pi_n^{Sh}(X)$ is a topological group and $\pi_n^{top}(X) \to \pi_n^{Sh}(X)$ is continuous, so is the adjoint $\pi_1^{\tau}(X) \to \pi_n^{Sh}(X)$. ∎

CHAPTER III

PATH COMPONENT SPACES

In this chapter, we study the path component spaces defined in Example 2.7 and used in Chapter 2.4. The *path component space* of a topological space X is the set of path components $\pi_0(X)$ of X with the quotient topology with respect to the canonical map $\pi_X : X \to \pi_0(X)$. We denote this space as $\pi_0^{top}(X)$ and remove or change the subscript of the map π_X when convenient. The following definitions are equivalent up to homeomorphism:

1. $\pi_0^{top}(X) = [*, X]$
2. $\pi_0^{top}(X) = [S^0, X]_*$ for any choice of basepoint in X.
3. $\pi_0^{top}(X)$ is the coequalizer of the maps $ev_0, ev_1 : P(X) \to X$ which are evaluation at 0 and 1.

It is then clear that $\pi_0^{top} :$ **Top** $\to$ **Top** is a functor which factors through the homotopy category **hTop**. If X has basepoint x, we choose the basepoint of $\pi_0^{top}(X)$ to be the path component of x in X. This gives a based version of the functor π_0^{top}, however, the presence of basepoint will be clear from context.

Example 3.1 Let $T \subseteq \mathbb{R}^2$ be the topologist's sine curve

$$\{(0,0)\} \cup \left\{(x,y) | y = \sin\left(\frac{1}{x}\right), 0 < x \leq 1\right\}$$

or closed topologist's sine curve $\{0\}\times[-1,1]\cup\{(x,y)|y=\sin\left(\frac{1}{x}\right), 0<x\leq 1\}$. It is easy to see that in both cases, $\pi_0^{top}(T)$ is homeomorphic to the *Sierpinski space* $\mathbb{S}=\{0,1\}$ with topology $\{\emptyset,\{1\},\{0,1\}\}$.

It is worthwhile to mention the remarkable fact, proved by D. Harris that every topological space is the path component space of some paracompact Hausdorff space.

Theorem 3.2 [Har80] *Every topological space Y is homeomorphic to the path component space of some paracompact Hausdorff space $\mathcal{H}(Y)$.*

Some properties and variants of the functor $\mathcal{H}$ are included in [Har80]. The next example indicates that subspaces of $\mathbb{R}$ appear quite naturally as path component spaces.

Example 3.3 Let X be the set $\mathbb{R}\times I$. We define a simple Hausdorff topology on X such that $\pi_0^{top}(A\times I)\cong A$ for subspaces $A\subseteq\mathbb{R}$ and $A\times I\subseteq X$. The topology on X has a basis consisting of sets of the form $\{a\}\times(s,t)$ and $\{a\}\times(t,1]\cup(a,b)\times I\cup\{b\}\times[0,s)$ for $0<s<t<1$ and $a<b$. This topology is a simple extension of the *ordered square* in [Mun00, §16, Example 3] and is the order topology given by the dictionary ordering on X. The path components of X are $\{z\}\times I$ for $z\in\mathbb{R}$ (see [Mun00, §24, Example 6]). It then suffices to show that for each $A\subseteq\mathbb{R}$, the projection $p_A:X_A\to A$ is quotient, where $X_A=A\times I$ has the subspace topology of X. Suppose U is open in X so that $U\cap A$ is open in A. Since $U\times I=p_{\mathbb{R}}^{-1}(U)$ is open in X and so $(U\times I)\cap X_A=(U\cap A)\times I=p_A^{-1}(U\cap A)$ is open in X_A. Therefore p_A is continuous. Now suppose $V\subseteq A$ such that $p_A^{-1}(V)=V\times I$ is open in X_A. For each $v\in V$, there

is an open neighborhood $\{v\} \times (t_v, 1] \cup (v, b_v) \times I \cup \{b_v\} \times [0, s_v)$ of $(v, 1)$ contained in $V \times I$. Since $V \times I$ is saturated with respect to p_A, we have $[v, b_v] \cap A \times I \subset V \times I$. Similarly, since $(v, 0) \in V$ for each $v \in V$ we can find a closed interval $[a_v, v]$ such that $[a_v, v] \cap A \times I \subset V \times I$. Therefore, for each $v \in V$, we have $v \in (a_v, b_v) \cap A \subseteq V$. Therefore V is open in A, p_A is quotient, and consequently $\pi_0^{top}(X_A) \cong A$.

Example 3.4 Using the previous example, we can find a space Y such that $\pi_0^{top}(Y) \cong S^1$. Let $\epsilon : \mathbb{R} \to S^1$ denote the exponential map and $X = \mathbb{R} \times I$ be the space defined in the previous example. Let Y be the set $S^1 \times I$ with the quotient topology with respect to $\epsilon \times id_I : X \to S^1 \times I$. It is easy to see that a basic open neighborhood in Y is $\epsilon(U)$ where U is a basic open neighborhood of X described above. Similarly, one can show that the projection $Y \to S^1$ is precisely the quotient map $\pi_Y : Y \to \pi_0^{top}(Y)$ and so $S^1 \cong \pi_0^{top}(Y)$.

We now observe some of the other basic properties of path component spaces. We will be particularly interested in the preservation of limits and colimits. The following will be very useful later on.

Proposition 3.5 π_0^{top} *preserves coproducts and quotients in* **Top** *and* $\mathbf{Top_*}$.

Proof. Clearly, $\pi_0^{top}(\coprod_\lambda X_\lambda) \cong \coprod_\lambda \pi_0^{top}(X_\lambda)$ for any family of spaces $\{X_\lambda\}$. If $q : X \to Y$ is a quotient map, then the diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ \pi_0^{top}(X) & \xrightarrow[f_*]{} & \pi_0^{top}(Y) \end{array}$$

commutes. The bottom map f_* is quotient by the Quotient Square Lemma (1.22). In the based case the quotient map $q : \coprod_\lambda X_\lambda \to \bigvee_\lambda X_\lambda$ induces a quotient map $q_* : \pi_0^{top}(\coprod_\lambda X_\lambda) \to \pi_0^{top}(\bigvee_\lambda X_\lambda)$ which makes the same identifications as the quotient map

$$\pi_0^{top}\left(\coprod_\lambda X_\lambda\right) \cong \coprod_\lambda \pi_0^{top}(X_\lambda) \to \bigvee_\lambda \pi_0^{top}(X_\lambda)$$

Therefore, there is a natural homeomorphism $\pi_0^{top}(\bigvee_\lambda X_\lambda) \cong \bigvee_\lambda \pi_0^{top}(X_\lambda)$. ∎

Though π_0^{top} preserves coproducts, unfortunately it fails to be cocontinuous. Since **Top** is cocomplete, it suffices to exhibit a coequalizer which is not preserved [Mac00, §V.4].

Example 3.6 Let $Y = \{1, \frac{1}{2}, \frac{1}{3}, ..., 0\} \subseteq \mathbb{R}$. We define parallel maps $f, g : \mathbb{Z}^+ \to Y$ by $f(n) = \frac{1}{n}$ and $g(n) = \frac{1}{n+1}$. It is easy to see that the coequalizer of these maps is homeomorphic to the Sierpinski space $\mathbb{S}$ of Example 3.1. The Sierpinski space is path connected since the function $\alpha : I \to \{0, 1\}$ given by $\alpha([0, \frac{1}{2}]) = 0$ and $\alpha((\frac{1}{2}, 1]) = 1$ is continuous. Therefore $\pi_0^{top}(\mathbb{S})$ is a one point space. Noting that both $\mathbb{Z}^+$ and Y are totally path disconnected (so $\pi_0^{top}(\mathbb{Z}^+) \cong \mathbb{Z}^+$ and $\pi_0^{top}(Y) \cong Y$), we find that $f = f_*$ and $g = g_*$. Therefore the coequalizer of f_* and g_* is $\mathbb{S}$ which is not a one point space. This means the path component space of the coequalizer of f and g is not homeomorphic to the coequalizer of f_* and g_*.

One might notice in the previous example that the path component space of the coequalizer is a quotient of the coequalizer of the induced maps. This phenomenon in fact generalizes to all (small) colimits.

Proposition 3.7 *Let J be a small category and $F : J \to$ **Top** be a diagram with colimit $colimF$. Suppose $colim(\pi_0^{top}F)$ is the colimit of diagram $\pi_0^{top} \circ F : J \to$ **Top**. There is a canonical quotient map $Q : colim(\pi_0^{top}F) \to \pi_0^{top}(colimF)$.*

Proof. By the colimit existence theorem [Mac00, §V.4], $colimF$ is the coequalizer of parallel maps f and g and $colim(\pi_0^{top}F)$ is the coequalizer of parallel maps f' and g' as seen in the diagram below. The coproducts on the left are over all morphisms $u : j \to k$ in J and the coproducts in the middle column are over all objects $i \in J$. The naturality of $\pi : X \to \pi_0^{top}(X)$ and the homeomorphisms $\pi_0^{top}(\coprod_\alpha X_\lambda) \cong \coprod_\lambda \pi_0^{top}(X_\lambda)$ of Proposition 3.5 gives the commutativity of the squares on the left and top right. By Proposition 3.5, q_* is a quotient map. Since $q \circ f = q \circ g$, we have $q_* \circ f_* = q_* \circ g_*$. Therefore $q_* \circ t \circ f' = q_* \circ f_* \circ s = q_* \circ g_* \circ s = q_* \circ t \circ g'$. By the universal property of $colim(\pi_0^{top}F)$, this induces a unique map $Q : colim(\pi_0^{top}F) \to \pi_0^{top}(colimF)$ such that $Q \circ q' = q_* \circ t$. Since t is a homeomorphism and q_* is a quotient map, Q is also a quotient map.

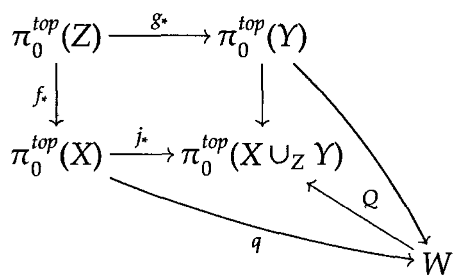
$$
\begin{array}{ccccc}
\coprod_{u:j\to k} F(j) & \overset{f}{\underset{g}{\rightrightarrows}} & \coprod_{i\in J} F(i) & \xrightarrow{\quad q \quad} & colimF \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\pi_0^{top}\left(\coprod_{u:j\to k} F(j)\right) & \overset{f_*}{\underset{g_*}{\rightrightarrows}} & \pi_0^{top}\left(\coprod_{i\in J} F(i)\right) & \xrightarrow{\quad q_* \quad} & \pi_0^{top}(colimF) \\
\uparrow{\scriptstyle s\,\cong} & & \uparrow{\scriptstyle t\,\cong} & & \uparrow{\scriptstyle \exists! Q} \\
\coprod_{u:j\to k} \pi_0^{top}(F(j)) & \overset{f'}{\underset{g'}{\rightrightarrows}} & \coprod_{i\in J} \pi_0^{top}(F(i)) & \xrightarrow{\quad q' \quad} & colim(\pi_0^{top}F)
\end{array}
$$

■

Corollary 3.8 *Let $X \cup_Z Y$ be the pushout of the diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$ where $\pi_0^{top}(Y)$*

is discrete and $g_* : \pi_0^{top}(Z) \to \pi_0^{top}(Y)$ *is surjective. The inclusion* $j : X \to X \cup_Z Y$ *induces a quotient map* $j_* : \pi_0^{top}(X) \to \pi_0^{top}(X \cup_Z Y)$ *on path component spaces.*

Proof. The pushout of the diagram $\pi_0^{top}(X) \xleftarrow{f_*} \pi_0^{top}(Z) \xrightarrow{g_*} \pi_0^{top}(Y)$ is the quotient space $W = \pi_0^{top}(X)/\sim$ where for each $P, Q \in \pi_0^{top}(Z)$ such that $g_*(P) = g_*(Q)$ we make the identification $f_*(P) \sim f_*(Q)$. Let $q : \pi_0^{top}(X) \to W$ be the quotient map. Consider the diagram

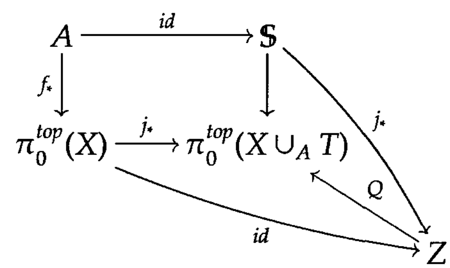


where Q is the canonical quotient map of Theorem 3.7 induced by the universal property of pushouts. Since both q and Q are quotient maps j_* is also a quotient map. ■

As in Example 3.1, let $T \subset \mathbb{R}^2$ be the topologist's sine curve, $a = (0,0)$, $b = (1, \sin(1))$, and $A = \{a, b\} \subseteq T$. Let $\mathbb{S} = \{a, b\}$ be the Sierpinski space with topology $\{\emptyset, \{b\}, \{a, b\}\}$. The next corollary illustrates the possibility of weakening the topology of path component spaces by attach the topologist's sine curve in the appropriate way and also gives an example of when pushouts are preserved by π_0^{top}.

Corollary 3.9 *Let* $f : A \to X$ *be a map such that* $f(a), f(b)$ *lie in distinct path components of Hausdorff space* X. *Let* $X \cup_A T$ *be the pushout of the diagram* $X \xleftarrow{f} A \xrightarrow{i} T$ *where* i *is inclusion. The inclusion* $j : X \to Z$ *induces a continuous bijection* $j_* : \pi_0^{top}(X) \to \pi_0^{top}(X \cup_A T)$ *which is not a homeomorphism and* $\pi_0^{top}(Z)$ *is canonically homeomorphic to* *the pushout of* $\pi_0^{top}(X) \xleftarrow{f} A \xrightarrow{id} \mathbb{S}$.

Proof. It is easy to see the map $j_* : \pi_0^{top}(X) \to \pi_0^{top}(X \cup_A T)$ is a bijection since we do not create any new connections between path components by attaching T in this way. Applying π_0^{top} to $X \xleftarrow{f} A \xrightarrow{i} T$ gives diagram $\pi_0^{top}(X) \xleftarrow{\pi_X \circ f} A \xrightarrow{id} \mathbb{S}$. The pushout Z of this diagram is simply $\pi_0(X)$ with a topology (strictly) weaker than that of $\pi_0^{top}(X)$. Applying Theorem 3.7 gives diagram



where Q is a quotient map. Since Q is a bijective quotient map, it is a homeomorphism. ∎

Now we observe the behavior of π_0^{top} on products.

Proposition 3.10 *Let* $\{X_\lambda\}$ *be a family of spaces and* $X = \prod_\lambda X_\lambda$. *Let* $\pi_\lambda : X_\lambda \to \pi_0^{top}(X_\lambda)$ *and* $\pi : X \to \pi_0^{top}(X)$ *be the canonical quotient maps and* $\prod_\lambda \pi_\lambda : X \to \prod_\lambda \pi_0^{top}(X_\lambda)$ *be the product map. There is a natural continuous bijection* $\Phi : \pi_0^{top}(X) \to \prod_\lambda \pi_0^{top}(X_\lambda)$ *such that* $\Phi \circ \pi = \prod_\lambda \pi_\lambda$.

Proof. The projections $pr_\lambda : \prod_\lambda X_\lambda \to X_\lambda$ induces maps $(pr_\lambda)_* : \pi_0^{top}(\prod_\lambda X_\lambda) \to \pi_0^{top}(X_\lambda)$ which in turn induce the map $\Phi : \pi_0^{top}(\prod_\lambda X_\lambda) \to \prod_\lambda \pi_0^{top}(X_\lambda)$. This is a bijection due to the basic fact that the non-topological functor π_0 preserves arbitrary products. ∎

Corollary 3.11 *$\Phi : \pi_0^{top}(X) \to \prod_\lambda \pi_0^{top}(X_\lambda)$ is a homeomorphism if and only if the product of quotients $\prod_\lambda \pi_\lambda : X \to \prod_\lambda \pi_0^{top}(X_\lambda)$ is itself a quotient map.*

Proof. This follows from the fact that π is a quotient map, Φ is a bijection, and $\Phi \circ \pi = \prod_\lambda \pi_\lambda$. ■

Corollary 3.12 *If $\pi_0^{top}(X_\lambda)$ is discrete for each λ, then $\Phi : \pi_0^{top}(X) \to \prod_\lambda \pi_0^{top}(X_\lambda)$ is a homeomorphism.*

Proof. If $\pi_0^{top}(X_\lambda)$ is discrete, then $\pi_\lambda : X_\lambda \to \pi_0^{top}(X_\lambda)$ is open. Since products of open maps are open, $\prod_\lambda \pi_\lambda : X \to \prod_\lambda \pi_0^{top}(X_\lambda)$ is open and must be quotient. By Corollary 3.11, Φ is a homeomorphism. ■

Of course, not all products (or even powers) of quotient maps are quotient. In light of Corollary 3.11, the facts in Chapter 1.2 provide some sufficient conditions on $X, Y, \pi_0^{top}(X), \pi_0^{top}(Y)$ to guarantee that $\pi_0^{top}(X \times Y) \cong \pi_0^{top}(X) \times \pi_0^{top}(Y)$. This, however, does not occur in general.

Corollary 3.13 *π_0^{top} does not preserve finite products.*

Proof. Let $\mathbb{R}_K$ be the real line with the K-topology. In this space the set $K = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$ is closed and sets of the form $(a, b), (a, b) - K$ form a basis. Let $\mathbb{Q}_K$ be the rational numbers with the subspace topology of $\mathbb{R}_K$. Now let $X = \mathbb{Q}_K \sqcup_K CK$ where CK is the cone on K. The path components of X are the singletons $\{a\}$ for $a \in \mathbb{Q}_K - K$ and the set CK. The path component space of X is $\pi_0^{top}(X) \cong \mathbb{Q}_K/K$ but the map

$\pi_X \times \pi_X : X \times X \to \pi_0^{top}(X) \times \pi_0^{top}(X)$ is not a quotient map [Mun00, §22]. By Prop. 3.10 the topology of $\pi_0^{top}(X \times X)$ is strictly finer than that of $\pi_0^{top}(X) \times \pi_0^{top}(X)$. ■

Other examples of this failure arise in the context of topological fundamental groups. In fact 2.34 and 2.39 imply that if $\pi_1^{top}(X)$ is not a topological group then $\pi_0^{top}(\Omega(X) \times \Omega(X)) \cong \pi_1^{top}(X \times X)$ is not homeomorphic to $\pi_0^{top}(\Omega(X)) \times \pi_0^{top}(\Omega(X)) = \pi_1^{top}(X) \times \pi_1^{top}(X)$.

We will also have need to consider path components of monoids and groups with topology. In particular, if multiplication in $M \in$ **MonwTop** is only continuous in each variable, then we do not have H-space structure and must do more to obtain monoid structure on $\pi_0(M) = [*, M]$. Recall the notion of semitopological and topological monoids (with continuous involution) from A.1 of the Appendix.

Proposition 3.14 Suppose M is a semitopological monoid and C_m denotes the path component of $m \in M$. If $a_i \in C_{m_i}$ for $i = 1, 2$, then $a_1 a_2 \in C_{m_1 m_2}$. Consequently, there is a well-defined multiplication $\pi_0(M) \times \pi_0(M) \to \pi_0(M)$, $(C_m, C_n) \mapsto C_{mn}$ making $\pi_0^{top}(M)$ a semitopological monoid. Moreover, if $s : M \to M$ is an involution on s, then $s_* : \pi_0(M) \to \pi_0(M)$ is a continuous involution on $\pi_0^{top}(M)$.

Proof. Let $p_i : I \to M$ be a path $p_i(0) = a_i$, $p_i(1) = m_i$. Let $\lambda_{a_1} : M \to M$ (resp. $\rho_{m_2} : M \to M$) be left (resp. right) multiplication by a_1 (resp. m_2). Consider the concatenation $q = (\lambda_{a_1} \circ p_2) * (\rho_{m_2} \circ p_1) : I \to M$. This is well-defined since $\lambda_{a_1} \circ p_2(1) = a_1 m_2 = \rho_{m_2} \circ p_1(0)$ and satisfies $q(0) = \lambda_{a_1} \circ p_2(0) = a_1 a_2$ and $q(1) = \rho_{m_2} \circ p_1(1) = m_1 m_2$. Clearly then $a_1 a_2 \in C_{m_1 m_2}$. Since multiplication in M is associative and unital (if e is the identity of M, then C_e is the identity of $\pi_0(M)$), the multiplication $C_m C_n = C_{mn}$ in $\pi_0^{top}(M)$ is both associative and unital. To see that $\pi_0^{top}(M)$ is a semitopological monoid, let $m \in M$ and $\lambda_m, \rho_m : M \to M$ be (continuous) right and left multiplication by m. The induced maps $(\lambda_m)_*, (\rho_m)_* : \pi_0^{top}(M) \to \pi_0^{top}(M)$ are left and right multiplication by C_m respectively and are continuous by the functorality of π_0^{top}. If $s : M \to M$ is a continuous involution on M, then $s_*(C_m) = C_{s(m)}$. Therefore, $s(C_e) = C_e$, $s(C_m C_n) = s(C_{mn}) = C_{s(mn)} = C_{s(n)s(m)} = C_{s(n)}C_{s(m)}$ and $s_*^2 = (s^2)_* = (id)_* = id$ proving that s_* is a continuous involution on $\pi_0^{top}(M)$. ■

Corollary 3.15 *If G is a semitopological (resp. quasitopological group), then so is $\pi_0^{top}(G)$.*

Proof. If $gg^{-1} = e = g^{-1}g$ in G, then $C_g C_{g^{-1}} = C_{gg^{-1}} = C_e = C_{g^{-1}g} = C_{g^{-1}}C_g$. Since inverses are given by $C_g^{-1} = C_{g^{-1}}$, $\pi_0^{top}(G)$ is a group whenever G is. If $inv : G \to G$ is continuous, then $(inv)_*$ is inversion on $\pi_0^{top}(G)$ and is continuous. ■

Proposition 3.16 *Let C be the category* **sTopMon**, **sTopMon***, **sTopGrp** *or* **qTopGrp**. *Then $\pi_0^{top} : C \to C$ is a functor.*

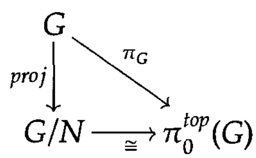
Proof. We have already observed that π_0^{top} is well-defined on objects for each case. It suffices to deal with morphisms. If $f : M \to N$ is a continuous homomorphism of semitopological monoids (or groups), then the continuous map $f_* : \pi_0^{top}(M) \to \pi_0^{top}(N)$, $f_*(C_m) = C_{f(m)}$ satisfies $f_*(C_m C_n) = f_*(C_{mn}) = C_{f(mn)} = C_{f(m)f(n)} = C_{f(m)}C_{f(n)} = f_*(C_m)f_*(C_n)$ so that f_* is indeed a homomorphism. This is enough for the first, third and fourth categories. For the last case, suppose $f : (M, s) \to (N, t)$ is a continuous, involution-preserving ($f \circ s = t \circ f$) morphism in **sTopMon***. The functorality of π_0^{top} (on spaces) gives that $t_* \circ f_* = f_* \circ s_*$. Therefore $f_* : (\pi_0^{top}(M), s_*) \to (\pi_0^{top}(N), t_*)$ is also a continuous, involution-preserving homomorphism. Preservation of composition and identity are immediate from the functorality of π_0^{top} : **Top** $\to$ **Top**. ■

Proposition 3.17 *If M is a semitopological monoid (resp. semitopological monoid with continuous involution, semitopological group), the path component of the identity e is a semitopological submonoid (resp. semitopological submonoid with continuous involution, normal semitopological subgroup) of M.*

Proof. Let N be the path component of e. Suppose $a, b \in M$ and $p, q : I \to M$ are paths with $p(0) = q(0) = e$ and $p(1) = a, q(1) = b$. Let $l_a : M \to M$ be continuous left multiplication by a. Then $r = (l_a \circ q) * p : I \to M$ is a path with $r(0) = p(0) = e$ and $r(1) = aq(1) = ab$. Therefore N is a submonoid of M. If $s : M \to M$ is an involution on M, and $p : I \to M$ is a path with $p(0) = e$ and $p(1) = a$, then $s \circ p : I \to M \to M$ is a path $p(0) = e$ and $p(1) = s(a)$. Therefore N is closed under the image of all continuous involutions $M \to M$. Therefore if $(M, s) \in$ **sTopMon***, then $(N, s|_N)$ is a subobject of (M, s). If M is a semitopological group, and $p : I \to M$ is a path from e to a, let $\lambda_{a^{-1}}$ be left multiplication by a^{-1} so that $\lambda_{a^{-1}} \circ p : I \to M$ is a path from a^{-1} to $a^{-1}a = e$. Therefore N is closed under inversion and is a subgroup. Clearly if inversion is continuous in M, it will be continuous in N with the subspace topology. To see that N is also normal we take $n \in N$ and $a \in M$. Let $p : I \to M$ be a path from e to n, l_a be left multiplication by a and $\rho_{a^{-1}}$ be right multiplication by a^{-1}. Then

$\rho_{a^{-1}} \circ \lambda_a \circ p : I \to M$ is a path from $aea^{-1} = e$ to ana^{-1}. Therefore $aNa^{-1} \subseteq N$. Since this also holds when we replace a with a^{-1}, it follows that $a^{-1}Na \subseteq N$ and therefore $N \subseteq aNa^{-1}$. ■

Corollary 3.18 *If G is a semi(quasi)topological group and N is the path component of the identity e, then there is an isomorphism $\pi_0^{top}(G) \cong G/N$ of semi(quasi)topological groups such that the following diagram commutes*



Proof. If a, b are in the coset gN, then $g^{-1}a, g^{-1}b \in N$ and we can find a path q from $g^{-1}a$ to $g^{-1}b$. If λ_g is left multiplication by g, $\lambda_g \circ q : I \to G$ connects the points a and b. Therefore each coset gN is path connected. Additionally, if $p : I \to G$ is a path with $p(0) = g$, $p(0) = h$ and $\lambda_{g^{-1}}$ is left multiplication by g^{-1}, then $q = \lambda_{g^{-1}} \circ p$ is a path $q(0) = e$ and $q(1) = g^{-1}h$. Therefore $g^{-1}h \in N$ and $h \in gN$. Therefore every path $p : I \to G$ must lie entirely within the coset $p(0)N$. So the path components of G are precisely the cosets gN. ■

Corollary 3.19 *Let G_α be a family of semi(quasi)topological groups and $G = \prod_\alpha G_\alpha$ be the product group with the product topology. There is a canonical isomorphism $\pi_0^{top}(G) \cong \prod_\alpha \pi_0^{top}(G_\alpha)$ of semi(quasi)topological groups.*

Proof. The projections $p_\alpha : G \to G_\alpha$ induce the canonical, continuous group isomorphism $\Phi : \pi_0^{top}(G) \to \prod_\alpha \pi_0^{top}(G_\alpha)$. It was noted in Corollary 3.11 that Φ is a homeomorphism if and only if $\prod_\alpha \pi_{G_\alpha} : \prod_\alpha G_\alpha \to \prod_\alpha \pi_0^{top}(G_\alpha)$ is quotient. If N_α is the path component of the identity in G_α, then the projection $G_\alpha \to \pi_0^{top}(G_\alpha) \cong G_\alpha/N_\alpha$ is a quotient map of semitopological groups. It is a basic fact of semitopological groups [AT08, Theorem 1.5.1] that these projections are also open. Since products of open maps are open, $\prod_\alpha G_\alpha \to \prod_\alpha G_\alpha/N_\alpha \cong \prod_\alpha \pi_0^{top}(G_\alpha)$ is open and therefore quotient. ■

Remark 3.20 If M is a topological monoid (with continuous involution), then $\pi_0^{top}(M)$ is a semitopological monoid (with continuous involution) but is NOT always a topological monoid. The diagram

$$\begin{array}{ccc} M \times M & \longrightarrow & M \\ \downarrow{\scriptstyle \pi_M \times \pi_M} & & \downarrow{\scriptstyle \pi_M} \\ \pi_0^{top}(M) \times \pi_0^{top}(M) & \longrightarrow & \pi_0^{top}(M) \end{array}$$

where the horizontal maps are multiplication commutes. If the product map $\pi_M \times \pi_M$ is quotient, then $\pi_0^{top}(M)$ is a topological monoid, however, this is not always the case. For an explicit example consider the monoid $\Omega^M(X)$ of Moore loops [May90] in a space X such that $\pi_1^{top}(X)$ is not a topological group. Since $\Omega(X) \simeq \Omega^M(X)$, multiplication in $\pi_0^{top}(\Omega^M(X)) \cong \pi_1^{top}(X)$ is not continuous.

Example 3.21 An example of particular importance to us is the path component space of the free topological monoid with continuous involution $M_T^*(X)$ defined in the Appendix A.1. Let X be an unbased space and $\pi_X : X \to \pi_0^{top}(X)$ be the quotient map identifying path components. Since π_X is a quotient map, the semitopological monoid $M^*_{\pi_X}(\pi_0^{top}(X))$ is well-defined. In particular its topology is characterized by the fact that the monoid homomorphism $M^*(\pi_X) : M_T^*(X) \to M^*_{\pi_X}(\pi_0^{top}(X))$ is a quotient map. Since the non-topological $\pi_0 :$ **Top** $\to$ **Set** preserves products and coproducts, there is a canonical monoid isomorphism $\psi : \pi_0(M_T^*(X)) \to M^*(\pi_0(X))$ defined as follows: The path component of $w = x_1^{\epsilon_1}...x_n^{\epsilon_n}$ is sent to $\psi(w) = P_1^{\epsilon_1}...P_n^{\epsilon_n}$ where P_i is the path component of x_i in X. This makes the diagram

$$\begin{array}{ccccc} & & M_T^*(X) & & \\ & \swarrow^{\pi} & \downarrow{\scriptstyle M^*(\pi_X)} & \searrow^{M_T^*(\pi_X)} & \\ \pi_0^{top}(M_T^*(X)) & \xrightarrow[\psi]{} & M^*_{\pi_X}(\pi_0^{top}(X)) & \xrightarrow[id]{} & M_T^*(\pi_0^{top}(X)) \end{array}$$

commutes in the category of semitopological monoids. Since the two non-horizontal maps in the left triangle are quotient we have:

Lemma 3.22 $\psi : \pi_0^{top}(M_T^*(X)) \to M^*_{\pi_X}(\pi_0^{top}(X))$ *is a natural (in X) isomorphism of semitopological monoids.*

Now we consider the case when G is a topological group. Since any quotient group of a topological group is itself a topological group with the quotient topology, it is clear from Corollary 3.18 that:

Proposition 3.23 *If G is a topological group, then so is $\pi_0^{top}(G)$. Moreover π_0^{top} :* **TopGrp** $\to$ **TopGrp** *is a functor.*

Proof. We have already shown that $\pi_0^{top}(G)$ is a quasitopological group (Corollary 3.15) so it suffices to check that multiplication is continuous. By Corollary 3.19, the canonical map $\pi_0^{top}(G \times G) \to \pi_0^{top}(G) \times \pi_0^{top}(G)$ is a homeomorphism. By Corollary 3.11, the product $\pi_G \times \pi_G : G \times G \to \pi_0^{top}(G) \times \pi_0^{top}(G)$ is quotient. Now by Remark 3.20, multiplication in $\pi_0^{top}(G)$ is continuous. ■

Theorem 3.24 *If G is a topological group, then $M_*(X, G)$ and $[X, G]_*$ are topological groups for any space X.*

Proof. Since G is a group-like space, the operation $M_*(X, G) \times M_*(X, G) \to M_*(X, G)$, $(f, g) \mapsto \mu \circ (f \times g) \circ \Delta$ studied in Chapter 2.2.1 is continuous. Here μ is the multiplication of G and $\Delta : X \to X \times X$ is the diagonal. With this multiplication $M_*(X, G)$ is a group where the identity is the constant map at the inverse of $f : X \to G$ is $f^{-1} : X \to G$, $x \mapsto f(x)^{-1}$. Since both multiplication and inversion are continuous in G, $M_*(X, G)$ is a topological group. As previously noted, the operation $[f] * [g] = [\mu \circ (f \times g) \circ \Delta]$ gives group structure on $[X, G]_*$. By Remark 2.52 there is a natural isomorphism $[X, G]_* \cong \pi_0^{top}(M_*(X, G))$ of quasitopological groups. Corollary 3.23 asserts that $\pi_0^{top}(M_*(X, G))$ is a topological group. ■

Corollary 3.25 *If G is a topological group, then $\pi_n^{top}(G)$ is a topological group for all $n \geq 0$ (abelian for $n \geq 1$).*

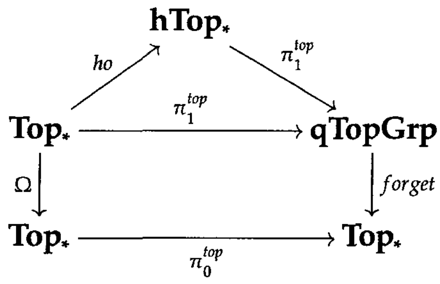
CHAPTER IV

THE TOPOLOGICAL FUNDAMENTAL GROUP

In this chapter we study the *topological fundamental group* $\pi_1^{top}(X, x) = [S^1, X]_*$ as defined in Example 2.9. Prior to this research and the independent work of Fabel [Fab09] it was thought that these groups are always topological groups. In fact, many authors asserted that $\pi_1^{top} : \mathbf{Top}_* \to \mathbf{TopGrp}$ was a well-defined functor under the false assumption that $\pi \times \pi : \Omega(X) \times \Omega(X) \to \pi_1^{top}(X) \times \pi_1^{top}(X)$ is always a quotient map. In this chapter, we provide counterexamples to this claim by computing π_1^{top} on a class of suspension spaces that resemble "non-discrete wedges of circles."

4.1 The topological properties of π_1^{top}

Many of the results in Chapter 2 give immediate results concerning the topological nature of π_1^{top}. For instance, since S^1 is a cogroup-like, $\pi_1^{top} : \mathbf{Top}_* \to \mathbf{qTopGrp}$ is a functor to the category of quasitopological groups (Theorem 2.30) which factors through the homotopy category. There is an additional factorization: $\pi_1^{top} \cong \pi_0^{top}\Omega$. Specifically, the following diagram of functors commutes up to natural homeomorphism:



We call upon other results from Chapter 2 as needed.

Lemma 4.1 [Bis02, Prop. 3.2] *If $\gamma : I \to X$ is a path, then $h_\gamma : \pi_1^{top}(X, \gamma(1)) \to \pi_1^{top}(X, \gamma(0))$, $h_\gamma([\alpha]) = [\gamma * \alpha * \gamma^{-1}]$ is an isomorphism of quasitopological groups.*

Proof. Consider the concatenation functions $\Gamma : \Omega(X, \gamma(1)) \to \Omega(X, \gamma(0))$, $\Gamma(\alpha) = \gamma * \alpha * \gamma^{-1}$ and $\Gamma' : \Omega(X, \gamma(0)) \to \Omega(X, \gamma(1))$, $\Gamma'(\beta) = \gamma^{-1} * \beta * \gamma$. These are continuous as the restrictions of the more general concatenation functions from Lemma 1.21. Applying π_0^{top} gives the continuous homomorphisms $h_\gamma : \pi_1^{top}(X, \gamma(1)) \to \pi_1^{top}(X, \gamma(0))$ and $h_{\gamma^{-1}} : \pi_1^{top}(X, \gamma(0)) \to \pi_1^{top}(X, \gamma(1))$, $\Gamma'_*([\beta]) = [\gamma^{-1} * \beta * \gamma]$ respectively. These are continuous inverses of each other. ■

The following, is a convenient description of the topology of topological fundamental groups.

Corollary 4.2 *For every based space Y, the canonical monoid homomorphism*

$$g : M_T(\Omega(Y)) = \bigoplus_{n \geq 0} \Omega(Y)^n \to \pi_1^{top}(Y),\ g(\alpha_1 \alpha_2 ... \alpha_n) = [\alpha_1 * \alpha_2 * \cdots * \alpha_n]$$

on the free topological monoid on $\Omega(Y)$ is quotient.

Proof. For each $n \geq 1$, concatenation $C_n : \Omega(Y)^n \to \Omega(Y)$ is continuous 1.21. These induce a map $C : M_T(\Omega(Y)) \to \Omega(Y)$ (The disjoint identity $\Omega(Y)^0 = *$ is taken to the constant map). Let $\sigma : \Omega(Y) \hookrightarrow \Omega(Y)^1 \subset M_T(\Omega(Y))$ be the universal arrow and $\pi : \Omega(Y) \to \pi_1^{top}(Y)$ be the quotient map. It follows that $g = \pi \circ C$ is a monoid homomorphism which is quotient since $\pi = \pi \circ C \circ \sigma$. ■

It worthwhile to note the results of Chapter 2.2.2 that provide conditions to guarantee that $\pi_1^{top}(X)$ is a topological group.

Theorem 4.3 *Let X be a path connected space.*

1. *If $\Omega(X)$ and $\pi_1^{top}(X)$ are first countable, then $\pi_1^{top}(X)$ is a topological group.*

2. *If X is metrizable, then $\pi_1^{top}(X)$ is first countable (and T_1) if and only if $\pi_1^{top}(X)$ is a pseudometrizable (metrizable) topological group.*

3. *If X is a separable metrizable space, then $\pi_1^{top}(X)$ is second countable (and T_1) if and only if $\pi_1^{top}(X)$ is a separable pseudometrizable (separable metrizable) topological group.*

4. *If $\pi_1^{top}(X)$ is locally compact Hausdorff, then it is a topological group.*

Proof. 1. If $\Omega(X)$ and $\pi_1^{top}(X)$ are first countable, then $\pi \times \pi : \Omega(X) \times \Omega(X) \to \pi_1^{top}(X) \times \pi_1^{top}(X)$ is a quotient map by Fact 1.24. By Corollary 2.34, $\pi_1^{top}(X)$ is a topological group. 2. and 3. are special cases of Theorem 2.35. 4. is a special case of Theorem 2.36. ■

4.1.1 Attaching cells

One of the great conveniences of the quotient topology on the fundamental group is that attaching n-cells to a space changes the topology in a rather convenient way. The following lemma appears in [Bis02]. The proof included here is slightly longer but is more intuitive than Biss'. For the statement and proof, we fix an $\epsilon \in (0,1)$ and let $c^n = B^n - B^n(\epsilon) = \{x \in \mathbb{R}^n | \epsilon < |x| \leq 1\}$ so that $int(c^n) = c^n - S^{n-1}$. If $n > 2$, $int(c^n) \simeq S^{n-1}$ is 1-connected. If $n = 2, a, b > 0$, and $R = \{(at, bt) | t \in [0, \infty)\} \subset \mathbb{R}^2$ is any ray emanating from the origin, then $int(c^n) - R$ is 1-connected.

Lemma 4.4 *Suppose Z is a based space, $n \geq 2$ an integer, and $f : S^{n-1} \to Z$ is a based loop. Let $Z' = Z \cup_f B^n$ be the space obtained by attaching a n-cell to Z via the attaching map f. The inclusion $j : Z \hookrightarrow Z'$ induces a group epimorphism (isomorphism when $n > 2$) $j_* : \pi_1^{top}(Z) \to \pi_1^{top}(Z')$ which is also a topological quotient map.*

Proof. Clearly j_* is a continuous surjection for all $n \geq 2$ as well as a group isomorphism when $n > 2$. Therefore it suffices to show j_* is a quotient map for all integers $n \geq 2$. We re-label our spaces by letting $Z_1 = Z \subseteq Z_2 = Z \cup c^n \subset Z_3 = Z'$ so that Z_2 is Z_1 with an "open collar." Clearly, the inclusion $j_1 : Z_1 \hookrightarrow Z_2$ is a homotopy equivalence and induces an isomorphism $(j_1)_* : \pi_1^{top}(Z_1) \hookrightarrow \pi_1^{top}(Z_2)$ of quasitopological groups. Suppose $j_2 : Z_2 \hookrightarrow Z_3$ so that $j_2 \circ j_1 = j$. Since Z_2 is open in Z_3, the map $(j_2)_\# : \Omega(Z_2) \hookrightarrow \Omega(Z_3)$ induced on loop spaces is an open embedding. Suppose $U \subseteq \pi_1^{top}(Z_3)$ such that $j_*^{-1}(U) = (j_1)_*^{-1}((j_2)_*^{-1}(U))$ is open in $\pi_1^{top}(Z_1)$. Immediately, we have that $(j_2)_*^{-1}(U)$ is open in $\pi_1^{top}(Z_2)$ since $(j_1)_*$ is a homeomorphism. For $k = 1, 2, 3$, let $\pi_k : \Omega(Z_k) \to \pi_1^{top}(Z_k)$ be quotient map identifying homotopy classes of loops. It suffices to show that $\pi_3^{-1}(U)$ is open in $\Omega(Z_3)$. Let

$\alpha \in \pi_3^{-1}(U)$. If α has image entirely in Z_2, then $V = (j_2)_\#^{-1}(\pi_3^{-1}(U)) = \pi_2^{-1}((j_2)_*^{-1}(U))$ is an open neighborhood of α in $\Omega(Z_2)$. Since $(j_2)_\#$ is an open embedding, $(j_2)_\#(V)$ is an open neighborhood of α contained in $\pi_3^{-1}(U)$ and we are done. Therefore, we suppose $\alpha(I) \cap B^n(\epsilon) \neq \emptyset$ and take the open pullback $\alpha^{-1}(E^n) = \coprod_{m \in M}(c_m, d_m) \subset I$ noting that only finitely many of the restrictions $\alpha|_{[c_m,d_m]} : [c_m, d_m] \to Z_3$ have image intersecting $B^n(\epsilon)$. Suppose $\alpha|_{[c_{m_1},d_{m_1}]}, ..., \alpha|_{[c_{m_k},d_{m_k}]}$ correspond to these restrictions. For each $i = 1, ..., k$, we find closed intervals $[a_i, b_i] \subseteq (c_{m_i}, d_{m_i})$ such that $a_i < b_i$ are rational numbers and $\alpha((c_{m_i}, d_{m_i}) - (a_i, b_i)) \subset int(c^n)$. Let $C = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$ and $D = [0, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_{k-1}, a_k] \cup [b_k, 1]$ (here we use the convention that $[s, t]$ is the singleton, if $s = t$). In terms of the compact-open topology, we have $\alpha \in \langle C, E^n \rangle \cap \langle D, Z_2 \rangle$.

Clearly, there is a loop $\beta : I \to Z_2$ such that $\beta|_D = \alpha|_D$ and $\beta(C) \subseteq int(c^n)$. These two conditions imply that $\alpha \simeq j_2 \circ \beta$ when $n > 2$. When $n = 2$, we must be careful that none of the $\beta|_{[a_i,b_i]}$ "go around" $int(c^2) \simeq S^1$. In this case, we find a loop $\beta : I \to Z_2$ such that $\beta|_D = \alpha|_D$ and $\beta([a_i, b_i]) \subseteq int(c^2) - R_i$ where $R_i \subset \mathbb{R}^2$ is some ray emanating from the origin. With these choices of β we have that that $\alpha \simeq j_2 \circ \beta$ for all $n \geq 2$. Note that

$$\mathscr{V}_n = \begin{cases} \langle C, int(c^n) \rangle & n > 2 \\ \bigcap_{i=1}^{k} \langle [a_i, b_i], int(c^2) - R_i \rangle & n = 2 \end{cases}$$

is an open neighborhood of β in $\Omega(Z_2)$. Additionally, since $[\alpha] = [j_2 \circ \beta] = (j_2)_*([\beta]) \in U$, β lies in the open set $\pi_2^{-1}((j_2)_*^{-1}(U))$ of $\Omega(Z_2)$. As asserted in Corollary 1.20, basic open neighborhoods of loops in $\Omega(Z_2)$ may be taken to be of the form $\bigcap_{l=1}^{M} \langle K_M^l, U_l \rangle$

where $K_M^l = \left[\frac{l-1}{M}, \frac{l}{M}\right]$ and U_l is open in Z_2. We find such a neighborhood $\mathscr{V}$ so that

$$\beta \in \mathscr{V} = \bigcap_{l=1}^{M} \langle K_M^l, U_l \rangle \subseteq \pi_2^{-1}((j_2)_*^{-1}(U)) \cap \mathscr{V}_n.$$

We choose M large enough so that Ma_i, Mb_i are integers for all i, or in other words, so that for each l either $K_M^l \subset C$ or $K_M^l \subset D$. Since E^n is locally path connected, we may assume that $U_l \subseteq int(c^n) \subset E^n$ whenever $K_M^l \subset C$. For $n = 2$ and $K_M^l \subset [a_i, b_i]$, we may also assume that $U_l \subseteq int(c^2) - R_i$. Now we can easily find the desired open neighborhood of α in $\Omega(Z_3)$, namely:

$$\mathscr{U} = \left(\bigcap_{K_M^l \subseteq D} \langle K_M^l, U_l \rangle \right) \cap \langle C, E^n \rangle$$

It is clear that $\alpha \in \mathscr{U}$ since $\alpha|_D = \beta|_D$ and $\alpha(C) \subseteq E^n$. $\mathscr{U}$ is open in $\Omega(Z_3)$ since Z_2 is open in Z_3 (so each U_l is open in Z_3). Suppose $\gamma \in \mathscr{U} \subset \Omega(Z_3)$. Clearly, there is a loop $\delta : I \to Z_2$ such that $\gamma|_D = \delta|_D$ and $\delta \in \mathscr{V} \subseteq \pi_2^{-1}((j_2)_*^{-1}(U))$. In other words δ agrees with γ on D and $\delta(K_M^l) \subseteq U_l$ for all the intervals $K_M^l \subset C$. This implies $\delta \in \langle C, int(c^n) \rangle$ and when $n = 2$ we have $\delta \in \bigcap_{i=1}^{k} \langle [a_i, b_i], int(c^2) - R_i \rangle$. It is a basic fact that if $p_1, p_2 : I \to S$ are paths into a 1-connected space Y such that $p_1(i) = p_2(i)$, $i = 0, 1$, then there is an endpoint preserving homotopy of paths $p_1 \simeq p_2$ in S. This guarantees a homotopy of loops $\gamma \simeq j_2 \circ \delta$ in Z_3. Consequently, $[\gamma] = [j_2 \circ \delta] = (j_2)_*([\delta]) \in U$ proving the inclusion $\mathscr{U} \subseteq \pi_3^{-1}(U)$. ■

Lemma 4.5 *Suppose Z is a based space, $n \geq 2$ an integer, and $f_\alpha : S^{n-1} \to Z$, $\alpha \in A$*

is a family of based maps. Let $Z' = Z \cup_{f_\alpha} B^n_\alpha$ *be the space obtained by attaching n-cells to* Z *via the attaching maps* f_α*. The inclusion* $j : Z \hookrightarrow Z'$ *induces a group epimorphism (isomorphism when* $n > 2$*)* $j_* : \pi_1^{top}(Z) \to \pi_1^{top}(Z')$ *which is also a topological quotient map.*

Proof. Clearly j_* is a continuous surjection for all $n \geq 2$ as well as a group isomorphism when $n > 2$. Therefore it suffices to show j_* is a quotient map for all integers $n \geq 2$. We re-label $Z = Z_1$ and $Z' = Z_4$ and take the approach of factoring the inclusion $j : Z_1 \hookrightarrow Z_4$ twice as $Z_1 \subseteq Z_2 \subseteq Z_3 \subseteq Z_4$. We will let $\pi_k : \Omega(Z_k) \to \pi_1^{top}(Z_k)$, $k = 1, 2, 3, 4$ denote the quotient maps identifying homotopy classes of maps. Consider the commutative diagram

$$\begin{array}{ccc} \Omega(Z_1) & \xrightarrow{j_\#} & \Omega(Z_4) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_4} \\ \pi_1^{top}(Z_1) & \xrightarrow[j_*]{} & \pi_1^{top}(Z_4) \end{array}$$

and suppose $U \subseteq \pi_1^{top}(Z_4)$ such that $j_*^{-1}(U)$ is open in $\pi_1^{top}(Z_1)$. It suffices to show that $\pi_4^{-1}(U)$ is open in $\Omega(Z_4)$ so we suppose $\beta \in \pi_4^{-1}(U)$. Since the image $\beta(I)$ is compact in Z_4 it may intersect only finitely many of the attached cells. Suppose $\alpha_1, ..., \alpha_N$ are the indices in A such that $\beta(I) \cap E^n_{\alpha_i} \neq \emptyset$. Let $Z_2 = Z_1 \cup_{\alpha_i} B^n_{\alpha_i} \subseteq Z_4$ be the subspace of Z_4 which is Z_1 with the cells $B^n_{\alpha_1}, ...B^n_{\alpha_N}$ attached. Additionally, for each $\alpha \in A - \{\alpha_1, ...\alpha_N\}$ we take a point $z_\alpha \in int(B^n)$ and let $Z_3 = Z_4 - \{z_\alpha | \alpha \in A - \{\alpha_1, ...\alpha_N\}\}$ be the open subspace of Z_4 with the chosen interior points removed. We know from Lemma 4.4 that the inclusion $j_1 : Z_1 \hookrightarrow Z_2$ induces a quotient map $(j_1)_* : \pi_1^{top}(Z_1) \to \pi_1^{top}(Z_2)$ since Z_2 is obtained from Z_1 by attaching only finitely many n-cells. Also the inclusion $j_2 : Z_2 \hookrightarrow Z_3$ is a homotopy equivalence and therefore induces an isomorphism of quasitopological groups $(j_2)_* : \pi_1^{top}(Z_2) \to \pi_1^{top}(Z_3)$. Lastly, since I is compact and Z_3 is open in Z_4, the inclusion $j_3 : Z_3 \hookrightarrow Z_4$ induces an open embedding $(j_3)_\# : \Omega(Z_3) \hookrightarrow \Omega(Z_4)$ on loop spaces. We now have that $j_3 \circ j_2 \circ j_1 = j$ and $(j_2 \circ j_1)_* = (j_2)_* \circ (j_1)_* : \pi_1^{top}(Z_1) \to \pi_1^{top}(Z_3)$ is a quotient map. The equality

$$j_*^{-1}(U) = (j_2 \circ j_1)_*^{-1}((j_3)_*^{-1}(U))$$

then implies that $(j_3)_*^{-1}(U)$ is open in $\pi_1^{top}(Z_3)$. Therefore, $V = \pi_3^{-1}((j_3)_*^{-1}(U)) = (j_3)_\#^{-1}(\pi_4^{-1}(U))$ is an open neighborhood of β in $\Omega(Z_3)$. Since $(j_3)_\# : \Omega(Z_3) \hookrightarrow \Omega(Z_4)$ is an open embedding, $(j_3)_\#(V)$ is an open neighborhood of β in $\Omega(Z_4)$. If $\gamma \in (j_3)_\#(V)$, then we have a loop $\gamma' \in V$ such that $j_3 \circ \gamma' = \gamma$. But this means $[\gamma'] \in (j_3)_*^{-1}(U)$, so that $[\gamma] = [j_3 \circ \gamma'] \in U$ and consequently $\gamma \in \pi_4^{-1}(U)$. This proves the inclusion $(j_3)_\#(V) \subseteq \pi_4^{-1}(U)$ and that $\pi_4^{-1}(U)$ is open in $\Omega(Z_4)$. ■

4.1.2 Discreteness and separation

In general, it is difficult to determine if the fundamental group of a space is a topological group. There are, however, instances when it is easy to answer in the affirmative, namely those spaces X for which $\pi_1^{top}(X)$ has the discrete topology. The following proposition is a consequence of 2.51 and 2.55.

Proposition 4.6 *For any based space (X, x), the following are equivalent:*

1. *$\pi_1^{top}(X)$ is a discrete group.*

2. *$\Omega(X)$ is semilocally 0-connected.*

3. *The singleton $\{[c_x]\}$ containing the identity (homotopy class of the constant loop) is open in $\pi_1^{top}(X)$.*

4. *Each null-homotopic loop $\alpha \in \Omega(X)$ has an open neighborhood containing only null-homotopic loops.*

These obvious characterizations are inconvenient in that they do not characterize discreteness in terms of the topological properties of X itself. The next theorem was proved independently in [CM09] and is a consequence of the general results in Chapter 2.4. Unfortunately, the general statements in Chapter 2 depend on Wada's proofs in [Wad54] which are omitted here. A direct proof of discreteness is greatly simplified in the case of the fundamental group and so it is provided here. This particular proof also appears in the independent work of [CM09].

Theorem 4.7 *Suppose X is path connected. If $\pi_1^{top}(X)$ is discrete, then X is semilocally 1-connected. If X is locally path connected and semilocally 1-connected, then $\pi_1^{top}(X)$ is discrete.*

Proof. We suppose $x \in X$ and by Lemma 4.1 may assume that $\pi_1^{top}(X)$ is discrete or equivalently that $\Omega(X)$ is semilocally 0-connected. This allows us to find open neighborhood W of the constant loop c_x in $\Omega(X)$ such that $\alpha \simeq c_x$ for each $\alpha \in W$. There is an open neighborhood U of x in X such that $c_x \in \langle S^1, U\rangle \subseteq W$. Since every loop $\alpha \in \langle S^1, U\rangle$ is null-homotopic in X, the inclusion $i : U \hookrightarrow X$ induces the trivial homomorphism $i_* : \pi_1(U, x) \to \pi_1(X, x)$. Thus X is semilocally 1-connected.

To prove the converse, suppose X is locally path connected and semilocally 1-connected and that $\alpha \in M((I,\{0,1\}),(X,\{x\}))$. We find an open neighborhood of α in $M((I,\{0,1\}),(X,\{x\}))$ containing only loops homotopic to α in X. This suffices to show that $\Omega(X)$ is semilocally 0-connected. For each $t \in I$, we find an open neighborhood U_t of $\alpha(t)$ in X such that the inclusion $u_t : U_t \hookrightarrow X$ induces the trivial homomorphism $(u_t)_* : \pi_1(U_t, \alpha(t)) \to \pi_1(X, \alpha(t))$. We then find a path connected, open neighborhood V_t of $\alpha(t)$ contained in U_t. Take a finite subcover $\{V_{t_1}, ..., V_{t_k}\}$ of $\alpha(I)$ and finite subdivisions of I to find an integer $m \geq 1$ such that $\alpha \in \bigcap_{j=1}^{m}\langle K_m^j, V_j\rangle$ where $V_j = V_{t_{i_j}}$ for not necessarily distinct $i_j \in \{1, ..., k\}$. For $j = 0, ..., m$, let $s_j = \frac{j}{m} \in I$. For each $j = 1, ..., m-1$ we have $\alpha(s_j) \in V_j \cap V_{j+1}$ and find a path connected, open neighborhood W_j such that $\alpha(s_j) \in W_j \subseteq V_j \cap V_{j+1}$. Now

$$\mathscr{U} = \bigcap_{j=1}^{m}\langle K_m^j, V_j\rangle \cap \bigcap_{j=1}^{m-1}\langle\{s_j\}, W_j\rangle$$

is an open neighborhood of α in $M((I,\{0,1\}),(X,\{x\}))$. We suppose $\gamma \in \mathscr{U}$ and construct a homotopy to α. We have $\gamma(s_j) \in W_j$ for $j = 1, ..., m-1$ allowing us to find paths $p_j : I \to W_j$ such that $p_j(0) = \alpha(s_j)$ and $p_j(1) = \gamma(s_j)$. Let $p_0 = p_m = c_x$ be the constant path at x. We now make use of our notation for restricted paths. For $j = 1, ..., m$ we define loops $\beta_j : I \to V_j$ based at $\alpha(s_{j-1})$ as the concatenations

$$\beta_j = p_{j-1} * \gamma_{K_m^j} * p_j^{-1} * \alpha_{K_m^j}^{-1}$$

Recall that $V_j = V_{t_{i_j}}$ where $\alpha(t_{i_j}) \in V_j$. Since V_j is path connected, the points

$\alpha(s_{j-1})$ and $\alpha(t_{i_j})$ lie in the same path component of U_j. Therefore the inclusion $u_j : U_j \hookrightarrow X$ induces the trivial homomorphism $(u_j)_* : \pi_1(U_j, \alpha(s_{j-1})) \to \pi_1(X, \alpha(s_{j-1}))$. Consequently, each loop β_j is homotopic (in X) to the constant loop at $\alpha(s_{j-1})$. The homotopies of loops $\beta_j \simeq c_{\alpha(s_{j-1})}$ give fixed endpoint homotopies of paths $\alpha_{K^j_m} \simeq p_{j-1} * \gamma_{K^j_m} * p_j^{-1}$. Now we have concatenations of homotopies

$$\alpha \simeq *_{j=1}^{m} \alpha_{K^j_m} \simeq *_{j=1}^{m} \left(p_{j-1} * \gamma_{K^j_m} * p_j^{-1}\right) \simeq p_0 * \left(*_{j=1}^{m} \gamma_{K^j_m}\right) * p_m^{-1} \simeq p_0 * \gamma * p_m^{-1} \simeq \gamma$$

This proves that $\mathscr{U}$ contains only loops homotopic to α in X, or in other words that the inclusion $\mathscr{U} \hookrightarrow M((I, \{0,1\}), (X, \{x\}))$ induces the constant function on path components. ■

Since a locally path connected space has a universal cover if and only if it is semilocally 1-connected, we have:

Corollary 4.8 *Let X be path connected and locally path connected. Then X is semilocally 1-connected $\Leftrightarrow \pi_1^{top}(X)$ is discrete $\Leftrightarrow X$ has a universal cover.*

Proposition 4.9 *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of spaces and $X = \prod_\lambda X_\lambda$ be the product space. Then $\pi_1^{top}(X)$ is discrete if and only if $\pi_1^{top}(X_\lambda)$ is discrete for each $\lambda \in \Lambda$ and $\pi_1^{top}(X_\lambda) = 0$ for all but finitely many $\lambda \in \Lambda$.*

Proof. Suppose $\pi_1^{top}(X_\lambda)$ is discrete for each $\lambda \in \Lambda$ and $\pi_1^{top}(X_\lambda) \neq 0$ only for $\lambda \in F$ where $F \subset \Lambda$ is finite. Then $\prod_{\lambda \in F} \pi_1^{top}(X_\lambda)$ is discrete. The continuity of the bijection $\phi : \pi_1^{top}(X) \to \prod_\lambda \pi_1^{top}(X_\lambda) \cong \prod_{\lambda \in F} \pi_1^{top}(X_\lambda)$ implies that $\pi_1^{top}(X)$ is discrete. Now suppose $\pi_1^{top}(X)$ is discrete. The projections $p_\lambda : X \to X_\lambda$ are retractions and induce retractions $(p_\lambda)_* : \pi_1^{top}(X) \to \pi_1^{top}(X_\lambda)$. But every retract of a discrete space is discrete. Therefore $\pi_1^{top}(X_\lambda)$ is discrete for each $\lambda \in \Lambda$. By 4.7, X is semilocally 1-connected. A basic neighborhood of a point in X is of the form

$$U = \prod_{\lambda \in F} U_\lambda \times \prod_{\lambda \in \Lambda - F} X_\lambda$$

where $F \subset \Lambda$ is finite and U_λ is open X_λ. But the inclusion $U \hookrightarrow X$ does not induces the trivial homomorphism on fundamental groups if $\pi_1(X_\lambda) \neq 0$ for infinitely many λ. Since this was for arbitrary U, we must have $\pi_1^{top}(X_\lambda) = 0$ for all but finitely many $\lambda \in \Lambda$. ■

Proposition 4.10 *Let $\{X_\lambda\}$ be a family of spaces such that $\pi_1^{top}(X_\lambda)$ is discrete, $X = \prod_\lambda X_\lambda$, and $p_\lambda : X \to X_\lambda$ be the projections. The natural map $\phi : \pi_1^{top}(X) \to \prod_\lambda \pi_1^{top}(X_\lambda)$, $\phi([f]) = ([p_\lambda \circ f])$ is an isomorphism of topological groups.*

Proof. It is a basic fact of algebraic topology that ϕ is a natural group isomorphism. By Lemma 1.6, $\psi : \Omega(X) \to \prod_\lambda \Omega(X_\lambda)$, $\psi(f) = (p_\lambda \circ f)$ is a homeomorphism. Applying π_0^{top} we get a homeomorphism $\pi_1^{top}(X) \to \pi_0^{top}(\prod_\lambda \Omega(X_\lambda))$. Since each $\pi_0^{top}(\Omega(X_\lambda)) = \pi_1^{top}(X_\lambda)$ is discrete, Corollary 3.12 applies and the canonical map $\pi_0^{top}(\prod_\lambda \Omega(X_\lambda)) \to \prod_\lambda \pi_1^{top}(X_\lambda)$ is a homeomorphism. Taking composites, we obtain the desired homeomorphism $\pi_1^{top}(X) \cong \prod_\lambda \pi_1^{top}(X_\lambda)$. ■

Example 4.11 If X is the countable product $X = \prod_{n \geq 1} S^1$, then $\pi_1^{top}(X)$ is isomorphic to the non-discrete topological group $\prod_{n\geq 1} \mathbb{Z}$ which is the countable product of discrete free cyclic groups. Interestingly, this space is not semilocally 1-connected.

Corollary 4.12 *If $\{X_\lambda\}$ is a (countable) family of spaces where each X_λ has the homotopy type of a CW-complex or manifold, then $\pi_1^{top}(\prod_\lambda X_\lambda)$ is a (metrizable) topological group.*

It is often difficult to determine the existence of separation properties in quasitopological groups. These complications are evident even in simple examples. In [Bis02] and [Fab05a], it is shown that the harmonic archipelago $\mathbb{HA}$ (a non-compact subspace of $\mathbb{R}^3$), introduced in [BS98], satisfies: $\pi_1^{top}(\mathbb{HA})$ is an uncountable, indiscrete group. The next example is a simple metric space with fundamental group isomorphic to the indiscrete group of integers.

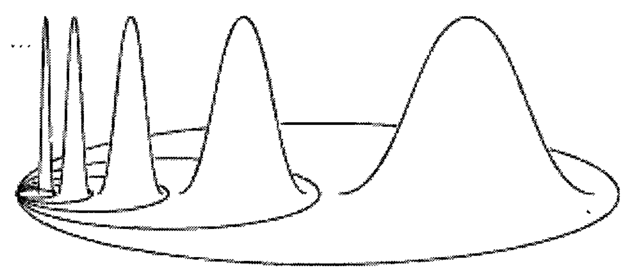


Figure 2: *The harmonic archipelago [BS98]*

Example 4.13 Let $S^1 = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$ be the unit circle in the xy-plane of $\mathbb{R}^3$. For all integers $n \geq 1$ let

$$C_n = \left\{(x, y, z) \in \mathbb{R}^3 | \left(x - \frac{1}{n}\right)^2 + y^2 + z^2 = \left(1 + \frac{1}{n}\right)^2\right\}$$

Now let $X = S^1 \cup (\bigcup_{n\geq 1} C_n)$ have basepoint $(-1, 0, 0)$. This may be viewed as a sequence of spheres whose equators converge to a circle where the circle and spheres all have exactly one point in common. This is a non-compact space and is weakly equivalent to the wedge of spheres $S^1 \vee (\bigvee_{n\geq 1} S^2)$ (which has discrete fundamental group). We have $\pi_1(X) \cong \mathbb{Z}$, however, every open neighborhood W of a loop $\alpha : S^1 \to S^1 \subset X$ contains a loop $\beta : S^1 \to \bigcup_{n\geq 1} C_n \subset X$ which is null homotopic. Therefore every open neighborhood of the class $[\alpha]$ in $\pi_1^{top}(X)$ contains the identity $[c_x]$. Thus $\overline{[c_x]} = \overline{[\alpha]}$ for each $[\alpha] \in \pi_1^{top}(X)$ and since every open neighborhood of $[c_x]$ contains $\overline{[c_x]} = \pi_1^{top}(X)$ the topology of $\pi_1^{top}(X)$ is the indiscrete topology. This example also illustrates how weakly equivalent spaces may have fundamental groups with non-isomorphic topological structure.

Since there are simple spaces with non-trivial, indiscrete fundamental group, we cannot take any separation properties for granted. The following is a simple characterization of spaces X for which $\pi_1^{top}(X)$ is T_1.

Proposition 4.14 *Suppose (X, x) is path connected and $\pi : \Omega(X) \to \pi_1^{top}(X)$ is the canonical quotient map. The following are equivalent:*

1. *Whenever $\alpha, \beta \in \Omega(X)$ such that $[\alpha] \neq [\beta]$, there are open neighborhoods A, B of α, β respectively, such that $\pi(A) \cap \pi(B) = \emptyset$.*

2. *For each loop $\alpha \in \Omega(X)$ which is not null-homotopic, there is an open neighborhood V of α such that V contains no null-homotopic loops.*

3. *The singleton containing the identity is closed in $\pi_1^{top}(X)$.*

4. *$\pi_1^{top}(X)$ is T_0.*

5. *$\pi_1^{top}(X)$ is T_1.*

Proof. 2. $\Leftrightarrow$ 3. follows from the definition of the quotient topology and 3. $\Leftrightarrow$ 4. $\Leftrightarrow$ 5. holds for all quasitopological groups (A.27). For 1. $\Rightarrow$ 2. suppose $\alpha \in \Omega(X)$ such that $\alpha \not\simeq c_x$. Then there is a neighborhood A of α and $B = \langle I, U\rangle$ of c_x such that $\pi(A) \cap \pi(B) = \emptyset$. Clearly $[c_x] \notin \pi(A)$ and so A contains no null-homotopic loops. Finally, to prove 2. $\Rightarrow$ 1. we suppose $\alpha, \beta \in \Omega(X)$ such that $[\alpha] \neq [\beta]$. Clearly, then $\alpha * \beta^{-1} \not\simeq c_x$. By assumption, there is an open neighborhood $V = \bigcap_{j=1}^{n}\langle K_n^j, U_j\rangle$ of $\alpha * \beta^{-1}$ containing no null-homotopic loops. We may assume that n is even. Now $A = V_{[0,\frac{1}{2}]}$ is a neighborhood of α and $B = V^{-1}_{[\frac{1}{2},1]}$ is a neighborhood of β. Suppose $\delta \in A$ and $\gamma \in B$. Since $\delta * \gamma^{-1} \in A^{[0,\frac{1}{2}]} \cap \left(B^{[\frac{1}{2},1]}\right)^{-1} = V$, the loop $\delta * \gamma^{-1}$ cannot be null-homotopic and therefore $[\delta] \neq [\gamma]$. Therefore $\pi(A) \cap \pi(B) = \emptyset$. ∎

Now we relate a modern concept (apparently introduced in [Zas99]) useful in the study of wild spaces to separation in topological fundamental groups.

Definition 4.15 A space X is *homotopy Hausdorff* at $x \in X$ if for each non-trivial class $g \in \pi_1(X, x)$, there is an open neighborhood U of x in X such that if $i : U \hookrightarrow X$ is the inclusion, then $g \notin i_*(\pi_1(U, x))$ (or equivalently U contains no loops $\alpha \in \Omega(U, x)$ with $[\alpha] = g$). If X is homotopy Hausdorff at all of its points then we say it is *homotopy Hausdorff*.

This notion also appears in [FZ07] and [BS98] and is useful for studying generalized universal covering spaces of locally path connected spaces. The term "Hausdorff" is appropriate because X is homotopy Hausdorff if and only if its generalized cover (in the sense of Fischer and Zastrow) is Hausdorff. It turns out this property is also a necessary condition for the existence of the T_1 separation axiom in $\pi_1^{top}(X)$.

Proposition 4.16 *If X is path connected and $\pi_1^{top}(X)$ is T_1, then X is homotopy Hausdorff.*

Proof. Suppose $\pi_1^{top}(X)$ is T_1 and X is not homotopy Hausdorff at $x \in X$. There is a non-trivial class $[\alpha] \in \pi_1^{top}(X)$ such for every open neighborhood U of x, there is a loop $\delta : I \to U$ based at x such that $[\delta] = [\alpha]$. Since $\pi_1^{top}(X)$ is T_1, if α represents $[\alpha]$, then there is an open neighborhood $\bigcap_{j=1}^{n}\langle K_n^j, U_j\rangle$ of α not containing any null homotopic loops. But U_1 is an open neighborhood of x and so there is a loop $\delta : I \to U_1$ based at x such that $[\delta] = [\alpha]$. Let $\beta : I \to X$ be the loop defined by $\beta_{K_{2n}^1} = \delta^{-1}$, $\beta_{K_{2n}^2} = \alpha_{K_n^1}$, and $\beta_{K_n^j} = \alpha_{K_n^j}$ for $j = 2, ..., n$. Now $\beta \in \bigcap_{j=1}^{n}\langle K_n^j, U_j\rangle$ and $[\beta] = [\delta^{-1} * \alpha] = [c_{x_0}]$ which is a contradiction. ■

Proposition 4.17 *The converse of Proposition 4.16 is not true even when X is a compact, locally path connected subspace of $\mathbb{R}^3$ or a compact, locally 1-connected subspace of $\mathbb{R}^3$.*

Proof. For the second statement, we refer to the space $A \subset \mathbb{R}^3$ in [CMR⁺08]. This space is locally path connected and homotopy Hausdorff but $\pi_1^{top}(A)$ is not T_1. Clearly every neighborhood of constant loop at the origin contains a loop homotopic to aba^{-1} as in [CMR⁺08, Corollary 3.2], where a has image in a fixed "connecting arc" touching the origin and b is an embedding $S^1 \to A$ of constant radius on the "surface." Since all such loops are homotopic, every neighborhood of the identity of $\pi_1^{top}(A)$ contains the class $[aba^{-1}]$ and so $\pi_1^{top}(A)$ is not T_1. For the second example we refer to the results to follow in Chapter 4.2. Take $T_C \subset \mathbb{R}^2$ to be the closed topologist sine curve so that $\pi_0^{top}(T_C)$ is homeomorphic to the Sierpinski space

$\mathbb{S} = \{0, 1\}$ with topology $\{\emptyset, \{1\}, \{0, 1\}\}$. The space $X = T_C \times S^1 / T_C \times \{(1, 0)\} \cong \Sigma((T_C)_+)$ is not locally path connected, but is compact, locally 1-connected, and embeds as a subspace of $\mathbb{R}^3$. The results of Chapter 4.2 indicate that $\pi_1^{top}(X)$ is a topological group isomorphic to the free topological group $F_M(\mathbb{S})$. As a group, this is the free product $\mathbb{Z} * \mathbb{Z} = \langle 0, 1 \rangle$. We also have that every open neighborhood of the generator 0 contains the generator 1. Clearly this group is not T_1 but X is homotopy Hausdorff since it is locally 1-connected. ■

The notion of homotopy Hausdorff also provides application to characterizations of discreteness. It is shown in [FZ07, 4.6] that if X is path connected, first countable, homotopically Hausdorff, and $\pi_1(X)$ is countable, then X is semilocally 1-connected. Adding the condition that X be locally path connected and applying 4.7 and 4.16, we obtain the following:

Corollary 4.18 *Let X be path connected, first countable, locally path connected, and homotopy Hausdorff. If $\pi_1(X)$ is countable, then $\pi_1^{top}(X)$ is discrete.*

It is well known that if X is compact Hausdorff, path connected, locally path connected, and semilocally 1-connected, then $\pi_1(X)$ is finitely generated. Adding the condition of compactness to the previous corollary, we find:

Corollary 4.19 *Let X be path connected, compact Hausdorff, first countable, locally path connected, and homotopy Hausdorff. Then $\pi_1(X)$ is finitely generated if and only if $\pi_1^{top}(X)$ is discrete.*

The following is an extension of [CL05, Theorem 2.1] using 4.7.

Corollary 4.20 *If X is a path connected, locally path connected, separable metric space, then X admits a universal cover $\Leftrightarrow$ X is homotopy Hausdorff and $\pi_1(X)$ is countable $\Leftrightarrow$ $\pi_1^{top}(X)$ is countable and T_1 $\Leftrightarrow$ $\pi_1^{top}(X)$ is discrete.*

As the continuity of multiplication is critical in proving that every T_0 topological group is Tychonoff, it can be difficult to recognize separation properties T_i, $i \geq 2$ in quasitopological groups. Additionally, the complex nature of homotopy as an equivalence relation further complicates our attempt to characterize stronger separation properties in fundamental groups with the quotient topology. To be able to make any general statement for when $\pi_1^{top}(X)$ is Hausdorff, it is necessary to use the basis constructed for arbitrary quotient spaces in Chapter 1.2. We apply this construction to the quotient map $\pi : \Omega(X) \to \pi_1^{top}(X)$.

Proposition 4.21 *For a path connected, based space (X, x), $\pi_1^{top}(X)$ is Hausdorff if and only if for each class $[\beta] \in \pi_1^{top}(X) - \{[c_x]\}$, there is a pointwise open covering $\mathscr{U} \in Cov(\Omega(X))$ such that $\mathscr{O}_\pi([c_x], \mathscr{U}) \cap \mathscr{O}_\pi([\beta], \mathscr{U}) = \emptyset$.*

Proof. If $\pi_1^{top}(X)$ is Hausdorff and $[\beta] \in \pi_1^{top}(X) - \{[c_x]\}$, we can find disjoint open neighborhoods W of $[c_x]$ and V of $[\beta]$. Now we may find pointwise open coverings $\mathscr{W} = \{W^\alpha\}_{\alpha \in \Omega(X)}, \mathscr{V} = \{V^\alpha\}_{\alpha \in \Omega(X)} \in Cov(\Omega(X))$ such that $\mathscr{O}_\pi([c_x], \mathscr{W}) \subset W$ and $\mathscr{O}_\pi([\beta], \mathscr{V}) \subset V$. We let $\mathscr{W} \cap \mathscr{V} = \{W^\alpha \cap V^\alpha\}_{\alpha \in \Omega(X)} \in Cov(\Omega(X))$ be the intersection of the two. Since $\mathscr{W}, \mathscr{V} \leq \mathscr{W} \cap \mathscr{V}$, we have $\mathscr{O}_\pi([c_x], \mathscr{W} \cap \mathscr{V}) \subseteq \mathscr{O}_\pi([c_x], \mathscr{W}) \subset W$ and $\mathscr{O}_\pi([\beta], \mathscr{W} \cap \mathscr{V}) \subseteq \mathscr{O}_\pi([\beta], \mathscr{V}) \subset V$. To prove the converse, we suppose that $[\beta_1]$ and $[\beta_2]$ are distinct classes in $\pi_1^{top}(X)$. Therefore $[\beta_1 * \beta_2^{-1}] \neq [c_x]$ and by assumption there is a $\mathscr{U} \in Cov(\Omega(X))$ such that $\mathscr{O}_\pi([c_x], \mathscr{U}) \cap \mathscr{O}_\pi([\beta_1 * \beta_2^{-1}], \mathscr{U}) = \emptyset$.

Since right multiplication by $[\beta_2]$ is a homeomorphism, we have that $\mathcal{O}_\pi([\beta_1 * \beta_2^{-1}], \mathcal{U})[\beta_2]$ is open containing $[\beta_1]$ and $\mathcal{O}_\pi([c_x], \mathcal{U})[\beta_2]$ is open containing $[\beta_2]$. But $\left(\mathcal{O}_\pi([\beta_1 * \beta_2^{-1}], \mathcal{U})[\beta_2]\right) \cap (\mathcal{O}_\pi([c_x], \mathcal{U})[\beta_2]) = \emptyset$ and so $\pi_1^{top}(X)$ is Hausdorff. ∎

Though this proposition is entirely general, it is quite difficult to apply. We obtain a more practical approach when we apply results from shape theory. The topological shape homotopy groups $\check{\pi}_n^{top}(X)$ defined in Chapter 2.5.3 are Hausdorff topological groups. The following is also noted:

Theorem 4.22 *If the canonical, continuous homomorphism $\Phi : \pi_1^{top}(X) \to \check{\pi}_1^{top}(X)$ is injective, then $\pi_1^{top}(X)$ is a functionally Hausdorff quasitopological group.*

Some recent results on the injectivity of $\Phi : \pi_1^{top}(X) \to \check{\pi}_1^{top}(X)$ include [CC06, EK98, FZ05, FG05]. Perhaps most notably, Φ is injective when X is a 1-dimensional compact Hausdorff space or an arbitrary subspace of $\mathbb{R}^2$. The converse of Theorem 4.22 is false.

Example 4.23 Consider the path connected, semilocally 1-connecetd but non-locally path connected space (Z^+, z_0) of [FG05, Example 2.4]. It is easy to see that $\pi_1^{top}(Z^+) \cong \pi_1^{\tau}(Z^+) \cong \mathbb{Z}$ is discrete, free cyclic (and therefore functionally Hausdorff). Fischer and Guilbault note $\check{\pi}_1(Z^+) \cong \mathbb{Z}$ but that $\Phi : \pi_1(Z^+) \to \check{\pi}_1(Z^+)$ is the trivial homomorphism.

Example 4.24 For integer $n \geq 1$, let $C_n \subset \mathbb{R}^2$ be the circle of radius $\frac{1}{n}$ centered at $\left(\frac{1}{n}, 0\right)$. The one point union $\mathbb{HE} = \bigcup_{n\geq 1} C_n$ is the *Hawaiian earring* and is one of the most fundamental examples in the study of π_1^{top}.

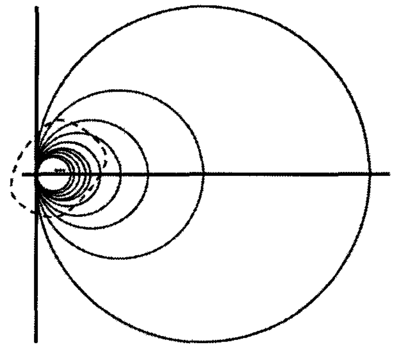


Figure 3: *The Hawaiian earring. Any open neighborhood of the origin contains all but finitely many of the C_n.*

It was first proven in [MM86] that $\Phi : \pi_1^{top}(\mathbb{HE}) \to \check{\pi}_1^{top}(\mathbb{HE})$ is injective. Though it is asserted in [Bis02] that Φ is a topological embedding, Fabel [Fab05b] has shown this to be false. Fabel has also shown that $\pi_1^{top}(\mathbb{HE})$ is not first countable [Fab06] and fails to be a topological group [Fab09].

In general, determining when topological fundamental groups are regular or Tychonoff remains a challenging problem. As noted in [Fab09], it is not even known if $\pi_1^{top}(\mathbb{HE})$ is regular.

4.1.3 Covering spaces and π_1^{top}

We strengthen Theorem 4.7 through application of covering spaces. A *covering map* is an open surjection $p : \tilde{X} \to X$ such that for each $x \in X$, there is an *evenly covered* neighborhood U of x, i.e. $p^{-1}(U) = \coprod_{\lambda} V_{\lambda}$ such that for each λ, the restriction $V_{\lambda} \to U$ of p is a homeomorphism. The space $\tilde{X}$ is a *cover* of X and we will always assume covers are path connected. A covering map is *trivial* if it is a homeomorphism. We refer to [Mun00] for basic facts regarding covering maps. It is easy to see that the collection of neighborhoods of the form V_{λ} form a basis

$\mathscr{B}_p$ for the topology of $\tilde{X}$ which is closed under finite intersection. This means the neighborhoods $\bigcap_{j=1}^{n}\langle K_n^j, V_j\rangle$, $V_j \in \mathscr{B}_p$ form a basis for the topology of the space of paths $P(\tilde{X})$. Recall that $P(X,x) = \{\alpha \in P(X) | \alpha(0) = x\}$ is the space of paths starting at x.

Lemma 4.25 *If $p : \tilde{X} \to X$ is a covering map and $p(\tilde{x}) = x$, then $p_\# : \Omega(\tilde{X}, \tilde{x}) \to \Omega(X, x)$, $\tilde{\alpha} \mapsto p \circ \tilde{\alpha} = \alpha$ is an open embedding.*

Proof. Note that $p_\#$ is continuous by functorality and injective by the uniqueness of lifts. Let $U = \bigcap_{j=1}^{n}\langle K_n^j, U_j\rangle \cap \Omega(\tilde{X}, \tilde{x})$ be a non-empty open neighborhood in $\Omega(\tilde{X}, \tilde{x})$ where each $U_j \in \mathscr{B}_p$. Clearly $p_\#(U) \subseteq \bigcap_{j=1}^{n}\langle K_n^j, p(U_j)\rangle \cap \Omega(X, x)$. Since U is non-empty, there is some $\tilde{\alpha} \in U$ that is the lift of $\alpha = p \circ \tilde{\alpha} \in \bigcap_{j=1}^{n}\langle K_n^j, p(U_j)\rangle \cap \Omega(X, x)$. The lift $\tilde{\alpha}$ is defined as follows: There is a homeomorphism $h_j : p(U_j) \to U_j$ such that $p \circ h_j$ is the identity of $p(U_j)$. For each $t \in K_n^j$, we have $\tilde{\alpha}(t) = h_j \circ \alpha(t)$. Note that $U_{j-1} \cap U_j$ is non-emtpy (since $\tilde{\alpha} \in U$) and evenly covers $p(U_{j-1}) \cap p(U_j)$. Therefore, if β is any other loop in $\bigcap_{j=1}^{n}\langle K_n^j, p(U_j)\rangle \cap \Omega(X, x)$, the unique lift $\tilde{\beta} \in P(\tilde{X}, \tilde{x})$ is defined in the same way, that is, for each $t \in K_n^j$, $\tilde{\beta}(t) = h_j \circ \beta(t)$. Since $\tilde{\beta}(1) \in p^{-1}(x) \cap U_n = \{\tilde{x}\}$, $\tilde{\beta}$ is a loop in U. Therefore $\beta \in p_\#(U)$ giving the equality $p_\#(U) = \bigcap_{j=1}^{n}\langle K_n^j, p(U_j)\rangle \cap \Omega(X, x)$.
■

The next few results indicate that the data of path connected covers of an arbitrary space are captured as special open subgroups of $\pi_1^{top}(X)$.

Theorem 4.26 *If $p : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering map, the induced homomorphism $p_* : \pi_1^{top}(\tilde{X}, \tilde{x}) \to \pi_1^{top}(X, x)$ is an open embedding of quasitopological groups.*

Proof. It is known that p_* is injective [Mun00, Theorem 54.6] and p_* is continuous by the functorality of π_1^{top}. Suppose U is an open neighborhood in $\pi_1^{top}(\tilde{X},\tilde{x})$. The diagram

$$\begin{array}{ccc} \Omega(\tilde{X},\tilde{x}) & \xrightarrow{p_\#} & \Omega(X,x) \\ \pi \downarrow & & \downarrow \pi \\ \pi_1^{top}(\tilde{X},\tilde{x}) & \xrightarrow[p_*]{} & \pi_1^{top}(X,x) \end{array}$$

commutes. Since $\pi : \Omega(X,x) \to \pi_1^{top}(X,x)$ is quotient, it suffices to show that $\pi^{-1}(p_*(U))$ is open in $\Omega(X,x_0)$. If $\alpha \in \pi^{-1}(p_*(U))$, $[\alpha]$ lies in the image of p_* and the unique lift $\tilde{\alpha} \in P(\tilde{X},\tilde{x})$ of α is a loop in $\Omega(\tilde{X},\tilde{x})$ [Mun00, Theorem 54.6]. Since $\tilde{\alpha} \in \pi^{-1}(U)$ and $p_\#$ is an open embedding (Lemma 4.25), $p_\#(\pi^{-1}(U))$ is an open neighborhood of $p_\#(\tilde{\alpha}) = p \circ \tilde{\alpha} = \alpha$ which is clearly contained in $\pi^{-1}(p_*(U))$. ■

Theorem 4.26 immediately provides a characterization of discreteness which is more general than that of Theorem 4.7 and [CM09] since it applies to many non-locally path connected spaces with 1-connected covers.

Corollary 4.27 *If X admits a path connected, 1-connected cover, then $\pi_1^{top}(X)$ is discrete.*

Proof. If $p : \tilde{X} \to X$ is a covering map and $\pi_1(\tilde{X}) = 1$, the inclusion $1 \to \pi_1^{top}(X)$ of the identity is an open embedding. Since the singleton containing the identity in $\pi_1^{top}(X)$ is open and translations in quasitopological groups are homeomorphisms, $\pi_1^{top}(X)$ must be discrete. ■

This gives a very general condition to imply countability in fundamental groups.

Corollary 4.28 *If X is a separable metric space with a 1-connected cover, then $\pi_1^{top}(X)$ is countable.*

Proof. By the previous corollary, $\pi_1^{top}(X)$ is discrete. It follows from Proposition

2.67 that if X is a separable metric space, then $\pi_1^{top}(X)$ is countable. ∎

Upon seeing Theorem 4.25, one might be tempted to extend the known classification of covers to all locally path connected spaces using open subgroups of the topological fundamental group, however Proposition 4.30 below indicates the unlikelihood that every open subgroup H will admit a covering map $p : \tilde{X} \to X$ such that $p_*(\pi_1^{top}(\tilde{X})) = H$.

Definition 4.29 Let H be a subgroup of $\pi_1(X, x)$. We say X is *semilocally H-connected* if every point $y \in X$ is contained in a neighborhood U such that for every $\gamma \in \Omega(U, y)$ and paths $\alpha, \beta : I \to X$ from x to y where $[\alpha * \beta^{-1}] \in H$, we have $[\alpha * \gamma * \beta^{-1}] \in H$.

Note that if H is the trivial subgroup, X is semilocally H-connected if and only if X is semilocally 1-connected in the usual sense.

Proposition 4.30 *If $p : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering map and $H = p_*\left(\pi_1^{top}(\tilde{X}, \tilde{x})\right)$, then X is semilocally H-connected.*

Proof. Let U be an evenly covered neighborhood of $y \in X$. Let $\gamma \in \Omega(U, y)$ and paths $\alpha, \beta : I \to X$ from x to y such that $[\alpha * \beta^{-1}] \in H$. Since $\alpha * \beta^{-1} \in H$, the lift of the loop $\alpha * \beta^{-1}$ is the loop $\tilde{\alpha} * \tilde{\beta}^{-1}$ in $\tilde{X}$ based at $\tilde{x}$. If $p^{-1}(U) = \coprod_\lambda V_\lambda$, then let V_{λ_0} be the V_λ which contains $\tilde{\alpha}(1) = \tilde{\beta}(1)$. Let $h_0 : U \to V_{\lambda_0}$ be the homeomorphism such that $p \circ h_0 = id_U$. Since γ has image in U, the lift of $\delta = \alpha * \gamma * \beta^{-1}$ is $\tilde{\delta} = \tilde{\alpha} * (h_0 \circ \gamma) * \tilde{\beta}^{-1}$. Since $\tilde{\delta} \in \Omega(\tilde{X}, \tilde{x})$, it follows that $p_*(\tilde{\delta}) = [p \circ \tilde{\delta}] = [\alpha * \gamma * \beta^{-1}] \in H$. ∎

If G is a quasitopological group and H is an open subgroup, the set of right cosets G/H with the quotient topology with respect to the projection $G \to G/H$, $g \mapsto gH$ is a discrete space since all cosets are open.

Corollary 4.31 *[Mun00, Theorem 54.6] Let* $p : (\tilde{X}, \tilde{x}) \to (X, x)$ *be a covering map and* $H = p_*\left(\pi_1^{top}(\tilde{X}, \tilde{x})\right)$*. The lifting correspondence* $\Phi : \pi_1^{top}(X, x)/H \to p^{-1}(x)$, $[\alpha]H \mapsto \tilde{\alpha}(1)$ *is a bijection of discrete spaces.*

Regarding spaces with indiscrete topological fundamental group, we have:

Corollary 4.32 *If X is path connected and* $\pi_1^{top}(X)$ *is non-trivial has the indiscrete topology, then every covering map* $p : \tilde{X} \to X$ *is trivial.*

Proof. Suppose $p : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering map such that the cardinality of $p^{-1}(x)$ is greater than 1 and $\pi_1^{top}(X, x)$ is non-trivial and indiscrete. Since $\pi_1^{top}(X, x)$ does not have the discrete topology $H = p_*\left(\pi_1^{top}(\tilde{X}, \tilde{x})\right)$ cannot be the trivial subgroup. Since $|\pi_1^{top}(X, x)/H| = |p^{-1}(x)| > 1$, H must be a proper subgroup. By Theorem 4.25, $p_*\left(\pi_1^{top}(\tilde{X}, \tilde{x})\right)$ is a non-trivial, proper open subgroup of $\pi_1^{top}(X, x)$ contradicting the fact that $\pi_1^{top}(X, x)$ has the indiscrete topology. ■

4.2 A computation of $\pi_1^{top}(\Sigma(X_+))$

In this chapter, we describe the isomorphism class of $\pi_1^{top}(\Sigma(X_+))$ in the category of quasitopological groups for an arbitrary space X.

4.2.1 The spaces $\Sigma(X_+)$

Let X be an arbitrary topological space and $X_+ = X \sqcup \{*\}$ be the based space with added isolated basepoint. Let

$$(\Sigma(X_+), x_0) = \left(\frac{X_+ \times I}{X \times \{0, 1\} \cup \{*\} \times I}, x_0\right)$$

be the reduced suspension of X_+ with canonical choice of basepoint and $x \wedge s$ denote the image of $(x, s) \in X \times I$ under the quotient map $X_+ \times I \to \Sigma(X_+)$. For subsets $A \subseteq X$ and $S \subseteq I$, let $A \wedge S = \{a \wedge s | a \in A, s \in S\}$. A subspace $P \wedge I$ where $P \in \pi_0(X)$ is a path component of X is called a *hoop* of $\Sigma(X_+)$.

Suppose $\mathscr{B}_X$ is a basis for the topology of X which is closed under finite intersections. For a point $x \wedge t \in X \wedge (0,1) = \Sigma(X_+) - \{x_0\}$, a subset $U \wedge (c, d)$ where $x \in U$, $U \in \mathscr{B}_X$ and $t \in (c, d) \subseteq (0, 1)$ is an open neighborhood of $x \wedge t$. Open neighborhoods of x_0 may be given in terms of open coverings of $X \times \{0, 1\}$ in $X \times I$. If $U^x \in \mathscr{B}_X$ is an open neighborhood of x in X and $t_x \in (0, \frac{1}{8})$, the set

$$\bigcup_{x \in X} (U^x \wedge [0, t_x) \cup (1 - t_x, 1])$$

is an open neighborhood of x_0 in $\Sigma(X_+)$. The collection $\mathscr{B}_{\Sigma(X_+)}$ of neighborhoods of the form $U \wedge (c, d)$ and $\bigcup_{x \in X} (U^x \wedge [0, t_x) \cup (1 - t_x, 1])$ is a basis for the topology $\Sigma(X_+)$ which is closed under finite intersection. The following are obvious facts regarding $\Sigma(X_+)$.

Remark 4.33 *For an arbitrary space X,*

1. $\Sigma(X_+)$ *is path-connected.*

2. $\Sigma(X_+) - \{x_0\} = X \wedge (0, 1) \cong X \times (0, 1)$.

3. *Every basic neighborhood* $V \in \mathscr{B}_{\Sigma(X_+)}$ *containing* x_0 *is arc connected and 1-connected.*

4. *For each* $t \in (0, 1)$*, the closed subspaces* $X \wedge [0, t]$ *and* $X \wedge [t, 1]$ *are homeomorphic to* CX *the cone of* X*, and are contractible to the basepoint point* x_0.

5. $\Sigma(X_+)$ *is Hausdorff if and only if* X *is Hausdorff, but the following holds for arbitrary* X*: For each point* $x \wedge t \in X \wedge (0,1)$*, there are disjoint open neighborhoods separating* $x \wedge t$ *and the basepoint* x_0.

Remark 4.34 It is a well-known fact that the reduced suspension functor $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$ is left adjoint to the loop space functor $\Omega : \mathbf{Top}_* \to \mathbf{Top}_*$. Additionally, adding isolated basepoint to an unbased space $(-)_+ : \mathbf{Top} \to \mathbf{Top}_*$ is left adjoint to the functor $U : \mathbf{Top}_* \to \mathbf{Top}$ forgetting basepoint. Taking composites, we see the construction $\Sigma((-)_+) : \mathbf{Top} \to \mathbf{Top}_*$ is a functor left adjoint to $U\Omega$. For a map $f : X \to Y$, the map $\Sigma(f_+) : \Sigma(X_+) \to \Sigma(Y_+)$ is defined by $\Sigma(f_+)(x \wedge s) = f(x) \wedge s$. The adjunction is illustrated by natural homeomorphisms

$$M_*(\Sigma(X_+), Y) \cong M_*(X_+, \Omega(Y)) \cong M(X, U\Omega(Y)).$$

This adjunction immediately gives motivation for our proposed computation of $\pi_1^{top}(\Sigma(X_+))$.

Proposition 4.35 *Every topological fundamental group* $\pi_1^{top}(Y)$ *is a quotient quasitopological group of* $\pi_1^{top}(\Sigma(X_+))$ *for some space* X.

Proof. Let $cu : \Sigma(\Omega(Y)_+) \to Y$ be the adjoint of the unbased identity of $\Omega(Y)$. The basic property of counits gives that the unbased map $U\Omega(cu) : \Omega(\Sigma(\Omega(Y)_+)) \to \Omega(Y)$ is a topological retraction. Applying the path component functor, we obtain a group epimorphism $\pi_1^{top}(\Sigma(\Omega(Y)_+)) \to \pi_1^{top}(Y)$ which is, by Remark 3.5, a quotient map of spaces. Take $X = \Omega(Y)$. ∎

Since a quotient group (with quotient topology) of a topological group is itself a topological group, the spaces $\Sigma(X_+)$ are prime candidates for counterexamples to the claim that π_1^{top} takes values in **TopGrp**.

It is convenient to view the spaces $\Sigma(X_+)$ as generalized of wedges of circles. Intuitively, one might think of $\Sigma(X_+)$ as a "wedge of circles parameterized by the space X." Let $\bigvee_X S^1$ be the wedge of circles indexed by the underlying set of X. Suppose $\epsilon : I \to S^1$ is the exponential map and a point in the x-th summand of the wedge is denoted as $\epsilon(t)_x$ for $t \in I$. The pushout property implies that every map $f : X \to Y$ induces a map $\bigvee_X S^1 \to \bigvee_Y S^1$ given by $\epsilon(t)_x \mapsto \epsilon(t)_{f(x)}$ for all $t \in I$, $x \in X$. It is easy to see that $\bigvee_{(-)} S^1 : \mathbf{Top} \to \mathbf{Top}_*$ is a functor which we may relate to $\Sigma((-)_+)$ in the following way.

Remark 4.36 There is a natural transformation $\gamma : \bigvee_{(-)} S^1 \to \Sigma((-)_+)$ where each component $\gamma_X : \bigvee_X S^1 \to \Sigma(X_+)$ given by $\gamma_X(\epsilon(t)_x) = x \wedge t$ is a continuous bijection. Moreover, γ_X is a homeomorphism if and only X has the discrete topology.

Proof. Clearly, if X is a discrete space, then γ_X is a homeomorphism. If X is not discrete, let dX denote the underlying set of X with the discrete topology. The identity $id : dX \to X$ is continuous and induces the bijection $\gamma_X = \gamma_{dX} \circ id : \bigvee_X S^1 \cong \Sigma(dX_+) \to \Sigma(X_+)$. Naturality follows from the equation $\Sigma(f_+)(x \wedge t) = f(x) \wedge t = \gamma_Y(\epsilon(t)_{f(x)})$. ■

According to this remark, if X has the discrete topology, then $\Sigma(X_+)$ is homeomorphic to a wedge of circles. By the van Kampen Theorem and Theorem 4.7, $\pi_1^{top}(\Sigma(X_+))$ must be isomorphic to the discrete free group $F(X)$. We will see later on that $\pi_1^{top}(\Sigma(X_+))$ is discrete if and only if X is semilocally 0-connected.

Example 4.37 Let $X = \mathbb{N}^* = \{1, 2, ..., \infty\}$ be the one-point compactification of the discrete space of natural numbers. The suspension $\Sigma((\mathbb{N}^*)_+)$ is a one-dimensional planar continuum which is not locally path connected. If dX is the underlying set of X with the discrete topology, the identity $dX \to X$ induces the continuous bijection $\gamma_X : \bigvee_X S^1 \to \Sigma(X_+)$ which is a weak equivalence but not a homotopy equivalence.

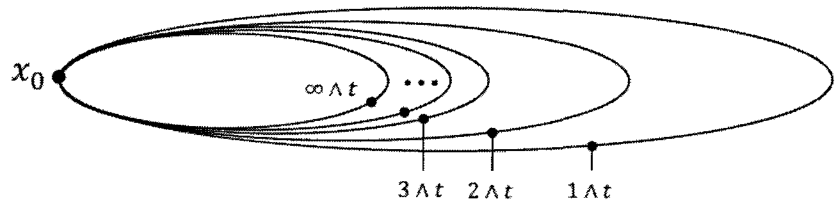


Figure 4: $\Sigma((\mathbb{N}^*)_+)$

4.2.2 The fundamental group $\pi_1(\Sigma(X_+))$

To compute $\pi_1(\Sigma(X_+))$, we relate free topological monoids to $\pi_1^{top}(\Sigma(X_+))$, via the unbased *James map* $u : X \to \Omega(\Sigma(X_+))$, $u(x)(t) = u_x(t) = x \wedge t$. Since u is the adjoint of the identity of $\Sigma(X_+)$, it is natural in X. Define a function $\mathscr{J} : M_T^*(X) \to \Omega(\Sigma(X_+))$ taking the empty word to the constant map and $\mathscr{J}(x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}) = *_{i=1}^n \left(u_{x_i}^{\epsilon_i}\right)$.

The following lemma is a direct consequence of the Lebesgue lemma, Theorem 1.20 and our choice of basis $\mathscr{B}_{\Sigma(X_+)}$.

Lemma 4.38 *A convenient neighborhood base of the loop u_x in $\Omega(\Sigma(X_+))$ is of the form $\mathscr{V} = \langle K_m^1, W\rangle \cap \bigcap_{j=2}^{m-1} \langle K_m^j, U \wedge (s_j, t_j)\rangle \cap \langle K_m^m, W\rangle$ where $W \in \mathscr{B}_{\Sigma(X_+)}$ is an open neighborhood of x_0 in $\Sigma(X_+)$ and $U \in \mathscr{B}_X$ is an open neighborhood of x in X.*

Corollary 4.39 *A convenient neighborhood base of the loop $\mathscr{J}(x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}) = *_{i=1}^n \left(u_{x_i}^{\epsilon_i}\right)$*

in $\Omega(\Sigma(X_+))$ is of the form $\mathscr{U} = \bigcap_{i=1}^{n}\left(\mathscr{V}_i^{\epsilon_i}\right)^{K_n^i}$ where $\mathscr{V}_i$ is an open neighborhood of u_{x_i} as in the previous lemma.

Proposition 4.40 *$\mathscr{J} : M_T^*(X) \to \Omega(\Sigma(X_+))$ is a topological embedding natural in X.*

Proof. Clearly, $\mathscr{J}$ is injective. The James map $u : X \to \Omega(\Sigma(X_+))$ induces a continuous homomorphism $M_T^*(u) : M_T^*(X) \to M_T^*(\Omega(\Sigma(X_+)))$, $x_1^{\epsilon_1}...x_n^{\epsilon_n} \mapsto u_{x_1}^{\epsilon_1}...u_{x_n}^{\epsilon_n}$. Let $c : M_T^*(\Omega(\Sigma(X_+))) \to \Omega(\Sigma(X_+))$ be concatenation $\alpha_1^{\epsilon_1}...\alpha_n^{\epsilon_n} \mapsto \alpha_1^{\epsilon_1} * \cdots * \alpha_n^{\epsilon_n}$. Since n-fold concatenation $\Omega(\Sigma(X_+))^n \to \Omega(\Sigma(X_+))$ and inversion $\Omega(\Sigma(X_+)) \to \Omega(\Sigma(X_+))$, $\alpha \mapsto \alpha^{-1}$ are continuous, c is also continuous. The composite $\mathscr{J} = c \circ M_T^*(u)$ is therefore continuous. Suppose $\mathscr{U} = \bigcap_{i=1}^{n}\left(\mathscr{V}_i^{\epsilon_i}\right)^{K_n^i}$ is an open neighborhood of $*_{i=1}^{n}\left(u_{x_i}^{\epsilon_i}\right)$ in $\Omega(\Sigma(X_+))$ where each $\mathscr{V}_i$ is as in Lemma 4.38. Then $U = U_1^{\epsilon_1}...U_n^{\epsilon_n}$ is an open neighborhood of $x_1^{\epsilon_1}x_2^{\epsilon_2}\dots x_n^{\epsilon_n}$ in $M_T^*(X)$ such that

$$\mathscr{J}(U) = \left\{*_{i=1}^{n}\left(u_{y_i}^{\epsilon_i}\right) | y_i \in U_i\right\} = \mathscr{U} \cap \mathscr{J}(M_T^*(X))$$

Therefore $\mathscr{J}$ is an embedding. To check naturality, we let $f : X \to Y$ be a map of spaces and check that the diagram

$$\begin{array}{ccc} M_T^*(X) & \xrightarrow{M_T^*(f)} & M_T^*(X) \\ \downarrow{\scriptstyle \mathscr{J}} & & \downarrow{\scriptstyle \mathscr{J}} \\ \Omega(\Sigma(X_+)) & \xrightarrow[\Omega(\Sigma(f_+))]{} & \Omega(\Sigma(Y_+)) \end{array}$$

commutes in **Top**. Let $w = x_1^{\epsilon_1}...x_n^{\epsilon_n} \in M_T^*(X)$ so that $\mathscr{J} \circ M_T^*(f) = *_{i=1}^{n}\left(u_{f(x_i)}^{\epsilon_i}\right)$. But we also have

$$\Omega(\Sigma(f_+)) \circ \mathscr{J}(w) = \Sigma(f_+) \circ \left(*_{i=1}^{n}\left(u_{x_i}^{\epsilon_i}\right)\right) = *_{i=1}^{n}\left(\Sigma(f_+) \circ u_{x_i}^{\epsilon_i}\right) = *_{i=1}^{n}\left(u_{f(x_i)}^{\epsilon_i}\right)$$

where the last equality follows from the naturality of $u : id \to \Omega(\Sigma((-)_+))$. ■

Remark 4.41 This construction follows the well-known James construction [CM95, 5.3] used originally by I.M. James to study the geometry of $\Omega(\Sigma Z, *)$ for a connected CW-complex Z.

Throughout the rest of this section, let $\pi_X : X \to \pi_0^{top}(X)$ and $\pi_\Omega : \Omega(\Sigma(X_+)) \to \pi_1^{top}(\Sigma(X_+))$ denote the canonical quotient maps. To compute $\pi_1^{top}(\Sigma(X_+))$, we must first understand the algebraic structure of $\pi_1(\Sigma(X_+))$. We begin by observing that the James map $u : X \to \Omega(\Sigma(X_+))$ induces a continuous map $u_* : \pi_0^{top}(X) \to \pi_0^{top}(\Omega(\Sigma(X_+))) = \pi_1^{top}(\Sigma(X_+))$ on path component spaces. The underlying function $u_* : \pi_0(X) \to \pi_1(\Sigma(X_+))$ induces a group homomorphism $h_X : F(\pi_0(X)) \to \pi_1(\Sigma(X_+))$ on the free group generated by the path components of X. In particular, h_X takes the reduced word $P_1^{\epsilon_1}P_2^{\epsilon_2}\dots P_k^{\epsilon_k}$ (where $P_i \in \pi_0(X)$ and $\epsilon_i \in \{\pm 1\}$) to the homotopy class $[u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \cdots * u_{x_k}^{\epsilon_k}]$ where $x_i \in P_i$ for each i. We show that h_X is a group isomorphism.

Remark 4.42 h_X is the unique group homomorphism making the following diagram commute.

$$\begin{array}{ccccc} M^*(X) & \xrightarrow{\mathscr{J}} & \Omega(\Sigma(X_+)) & & \\ {\scriptstyle M^*(\pi_X)}\big\downarrow & & & \searrow^{\pi_\Omega} & \\ M^*(\pi_0(X)) & \xrightarrow[R]{} & F(\pi_0(X)) & \xrightarrow[h_X]{} & \pi_1(\Sigma(X_+)) \end{array}$$

Definition 4.43 A loop $\alpha \in M((I, \{0,1\}), (Y, \{y\}))$ is *simple* if $\alpha^{-1}(y) = \{0,1\}$. The subspace of $M((I, \{0,1\}), (Y, \{y\}))$ consisting of simple loops is denoted $\Omega_s(Y)$.

Remark 4.44 Ω_s is not a functor since it is not well-defined on morphisms. It is easy to see, however, that $\Omega_s(\Sigma((-)_+)) : \mathbf{Top} \to \mathbf{Top}$ is a functor.

The map $X \to \{*\}$ collapsing X to a point induces a retraction $r : \Sigma(X_+) \to \Sigma S^0 \cong S^1$. This, in turn, induces a retraction $r_* : \pi_1^{top}(\Sigma(X_+)) \to \pi_1^{top}(S^1) \cong \mathbb{Z}$ onto the discrete group of integers. By the previous remark, if $\alpha \in \Omega_s(\Sigma(X_+))$, then $r \circ \alpha : I \to S^1$ is a simple loop in S^1. But the homotopy class of a simple loop in S^1 is either the identity or a generator of $\pi_1^{top}(S^1)$. Therefore $r_*([\alpha])$ must take on the value $1, 0$ or -1.

Definition 4.45 A simple loop $\alpha \in \Omega_s(\Sigma(X_+))$ has *positive (resp. negative) orientation* if $[\alpha] \in r_*^{-1}(1)$ (resp. $[\alpha] \in r_*^{-1}(-1)$). If $[\alpha] \in r_*^{-1}(0)$, then we say α has no orientation and is *trivial*. The subspaces of $\Omega_s(\Sigma(X_+))$ consisting of simple loops with positive, negative, and no orientation are denoted $\Omega_{+s}(\Sigma(X_+))$, $\Omega_{-s}(\Sigma(X_+))$, and $\Omega_{0s}(\Sigma(X_+))$ respectively.

The fact that $\mathbb{Z}$ is discrete, allows us to write the loop space $\Omega(\Sigma(X_+))$ as the disjoint union of the subspaces $\pi_\Omega^{-1}(r_*^{-1}(n))$, $n \in \mathbb{Z}$. Consequently, we may write

$\Omega_s(\Sigma(X_+))$ as disjoint union

$$\Omega_s(\Sigma(X_+)) = \Omega_{+s}(\Sigma(X_+)) \sqcup \Omega_{0s}(\Sigma(X_+)) \sqcup \Omega_{-s}(\Sigma(X_+)).$$

We also note that $\Omega_{-s}(\Sigma(X_+)) = \Omega_{+s}(\Sigma(X_+))^{-1}$. Thus loop inversion give a homeomorphism $\Omega_{+s}(\Sigma(X_+)) \cong \Omega_{-s}(\Sigma(X_+))$. The next two lemmas are required to prove the surjectivity of h_X.

Lemma 4.46 *A simple loop $\alpha \in \Omega_s(\Sigma(X_+))$ is null-homotopic if and only if it is trivial.*

Proof. By definition, a simple loop which has orientation is not null-homotopic. Therefore, it suffices to show that any trivial loop is null-homotopic. If α is trivial, then α does not traverse any hoop of $\Sigma(X_+)$, i.e. there is a $t \in (0,1)$ such that α has image in either $X \wedge [0,t]$ or $X \wedge [t,1]$. By Remark 4.33.4, α is null-homotopic. ■

The subspaces $P \wedge (0,1)$, $P \in \pi_0(X)$ are precisely the path components of $X \wedge (0,1)$. Therefore, if $p : I \to \Sigma(X_+)$ is a path such that $p(0) \in P_1 \wedge (0,1)$ and $p(1) \in P_2 \wedge (0,1)$ for distinct $P_1, P_2 \in \pi_0(X)$ (i.e. the endpoints of p lie in distinct hoops and are not the basepoint x_0), then there is a $t \in (0,1)$ such that $p(t) = x_0$. This implies that the image of each simple loop lies entirely within a single hoop.

Lemma 4.47 *If simple loops α and β have the same orientation and have image in the same hoop $P \wedge I$, then they are homotopic.*

Proof. Suppose α and β have positive orientation and image in $P \wedge I$. Since $P \wedge (1,0)$ is a path component of $X \wedge (0,1)$, we may find a $t \in (0,1)$ and a path $p : I \to X \wedge (0,1)$

such that $p(0) = \alpha(t)$ and $p(1) = \beta(t)$. Now

$$\alpha_{[0,t]} * p * \beta^{-1}_{[0,t]} \text{ and } \beta^{-1}_{[t,1]} * p^{-1} * \alpha_{[t,1]}$$

are trivial simple loops which by the previous lemma must be null homotopic. This gives fixed endpoint homotopies of paths

$$\alpha_{[0,t]} \simeq \beta_{[0,t]} * p^{-1} \text{ and } \alpha_{[t,1]} \simeq p * \beta_{[t,1]}$$

The concatenation of these two gives

$$\alpha \simeq \alpha_{[0,t]} * \alpha_{[t,1]} \simeq \beta_{[0,t]} * p^{-1} * p * \beta_{[t,1]} \simeq \beta_{[0,t]} * \beta_{[t,1]} \simeq \beta.$$

One may simply invert loops to prove the case of negative orientation. ∎

The next two statements are required to prove the injectivity of h_X.

Lemma 4.48 *If $w = P_1^{\epsilon_1} \dots P_n^{\epsilon_n} \in F(\pi_0(X))$ is a non-empty reduced word such that $\sum_{i=1}^n \epsilon_i \neq 0$, then $h_X(w)$ is not the identity of $\pi_1(\Sigma(X_+))$.*

Proof. The retraction $r : \Sigma(X_+) \to S^1$ induces an epimorphism $r_* : \pi_1(\Sigma(X_+)) \to \mathbb{Z}$ on fundamental groups, where $r_*([u_x]^\epsilon) = \epsilon$ for each $x \in X$ and $\epsilon \in \{\pm 1\}$. Therefore, if $\sum_{i=1}^n \epsilon_i \neq 0$, then $r_*(h_X(w)) = r_*([u_{x_1}^{\epsilon_1} * \dots * u_{x_n}^{\epsilon_n}]) = \sum_{i=1}^n \epsilon_i \neq 0$ (where $x_i \in P_i$) and $h_X(w)$ cannot be the identity of $\pi_1(\Sigma(X_+))$. ∎

Remark 4.49 Let $P_1^{\epsilon_1} \dots P_n^{\epsilon_n} \in F(\pi_0(X))$ be a reduced word.

1. If $1 \leq k \leq m \leq n$, the subword $P_k^{\epsilon_k} \dots P_m^{\epsilon_m}$ is also reduced.

2. If $n \geq 2$ and $\sum_{i=1}^{n} \epsilon_i = 0$, then there are $i_0, i_1 \in \{1, 2, ..., n\}$ such that $P_{i_0} \neq P_{i_1}$.

Theorem 4.50 *$h_X : F(\pi_0(X)) \to \pi_1(\Sigma(X_+))$ is a natural isomorphism of groups.*

Proof. To show that h_X is surjective, we suppose $\alpha \in \Omega(\Sigma(X_+))$ is an arbitrary loop. The pullback $\alpha^{-1}(\Sigma(X_+) - \{x_0\}) = \coprod_{m \in M}(c_m, d_m)$ is an open subset of $(0, 1)$. Each restriction $\alpha_m = \alpha_{[c_m, d_m]}$ is a simple loop, and by 4.33.5, all but finitely many of the α_m have image in the 1-connected neighborhood $X \wedge \left[0, \frac{1}{8}\right) \sqcup \left(\frac{7}{8}, 1\right]$. Therefore α is homotopic to a finite concatenation of simple loops $\alpha_{m_1} * \alpha_{m_2} * \dots * \alpha_{m_n}$. By Lemma 4.46, we may suppose that each α_{m_i} has orientation $\epsilon_i \in \{\pm 1\}$ and image in hoop $P_i \wedge I$. Lemma 4.47 then gives that $\alpha_{m_i} \simeq u_{x_i}^{\epsilon_i}$ for any $x_i \in P_i$. But then

$$h_X(P_1^{\epsilon_1} P_2^{\epsilon_2} \dots P_n^{\epsilon_n}) = [u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \dots * u_{x_n}^{\epsilon_n}] = [\alpha_{m_1} * \alpha_{m_2} * \dots * \alpha_{m_n}] = [\alpha].$$

For injectivity, we suppose $w = P_1^{\epsilon_1} P_2^{\epsilon_2} \dots P_n^{\epsilon_n}$ is a non-empty reduced word in $F(\pi_0(X))$. It suffices to show that $h_X(w) = \left[u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \dots * u_{x_n}^{\epsilon_n}\right]$ is non-trivial when $x_i \in P_i$ for each i. We proceed by induction on n and note that Lemma 4.48 gives the first step of induction $n = 1$. Suppose $n \geq 2$ and $h_X(v)$ is non-trivial for all reduced words $v = Q_1^{\delta_1} Q_2^{\delta_2} \dots Q_j^{\delta_j}$ of length $j < n$. By Lemma 4.48, it suffices to show that $h_X(w) = \left[u_{x_1}^{\epsilon_1} * u_{x_1}^{\epsilon_1} * \dots * u_{x_n}^{\epsilon_n}\right]$ is non-trivial when $\sum_{i=1}^{n} \epsilon_i = 0$. We suppose otherwise, i.e. that there is a homotopy of based loops $H : I^2 \to \Sigma(X_+)$ such that $H(t, 0) = x_0$ and $H(t, 1) = \left(u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \dots * u_{x_n}^{\epsilon_n}\right)(t)$ for all $t \in I$. For $j = 0, 1, ..., 2n$, we let $b_j = \left(\frac{j}{2n}, 1\right) \in I^2$. Remark 4.33.5 indicates that the singleton $\{x_0\}$ is closed in $\Sigma(X_+)$ so that $H^{-1}(x_0)$ is a compact subset of I^2. Since each $u_{x_i}^{\epsilon_i}$ is simple we have that

$$H^{-1}(x_0) \cap \partial(I^2) = \{0,1\} \times I \cup I \times \{0\} \sqcup \coprod_{i=1}^{n-1} \{b_{2i}\}$$

where ∂ denotes boundary in $\mathbb{R}^2$. We also have that $H(b_{2i-1}) = u_{x_i}^{\epsilon_i}(\frac{1}{2}) = x_i \wedge \frac{1}{2} \neq x_0$ for each $i = 1, ..., n$. This allows us to find an $r_0 > 0$ so that when $U_i = B(b_{2i-1}, r_0) \cap I^2$ is the ball of radius r_0 about b_{2i-1} in I^2, we have $H^{-1}(x_0) \cap \bigcup_{i=1}^{n} U_j = \emptyset$. Now we find an $r_1 \in (0, r_0)$ and cover $H^{-1}(x_0)$ with finitely many open balls $V_l = B(z_l, r_1) \cap I^2$ so that

$$\left(\bigcup_l \overline{V_l}\right) \cap \left(\bigcup_{i=1}^{n} \overline{U_i}\right) = \emptyset \text{ and } H\left(\bigcup_l V_l\right) \subseteq \left[0, \frac{1}{8}\right) \sqcup \left(\frac{7}{8}, 1\right] \wedge X$$

(which is possible since H is continuous). Note that if $q : I \to \bigcup_l V_l$ is a path with endpoints $q(0), q(1) \in H^{-1}(x_0)$, then the loop $H \circ q : I \to \Sigma(X_+)$ is based at x_0 and has image in the 1-connected neighborhood $X \wedge \left[0, \frac{1}{8}\right) \sqcup \left(\frac{7}{8}, 1\right]$, and therefore must be null-homotopic. We note that there is no path $q : I \to \bigcup_l V_l$ such that $q(0) = b_{2k}, q(1) = b_{2m}$ for $1 \le k < m \le n$. If $q : I \to \bigcup_l V_l$ is such a path, the concatenation $u_{x_{k+1}}^{\epsilon_{k+1}} * u_{x_{k+2}}^{\epsilon_{k+2}} * \cdots * u_{x_m}^{\epsilon_m}$ is null-homotopic since $H \circ q$ is null-homotopic and $(H \circ q) \simeq u_{x_{k+1}}^{\epsilon_{k+1}} * u_{x_{k+2}}^{\epsilon_{k+2}} * \cdots * u_{x_m}^{\epsilon_m}$. This means that $h_X\left(P_{k+1}^{\epsilon_{k+1}} P_{k+2}^{\epsilon_{k+2}} \dots P_m^{\epsilon_m}\right) = \left[u_{x_{k+1}}^{\epsilon_{k+1}} * u_{x_{k+2}}^{\epsilon_{k+2}} * \cdots * u_{x_m}^{\epsilon_m}\right]$ is the identity of $\pi_1(\Sigma(X_+))$. But by Remark 4.49.1 $P_{k+1}^{\epsilon_{k+1}} P_{k+2}^{\epsilon_{k+2}} \dots P_m^{\epsilon_m}$ is a reduced word of length $< n$ and so by our induction hypothesis $h_X\left(P_{k+1}^{\epsilon_{k+1}} P_{k+2}^{\epsilon_{k+2}} \dots P_m^{\epsilon_m}\right)$ cannot be the identity.

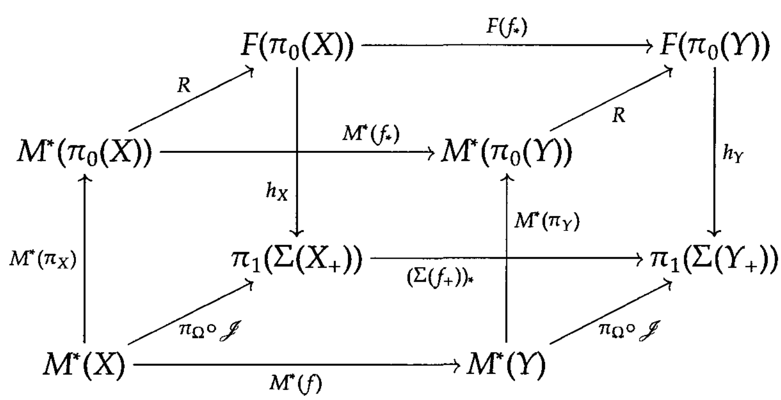
Since such paths q do not exist, each b_{2i} lies in a distinct path component (and consequently connected component) of $\bigcup_l V_l$ for each $i = 1, ..., n$. Let $C_i = \bigcup_{m=1}^{M_i} V_{l_m}^i$ be the path component of $\bigcup_l V_l$ containing b_{2i}. But this means the b_{2i-1}, $i = 1, ..., n$

all lie in the same path component of $I^2 - \bigcup_l V_l$. Specifically, the subspace

$$\left(\partial(I^2) - \bigcup_l V_l\right) \cup \left(\partial\left(\bigcup_{i=1}^{n} C_i\right) - \partial(I^2)\right)$$

is path connected and contains each of the b_{2i-1}. Since we were able to assume that $\sum_i \epsilon_i = 0$, we know by Remark 4.49.2 that there are $i_0, i_1 \in \{1, ..., n\}$ such that $P_{i_0} \neq P_{i_1}$. We have shown that there is a path $p : I \to I^2 - \bigcup_l V_l$ with $p(0) = b_{2i_0-1}$ and $p(1) = b_{2i_1-1}$. But then $H \circ p : I \to \Sigma(X_+)$ is a path with $x_0 \notin H \circ p(I)$, $H(p(0)) = \frac{1}{2} \wedge x_{i_0}$, and $H(p(1)) = \frac{1}{2} \wedge x_{i_1}$. But this is impossible as $H(p(0))$ and $H(p(1))$ lie in different hoops of $\Sigma(X_+)$. Therefore $u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \cdots * u_{x_n}^{\epsilon_n}$ cannot be null-homotopic.

To check the naturality of h_X we use the following cubical diagram:



in **Mon**. Here $f : X \to Y$ is a map of spaces and $f_* : \pi_0(X) \to \pi_0(Y)$ is the induced function. The left and right squares commute by Remark 4.42. The top (resp. bottom) square commutes by the naturality of R (resp. $\pi_\Omega \circ \mathscr{J}$). The front square commutes by the functorality of M^* and the naturality of $\pi : id \to \pi_0$. Since $R \circ M^*(\pi_X) : M^*(X) \to F(\pi_0(X))$ is surjective, the back square commutes. This is precisely the naturality of h_X. In particular, if $W \in F(\pi_0(X))$ and $R \circ M^*(\pi_X)(w) = W$, then

$$h_Y \circ F(f_*)(W) = h_Y \circ R \circ M^*(f_*)(M^*(\pi_X)(w)) = h_Y \circ R \circ M^*(\pi_Y) \circ M^*(f)(w) =$$

$$= \mathscr{J} \circ M^*(f)(w) = (\Sigma(f_+))_* \circ \mathscr{J}(w) = (\Sigma(f_+))_* \circ h_X \circ R \circ M^*(\pi_X)(w) =$$

$$= (\Sigma(f_+))_* \circ h_X(W)$$

■

Corollary 4.51 *The fibers of the map $\pi_\Omega \circ \mathscr{J} : M_T^*(X) \to \pi_1^{top}(\Sigma(X_+))$ are equal to those of $R \circ M_T(\pi_X) : M_T^*(X) \to M_T^*(\pi_0^{top}(X)) \to F(\pi_0(X))$.*

Since $u_x \simeq u_y$ if and only if x and y lie in the same path component of X, we denote the homotopy class of u_x by $[u_P]$ where P is the path component of x in X. Thus $\{[u_P] | P \in \pi_0(X)\}$ freely generates $\pi_1(\Sigma(X_+))$ and the map $u_* : \pi_0^{top}(X) \to \pi_1^{top}(\Sigma(X_+))$ is the canonical injection of generators.

The James map $u : X \to \Omega(\Sigma(X_+))$ has image in $\Omega_{+s}(\Sigma(X_+))$ and the map $u : X \to \Omega_{+s}(\Sigma(X_+))$ with restricted codomain induces a continuous bijection $u_* : \pi_0^{top}(X) \to \pi_0^{top}(\Omega_{+s}(\Sigma(X_+)))$ on path component spaces. The fact that this bijection is also a homeomorphism is an obvious consequence of the next lemma which is to be used in the proof of Theorem 4.53. For a map $f : X \to Y$, let $f_{**} = \pi_0^{top}(M_T^*(f))$ be the induced, continuous, involution-preserving monoid homomorphism.

Lemma 4.52 *The James map $u : X \to \Omega_{+s}(\Sigma(X_+))$ induces a natural isomorphism of semi-topological monoids with continuous involution $u_{**} : \pi_0^{top}(M_T^*(X)) \to \pi_0^{top}\left(M_T^*(\Omega_{+s}(\Sigma(X_+)))\right)$.*

Proof. We note that on generators u_{**} is given by $u_{**}(P) = [u_P]$. The naturality of $\psi : \pi_0(M^*(-)) \to M^*(\pi_0(-))$ applied to the James map makes the following diagram commute in the category of monoids (without topology)

$$\begin{array}{ccc}
\pi_0(M_T^*(X)) & \xrightarrow[\cong]{\psi_X} & M^*(\pi_0(X)) \\
{\scriptstyle u_{**}}\downarrow & & \cong \downarrow {\scriptstyle M^*(u_*)} \\
\pi_0(M_T^*(\Omega_{+s}(\Sigma(X_+)))) & \xleftarrow[\psi^{-1}_{\Omega_{+s}(\Sigma(X_+))}]{\cong} & M^*(\pi_0(\Omega_{+s}(\Sigma(X_+))))
\end{array}$$

Since u_* is a bijection, $M^*(u_*)$ is a monoid isomorphism. Therefore $u_{**} : \pi_0^{top}(M_T^*(X)) \to \pi_0^{top}\left(M_T^*(\Omega_{+s}(\Sigma(X_+)))\right)$ is a continuous, involution-preserving monoid isomorphism and it suffices to show the inverse is continuous. Let $r : \Omega_{+s}(\Sigma(X_+)) \to M((0,1), (0,1)\times X)$ be the map taking each positively oriented simple loop $\alpha : I \to \Sigma(X_+)$ to the restricted map $\alpha|_{(0,1)} : (0,1) \to X \wedge (0,1) \cong X \times (0,1)$ and $p : M((0,1), X \times (0,1)) \to M((0,1), X)$ be post-composition with the projection $X\times(0,1) \to X$. For any $t \in (0,1)$, consider the composite map

$$v : \Omega_{+s}(\Sigma(X_+)) \xrightarrow{j_t} (0,1) \times \Omega_{+s}(\Sigma(X_+)) \xrightarrow{id\times(p\circ r)} (0,1) \times M((0,1), X) \xrightarrow{ev} X$$

where $j_t(\alpha) = (t, \alpha)$ and ev is the evaluation map $ev(t, f) = f(t)$. If $\alpha \in \Omega_{+s}(\Sigma(X_+))$ such that $\alpha(t) = x \wedge s$, then $v(\alpha) = x$. It is easy to check that the continuous homomorphism $v_{**} : \pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+)))) \to \pi_0^{top}(M_T^*(X))$ is the inverse of u_{**} since on the generator $[u_P]$ of $\pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+))))$, we have $v_{**}([u_P]) = v_{**}([u_x]) = P$

(where $x \in P$). ■

4.2.3 $\pi_1^{top}(\Sigma(X_+)) \cong F_R^{\pi_X}\left(\pi_0^{top}(X)\right)$

Recall the definition of $F_R^q(Y)$ for quotient map $q : X \to Y$ from the Appendix A.3. We are interested in the case $q = \pi_X$ which results in the quasitopological group $F_R^{\pi_X}\left(\pi_0^{top}(X)\right)$. By construction $F_R^{\pi_X}\left(\pi_0^{top}(X)\right)$ is the quotient space of $M_T^*(X)$ with respect to the canonical map $M_T^*(X) \to F(\pi_0(X))$. The main theorem of this chapter is

Theorem 4.53 $h_X : F_R^{\pi_X}\left(\pi_0^{top}(X)\right) \to \pi_1^{top}(\Sigma(X_+))$ *is a natural isomorphism of quasitopological groups.*

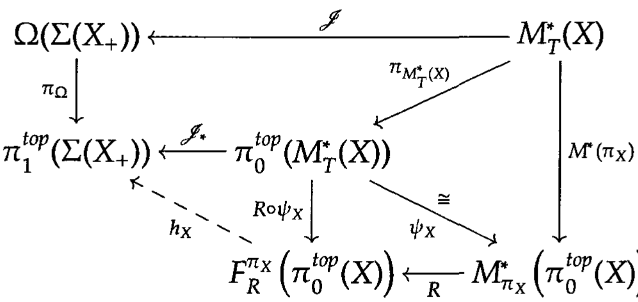
This theorem is particularly powerful in that X may be *any* topological space. Since $F_R^q(Y)$ is a topological group if and only if $id : F_R^q(Y) \cong F_M(Y)$, a direct consequence of this theorem is that:

Corollary 4.54 $\pi_1^{top}(\Sigma(X_+))$ *either fails to be a topological group or is the free topological group* $F_M(\pi_0^{top}(X))$ *on the path component space* $\pi_0^{top}(X)$.

Remark 4.55 This description of $\pi_1^{top}(\Sigma(X_+))$ becomes remarkably simple when X is totally path disconnected (i.e. $\pi_X : X \cong \pi_0^{top}(X)$). In this case we have

$$\pi_1^{top}(\Sigma(X_+)) \cong F_R^{\pi_X}\left(\pi_0^{top}(X)\right) \cong F_R\left(\pi_0^{top}(X)\right) \cong F_R(X).$$

We use the following commutative diagram to prove h_X is continuous.



Here $\mathscr{J}_* = \pi_0^{top}(\mathscr{J})$ is the map induced by $\mathscr{J}$ on path component spaces. Recall from Corollary 4.51, that the fibers of the composites $\pi_\Omega \circ \mathscr{J} : M_T^*(X) \to \pi_1^{top}(\Sigma(X_+))$ and $R \circ M^*(\pi_X) = R \circ (\psi_X \circ \pi_{M_T^*(X)}) : M_T^*(X) \to F_R^{\pi_X}\left(\pi_0^{top}(X)\right)$ are equal. Since

$$R \circ M^*(\pi_X) : \bigoplus_{n \geq 0} (X \oplus X^{-1})^n \to F_R^{\pi_X}\left(\pi_0^{top}(X)\right)$$

is quotient, the group isomorphism $h_X : F_R^{\pi_X}\left(\pi_0^{top}(X)\right) \to \pi_1^{top}(\Sigma(X_+))$ is always continuous Theorem 4.53 is equivalent to the assertion that h_X is open.

Outline of Proof 4.56 We have already proven that h_X is a continuous, group isomorphism. To prove that h_X is open, we take the following approach: It is shown in the proof of Theorem 4.50 that for $\alpha \in \Omega(\Sigma(X_+))$, all but finitely many of the restrictions $\alpha_{[c_m, d_m]}$ which are simple loops have image in the contractible neighborhood $X \wedge \left[0, \frac{1}{8}\right) \sqcup \left(\frac{7}{8}, 1\right]$. We assign to α, the word $\mathscr{D}(\alpha) = \alpha_{[c_{m_1}, d_{m_1}]} \ldots \alpha_{[c_{m_n}, d_{m_n}]}$ where the letters $\alpha_{[c_{m_i}, d_{m_i}]}$ are the non-trivial simple loops. This gives a "decomposition" function $\mathscr{D} : \Omega(\Sigma(X_+)) \to M_T^*(\Omega_{+s}(\Sigma(X_+)))$ to the free topological monoid on the space $\Omega_{+s}(\Sigma(X_+)) \sqcup \Omega_{+s}(\Sigma(X_+))^{-1} = \Omega_{+s}(\Sigma(X_+)) \sqcup \Omega_{-s}(\Sigma(X_+))$ of non-trivial simple loops. The use of this free topological monoid provides a convenient setting for forming neighborhoods of arbitrary loops from strings of neighborhoods of simple loops and is the key to proving Theorem 4.53 for arbitrary X.

Step 1: The topology of simple loops

Throughout the rest of this section let $U = X \wedge \left[0, \frac{1}{8}\right) \sqcup \left(\frac{7}{8}, 1\right]$. This is an arc-connected, contractible neighborhood and by the definition of $\mathscr{B}_{\Sigma(X_+)}$ contains all basic open neighborhoods of the basepoint x_0. We now prove a basic property of open neighborhoods of simple loops in the free path space $M(I, \Sigma(X_+))$. Recall that basic open neighborhoods in $M(I, \Sigma(X_+))$ are those described in Lemma 1.20 with respect the basis $\mathscr{B}_{\Sigma(X_+)}$.

Lemma 4.57 *Suppose $0 < \epsilon < \frac{1}{8}$ and $W = \bigcap_{i=1}^{m} \langle K_m^i, W_i \rangle$ is a basic open neighborhood of simple loop $\alpha : I \to \Sigma(X_+)$ in the free path space $M(I, \Sigma(X_+))$. There is a basic open neighborhood V_0 of x_0 in $\Sigma(X_+)$ contained in $X \wedge [0, \epsilon) \sqcup (1 - \epsilon, 1]$ and a basic open neighborhood $V = \bigcap_{j=1}^{n} \langle K_n^j, V_j \rangle$ of α in $M(I, \Sigma(X_+))$ contained in W such that:*

1. *$V_0 = V_1 = V_2 = \cdots = V_l = V_k = V_{k+1} = \cdots = V_n$ for integers $1 \leq l < k \leq n$.*
2. *The open neighborhoods $V_{l+1}, \ldots, V_{k-1}$ are of the form $A \wedge (a, b)$ where $A \in \mathscr{B}_X$ and $b - a < \epsilon$.*

Proof. Let $V_0 = (W_1 \cap W_m) \cap (X \wedge [0, \epsilon) \sqcup (1 - \epsilon, 1]) \subset U$. Since $\mathscr{B}_{\Sigma(X_+)}$ is closed under finite intersection $V_0 \in \mathscr{B}_{\Sigma(X_+)}$. There is an integer $M > 3$ such that m divides M and $\alpha(K_M^1 \sqcup K_M^M) \subseteq V_0$. Since α is simple we have $\alpha([\frac{1}{M}, \frac{M-1}{M}]) \subset X \wedge (0, 1)$. When $p = 2, ..., M - 1$ and $K_M^p \subseteq K_m^i$ we may cover $\alpha(K_M^p)$ with finitely many open neighborhoods contained in $W_i \cap (X \wedge (0,1))$ of the form $A \wedge (a,b)$ where $A \in \mathscr{B}_X$ and $b - a < \epsilon$. We then apply the Lebesgue lemma to take even subdivisions of I to find open neighborhoods $Y_i = \bigcap_{q=1}^{N_p} \langle K_{N_p}^q, Y_{p,q}^i \rangle \subseteq \langle I, W_i \rangle$ of the restricted path $\alpha_{K_M^p}$. Here each $Y_{p,q}^i$ is one of the open neighborhoods $A \wedge (a,b) \subseteq W_i$. We now use the induced neighborhoods of section 1.2 to define

$$V = \langle K_M^1 \sqcup K_M^M, V_0 \rangle \cap \bigcap_{p=2}^{M-1} \left((Y_i)^{K_M^p} \right).$$

This is an open neighborhood of α by definition, and it suffices to show that $V \subseteq W$. We suppose $\beta \in V$ and show that $\beta(K_m^i) \subseteq W_i$ for each i. Clearly, $\beta(K_M^1 \sqcup K_M^M) \subseteq W_1 \cap W_m$. If $p = 2, \ldots, M-1$ and $K_M^p \subseteq K_m^i$, then $\beta_{K_M^p} \in V_{K_M^p} \subseteq Y_i$ and $\beta(K_M^p) = \beta_{K_M^p}(I) \subseteq \bigcup_{q=1}^{N_p} Y_{p,q}^i \subseteq W_i$. We may write V as $V = \bigcap_{j=1}^{n} \langle K_n^j, V_j \rangle$ simply by finding an integer n which is divisible by M and every N_p and re-indexing the open neighborhoods V_0 and $Y_{p,q}^i$. In particular, we can set $V_j = V_0$ when $K_n^j \subseteq K_M^1 \cup K_M^M$. Additionally, if $H_{K_M^p}^{-1} : I \to K_M^p$ is the unique linear homeomorphism (as in section 1.2), then we let $V_j = Y_{p,q}^i$ whenever

$$K_n^j \subseteq H_{K_M^p}^{-1}\left(K_{N_p}^q\right) \subseteq K_M^p \subseteq K_m^i.$$

It is easy to see that both 1. and 2. in the statement are satisfied by V. ∎

We note some additional properties of the neighborhood V constructed in the previous lemma:

Remark 4.58 For each path $\beta \in V$ we have,

$$\beta\left(\left(\bigcup_{j=1}^{l} K_n^j\right) \cup \left(\bigcup_{j=k}^{n} K_n^j\right)\right) \subseteq U \text{ and } x_0 \notin \beta\left(\bigcup_{j=l+1}^{k-1} K_n^j\right)$$

This follows directly from the conditions 1. and 2. in the lemma.

Remark 4.59 It is previously noted that there are disjoint open neighborhoods W_+, W_0, and W_- in $M(I, \Sigma(X_+))$ containing $\Omega_{+s}(\Sigma(X_+))$, $\Omega_{0s}(\Sigma(X_+))$, and $\Omega_{-s}(\Sigma(X_+))$ respectively. Consequently, if α has positive orientation, then we may take $V \subseteq W_+$ such that $V \cap \Omega_s(\Sigma(X_+)) \subseteq \Omega_{+s}(\Sigma(X_+))$, i.e. all simple loops in V also have positive orientation. The same holds for the negative and trivial case. In some sense, this means that V, when thought of as an instruction set, is "good enough" to distinguish orientations of simple loops.

Remark 4.60 We now give a construction necessary for Step 4 which produces a simple loop $\mathscr{S}_V(\beta) \in V$ for each path $\beta \in V$. For brevity, we let $[0, r] = \bigcup_{j=1}^{l} K_n^j$, $[r, s] = \bigcup_{j=l+1}^{k-1} K_n^j$, and $[s, 1] = \bigcup_{j=k}^{n} K_n^j$ and define $\mathscr{S}_V(\beta)$ piecewise by letting $\mathscr{S}_V(\beta)$ be equal to β on the middle interval $[r, s]$ (i.e. $\mathscr{S}_V(\beta)_{[r,s]} = \beta_{[r,s]}$). We then demand that $\mathscr{S}_V(\beta)$ restricted to $[0, r]$ is an arc in V_0 connecting x_0 to $\beta(r)$ and similarly $\mathscr{S}_V(\beta)$ restricted to $[s, 1]$ is an arc in V_0 connecting $\beta(s)$ to x_0. Since the image of $\mathscr{S}_V(\beta)$ on $[0, r] \cup [s, 1]$ remains in V_0, it follows that $\mathscr{S}_V(\beta) \in V$. Additionally, Remark 4.58 and the use of arcs to define $\mathscr{S}_V(\beta)$ means that $\mathscr{S}_V(\beta)$ is a simple loop.

Step 2: Decomposition of arbitrary loops

Here we assign to each loop in $\Sigma(X_+)$, a (possibly empty) word of simple loops with orientation. We again use the observation, that $\Omega_{+s}(\Sigma(X_+))$ and $\Omega_{-s}(\Sigma(X_+)) = \Omega_{+s}(\Sigma(X_+))^{-1}$ are disjoint homeomorphic subspaces of $M(I, \Sigma(X_+))$. The free topological monoid on $\Omega_{+s}(\Sigma(X_+)) \sqcup \Omega_{-s}(\Sigma(X_+)) = \Omega_{+s}(\Sigma(X_+))^{-1}$ is just the free topological monoid with continuous involution $M_T^*(\Omega_{+s}(\Sigma(X_+)))$. We make no distinction between the one letter word α^{-1} in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$ and the reverse loop $\beta = \alpha^{-1} \in \Omega_{+s}(\Sigma(X_+))^{-1}$. Similarly, a basic open neighborhood of α^{-1} in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$ corresponds to an open neighborhood of β in $\Omega_{+s}(\Sigma(X_+))^{-1}$. We now define the "decomposition" function $\mathscr{D} : \Omega(\Sigma(X_+)) \to M_T^*(\Omega_{+s}(\Sigma(X_+)))$. In step 4 we refer to the details described here.

Decomposition 4.61 Suppose $\beta \in M(I, \{0,1\}; \Sigma(X_+), \{x_0\})$ is an arbitrary loop. First, if β has image contained in U (i.e. $\beta \in \langle I, U \rangle$), then we let $\mathscr{D}(\beta) = e$ be the empty word. Suppose then that $\beta(I) \not\subseteq U$. The pullback $\beta^{-1}(X \wedge (0,1)) = \coprod_{m \in M}(c_m, d_m)$ is open in I where M is a countable indexing set with ordering induced by the ordering of I. Each restricted loop $\beta_m = \beta_{[c_m, d_m]} : I \to \Sigma(X_+)$ is a simple loop. Remark 4.33.5 implies that all but finitely many of these simple loops have image in U and so we may take $m_1 < ... < m_k$ to be the indices of M corresponding to those $\beta_{m_1}, ..., \beta_{m_k}$ with image not contained in U. Note that if $C = I - \bigcup_{i=1}^{k}(c_{m_i}, d_{m_i})$, then $\beta \in \langle C, U \rangle$. If none of the β_{m_i} have orientation, we again let $\mathscr{D}(\beta) = e$. On the other hand, if one of the β_{m_i} has orientation, we let $m_{i_1} < ... < m_{i_n}$ be the indices corresponding to the simple loops $\beta_j = \beta_{m_{i_j}}$ which have either positive or negative orientation. We then let $\mathscr{D}(\beta)$ be the word $\beta_1\beta_2...\beta_n$ in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$.

Remark 4.62 Informally, $\mathscr{D}(\beta)$ denotes the word composed of the simple loops of

β which contribute a letter in the unreduced word of the homotopy class $[\beta]$. We may suppose that β_j has image in $P_j \wedge I$ and orientation $\epsilon_j \in \{\pm 1\}$, or equivalently that $[\beta_j] = [u_{P_j}]^{\epsilon_j}$. Clearly $\beta \simeq *_{j=1}^{n}\beta_j$ and $h_X^{-1}([\beta]) = R(P_1^{\epsilon_1}P_n^{\epsilon_n}\dots P_n^{\epsilon_n}) \in F(\pi_0(X))$.

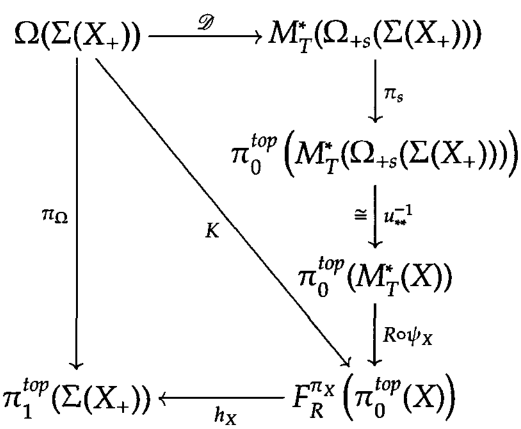
Step 3: Factoring π_Ω

We factor the quotient map $\pi_\Omega : \Omega(\Sigma(X_+)) \to \pi_1^{top}(\Sigma(X_+))$ into a composite using the following functions:

1. The decomposition function $\mathscr{D} : \Omega(\Sigma(X_+)) \to M_T^*(\Omega_{+s}(\Sigma(X_+)))$
2. The quotient map $\pi_s : M_T^*(\Omega_{+s}(\Sigma(X_+))) \to \pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+))))$ identifying path components (homotopy classes of positively oriented simple loops)
3. The natural homeomorphism $u_{**}^{-1} : \pi_0^{top}\left(M_T^*(\Omega_{+s}(\Sigma(X_+)))\right) \to \pi_0^{top}(M_T^*(X))$ of Lemma 4.52
4. The quotient map $R \circ \psi_X : \pi_0^{top}(M_T^*(X)) \to F_R^{\pi_X}\left(\pi_0^{top}(X)\right)$
5. The continuous, group isomorphism $h_X : F_R^{\pi_X}\left(\pi_0^{top}(X)\right) \to \pi_1^{top}(\Sigma(X_+))$

We let $K = R \circ \psi_X \circ u_{**}^{-1} \circ \pi_s \circ \mathscr{D} : \Omega(\Sigma(X_+)) \to F_R^{\pi_X}\left(\pi_0^{top}(X)\right)$ be the composite of 1.-4. and $K' = R \circ \psi_X \circ u_{**}^{-1} \circ \pi_s : M_T^*(\Omega_{+s}(\Sigma(X_+))) \to F_R^{\pi_X}\left(\pi_0^{top}(X)\right)$ be the continuous (and even quotient) composite of 2.-4.

Lemma 4.63 *The following diagram commutes:*



The function $\mathscr{D}$ will not be continuous even when X contains only a single point (i.e. $\Sigma(X_+) \cong S^1$). This is illustrated by the fact that any open neighborhood of a concatenation $\alpha * \alpha^{-1}$ for simple loop α contains a trivial simple loop β which may be found by "pulling" the middle of $\alpha * \alpha^{-1}$ off of x_0 within a sufficiently small neighborhood of x_0.

Step 4: Continuity of K

Lemma 4.64 $K : \Omega(\Sigma(X_+)) \to F_R^{\pi X}\left(\pi_0^{top}(X)\right)$ *is continuous.*

Proof. Suppose W is open in $F_R^{\pi X}\left(\pi_0^{top}(X)\right)$ and $\beta \in K^{-1}(W)$. We now refer to the details of the decomposition of β in step 2. If β has image in U, then clearly $\beta \in \langle I, U\rangle \subseteq \mathscr{D}^{-1}(e) \subseteq K^{-1}(W)$. Suppose, on the other hand, that some simple loop restriction β_{m_i} has image intersecting $\Sigma(X_+) - U$ and $\mathscr{D}(\beta) = \beta_1\beta_2...\beta_n$ is the (possibly empty) decomposition of β. Recall from our the notation in step 2, that $\beta_j = \beta_{m_{i_j}}$, $j = 1, ..., n$ are the β_{m_i} with orientation. Since $K' = R \circ \psi_X \circ u_{**}^{-1} \circ \pi_s$ is continuous, $(K')^{-1}(W)$ is an open neighborhood of $\mathscr{D}(\beta)$ in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$.

Recall that $\beta \in \langle C, U \rangle$, where $C = I - \bigcup_{i=1}^{k}(c_{m_i}, d_{m_i})$. We construct the rest of desired open neighborhood of β by defining an open neighborhood of each β_{m_i} and taking the intersection of the induced neighborhoods.

If $i \neq i_j$ for any $j = 1, ..., n$, then β_{m_i} does not appear as a letter in the decomposition of β and must be trivial. We apply Lemma 4.57, to find an open neighborhood $V_i = \bigcap_{l=1}^{N_i} \langle K_{N_i}^l, V_l^i \rangle$ of β_{m_i} in $M(I, \Sigma(X_+))$ which satisfies both 1. and 2. in the statement. By Remark 4.59, we may also assume that $V_i \cap \Omega_s(\Sigma(X_+)) \subseteq \Omega_{0s}(\Sigma(X_+))$, i.e all simple loops in V_i are trivial.

If $i = i_j$ for some $j = 1, ..., n$, then $\beta_j = \beta_{m_{i_j}}$ has orientation ϵ_j. Since $(K')^{-1}(W)$ is an open neighborhood of $\mathscr{D}(\beta) = \beta_1\beta_2...\beta_n$ in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$ and basic open neighborhoods in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$ are products of open neighborhoods in $\Omega_{+s}(\Sigma(X_+))$ and $\Omega_{+s}(\Sigma(X_+))^{-1}$, we can find basic open neighborhoods $V_{i_j} = \bigcap_{l=1}^{N_{i_j}} \langle K_{N_{i_j}}^l, U_l^{i_j} \rangle$ of β_j in $M(I, \Sigma(X_+))$ such that

$$W_j = V_{i_j} \cap \Omega_{+s}(\Sigma(X_+))^{\epsilon_j} \text{ and } \beta_1\beta_2...\beta_n \in W_1W_2...W_n \subseteq (K')^{-1}(W).$$

We assume each V_{i_j} satisfies 1. and 2. of Lemma 4.57 and by Remark 4.59 that $V_{i_j} \cap \Omega_s(\Sigma(X_+)) \subseteq \Omega_{+s}(\Sigma(X_+))^{\epsilon_j}$.

Let

$$\mathscr{U} = \langle C, U \rangle \cap \left(\bigcap_{i=1}^{k} \left(V_i^{[c_{m_i}, d_{m_i}]} \right) \right)$$

so that $\mathscr{V} = \mathscr{U} \cap \Omega(\Sigma(X_+))$ is an open neighborhood of β in the loop space. We claim that each loop $\gamma \in \mathscr{V}$ is homotopic to a loop γ' such that $\mathscr{D}(\gamma') \in (K')^{-1}(W)$. If this is done, we have

$$h_X(K(\gamma)) = \pi_\Omega(\gamma) = \pi_\Omega(\gamma') = h_X(K(\gamma'))$$

and since h_X is a bijection,

$$K(\gamma) = K(\gamma') = K'(\mathscr{D}(\gamma')) \in W$$

This gives $K(\mathscr{V}) \subseteq W$, proving the continuity of K.

We define γ' piecewise and begin by setting $\gamma'(C) = x_0$. The restricted path $\gamma_i = \gamma_{[c_{m_i}, d_{m_i}]} : I \to \Sigma(X_+)$ lies in the open neighborhood $\mathscr{U}_{[c_{m_i}, d_{m_i}]} \subsetneq V_i$. We now define γ' on $[c_{m_i}, d_{m_i}]$ by using the construction of Remark 4.60. We set

$$\gamma_i' = (\gamma')_{[c_{m_i}, d_{m_i}]} = \mathscr{S}_{V_i}(\gamma_i)$$

which by construction is a simple loop in V_i. Intuitively, we have replaced the portions of γ which are close to x_0 ("close" meaning with respect to $\mathscr{V}$) with arcs and constant paths. Since $\gamma_i' = \gamma'(C) = x_0 \in U$ and $(\gamma')_{[c_{m_i}, d_{m_i}]} \in V_i$ for each i, it follows that $\gamma' \in \mathscr{V}$. Moreover, since $\gamma(t) \neq \gamma'(t)$ only when $\gamma(t)$ and $\gamma'(t)$ both lie in the path connected, contractible neighborhood U, it is obvious that $\gamma \simeq \gamma'$. It now suffices to show that $\mathscr{D}(\gamma') \in (K')^{-1}(W)$. We begin by checking which of the simple loops $\gamma_i' \in V_i \cap \Omega(\Sigma(X_+))$ have orientation and will appear in the word $\mathscr{D}(\gamma')$. If $i \neq i_j$ for any j, all simple loops in V_i, including γ_i' are trivial. Therefore γ_i' has no orientation and will not appear as a letter in $\mathscr{D}(\gamma')$. If this is the case for all i so that $\mathscr{D}(\beta)$ is the empty word, then $\mathscr{D}(\gamma')$ must also be the empty word $e \in (K')^{-1}(W)$. Suppose on the other hand that $\mathscr{D}(\beta) = \beta_1\beta_2...\beta_n \neq e$ and $i = i_j$ for some j. The neighborhood $V_i = V_{i_j}$ was chosen so that all simple loops in V_{i_j} have orientation ϵ_j. Since $\gamma_i' \in V_{i_j}$ is simple, it has orientation ϵ_j and we have $\gamma_i' \in V_{i_j} \cap \Omega_{+s}(\Sigma(X_+))^{\epsilon_j} = W_j$. Therefore $\mathscr{D}(\gamma') = \gamma_{i_1}'\gamma_{i_2}'...\gamma_{i_n}' \in W_1W_2...W_n \subseteq (K')^{-1}(W)$. ■

Since K is continuous, $\pi_\Omega = h_X \circ K$ is quotient, and h_X is bijective, h_X is a homeomorphism.

4.2.4 The weak suspension spaces $w\Sigma(X_+)$ and $\pi_1^{top}(w\Sigma(X_+))$

We pause here to note a deficiency of the suspension spaces $\Sigma(X_+)$: $\Sigma(X_+)$ is not always first countable at x_0. This occurs particularly when X is not compact.

Proposition 4.65 *If X is compact, then there is a countable neighborhood base at the basepoint x_0 consisting of neighborhoods of the form $X \wedge \left[0, \frac{1}{n}\right) \sqcup \left(\frac{n-1}{n}, 1\right]$.*

Proof. If X is compact and $V = \bigcup_{x\in X} (U^x \wedge [0, t_x) \cup (1 - t_x, 1])$ is a basic open neighborhood of x_0 in $\Sigma(X_+)$ for pointwise open cover $\{U^x\}_{x\in X}$ of X, then we may find $x_1, .., x_n$ such that $\{U^{x_1}, ..., U^{x_n}\}$ is a finite subcover. Find an integer n such that $0 < \frac{1}{n} \leq \min_i t_{x_i}$. We now have

$$X \wedge \left[0, \frac{1}{n}\right) \sqcup \left(\frac{n-1}{n}, 1\right] = \bigcup_{i=1}^{n} \left(U^{x_i} \wedge \left[0, \frac{1}{n}\right) \sqcup \left(\frac{n-1}{n}, 1\right]\right) \subset V$$

■

Unfortunately, if X is a non-compact, first-countable (resp. metric) space such as $\mathbb{Q}$, then $\Sigma(X_+)$ may not be first countable (resp. a metric space). For this reason we consider a slightly weaker topology on the underlying set of $\Sigma(X_+)$, and denote the resulting space as $w\Sigma(X_+)$. A basis for the topology of $w\Sigma(X_+)$ is given by subsets of the form $V \wedge (a,b)$ and $X \wedge \left[0,\frac{1}{n}\right) \sqcup \left(\frac{n-1}{n},1\right]$, for $V \in \mathscr{B}_X$ and integer $n \geq 2$. The identity function $id : \Sigma(X_+) \to w\Sigma(X_+)$ is continuous since the topology of $w\Sigma(X_+)$ is coarser than that of $\Sigma(X_+)$. The *weak suspension* $w\Sigma(X_+)$ provides us with some of our most interesting examples and has a few advantages over $\Sigma(X_+)$.

Fact 4.66 If X is a subspace of $\mathbb{R}^n$, then $w\Sigma(X_+)$ may be embedded as a subspace of $\mathbb{R}^{n+1}$. In particular, we may suppose $X \subseteq [1,2]^n \times \{0\} \subset \mathbb{R}^{n+1}$. For $a = (a_1, ..., a_n, 0) \in X$, let $C_a \subset \mathbb{R}^{n+1}$ be the circle which is the intersection of the n-sphere $\left\{x = (x_1, ..., x_{n+1}) | \left\|x - \frac{a}{2}\right\| = \left\|\frac{a}{2}\right\|\right\}$ and the plane spanned by vectors a and $(0, ..., 0, 1)$. One can define a homeomorphism $w\Sigma(X_+) \to \bigcup_{x\in X} C_x$ quite easily. It is not necessarily true that $\Sigma(X_+)$ is homeomorphic to $\bigcup_{x\in X} C_x$ if X is not compact.

The arguments in this section may be repeated to compute $\pi_1^{top}(w\Sigma(X_+))$ or one may prove the following theorem.

Theorem 4.67 *The identity map $id : \Sigma(X_+) \to w\Sigma(X_+)$ is a homotopy equivalence and therefore induces a natural isomorphism of quasitopological groups $id : \pi_1^{top}(\Sigma(X_+)) \to \pi_1^{top}(w\Sigma(X_+))$.*

Proof. For $0 < \epsilon < \frac{1}{3}$, let $H_\epsilon : [\epsilon, 1-\epsilon] \to I$ be the unique increasing linear homeomorphism and $c_\epsilon : w\Sigma(X_+) \to \Sigma(X_+)$ be the map collapsing the contractible subspace $X \wedge [0,\epsilon) \sqcup (1-\epsilon, 1]$ to x_0 and taking $x \wedge t$ to $x \wedge H_\epsilon(t)$ for $t \in [\epsilon, 1-\epsilon]$. Let $c_0 = id_{\Sigma(X_+)}$. We check that the composites $c_{\frac{1}{3}} \circ id_{w\Sigma(X_+)} : \Sigma(X_+) \to \Sigma(X_+)$ and $id_{w\Sigma(X_+)} \circ c_{\frac{1}{3}} : w\Sigma(X_+) \to w\Sigma(X_+)$ are homotopic to the respective identities. Consider the map $h : X \times I \times I \to X \times I$ given by $h(x,s,t) = \left(x, H_{\frac{1-t}{3}}(s)\right)$. Composing with the quotient map $q : X \times I \to \Sigma(X_+)$, we see that $q \circ h : X \times I \times I \to \Sigma(X_+)$ sends $A = X \times \{0,1\} \times I$ to the basepoint x_0. This induces the map $G : \Sigma(X_+) \times I \cong X \times I \times I/A \to \Sigma(X_+)$ which is the homotopy $c_{\frac{1}{3}} \circ id_{w\Sigma(X_+)} \simeq id_{\Sigma(X_+)}$. It is obvious that $G : w\Sigma(X_+) \times I \to w\Sigma(X_+)$ is continuous on points in $w\Sigma(X_+) \times I - G^{-1}(x_0)$. Let $U = X \wedge [0,r) \sqcup (1-r,1]$, $r < \frac{1}{2}$ be a basic open neighborhood of x_0 in $w\Sigma(X_+)$. Then

$$G^{-1}(U) = \{(x \wedge t, s) | (2r-1)(t-1) > 3(s-r)\} \cup \{(x \wedge t, s) | (1-2r)(t-1) < 3(s+r-1)\}$$

is clearly open in $w\Sigma(X_+) \times I$. ■

4.3 The topological properties of $\pi_1^{top}(\Sigma(X_+))$

We now use the computation of the previous section and the results from the Appendix to study the topology of $\pi_1^{top}(\Sigma(X_+))$. In particular, we are interested in classifying the spaces X for which $\pi_1^{top}(\Sigma(X_+))$ is a topological group. The main theorem here (Theorem 4.75) characterizes when $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group. In the results preceding Theorem 4.75, $\pi_1^{top}(\Sigma(X_+))$ need not be Hausdorff.

Proposition 4.68 *(See Lemma A.50) The following are equivalent for an arbitrary space X:*

1. *$\pi_1^{top}(\Sigma(X_+))$ is a topological group.*

2. *$\pi_1^{top}(\Sigma(X_+))$ is isomorphic to the free topological group $F_M\left(\pi_0^{top}(X)\right)$.*

3. *The identities* $F_R^{\pi_X}\left(\pi_0^{top}(X)\right) \to F_R\left(\pi_0^{top}(X)\right) \to F_M\left(\pi_0^{top}(X)\right)$ *are homeomorphisms.*

Corollary 4.69 *(See Corollary A.51) If* $F_R(X)$ *is a topological group, then so is* $\pi_1^{top}(\Sigma(X_+))$.

Corollary 4.70 *(See Corollary A.52) If all powers of the quotient map* $\pi_X : X \to \pi_0^{top}(X)$ *are quotient and* $F_R\left(\pi_0^{top}(X)\right)$ *is a topological group, then* $\pi_1^{top}(\Sigma(X_+))$ *is a topological group.*

Corollary 4.71 *If X is a Tychonoff,* k_ω*-space (defined in 1.25), then* $\pi_1^{top}(\Sigma(X_+))$ *is a topological group.*

Proof. It is shown in [MMO73] that if X is a Tychonoff, k_ω-space, then (1) $F_M(Y) \cong \varinjlim_n F_M(Y)_n$ and (2) $R_n : \bigoplus_{i=0}^n (X \oplus X^{-1})^i \to F_M(X)_n$ is a quotient map for each $n \geq 1$. Sipacheva shows in [Sip05, Statement 5.1] that if (1) and (2) are true, then $R : M_T^*(X) \to F_M(X)$ is quotient. Whenever $R : M_T^*(X) \to F_M(X)$, $F_R(X)$ is a topological group. By Corollary 4.69, $\pi_1^{top}(\Sigma(X_+))$ is a topological group. ■

Corollary 4.72 *(See Example A.53) If X is an A-space, then* $\pi_1^{top}(\Sigma(X_+))$ *is a topological group which is an A-space.*

Corollary 4.73 *If X is first countable and* $\pi_0^{top}(X)$ *is an A-space, then* $\pi_1^{top}(\Sigma(X_+))$ *is a topological group which is an A-space.*

Proof. Since X and $\pi_0^{top}(X)$ are first countable, Fact 1.24.5 implies that all powers of π_X are quotient. By Corollary A.52, $\pi_1^{top}(\Sigma(X_+)) \cong F_R^{\pi_X}\left(\pi_0^{top}(X)\right) \cong F_R\left(\pi_0^{top}(X)\right)$. Since $\pi_0^{top}(X)$ is an A-space, $F_R\left(\pi_0^{top}(X)\right)$ is a topological group which is an A-space by Example A.53. ■

Example 4.74 Let X be either the topologists sine curve

$$T = \{(0,0)\} \cup \left\{(x,y) \in \mathbb{R}^2 | y = \sin\left(\frac{1}{x}\right), 0 < x \leq \pi\right\}$$

or the closed topologist's sine curve $T_C = T \cup (\{0\} \times [-1,1])$. We noted in Chapter 3 that $\pi_0^{top}(T) \cong \pi_0^{top}(T_C) \cong \mathbb{S}$ where $\mathbb{S} = \{0,1\}$ is the Sierpinski space with topology $\{\emptyset, \{1\}, \{0,1\}\}$. By Fact 4.66, $w\Sigma(X_+)$ embeds into $\mathbb{R}^3$. It is locally 1-connected but not locally path connected. Since X is first countable and $\mathbb{S}$ is an A-space (it is finite), the previous Corollary indicates that there are isomorphism $\pi_1^{top}(w\Sigma(X_+)) \cong \pi_1^{top}(\Sigma(X_+)) \cong F_M(\mathbb{S})$ of topological groups.

A somewhat more practical characterization appears the following Theorem which is a special case of A.67. This theorem reduces the characterization of X for which $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group to a separation property and three well-known classification problems in topology.

Theorem 4.75 *Let X be Hausdorff. Then $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group if and only if the following three conditions hold:*

1. *$\pi_0^{top}(X)$ is Tychonoff.*

2. $F_M\left(\pi_0^{top}(X)\right) \cong \varinjlim_n F_M(\pi_0^{top}(X))_n$.

3. *$R_n : \bigoplus_{i=0}^{n}(\pi_0^{top}(X) \oplus \pi_0^{top}(X)^{-1})^i \to F_M(\pi_0^{top}(X))_n$ is a quotient map for each $n \geq 1$.*

4. *$\pi_X^n : X^n \to \pi_0^{top}(X)^n$ is a quotient map for each $n \geq 1$.*

The following example is perhaps the most interesting example arising from this Theorem:

Example 4.76 Let $X = (0,1) \cap \mathbb{Q} \subset \mathbb{R}$ and $Y = \bigcup_{a \in X} C_a \subset \mathbb{R}^2$ be as in Fact 4.66. This space is a locally 1-connected (but non locally path connected), planar subspace (whereas $\Sigma(X_+)$ is not even a metric space). We have $Y \cong w\Sigma(X_+) \simeq \Sigma(X_+)$ and so $\pi_1^{top}(Y) \cong \pi_1^{top}(\Sigma(X_+)) \cong F_R^{\pi_X}\left(\pi_0^{top}(X)\right) \cong F_R(X) \cong F_R(\mathbb{Q})$. The last two isomorphisms (in **qTopGrp**) come from the fact that $X \cong \mathbb{Q}$ is totally path disconnected. Since X is Tychonoff, Theorem A.67 that $F_R(X)$ is a topological group if and only if $F_M(X) \cong \varinjlim_n F_M(X)_n$ and $R_n : \bigoplus_{i=1}^{n}(X \oplus X^{-1})^i \to F_M(X)_n$ is quotient for all $n \geq 1$. However, it is shown in [FOT79] that both of these conditions fail for $X = \mathbb{Q}$. Therefore, $\pi_1^{top}(Y) \cong F_R(\mathbb{Q})$ is not a topological group.

Corollary 4.77 π_1^{top} : **hTop**$_*$ $\to$ **qTopGrp** *does not preserve finite products.*

Proof. It is a direct consequence of 2.34 and 2.39 that if the canonical (continuous) group isomorphism $\pi_1^{top}(X \times X) \to \pi_1^{top}(X) \times \pi_1^{top}(X)$ is a homeomorphism, then $\pi_1^{top}(X)$ is a topological group. Example 4.76 shows that this cannot always happen. ■

Applying what we know about powers of quotient maps from Chapter 1.2, we have:

Corollary 4.78 *If X is first countable and $\pi_0^{top}(X)$ is a first countable, Tychonoff, k_ω-space, then $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group. This also holds when we replace "first countable" with "locally compact Hausdorff."*

Proof. If $X, \pi_0^{top}(X)$ are first countable (locally compact Hausdorff) then the powers $\pi_X^n : X^n \to \pi_0^{top}(X)^n$ are all quotient. As mentioned in the proof of Corollary 4.71, if $\pi_0^{top}(X)$ is Tychonoff and k_ω, then conditions 2. and 3. of Theorem 4.75 hold. Since all conditions of Theorem 4.75 are satisfied, $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group. ■

We have the following simplification (due to A.65) when X is totally path disconnected, i.e. $\pi_X : X \cong \pi_0^{top}(X)$. Recall that we have $\pi_1^{top}(\Sigma(X_+)) \cong F_R(X)$ in this case.

Corollary 4.79 *Let X be a totally path disconnected Hausdorff space. Then $\pi_1^{top}(\Sigma(X_+)) \cong F_R(X)$ is a topological group if and only if the following conditions hold:*

1. *X is Tychonoff.*

2. *$F_M(X) \cong \varinjlim_n F_M(X)_n$.*

3. *$R_n : \bigoplus_{i=0}^n (X \oplus X^{-1})^i \to F_M(X)_n$ is a quotient map for each $n \geq 1$.*

Example 4.80 If X is the cantor set, $\beta\mathbb{N}$, the one-point compactification of a discrete space, or any other totally disconnected, compact space, then $\pi_1^{top}(\Sigma(X_+))$ is isomorphic to the free topological group $F_M(X)$.

We may also give a nice characterization of discreteness.

Corollary 4.81 *The following are equivalent:*

1. *$\pi_1^{top}(\Sigma(X_+))$ is a discrete group.*

2. *$\pi_0^{top}(X)$ is discrete.*

3. *X is semilocally 0-connected.*

Proof. 1. $\Leftrightarrow$ 2. follows from Corollary A.54. 2. $\Leftrightarrow$ 3. was proven in Proposition 2.54. ■

Proposition 4.82 *(See Theorem A.63) The following are equivalent:*

1. $\pi_1^{top}(\Sigma(X_+))$ *is* T_1.

2. $\pi_X : X \to \pi_0^{top}(X)$ *is separating, i.e. for each* $x_1, x_2 \in X$ *with distinct path components* $\pi_X(x_1), \pi_X(x_2)$, *there are open neighborhood* U_i *of* x_i *in* X *such that* $\pi_X(U_1) \cap \pi_X(U_2) = \emptyset$.

3. *The canonical injection* $\pi_0^{top}(X^n) \to \pi_1^{top}(\Sigma(X_+))$ *is a closed embedding for each* $n \geq 1$.

Corollary 4.83 *If* $\pi_0^{top}(X)$ *is Hausdorff, then* $\pi_1^{top}(\Sigma(X_+))$ *is* T_1.

Corollary 4.84 *Let (P) be a topological property hereditary to closed subspaces. Then if* $\pi_1^{top}(\Sigma(X_+))$ *is* T_1 *and has property (P), then for each* $n \geq 1$, $\pi_0^{top}(X^n)$ *also has property (P). For example, if* $\pi_1^{top}(\Sigma(X_+))$ *is Hausdorff (resp.* T_1 *and regular,* T_1 *and normal,* T_1 *and paracompact), then so is* $\pi_0^{top}(X^n)$.

Corollary 4.85 *(See Proposition 4.82 and Theorem A.71) Let X be such that* $\pi_1^{top}(\Sigma(X_+))$ *is* T_1. *Then* $\pi_1^{top}(\Sigma(X_+))$ *is first countable if and only if* $\pi_1^{top}(\Sigma(X_+))$ *is discrete.*

Example 4.86 Let $\mathbb{Q}_K$ denote the rational numbers with the subspace topology of the real line with the K-topology [Mun00]. Then $\pi_0^{top}(\mathbb{Q}_K) \cong \mathbb{Q}_K$ is Hausdorff and totally path disconnected but is not regular. Since $\mathbb{Q}_K$ is Hausdorff, $\pi_1^{top}(\Sigma(\mathbb{Q}_K)_+) \cong$

$F_R(\mathbb{Q}_K)$ is T_1. If it is a topological group, it must also be regular. But the non-regular subspace $\mathbb{Q}_K$ embeds in $F_R(\mathbb{Q}_K)$. Therefore $F_R(\mathbb{Q}_K)$ is not a topological group.

Example 4.87 Using Remark 3.2, we can produce a large class of spaces each with topological fundamental group failing to be a topological group. For every Hausdorff, non-completely regular space Y, there is a paracompact Hausdorff space $X = \mathcal{H}(Y)$ such that $\pi_0^{top}(X) \cong Y$. Since $\pi_0^{top}(X)$ is Hausdorff, $\pi_1^{top}(\Sigma(X_+))$ is T_1 but by the previous corollary $\pi_1^{top}(\Sigma(X_+))$ cannot be a topological group.

Proposition 4.88 *$\pi_1^{top}(\Sigma(X_+))$ is functionally Hausdorff if and only if $\pi_0^{top}(X)$ is functionally Hausdorff.*

Proof. If $\pi_0^{top}(X)$ is functionally Hausdorff, then by Lemma A.37 so is $F_M\left(\pi_0^{top}(X)\right)$. Since $\pi_1^{top}(\Sigma(X_+)) \cong F_R^{\pi x}\left(\pi_0^{top}(X)\right) \to F_M\left(\pi_0^{top}(X)\right)$ is a continuous group isomorphism, $\pi_1^{top}(\Sigma(X_+))$ must also be functionally Hausdorff. Conversely, if $\pi_1^{top}(\Sigma(X_+))$ is functionally Hausdorff the fact that $u_* : \pi_0^{top}(X) \to \pi_1^{top}(\Sigma(X_+))$ is a continuous injection implies that $\pi_0^{top}(X)$ is functionally Hausdorff. ■

Certainly 4.84 implies that whenever $\pi_1^{top}(\Sigma(X_+))$ is normal and T_1, then $\pi_0^{top}(X^n)$ is normal for each n. The author does not know of a partial converse simpler than the following.

Proposition 4.89 *Suppose $\pi_1^{top}(\Sigma(X_+))$ is T_1. Then $\pi_1^{top}(\Sigma(X_+))$ is normal if and only if the closed subspace $F_R^{\pi x}(\pi_0^{top}(X))_n$ of $F_R^{\pi x}(\pi_0^{top}(X))$ consisting of words of length at most n is normal for each $n \geq 1$.*

Proof. Suppose $\pi_1^{top}(\Sigma(X_+))$ is normal. It is shown in the Appendix A.60 that $F_R^{\pi x}(\pi_0^{top}(X))_n$ is closed in $\pi_1^{top}(\Sigma(X_+)) \cong F_R^{\pi x}(\pi_0^{top}(X))$. Therefore each $F_R^{\pi x}(\pi_0^{top}(X))_n$

is normal. Conversely, suppose each $F_R^{\pi x}(\pi_0^{top}(X))_n$ is normal. It is shown in the Appendix A.61.1 that $F_R^{\pi x}(\pi_0^{top}(X)) \cong \varinjlim_n F_R^{\pi x}(\pi_0^{top}(X))_n$ and it is well-known that a space which is the inductive limit of closed normal subspaces is normal [Dug66, pg. 158]. ∎

4.4 The topological properties of π_1^τ

Recall from Chapter 2.5.1 that $\pi_1^\tau(X)$ is the topological group $\tau(\pi_1^{top}(X))$ where τ : **GrpwTopTopGrp** is left adjoint to the inclusion **TopGrp** $\rightarrow$ **GrpwTop**. Its topology is characterized by the fact that the map $m : F_M(\pi_1^{top}(X)) \rightarrow \pi_1^\tau(X)$ induced by the identity of $\pi_1(X)$ is quotient. A useful description of the universal property of $\pi_1^\tau(X)$ is:

Corollary 4.90 *If $\Phi : \pi_1^\tau(X) \rightarrow G$ is a homomorphism to a topological group G such that $\Phi \circ \pi : \Omega(X) \rightarrow \pi_1^\tau(X) \rightarrow G$ is continuous, then Φ is also continuous.*

Some of the basic properties of these topological groups follow from the general results in Chapter 2.5.1. One of the most interesting is the following special case of 2.74:

Corollary 4.91 *The identity $\pi_1^{top}(X) \rightarrow \pi_1^\tau(X)$ is continuous and is a homeomorphism if and only if $\pi_1^{top}(X)$ is a topological group.*

The development of the theory of the quotient topology is important to the study of the topology of $\pi_1^\tau(X)$.

Corollary 4.92 *If x_0, x_1 lie in the same path component of X, then $\pi_1^\tau(X, x_0) \cong \pi_1^\tau(X, x_1)$.*

Proof. Apply τ to the isomorphism of 4.1. ■

Corollary 4.93 *If X has a 1-connected cover, then $\pi_1^\tau(X)$ is discrete.*

Proof. If X has a 1-connected cover, then $\pi_1^{top}(X)$ is discrete by Corollary 4.27. But $\pi_1^{top}(X)$ is discrete if and only if $\pi_1^\tau(X)$ is discrete by 2.74. ■

Corollary 4.94 *$\pi_1^\tau(X)$ is discrete for any CW-complex or manifold X.*

Since we construct $\pi_1^\tau(X)$ by removing open sets from the topology of $\pi_1^{top}(X)$ there may be concern over when $\pi_1^\tau(X)$ is Hausdorff. We now give some conditions involving the existence of this separation property.

Theorem 4.95 *If $\pi_1^\tau(X)$ is Hausdorff, then $\pi_1^{top}(X)$ is functionally Hausdorff and X is homotopy Hausdorff. If the canonical homomorphism $\Phi : \pi_1(X) \to \check{\pi}_1(X)$ is injective, then $\pi_1^\tau(X)$ is Hausdorff.*

Proof. Every Hausdorff topological group is functionally Hausdorff. Since the identity $\pi_1^{top}(X) \to \pi_1^\tau(X)$ is continuous, $\pi_1^{top}(X)$ is functionally Hausdorff whenever $\pi_1^\tau(X)$ is. It is proven in 4.16 that X is homotopy Hausdorff whenever $\pi_1^{top}(X)$ is T_1. It is observed in Chapter 2.5.3 that $\Phi : \pi_1^{top}(X) \to \check{\pi}_1^{top}(X)$ is continuous. Since $\check{\pi}_1^{top}(X)$ is a Hausdorff topological group, the universal property of $\pi_1^\tau(X)$ implies that $\Phi : \pi_1^\tau(X) \to \check{\pi}_1^{top}(X)$ is continuous. Therefore, if Φ is injective, then $\pi_1^\tau(X)$ continuously injects into a Hausdorff group as must be Hausdorff. ■

It is a basic and useful fact that any free group may be realized as the fundamental group of a wedge of circles. We show here that any free topological group may be realized as the fundamental group of a generalized wedge $\Sigma(X_+)$.

Theorem 4.96 *There is a natural isomorphism $h_X : F_M(\pi_0^{top}(X)) \to \pi_1^{\tau}(\Sigma(X_+))$ of topological groups.*

Proof. Let $u : X \to \Omega(\Sigma(X_+))$ be the unbased unit of the adjunction $\mathbf{Top}_*(\Sigma(X_+), Y) \cong \mathbf{Top}(X, \Omega Y)$. This map induces a continuous injection $u_* : \pi_0^{top}(X) \to \pi_1^{top}(\Sigma(X_+))$ on path component spaces. In Chapter 4.2, it is shown that u_* induces a natural group isomorphism $h_X : F(\pi_0(X)) \to \pi_1(\Sigma(X_+))$ so that $h_X^{-1} \circ u_*$ is the canonical injection of generators. Moreover, $h_X^{-1} : \pi_1^{top}(\Sigma(X_+)) \to F_M(\pi_0^{top}(X))$ is continuous for an arbitrary space X. Since $F_M(\pi_0^{top}(X))$ is a topological group, $h_X^{-1} : \pi_1^{\tau}(\Sigma(X_+)) \to F_M(\pi_0^{top}(X))$ is continuous by the universal property of $\pi_1^{\tau}(\Sigma(X_+))$. The continuous injection $id \circ u_* : \pi_0^{top}(X) \to \pi_1^{top}(\Sigma(X_+)) \to \pi_1^{\tau}(\Sigma(X_+))$ induces (by the universal property of free topological groups) the continuous inverse $h_X : F_M(\pi_0^{top}(X)) \to \pi_1^{\tau}(\Sigma(X_+))$. ■

Example 4.97 If Y is an space, there is a paracompact Hausdorff space X such that $\pi_0^{top}(X) \cong Y$ (3.2). Since $\pi_1^{\tau}(\Sigma(X_+)) \cong F_M(Y)$ we realize every free topological group as a fundamental group. Some interesting examples come when $Y \subseteq \mathbb{R}$ (See 3.3) or $Y = S^1$ (See 3.4).

Example 4.98 The case when $X = \Omega Y$ for a based space Y gives a natural isomorphism of topological groups $h_{\Omega Y} : F_M(\pi_1^{top}(Y)) \to \pi_1^{\tau}(\Sigma((\Omega Y)_+))$. The counit map

$\Sigma((\Omega Y)_+) \to Y$ induces the multiplication map $F_M(\pi_1^{top}(Y)) \to \pi_1^{\tau}(Y)$ used to define the topology of $\pi_1^{\tau}(Y)$.

Corollary 4.99 *For any unbased space X, the following are equivalent:*

1. *$\pi_1^{\tau}(\Sigma(X_+))$ is Hausdorff.*

2. *$\pi_1^{top}(\Sigma(X_+))$ is functionally Hausdorff.*

3. *$\pi_0^{top}(X)$ is functionally Hausdorff.*

Proof. 1. $\Rightarrow$ 2. follows from Theorem 4.95. 2. $\Rightarrow$ 3. follows from the fact that $u_* : \pi_0^{top}(X) \to \pi_1^{top}(\Sigma(X_+))$ is a continuous injection. 3. $\Rightarrow$ 1. If $\pi_0^{top}(X)$ is functionally Hausdorff, then $F_M(\pi_0^{top}(X)) \cong \pi_1^{\tau}(\Sigma(X_+))$ is Hausdorff (Lemma A.37). ∎

Corollary 4.100 *A quotient map $q : X \to Y$ induces a quotient map $q_* : \pi_1^{\tau}(\Sigma(X_+)) \to \pi_1^{\tau}(\Sigma(Y_+))$ of topological groups.*

Proof. Both the functors F_M and π_0^{top} preserve quotients and $F_M \circ \pi_0^{top} \cong \pi_1^{\tau}(\Sigma((-)_+))$. ∎

Example 4.101 If X is a totally path disconnected (for instance, if X is zero-dimensional), then $\pi_1^{\tau}(\Sigma(X_+)) \cong F_M(X)$.

The fact that every group is realized as a fundamental group is easily arrived at by attaching 2-cells to wedges of circles. Similarly, we attach 2-cells to a generalized wedge $\Sigma(X_+)$ to realize every topological group as a fundamental group.

Theorem 4.102 *Every topological group G is isomorphic to the fundamental group $\pi_1^{\tau}(Z)$ of a space Z obtained by attaching 2-cells to a space of the form $\Sigma(X_+)$. Moreover, one may continue to attach cells of dimension > 2 to obtain a space Y which resembles a $K(G,1)$ in that $\pi_1^{\tau}(Y) \cong G$ and $\pi_n^{\tau}(Y) = 0$ for all $n > 1$.*

Proof. Suppose G is a topological group. According to Theorem 3.2, there is a (paracompact Hausdorff) space X such that $\pi_0^{top}(X)$ is homeomorphic to the underlying space of G. If G is totally path disconnected, then we may take $X = G$. This gives $h_G : \pi_1^{\tau}(\Sigma(X_+)) \cong F_M(\pi_0^{top}(X)) \cong F_M(G)$. The identity $G \to G$ induces the retraction $m_G : F_M(G) \to G$ so that $m_G \circ h_G : \pi_1^{\tau}(\Sigma(X_+)) \cong F_M(G) \to G$ is a topological quotient map. For each $\alpha \in \ker(m_G \circ h_G)$ choose a representative loop $f_\alpha : S^1 \to \Sigma(X_+)$ and attach a 2-cell to $\Sigma(X_+)$. The resulting space is $Z = \Sigma(X_+) \sqcup_\alpha e_\alpha^2$ and by Lemma 4.5, the inclusion $j : \Sigma(X_+) \hookrightarrow Z$ induces a quotient map $j_* : \pi_1^{top}(\Sigma(X_+)) \to \pi_1^{top}(Z)$. Since the functor τ preserves quotients $j_* : \pi_1^{\tau}(\Sigma(X_+)) \to \pi_1^{\tau}(Z)$ is also quotient. Since $\ker(m_G \circ h_G) = \ker j_*$ and both $m_G \circ h_G$ and j_* are quotient, $\pi_1^{\tau}(Z) \cong G$ as topological groups. The second statement follows by the usual process of inductively killing the n-th homotopy group by attaching cells of dimension $n+1$. The fact that the inclusions at each step induce group isomorphisms on the fundamental group (which are topological quotients by Lemma 4.5 and therefore homeomorphisms) means that the direct limit space Y will satisfy $\pi_1^{\tau}(Y) \cong G$ and $\pi_n^{\tau}(Y) = 0$ for all $n \neq 1$. ∎

In the construction of Y in the previous theorem one will notice that Y is a CW-complex (and therefore a proper $K(G,1)$) if and only if $X = G$ is a discrete group. This theorem permits the odd phenomenon of taking non-trivial fundamental groups of fundamental groups. For instance, there is a space X such that $\pi_1^{\tau}(X) \cong S^1$. Taking the identity to be the basepoint of $\pi_1^{\tau}(X)$, we find the discrete group of integers as $\pi_1^{\tau}(\pi_1^{\tau}(X)) \cong \mathbb{Z}$.

A topological van Kampen theorem 4.103 We now prove a result analogous to the classical Seifert-van Kampen Theorem for fundamental groups. We assume all based spaces are Hausdorff. Unfortunately, the general statement:

Statement 4.104 *If $\{U_1, U_2, U_1 \cap U_2\}$ is an open cover of X consisting of path connected neighborhoods, the diagram*

$$\begin{array}{ccc} \pi_1^{\tau}(U_1 \cap U_2) & \longrightarrow & \pi_1^{\tau}(U_1) \\ \downarrow & & \downarrow \\ \pi_1^{\tau}(U_2) & \longrightarrow & \pi_1^{\tau}(X) \end{array}$$

induced by inclusions is a pushout in the category of topological groups.

is not always true. To see why, we study an example.

Example 4.105 Let $X = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of the discrete space of natural numbers and consider $\Sigma(X_+)$ as in Example 4.37. We construct a space Y by attaching 1-cells to $\Sigma(X_+)$. For each $x \in X$, let $f_x : S^0 \to \Sigma(X_+)$ be the map given by $f(-1) = x_0$ and $f(1) = x \wedge \frac{1}{2}$. Let $Y = \Sigma(X_+) \sqcup_{f_x} e_x^1$ be the space obtained by attaching a copy of the unit 1-disc $e_x^1 = [-1, 1]$ for each x via the attaching map f_x.

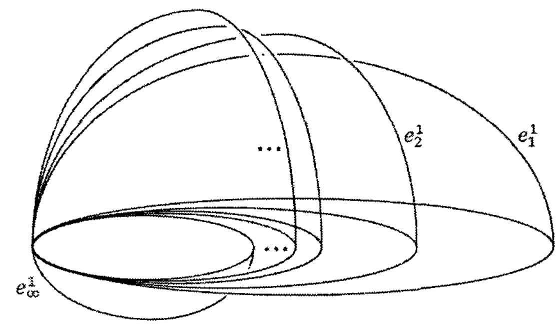


Figure 5: $Y = \Sigma(X_+) \sqcup_{f_x} e_x^1$

Note that any open neighborhood of the loop $\alpha : I \to \Sigma(X_+) \subset Y$, $\alpha(t) = \infty \wedge t$ contains loops which are not homotopic to α. By Proposition 4.7, $\pi_1^{top}(Y)$ is not discrete. Since the identity $\pi_1^{top}(Y) \to \pi_1^{\tau}(Y)$ is continuous, $\pi_1^{\tau}(Y)$ is not discrete. Define an open cover of Y by letting

$$U_1 = \left(X \wedge \left(\left[0, \frac{1}{6}\right) \cup \left(\frac{2}{6}, 1\right]\right)\right) \cup \bigcup_{x \in X} e_x^1 \text{ and } U_2 = \left(X \wedge \left(\left(\frac{5}{6}, 1\right] \cup \left[0, \frac{4}{6}\right)\right)\right) \cup \bigcup_{x \in X} e_x^1$$

Note that $U_1 \cong U_2$.

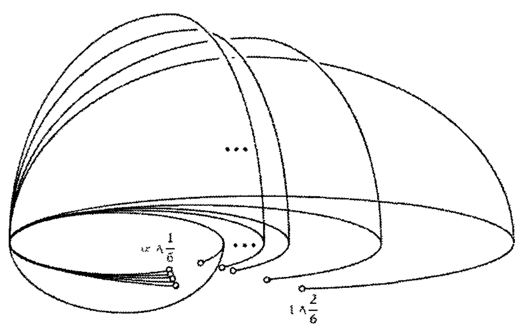


Figure 6: *The open set* $U_1 \subset Y$

Collapsing the set $\Sigma(X_+) \cap U_1$ to a point gives map $U_1 \to \bigvee_X S^1$ to a countable wedge of circles which induces an isomorphism $\pi_1^{\tau}(U_1) \to \pi_1^{\tau}\left(\bigvee_X S^1\right)$ of discrete topological groups. Consequently, $\pi_1^{\tau}(U_1) \cong \pi_1^{\tau}(U_2)$ is the discrete free group on countably many generators. Now we have

$$U_1 \cap U_2 = \left(X \wedge \left[0, \frac{1}{6}\right) \cup \left(\frac{2}{6}, \frac{4}{6}\right) \cup \left(\frac{5}{6}, 1\right]\right) \cup \bigcup_{x \in X} e_x^1$$

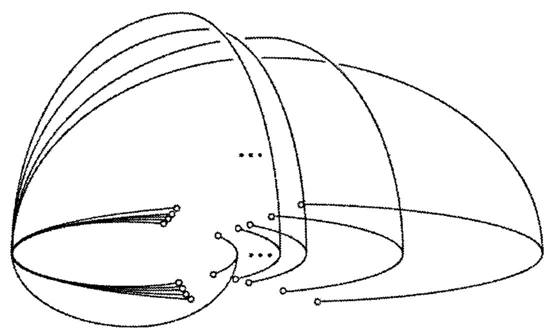


Figure 7: *The intersection* $U_1 \cap U_2$

Clearly $\pi_1(U_1 \cap U_2) = 0$. If the square

$$\begin{array}{ccc} \pi_1^\tau(U_1 \cap U_2) & \longrightarrow & \pi_1^\tau(U_1) \\ \downarrow & & \downarrow \\ \pi_1^\tau(U_2) & \longrightarrow & \pi_1^\tau(Y) \end{array}$$

is a pushout in the category of topological groups, then $\pi_1^\tau(Y)$ is the free topological product of two discrete groups and must also be discrete. This contradiction indicates that Statement 4.104 cannot be true in full generality. The complication arising here motivates the following definition.

Definition 4.106 A path $p : I \to X$ is *locally well-ended* if for every open neighborhood U of p in $P(X)$ there are open neighborhoods V_0, V_1 of $p(0), p(1)$ in X respectively such that for every $a \in V_0, b \in V_1$ there is a path $q \in U$ with $q(0) = a$

and $q(1) = b$. A space X is *lwe-path connected,* if every pair of points in X can be connected by a locally well-ended path.

Example 4.107 It is easy to see that $U_1 \cap U_2$ in Example 4.105 fails to be lwe-path connected since there is no locally well-ended path from the basepoint to $\infty \wedge \frac{1}{2}$. It will turn out that this is the reason why Statement 4.104 fails to hold in full generality. On the other hand Y is lwe-path connected and so it is not true that a path connected, open subspace of a lwe-path connected space must also be lwe-path connected.

Since Definition 4.106 does not seem to appear elsewhere, we consider some qualities of locally well-ended paths and lwe-path connected spaces.

Remark 4.108 It is necessary to specify the codomain X since a path $p : I \to A$ in a subspace $A \subseteq X$ may be locally well-ended whereas the path $p : I \to A \hookrightarrow X$ is not. The next proposition indicates that this complication does not arise when A is an open subset of X since, whenever A is open, $P(A)$ is an open subspace of $P(X)$.

Proposition 4.109 *If A is open in X and $p : I \to A$ is a path, then $p : I \to A$ is locally well-ended if and only if $p : I \to A \hookrightarrow X$ is locally well-ended.*

Lemma 4.110 *The concatenation of locally well-ended paths is locally well-ended. The reverse of a locally well-ended path is locally well-ended.*

Proof. If $U = \bigcap_{j=1}^{n} \langle K_n^j, V_j \rangle$ is a basic open neighborhood of concatenation $p * q$, then $U_{[0,\frac{1}{2}]}$ is an open neighborhood of p and $U_{[\frac{1}{2},1]}$ is an open neighborhood of

q. If p and q are locally well-ended, there are open neighborhoods P_0, P_1, Q_0, Q_1 of $p(0), p(1), q(0), q(1)$ respectively with the property indicated in the Definition 4.106. Suppose $a \in P_0$ and $b \in Q_1$. There is a path $p' \in U_{[0,\frac{1}{2}]}$ from a to $p(1)$ and a path $q' \in U_{[\frac{1}{2},1]}$ from $q(0)$ to b. Since $p(1) = q(0)$, the concatenation $p' * q'$ is a well defined path and is an element of

$$\left(U_{[0,\frac{1}{2}]}\right)^{[0,\frac{1}{2}]} \cap \left(U_{[\frac{1}{2},1]}\right)^{[\frac{1}{2},1]} = \mathcal{U}.$$

The fact that p^{-1} is locally well-ended whenever p is follows from the symmetry of the unit interval. ■

Corollary 4.111 *If X is path connected and $\{U_\alpha\}$ is an open cover of X consisting of lwe-path connected neighborhoods, then X is also lwe-path connected.*

Proof. Let $a, b \in X$ and $p : I \to X$ be an path from a to b. Find an integer $n > 1$ such that the restricted path $\alpha_{K_n^j}$ has image in U_{α_j} for $j = 1, ..., n$. Let $x_j = \alpha\left(\frac{j}{n}\right)$ for $j = 0, 1, ..., n$. Since $x_{j-1}, x_j \in U_{\alpha_j}$ and U_{α_j} is lwe-path connected, there is a locally well-ended path $\beta_j : I \to U_{\alpha_j}$ from x_{j-1} to x_j. Since each U_α is open, by Proposition 4.109, each $\beta_j : I \to U_{\alpha_j} \hookrightarrow X$ is locally well-ended. Now $(((\beta_1 * \beta_2) * \beta_3) * \cdots * \beta_{n-1}) * \beta_n$ is a path from a to b, which is a concatenation of locally well-ended paths in X and so must be locally well-ended. ■

Our interest in path connected, based spaces, motivates the inclusion of the following fact.

Proposition 4.112 *For path connected space (X, x_0), the following are equivalent:*

1. *X is lwe-path connected.*

2. *For each $x \in X$, there is a path $p : I \to X$ from x_0 to x such that for every open neighborhood U of p in $M((I,0),(X,x_0))$ there is an open neighborhood V of x such that for each $y \in V$ there is a path $q \in U$ from x_0 to y.*

Proof. 1. $\Rightarrow$ 2. follows from the fact that $M((I,0),(X,x_0))$ is a subspace of $P(X)$. 2. $\Rightarrow$ 1. If $a, b \in X$ and $p, q : I \to X$ are paths $p(0) = q(0) = x_0$, $p(1) = a$, and $p(1) = b$ satisfying the conditions in 2. it is easy to see that $p * q^{-1}$ is a locally well-ended path from a to b. ■

We now observe that being lwe-path connected is not a rare quality in a space.

Proposition 4.113 *Every path $p : I \to X$ in a locally path connected space X is locally well-ended. Consequently, all locally path connected spaces are lwe-path connected.*

Proof. Suppose X is locally path connected, $p : I \to X$ is path, and $U = \bigcap_{j=1}^{n} \langle K_n^j, U_j \rangle$ is a basic open neighborhood of p in $P(X)$. Find a path connected neighborhood V_0, V_1 of $p(0), p(1)$ respectively such that $V_0 \subseteq U_1$ and $V_1 \subseteq U_n$. For points $a \in V_0, b \in V_1$, we take paths $\alpha : I \to V_0$ from a to $p(0)$ and $\beta : I \to V_1$ from $p(1)$ to b. Now we define a path $q \in U$ from a to b by

$$q_{K_1^{2n}} = \alpha, q_{K_{2n}^2} = p_{K_n^1}, q_{[\frac{1}{n},\frac{n-1}{n}]} = p_{[\frac{1}{n},\frac{n-1}{n}]}, q_{K_{2n}^{2n-1}} = p_{K_n^{n-1}}, \text{ and } q_{K_{2n}^{2n}} = \beta$$

■

There are many non-locally path connected spaces which are lwe-path connected.

Example 4.114 For every space X, the (unreduced) suspension SX, the cone CX, and $\Sigma(X_+)$ are lwe-path connected but not necessarily locally path connected.

For the next proposition, we use the following convention: If an inclusion map $V \hookrightarrow U$ induces the constant function $\pi_0(V) \to \pi_0(U)$ on path components, we write $V \subseteq_0 U$.

Proposition 4.115 *The following are equivalent.*

1. *For every point $x \in X$, the constant path $c_x : I \to X$ at x is locally well-ended.*

2. *For each $x \in X$ and open neighborhood U of x there is a neighborhood V of x such that $V \subseteq_0 U$.*

3. *X is locally path connected.*

Proof. 1. $\Leftrightarrow$ 2. follows easily from two observations: (a) The neighborhoods $\langle I, U\rangle$ where U is an open neighborhood of x in X form a neighborhood base at c_x and (b) $V \subseteq_0 U$ if and only if every pair of points in V can be connected by a path in U.

3. $\Rightarrow$ 2. is obvious.

2. $\Rightarrow$ 3. Suppose 2. holds and U is an open neighborhood of x in X. Let $\pi_U : U \to \pi_0(U)$ be the function identifying path components. By assumption, each $y \in U$ lies in an open neighborhood V_y such that $V_y \subseteq_0 U$. For a set $A \subset U$, let $V(A) = \bigcup_{y \in A} V_y$. It is easy to check that if $A \subseteq_0 U$, then $V(A) \subseteq_0 U$. We also note that if $A \subseteq_0 U$, then $\pi_U^{-1}(\pi_U(A)) \subseteq_0 U$ since $\pi_U(A)$ is a singleton. Let $W_1 = V_x$

and inductively we define $W_{n+1} = V(\pi_U^{-1}(\pi_U(W_n)))$. By the previous observations, it is clear that $W_n \subseteq_0 U$ for each $n \geq 1$. We have inclusions $W_1 \subseteq W_2 \subseteq \ldots$ and let $W = \bigcup_{n \geq 1} W_n \subseteq U$. The set W is an open neighborhood of x contained in U since if $y \in W_n$, then $y \in V_y \subseteq W_{n+1} \subseteq W$. Additionally, we have constructed W to be saturated with respect to π_U, i.e. $W = \pi_U^{-1}(\pi_U(W))$. To check that W is path connected, we suppose $y, z \in W$. We have $y, z \in W_N$ for some $N \geq 1$. Since $W_N \subseteq_0 U$ there is a path $\alpha : I \to U$ with $\alpha(0) = y$ and $\alpha(1) = z$. Since W is saturated with respect to π_U and $\alpha(t)$ lies in the same path component of U as y and z it follows that $\alpha(t) \in W$ for each $t \in [0, 1]$. Therefore α is a path in W from y to z. ■

Statement 4.104 is proved here in the case that the intersection $U_1 \cap U_2$ is lwe-path connected.

van Kampen Theorem 4.116 *Let (X, x_0) be a based space and $\{U_1, U_2, U_1 \cap U_2\}$ an open cover of X consisting of path connected open neighborhoods each containing x_0. Let $k_i : U_1 \cap U_2 \hookrightarrow U_i$ and $l_i : U_i \hookrightarrow X$ be the inclusions. If $U_1 \cap U_2$ is lwe-path connected, the induced diagram of continuous homomorphisms*

$$\begin{array}{ccc} \pi_1^\tau(U_1 \cap U_2) & \xrightarrow{(k_1)_*} & \pi_1^\tau(U_1) \\ \downarrow{\scriptstyle (k_2)_*} & & \downarrow{\scriptstyle (l_1)_*} \\ \pi_1^\tau(U_2) & \xrightarrow[(l_2)_*]{} & \pi_1^\tau(X) \end{array}$$

is a pushout in the category of topological groups. In other words, there is a canonical isomorphism

$$\pi_1^\tau(X) \cong \pi_1^\tau(U_1) *_{\pi_1^\tau(U_1 \cap U_2)} \pi_1^\tau(U_2)$$

of topological groups.

Proof. We show that if G is a topological group and $f_i : \pi_1^{\tau}(U_i) \to G$ are continuous homomorphisms such that $f_1 \circ (k_1)_* = f_2 \circ (k_2)_*$, then there is a unique, continuous homomorphism $\Phi : \pi_1^{\tau}(X) \to G$ such that $\Phi \circ (l_i)_* = f_i$. The classical van Kampen theorem [Mun00, Theorem 70.1] guarantees the existence and uniqueness of the homomorphism Φ and so it suffices to show that Φ is continuous. To do this, we show that the composite $\phi = \Phi \circ \pi : \Omega(X) \to \pi_1^{\tau}(X) \to G$ is continuous. If this can be done then the universal property of $\pi_1^{\tau}(X)$ immediately gives the continuity of $\Phi : \pi_1^{\tau}(X) \to G$.

Suppose W is open in G and $\alpha \in \phi^{-1}(W)$. We construct an open neighborhood on restricted paths of α and combine these to form an open neighborhood of α. To find the appropriate restrictions, we recall how Φ is defined. Since $U_1 \cap U_2$ is lwe-path connected, Proposition 4.112 tells us that for each point $x \in U_1 \cap U_2$ there is a locally well-ended path $p_x : I \to U_1 \cap U_2$ from x_0 to x. Even though $U_1 \cap U_2$ may not be locally path connected at x_0 (recall Proposition 4.115), we take p_{x_0} to be the constant path c_{x_0}. Now for any path $q : I \to U_i$ such that $q(0), q(1) \in U_1 \cap U_2$, we define a loop $L(q) \in \Omega U_i$ by $L(q) = p_{q(0)} * q * p_{q(1)}^{-1}$. Find a subdivision $0 = t_0 < t_1 < ... < t_n = 1$ such that for each $j = 1, ..., n$, the restricted path $\alpha_j = \alpha_{[t_{j-1}, t_j]}$ has image in U_{i_j}, $i_j \in \{1, 2\}$, and such that $\alpha_j(0), \alpha_j(1) \in U_1 \cap U_2$. For convenience, let $a_j = \alpha_{j+1}(0) = \alpha(t_j)$ for $j = 0, 1, ..., n$. Now $\Phi([\alpha])$ is defined as the product in G:

$$\Phi([\alpha]) = f_{i_1}([L(\alpha_1)])f_{i_2}([L(\alpha_2)]) \dots f_{i_n}([L(\alpha_n)]) \in W.$$

The homomorphism Φ is well defined, since whenever α_j has image in $U_1 \cap U_2$, we have $f_1 \circ (k_1)_*([L(\alpha_j)]) = f_2 \circ (k_2)_*([L(\alpha_j)])$. It is important to note that the choice of paths p_x in $U_1 \cap U_2$ is irrelevant to the definition of Φ.

Since G is a topological group, there open neighborhoods W_j of $f_{i_j}([L(\alpha_j)])$ in G such that $W_1 W_2 ... W_n \subseteq W$. Since the composites $f_i \circ \pi_i : \Omega U_i \to \pi_1^\tau(U_i) \to G$ are continuous, we can find a basic open neighborhood $V_j = \bigcap_{m=1}^{M_j} \langle K_{M_j}^m, A_m^j \rangle$ of $L(\alpha_j) = p_{a_{j-1}} * \alpha_j * p_{a_j}^{-1}$ contained in $\pi_{i_j}^{-1}(f_{i_j}^{-1}(W_j)) \subseteq \Omega U_{i_j}$. We may assume that M_j is divisible by 3 and that $A_m^j \subseteq U_1 \cap U_2$ whenever $L(\alpha_j)(K_{M_j}^m) \subseteq U_1 \cap U_2$. Since each p_{a_j} has image in $U_1 \cap U_2$ this automatically implies that $A_m^j \subseteq U_1 \cap U_2$ for each $K_{M_j}^m \subseteq \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Taking restricted neighborhoods (in $P(X)$), we find that $\left(V_j\right)_{[0,\frac{1}{3}]}$ is an open neighborhood of $p_{a_{j-1}}$, $\left(V_j\right)_{[\frac{1}{3},\frac{2}{3}]}$ is an open neighborhood of α_j and $\left(V_j\right)_{[\frac{2}{3},1]}$ is an open neighborhood of $p_{a_j}^{-1}$. Since $\left(\left(V_j\right)_{[\frac{2}{3},1]}\right)^{-1}$ and $\left(V_{j+1}\right)_{[0,\frac{1}{3}]}$ are both neighborhoods of p_{a_j}, we may assume that they are equal for $j = 1, ..., n-1$.

Since $p_{a_j} : I \to U_1 \cap U_2$ is locally well-ended for each $j = 1, ..., n-1$ and $p_{a_j}(0) = x_0$, $p_{a_j}(1) = a_j = \alpha(t_j)$, Proposition 4.112 allows us to find an open neighborhood B_j of a_j in $U_1 \cap U_2$ such that for each $x \in B_j$ there is a path $\delta \in \left(V_{j+1}\right)_{[0,\frac{1}{3}]}$ from x_0 to x. We consider the neighborhood

$$\mathscr{U} = \bigcap_{j=1}^{n} \left(\left(V_j\right)_{[\frac{1}{3},\frac{2}{3}]}\right)^{[t_{j-1},t_j]} \cap \bigcap_{j=1}^{n-1} \langle \{t_j\}, B_j \rangle$$

of α in $\Omega(X)$. For any loop $\gamma \in \mathscr{U}$, we notice that

- For each $j = 1, \dots, n$,

$$\left(V_j\right)_{[\frac{1}{3},\frac{2}{3}]} = \left(\left(\left(V_j\right)_{[\frac{1}{3},\frac{2}{3}]}\right)^{[t_{j-1},t_j]}\right)_{[t_{j-1},t_j]}$$

  is an open neighborhood of $\gamma_{[t_{j-1},t_j]}$ in $P(X)$.

- If α_j has image in $U_1 \cap U_2$, then so does $\gamma_{[t_{j-1},t_j]}$.

- For $j = 1, \dots, n-1$, since $\gamma(t_j) \in B_j$, there is a path

$$\delta_j \in \left(V_{j+1}\right)_{[0,\frac{1}{3}]} = \left(\left(V_j\right)_{[\frac{2}{3},1]}\right)^{-1} \subseteq \langle I, U_1 \cap U_2 \rangle$$

  from x_0 to $\gamma(t_j)$.

Let $\delta_0 = \delta_n = c_{x_0}$ and define a loop β by demanding that $\beta_{[t_{j-1},t_j]}$ is the loop $\delta_{j-1} * \gamma_{[t_{j-1},t_j]} * \delta_j^{-1} \in \Omega U_{i_j}$ for $j = 1, \dots, n$. We note that if α_j has image in $U_1 \cap U_2$, then so does $\beta_{[t_{j-1},t_j]}$. Certainly

$$\beta \simeq \left(\delta_0 * \gamma_{[t_0,t_1]} * \delta_1^{-1}\right) * \cdots * \left(\delta_{j-1} * \gamma_{[t_{j-1},t_j]} * \delta_j^{-1}\right) * \left(\delta_j * \gamma_{[t_j,t_{j+1}]} * \delta_{j+1}^{-1}\right) * \cdots * \left(\delta_{n-1} * \gamma_{[t_{n-1},t_n]} * \delta_n\right)$$

is homotopic to γ. Moreover, for $j = 1, \dots, n$, we have $\left(\beta_{[t_{j-1},t_j]}\right)_A \in (V_j)_A$ for $A = \left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right]$. Therefore

$$\beta_{[t_{j-1},t_j]} \in \bigcap_A \left((V_j)_A\right)^A = V_j \subseteq \pi_{i_j}^{-1}(f_{i_j}^{-1}(W_j)) \subseteq \Omega U_{i_j}$$

All together, we see that

$$\phi(\gamma) = \Phi([\gamma]) = \Phi([\beta]) = f_{i_1}([\beta_{[t_0,t_1]}])f_{i_2}([\beta_{[t_1,t_2]}])\cdots f_{i_n}([\beta_{[t_{n-1},t_n]}]) \in W_1W_2\ldots W_n \subseteq W.$$

Since the choice of the paths p_x is irrelevant to the definition of Φ (we may replace p_{a_j} by δ_j) and $\beta_{[t_{j-1},t_j]}$ has image in U_{i_j}, the third equality makes sense. This proves the inclusion $\mathscr{U} \subseteq \phi^{-1}(W)$ and the continuity of ϕ. ∎

Remark 4.117 While the condition that $U_1 \cap U_2$ be lwe-path connected is sufficient for the van Kampen theorem to hold, it is certainly not a necessary condition. For any path connected, non-lwe-path connected space X, the unreduced suspension SX is quotient of $X \times I$ by collapsing $X \times \{0\}$ and $X \times \{1\}$ to a point. Let U_1 and U_2 be the image of $X \times \left[0, \frac{2}{3}\right)$ and $X \times \left(\frac{1}{3}, 1\right]$ in the quotient respectively. The open sets U_1, U_2 are contractible and the van Kampen theorem holds trivially even though $U_1 \cap U_2$ is not lwe-path connected.

Example 4.118 Here we compute $\pi_1^{\tau}(Y)$ from Example 4.105 by choosing an appropriate cover. For each $x \in X = \mathbb{N} \cup \{\infty\}$ let 0_x denote 0 in $e_x^1 = [-1, 1]$. Note that $U_3 = Y - \bigcup_{x \in X}\{0_x\}$ is homotopy equivalent to $\Sigma(X_+)$ so $\pi_1^{\tau}(U_3) \cong F_M(X)$. Since $X = U_1 \cup U_3$ and $U_1 \cap U_3$ is lwe-path connected and 1-connected the van Kampen theorem applies and gives

$$\pi_1^{\tau}(Y) \cong \pi_1^{\tau}(U_1) * \pi_1^{\tau}(U_3) \cong F_M(\mathbb{N}) * F_M(X) \cong F_M(\mathbb{N} \sqcup X) \cong F_M(X)$$

Corollary 4.119 *Given the hypothesis of Theorem 4.116, the homomorphism* $F_M(\Omega(U_1)) * F_M(\Omega(U_2)) \to \pi_1^{\tau}(X)$ *induced by the canonical maps* $\Omega(U_i) \to \pi_1^{\tau}(X)$, $i = 1, 2$ *is a topological quotient map.*

Proof. Since $F_M(\Omega(U_1)) * F_M(\Omega(U_2)) \cong F_M(\Omega(U_1) \oplus \Omega(U_2))$ (here $\oplus$ is the coproduct in **Top**) is suffices to show that $Q : F_M(\Omega(U_1) \oplus \Omega(U_2)) \to \pi_1^{\tau}(X)$, $Q(\alpha_1 ... \alpha_n) = [\alpha_1 * \cdots * \alpha_n]$ is quotient. Let $\pi_i : \Omega(U_i) \to \pi_1^{top}(U_i)$ be the quotient map identifying path components. Since F_M preserves quotients, $F_M(\pi_1 \oplus \pi_2)$ is quotient. The map

$$k : F_M(\pi_1^{top}(U_1) \oplus \pi_1^{top}(U_2)) \to \pi_1^{\tau}(U_1) * \pi_1^{\tau}(U_2)$$

of Proposition A.44 is also quotient. Additionally, the canonical homomorphism $k' : \pi_1^{\tau}(U_1) * \pi_1^{\tau}(U_2) \to \pi_1^{\tau}(U_1) *_{\pi_1^{\tau}(U_1 \cap U_2)} \pi_1^{\tau}(U_2)$ is always quotient. Let $h : \pi_1^{\tau}(U_1) *_{\pi_1^{\tau}(U_1 \cap U_2)} \pi_1^{\tau}(U_2) \cong \pi_1^{\tau}(X)$ be the isomorphism of Theorem 4.116. The composite $Q = h \circ k' \circ k \circ F_M(\pi_1 \oplus \pi_2)$ is quotient since it is the composite of quotient maps. ■

Corollary 4.120 *Let* X, Y *be path connected spaces each of which has a neighborhood base (one of which is countable) of path connected, 1- connected neighborhoods at its basepoint. Then there is a canonical isomorphism* $\pi_1^{\tau}(X \vee Y) \cong \pi_1^{\tau}(X) * \pi_1^{\tau}(Y)$ *of topological groups.*

Proof. We first recall a theorem of Griffiths [Gri54] which says that if W_1, W_2 are based spaces, one of which has a countable base of 1-connected neighborhoods at its basepoint, then the inclusions $W_i \hookrightarrow W_1 \vee W_2$ induce an isomorphism $\pi_1(W_1) * \pi_1(W_2) \to \pi_1(W_1 \vee W_2)$ of groups. Let A (resp. B) be a path connected, 1-connected neighborhood of the basepoint in X (resp. Y). Since X and Y are locally path connected and locally 1-connected at their basepoints, so are A,B, and $A \vee B$. Griffiths theorem implies that $\pi_1(A \vee B) = 0$. Let $U_1 = X \vee B$ and $U_2 = A \vee Y$ so that $U_1 \cap U_2 = A \vee B$. The van Kampen theorem applies and we have an isomorphism $\pi_1^{\tau}(X \vee Y) \cong \pi_1^{\tau}(X \vee B) * \pi_1^{\tau}(A \vee Y)$ of topological groups. The inclusions $X \hookrightarrow X \vee B$ and $Y \hookrightarrow A \vee Y$ induce continuous group isomorphisms $\pi_1^{\tau}(X) \to \pi_1^{\tau}(X \vee B)$ and $\pi_1^{\tau}(Y) \to \pi_1^{\tau}(A \vee Y)$. These group isomorphisms are also homeomorphisms since their inverses are induced by the retractions $X \vee B \to X$ and $A \vee Y \to Y$. All together, we have a canonical isomorphism of topological groups

$$\pi_1^{\tau}(X \vee Y) \cong \pi_1^{\tau}(X \vee B) * \pi_1^{\tau}(A \vee Y) \cong \pi_1^{\tau}(X) * \pi_1^{\tau}(Y)$$

■

CHAPTER V

CONCLUSIONS AND FUTURE DIRECTIONS

This computation and analysis of $\pi_1^{top}(\Sigma(X_+))$ offers new insight into the nature of topological fundamental groups and provides a geometric interpretation of many quasitopological and free topological groups. We note here how these ideas may be extended to higher dimensions and abelian groups, i.e. to the higher topological homotopy groups $\pi_n^{top}(X, x) = \pi_0^{top}(\Omega^n(X, x))$ and free abelian topological groups. These quasitopological abelian groups were first studied in [GHMM08] and [GH09], however, these authors assert that $\pi_n^{top}(X, x)$ is a topological group without sufficient proof. This misstep is noted in [GHB10] and the following problem remains open.

Problem 5.1 For $n \geq 2$, is π_n^{top} a functor to the category abelian topological groups?

As mentioned in the introduction, Fabel has shown that the topological fundamental group of the Hawaiian earring fails to be a topological group. This particular complication seems to disappear in higher dimensions since, for $n \geq 2$, the n-th topological fundamental group of the n-dimensional Hawaiian earring is indeed a topological group [GHB10]. The results in this paper, however, indicate that Problem 5.1 is likely to have a negative answer. Just as in Proposition 4.35, we have

Proposition 5.2 *For every based space Y, $\pi_n^{top}(Y)$ is a topological quotient group of $\pi_n^{top}(\Sigma^n(\Omega^n(Y)_+))$.*

Therefore, if $\pi_n^{top}(\Sigma^n(X_+))$ is a topological group for every X, then $\pi_n^{top}(Y)$ is a topological group for every Y. Consequently, the spaces $\Sigma^n(X_+)$ are prime candidates for producing counterexamples to Problem 5.1. Let $Z_R^q(Y)$ (resp. $Z_R(Y)$) be the free abelian group on the underlying set of Y viewed as the quotient space of $F_R^q(Y)$ (resp. $F_R(Y)$) with respect to the abelianization map. These groups have many of the same topological properties as their non-abelian counterparts. In particular, $Z_R^q(Y)$ (resp. $Z_R(Y)$) either fails to be a topological group or is the free abelian topological group $Z_M(Y)$ on Y. The results of this dissertation indicate the likelihood of the following statement:

Conjecture 5.3 For an arbitrary space X, the canonical map $\pi_0^{top}(X) \to \pi_n^{top}(\Sigma^n(X_+))$ induces an isomorphism $h_X : Z_R^{\pi_X}(\pi_0^{top}(X)) \to \pi_n^{top}(\Sigma^n(X_+))$ of quasitopological groups which are not topological groups.

If this is indeed the case, then π_n^{top} will be a functor to the category of abelian quasitopological groups but not to the category of topological abelian groups. A computation of $\pi_n^{top}(\Sigma^n(X_+))$ for $n \geq 2$ should then provide an answer to Problem 5.1.

APPENDIX

A great deal of the topological algebra required for this dissertation cannot be found in the literature. In this appendix, we included a number of these useful constructions and results. Since we may only assume that multiplication in homotopy mapping spaces is continuous *in each variable*, much of this content focuses on such objects. A nice reference for the theory of monoids and groups with topology is [AT08].

A.1 Monoids with topology

A monoid M endowed with a topology (no restrictions on the continuity of operation) will be referred to simply as a *monoid with topology*. Let **MonwTop** be the category of monoids with topology and continuous monoid homomorphisms.

Definition A.1 A *semitopological monoid* is a monoid with topology M such that multiplication $\mu : M \times M \to M$ is continuous in each variable. If, in addition, μ is continuous, then M is a *topological monoid*. The category of semitopological (resp. topological) monoids is the full subcategory **sTopMon** (resp. **TopMon**) of **MonwTop**.

Recall that an *involution* on a monoid M is a function $s : M \to M$ such that $s^2 = id_M$, $s(mn) = s(n)s(m)$, and $s(e) = e$.

Definition A.2 A *semitopological monoid with continuous involution* is a pair (M, s) where M is a semitopological monoid and $s : M \to M$ is a continuous involution on M. If, in addition, M is a topological monoid, then (M, s) is a *topological monoid with continuous involution*. A morphism $f : (M_1, s_1) \to (M_2, s_2)$ of two such pairs is a continuous homomorphism $f : M_1 \to M_2$ such that f preserves involution, i.e. $f \circ s_1 = s_2 \circ f$. Let **sTopMon*** be the category of semitopological monoids with continuous involution and continuous, involution-preserving homomorphisms. Let **TopMon*** be the full subcategory of **sTopMon*** consisting of topological monoids.

Remark A.3 Since group inversion is an involution on G, there are forgetful functors **TopGrp** $\to$ **TopMon*** $\to$ **TopMon** $\to$ **sTopMon**.

Universal Construction A.4 For any (unbased) space X, the topological sum $M_T(X) = \bigoplus_{n\geq 0} X^n$ (where $X^0 = \{e\}$ is a singleton) is a topological monoid with identity e called the *free topological monoid on* X. An element $(x_1, ..., x_n) \in X^n$ in $M_T(X)$ will be written as a word $w = x_1...x_n$ and multiplication is simply word concatenation. The length of a word $w = x_1 \dots x_n$ is $|w| = n$ and we let $|e| = 0$. A basic open neighborhood of w is a product $U_1 \dots U_n = \{u_1...u_n | u_i \in U_i\}$ where U_i is an open neighborhood of x_i in X. It is well known that M_T : **Top** $\to$ **TopMon** is a functor left adjoint to the functor U : **TopMon** $\to$ **Top** forgetting monoid structure. Equivalently, the canonical inclusion $\sigma : X \hookrightarrow M_T(X)$ is universal in the sense that any continuous function $f : X \to N$ to a topological monoid N induces a unique continuous monoid homomorphism $\tilde{f} : M_T(X) \to N$ such that $\tilde{f} \circ \sigma = f$. In particular $\tilde{f}(x_1...x_n)$ is the product $f(x_1)\dots f(x_n)$ in N. The underlying monoid of $M_T(X)$ will be denoted $M(X)$.

Let X^{-1} be a homeomorphic copy of X (with elements written as $x^{-1} \in X^{-1}$). Let $M_T^*(X) = M_T(X \oplus X^{-1}) = \bigoplus_{n\geq 0}(X \oplus X^{-1})^n$ be the free topological monoid on two copies of X and $M^*(X)$ denote the underlying monoid. A typical element of $M_T^*(X)$

is written as $x_1^{\epsilon_1}...x_n^{\epsilon_n}$ with $x_i \in X$ and $\epsilon_i \in \{\pm 1\}$ and a basic open neighborhood of w is a product of neighborhood $U_1^{\epsilon_1}\dots U_n^{\epsilon_n}$ where U_i is an open neighborhood of x_i in X. There is a canonical, continuous involution $M_T^*(X) \to M_T^*(X)$ on $M_T^*(X)$ given by $w = x_1^{\epsilon_1}...x_n^{\epsilon_n} \mapsto w^{-1} = x_n^{-\epsilon_n}...x_1^{-\epsilon_n}$ making the pair $(M_T^*(X),^{-1})$ a topological monoid with continuous involution.

A word $w \in M_T^*(X)$ is *reduced* if $x_i = x_{i+1}$ implies $\epsilon_i = \epsilon_{i+1}$ for each $i = 1, 2, ..., n-1$. The empty word is vacuously reduced. The collection of reduced words, of course, forms the free group $F(X)$ generated by the underlying set of X and the monoid epimorphism $R : M^*(X) \to F(X)$ denotes the usual reduction of words. In other words, $F(X)$ is the quotient monoid of $M^*(X)$ by the equivalence relation generated by $xx^{-1} \sim e \sim x^{-1}x$ for each $x \in X$. The following justifies calling $M_T^*(X)$ the *free topological monoid with continuous involution on X.*

Proposition A.5 M_T^* : **Top** $\to$ **TopMon*** *is a functor left adjoint to the functor* U : **TopMon*** $\to$ **Top** *forgetting monoid structure. Equivalently, the canonical inclusion* $\sigma : X \to M_T^*(X)$ *is universal in the sense that for each continuous function* $f : X \to N$ *where* (N, t) *is a topological monoid with continuous involution, there is a unique, continuous, involution-preserving homomorphism* $\tilde{f} : (M_T^*(X),^{-1}) \to (N, t)$ *such that* $\tilde{f} \circ \sigma = f$.

Proof. Let $g : X \oplus X^{-1} \to N$ be given by $g(x) = f(x)$ and $g(x^{-1}) = t(f(x))$. Note that $t(g(x^{-\epsilon}) = g(x^{\epsilon})$ for all $x \in X$ and $\epsilon \in \{\pm 1\}$. Since $M_T^*(X)$ is the free topological monoid on $X \oplus X^{-1}$, there is a unique continuous homomorphism $\tilde{f} : M_T^*(X) \to N$ given by $\tilde{f}(x_1^{\epsilon_1}...x_n^{\epsilon_n}) = g(x_1^{\epsilon_1})...g(x_n^{\epsilon_n})$. Therefore, it suffices to check that $\tilde{f}$ preserves involution (i.e. that $t\tilde{f}(w^{-1}) = \tilde{f}(w)$). This is done by the equation: $t(\tilde{f}(x_n^{-\epsilon_n}...x_1^{-\epsilon_1})) =$

$t(g(x_n^{-\epsilon_n})...g(x_1^{-\epsilon_1})) = t(g(x_1^{-\epsilon_1}))...t(g(x_n^{-\epsilon_n})) = g(x_1^{\epsilon_1})...g(x_n^{\epsilon_n}) = \tilde{f}(x_1^{\epsilon_1}...x_n^{\epsilon_n})$ ■

One of the reasons multiplication fails to be continuous in homotopy mapping spaces is the fact that a power $q^n : X^n \to Y^n$ of a quotient map $q : X \to Y$ may not be quotient. This deficiency of **Top** also means the functors M_T and M_T^* do not preserve quotients. The following is a simple characterization of this failure.

Proposition A.6 *The following are equivalent for a quotient map $q : X \to Y$.*

1. *The power $q^n : X^n \to Y^n$ is a quotient map for all $n \geq 1$.*

2. *$M_T(q) : M_T(X) \to M_T(Y)$ is a topological quotient map.*

3. *$M_T^*(q) : M_T^*(X) \to M_T^*(Y)$ is a topological quotient map.*

Proof. If $\{q_\alpha\}$ is a set of continuous surjections, then q_α is quotient for each α if and only if $\bigoplus_\alpha q_\alpha$ is quotient. 1. $\Leftrightarrow$ 2. and 1. $\Leftrightarrow$ 3. then follow from the simple observation that $M_T(q) = \bigoplus_{n\geq 0} q^n$ and $M_T^*(q) = M_T(q \oplus q) = \bigoplus_{n\geq 0}(q \oplus q)^n$. ■

Let $q : X \to Y$ be a continuous surjection. The induced homomorphism $M(q) : M(X) \to M(Y)$ is an epimorphism and we may give the monoid $M(Y)$ the quotient topology with respect to $M(q) : M_T(X) \to M(Y)$. We can do the same in the involuted case, by letting $M_q^*(Y) = M_{q\oplus q}(Y \oplus Y^{-1})$. We note that $M_q(Y)$ is not necessarily a topological monoid though if $q = id_Y$, then $M_{id_Y}(Y) = M_T(Y)$.

Proposition A.7 *$M_q(Y)$ is a semitopological monoid and $(M_q^*(Y),^{-1})$ is a semitopological monoid with continuous involution.*

Proof. We use the diagram of monoids with topology

$$\begin{array}{ccc} M_T(X) & \longrightarrow & M_T(X) \\ {\scriptstyle M(q)}\downarrow & & \downarrow{\scriptstyle M(q)} \\ M_q(Y) & \longrightarrow & M_q(Y) \end{array}$$

where the vertical maps are quotient maps. The diagram commutes when the top map is left (right) multiplication by a word $w = x_1...x_n$ and the bottom map is left (right) multiplication by $M(q)(w) = q(x_1)...q(x_2)$. Since left (right) multiplication by w in $M_T(X)$ is continuous, the Quotient Square Lemma implies the bottom multiplication maps are continuous. Since $M(q)$ is surjective, this accounts for all words in $M(Y)$. Therefore $M_q(Y)$ is a semitopological monoid. We have already shown that $M_q^*(Y) = M_{q\oplus q}(Y \oplus Y^{-1})$ is a semitopological monoid. For the involution, replace $M_T(X)$ by $M_T^*(X)$, $M_q(Y)$ by $M_q^*(Y)$, and $M(q)$ by the quotient map $M(q \oplus q) : M_T^*(X) \to M_q^*(Y)$ in the above square. Letting the top and bottom maps be the canonical involutions, the Quotient Square Lemma again may be used to prove the continuity of the involution of $M_q^*(Y)$. ■

The construction of $M_q(Y)$ is functorial in the following manner: Let **Quo(Top)** be the category of quotient maps $q : X \to Y$ in **Top**. A morphism between two such surjections is a commuting square

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

that we write as a pair of maps (f, g).

Proposition A.8 *The assignment* $(q : X \to Y) \mapsto M_q(Y)$ *on objects and* $(f, g) \mapsto M(g)$ *on morphisms is a functor* **Quo(Top)** $\to$ **sTopMon** *and the assignment* $q \mapsto q \oplus q \mapsto M_q^*(Y)$ *on objects and* $(f, g) \mapsto M^*(g)$ *on morphisms is a functor* **Quo(Top)** $\to$ **sTopMon***.

Proof. We have already shown that these functors are well-defined on objects. Suppose $(f, g) : q_1 \to q_2$ is a morphism of quotient maps $q_i : X_i \to Y_i$. It suffices to show that the induced monoid homomorphism $M(g) : M_{q_1}(Y_1) \to M_{q_2}(Y_2)$ is continuous. The functorality of free (topological) monoids implies that $M_T(f) : M_T(X_1) \to M_T(X_2)$ is a continuous monoid homomorphism such that the diagram

$$\begin{array}{ccc} M_T(X_1) & \xrightarrow{M_T(f)} & M_T(X_2) \\ {\scriptstyle M(q_1)}\downarrow & & \downarrow{\scriptstyle M(q_2)} \\ M_q(Y_1) & \xrightarrow[M(g)]{} & M_q(Y_2) \end{array}$$

commutes in **sTopMon**. Since the vertical maps are quotient, the homomorphism $M(g) : M_{q_1}(Y_1) \to M_{q_2}(Y_2)$ is continuous by the Quotient Square Lemma. The involuted case follows in the same manner. ■

Proposition A.9 *Let* $q : X \to Y$ *and* $r : Y \to Z$ *be quotient maps. The identity* $id : M_{r \circ q}(Z) \to M_r(Z)$ *is continuous.*

Proof. The diagram on the left induces the diagram on the right.

$$
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\Big\downarrow{\scriptstyle r\circ q} & & \Big\downarrow{\scriptstyle q} \\
 & & Y \\
 & & \Big\downarrow{\scriptstyle r} \\
Z & \xrightarrow[id]{} & Z
\end{array}
\qquad
\begin{array}{ccc}
M_T(X) & \xrightarrow{id} & M_T(X) \\
\Big\downarrow{\scriptstyle M(r\circ q)} & & \Big\downarrow{\scriptstyle M_T(q)} \\
 & & M_T(Y) \\
 & & \Big\downarrow{\scriptstyle M(r)} \\
M_{r\circ q}(Z) & \xrightarrow[id]{} & M_r(Z)
\end{array}
$$

Since the left vertical map in the right diagram is quotient, then bottom map is continuous. ■

Lemma A.10 *Let $q : X \to Y$ be a quotient map.*

1. *The canonical injections $\sigma : Y \to M_q(Y)$ and $\sigma : Y \to M_q^*(Y)$ are topological embeddings.*

2. *The identity $id : M_q(Y) \to M_{id_Y}(Y) = M_T(Y)$ is continuous.*

3. *The following are equivalent:*

   (a) $M_q(Y)$ is a topological monoid.

   (b) $M_q^*(Y)$ is a topological monoid.

   (c) The identity $M_q(Y) \to M_T(Y)$ is a homeomorphism.

   (d) The identity $M_q^*(Y) \to M_T^*(Y)$ is a homeomorphism.

   (e) The power $q^n : X^n \to Y^n$ is a quotient map for each $n \geq 2$.

Proof. 1. This follows from the fact that the quotient map $q : X \to Y$ occurs as a summand of the quotient map $M(q) : M_T(X) \to M_q(Y)$ (resp. $M^*(q) : M_T^*(X) \to$

$M_q^*(Y)$).

2. Take $r = id_Y$ and apply Prop. A.9.

3. (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (b) are obvious. We know from 2. that the identity morphisms $id : M_q(Y) \to M_{id}(Y) = M_T(Y)$ and $id : M_q^*(Y) \to M_{id}^*(Y) = M_T^*(Y)$ are continuous. The continuous injection $Y \hookrightarrow M_q(Y)$ (resp. $Y \hookrightarrow M_q^*(Y)$) induces the inverse $id : M_T(Y) \to M_q(Y)$ (resp. $id : M_T^*(Y) \to M_q^*(Y)$) whenever $M_q(Y)$ (resp. $M_q^*(Y)$) is a topological monoid. This gives (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d). Consider the diagram

$$\begin{array}{ccc} M_T(X) & \xrightarrow{M_T(q)} & M_T(Y) \\ {\scriptstyle M_T(q)}\downarrow & & \downarrow{\scriptstyle id} \\ M_q(Y) & \xrightarrow[id]{} & M_T(Y) \end{array}$$

in **sTopMon**. It is easy to see that $id : M_q(Y) \to M_T(Y)$ is a homeomorphism if and only if $M_T(q) : M_T(X) \to M_T(Y)$ is quotient. But by Prop. A.6, $M_T(q)$ is quotient if and only if q^n is a quotient map for all $n \geq 1$. Thus (c) $\Leftrightarrow$ (e) is proven. The analogous diagram in the involuted case and Prop. A.6 give (d) $\Leftrightarrow$ (e). ■

A.2 Groups with topology

Topological groups are widely studied objects. The groups we consider here are not quite as well studied though some of these results appear in [AT08].

Definition A.11 A group G with a topology (with no restriction on the continuity of operations) is a *group with topology*. Let **GrpwTop** denote the category of groups with topology and continuous group homomorphisms. If, in addition, multiplication $\mu : G \times G \to G$ is continuous in each variable (equivalently all translations are continuous) then G is a *semitopological group*. If inversion $G \to G$, $g \mapsto g^{-1}$ is also continuous, then G is a *quasitopological group*. If multiplication and inversion are continuous, G is a *topological group*. Let **TopGrp** $\subset$ **qTopGrp** $\subset$ **sTopGrp** be the full subcategories of **GrpwTop** consisting of topological, quasitopological, and semitopological groups.

For each $g \in G$, we will denote the continuous restrictions of multiplication $\mu : G \times G \to G$ to $G \times \{g\}$ as map $\rho_g : G \to G$, $\rho_g(h) = hg$ and to $\{g\} \times G$ as $\lambda_g : G \to G$, $\lambda_g(h) = gh$. If G is a semitopological group, these restrictions are homeomorphisms since they have inverses $\rho_{g^{-1}}$ and $\lambda_{g^{-1}}$ respectively. If G is a quasitopological group then inversion $a \mapsto a^{-1}$ is clearly a homeomorphism. For this reason, a neighborhood base at the identity e of semitopological group G suffices to characterize the topology of G. If G is a quasitopological group, we may assume this basis consists of symmetric neighborhoods $U \cap U^{-1}$.

Note that an isomorphism in **sTopGrp** and **qTopGrp** is a group isomorphism which is also a homeomorphism of spaces.

Remark A.12 Since group inversion is an involution on G, there is a forgetful functor **qTopGrp** $\to$ **sTopMon***.

Example A.13 Let G be any group with the finite complement topology $\mathcal{T} = \{U \subseteq G \,||G - U| < \infty\}$. If U is an open neighborhood of product g, then aU and Ub are open in G for any $a, b \in G$ since $G - aU$ and $G - Ub$ are finite sets. Therefore G is a semitopological group. Additionally, if U is open then $U^{-1} = \{g^{-1} | g \in U\}$ has finite complement and is open in G making G a quasitopological group. If G is finite, then, of course, G is a discrete group. If G is infinite then G is T_1 but not Hausdorff and cannot be a topological group.

Some basic facts. Multiplication of sets $A, B \subseteq G$ will be denoted $AB = \{ab \in G | a \in A, b \in B\}$.

Proposition A.14 *Let G be a quasitopological group with multiplication $\mu : G \times G \to G$.*

1. *G is a homogeneous space.*

2. *If $A \subseteq G$ and U is open in G, then AU and UA are open in G. Therefore μ is open.*

3. *If a subgroup H of G contains an non-empty open subset U, then H is open in G.*

4. *If H is an open subgroup of a quasitopological group G, then H is also closed in G.*

Proof. 1. If $g, h \in G$, then the translation $\rho_{g^{-1}h} : G \to G, x \mapsto xg^{-1}h$ is a homeomorphism such that $g \mapsto h$.

2. $AU = \bigcup_{a \in A} aU = \bigcup_{a \in A} \lambda_a(U)$ and $UA = \bigcup_{a \in A} Ua = \bigcup_{a \in A} \rho_a(U)$ are both open in G.

3. If $U \subseteq H$, then $Ua \subseteq H$ is open for every $a \in H$. Therefore $H = UH = \bigcup_{a \in H} Ua$ is open.

4. If H is an open subgroup of G, then each coset aH is open in G. But G is the disjoint union of cosets $G = \coprod_{aH \in G/H} aH$ and therefore $eH = H$ is closed in G. ■

If G is a quasitopological group and $H \leq G$ is a subgroup, H with the subspace topology of G becomes a quasitopological group. We say H is a *quasitopological subgroup* of G.

Proposition A.15 *Suppose $f : G \to H$ is a homomorphism of semitopological groups. If f is continuous at the identity e_G of G, then f is continuous.*

Proof. Suppose U is open in H and $g \in f^{-1}(U)$. Then $f(g) \in U$ and $f(g)^{-1}U$ is an open neighborhood of the identity e_H of H. If f is continuous at e_G, then there is an open neighborhood V of e_G such that $f(V) \subseteq f(g)^{-1}U$. But gV is an open neighborhood of V such that $f(gV) = f(g)f(V) \subseteq U$. ∎

If a G is semitopological group and H is a subgroup, let G/H be the set of left cosets in G with the quotient topology of G. This makes the canonical projection $\pi : G \to G/H$ a topological quotient map.

Proposition A.16 *Suppose H is a subgroup of semitopological group G. The canonical projection $\pi : G \to G/H$ onto the space of left cosets with the quotient topology is open.*

Proof. If U is open in G, then $\pi^{-1}(\pi(U)) = \{gH | g \in U\} = UH$. Since π is quotient by assumption and UH is open in G by A.14, $\pi(U)$ is open in G/H. ∎

Corollary A.17 *For $\alpha \in A$, let H_α be a subgroup of semitopological group G_α. The product of projections $\prod_\alpha \pi_\alpha : \prod_\alpha G_\alpha \to \prod_\alpha G_\alpha / H_\alpha$ is open.*

Corollary A.18 *If G is a topological group and H is a normal subgroup, then the quotient group G/H with the quotient topology is a topological group.*

Proof. We use the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/H \times G/H & \xrightarrow[m]{} & G/H \end{array}$$

where π is the projection and μ and m are multiplication operations. Since $\pi \times \pi$ is open, it is quotient. Since $\pi \circ \mu$ is continuous, m is continuous by the universal property of quotient spaces. ■

If H is normal in semi(quasi)topological group G, then G/H becomes a semi(quasi)-topological group. It is a *quotient semi(quasi)topological group* of G. It is easy to see that the universal properties of quotient groups and quotient spaces characterize such groups: If $\pi : G \to N$ is a group epimorphism and a quotient map spaces and $f : G \to H$ is such that $\ker \pi \subseteq \ker f$, then there is a unique, continuous homomorphism $g : N \to H$ such that $g \circ \pi = f$.

Proposition A.19 *Let H be a normal subgroup of semitopological group G. Then G/H is T_1 if and only if H is closed in G.*

Proof. If $\pi : G \to G/H$ is the projection and $e_{G/H} = H$ is the identity of G/H, then $H = \pi^{-1}(e_{G/H})$ is closed in G if and only if the singleton $\{e_{G/H}\}$ is closed in the quasitopological group G/H. Since all translations are homeomorphisms, this occurs precisely when G/H is T_1. ■

Proposition A.20 *Suppose $\phi : G \to H$ is a continuous homomorphism of semitopological groups. There is a continuous group isomorphism $\Psi : G/\ker\phi \to Im(\phi)$ which is a homeomorphism if and only if $\phi : G \to Im(\phi)$ is a quotient map.*

Proof. Both of statements follow from the commuting diagram

$$\begin{array}{ccc} G & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\phi} & \\ G/\ker\phi & \xrightarrow[\Psi]{} & Im(\phi) \end{array}$$

and the fact that the projection $\pi : G \to G/\ker\phi$ is a topological quotient map. ■

Closure in quasitopological groups. We observe a few elementary properties of the closure operator in quasitopological groups. For an element $g \in G$, let $\overline{g}$ denote the closure of the singleton $\{g\}$ in G. If necessary, we use superscript to distinguish the space where closure is being taken.

Proposition A.21 *Let G be a quasitopological group.*

1. *For each $g \in G$, $g\overline{e} = \overline{g} = \overline{e}g$.*

2. *The closure of the identity $\overline{e}$ is a normal subgroup of G.*

3. *For each $g \in G$ and open neighborhood U of g, $\overline{g} \subseteq U$.*

Proof. 1. We have that $g\overline{e} \cap \overline{e}g$ is closed and contains g, so $\overline{g} \subseteq g\overline{e} \cap \overline{e}g$. Now suppose $h \in g\overline{e} \cup \overline{e}g$ and U is any open set containing h. Then either $g^{-1}h \in \overline{e}$ or $hg^{-1} \in \overline{e}$. But $g^{-1}h \in g^{-1}U$ and $hg^{-1} \in Ug^{-1}$ and so we have $e \in g^{-1}U \cup Ug^{-1}$. This implies that $g \in U$ and $h \in \overline{g}$. Since $g\overline{e} \cup \overline{e}g \subseteq \overline{g} \subseteq g\overline{e} \cap \overline{e}g$, the equality is clear.

2. Suppose $a, b \in \overline{e}$ and U is an open neighborhood of ab^{-1}. Then $b \in U^{-1}a$ and consequently $e \in U^{-1}a$ and $a^{-1} \in U^{-1}$. Since $a \in U$ and $a \in \overline{e}$ we have $e \in U$. Therefore $ab^{-1} \in \overline{e}$ so that $\overline{e}$ is a group. $\overline{e}$ is normal by part 1.

3. We begin with the identity $g = e \in U$. Suppose $a \in \overline{e}$ and let $V = U \cap U^{-1}$. Then

$a \in aV$ and since aV is open and $a \in \bar{e}$ we have, $e \in aV$. Thus $a^{-1} \in V$ but V was symmetric, so $a \in V \subseteq U$. Now take any element $g \in U \subseteq G$ and $a \in \bar{g} = g\bar{e}$. We have $g^{-1}a \in \bar{e}$ and $e \in g^{-1}U$ so that $g^{-1}a \in g^{-1}U$ by the first part of the proof. This gives $a \in U$ so that $\bar{g} \subseteq U$. ■

Note that 3. of this last proposition indicates that two elements $g, h \in G$ are topologically indistinguishable if and only if $\bar{g} = \bar{h}$. 1. and 2. then give that

Corollary A.22 *For every $g, h \in G$, $g \in \bar{h}$ if and only if $\bar{g} = \bar{h}$.*

Recall that the *Kolmogorov quotient* of a topological space X is the T_1 quotient space $X/\sim$ where $x \sim y$ if and only if $\bar{x} = \bar{y}$.

Corollary A.23 *The quotient group $G/\bar{e}$ of quasitopological group G is the Kolmogorov quotient of G. All open neighborhoods $U \subseteq G$ are saturated with respect to the projection $p : G \to G/\bar{e}$. Consequently G has the initial topology with respect to $p : G \to G/\bar{e}$.*

Proof. The first statement follows directly from A.21 and A.22. If U is open in G, and $g \in G$, then $\bar{g} \subseteq U$. Therefore $p^{-1}(p(U)) = U$. Since the topology of G consists of neighborhoods of the form $p^{-1}(A)$ for A open in $G/\bar{e}$, G has the initial topology with respect to $p : G \to G/\bar{e}$. ■

Remark A.24 Since $\bar{e} \times \bar{e} = \overline{(e, e)}$ in $G \times G$, and $p \times p : G \times G \to G/\bar{e} \times G/\bar{e}$ is quotient, $G/\bar{e} \times G/\bar{e} \cong G \times G/\overline{(e, e)}$ is the Kolmogorov quotient of $G \times G$. Consequently, if U is an open neighborhood of (g, h) in $G \times G$, then $\bar{g} \times \bar{h} = \overline{(g, h)} \subseteq U$.

Proposition A.25 *Let G by a quasitopological group and $p : G \to G/\bar{e}$ be the projection. Then*

1. *G is a topological group $\Leftrightarrow$ $G/\overline{e}$ is a topological group.*

2. *G is first countable $\Leftrightarrow$ $G/\overline{e}$ is first countable.*

3. *G is pseudometrizable $\Leftrightarrow$ $G/\overline{e}$ is metrizable.*

Proof. 1. One direction is obvious by A.18. We observe commuting square

$$\begin{array}{ccc} G \times G & \xrightarrow{\;m\;} & G \\ {\scriptstyle q\times q}\downarrow & & \downarrow{\scriptstyle q} \\ G/\overline{e} \times G/\overline{e} & \xrightarrow[m']{} & G/\overline{e} \end{array}$$

where m and m' are the respective multiplication functions. Suppose m' is continuous and $V \subseteq G$ is open. By A.23, $V = p^{-1}(U)$ for open U in $G/\overline{e}$. Therefore $(p \times p)^{-1}\left(m'^{-1}(U)\right) = m^{-1}\left(p^{-1}(U)\right) = m^{-1}(V)$ is open in $G \times G$.

2. Since G is a quasitopological group, it suffices to work with neighborhood bases at the identities. If U_n is a countable base at $e \in G$, then $U_n = p^{-1}(V_n)$ for open V_n containing the identity $\overline{e}$ of $G/\overline{e}$. If W is an open neighborhood of $\overline{e}$ in $G/\overline{e}$, then $p^{-1}(W)$ is an open neighborhood of e in G. By assumption, there is an $U_m = p^{-1}(V_n) \subseteq p^{-1}(W)$. Clearly $V_n \subseteq W$. Conversely, if V_n is a countable base at $\overline{e} \in G/\overline{e}$, then $U_n = p^{-1}(V_n)$ is an open neighborhood of e in G. If W' is an open neighborhood of e in G, then by A.23, $W' = p^{-1}(W)$ for some open W in $G/\overline{e}$ containing $\overline{e}$. There is a $V_m \subseteq W$ and therefore $U_m = p^{-1}(V_m) \subseteq p^{-1}(W) = W'$.

3. In general, if X has the initial topology with respect to a map $f : X \to Y$ where the topology of Y is induced by a metric $d : Y \times Y \to [0, \infty)$, then $\rho = d \circ (f \times f)$ is a pseudometric on X which induces a topology that agrees with the initial topology

(See the proof of Prop. 2.86). Therefore, by A.23, if $d : G/\overline{e} \times G/\overline{e} \to [0, \infty)$ is a metric on $G/\overline{e}$, then $\rho = d \circ (p \times p)$ is a pseudometric which induces a topology agreeing with that of G. Now let $\rho : G \times G \to [0, \infty)$ be a pseudometric generating the topology of G. Specifically, the open balls $B_\rho(g, r) = \{h \in G | \rho(g, h) < r\}$, $g \in G$, $r > 0$ generate the topology of G. We induce a metric on $G/\overline{e}$ by showing that ρ is constant on the fibers of the quotient map $p \times p$. If $(g, h) \in \ker(p \times p)$, then $\overline{g} = \overline{e} = \overline{h}$. If $\rho(g, h) \neq 0$, then $\rho(g, h)$ lies in an interval $A = (a, b)$ where $a > 0$. Then $\rho^{-1}(A)$ is an open neighborhood of (g, h) such that $(e, e) \notin \rho^{-1}(A)$. Remark A.24 gives that

$$(e, e) \in \overline{e} \times \overline{e} = \overline{g} \times \overline{h} \subseteq \rho^{-1}(A).$$

which is a contradiction. Since $\rho(g, h) = 0$ whenever $\overline{g} = \overline{h}$, there is an induced map $d : G/\overline{e} \times G/\overline{e} \to [0, \infty)$ such that $d \circ (p \times p) = \rho$. We first check that d is a metric. Certainly, $d(\overline{g}, \overline{g}) = \rho(g, g) = 0$. If $d(\overline{g}, \overline{h}) = 0$, then $\rho(g, h) = 0$ and $h \in B_\rho(g, r)$ for every $r > 0$. This implies $g \in \overline{h}$ and A.22 gives $\overline{g} = \overline{h}$. It is also clear that

$$d(\overline{a}, \overline{b}) = \rho(a, b) = \rho(b, a) = d(\overline{b}, \overline{a})$$

and

$$d(\overline{a}, \overline{c}) = \rho(a, c) \leq \rho(a, b) + \rho(b, c) = d(\overline{a}, \overline{b}) + d(\overline{b}, \overline{c})$$

for all $a, b, c \in G$. Since d is a metric on $G/\overline{e}$, it suffices to show the topology of $G/\overline{e}$ agrees with the topology generated by the open balls $B_d(\overline{g}, r) = \{\overline{h} \in G/\overline{e} | d(\overline{g}, \overline{h}) < r\}$, $g \in G$, $r > 0$. Since p is quotient and G also has the initial topology with respect to

p, it suffices to check the equality $B_\rho(g,r) = p^{-1}(B_d(\overline{g},r))$ for $g \in G$, $r > 0$. But this follows immediately from the fact that $\rho(g,h) = d(\overline{g},\overline{h})$. ■

Since we have interest in non-Hausdorff topological groups, we prove the following corollary without generalizing the usual proof that a first countable, Hausdorff topopological group is metrizable.

Corollary A.26 *If G is a first countable topological group, then G is pseudometrizable.*

Proof. If G is a first countable topological group, then $G/\overline{e}$ is a T_1 (and therefore Hausdorff) topological group which is first countable by 2. of A.25. All such groups are metrizable [AT08, 3.3.12]. By 3. of A.25, G is pseudometrizable. ■

While T_1 need not imply T_2 in quasitopological groups (see Example A.13), the following equivalence is useful.

Proposition A.27 *In a quasitopological group G, the following are equivalent:*

1. *G is* T_0.
2. *G is* T_1.
3. *The singleton containing the identity is closed (i.e.* $\overline{e} = \{e\}$*).*

Proof. 3. $\Leftrightarrow$ 2. $\Rightarrow$ 1. is clear since singletons are closed in T_1 spaces and all translations are homeomorphisms. To show 1. $\Rightarrow$ 3. we suppose G is T_0 and let $a \in \overline{e}$ with $a \neq e$. Since G is T_0 there is an open set U such that either $a \in U$ and $e \notin U$ or $e \in U$ and $a \notin U$. The first case cannot happen, as $a \in \overline{e}$. Therefore $e \in U$ and $a \notin U$ and moreover $e \in U \cap U^{-1} \subset U$ and $a \notin U \cap U^{-1}$. But now $a(U \cap U^{-1})$ is an open neighborhood of a. This gives $e \in a(U \cap U^{-1}) \Rightarrow a^{-1} \in U \cap U^{-1} \Rightarrow a \in U \cap U^{-1}$.

This, however, is a contradiction and so $a = e$. Thus $\bar{e} \subset e$. ■

An alternative description of $\bar{e}$ may be given as follows. Let $\mathcal{U}_G$ be the intersection of all open neighborhoods in G containing the identity.

Proposition A.28 $\bar{e} = \mathcal{U}_G$

Proof. By Proposition A.21.3, we have the inclusion $\bar{e} \subseteq \mathcal{U}_G$. If every open neighborhood of e contains $a \in$ G (i.e. $a \in \mathcal{U}_G$), then $e \in \bar{a}$. But Corollary A.22 gives that $a \in \bar{a} = \bar{e}$. ■

Recall that an *A-space* (or Alexandrov space) is a topological space whose topology is closed under arbitrary intersections. An interesting and useful fact about A-spaces is the following:

Remark A.29 Let **A − space** be the full subcategory of **Top** consisting of A-spaces. **A − space** is closed under quotients, arbitrary coproducts (disjoint unions), and finite products. For instance, if X is an A-space, then $M_T(X), M_T^*(X), M_q(X)$, and $M_q^*(X)$ are all A-spaces.

Corollary A.30 *For a quasitopological group G, the following are equivalent.*

1. *G is an A-space.*

2. *$\bar{e}$ is open.*

3. *$G/\bar{e}$ is discrete.*

Proof. 1. ⇒ 2. If G is an A-space, then $\bar{e}$ is open as an intersection of open neighborhoods.

2. $\Rightarrow$ 1. Suppose $V = \bigcap_\alpha V_\alpha$ is any non-empty intersection of open neighborhoods in G. Pick any point $a \in V$. Now $a^{-1}V_\alpha$ is an open neighborhood of e in G for each α. We therefore have $e \in \overline{e} = \mathcal{U}_G \subseteq a^{-1}V = \bigcap_\alpha a^{-1}V_\alpha$ and $a \in a\overline{e} \subseteq V$. Therefore V is open in G.

2. $\Leftrightarrow$ 3. $\overline{e}$ is open in G if and only if the singleton containing the identity of $G/\overline{e}$ is open if and only if $G/\overline{e}$ is discrete (since it is a semitopological group. ■

Corollary A.31 *If a quasitopological group is an A-space, then it is also a topological group.*

Proof. If quasitopological group G is an A-space, then $G/\overline{e}$ is discrete by Corollary A.30 and is therefore a topological group. By Prop. A.25, G must then be a topological group. ■

Continuity of multiplication. By definition a quasitopological group G is a topological group if and only if the multiplication map $G \times G \to G$ is continuous. We now give a few convenient simplifications of this condition.

Proposition A.32 *For each quasitopological group G the following are equivalent:*

1. *G is a topological group.*

2. *For each $(a, b) \in G \times G$ and open neighborhood V of ab in G there is an open neighborhood A of a and B of b such that $AB \subseteq V$.*

3. *For each open neighborhood $W \subset G$ of the identity e, there is an open neighborhood $U \subseteq W$ of e such that $U^2 \subseteq W$.*

4. *There is a pair $(a, b) \in G \times G$ such that for each neighborhood V of ab in G, there is an open neighborhood A of a and B of b such that $AB \subseteq V$.*

Proof. 1. $\Leftrightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. is obvious.

3. $\Rightarrow$ 2. Suppose $(a, b) \in G \times G$ and V is and open neighborhood of ab. Now $W = a^{-1}Vb^{-1}$ is an open neighborhood of the identity e. If 3. holds, then there is an open neighborhood U of e such that $U^2 = UU \subseteq a^{-1}Vb^{-1}$. Therefore $aUUb \subseteq V$. If we let $A = aU$ and $B = Ub$, we have $a \in A$, $b \in B$ and $AB = (aU)(Ub) \subseteq V$.

4. $\Rightarrow$ 3. If W is an open neighborhood of e, and 4. holds for the pair (a, b), then aWb is an open neighborhood of ab. This allows us to find an open neighborhood A of a and B of b such that $AB \subseteq aWb$. Clearly $U = (a^{-1}A) \cap (Bb^{-1})$ is an open neighborhood of e. It suffices to check that $U^2 \subseteq W$. If $g, h \in U$, then $ag \in A$ and $hb \in B$. Therefore $aghb \in AB \subseteq aWb$ and consequently $gh \in W$. ∎

Corollary A.33 *If G is a quasitopological group such that there is a neighborhood base of open subgroups at the identity, then G is a topological group.*

Proof. If U is an open neighborhood of the identity, there is an open subgroup H such that $H^2 = H \subseteq U$. By Proposition A.32, G is a topological group. ∎

Corollary A.34 *If G is a quasitopological group having open subgroup H which becomes a topological group with the subspace topology, then G is a topological group.*

Proof. Suppose U is an open neighborhood of the identity e in G. Since H is a topological group there is an open neighborhood V of e such that $VV \subseteq H \cap U \subseteq U$.

Since H is open in G, V is open in G. ■

Miscellaneous facts about topological groups.

Proposition A.35 *Let $f : G \to H$ be a homomorphism where H is a topological group and G has the initial topology with respect to f. Then G is a topological group.*

Proof. Suppose $U = f^{-1}(V)$ is an open neighborhood of $gg' \in G$ where V is open in H. There are open neighborhoods A, B of $f(g), f(g')$ respectively such that $f(gg') = f(g)f(g') \in AB \subseteq V$. If $a \in f^{-1}(A)$ and $b \in f^{-1}(B)$ then $f(ab) = f(a)f(b) \in AB \subseteq V$ and so $ab \in f^{-1}(V) = U$. Therefore $f^{-1}(A)$ and $f^{-1}(B)$ are open neighborhoods of g and g' such that $f^{-1}(A)f^{-1}(B) \subseteq U$. Therefore multiplication in G is continuous. It is easy to see that if $U = f^{-1}(V)$ is open in G, then so is $U^{-1} = f^{-1}(V)^{-1} = f^{-1}(V^{-1})$ since inversion is a homeomorphism in H and f is a homomorphism. Therefore inversion in G is continuous. ■

Proposition A.36 *Suppose*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ G' & \xrightarrow[f']{} & H' \end{array}$$

is a diagram in **Grp** *where H, H' are topological groups and G, G' have the initial topology with respect to f, f'. If h is continuous, then so is g.*

Proof. Suppose $U = (f')^{-1}(V)$ is open in G' where V is open in H'. Then $g^{-1}(U) = g^{-1}((f')^{-1}(V)) = (f)^{-1}(h^{-1}(V))$ is open in G since the diagram commutes and h and f are continuous. ■

A.3 Topologies on free groups

A.3.1 Free topological groups $F_M(X)$

The *free (Markov) topological group* on an unbased space Y is the unique topological group $F_M(Y)$ with a continuous map $\sigma : Y \to F_M(Y)$ universal in the sense that for any map $f : Y \to G$ to a topological group G, there is a unique continuous homomorphism $\tilde{f} : F_M(Y) \to G$ such that $f = \tilde{f} \circ \sigma$. Using Taut liftings [Por91] or the Freyd special adjoint functor theorem [Fre66, Kat44], it can be shown that $F_M(Y)$ exists for every space Y and that $F_M : \mathbf{Top} \to \mathbf{TopGrp}$ is a functor left adjoint to the forgetful functor $\mathbf{TopGrp} \to \mathbf{Top}$. Moreover, the underlying group of $F_M(Y)$ is simply the free group $F(Y)$ on the underlying set of Y and $\sigma : Y \to F_M(Y)$ is the canonical injection of generators. There is a vast literature on free topological groups and we do make use of some of this theory. The reader is referred to [Tho74] for proofs of the following lemma. Recall that a space Y is functionally Hausdorff if for each pair of distinct points $a, b \in Y$ there is a continuous, real-valued function $f : Y \to \mathbb{R}$ such that $f(a) \neq f(b)$.

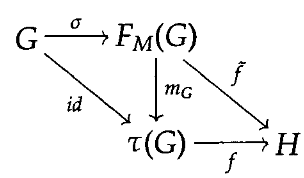
Lemma A.37 *[Tho74] Let Y be a topological space.*

1. *$F_M(Y)$ is Hausdorff if and only if Y is functionally Hausdorff.*
2. *$\sigma : Y \to F_M(Y)$ is an embedding if and only if Y is completely regular.*

Lemma A.38 *F_M preserves colimits and quotient maps.*

Proof. As a left adjoint, F_M preserves all colimits. Suppose $q : X \to Y$ is a quotient map and $\sigma_X : X \to F_M(X)$ and $\sigma_Y : Y \to F_M(Y)$ are the canonical injections. Let G be a topological group and $\tilde{f} : F_M(X) \to G$ be a continuous homomorphism such that $\tilde{f}(\ker F_M(q)) = 0$. Suppose $x, x' \in X$ such that $q(x) = q(x')$. Since $F_M(q)(\sigma_X(x)) = \sigma_Y(q(x)) = \sigma_Y(q(x')) = F_M(q)(\sigma_X(x'))$ and $\tilde{f}$ is constant on the fibers of $F_M(q)$, it follows that $f = \tilde{f} \circ \sigma_X : X \to G$ is constant on the fibers of q. This induces a map $k : Y \to G$ such that $k \circ q = f$. But then k induces a continuous homomorphism $\tilde{k} : F_M(Y) \to G$ such that $\tilde{k} \circ \sigma_Y = k$. But then $\tilde{f} \circ \sigma_X = f = \tilde{k} \circ \sigma_Y \circ q = \tilde{k} \circ F_M(q) \circ \sigma_X$ and the uniqueness of $\tilde{f}$ gives that $\tilde{f} = \tilde{k} \circ F_M(q)$. ■

Another useful construction which makes use of free topological groups is the following: Given any group with topology G, the identity $id : G \to G$ induces the multiplication epimorphism $m_G : F(G) \to G$ on the free group. We may now give G the quotient topology with respect to $m_G : F_M(G) \to G$ and denote the resulting group with topology as $\tau(G)$. Since any quotient group of a topological group with the quotient topology from the projection is a topological group (A.18), $\tau(G)$ is also a topological group. The identity function $G \to \tau(G)$ is continuous since it is the composite $m_G \circ \sigma : G \to F_M(G) \to \tau(G)$. Moreover, any continuous homomorphism $f : G \to H$ to a topological group H induces a continuous homomorphism $\tilde{f} : F_M(G) \to H$ such that the diagram



commutes.Since m_G is quotient, $f : \tau(G) \to H$ is continuous. Stated entirely in categorical terms this amounts to the fact that **TopGrp** is a full reflective subcategory of **GrpwTop**.

Functorality A.39 τ : **GrpwTop** $\to$ **TopGrp** *is a functor left adjoint to the inclusion functor* U : **TopGrp** $\to$ **GrpwTop**. *Moreover, each reflection map* $r_G : G \to \tau(G)$ *is the continuous identity homomorphism.*

Proof. For a continuous homomorphism $f : G \to H$ of groups with topology, $F_M(f) : F_M(G) \to F_M(H)$ is a continuous homomorphism such that the square

$$\begin{array}{ccc} F_M(G) & \xrightarrow{F_M(f)} & F_M(H) \\ {\scriptstyle m_G}\downarrow & & \downarrow{\scriptstyle m_H} \\ \tau(G) & \xrightarrow[\tau(f)=f]{} & \tau(H) \end{array}$$

commutes. The left vertical map is quotient and so the bottom map $\tau(f)$ is continuous. Since τ is the identity functor on the underlying algebraic groups, the rest of the conditions to be a functor are satisfied. The natural bijection of the adjunction is **TopGrp**$(\tau(G), H) \cong$ **GrpwTop**$(G, U(H))$, $f \mapsto f \circ r_G$. ∎

Lemma A.40 τ *preserves colimits, finite products, and quotient maps.*

Proof. As a left adjoint τ preserves all colimits. If $f : G \to G'$ is a group homomorphism of groups with topology such that f is also a topological quotient, then

$F_M(f) : F(G) \to F(G')$ is also a topological quotient. Since the diagram

$$\begin{array}{ccc} F_M(G) & \xrightarrow{F_M(f)} & F_M(G') \\ {\scriptstyle m_G}\downarrow & & \downarrow{\scriptstyle M_{G'}} \\ \tau(G) & \xrightarrow[\tau(f)]{} & \tau(G') \end{array}$$

commutes in **TopGrp** and the top and vertical maps are topological quotients, $\tau(f)$ must also be a topological quotient map by the Quotient Square Lemma. To check that τ preserves finite products we take $G, H \in$ **GrpwTop**. Clearly the projections of $G \times H$ induce the continuous group isomorphism $\tau(G \times H) \to \tau(G) \times \tau(H)$. The maps $i : G \to G \times H$, $i(g) = (g, e_h)$ and $j : H \to G \times H$, $j(h) = (e_G, h)$ are embeddings of groups with topology. Let μ be the continuous multiplication of $\tau(G \times H)$. The continuous composite

$$\mu \circ (\tau(i) \times \tau(j)) : \tau(G) \times \tau(H) \to \tau(G \times H) \times \tau(G \times H) \to \tau(G \times H)$$

is given by $(g, h) \mapsto (g, e_H)(e_G, h) = (g, h)$ and is therefore the identity. Thus $id : \tau(G \times H) \cong \tau(G) \times \tau(H)$. ∎

Corollary A.41 *Let G be a group with topology. Then G is a topological group if and only if $G \cong \tau(G)$.*

Proof. One direction is obvious. If G is a topological group, then the identity $id : G \to G$ induces the continuous identity $\tau(G) \to G$. ∎

Corollary A.42 *G is discrete if and only if $\tau(G)$ is discrete.*

Proof. Since $r_G : G \to \tau(G)$ is continuous, G is discrete whenever $\tau(G)$ is. If G is discrete, then so is $F_M(G)$ and the quotient $\tau(G)$. ■

The fact that every Hausdorff topological group is functionally Hausdorff implies the following corollary. The author does not know if the converse holds.

Corollary A.43 *If $\tau(G)$ is Hausdorff, then G is functionally Hausdorff.*

The category **TopGrp** is cocomplete and $A *_G B$ denotes the pushout (free topological product with amalgamation) of a diagram $A \leftarrow G \to B$. If $G = \{*\}$, then this is simply the *free topological product* $A * B$. Universal properties quickly verify that $A * B$ has the quotient topology with respect to the canonical homomorphism $k_{A,B} : F_M(A \oplus B) \to A * B$ (here $\oplus$ denotes the coproduct in **Top**) and $A *_G B$ has the quotient topology with respect to the canonical map $A * B \to A *_G B$. Free topological products are related to the functor τ in the following way:

Proposition A.44 *For groups with topology A, B, the canonical epimorphism $k_{A,B}$: $F_M(A \oplus B) \to \tau(A) * \tau(B)$ is a topological quotient map.*

Proof. The following diagram commutes in the category of topological groups.

$$\begin{array}{ccccc} F_M(F_M(A) \oplus F_M(B)) & \xrightarrow{k_{F_M(A),F_M(B)}} & F_M(A) * F_M(B) & \xrightarrow{\cong} & F_M(A \oplus B) \\ \downarrow{\scriptstyle F_M(m_A \oplus m_B)} & & & & \downarrow{\scriptstyle k_{A,B}} \\ F_M(\tau(A) \oplus \tau(B)) & & \xrightarrow[k_{A,B}]{} & & \tau(A) * \tau(B) \end{array}$$

Since m_A, m_B are quotient and F_M preserves quotients, all maps except for the right vertical map are known to be quotient. By the universal property of quotient spaces, $k_{A,B} : F_M(A \oplus B) \to \tau(A) * \tau(B)$ must also be quotient. ■

A.3.2 Reduction topologies $F_R^q(Y)$

For every unbased space X, there is a canonical monoid epimorphism $R : M^*(X) \to F(X)$ (See A.1) reducing words by the relations $xx^{-1} \sim e \sim x^{-1}x$. We note that R also satisfies $R(w^{-1}) = R(w)^{-1}$ for each word $w \in M^*(X)$. Give $F(X)$ the quotient topology with respect to $R : M_T^*(X) \to F(X)$ and denote the resulting group with topology as $F_R(X)$. This quotient topology on $F(X)$ will be called the *reduction topology*.

Functorality A.45 *For each space X, $F_R(X)$ is a quasitopological group. Moreover, F_R :* **Top** $\to$ **qTopGrp** *is a functor and $R : M_T^* \to F_R$ is a natural transformation each component of which is a monoid epimorphism and topological quotient map.*

Proof. For this proof all we need is the fact that $M_T^*(X)$ is a semitopological monoid with continuous involution. By the Quotient Square Lemma, if the diagram

$$\begin{array}{ccc} M_T^*(X) & \xrightarrow{f} & M_T^*(Y) \\ \downarrow R & & \downarrow R \\ F_R(X) & \xrightarrow[f']{} & F_R(Y) \end{array}$$

commutes where f is continuous, then f' is also continuous. The diagram commutes when we let $X = Y$, f be left multiplication by word w in $M_T^*(X)$ (resp. right multiplication by w in $M_T^*(X)$, the involution $w \mapsto w^{-1}$ in $M_T^*(X)$) and f' be left multiplication by $R(w)$ in $F_R(X)$ (resp. right multiplication by $R(w)$ in $F_R(X)$, group inversion $w \mapsto w^{-1}$ in $M_T^*(X)$). Since $M_T^*(X)$ is a topological monoid with continuous involution, f is continuous in each of these cases. Therefore right and left multiplication by fixed words and inversion are continuous making $F_R(X)$ a quasitopological group. Moreover, a map $g : X \to Y$ in **Top** induces a continuous homomorphism $f = M_T^*(g) : M_T^*(X) \to M_T^*(Y)$. The diagram commutes when we let f' be the homomorphism $F(g) : F_R(X) \to F_R(Y)$ induced on free groups. By the same argument $F(g)$ is continuous and F_R is a well-defined functor (preservation of identity and composition follows from the functorality of the free group). The above diagram also illustrates the naturality of R. ■

Let $\sigma : X \to F_R(X)$ be the continuous injection of generators. Though we will see that $F_R(X)$ is not always a topological group, a nice property of F_R is the following:

Universal Property A.46 *Let X be a space, (M, s) be a topological monoid with continuous involution, and G be a quasitopological group. If $f : X \to M$ is a continuous function and $g : (M, s) \to (G, ^{-1})$ is a continuous, involution preserving homomorphism, then there is a unique, continuous group homomorphism $h : F_R(X) \to G$ such that $h \circ \sigma = g \circ f$.*

Proof. The canonical embedding of generators $\sigma' : X \to M_T^*(X)$ satisfies $R \circ \sigma' = \sigma$. The map f induces a continuous, involution-preserving monoid homomorphism $\tilde{f} : (M_T^*(X), ^{-1}) \to (M, s)$ such that $\tilde{f} \circ \sigma' = f$ which is $\tilde{f}(x) = f(x)$ and $\tilde{f}(x^{-1}) = s(f(x))$ on generators. Since g preserves involution, we have $g(s(f(x))) = g(f(x))^{-1}$. Since $g\tilde{f}(xx^{-1}) = gf(x)gf(x)^{-1} = gf(x)^{-1}gf(x) = g\tilde{f}(x^{-1}x)$ is the identity of G, $g \circ \tilde{f} : M_T^*(X) \to G$ is constant on the fibers of $R : M_T^*(X) \to F_R(X)$, there is a unique, continuous group homomorphism $h : F_R(X) \to G$ such that $h \circ R = g \circ \tilde{f}$. Consequently,

$h \circ \sigma = h \circ R \circ \sigma' = g \circ \tilde{f} \circ \sigma' = g \circ f.$ ■

Corollary A.47 *If X is a space and $f : X \to G$ is a continuous function to a topological group, then there is a unique continuous group homomorphism $\tilde{f} : F_R(X) \to G$ such that $\tilde{f} \circ \sigma = f$.*

Corollary A.48 *For any space X, the identity $F_R(X) \to F_M(X)$ is continuous and is a homeomorphism if and only if $F_R(X)$ is a topological group. Moreover, the identity $\tau(F_R(X)) \to F_M(X)$ is an isomorphism of topological groups.*

Proof. The continuous injection $\sigma : X \to F_M(X)$ induces the continuous identity $F_R(X) \to F_M(X)$ by A.47. Since $F_M(X)$ is a topological group, the universal property of $\tau(F_R(X))$ gives that $\tau(F_R(X)) \to F_M(X)$ is continuous. The continuous inverse $F_M(X) \to \tau(F_R(X))$ is induced by the map $\sigma = r_{F_R(X)} \circ \sigma : X \to F_R(X) \to \tau(F_R(X))$ from the universal property of free topological groups. Since $\tau(F_R(X)) \cong F_M(X)$ it follows that $F_R(X)$ is a topological group if and only if $id : F_R(X) \cong F_M(X)$. ■

Now we consider a construction which generalizes F_R. This construction plays a key role in recognizing the isomorphism class of the quasitopological group $\pi_1^{top}(\Sigma(X_+))$ as in Chapter 4.2. Fix a quotient map $q : X \to Y$ in **Quo(Top)**. We generalize the previous construction of $F_R(Y)$ by replacing the topological monoid $M_T^*(Y)$ with the semitopological monoid $M_q^*(Y)$. Specifically, let $F(Y)$ have the quotient topology with respect to the reduction map $R : M_q^*(Y) \to F(Y)$ and denote the resulting group with topology as $F_R^q(Y)$. We will refer to this quotient topology as the *q-reduction topology*. Since $Q = M^*(q) : M_T^*(X) \to M_q^*(Y)$ is quotient by definition, the composite $RQ : M_T^*(X) \to F_R^q(Y)$ is quotient. Note that when $q = id_Y$, we have $F_R^{id}(Y) = F_R(Y)$. Functorality follows similarly to that of M_q^* and F_R.

Functorality A.49 *For each $q : X \to Y$ in* **Quo(Top)**, *$F_R^q(Y)$ is a quasitopological group. There is a functor* **Quo(Top)** $\to$ **qTopGrp** *given by $(q : X \to Y) \mapsto F_R^q(Y)$ on objects and $(f, g) \mapsto F(g)$ on morphisms. Additionally, $R : M_q^* \to F_R^q$ is a natural transformation each component of which is a quotient map of semitopological monoids with continuous involution.*

Lemma A.50 *Let $q : X \to Y$ be a quotient map.*

1. *The canonical injection of generators $\sigma : Y \to F_R^q(Y)$ is continuous.*

2. *The identity $id : F_R^q(Y) \to F_R(Y)$ is continuous and is a homeomorphism if and only if $F_R(q) : F_R(X) \to F_R(Y)$ is a topological quotient map.*

3. *The following are equivalent:*

   (a) *$F_R^q(Y)$ is a topological group*

   (b) *$id : F_R^q(Y) \cong F_R(Y)$ and $id : F_R(Y) \cong F_M(Y)$*

   (c) *$F(q) : F_R(X) \to F_M(Y)$ is a topological quotient map.*

4. *The identity $\tau(F_R^q(Y)) \to F_M(Y)$ is an isomorphism of topological groups.*

Proof. 1. Since σ is the composite of $R : M_q^*(Y) \to F_R^q(Y)$ and the embedding $Y \hookrightarrow M_q^*(Y)$ (A.10.1), σ is continuous.

2. Consider the diagram

$$\begin{array}{ccccc} M_T^*(X) & \xrightarrow{R} & F_R(X) & \xrightarrow{id} & F_M(X) \\ {\scriptstyle R\circ M^*(q)}\downarrow & & {\scriptstyle F_R(q)}\downarrow & & {\scriptstyle F_M(q)}\downarrow \\ F_R^q(Y) & \xrightarrow[id]{} & F_R(Y) & \xrightarrow[id]{} & F_M(Y) \end{array}$$

commuting in **sTopMon***. 2. follows immediately from the fact that the left vertical map in the left square is quotient.

3. (b) $\Rightarrow$ (a) is obvious. (a) $\Rightarrow$ (b) If $F_R^q(Y)$ is a topological group, $\sigma : Y \to F_R^q(Y)$ induces $id : F_M(Y) \to F_R^q(Y)$. Since the identity $F_R^q(Y) \to F_R(Y) \to F_M(Y)$ is continuous, the three topologies on $F(Y)$ must agree. For (b) $\Leftrightarrow$ (c) it suffices to observe that the top and left maps in the left square of the above diagram are quotient.

4. The map $r_{F_R^q(Y)} \circ \sigma : Y \to F_R^q(Y) \to \tau(F_R^q(Y))$ is continuous and induces the continuous identity $id : F_M(Y) \to \tau(F_R^q(Y))$. The continuous identity $id : F_R^q(Y) \to F_M(Y)$ induces $id : \tau(F_R^q(Y)) \to F_M(Y)$ which is continuous by the universal property of $\tau(F_R^q(Y))$. ∎

Corollary A.51 *If $F_R(X)$ is a topological group and $q : X \to Y$ is quotient, then $F_R^q(Y)$ is a topological group.*

Proof. If $F_R(X)$ is a topological group, then $id : F_R(X) \cong F_M(X)$ by Corollary A.48. Since F_M preserves quotients, $F_M(q) : F_M(X) \to F_M(Y)$ is quotient. Therefore, the composite $F(q) : F_R(X) \to F_M(Y)$ is quotient and 3. of Lemma A.50 implies that $F_R^q(Y)$ is a topological group. ∎

Corollary A.52 *If all powers of the quotient map $q : X \to Y$ are quotient, then $id : F_R^q(Y) \cong F_R(Y)$.*

Proof. If $q^n : X^n \to Y^n$ is quotient for each $n \geq 1$, then $id : M_q^*(Y) \cong M_T^*(Y)$ by Lemma A.10. Therefore, the quotients $F_R^q(Y)$ and $F_R(Y)$ are homeomorphic. ∎

Example A.53 Let X be an A-space and $q : X \to Y$ be any quotient map. By A.29, $M_T^*(X)$ is also an A-space. Since the category of A-spaces is closed under quotients, $F_R^q(Y)$ must also be an A-space. Moreover, A.30 implies that $F_R^q(Y)$ is a topological group. In particular, if X is an A-space, then $F_R(X)$ is a topological group which is an A-space.

Corollary A.54 *The following are equivalent for any quotient map $q : X \to Y$:*

1. *Y is a discrete space.*

2. *$F_R^q(Y)$ is a discrete group.*

3. *$F_R(Y)$ is a discrete group.*

Proof. 3. $\Rightarrow$ 2. $\Rightarrow$ 1. is obvious since we have continuous injections $id : F_R^q(Y) \to F_R(Y)$ and $\sigma : Y \to F_R^q(Y)$. For 1. $\Rightarrow$ 3. suppose Y is discrete. Then $M_T^*(Y)$ is discrete and the quotient $F_R(Y)$ is discrete. ∎

Now we study the topological properties of $F_R^q(Y)$ in more detail. The following definition is reminiscent of the first condition in Lemma 4.14.

Definition A.55 We say a continuous function $f : X \to Y$ is *separating* if whenever $f(x_1) = y_1 \neq y_2 = f(x_2)$ there are open neighborhoods U_i of x_i in X such that $f(U_1) \cap f(U_2) = \emptyset$.

Remark A.56 For any quotient map $q : X \to Y$, Y is Hausdorff $\Rightarrow$ q is separating $\Rightarrow$ Y is T_1.

The following definition makes sense for a fixed quotient map $q : X \to Y$.

Definition A.57 A neighborhood $U = U_1^{\epsilon_1} \dots U_n^{\epsilon_n}$ of $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ in $M_T^*(X)$ is *q-separating* if $q(U_i) \cap q(U_j) = \emptyset$ whenever $q(x_i) \neq q(x_j)$. We say U is *separating* in the case $q = id_Y$

Remark A.58 If $U_1^{\epsilon_1} \dots U_n^{\epsilon_n}$ is a q-separating neighborhood of $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$, then

$$U_1^{\epsilon_1} \dots U_{i-1}^{\epsilon_{i-1}} U_{i+1}^{\epsilon_{i+1}} \dots U_n^{\epsilon_n}$$

is a q-separating neighborhood of $x_1^{\epsilon_1} \dots x_{i-1}^{\epsilon_{i-1}} x_{i+1}^{\epsilon_{i+1}} \dots x_n^{\epsilon_n}$. This will be particularly useful when we remove letters by word reduction.

Let $Q = M^*(q) : M_T^*(X) \to M_q^*(Y)$ be the induced monoid homomorphism which takes word $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ to $Q(w) = q(x_1)^{\epsilon_1} \dots q(x_n)^{\epsilon_n}$ and is quotient by definition. Additionally, the composite $RQ : M_T^*(X) \to M_q^*(Y) \to F_R^q(Y)$ is quotient.

Lemma A.59 *Let $q : X \to Y$ be a separating quotient map, $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ be a non-empty word in $M_T^*(X)$, $y_i = q(x_i)$, and W be an open neighborhood of w.*

1. *There is a q-separating neighborhood $U = U_1^{\epsilon_1} \dots U_n^{\epsilon_n}$ of w contained in W.*

2. *$Q(w)$ is reduced if and only if $Q(v)$ is reduced for each $v \in U$.*

3. *If $v \in U$, then $|RQ(w)| \leq |RQ(v)| \leq |Q(w)|$.*

Proof. 1. If all of the y_i are the same, there is an open $V = U_i$ of x_i such that $U = U_1^{\epsilon_1} \dots U_n^{\epsilon_n} \subseteq W$. On the other hand, if $y_i \neq y_j$, there are open neighborhoods $V_{i,j}$ of x_i and $V_{j,i}$ of x_j such that $q(V_{i,j}) \cap q(V_{j,i}) = \emptyset$. For each i, take open neighborhood $U_i \subseteq \bigcap_{q(x_i) \neq q(x_j)} V_{i,j}$ of x_i such that $U = U_1^{\epsilon_1} \dots U_n^{\epsilon_n} \subseteq W$. Clearly U is a q-separating neighborhood of w contained in W.

2. Suppose first that $Q(w)$ is a reduced word. Then for each $i \in \{1, ..., n-1\}$ either $y_i \neq y_{i+1}$ or $\epsilon_i = \epsilon_{i+1}$. Suppose $v = z_1^{\epsilon_1} \dots z_n^{\epsilon_n}$ lies in the q-separating neighborhood U and $i \in \{1, ..., n-1\}$ such that $\epsilon_i = -\epsilon_{i+1}$. If $q(z_i) = q(z_{i+1})$, then we must have $y_i = q(x_i) = q(x_{i+1}) = y_{i+1}$. But this cannot be since $Q(w)$ is reduced. Therefore $Q(v)$ is reduced. The converse is obvious since if $Q(w)$ is not reduced then U already contains $Q(w)$.

3. Suppose $v = z_1^{\epsilon_1} \dots z_n^{\epsilon_n} \in U$. The second inequality is obvious since $|RQ(v)| \leq |Q(v)| = |Q(w)|$. To prove the first inequality, Remark A.58 indicates that it suffices prove that for every reduction in $Q(v)$, there is a corresponding reduction in $Q(w)$. This follows directly from 2. ■

Let $F_R^q(Y)_n$ denote $F(Y)_n = \{w \in F(Y) | |w| \leq n\}$ with the subspace topology of $F_R^q(Y)$.

Corollary A.60 *If $q : X \to Y$ is separating and $n \geq 0$, then $F_R^q(Y)_n$ is closed in $F_R^q(Y)$.*

Proof. Suppose $w \in M_T^*(X)$ such that $|RQ(w)| > n$. Now take any q-separating neighborhood U of w in $M_T^*(X)$. Lemma A.59 asserts that if $v \in U$, then $|RQ(v)| \geq$

$|RQ(w)| > n$. Consequently, $w \in U \subset M_T^*(X) - RQ^{-1}(F_R^q(Y)_n)$ proving that $RQ^{-1}\left(F_R^q(Y)_n\right)$ is closed in $M_T^*(X)$. Since RQ is quotient $F_R^q(Y)_n$ is closed in $F_R^q(Y)$. ■

We now observe some properties of $F_R^q(Y)$ which are often desirable in free topological groups. Let Z denote the set of all finite sequences $\zeta = \epsilon_1, ..., \epsilon_n$ with $\epsilon_i \in \{\pm 1\}$, including the empty sequence. For each $\zeta = \epsilon_1, ..., \epsilon_n \in Z$, let $X^\zeta = \{x_1^{\epsilon_1} \dots x_n^{\epsilon_n} | x_i \in X\}$ and recall that $M_T^*(X) = \coprod_{\zeta \in Z} X^\zeta$. Let $|\zeta|$ denote the length of each sequence and $(RQ)_\zeta : X^\zeta \to F_R^q(Y)_{|\zeta|}$ be the restriction of the quotient map $RQ : M_T^*(X) \to F_R^q(Y)$. The proof of the next proposition is based on that of Statement 5.1 in [Sip05].

Proposition A.61 *Let $q : X \to Y$ be separating.*

1. *$F_R^q(Y)$ has the inductive limit topology of the sequence of closed subspaces $\{F_R^q(Y)_n\}_{n \geq 0}$, i.e. $F_R^q(Y) \cong \varinjlim_n F_R^q(Y)_n$.*
2. *For each $n \geq 0$, the restriction $(RQ)_n : \bigoplus_{i=0}^n (X \oplus X^{-1})^i \to F_R^q(Y)_n$ of RQ is a quotient map.*

Proof. 1. Suppose $C \subseteq F_R^q(Y)$ such that $C \cap F_R^q(Y)_n$ is closed in $F_R^q(Y)_n$ for each $n \geq 0$. Since $(RQ)_\zeta$ is continuous $Q^{-1}(R^{-1}(C)) \cap X^\zeta = (RQ)_\zeta^{-1}(C) = (RQ)_\zeta^{-1}(C \cap F_R^q(Y)_{|\zeta|})$ is closed in X^ζ for each ζ. But $M_T^*(X)$ is the disjoint union of the X^ζ and so $(RQ)^{-1}(C)$ is closed in $M_T^*(X)$. Since RQ is quotient, C is closed in $F_R^q(Y)$.
2. Suppose $A \subseteq F_R^q(Y)_n$ such that $(RQ)_n^{-1}(A)$ is closed in $\bigoplus_{i=0}^n (X \oplus X^{-1})^i = \bigoplus_{|\zeta| \leq n} X^\zeta$. Since $F_R^q(Y)_n$ is closed in $F_R^q(Y)$ and RQ is a quotient map, it suffices to show that

$$RQ_\zeta^{-1}(A) = \{a = a_1^{\epsilon_1}...a_k^{\epsilon_k} \in M_T^*(X) | RQ(a) \in A\} = RQ^{-1}(A) \cap X^\zeta$$

is closed in X^ζ for each $\zeta = \epsilon_1, ..., \epsilon_k \in Z$. If $|\zeta| \le n$, then $RQ_\zeta^{-1}(A) \cap X^\zeta = RQ_n^{-1}(A) \cap X^\zeta$ is closed by assumption. For $|\zeta| > n$, we proceed by induction, and suppose $RQ_\delta^{-1}(A) \cap X^\delta$ is closed in X^δ for all $\delta \in Z$ of length $|\delta| = n, n+1, ..., |\zeta| - 1$. Let $w = x_1^{\epsilon_1}...x_k^{\epsilon_k} \in X^\zeta$ such that $RQ(w) \notin A$. Let $y_i = q(x_i)$. If $Q(w) = y_1^{\epsilon_1}...y_k^{\epsilon_k}$ is reduced, U is a q-separating neighborhood of w, and $v \in U$, then by Lemma A.59, $n < |\zeta| = |RQ(w)| = |RQ(v)|$. Thus $RQ(U) \cap F_R^q(Y)_n = \emptyset$ and since $A \subseteq F_R^q(Y)_n$ we have $U \cap RQ_\zeta^{-1}(A) = \emptyset$. Therefore we may suppose that $Q(w)$ is not reduced. For each $i \in \{1, ..., k-1\}$ such that $y_i = y_{i+1}$ and $\epsilon_i = -\epsilon_{i+1}$ find a q-separating neighborhood U_i of w in the following way. Remove the two letters $x_i^{\epsilon_i}, x_{i+1}^{\epsilon_{i+1}}$ from w to obtain the word $w_i = x_1^{\epsilon_1}...x_{i-1}^{\epsilon_{i-1}}x_{i+2}^{\epsilon_{i+2}}...x_k^{\epsilon_k}$ which satisfies $RQ(w_i) = RQ(w) \notin A$. Let $\zeta_i = \epsilon_1, ..., \epsilon_{i-1}, \epsilon_{i+2}, ..., \epsilon_k$ so that $|\zeta_i| = |\zeta| - 2$ and $w_i \in X^{\zeta_i} - RQ_{\zeta_i}^{-1}(A)$. By our induction hypothesis $RQ_{\zeta_i}^{-1}(A)$ is closed in X^{ζ_i} and so we may find a q-separating neighborhood $V_i = A_1^{\epsilon_1}...A_{i-1}^{\epsilon_{i-1}}A_{i+2}^{\epsilon_{i+2}}...A_k^{\epsilon_k}$ of w_i such that $V_i \cap RQ_{\zeta_i}^{-1}(A) = \emptyset$. We may then find neighborhoods A_i, A_{i+1} of x_i, x_{i+1} respectively such that

$$U_i = A_1^{\epsilon_1}...A_{i-1}^{\epsilon_{i-1}}A_i^{\epsilon_i}A_{i+1}^{\epsilon_{i+1}}A_{i+2}^{\epsilon_{i+2}}...A_k^{\epsilon_k}$$

is a q-separating neighborhood of w. Now take a q-separating neighborhood U of w such that $w \in U \subseteq \bigcap_i U_i$ (the intersection ranges over i such that $y_i = y_{i+1}$ and $\epsilon_i = -\epsilon_{i+1}$). It now suffices to show that $RQ(v) \notin A$ whenever $v = a_1^{\epsilon_1}...a_k^{\epsilon_k} \in U$. If $Q(v)$ is reduced, then $n < |\zeta| = |Q(w)| = |Q(v)| = |RQ(v)|$. Thus $RQ(v) \notin F_R^q(Y)_n$ and we are done. If $Q(v)$ is not reduced, then there is an $i_0 \in \{1, ..., k-1\}$ such that $q(a_{i_0}) = q(a_{i_0+1})$ and $\epsilon_{i_0} = -\epsilon_{i_0+1}$. Since U is a q-separating neighborhood of w

we must also have $y_{i_0} = y_{i_0+1}$. If $v_{i_0} = a_1^{\epsilon_1}...a_{i_0-1}^{\epsilon_{i_0-1}}a_{i_0+2}^{\epsilon_{i_0+2}}...a_k^{\epsilon_k}$ is the word obtained by removing $a_{i_0}^{\epsilon_{i_0}}, a_{i_0+1}^{\epsilon_{i_0+1}}$, we have $RQ(v) = RQ(v_{i_0})$ and $v_{i_0} \in V_{i_0}$. But $V_{i_0} \cap RQ_{\zeta_{i_0}}^{-1}(A) = \emptyset$ and so $RQ(v) = RQ(v_{i_0}) \notin A$. ■

For each $n \geq 1$, let Y_q^n denote the product Y^n with the quotient topology from the product function $q^n : X^n \to Y^n$. Of course, since q is quotient, $Y_q^1 = Y$ and if $q = \pi_X : X \to \pi_0^{top}(X)$, then $Y_q^n \cong \pi_0^{top}(X^n)$. Similarly, denote Y_q^{ζ} and $(Y \oplus Y^{-1})_q^n$ as the quotients of X^{ζ} and $(X \oplus X^{-1})^n$ with respect to q and its powers and sums. In these terms, we have

$$M_q^*(Y) = \bigoplus_{n \geq 0} (Y \oplus Y^{-1})_q^n = \bigoplus_{\zeta \in Z} Y_q^{\zeta}$$

Let $Q_n : \bigoplus_{i=0}^{n} (X \oplus X^{-1})^i \to \bigoplus_{i=0}^{n} (Y \oplus Y^{-1})_q^i$ and $R_n : \bigoplus_{i=0}^{n} (Y \oplus Y^{-1})_q^i \to F_R^q(Y)_n$ be the respective restrictions of R and Q. Since $R_n \circ Q_n = (RQ)_n$, the previous proposition implies:

Corollary A.62 *If $q : X \to Y$ is separating and $n \geq 0$, the restriction $R_n : \bigoplus_{i=0}^{n} (Y \oplus Y^{-1})_q^i \to F_R^q(Y)_n$ of reduction $R : M_q^*(Y) \to F_R^q(Y)$ is a quotient map.*

Theorem A.63 *The following are equivalent.*

1. *$q : X \to Y$ is separating.*

2. *$F_R^q(Y)$ is T_1.*

3. *For each $n \geq 1$, the canonical map $\sigma_n : Y_q^n \to F_R^q(Y)$ taking $(y_1, ..., y_n)$ to the word $y_1..., y_n$ is a closed embedding.*

Proof. 1. $\Rightarrow$ 2. If $q : X \to Y$ is separating, the singleton $F_R^q(Y)_0 = \{e\}$ containing the identity is closed by Corollary A.60. Since $F_R^q(Y)$ is a quasitopological group, it is

T_1.

2. $\Rightarrow$ 1. Suppose $q : X \to Y$ is not separating. There are distinct $y_1, y_2 \in Y$ such that whenever $q(x_i) = y_i$ and U_i is an open neighborhood of x_i, then $q(U_1) \cap q(U_2) \neq \emptyset$. Suppose W is any open neighborhood of reduced word $y_1 y_2^{-1}$ in $F_R^q(Y)$ and $x_i \in q^{-1}(y_i)$. Since RQ is continuous, there are open neighborhoods U_i of x_i such that $x_1 x_2^{-1} \in U_1 U_2^{-1} \subset RQ^{-1}(W)$. But there is a $y_3 \in q(U_1) \cap q(U_2)$ by assumption and so $Q(U_1 U_2^{-1}) \subset R^{-1}(W)$ contains the word $y_3 y_3^{-1}$. Therefore $e = R(y_3 y_3^{-1}) \in W$. But if every neighborhood of $y_1 y_2^{-1}$ in $F_R^q(Y)$ contains the identity, then $F_R^q(Y)$ is not T_1.

1. $\Rightarrow$ 3. Suppose A is a closed subspace of Y_q^n and q is separating. Let $j : Y_q^n \hookrightarrow \coprod_{i=0}^{n}(Y \sqcup Y^{-1})_q^i$ be given by $j(y_1, ..., y_n) = y_1 ... y_n$ so that $R_n \circ j = \sigma_n$. Since j is a closed embedding, $R_n^{-1}(\sigma_n(A)) = j(A)$ is closed in $\coprod_{i=0}^{n}(Y \sqcup Y^{-1})_q^i$. But R_n is quotient by Corollary A.62 and $F_R^q(Y)_n$ is closed in $F_R^q(Y)$. Therefore $\sigma_n(A)$ is closed in $F_R^q(Y)$.

3. $\Rightarrow$ 1. If q is not separating, the argument for 2. $\Rightarrow$ 1. implies that there are distinct $y_1, y_2 \in Y$ such that any open neighborhood of the three letter word $y_1 y_2 y_1^{-1}$ in $F_R^q(Y)$ contains the one letter word y_1 which lies in the image of σ_1. Therefore, if q is not separating, the image of σ_1 cannot be closed. ■

Overall, we wish to characterize the quotient maps $q : X \to Y$ for which $F_R^q(Y)$ is a Hausdorff topological group. One such characterization is the following.

Theorem A.64 *Let $q : X \to Y$ be a quotient map. Then $F_R^q(Y)$ is a Hausdorff topological group if and only if the following three conditions hold:*

1. *Y is Tychonoff.*
2. $F_M(Y) \cong \varinjlim_n F_M(Y)_n$.

3. $RQ_n : \bigoplus_{i=0}^{n}(X \oplus X^{-1})^i \to F_M(Y)_n$ *is a quotient map for each* $n \geq 1$.

Proof.

Suppose $F_R^q(Y)$ is a Hausdorff topological group. Since q is quotient, we have $id : F_R^q(Y) \cong F_M(Y)$ by A.50. Since $F_M(Y)$ is Hausdorff, Y must be functionally Hausdorff by Lemma A.37. Consequently, q is separating. Since q is quotient and separating, A.63 implies that $\sigma : Y \to F_R^q(Y) \cong F_M(Y)$ is actually and embedding. By Lemma A.37, Y must be Tychonoff. Since $id : F_R^q(Y)_n \cong F_M(Y)_n$ for each n, it follows that

$$F_M(Y) \cong F_R^q(Y) \cong \varinjlim_n F_R^q(Y)_n \cong \varinjlim_n F_M(Y)_n$$

where the second isomorphism comes from A.61.1. The fact that $RQ_n : \bigoplus_{i=0}^{n}(X \oplus X^{-1})^i \to F_R^q(Y)_n \cong F_M(Y)_n$ is quotient follows from A.61.2.

Now suppose conditions 1.-3. hold. Since Y is Tychonoff, q is separating. Since $RQ_n : \bigoplus_{i=0}^{n}(X \oplus X^{-1})^i \to F_M(Y)_n$ is quotient by assumption and $RQ_n : \bigoplus_{i=0}^{n}(X \oplus X^{-1})^i \to F_R^q(Y)_n$ is quotient by A.61.2, we have $id : F_R^q(Y)_n \cong F_M(Y)_n$ for each n. Therefore

$$id : F_M(Y) \cong \varinjlim_n F_M(Y)_n \cong \varinjlim_n F_R^q(Y)_n \cong F_R^q(Y)$$

where the last isomorphism is from A.61.1. Lemma A.50 implies that $F_R^q(Y)$ is a topological group. Since q is separating $F_R^q(Y)$ is T_1 A.63 and every T_1 topological group is Hausdorff. ■

The proof of the following simplification when $X = Y$ and $q = id_Y$ is the same.

Proposition A.65 *For a Hausdorff space* Y, $F_R(Y)$ *is a topological group if and only if the* *following three conditions hold:*

1. *Y is Tychonoff*

2. $F_M(Y) \cong \varinjlim_n F_M(Y)_n$

3. $R_n : \bigoplus_{i=0}^{n}(Y \oplus Y^{-1})^i \to F_M(Y)_n$ *is a quotient map for each* $n \geq 1$.

It should be noted that the conditions 2. and 3. in the previous proposition have received a great deal of attention in topological algebra. Consequently, this characterization is quite useful for determining when our constructions result in a topological group. Full characterizations of spaces Y for which 2. and 3. hold individually remain open problems. See sections 5-8 of [Sip05] for recent results on these characterization problems.

Recall from Lemma A.50 that for quotient $q : X \to Y$, $id : F_R^q(Y) \cong F_M(Y)$ if and only if $id : F_R^q(Y) \cong F_R(Y)$ and $F_R(Y) \cong F_M(Y)$. This allows us to give alternative characterizations by considering two separate characterizations.

Theorem A.66 *Let $q : X \to Y$ be a continuous surjection. If $q^n : X^n \to Y^n$ is a quotient map for all $n \geq 1$, then the induced, continuous epimorphism $F_R(q) : F_R(X) \to F_R(Y)$ is a topological quotient map. If X and Y are Hausdorff spaces, the converse holds.*

Proof. If $q^n : X^n \to Y^n$ is a quotient map for each $n \geq 1$, then so is $M_T^*(q) = \bigoplus_{n \geq 0}(q \oplus q)^n : M_T^*(X) \to M_T^*(Y)$. Since the diagram

$$\begin{array}{ccc} M_T^*(X) & \xrightarrow{M_T^*(q)} & M_T^*(Y) \\ {\scriptstyle R_X}\downarrow & & \downarrow{\scriptstyle R_Y} \\ F_R(X) & \xrightarrow[F_R(q)]{} & F_R(Y) \end{array}$$

commutes and the reduction maps are quotient maps, $F_R(q)$ is a quotient map.

To prove the converse, let $\sigma_n^X : X^n \to F_R(X)$ and $\sigma_n^Y : Y^n \to F_R(Y)$ be the canonical, closed embeddings of Theorem A.63 and $\tilde{X}^n = \sigma_n^X(X^n)$ and $\tilde{Y}^n = \sigma_n^Y(Y^n)$ be their images.

We show the restriction $p = F_R(q)|_{\tilde{X}^n} : \tilde{X}^n \to \tilde{Y}^n$ is a quotient map using the commutative diagram

$$\begin{array}{ccccc} X^n & \xrightarrow{\sigma_X^n} & F_R(X) & \xleftarrow{R_X} & M_T^*(X) \\ {\scriptstyle q^n}\big\downarrow & & \big\downarrow{\scriptstyle F_R(q)} & & \big\downarrow{\scriptstyle M_T^*(q)} \\ Y^n & \xrightarrow[\sigma_Y^n]{} & F_R(Y) & \xleftarrow[R_Y]{} & M_T^*(Y) \end{array}$$

where the reduction maps are distinguished with subscripts. To see that p being quotient implies q^n is quotient, take $C \subseteq Y^n$ such that $(q^n)^{-1}(C)$ is closed in X^n. Then $\sigma_n^X((q^n)^{-1}(C)) = p^{-1}(\sigma_n^Y(C))$ is closed in $\tilde{X}^n$ and consequently $\sigma_n^Y(C)$ is closed in $\tilde{Y}^n$. Since σ_n^Y is a continuous injection, C is closed in Y^n.

Suppose $A \subseteq \tilde{Y}^n$ such that $p^{-1}(A)$ is closed in $\tilde{X}^n$. Since R_X is quotient and $F_R(q)$ is assumed to be quotient and $\tilde{Y}^n$ is closed in $F_R(Y)$, it suffices to show that

$$B^\zeta = R_X^{-1}(F_R(q)^{-1}(A) \cap X^\zeta = \{x = x_1^{\epsilon_1}...x_k^{\epsilon_k} | R_Y(M_T^*(q)(x)) = R_Y(q(x_1)^{\epsilon_1}...q(x_k)^{\epsilon_k}) \in A\}$$

is closed in X^ζ for each $\zeta = \epsilon_1, ..., \epsilon_k$. We proceed by induction on $|\zeta| = k$. It is clear that if $|\zeta| < n$, then $B^\zeta = \emptyset$. Additionally, if $|\zeta| = n$ and $\zeta \neq 1, 1, ..., 1$, then $B^\zeta = \emptyset$. On the other hand, if $|\zeta| = n$ and $\zeta = 1, 1, ..., 1$, then $B^\zeta = \{x_1...x_n | q(x_1)...q(x_n) \in A\} = R_X^{-1}(p^{-1}(A)) \cap X^\zeta$ is closed by assumption. Now we suppose that $|\zeta| > n$ and B^ζ is closed in X^{δ} for all δ such that $|\delta| = n, n+1, ..., |\zeta| - 1$. Let $x = x_1^{\epsilon_1}...x_k^{\epsilon_k} \in X^{\zeta} - B^{\zeta}$ and $y = M_T^*(q)(x) = q(x_1)^{\epsilon_1}...q(x_k)^{\epsilon_k}$. Since $x \notin B^{\zeta}$, we have $R_Y(y) \notin A$. Let $E = E_1^{\epsilon_1}...E_k^{\epsilon_k}$ be a separating neighborhood of y in $M_T^*(Y)$. Since $M_T^*(q)$ is continuous, there is a separating neighborhood $D = D_1^{\epsilon_1}...D_k^{\epsilon_k}$ of x, such that $q(D_i) \subseteq E_i$ for each $i \in \{1, ..., k\}$. Since E is a separating neighborhood, if $q(x_i) \neq q(x_j)$, then $q(D_i) \cap q(D_j) = \emptyset$. Now we consider the cases when y is and is not reduced.

If y is reduced and $v \in D$, then $M_T^*(q)(v) \in E$ must also be reduced by A.59. Therefore $n < |\zeta| = |y| = |R_Y(M_T^*(v))|$, i.e. the reduced word of $M_T^*(q)(v)$ has length greater than n and cannot lie in $A \subseteq \tilde{Y}^n$. Therefore $D \cap B^{\zeta} = \emptyset$.

If y is not reduced, then for each $i \in \{1, ..., k-1\}$ such that $q(x_i) = q(x_{i+1})$ and $\epsilon_i = -\epsilon_{i+1}$, we let $w_i = q(x_1)^{\epsilon_1} \dots q(x_{i-1})^{\epsilon_{i-1}} q(x_{i+2})^{\epsilon_{i+2}} \dots q(x_k)^{\epsilon_k} \in M_T^*(Y)$ and $u_i = x_1^{\epsilon_1} \dots x_{i-1}^{\epsilon_{i-1}} x_{i+2}^{\epsilon_{i+2}} \dots x_k^{\epsilon_k}$ be the words obtained by removing the i-th and (i+1)-th letters from y and x respectively. We also let $\zeta_i = \epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+2}, ...\epsilon_k$. This gives $F_R(q)(R_X(u_i)) = R_Y(M_T^*(u_i)) = R_Y(w_i) = R_Y(y) \notin A$ and consequently $u_i \in X^{\zeta_i} - B^{\zeta_i}$. We know by our induction hypothesis that B^{ζ_i} is closed in X^{ζ_i} and so we may find a separating neighborhood $V_i = A_1^{\epsilon_1} \dots A_{i-1}^{\epsilon_{i-1}} A_{i+2}^{\epsilon_{i+2}} \dots A_k^{\epsilon_k}$ of u_i contained in $X^{\zeta_i} - B^{\zeta_i}$. Let $A_i = A_{i+1} = X$ so that

$$U_i = A_1^{\epsilon_1}...A_{i-1}^{\epsilon_{i-1}} A_i^{\epsilon_i} A_{i+1}^{\epsilon_{i+1}} A_{i+2}^{\epsilon_{i+2}}...A_k^{\epsilon_k}$$

is an open neighborhood of x. Now take a separating neighborhood U of x such that $U \subseteq D \cap \bigcap_i U_i$ where the intersection ranges over the $i \in \{1, \dots, k-1\}$ such that $q(x_i) = q(x_{i+1})$ and $\epsilon_i = -\epsilon_{i+1}$. It now suffices to show that $F_R(q)(R_X(v)) =$

$R_Y(q(z_1)^{\epsilon_1}\dots q(z_k)^{\epsilon_k}) \notin A$ whenever $v = z_1^{\epsilon_1}\dots z_k^{\epsilon_k} \in U$. If $M_T^*(q)(v) = q(z_1)^{\epsilon_1}\dots q(z_k)^{\epsilon_k}$ is reduced, then $n < |\zeta| = |x| = |R_Y(M_T^*(q))|$ and $R_Y(M_T^*(q)(v)) \notin A$. On the other hand, suppose $q(z_1)^{\epsilon_1}\dots q(z_k)^{\epsilon_k}$ is not reduced. There is an $i_0 \in \{1,\dots,k-1\}$ such that $q(z_{i_0}) = q(z_{i_0+1})$ and $\epsilon_{i_0} = -\epsilon_{i_0+1}$. But $z_{i_0} \in D_{i_0}$ and $z_{i_0+1} \in D_{i_0+1}$, so we must have $q(x_{i_0}) = q(x_{i_0+1})$. Since $v \in U \subseteq U_{i_0}$, we have

$$v_{i_0} = z_1^{\epsilon_1}\dots z_{i_0-1}^{\epsilon_{i_0-1}} z_{i_0+2}^{\epsilon_{i_0+2}}\dots z_k^{\epsilon_k} \in V_{i_0} \subseteq X^{\zeta_{i_0}} - B^{\zeta_{i_0}}$$

Therefore

$$F_R(q)(R_X(v)) = R_Y(M_T^*(q)(v)) = R_Y(M_T^*(q)(v_{i_0})) = F_R(q)(R_X(v_{i_0})) \notin A$$

proving that $U \cap B^\zeta = \emptyset$ and B^ζ is closed in X^ζ. ■

Putting all of the previous results together, we obtain the following classification theorem.

Theorem A.67 *Let X be Hausdorff and $q : X \to Y$ be a quotient map. The following are equivalent:*

1. *$F_R^q(Y)$ is a Hausdorff topological group.*
2. *$id : F_R^q(Y) \cong F_R(Y) \cong F_M(Y)$ and q is separating.*
3. *$RQ : M_T^*(X) \to F_M(Y)$ is a topological quotient map and q is separating.*
4. *The following three conditions hold:*
   - *(a) Y is Tychonoff.*

(b) $F_M(Y) \cong \varinjlim_n F_M(Y)_n$.

(c) $R_n : \bigoplus_{i=0}^{n}(Y \oplus Y^{-1})^i \to F_M(Y)_n$ *is a quotient map for each* $n \geq 1$.

(d) $q^n : X^n \to Y^n$ *is a quotient map for each* $n \geq 1$.

Proof. In A.50 it is shown for arbitrary q that $F_R^q(Y)$ is a topological group $\Leftrightarrow$ $id : F_R^q(Y) \cong F_R(Y) \cong F_M(Y) \Leftrightarrow RQ : M_T^*(X) \to F_M(Y)$ is a topological quotient map. Also q is separating $\Leftrightarrow F_R^q(Y)$ is T_1 (A.63) and any topological group is T_1 if and only if it is Hausdorff. Therefore, we have 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3.

1. $\Leftrightarrow$ 4. follows from Theorem A.64 and the fact that when X and Y are Hausdorff, $id : F_R^q(Y) \cong F_R(Y) \Leftrightarrow F_R(q)$ is quotient $\Leftrightarrow q^n$ is quotient for each $n \geq 1$. ■

The arguments used to prove the next statements are based on the arguments used by Fabel [Fab06] to show that the Hawaiian earring group $\pi_1^{top}(\mathbb{HE})$ is not first countable. Given a sequence of integers N_m, we write $\lim_{m\to\infty} N_m = \infty$ when for each $M \geq 1$, there is an m_0 such that $N_m \geq M$ for all $m \geq m_0$.

Lemma A.68 *If* $q : X \to Y$ *is separating and* w_m *is a sequence of reduced words in* $F_R^q(Y)$ *such that* $\lim_{m\to\infty} |w_m| = \infty$*, then the set* $\{w_m\}_{m\geq 1}$ *is closed in* $F_R^q(Y)$.

Proof. Let $C = \{w_m\}_{m\geq 1} \subset F_R^q(Y)$. Since $RQ : M_T^*(X) \to F_R^q(Y)$ is quotient, it suffices to show that $RQ^{-1}(C)$ is closed in $M_T^*(X)$. Let $z_k \in RQ^{-1}(C)$, $k \in K$ be a net ($(K, \geq)$ is a directed set) in $M_T^*(X)$ converging to $z \in X^{\zeta_0} \subset M_T^*(X)$ such that $RQ(z) \notin C$. For each $k \in K$, we write $RQ(z_k) = w_{m_k}$, which implies $|z_k| \geq |w_{m_k}|$. Since X^{ζ_0} is open in $M_T^*(X)$, there is a $k_0 \in K$ such that $z_k \in X^{\zeta_0}$ (and consequently $|z_k| = |z|$) for every $k \geq k_0$. If the net of integers m_k is bounded by integer M, then $RQ(z_k) \in \{w_1, w_2, ..., w_M\}$ for each $k \in K$. But $F_R^q(Y)$ is T_1 by Theorem A.63 and so the finite set $\{w_1, w_2, ..., w_M\}$ is closed in $F_R^q(Y)$. Since $RQ(z_k) \to RQ(z)$, we must have $RQ(z) \in \{w_1, w_2, \dots, w_M\} \subseteq C$ but this is a contradiction. Suppose, on the other hand, that m_k is unbounded and $k_0 \in K$. Since $\lim_{m\to\infty} |w_m| = \infty$, there is an m_0 such that $|w_m| > |z|$ for all $m \geq m_0$. Since m_k is unbounded, there is a $k_1 \geq k_0$ such that $m_{k_1} > m_0$. But this means

$$\left|z_{k_1}\right| \geq \left|w_{m_{k_1}}\right| > |z|$$

This contradicts that $|z_k|$ is eventually $|z|$. Therefore we must have that $RQ(z) \in C$ which again is a contradiction. Since any convergent net in $RQ^{-1}(C)$ has limit in $RQ^{-1}(C)$, this set must be closed in $M_T^*(X)$. ■

Corollary A.69 *Let $q : X \to Y$ be separating and w_m be a sequence in $F_R^q(Y)$ such that* $\lim_{m\to\infty} |w_m| = \infty$. *Then w_m does not have a subsequence which converges in $F_R^q(Y)$.*

Proof. If $\lim_{m\to\infty} |w_m| = \infty$, then $\lim_{m\to\infty} |w_{m_j}| = \infty$ for any subsequence w_{m_j}. Therefore, it suffices to show that w_m does not converge in $F_R^q(Y)$ whenever $\lim_{m\to\infty} |w_m| = \infty$. Suppose $w_m \to v$ for some $v \in F_R^q(Y)$. There is a subsequence w_{m_j} of w_m such that $|w_{m_j}| > |v|$ for each $j \geq 1$. But $\lim_{m\to\infty} |w_{m_j}| = \infty$ and so $C = \{w_{m_j}\}_{j\geq 1}$ is closed in $F_R^q(Y)$ by Lemma A.68. This implies $v \in C$ which is impossible. ■

Corollary A.70 *If $q : X \to Y$ is separating and K is a compact subset of $F_R^q(Y)$, then $K \subseteq F_R^q(Y)_n$ for some $n \geq 1$.*

Proof. Suppose $K \not\subseteq F_R^q(Y)_n$ for any $n \geq 1$. Take $w_1 \in K$ such that $|w_1| = n_1$.

Inductively, if we have $w_m \in K \cap F_R^q(Y)_{n_m}$, there is an $n_{m+1} > n_m$ and a word $w_{m+1} \in K \cap \left(F_R^q(Y)_{n_{m+1}} - F_R^q(Y)_{n_m}\right)$. Now we have a sequence $w_m \in K$ such that $|w_1| < |w_2| < ...$ which clearly gives $\lim_{m\to\infty} |w_m| = \infty$. Corollary A.69 then asserts that w_m has no converging subsequence in $F_R^q(Y)$, however, this contradicts the fact that K is compact. ■

Theorem A.71 *Let $q : X \to Y$ be a separating quotient map. The following are equivalent:*

1. *Y is a discrete space.*
2. *$F_R^q(Y)$ is a discrete group.*
3. *$F_R^q(Y)$ is first countable.*

Proof. 1. $\Leftrightarrow$ 2. was proven in A.54 and 2. $\Rightarrow$ 3. is clear. To prove 3. $\Rightarrow$ 1., we suppose Y is non-discrete and $F_R^q(Y)$ is first countable. Since q is quotient and separating, Y must be T_1. Let $y_0 \in Y$ such that the singleton $\{y_0\}$ is not open. Since q is quotient $q^{-1}(y_0)$ is not open in X. There is an $x_0 \in q^{-1}(y_0)$ such that every open neighborhood U of x_0 in X satisfies $q(U) \neq \{y_0\}$. In fact, $q(U)$ must be infinite, since if $q(U) = \{y_0, y_1, ..., y_m\}$, then $U \cap \bigcap_{i=1}^{m}(X - q^{-1}(y_i))$ is an open neighborhood of x_0 contained in $q^{-1}(y_0)$. Suppose $\{B_1, B_2, ...\}$ is a countable basis of open neighborhoods at the identity e in $F_R^q(Y)$ where $B_{i+1} \subseteq B_i$ for each i. Choose any $z \in X$ such that $q(z) \neq q(x_0) = y_0$ and let $w_n = \left(x_0 z x_0 x_0^{-1} z^{-1} x_0^{-1}\right)^n \in M_T^*(X)$. It is clear that $RQ(w_n) = R((y_0 q(z) y_0 y_0^{-1} q(z)^{-1} y_0^{-1})^n) = e$ and therefore $w_n \in RQ^{-1}(B_i)$ for all $i, n \geq 1$. Since q is separating, there is an open neighborhood U_n of x_0 and

V_n of z such that $\mathscr{U}_n = (U_n V_n U_n U_n^{-1} V_n^{-1} U_n^{-1})^n$ is a q-separating neighborhood of w_n contained in $RQ^{-1}(B_n)$. Recall that $\mathscr{U}_n$ being a q-separating neighborhood means $q(U_n) \cap q(V_n) = \emptyset$. Since $q(U_n)$ is infinite, we can find $y_n \in q(U_n)$ distinct from y_0 and $x_n \in U_n \cap q^{-1}(y_n)$. Since $q(U_n) \cap q(V_n) = \emptyset$, the three elements $y_0, y_n, q(z)$ of Y are distinct for each $n \geq 1$. Now we have

$$v_n = \left(x_0 z x_0 x_n^{-1} z^{-1} x_n^{-1}\right)^n \in \mathscr{U}_n \subseteq RQ^{-1}(B_n)$$

which satisfies

$$RQ(v_n) = R\left(\left(y_0 q(z) y_0 y_n^{-1} q(z)^{-1} y_n^{-1}\right)^n\right) = \left(y_0 q(z) y_0 y_n^{-1} q(z)^{-1} y_n^{-1}\right)^n \in B_n.$$

Note that $|RQ(v_n)| = 6n$ and so $\lim_{n\to\infty} |RQ(v_n)| = \infty$. By Corollary A.69, the sequence $RQ(v_n)$ cannot converge to the identity of $F_R^q(Y)$. But since $\{B_i\}$ is a countable basis at e and $RQ(v_n) \in B_n$, we must have $RQ(v_n) \to e$. This is a contradiction.

■

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