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The process of making meaning: The interplay between teachers' knowledge of mathematical proofs and their classroom practices

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THE PROCESS OF MAKING MEANING: THE INTERPLAY BETWEEN TEACHERS' KNOWLEDGE OF MATHEMATICAL PROOFS AND THEIR CLASSROOM PRACTICES

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DEDICATION

This dissertation is dedicated to my family and friends. You surrounded me with laughter and love. You supported me through the tough times and helped me celebrate the good times. I am truly grateful to all of you.

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THE PROCESS OF MAKING MEANING:

THE INTERPLAY BETWEEN TEACHERS' KNOWLEDGE OF MATHEMATICAL PROOFS AND THEIR CLASSROOM PRACTICES

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The purpose of this study was to investigate and describe how middle school mathematics teachers *make meaning* of proofs and the process of proving in the context of their classroom practices. A framework of *making meaning*, created by the researcher, guided the data collection and analysis phases of the study. This framework describes the five central aspects of the process of *making meaning*: knowledge, beliefs, utilization of knowledge, interconnections of practice and knowledge, and making sense of past knowledge and current experiences. The utilization of a qualitative research methodology that combined ethnographic fieldwork and discourse analysis allowed the researcher to consider the interplay of individual knowledge and action with contextual influences.

Data was gathered through participant observations, conducted in grade levels 5-8 classrooms, and interviews were conducted with teachers and administrators. The participants in the study include six middle grades teachers and two administrators from a

New Hampshire public school. The teachers' decision making processes, understandings of proofs, and the connections formed between their past knowledge and current experiences were analyzed. Data analysis was conducted using open coding, the creation of episodic threads, and the development and examination of themes.

Findings from this study suggest that: (1) these teachers use proofs and the process of proving in their classrooms often and in meaningful ways, (2) the teachers hold a dual understanding of proofs and the process of proving: one related to their own education experiences and one related to their students' education, (3) the teachers use alternative resources to *make meaning* of proofs and the process of proving in relation to their professional experiences, and (4) the use of alternative resources has allowed the teachers to disconnect their view of proofs and the process of proving in relation to their students education from their own past experiences, which they view as negative.

Implications for future research related to teachers' knowledge of proofs and the process of *making meaning,* as well as implications for the education and professional development of middle school mathematics teachers are presented and discussed.

CHAPTER 1

INTRODUCTION

Two critical topics in current mathematics education research are teacher knowledge and the use of mathematical proofs in precollege mathematics courses. Focusing on a combination of these important topics is imperative to the future of mathematics education, and is the central concern of this study. This study is concentrated on the interaction of middle school mathematics teachers' knowledge of mathematical proofs and the use of proofs in their classroom practices. In this chapter a rationale for this study will be given, followed by a discussion of the purposes of this study and an explanation of the research questions. Finally, an overview of the research conducted will be presented and the contents of the dissertation will be outlined.

Rationale

In the National Council of Teachers of Mathematics [NCTM's] *Principles and Standards for School Mathematics* (2000), one of the five process standards for teaching and learning mathematics is *Reasoning & Proof.* The importance of this standard is made clear by the description of mathematical reasoning as "a habit of mind" that "must be developed through consistent use in many contexts and from the earliest grades." Included in the overview of the Reasoning and Proof Process Standard the NCTM (2000) states "instructional programs from prekindergarten through grade 12 should enable all students to-

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof" (p. 56).

Yackel and Hanna (2003) state that the NCTM's decision to place "emphasis on reasoning as a central aspect in all areas and at all levels of mathematics instruction is a deliberate choice that mathematics educators have made as a result of a better understanding of how individuals come to know" (p. 227).

Ross (1998) asserts that, because reasoning and proof are essential to mathematics and lacking the ability to reason and create proofs leaves a student with just memorization and procedures to follow, mathematics teachers should,

> make it their aim to explain everything in mathematics to the extent that this is reasonable and effective at the student's level of mathematical knowledge. The important thing is to be honest; if only illustrations and a plausibility argument are supplied, the student should be reminded that a logical reason or proof is needed (p. 254).

The position of the NCTM and many researchers in the field of mathematics education (Ball, 1991; Hanna, 1989; Hersh, 1993; Stylianides, 2007; Yackel & Hanna, 2003) to make proofs a central aspect of all students' mathematics education can be represented by the following statement, "If you believe, as many do, that proof is math and math is proof, then in a math course you're duty bound to prove something" (Hersh, 1993, p. 396). The support for the inclusion of proofs and the process of proving in precollege mathematics courses is clear. However, the knowledge and resources needed by teachers to implement such recommendations is unclear and a growing topic in the field of mathematics education research.

In the NCTM's *Principles and Standards for School Mathematics* (2000) the teachers' role in developing reasoning and proof in the middle grades is described to include the ability to, "help students appreciate and use the power of mathematical reasoning by regularly engaging students in thinking and reasoning in the classroom" (p. 265). This clearly implies that the teachers must know and understand proofs and the process of proving, since that is the heart and soul of mathematical reasoning. The question of how teachers come to learn and understand proofs in such a way that they are able to carry out this recommendation is an area of interest in the mathematics education community. Moreover, the fact that "the emphasis that teachers place on justification and proof no doubt plays an important role in shaping students' proof schemes" (Harel and Sowder, 2007, p. 827) underscores the concern for understanding the resources that effect teachers' abilities to utilize proofs in their classrooms.

Some work done in this area suggests that a teacher's ability to use proofs as a form of explanation relies heavily on their ability to explain why something is true, which relies on their explicit knowledge. "Explicit knowledge involves reasons and relationships: being able to explain why, as well as being able to relate particular ideas or procedures to others within mathematics" (Ball, 1991, p. 16). Other work done around the area of teachers' knowledge of proof has focused on teachers' conceptions of proofs (Borko, Peressini, Romagnano, et al, 2000; and, Cyr, 2004), their abilities to understand the validity of proofs (Martin and Harel, 1989; and, Stylianides, Stylianides, and Phillippou, 2007), the interplay of teacher and student interaction during the process of learning to prove (Martin, McCrone, Brower, and Dindyal, 2005), and the teachers' role in facilitating the proof related events (Stylianides, 2007). These studies will be discussed further in Chapter 2.

The present study attempts to describe the process teachers go through as they make sense of their knowledge of proofs and the process of proving in the context of their classroom. Considering the recommendations that proofs be a central aspect of students mathematical education, the teacher's role in implementing these recommendations, and our lack of understanding related to what knowledge and resources teachers need in order to implement such recommendations, this study's focus on teachers' knowledge and proofs has a significant position in mathematics education research. Moreover, this study is compelling because it describes a complex and personal process by revealing how mathematics teachers' conceptions and knowledge inform their classroom practices.

Purpose

The purpose of this study is to describe how teachers *make meaning* of their knowledge of proofs and the process of proving in the context of their classroom practices. In order to do this a framework for viewing the process of *making meaning* was created. The aspects of *making meaning* include: *knowledge*, *beliefs*, *utilization of knowledge*, *making sense of past knowledge and current experiences*, and *the interconnections of practices and knowledge*. This framework will be discussed in Chapter 2. The purpose of this study is to explore each of these aspects and then analyze *making meaning* as a unified process. Thus, the goals of this study include:

- Identify and describe patterns of classroom practices that are related to proofs and the process of proving (*utilization of knowledge*).

- Identify and describe different ways teachers make decisions about how they use proof related methods in their classroom (*utilization of knowledge* and *making sense of part knowledge and current experiences*).
- Identify and describe the teachers' knowledge and beliefs about proofs in relation to their past experiences (*knowledge* and *beliefs*).
- Identify and describe the teachers' knowledge and beliefs about proofs in relation to their students' education (*knowledge* and *beliefs*).
- Identify and describe the teachers' decision making process as one that is influenced by both their past and on-the-spot experiences (*making sense of past knowledge and current experiences*).
- Identify and describe teachers' negotiations between their knowledge and beliefs about proofs and their professional practices (*interconnections of practice and knowledge*).
- Identify and describe resources utilized by the teachers in *making meaning* of their knowledge of proofs and the process of proving and their classroom practices (*making sense of past knowledge and current experiences* and *interconnections of practice and knowledge*).

Each of these goals is in direct relation to the concept of *making meaning* and will be attended to during this study by addressing the following research questions.

Research Questions

The main research question guiding this study is: *How do middle school mathematics teachers make meaning of their knowledge of proofs and the process of* *proving in their classroom practices?* In order to address this question the following three topical questions were explored:

1. *How do teachers make decisions about whether or not to include the use of proof (or the process of proving) in their classrooms? And in what ways?*
2. *How do teachers think about or understand proofs and the process of proving?*
3. *How do teachers form connections between their understanding of proofs (and the process of proving) and the incorporation of certain teaching methods into their classrooms?*

Each of these questions addresses aspects of *making meaning*. The first topical question addresses the aspects of *utilization of knowledge* and *making sense of past knowledge and current experiences*. The second topical question addresses the aspects of *knowledge*, *beliefs*, and *making sense of past knowledge and current experiences*. The third topical question addresses the aspects of *interconnections of practice and knowledge* and *making sense of past knowledge and current experiences*. The use of these questions to address the aspects of *making meaning* will be discussed further in Chapter 2.

Overview of Research and the Structure of this Dissertation

This study focuses on the teachers' past experiences and the knowledge that they bring to the classroom as individuals, as well as the social learning environment of the classroom itself. Using a social constructivist lens offered the best way to view the teacher participants as individuals who construct their knowledge based on past experiences and understandings, and as members of the social learning environment found in their classrooms (Ernest, 1999). The use of a social constructivist lens will be defended and clarified in Chapter 2.

In order to address my research questions, four concepts were clarified. These concepts are discussed in Chapter 2. First, the concepts of *proof* and for *the process of proving* are defined and characterized based on the work done by Harel and Sowder (2007), Recion and Godino (2001), and Stylianides (2007). Second, a framework of *the functions of proofs and the process of proving* was generated based on the work done by de Villiers (1999), Ellis (2007), Hanna (1989), Hanna and Jahnke (1993), Harel and Sowder (2007), Hersh (1993), Lakatos (1976), Larson and Zandieh (2005), NCTM (1991, 2000), Stylianides and Silver (2004), and Yackel and Hanna (2003). Finally, in order to address questions related to teachers' knowledge it is imperative to recognize the complexity of the knowledge base that teachers use in their profession. Thus, a framework for viewing the process of *making meaning* was created. This framework was based partly on my own conception of *making meaning* and partly on the work done by Ball and Bass (2000), Ball et al (2001), Borko et al (2000), Cyr (2004), Ernest (1999), Martin and Harel (1989), Martin et al (2005), Shulman (1986), Stylianides (2007), Stylianides, Stylianides, and Phillippou (2007), and Thompson (1984).

The conceptual and practical concerns inherent in my attempt to understand the process of *making meaning* demanded a methodology that would allow me to explore the interplay of individual knowledge and action with contextual influences. Qualitative research methods based in ethnographic fieldwork allowed me to incorporate a social constructionist lens with an analytic framework based on concepts and methods from discourse analysis (Agar, 2006, Emerson, Fretz, & Shaw, 1995; Gee, 2005). Data was collected through participant observation and interviews with six teachers. The teachers all work at the same school, and taught mathematics in grades 5 – 8. Participant observations lasted at least six weeks with each teacher, and for each teacher a series of up to three interviews was conducted. Interviews were also conducted with the school's principal and curriculum coordinator. Methods of data collection and participant information will be discussed in Chapter 3.

The analytic framework for this study is based on the concepts and methods of discourse analysis as described by Gee (2005). The meaning and context questions that arise from discourse analysis helped make the familiar strange and strange familiar. The use of tools from discourse analysis will be discussed in Chapter 3.

Analysis was conducted in three distinct, yet interrelated phases: coding fieldnotes and interview transcripts using an open coding method, and creating episodic threads, as described by Emerson, Fretz, & Shaw (1995); developing themes at the individual and general levels; and, developing the text. The methods of analysis will be discussed in Chapter 3.

The findings from this study are presented in Chapter 4, which is separated into three parts. Part I addresses the teachers' *utilization of knowledge* by considering the first topical question. The findings will show that each of the teachers was incorporating activities and discussions utilizing proofs and the process of proving into their classrooms. Part II addresses the teachers' *beliefs* and *knowledge* about proofs and the process of proving. It also starts to address how the teachers *make sense of past knowledge and their current practices.* This part of the chapter considers the second topical question. The findings will show that some of the teachers hold dual understandings of proof. One related to their own educational experiences and one related to the role of proofs and the process of proving used in their classrooms. Part III of this chapter will continue to address the *connections between the teachers' knowledge and practices*. In this part the third topical question is considered. Discontinuities found in the first two parts of this chapter will be explored and resources utilized by teachers in *making meaning* of proofs and the process of proving will be presented.

The final chapter will discuss the findings presented in Chapter 4 and address my main research question. I will then discuss possible implications of this study and its limitations along with directions for future research.

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

This chapter reviews the literature relevant to this study and situates the dissertation within this literature. The concepts and theoretical frameworks used in developing this study, executing the data collection, analyzing the data, and coming to conclusions based on the analysis will be presented. To begin, the use of a social constructivist lens to view the teachers as learners in their classroom will be defined and defended. Then using particular theories and research the concepts of *proofs* and *the process of proving* will be explored. A framework of *the functions of proofs and the process of proving* will also be offered and supported. A review of the literature related to teachers' knowledge, particularly their knowledge of proof and the process of proving will be presented along with a framework of how this knowledge was viewed during this study. Finally, the topical questions guiding this study will be explored as my characterization of the term *making meaning* is discussed and clarified.

Social Constructivist Perspective

This dissertation focuses on teachers' past experiences and the knowledge that they bring to the classroom as individuals, as well as the social learning environment of the classroom itself. Social constructivism finds its origins in John Dewey's *Pedagogic Creed* written in 1897. Dewey (1897) describes the educational process as having "two sides – one psychological and one sociological; and that neither can be subordinated or neglected without evil results following" (p. 77). Using a social constructivist lens offered the best way to view the teacher participants as individuals who construct their knowledge based on past experiences and understandings, and as members of the social learning environment found in their classrooms. As suggested by Dewey, the social constructivist perspective values both the individual's construction of knowledge and the impact of the social world on that construction. Cobb (1994) argues "mathematical learning should be viewed as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society (p.13)." He goes on to say that the issue facing mathematics educators is not that of deciding whether to take a constructivist or sociocultural perspective, but to find ways of coordinating the two.

Focusing on the processes by which teachers *make meaning* of their knowledge emphasizes that the classroom is an environment in which the teacher is also a learner. During a study conducted by Cobb, Wood, and Yackel (1990) these researchers found that "in the course of listening to their [the students'] solutions the teacher modified her beliefs about mathematics and extended her understanding of children's learning of mathematics (p. 139)." Thus, their research supports the idea that the classroom was a learning environment for teachers. The topic of this study demands a specific focus on teachers' learning of mathematical proofs and the process of proving in the context of their classroom as a learning environment. The social constructivist lens allows one to view "the concepts of mathematics [as] derived by abstraction from direct experience of the physical world, from the generalization and reflective abstraction of previously constructed concepts, by negotiating meanings with others during discourse, or by some combination of these means (Ernest, 1999, pg 4)." Therefore, theories about teachers' knowledge and knowledge of mathematical proofs are needed to frame this study. Moreover, a frame for viewing the teachers' meaning making will needed; as well as, an analytic framework for analyzing discourse analysis, which will be discussed in the next chapter.

Proofs and The Process of Proving

In this section research related to the concepts of *proofs*, *the process of proving*, and *the functions of proof and the process of process of proving* will be presented, including students' knowledge and use of these concepts. Teachers' knowledge and use of these concepts will be discussed in the following section. The majority of this section will be used to present the frameworks used to view these three concepts.

Over the past several years the topic of proofs has been the focus of a large number of research studies. The majority of research in this area has been focused on students' conceptions of proof. Most of the research done about elementary students and proofs has been about specific situations where students were given opportunities to use proof methods in their classrooms and the effectiveness of these situations has been analyzed. These studies have all found that when students are part of a classroom community that puts an emphasis on mathematical proof they are able to formulate conjectures, participate in meaningful mathematical discourse, and learn about mathematics through the use of proof and the process of proving (Ball, 1991; Ball & Bass, 2003; Stylianides, 2007). Schoenfeld (1994), remarking on his observations of a third grade classroom states, "it is possible to have mathematics classes be communities in which mathematical sense-making takes place. And when that happens, proof will be a necessary component of the sense-making and discourse process" (p. 78).

Research focused on middle grades and proof has showed that students depend on the use of examples to both formulate and evaluate mathematical arguments (Healy & Hoyles, 2000; Knuth, Slaughter, Choppin, & Sutherland, 2000; Knuth & Sutherland, 2004). In Knuth and Sutherland (2004) the results indicate that "students demonstrated an overwhelming reliance on the use of examples as a means of demonstrating and/or verifying the truth of a statement" (p. 562). The use of examples to formulate mathematical arguments that students consider to be proof is not only a problem at the middle school level. Studies about high school students and mathematical proofs found similar results to those about middle school students (Martin & McCrone, 2001). At the high school level there is also *existence research.* Tarlow (2004) found that "when given challenging problems in an appropriately supportive environment, these students can, and did construct…sophisticated mathematical proofs" (p. 652).

The use of proofs in geometry has been long-standing and thus more research has been done at the secondary level. The research at this level shows that students do not demonstrate a strong understanding of proofs (McCrone &Martin, 2004; McCrone, Martin, Dindyal, & Wallace, 2002; Schoenfeld, 1989). Schoenfeld (1989) reported his findings from a study about high school students and proofs in geometry. He states:

> Despite their [the students'] claims that proofs and constructions are closely related, they behave on construction problems as though their proof-related knowledge were nonexistent. Despite their assertions that mathematics helps one to think logically and that one can be creative in mathematics, they claim that mathematics is best learned by memorization – and in the case of memorization (p. 348-349).

Although the research at the high school level helped to identify the difficulties students have with the notion of proof, since the focus of this study is on the middle grades, this research will not be reviewed in detail, here.

The results of studies focused on proofs at the college level show similar difficulties to those found at the precollege level (Moore, 1994; Selden & Selden, 2003; Weber, 2001). The results from these studies show that students have a great need for examples, and an inability to write, formulate, or understand formal mathematical proofs. Part of these difficulties could be due to the fact that students entering college still have difficulty in understanding what constitutes a mathematical proof and therefore have difficulty with the process of proving (Weber, 2001). Just as with the research related to high school students, this research identified difficulties with proofs but will not be reviewed here since the focus of this study is on the middle grades.

Schoenfeld (1994) claims that "Proof is one of the most misunderstood notions of the mathematics curriculum, and we really need to sort it out" (p. 74). Schoenfeld, like many others, feels that proof needs to be viewed as not just a final rigid product, but instead as a process of discovering, understanding, explaining, communicating, and developing absolute certainty, along with other similar functions (de Villiers, 1999; Healy & Hoyles, 2000; Knuth & Sutherland, 2004; Ross, 1998; Weber, 2001; Yackel & Hanna, 2003). These functions will be addressed and classified in the framework used during this study, and can be found toward the end of this section. The first concept addressed in this section will be the concept of proof.

The concept of *proof* has taken on many different definitions. To begin, the institutional meanings of proofs presented by Recion and Godino (2001) will be summarized. The authors describe four different meanings of proof as it is understood in the institutional contexts of daily life, empirical science, professional mathematics, and logic and foundations of mathematics. Recion and Godino (2001) explain that the meaning of proof in daily life is based on informal argumentation and that "this type of informal argumentation does not necessarily produce truth, since it is based on local value consideration, which lack the objective features of proof" (p. 92). In the context of experimental science the authors argue that the "intuitive argumentation of daily life is replaced by experimental proof; beliefs are replaced by theories, which are experimentally validated" (p. 93). The proofs found in this context are used in mathematics as a "first validating step, where some particular cases of the proposition to be proved are experimentally verified" (p. 93). In mathematics these proofs are called *empirical-inductive proofs* (p. 93). In defining proofs within the context of professional mathematics these authors describe mathematical proof as "the argumentative process that mathematicians develop to justify the truth of mathematical propositions, which is essentially a logical process" (p. 94). The reference here to proofs being associated with a process, demonstrates the difficultly to define *proof* separate from *the process of proving*. Although this definition is presented as means for clarifying the concept of *proof*, for the purpose of this study Recion and Godino's (2001) concept of *mathematical proof* will be classified as a concept of *the process of proving*. The final context discussed by Recion and Godino (2001) is that of logic and foundations of mathematics. The authors explain that in this context "the notion of proof appears linked to deduction and formal systems. Logical argumentation is essentially a deduction argumentation... proof is a sequence of propositions, each of which is an axiom or a proposition that has been derived from axioms by inference rules" (p. 94-95). The authors' definition of mathematical proof will be discussed again later as a means to help define *the process of proving.*

The most important context to consider when defining proof for this study is the context of middle school classrooms. Stylianides (2007) analyzed the process of proving that third grade students were engaged in and the teacher's role in facilitating the proof related events that occurred. This study will be reviewed in the following section. Although his work is focused on elementary students, as stated by the author, the following definition

> is acceptable across the whole spectrum of students' mathematical education [because] (1) it considers both mathematics as a discipline and students as mathematics learners; (2) it promotes a consistent meaning of proof throughout the grades; (3) it prevents empirical arguments from being considered as proofs; (4) it supports analysis of classroom instruction relate to proof and study of the role of teachers in managing their students' proving activities (p. 293- 294).

The definition of proof in the context of a mathematics classroom presented by Stylianides (2007) is vital to the frame of proof used in this study and is given below:

> Proof is a mathematical argument, that is, a logically-connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> (i) it uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
>
> (ii) it employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
>
> (iii) it is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291).

This definition of proof was used not only in framing the concept of proof for the purposes of data collection and analysis, it was also shared with the teachers before final interviews as a means to discuss their beliefs and knowledge of proofs and their use of proofs in their classroom practices. This will be discussed further in Chapter 3. As stated above it is difficult to define *proof* separate from *the process of proving*. For this reason, before presenting a frame of *proof*, the concept of *the process of proving* will be explored.

Although final proofs are extremely important to the discipline of mathematics, "the essential mathematical activity is finding the proof" (Hersh, 1993, p. 390). The activity of finding or developing a proof is at the heart of what will be defined as *the process of proving*. The work done by Harel and Sowder (1998, 2007) is central to the notion of both *proofs* and *the process of proving* used in this study. Harel and Sowder (2007) provide a comprehensive perspective on proofs. Clarifying for the reader their concept of "proof" the authors explain that, "in our perspective 'proof' is interpreted subjectively; a proof is what establishes truth for a person or a community" (p. 806). This idea of audience is essential to the use of proofs in K-12 mathematics and is reflected in Stylianides' (2007) definition of proof given above.

The definition used to define *the process of proving* for this study is taken directly from Harel and Sowder (2007). To begin we will look at their definition of conjecture and fact, as they are the foundation for the definition *proving*. A *conjecture* is defined to be "an assertion made by an individual who is uncertain of its truth" (p. 808). A *fact* is subjective and is the product of a conjecture and established truth. A conjecture "becomes a fact in the person's view once he or she becomes certain of its truth" (p. 808). The process of becoming certain of a truth is *the process of proving* and will be defined as "the process employed by an individual (or community) to remove doubts about the truth of an assertion" (p. 808). Looking back at the concept of mathematical proof given by Recion and Godino (2001) as "the argumentative process that mathematicians develop to justify the truth of mathematical propositions" (p. 94) there are clear similarities between the two definitions. Both are focused on "truth" and the process used in establishing this truth. However, for the purposes of this study the definition given by Harel and Sowder (2007) offers the most useful view because of its focus on both the individual and the community which is further represented in their definitions of the subprocesses of proving: *ascertaining* and *persuading*. *Ascertaining* "is the process an individual (or community) employs to remove her or his (or its) own doubts about the truth of an assertion" (p. 808). *Persuading* "is the process an individual or a community employs to remove others' doubts about the truth of an assertion" (p. 808). This focus on both the individual and the community is consistent with the social constructivist lens. Having established the definitions for *the process of proving* and the two subprocesses, *ascertaining* and *persuading*, Harel and Sowder's concept of "proof schemes" will be explored.

Harel and Sowder (2007) continue to focus on the individual and community as subjectively determining what constitutes a proof and present the following definition: "a person's (or a community's) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)" (p. 809). Considering the different proof schemes that either a person or community might work under, the authors have created taxonomy consisting of three different classifications of proof schemes. As they describe these classifications the authors focus on the individual as the student. For the purpose of this study the focus is not only on the students' proof schemes but even more so on the teachers' proof schemes. Using this model to help make sense of the teachers' knowledge is consistent with the view that the teacher is also a learner in the classroom. This view is based on the work done by Cobb, Wood, and Yackel (1990) described above. The first classification presented by Harel and Sowder is *external conviction proof schemes.* This class of schemes "depends (a) on an authority such as a teacher or a book, (b) on strictly the appearance of the argument (for example, proofs in geometry must have a two-column format), or (c) on symbol manipulations, with the symbols of the manipulations having no potential coherent system of referents" (pg. 809). The second classification, *empirical proof schemes*, relies on "either (a) evidence from examples (sometimes just one example) of direct measurement of quantities, substitutions of specific numbers in algebraic expressions, and so forth, or (b) perceptions" (p. 809).

The third classification, *deductive proof schemes*, has two subcategories, *transformational proof schemes* and *axiomatic proof schemes.* Both of these categories share the following three characteristics *generality*, *operational thought*, and *logical inference*. An *axiomatic proof scheme* has more characteristics, however, since these proof schemes will not be used as part of the framework for this study this classification will not be further explored. The three characteristics of *transformational proof schemes* are defined as follows: *Generality* is "an individual understanding that the goal is to justify a 'for all' argument, not isolated cases and no exception is accepted" (p. 809); *operational thought* is "when an individual forms goals and subgoals and attempts to anticipate their outcomes during the evidencing process" (p. 809); and, *logical inference* is "when an individual understands that justifying in mathematics must ultimately be based on logical inference rules" (p. 809 – 810). For a proof to be an expression of a *transformational proof scheme* it must demonstrate all three of these characteristics. During this study however, students and teachers were sometimes found to be engaging with proof schemes that relied on one or two of the above. During these times in the analysis the expressed characteristics are specified and investigated.

A summary of the research and concepts discussed above can be found in Table 2.1 and Table 2.2. In Table 2.1 the concepts used to view *proofs* during this study are listed along with a description of each term, the context in which each concept of proof is most commonly found, and the sources where these concepts were pulled from. These concepts of proof were used during this study to help focus on specific classroom events, interview topics, and as an aid in analyzing both the use of proofs and the teachers' understanding of proofs. This will be discussed more in the following chapter.

In Table 2.2 the terms and concepts used to view *process of proving* during this study are summarized. Because of the interconnections between *proofs* and *the process of proving* there is some overlap in the two tables. Even though the connections between these two concepts are considerable, they are two distinct concepts and each needed to be clarified separately. Since, particularly at the level of middle grades mathematics, it is likely to see the students and teachers engaging in *the process of proving*, without necessarily seeing a *proof*. The concepts of the *process of proving* used in this study are based mainly on the work done by Harel and Sowder. The concepts of *the process of proving* were used during this study to help focus on specific classroom events, interview topics, and as an aid in analyzing data from both classroom observations and interviews. This will be discussed more in the following chapter.

Table 2.1: Proofs

Term	Description	Context	Proof Scheme	Source (s)
Informal Arguments	Based on the everyday use of the word "proof" rather then the mathematical process associated with a proof. "this type of informal argumentation does not necessarily produce truth, since it is based on local value consideration, which lacks the objective features of proof"	Daily Life	N/A	Recion and Godino (2001)
Empirical-inductive proof	Experimental proofs that are the "first validating step, where some particular case of the proposition to be proved are experimentally verified"(p. 93). Proofs that express an *Empirical Proof Scheme* and rely on "either (a) evidence from examples (sometimes just one example) of direct measurement of quantities, substitutions of specific numbers in algebraic expressions, and so forth, or (b) perceptions" (p. 809).	Experimental Sciences and Mathematics	Empirical	Recion and Godino (2001) Harel and Sowder (2007)
External Conviction Proof	Proofs that express an *External Conviction Proof Scheme* and thus depend on one of the following: (a) on an authority such as a teacher or a book (b) on strictly the appearance of the argument (for example, proofs in geometry must have a two-column format), or (c) on symbol manipulations, with the symbols of the manipulations having no potential coherent system of referents (pg. 809).	Mathematics and Mathematics Classroom	External Conviction	Harel and Sowder (2007)

Table 2.1 Continued: Proof

Term	Description	Context	Proof Scheme	Source(s)
Stylianides' Definition of Proof	Proof is a mathematical argument, that is, a logically-connected sequence of assertions for or against a mathematical claim, with the following characteristics: (i) it uses statements accepted by the classroom community that are true and available without further justification; (ii) it employs forms of reasoning that are valid and known to, or within the conceptual reach of, the classroom community; and (iii) it is communicated with forms of expression that are appropriate and know to, or within the conceptual reach of, the classroom community (p. 291).	Mathematics Classroom	Transformational	Stylianides (2007) Harel and Sowder (2007)
Transformational Proof	Proofs that express a *Transformational Proof Scheme* and thus demonstrate the following: *Generality* – "an individual understanding that the goal is to justify a 'for all' argument, not isolated cases and no exception is accepted" (p. 809). *Operational Thought* - "when an individual forms goals and subgoals and attempts to anticipate their outcomes during the evidencing process" (p. 809). *Logical Inference* - "when an individual understands that justifying in mathematics must ultimately be based on logical inference rules" (p. 809 – 810).	Mathematics and Mathematics Classroom	Transformational	Harel and Sowder (2007)

Table 2.2: The Process of Proving

Term	Description/Definition	Source
Ascertaining	Defined as "the process an individual (or community) employs to remove her of his (or its) own doubts about the truth of an assertion" (p. 808).	Harel and Sowder (2007)
Persuading	Defined as "the process an individual or a community employs to remove others' doubts about the truth of an assertion" (p. 808).	Harel and Sowder (2007)
Informal Argumentation	Argumentation based on the everyday use of the word *proof* and is defined as the process one engages in when utilizing *informal arguments.*	Recion and Godino (2001)
Empirical-inductive	Defined as *ascertaining* or *persuading* by using evidence from examples. In other words, the process used when working under an *Empirical Proof Scheme*	Recion and Godino (2001) Harel and Sowder (2007)
External Conviction	Defined as *ascertaining* or *persuading* based on the expectations of external sources such as teachers, or textbooks. It is also the act of proving inline with a format such as two –column proofs, with less regard for actual argument then the format itself. In other words, the process used when working under an *External Conviction Proof Scheme.*	Harel and Sowder (2007)
Generality	Defined as *ascertaining* or *persuading* with the purpose of justifying a for-all statement.	Harel and Sowder (2007)
Logical Inference	Defined as *ascertaining* or *persuading* with the purpose of justifying using logical inference rules.	Harel and Sowder (2007)
Transformational	Defined as *ascertaining* or *persuading* when working under a *Transformational Proof Scheme.* Includes: *Generality, Logical Inference, and Operational Thought.*	Harel and Sowder (2007)

The Functions of Proofs and the Process of Proving Framework

In this section the literature related to *the functions of proofs and the process of proving* will be explored followed by a table that will summarize the extensive discussion about these functions. The functions of proofs and the process of proving are essential to both the mathematician and students of mathematics and have been characterized by many researchers in the field of mathematics education. For the purposes of this study the different functions described by some researchers will be compiled in one list and each of these functions with be explored.

Yackel and Hanna (2003) describe the functions of proof in mathematics as including "verification, explanation, systematization, discovery, communication, construction of empirical theory, exploration of definition and of the consequences of assumptions, and incorporation of a well-known fact into a new framework." (p. 228). Stylianides & Silver (2004) describe the ways which proof can be used in the mathematics curriculum as explanation, verification, falsification, and generalization of a new law. de Villiers (1999) explains that there are six interconnected roles of proof: verification, explanation, discovery, systematization, intellectual challenge, and communication. Although others in the field of mathematics education may have included functions of proof that are mentioned above, for this study the three above will be the basis for the framework of *the functions of proof and the process of proving.*

The functions of proofs and the process of proving will therefore be defined as follows:

- Verification
- Explanation
- Communication
- Discovery
- Exploration of Definitions
- Generalization
- Systematization*
- Justification, dismissal or modification of a conjecture and the consequences of assumptions
- Intellectual challenge*

Although some of these functions were not present during this study each of them is imperative for a full description of the functions of proofs and the process of proving. Therefore each of these will be clarified below, with more attention to those which were the most useful in this study. The functions marked with * are those that will not be part of the final framework because they were not useful during this study. One function of proofs and the process of proving not mentioned by the above authors that will be part of this framework is *inquiry*. This function will be described last and the purpose for its inclusion will be made clear.

Verification

Verification is the most common, although incomplete, notion of the functions of proofs and the process of proving (de Villers, 1999). Verification "is the role of proof as a means to demonstrate the truth of an assertion according to a predetermined set of rules of logic and premises" (Harel and Sowder, 2007, p. 819). de Villiers (1999) attributes this function with that of being "concerned with the truth of a statement" (p. 4). Hersh (1993) describes two functions of proofs, *convincing* and *explaining*. His discussion concerning the role of *convincing* is similar to that of de Villiers' and Harel and Sowder's notion of verification. He explains that the role of proofs in mathematics research is that of *convincing,* and that in journals and textbooks, "proof functions as the last judgment" (Hersh, 1993, p. 390). However, as will be discussed below Hersh does not believe that this notion of proof is one that encompasses all that the mathematician does or what the function of proofs is for students.

Relying on the work done by the above authors the function of *verification* will be defined as that which serves as a means to establish truth.

Explanation

The function of proofs as a method of explanation is valuable to the mathematical education of students (Hersh, 1993). For mathematicians, the use of proofs as explanation holds value, however their final or published proofs can sometimes fail to explain. "To ensure correctness of their proofs, [mathematicians] have consciously emphasized the deductive mechanism at the expense of the mathematical ideas" (Hanna, 1989, p. 49). However, when evaluating mathematical proofs, at least to some mathematicians, proofs that explain are more valuable than proofs that simply verify. "More than whether a conjecture is correct, mathematicians want to know why it is correct" (Hersh, 1993, p. 390). "In fact, for many mathematicians the clarification/explanation aspect of a proof is of greater importance than the aspect of verification" (de Villiers, 1999, p. 5). According to Harel and Sowder (2007) "*explanation* is different from verification in that for a mathematician it is usually insufficient to know only that a statement is true. He or she is likely to see insight in why the assertion is true" (p. 819). de Villiers (1999) attributes this function with that of "providing insight into why [a statement or conjecture] is true" (p. 4). Yackel and Hanna (2003) suggest that "the functions of proof that may have the most promise for mathematics education are those of explanation and communication" (p. 228).

Relying on the work done by the above authors the function of *explanation* will be defined as that which provides insight as to why a statement is true, as well as, how and why the verification demonstrates this truth.

Communication

Communication is an important function of proofs in both mathematics and mathematics education. "Communication in scholarly mathematics serves mainly to cope with mathematical complexity, while communication at school serves more to cope with epistemological complexity" (Hanna & Jahnke, 1993). Hersh (1993) explains that mathematicians also use publications in journals as a means of communicating new knowledge, adding knowledge to the discipline of mathematics, and as a way for their work to be critiqued or to critique the work of their colleagues. As stated above, Yackel and Hanna (2003) argue that communication may be one of the most important functions of proof in the mathematics classroom. de Villiers (1999) attributes this function with the process of "transmission of mathematical knowledge" (p. 4) and states that "proof is a unique way of communicating mathematical results between professional mathematicians, between teachers and students, and among students themselves" (p. 7). Harel and Sowder (2007) explain that

> *Communication* refers to the social interaction about the meaning, validity, and importance of the mathematical knowledge offered by the proof produced. Communication can be viewed in the context of the two subprocesses that define proving: *ascertaining* and *persuading* (p. 819).

Relying on the work done by the above authors the function of *communication* will be defined as that which fosters social interaction related to the meaning, validity, and importance of mathematical results. Moreover, it will be viewed as the context of *ascertaining* and *persuading*.

Discovery

Discovery is both a function of proof and part of the process of proving. One of the major steps in a mathematical proof is forming a conjecture. NCTM (2000) explains that "conjecture is a major pathway to discovery" (p. 57). Whether one is working to prove a new idea, or proving an existing idea for oneself, discovering the reasoning behind a statement's validity is a vital function of proofs and the process of proving. Lakatos (1976) stated that during the process of proving, there exists "a simple pattern of mathematical discovery – or of the growth of informal mathematical theories" (p. 127). This pattern of discovery starts with a conjecture, during the process of proving the conjecture one may find counterexamples or pieces of the conjecture that are not true, Lakatos refers to these as refutations. The conjecture must then be reworked and improved to compensate for these falsities and the process starts all over. Harel and Sowder (2007), who give credit to Lakatos for the best illustration of this process, explain that, "*discovery* refers to the situations where through the process of proving, new results may be discovered" (p. 819). de Villiers (1999) attributes this function with "the discovery or invention of new results" (p. 4). He states that

> It is often said that theorems are most often first discovered by means of intuition and/or quasi-empirical methods, before they are verified by the production of proofs... [and proofs are] not merely a means of verifying an already-discovered result, but often also a means of exploring, analyzing, discovering and inventing new results" (p. 5).

In terms of the process of proving found in the mathematics classroom, the "new results" discussed by de Villiers (1999) and Harel and Sowder (2007) need to be considered as new results for the individual, not necessarily the mathematics community at large.

Relying on the work done by the above authors the function of *discovery* will be defined as situations where through the process of proving, results that are new to the individual or community are invented or discovered.

Exploration of definitions

The function of proofs and the process of proving in students' learning of definitions is described by Larson and Zandieh (2005). The authors summarize the findings of Zandieh and Rasmussen (in preparation),

> Defin[ing] include[s] not just formulating a definition but also activities such as negotiating and revising definitions. These activities may involve generating conjectured definitions, creating examples to test the conjectures, and trying to prove whether or not a conjectured definition "works" in the sense of doing the job that the definition is being created to do (Larson and Zandieh, 2005, p. 1).

Larson and Zandieh (2005) found during their own study that, "The role of proof in defining is to 1) tell you what job the definition needs to do, 2) suggest what the definition ought to look like in order for it to do that job, and 3) to let you determine whether it actually does the job it is supposed to do" (p. 7).

The relationship between activities involving the exploration of definitions and the process of proving is made clear by NCTM (2000) in the Reasoning and Proof Process Standard:

> Students should move from considering *individual* mathematical objects – this triangle, this number, this data point – to thinking about *classes* of objects…they should be developing descriptions and mathematical statements about relationships between these classes of objects, and they can begin to understand the role of definition in mathematics (pg. 188).

The above recommendation by NCTM also demonstrates the function of generalization in proofs and the process of proving. This will be discussed further below.

Relying on the work done by the above authors, the function of exploring definitions will be defined as generating conjectured definitions, testing and proving conjectured definitions, and using definitions to make logical inferences in order to prove the correctness of a statement.

Generalization

NCTM's (2000) Reasoning and Proof Standard for grades 6 – 8 refers to the function of generalization in their description about what reasoning and proof should look like in these grades:

> Students should have frequent and diverse experience with mathematics reasoning as they –
>
> - examine patterns and structures to detect regularities;
> - formulate generalizations and conjectures about observed regularities;
> - evaluate conjectures;
> - construct and evaluate mathematical arguments (p. 262).

As early as grades 3 – 5, NCTM (2000) recommends that as part of students' learning of reasoning and proof "teachers should look for opportunities for students to revise, expand, and update generalizations they have" (p. 192).

Ellis (2007) found a deep connection between generalization and justification in middle school students' mathematical work. She states, " The nature of this interaction between generalizing and justifying highlights the developmental importance of student initial, limited general statements and proofs" (p. 223-224). During this study she conducted a teaching experiment with middle school students aimed at understanding the relationship between generalization and justification. She found different mechanisms where students' generalizations and justifications "influenced one another to support development of more sophisticated reasoning" (p. 208). These mechanisms will not be described here. However, the fact that Ellis (2007) centralizes the role of generalization in the process of proving and provides evidence that students' work with justification and generalization are interrelated and inseparable aided in defining this function of proof.

Relying on the work done by the above authors, the function of generalization will be defined as the process of creating generalized statements, rules, classifications, conjectures, or definitions based on patterns or through the work done during the process of justification.

Systematization

Although systematization is recognized here as a function of proofs and the process of proving, it will not appear in the final framework for this study because it was not useful to the data collection or analysis conducted. Harel and Sowder (2007) describe the function of systemization as that which

> Refers to the presentation of verifications in organized forms, where each result is derived sequentially from previously established results, definitions, axioms, and primary terms… a case of axiomatic proof scheme. The difference between systemization and verification is in the extent of formality (p. 819).

de Villiers (1999) further explains the relationship and differences between systematization and verification stating

> Although some elements of verification are obviously present here, the main objective is not "*to check whether certain statements are really true*' but to organize logically unrelated individual statements that are already known to be true into *a coherent unified whole* (p. 7).

The *extent of formality* and connection to *a coherent unified whole*, was beyond the use of proofs and the process of proving studied here. NCTM (2000) describe the lack of formality in the middle grades:

> Although mathematical argument at this level lacks the formalism and rigor often associated with mathematical proof, it shares many of its important features, including formulating a plausible conjecture, testing the conjecture, and displaying the associated reasoning for evaluation by others (p. 264).

While Ellis (2007) explains that:

> Although correct algebraic generalizations and deductive forms of proof remain a critical instructional goal, this study [the one described previously] suggests that students' incorrect, nondeductive generalizations and proofs may serve as an important bridge toward this goal (p. 224).

The analysis and data collection conducted during this study focused on the less formal mathematical arguments described by NCTM (2000) and Ellis (2007).

Justification, modification, or dismissal of a conjecture

These functions were classified together here to reiterate the fact that working with a conjecture is an ongoing process that includes justification, modification, and possibly dismissal. The best explanation of this process can be found in Lakatos (1976), and was summarized above when describing the function of discovery. NCTM (2000) states,

> posing conjectures and trying to justify them is an expected part of students' mathematical activity...Sometimes students' conjectures about mathematical

> properties and relationships will turn out to be wrong. Part of mathematical reasoning is examining and trying to understand why something that looks and seems as if it might be true is not and to begin to use counterexamples in this context (p. 191).

As part of the process of understanding why an assumption or conjecture may not be true, students have the opportunity to learn about *the consequences of assumptions.*

The function of proofs and the process of proving associated with the consequence of assumptions is listed separately from the function of *justification, modification, or dismissal of a conjecture* above. However, for the purposes of this study *the consequences of assumptions* will be framed as part of the function of *justification, modification, or dismissal of a conjecture.* One of the major themes in NCTM's (2000) Reasoning and Proof Standard for grades 6 – 8 is that students should be aware of the limitations of inductive reasoning:

> In order to use inductive reasoning appropriately, students need to know its limitations as well as its possibilities. Because many elementary and middle-grades tasks rely on inductive reasoning, teachers should be aware that students might develop an incorrect expectation that patterns always generalize in ways that would be expected on the basis of the regularities found in the first few terms (p. 265).

During the process of justifying, modifying, or dismissing a conjecture, students have the chance to learn about the limitations and power of inductive reasoning and thus, they can engage in learning about the consequences of assumptions.

Intellectual challenge

de Villiers (1999) explains, "to mathematicians proof is a mathematical challenge that they find as appealing as other people may find puzzles of other creative hobbies or endeavors" (p. 8). And thus, the function of intellectual challenge is that of "*self-*

realization and *fulfillment*" (p. 8). Harel and Sowder (2007) reiterate that this function "refers to the mental state of self-realization and fulfillment" (p. 819) and explain that "this role does not correspond to any of our proof schemes" (p. 819). As with the function of systemization, *intellectual challenge* is recognized here as a function of proofs and the process of proving, but will not appear in the final framework for this study because it was not useful to the data collection or analysis conducted.

Inquiry

I have characterized three major ways I believe one can use inquiry in the mathematics classroom:

1) Inquiring about mathematics through "real world" context;
2) Inquiring about mathematics through science; and
3) Inquiring about mathematics through mathematics.

I define *Inquiring about mathematics through mathematics* as the process of working to inquire about a mathematical concept using previously learned mathematical concepts, or using mathematically sound arguments to explore the truth or falsity of a new concept. This type of inquiry is one of the functions of proofs and the process of proving. NCTM (1991) explains that all classrooms should have students participating in inquiry, and that this inquiry should including proposing hypotheses and supporting and challenging hypotheses. The NCTM's use of hypothesis here instead of conjecture is insignificant, and the connections between inquiry and the process of proving lie in the use of inquiry to create a conjecture, and explore it's certainty by engaging with challenges and endorsements (or support).

The framework used to view *the functions of proof and the process of proving* is summarized in Table 2.3. This framework pulled together the work done by the above researchers and is focused on *the functions of proofs and the process of proving* that are most relevant to the middle grades and this study. The framework was used during this study to help focus data collection and as a tool for analysis. The uses of this framework will be discussed in the following chapter.

Table 2.3: Framework of the Functions of Proof and the Process of Proving

Verification	A means to establish truth.	de Villiers (1999); Harel & Sowder (2007); Hersh (1993); and, Yackel & Hanna (2003)
Explanation	Provides insight as to why a statement is true, as well as, why the verification demonstrates this truth.	de Villiers (1999); Hanna (1989); Harel & Sowder (2007); Hersh (1993); Stylianides & Silver (2004); **and,** Yackel & Hanna (2003)
Communication	The fostering of social interaction related to the meaning, validly, and importance of mathematical results. The context of *ascertaining* and *persuading*.	de Villiers (1999); Hanna & Jahnke (1993); Harel & Sowder (2007); Hersh (1993); **and,** Yackel & Hanna (2003)
Discovery	Situations where through the process of proving, results that are new to the individual or community, are invented or discovered.	de Villiers (1999); Harel & Sowder (2007); Lakatos (1976); NCTM (2000); and, Yackel & Hanna (2003)
Exploration of Definitions	The process of generating conjectured definitions, testing and proving conjectured definitions, and using definitions to make logical inferences in order to prove the correctness of a statement.	Larson & Zandieh (2005); NCTM (2000); and, Yackel & Hanna (2003)
Generalization	The process of creating generalized statements, rules, classifications, conjectures, or definitions based on patterns or through the work done during the process of justification.	Ellis (2007); NCTM (2000); Stylianides & Silver (2004); and, Yackel & Hanna (2003)
Justification, dismissal or modification of a conjecture and the consequences of assumptions	The process of justifying, modifying, or dismissing a conjecture. Learning about the limitations and power of inductive reasoning. And engage with assumptions as well as the consequences of assumptions.	NCTM (2000); Lakatos (1976); Stylianides & Silver (2004); Yackel and Hanna (2003)
Inquiry	Inquiring about mathematics through mathematics.	NCTM (1991)

Making Meaning and Teacher Knowledge

In this section the term *make meaning* will be explored, the aspects of *making meaning* will be connected to the research related to teacher knowledge that influenced this study, and a framework for viewing the process of *making meaning* will be presented. In order to address my frame of *making meaning*, the topical questions guiding this study will be analyzed and the aspects of *making meaning* will be presented. One topical question, *How do teachers think about or understand proofs and the process of proving?*, focuses on the teachers' knowledge and beliefs. *Knowledge* and *beliefs* are two of the five aspects I have defined to be part of the process of *making meaning.*

There are two types of knowledge I was most concerned with for this study. One is the teachers' pedagogical content knowledge, specifically related to proofs and the process of proving; and the second is the teachers' knowledge of proofs and the process of proving as it relates to mathematics and their own mathematics education. To address the aspect of *knowledge*, I will first describe teachers' pedagogical content knowledge. I will then review the literature related to teachers' knowledge of proofs and the process of proving.

Over the past twenty years, those interested in mathematics education have had an increasing interest in the knowledge required for a teacher to be successful in the mathematics classroom. In 1986, Shulman identified and described a new perspective for viewing teacher knowledge. He separated teacher knowledge into content knowledge and general pedagogical knowledge, and separated teacher content knowledge into three categories: subject-matter content knowledge; pedagogical content knowledge; and curricular knowledge. Pedagogical content knowledge can be characterized as the kind of knowledge:

> which goes beyond knowledge of a subject matter per se to the dimension of subject matter knowledge for teaching. I still speak of content knowledge here, but of the particular form of content knowledge that embodies the aspects of content most germane to its teachability...the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the way of representing and formulating the subject that make it comprehensible to others...[it] also includes an understanding of what makes the learning of specific topics easy or difficult...teachers need knowledge of strategies most likely to be fruitful in reorganizing the understanding of learners (Shulman, 1986, p. 9-10).

This knowledge of content for teaching builds on the teacher's subject-matter knowledge. Although mathematicians need to be able to communicate their research, this is different from being able to teach mathematics. Pedagogical content knowledge "is not something a mathematician would have by virtue of having studied advanced mathematics" (Ball et al., 2001).

In Ball et al. (2001) a short history of research on mathematics teaching tells us that before Shulman's notion of pedagogical content knowledge was introduced, most of the research and assessment of mathematics teachers focused on either their subject-matter content knowledge or their general pedagogical knowledge. In other words, either teachers' own mathematical skills or their own mathematical education was taken into account, or their knowledge about general classroom practices were analyzed. This research is important and has helped make some progress in understanding what is needed for a teacher to be successful. However, as Ball et al. (2001) point out, it is of much more importance that we investigate what kind of knowledge of mathematics is needed for teaching:

> Ultimately, teachers must be able to know and use mathematics in practice, not merely do well in courses or answer pedagogically contextualized questions in interviews. This conclusion suggests the need to redefine the problem from one about *teachers* and *what teachers know* to one about *teaching and what it takes to teach* (p.451-452).

I will address the notion of using mathematics in practice more while exploring the next topical question. Focusing on teachers' pedagogical content knowledge helped this study focus on the teachers' process of making meaning as it occurred during their classroom practices. Since pedagogical content knowledge builds on subject-matter knowledge, this frame also focused this study on the interplay between teachers' knowledge and past experiences with mathematics and their classroom practices.

Although there is a growing amount of research related to teachers' knowledge of proofs and research related to teachers' pedagogical content knowledge, there has been little research that has explored specifically teachers' pedagogical content knowledge of proofs and the process of proving.

In Cyr (2004), 309 preservice teachers were given a questionnaire designed to identify their conceptions about proof. It was found that the participants thought of proofs as formal and rigorous. They viewed proof as "a mandatory ritual and the mathematician's main duty" (p. 571) and "showed very little interest in recognizing the role proof plays when convincing students of the exactitude of theorems taught in class" (p. 572). Borko, Peressini, Romagnano, et al (2000) analyzed one teacher's understanding of mathematical proof and found that although she thought of proofs as formal symbolic manipulation, she had a different understanding of what constituted a proof in her classroom and how she used the word proof with her students. She believed that the audience of a proof is just as important as the writer of the proof. For this teacher, an argument is considered to be a proof depending on the argument's ability to convince the audience, and its ability to convince determines its usefulness in the classroom. These studies highlight three issues related to answering my topical question related to teachers' knowledge and beliefs. First, they show a need to learn about the teachers' knowledge and understanding of the role proofs and the process of proving play in mathematics and their own education. Second, it shows a need to learn about the teachers' knowledge and understanding of the role proofs and the process of proving play in their students' learning of mathematics. Third, it shows a need to focus on the teachers' understanding of audience as it relates to proofs.

Research has shown us that both preservice teachers and in-service teachers possess a minimal understanding about proofs and the process of proving. Martin and Harel (1989) conducted a study on preservice teachers and their understanding about mathematical proof. In this study 101 preservice elementary teachers were evaluated on their ability to identify the correctness of both inductive and deductive statement verifications. Preservice teachers were, at best, weak in judging whether or not a proof was correct. The authors also found that it was common for their participants to not accept deductively proven arguments as facts, and insisted on *further proof* through the use of example. Stylianides, Stylianides, and Phillippou (2007) conducted a study analyzing written responses from 95 preservice teachers and interviews with 11 of these participants. The participants in this study were senior university students in both elementary education and secondary education. The elementary program at the university where the participants studied included four mathematics courses where there was "a fair amount of knowledge about different methods of proofs, including proof by mathematical induction" (p. 150). The secondary majors were in the mathematics program at this university and "proofs had a predominate place in the program of study" (p. 150). The purpose of the study was to learn about their knowledge of proof by induction. As with most of the research related to teachers' understanding of proofs, these authors found that their participants demonstrated major difficulties. The difficulties found were categorized into: "the essence of the base step...the meaning associated with the inductive step... and the inferences that can be drawn from this proof, ...[and] the possibility of the truth set of a sentence to include values outside its domain of discourse" (p. 162). The difficulties seen in this analysis are concerning for the future of mathematics education. As Stylianides, Stylianides, and Phillippou point out:

> If preservice teachers' difficulties remain tacit and pass unchallenged through mathematics teacher education, they are likely to become sources of misconceptions or reasons underlying fragile instruction of proof in school mathematics (p. 163).

Although this dissertation will not evaluate teachers' ability to write or assess proofs, the studies conducted by Martin and Harel (1989) and Stylianides, Stylianides, and Phillippou (2007) highlight the potential of using interviews to discuss instances where the teachers made judgments about students' proofs or demonstrated fluencies and difficulties with using proofs during classroom events.

Relying on the work done by the above authors, *knowledge*, as an aspect of *making meaning* for this study, will concentrate mainly on the teachers' pedagogical content knowledge and will be focused on (a) the teachers' knowledge and understanding of the role of proofs and the process of proving in mathematics and their own education, (b) knowledge and understanding of the role of proofs and the process of proving in their students' learning of mathematics, (c) understanding of audience as it relates to proofs, and (d) their responses to judgments about students' proofs or demonstrated fluencies and difficulties with using proofs during classroom events.

To address *beliefs* as an aspect of *making meaning* for this study, I will first rely on work done by Thompson (1984) to show that teachers' beliefs influence their teaching practices. I will then refer back to the some of the studies above in order to characterize and substantiate the place of *beliefs* as an aspect of *making meaning*. Thompson (1984) conducted a case study with three junior high school teachers. As part of her analysis she looked at the connection between teachers' beliefs and their classroom practices. She concluded that:

> The observed consistency between the teachers' professed conceptions of mathematics and the manner in which they typically presented the content strongly suggests that the teachers' views, beliefs, and preferences about mathematics do influence their instructional practices (p. 125).

Thompson's (1984) study illustrates the importance of beliefs in the process of *making meaning*. In Cyr (2004) and Borko et al (2000) described above, we see that the teachers' beliefs about proofs and the process of proving were discussed as part of their understanding. Considering the link provided by Thompson (1984) we can also see how beliefs can influences teachers' decision making processes in the studies conducted by Martin, McCrone, Brower, and Dindyal (2005) and Stylianides (2007) discussed below.

It is difficult to differentiate between knowledge and beliefs. For the purposes of this study the aspect of *beliefs* will be characterized as the values teachers placed on proofs and the process of proving in their education, research in mathematics, and the education of their students.

The second topical question I will explore focuses on the teachers' utilization of their knowledge, *How do teachers make decisions about whether or not to include the* *use of proof (or the process of proving) in their classrooms? And in what ways?* *Utilization of knowledge* is one of the five aspects I have defined to be part of the process of *making meaning.* This aspect is directly linked to pedagogical content knowledge. As described by Ball & Bass (2000) pedagogical content knowledge is that which "highlights the close interweaving of subject matter and pedagogy in teaching." In order to address the aspect of *utilization of knowledge* I will review two studies which focus on the use of proofs and the process of proving. The first is set at the high school level, while the second is focused on the elementary grades.

Martin, McCrone, Brower, and Dindyal (2005) examined the interplay of teacher and student interaction during the process of learning to prove in a high school geometry class. The authors analyzed the classroom dialog and found patterns between the teachers' actions and the students' actions. The most notable teacher actions that promoted students' understanding of proofs and the process of proving were, *revoicing* and *coaching*. Based on the work done by (Brown and Renshaw, 2000) Martin et al. (2005) describe *revoicing* as "an utterance in which one person summarizes and rephrases statements made by others" (p. 105) and *coaching* as actions of coaxing and encouraging, specifically when used by the teacher to engage students in " 'ventur[ing] a guess' or try[ing] to develop reasoned arguments of their own" (p. 106). These actions earned their notability because:

> when the teacher uses these techniques to draw students into the action of class-negotiated conjecture development and proof construction, then the students have an opportunity to learn the rules of the game (of formal proof development) by playing the game, rather than by watching others play (p. 121).

The pedagogical choices made by this teacher to use techniques such as *revoicing* and *coaching* highlight the need for this dissertation to pay specific attention to the pedagogical choices made by teachers when involving students in proof related activities.

Stylianides (2007) analyzed three classroom episodes from the Mathematics Teaching and Learning to Teach Project at The University of Michigan. The author analyzed the process of proving third grade students were engaged in and the teachers' role in facilitating the proof related events that occurred. For the purposes of this review, I will focus on the results related to the teacher's role in the classroom. Stylianides found that through *instruction interventions* the teacher was able to foster a high level of mathematically rigorous argumentation. He defines *instruction interventions* as courses of action taken by a teacher in "trying to help their students improve their mathematical recourses related to the development of a proof" (p. 314). Results from this study show that:

> an active role of teachers in managing their students proving activity involves judgments on whether certain arguments qualify as proofs, decisions on what arguments could count as proofs, and selection from a repertoire of courses of action in designing instructional interventions to advance students' mathematical resources related to proof (p. 318).

This result implies that teachers need a high level of knowledge of proofs and the process of proving. Moreover, it shows that teachers need to have knowledge of their students' abilities to work with different forms of reasoning in order to utilize their knowledge of proofs and the process of proving in meaningful ways. This study, as well as that conducted by Martin, et al. (2005), highlights the concept of *utilization of knowledge* as an aspect of the process of *making meaning* that informed this study. Moreover, these results show the need for this dissertation to pay close attention to the pedagogical choices and instructional interventions as major sources of insight into how teachers *make meaning* of their knowledge of proof and the process of proving during classroom events.

The third topical question, *How do teachers form connections between their understanding of proofs (and the process of proving) and the incorporation of certain teaching methods into their classrooms?,* focuses on the interaction between the teachers' practices and knowledge. Using a social constructivist lens to view the teacher as a learner in the classroom who is negotiating between her past knowledge and experiences, and classroom events or discourse is at the forefront of the notion of *making meaning* (Ernest, 1999). Thus, the fourth aspect of making meaning is *the interconnections of practice and knowledge*. As part of this study it is therefore imperative to learn about the teachers' knowledge and beliefs (as described above) and gain insight into how the teachers negotiate their knowledge and beliefs into their professional practices.

Building on the *interconnection of practice and knowledge* is the fifth aspect of *making meaning, making sense of past knowledge and current experiences. Making sense of past knowledge and current experiences* reflects the teachers' decision making process as one that is influenced by both their past and on-the-spot experiences. As noted by Martin et al (2005) "teachers actions may result from carefully considered pedagogical choices or from spontaneous reactions to classroom events" (p. 98). Focusing on this aspect of *making meaning* helped to structure the data collection and analysis by highlighting the need to learn about the teacher participants during classroom observation, discussions about their past experiences, and discussions related to their understandings of how they made sense of their past experiences and knowledge during classroom events which occurred during observations.

In summary, the aspects of *making meaning* are viewed as: *knowledge*, *beliefs, utilization of knowledge*, *making sense of past knowledge and current experiences*, and *the interconnections of practices and knowledge*. All of these aspects are situated in the context of the teacher's classroom, are viewed best through a social constructivist lens, demonstrate a close relationship to pedagogical content knowledge, and are the major focus of this study. A framework of the aspects of *making meaning* can be found in Table 2.4.

Table 2.4: Framework of Making Meaning

Topical Questions:

(1) *How do teachers make decisions about whether or not to include the use of proof (or the process of proving) in their classrooms? And in what ways?*

(2) *How do teachers think about or understand proofs and the process of proving?*

(3) *How do teachers form connections between their understanding of proofs (and the process of proving) and the incorporation of certain teaching methods into their classrooms?*

Aspect	**Description**	**Topical Question(s) Addressing this Aspect**	**Resources**
Knowledge	Pedagogical Content Knowledge Knowledge of Proofs and the Process of Proving: • In their own education • In mathematics research • In their students education Understanding of audience as it pertains to proofs Judgments about students' proofs or demonstrated fluencies and difficulties with using proofs during classroom events. Understanding of students' abilities to use different forms of reasoning.	(2)	Shulman (1986) Ball et al. (2001) Cyr (2004) Borko et al (2000) Martin & Harel (1989) Stylianides, Stylianides, & Phillippou (2007) Ball & Bass (2000)
Beliefs	Beliefs of Proofs and the Process of Proving: • In their own education • In mathematics research • In their students education Values placed on proofs and the process of proving by teachers in different contexts.	(2)	Thompson (1984) Cyr (2004) Borko et al (2000) Martin et al (2005) Stylianides (2007)
Utilization of Knowledge	Pedagogical Content Knowledge Pedagogical choices and teachers' actions: • Coaching • Revoicing Instructional Interventions	(1)	Ball & Bass (2000) Martin et al (2005) Stylianides (2007)
Interconnections of Practice and Knowledge	How the teachers negotiate their knowledge and beliefs as part of their professional practices.	(3)	Ernest, 1999, and above authors
Making Sense of Past Knowledge and Current Experiences	Reflects the teachers' decision making process as one that is influenced by both their past and on-the-spot experiences.	(1), (2), and (3)	Ernest, 1999 Martin et al (2005), and above authors

CHAPTER 3

METHODOLOGY

The purpose of this study is to illuminate the process teachers engage in when they relate their knowledge of mathematical proofs to their professional responsibilities. This process is a complex and dynamic aspect of teachers' lives. The conceptual and practical concerns inherent in my attempt to understand this process demands a methodology that permits me to explore the interplay of individual knowledge and action with contextual influences. Ethnographic fieldwork provides an appropriate approach for these interests, allowing me to incorporate a social constructivist lens with an analytic framework based on concepts and methods from discourse analysis (Agar, 2006, Emerson, Fretz, & Shaw, 1995; Gee, 2005). Foregrounding this study in meaning and context questions fostered through a combination of ethnography and discourse analysis creates the opportunity to generate descriptive accounts of individual actions and specific classroom practices while building links to the broader social and cultural influences upon those actions.

Ethnographic fieldwork is rooted in the researcher's attempt to translate the lived experiences and actions of a group of individuals to others outside of those experiences. Agar (1996) explains that the ethnographer's job is to work with data based on an individual's or group's daily activities and the defining strategy of ethnography is participant observation since, "the only way to access those activities is to establish relationships with people, participate with them in what they do, and observe what is going on (pg. 18)". Since this study is focused on the daily lives of teachers in their classrooms, the activities they participate in, and how their knowledge and these actions interact with each other, using ethnographic fieldwork as an overarching research method was the most apt choice.

Although participant observation is the defining strategy of ethnography, this study also utilized interviewing as a means to collect data. The interests of this study lie at the heart of teachers' understanding and lived experiences. "At the root of in-depth interviewing is an interest in understanding the lived experiences of other people and the meaning they make of that experience (Seidman, 2006, p. 9)." Thus, this study would not be complete without engaging with participants in in-depth interviews. These interviews were structured to elicit narratives about the teachers' past experiences and their current experiences in the classroom (Davidson, 2003) and learn about the processes that they experience when they are making meaning during classroom events (Riessman, 2008).

Participant Observation and Fieldnotes

Participant observation relies on researchers immersing themselves in the everyday practices of the group of people they are studying (Emerson et al., 1995). During this study I observed teachers in their classrooms. Using the frame of mathematical proofs discussed in Chapter 2, I focused my observations on proof related events such as: students discussing proofs with each other and/or the teacher; activities which ask the students to conjecture about mathematical concepts; assignments where students are asked to justify or prove their ideas; the teacher presenting a proof to students; or any classroom discourse that focuses on the functions of proofs and the process of proving (as described in the previous chapter).

The amount of time spent doing ethnographic fieldwork varies. It is clear to me that no matter how long one spends immersed in the everyday lives of others one will never have complete understanding. "You build enough to get from where you started to where you end up when you can understand and operate in a new world. That's all. That's enough. That's a lot (Agar, 1994, p. 136)." Although it might seem over the top to consider a classroom as a *new world*, this is precisely the type of mindset one must have when entering an ethnographic study. I used the meaning and the context questions that arise from discourse analysis to push for these differences and view the classroom as a *new world*. These questions will be explained later in the analytical framework section of this chapter.

Based on a pilot study I conducted approximately 6 months prior to the current study, I was able to make informed decisions about the participant observations. First, I decided to spend at least three weeks with participants before I engaged with them in formal interviews. Second, the time I spent in the classroom should be structured so that I began observations at the start of a new unit or development of a concept. Although the kind of questions I am asking in this study do not rely on what mathematical concepts are being taught, it was important for me to see the progression that the teacher works through while she is engaging her students with a mathematical concept.

Fieldnotes are inscriptions that are "products of and reflect conventions for transforming witnessed events, persons, and places into words on paper (Emerson et al., 1995, p. 9)". During the process of transcribing observed events the researcher is already making sense and interpreting what is happening. There are several methods suggested by Emerson et al. (1995) to ensure quality of fieldnotes. During this study I followed their suggestions as closely as possible. First, fieldnotes need to be written as close in time to the observation as possible. As I was in the classroom everyday it became a nightly routine for me to convert the notes I took that day into fieldnotes. Emerson et al. also urge researchers to not talk with anyone in between their observations and when they write up their fieldnotes. Since I was teaching a course two nights a week and needed to hold office hours after my observations on another night, I was not always able to accomplish this. They also suggest that researchers leave the field after three or four hours. Since I was observing up to two teachers a day, I was in the field from the beginning of the school day, 8:30, to the end of the school day, 3:30. However, I did make sure to step away from the classrooms at least once every 3 – 4 hours to either have lunch, or to sit alone and reflect on what I was observing.

When writing fieldnotes the researcher is constantly making decisions about what and how to write. "While some of these decisions are straightforward, others are more implicit, arising from the particular stance (Emerson et al., 1995, p. 42)." One of the major impacts on how I wrote my fieldnotes came from the audience I had in mind, other researchers who did not necessarily have a strong vision of mathematics teaching. Emerson et al. recommend envisioning and writing for this type of audience, since "notes will be richer" and "will provide more background, context, and detail" (p. 45). Other suggestions from Emerson et al. that I followed when writing up my fieldnotes included: writing in the first person, focusing on my writing as a way to learn about and start to understand my observations and experiences, and writing my fieldnotes as a record of what I had observed.

Participant observation also includes: document analysis, reflection, analysis, and interpretation (Schwandt, 2001). During this study I collected copies of lesson plans, teachers' feedback on students' written work, and other classroom materials such as quizzes, tests, and worksheets. These documents were used during interviews as ways to build descriptions about teachers' knowledge, and beliefs; and as a way of demonstrating the presence of proof related activities.

Interviews

Interviews are a formal way for the researcher and participant to engage in conversation. It is important to think of the data from interviews as coming from "two *active* participants who jointly construct narrative and meanings (Riessman, 2008, p. 23)." Interviews are not merely a way for the participant to tell things exactly how they were/are. Instead they are a way for researchers to work with participants to generate detailed descriptions in a socially situated context. Through narrative interviews I gathered details about the teacher participants' experiences with mathematics, teaching, and other factors that may have influenced the process they experience when making meaning of proofs in their classrooms. We discussed classroom events, their process for writing lesson plans, students' work, their teaching philosophy, the schools' structure and philosophy, and other topics that were relevant to the individual teachers.

With all types of qualitative inquiry the questions for which we are seeking answers are those of *how* and *what*, not *why*. Coming from a mathematics background where I have trained myself to push the question of *why* persistently from both my students and myself, I needed to be very careful to now train myself to ask questions of *how* and *what*. Questions of *why* are asking for a justification. And although *why* questions lead to good things in mathematics where justification is constructive and encouraged, asking someone to justify an action during a qualitative interview is just the opposite. During my pilot study I learned how quickly this could put a participant on the defensive. Davidson (2003) suggests that to elicit narratives, we ask open-ended questions and questions that begin with "How" and "In what ways". Second, I learned to use the strategy of repeating the last phrase stated by the participant to push their explanations further and learn more about a particular topic or event. Finally, I found that although interviews are a socially situated conversation between two individuals, I needed to talk less and listen more.

To learn about the teachers' decision making process and their understanding of certain events and actions I used my experience from participant observations, referencing sections of fieldnotes and collected documents to discuss events or actions taken by the teacher. Working ethnographically with participants offers an effective context for which to optimize data collection and analysis from interviews (Riessman, 2008). Although three weeks is not a long time, it offered me a chance to build an understanding of the teachers' classroom practices. It also allowed time for the participant and myself to build a rapport.

Finding and Selecting Participants

Middle school teachers became the focus of this study for a number of reasons. First, because of their differing certifications this group of teachers shows a great diversity in their mathematical background. Teachers for the grade levels this study is focused on, grades 5 – 8, can have very different levels of mathematical background. Specifically in New Hampshire, elementary teachers who have earned their degrees through approved college programs may have only taken one course in mathematical content for teaching elementary students. In contrast, those who earn their degrees in secondary mathematics education through approved college programs have very similar mathematical coursework as students graduating with a bachelor's degree in mathematics. Degrees in the middle level mathematics education are not as heavily loaded with mathematics as secondary mathematics education degrees, however the mathematical coursework for this certification level is typically much more extensive than that required of elementary teachers. The range of mathematical backgrounds of certified teachers in grade levels 5-8 makes middle school teachers an interesting group to study.

Second, because of the mathematical complexity of the middle school curriculum these grade levels are of particular interest. The recommendations from the National Council of Teachers of Mathematics (1991, 2006), the Conference Board of the Mathematical Sciences (2001), and The National Mathematics Advisory Panel (2008) illustrate that because of the mathematical complexity found in the middle grades, teachers need to have a strong mathematical background, and in particular an understanding of the functions of proofs and the process of proving in both mathematics and in the mathematics classroom.

Third, before starting this dissertation I had worked with a number of middle school teachers during workshops that were designed to help increase their content knowledge of the middle school mathematics curriculum. During this time I gained an appreciation for both the teachers and for the sophistication of the mathematics their students were learning. My interests in researching and working with teachers at this grade level were heightened and thus became the focus of this dissertation.

When I began this study my goal was to have four participants. I was looking for middle school mathematics teachers, preferably with varying backgrounds in education and teaching experience. Because of time constraints I was looking for two sites, with two teachers at each site. This study is not part of any grants and I had no funding to offer teachers as compensation for their time.

After a search for willing participants and supporting schools, I heard back from both the 7th and 8th grade teachers at one school, which will be referred to as Light School. Both were interested and after meeting with them to discuss the study both agreed to participate. I began observations on the second day of classes. Although I had planned for two more participants from a different school, as the study progressed I realized that I would learn more if I stayed at Light School and expanded my study to include teachers from Grade 5, Grade 6, the principal, the curriculum coordinator, and others in this school who are involved with the mathematical education of these students. This was approved by my dissertation committee and welcomed by the Light School community.

Setting

Light School is a public school in New Hampshire serving about 500 students in preschool up to grade 8. The median household income in the town is $66,696 compared to the state's $48,957. Statistical information about the students' family background and average income is not available. The schools' population is 97% Caucasian, 4 points higher than the state's average. The average annual cost per pupil is $13,625.03 compared to the state average of $12,820.26. The school received a Blue Ribbon Award from the No Child Left Behind Blue Ribbon Schools, and has earned "Acceptable Status" in the content areas of both mathematics and reading. This is the highest accountability status offered in New Hampshire. Every core course at this school is taught by a teacher who has earned "highly-qualified" status and only one teacher at the school was hired under emergency or provisional credentials. The school-wide teacher to student ratio is 1:12, however in most of the classrooms observed in this study this ratio was closer to 1:9. The particular ratios for each of the classrooms will be provided in the next section.

Light School has some fundamental beliefs that they use as building blocks in how they conduct the business of education. Below are the school's Mission Statement, Guiding Beliefs and Vision Statement (website, 2008):

> **Our Mission**
>
> The purpose of the Light School is to nurture the intellectual, social, emotional and physical development of all students in a child-centered environment.
>
> **Guiding Beliefs**
>
> We believe that all students have diverse natures, interests, abilities, and capabilities that should be developed to their full potential.

We accept the challenge to address each student as an individual. We are dedicated to providing instructional opportunities which are developmentally and cognitively appropriate to ensure that all students are challenged and inspired. We strive to engage students in meaningful activities that connect learning to life experience. We value creativity and the arts. We understand the importance of nurturing a wide variety of student strengths and teaching styles. We are committed to maintaining high academic and behavioral expectations.

We believe that we learn and work best in an environment which is safe, supportive, and based upon trust and respect.

We acknowledge the necessity of physical and emotional well-being as a foundation for all learning. We value diversity and appreciate differences. We encourage collaboration, cooperation, and community service. We are committed to providing students with the necessary skills to become self-directed learners who value themselves and others.

We believe in the exponential power of teamwork and value the entire community's role in educating our students.

Our school is made up of a community of learners that is focused on the students, but includes staff, parents, and residents alike. We value communication and a partnership that comes together to listen, share knowledge, and plan in order to achieve our common goals. We continually reflect on our practice, and thoughtfully consider changes based on what has worked in the past and what current research clearly supports.

Vision Statement

Light School is a community of learners that respects the individual interests, abilities and learning styles of its members.

The curriculum ensures the continual development of every aspect of the child. Learners are challenged to explore, question, problem solve and apply knowledge to life situations. Technology, as well as all other available resources, is utilized to connect with the local and global community.

The school, as the center of the community, provides learning opportunities for all residents. The interaction between children and adults creates respect for each other and a shared responsibility for continued learning.

These statements and beliefs reflect the passion of the principal as well as the teachers and curriculum coordinator who participated in this study. I have included them here to give a sense for the school and because they were referenced during interviews with the teachers and the principal, which will be discussed in Chapter 4.

Light School has a strong dedication to the mathematical education of their students. One of the ways this is reflected is in its current initiative for every student to graduate the eighth grade having completed the requirements for ninth grade algebra. Other ways they have shown this dedication is in their hiring of a mathematics coordinator, having two certified mathematics teachers in the eighth grade classroom, and through the creation of their own mathematics curriculum. The Light School uses no published curriculum for any content areas, including mathematics. This decision was made based on the belief that every student learns in individual and different ways. This belief is evident in both the Guiding Beliefs and Vision Statement provided above. After working with a member of the staff at Children's Hospital in Boston who specialized in children's learning strategies, the faculty and administrators concluded that none of the published mathematics curriculum programs would allow them to facilitate the individual learning needs of their students. After this realization, the school hired their current curriculum coordinator as the mathematics curriculum coordinator in order to supervise and manage the creation of their own curriculum. The creation of their own curriculum will be discussed in Chapter 4 as one of the major ways teachers were able to *make meaning* of proofs and the process of proving.

The mathematics teachers and the curriculum coordinator created their curriculum and continually revise the program. According to the teachers and the Curriculum Coordinator, the curriculum is based on the NCTM *Principles and Standards* (2000) and the New Hampshire Grade Level Expectations. For the past four years, the curriculum has been modified to incorporate more algebra in every grade, particularly in grades 5 – 8, in order for the students to complete algebra in their eighth grade year. This was the first year that the students in eighth grade had been part of the modified curriculum. The principal is hoping that 80% of these students will be able to pass the algebra placement test given by the local high school. Before this year algebra was taught to a select number of students during a "before school" program.

The teachers work to create and revise their curriculum throughout the year and are in continuous communication with each other and the curriculum coordinator to ensure that the material covered is being taught in ways that are in line with the individual needs of each student. At the heart of this curriculum is the concept of teaching for understanding. The teachers' and curriculum coordinator's experiences with both creating and using this program will be discussed in the following chapter.

Participants

In this section short biographies of each participant will be provided as well as further information about each classroom. There were six teacher participants and two administrator participants in this study. Although others were interviewed, the data from those interviews did not become part of the analysis and so they will not be included in the set of participants discussed here.

Fifth Grade

The fifth grade classroom I observed during this study had 19 students. Mrs. White and Ms. Sage were the two teachers for this class. Mrs. White was working part time as the lead teacher of mathematics and science. Ms. Sage was working full time as the lead teacher for all other content areas and as the Educational Associate during mathematics and science, meaning that, during mathematics classes, Ms. Sage works mainly with students who are in need of accommodations because of medical reasons including ADHD and diabetes. This situation created a teacher to student ratio of about 1:9 during mathematics classes. The focus of my data collection in the fifth grade was centered on Mrs. White, and thus, Ms. Sage will be discussed minimally throughout the rest of this dissertation. At least one hour was dedicated to mathematics everyday. On Mondays and Wednesdays the time allotted to mathematics was an hour and forty-five minutes; on Tuesdays the time allotted was an hour; and on Thursdays and Fridays the time allotted was two hours. The amount of time these teachers dedicated to mathematics reflects their belief that learning mathematics was valuable for their students and required time to learn conceptually and not just procedurally.

Mrs. White earned her bachelor's degree in Landscape Architecture. She spent three years working in landscape design before coming to teaching through an alternative certification program. Her focus was on literacy and she spent three years teaching in the United States before deciding to work as a math and science teacher in Italy for a year. During this time she explained that she had "no textbooks, no support, no nothing...I did a lot of project-based learning, and figured out how to make it apply to the kids' lives. I just used it the way that I use math." After teaching in Italy she returned to the US and taught mathematics in a sixth grade class for two years at another school before coming to Light School. She has been working as a fifth grade teacher at Light School for the past nine years. During her second year at Light School she participated in a college course where she restructured her mathematics curriculum as her final project. During this restructuring she focused on making it "more applicable", "more pertinent" and "more exciting". This was Mrs. White's fourth year working part time. The first three years she was the lead teacher during literacy and this was her first year back teaching mathematics.

During interviews Mrs. White emphasized the importance of relating mathematics to the lived experiences of her students. She believes that doing mathematics is, "making sense of numbers in the world, making sense of how numbers fit together, how people might use numbers...[and] how you can make life easier if you do understand numbers." She explained that her teaching style is based on using open-ended questions and lots of projects, and in doing this one of her major goals is to keep her students engaged.

Sixth Grade

All the sixth grade teachers at this school teach a specialty content area and social studies. Ms. Blue was the sixth grade math teacher and taught three sections of mathematics as well as a section of social studies. There were 18-20 students in each section and there was always a teacher's aide or special educator assisting Ms. Blue. This made the teacher-to-student ratio at most 1:20 and the adult-to-student ratio at most 1:10. The length of each math class varied from 45 minutes to an hour. On Mondays and Thursdays classes were 45 minutes each. On Tuesdays, Wednesdays and Fridays classes were an hour.

Ms. Blue holds bachelor's degrees in both Elementary and Middle School Mathematics Education. She continued her schooling and earned her Masters' degree in

Elementary Education. As part of her bachelor's degree in Middle School Mathematics Education she took a wide variety of mathematics courses including Mathematical Proof, Linear Algebra, and Geometry. She also took mathematics methods courses for elementary school and middle school. As part of her master's degree she interned for a year at Light School in the fifth grade and worked with Mrs. Yellow, one of the eighth grade teachers in this study, teaching algebra before school. During her first year as a teacher, Ms. Blue and Mrs. Green, one of the eighth grade teachers in this study, co-taught mathematics in the sixth grade at Light School. Before then the sixth grade teachers each taught mathematics. Ms. Blue and Mrs. Green revised the sixth grade mathematics curriculum during this year. Ms. Blue explained that they "completely reworked the curriculum" and "kind of started from scratch." She went on to explain that she and Mrs. Green co-taught that year because of the work involved in revising the curriculum. After co-teaching with Mrs. Green for a year, Ms. Blue has been teaching sixth grade mathematics as the only classroom teacher for two years. She explains that the most important aspect of mathematics that she wants her students to understand at the end of the day is "to be flexible with it, and be able to enjoy it." She expects her students to "understand that there are a bunch of different ways to get the answer" and "understand how to do it, as opposed to just knowing the exact rules."

Seventh Grade

All the seventh grade teachers at this school teach a specialty content area and social studies. Mrs. Red was the seventh grade mathematics teacher and taught three sections of mathematics and one section of social studies. There were 19 students in each section. Mr. Orange was one of the seventh grade special education teachers. Mrs. Red explained that sometimes it felt as if she co-taught with Mr. Orange. His desk was located in Mrs. Red's classroom and he was part of almost every mathematics class she taught. Mrs. Ginger, another one of the seventh grade special education teachers, also had her desk in this classroom and was often part of the mathematics classes as well. Either Mr. Orange or Ms. Ginger was always there to aid in the mathematical instruction. This made the teacher to student ratio about 1:9, and more often then not, about 1:6. The length of classes varied from 45 minutes to an hour. On Mondays Tuesdays, and Thursdays classes were each an hour long. On Wednesdays and Fridays classes were each 45 minutes.

Mrs. Red earned her bachelor's and master's degrees in secondary mathematics education. She started college as a mathematics major and switched to mathematics education. As part of her master's degree she interned at Light School in Mrs. Yellow's class. The following year she started working as the seventh grade teacher at Light School and was in her fourth year teaching at the time of this study. Mrs. Red described her teaching as "looking for what works for the kids, hoping there's a choice. I like more discovery. I like kids to get into it." She explained that she structures some classes to have stations, so that students have choice about what to work on and that these stations have discovery-based projects and open-ended questions where students work as individuals or in groups. She expects her students to be able to solve problems and then argue their answers mathematically (the idea of arguing mathematically will be discussed in the next chapter).

Eighth Grade

The eighth grade teachers at this school teach a specialty content area and have other responsibilities, such as advisory groups. Mrs. Yellow is the lead mathematics teacher in the eighth grade. Mrs. Green is a part time teacher who co-taught mathematics with Mrs. Yellow. Mrs. Yellow and Mrs. Green share the responsibilities of teaching the mathematics courses. They teach three sections with 17 students in each section. With two certified mathematics teachers, this makes the teacher to student ratio about 1:8. Mrs. Yellow also had a student teacher for the first semester. This made the adult to student ratio less than 1:6. This was the sixth time Mrs. Yellow had a student teacher in her class. She had supervised five year-long interns before this year. The length of classes varied from 50 minutes to an hour and a half. On Mondays classes were an hour long. On Tuesdays they were 50 minutes. On Wednesday and Thursday they were each an hour and a half. On Fridays they were 55 minutes.

As mentioned above there is a school-wide goal for every student to graduate from eighth grade having finished ninth grade algebra. The students at Light School have spent time building up their algebra skills since kindergarten, and especially since fifth grade. However, the majority of the time spent in this eighth grade mathematics class is focused on algebra.

Mrs. Yellow earned her bachelor's and master's degrees in secondary mathematics education. After she finished her bachelor's degree and before starting her internship, she participated in an experiential summer education program. The program ran for seven weeks during which participants designed, planed, and implemented an experimental-based course for local K-12 students. Mrs. Yellow created and taught her course for students who had failed government or economics the year before. She is now an adviser for this program. Mrs. Yellow interned in the seventh grade at Light School and became the eighth grade mathematics teacher the following year. At the time of this study she was in her tenth year of teaching.

Mrs. Yellow believed that her students should understand "the big ideas", "the larger concepts." She felt it was her job as the teacher to "help make the connection from what they know to what they don't know." Mrs. Yellow believed that her students should engage in open-end problems that extended their thinking about mathematics.

Mrs. Green graduated with a bachelor's degree in economics. After being in the corporate world for two years she decided she wanted to do something "more meaningful". She thought about becoming a teacher and participated in an experiential summer education program. This is the same program that Mrs. Yellow attended and where she serves as an advisor. After that experience she felt a strong pull towards becoming an educator. She completed her master's degree in Middle School Mathematics Education. As part of her course work she needed to take more mathematics courses. She satisfied these requirements by completing courses in a Master of Science for Teachers Program. The courses were content based and taught as master's level mathematics courses. She did her internship at Light School and her supervising teacher was Mrs. Yellow. She spent her first year teaching at Light School as a co-teacher with Mrs. Blue in the sixth grade for one year, then took a year off, and is now working with Mrs. Yellow in the eighth grade.

Mrs. Green believes in giving her students a chance to "come up with conjectures… let them explore it a little bit before we get into the 'this is how it works'."

She expects her students to learn life skills such as "problem solving" and "learning to ask questions" that they will be able to apply to anything in their lives, not just mathematics.

Administrators

Mrs. Pink is the school's Curriculum Coordinator. She holds a bachelor's degree in both English and Education. She also holds a master's degree in Curriculum and Instruction. She is also a certificated elementary teacher with a specialty in mathematics. Before coming to Light School Mrs. Pink had a number of different teaching experiences. She taught elementary school for ten years in places like New Hampshire, Vermont, and Cairo, Egypt. She taught middle school mathematics in Vermont for ten years and then became a Faculty in Residence at a state university for three years. She started at Light School as the Mathematics Coordinator and became the Curriculum Coordinator after her second year. At the time of this study Mrs. Pink was in her sixth year at Light School.

Mrs. Pink explained the school wide approach to teaching mathematics as:

> teaching for understanding. It isn't rote teaching...It is about thinking about what kids need to know, looking at what they do know, and then building on what they know to get to where you want to go while making sure that the understanding is not surface level...you have to be seriously critical of your own teaching, in terms of looking at, are the kids really understanding.

She is concerned that teachers in the elementary grades have "not had the opportunity to spend enough time thinking about their understanding of mathematics and in some cases, they carry misunderstandings." She believes her job is to start discussions with the teachers about curriculum choices and to be available to give them advice. Every teacher in this study sang Mrs. Pink's praises. The principal even said, "every school should have a Mrs. Pink, she is perfect for us."

Mr. Purple is the Principal at Light School. I asked Mr. Purple to participate in this study after working with the eighth and seventh grade teachers and learning that his ideas about mathematics education and education in general was seen by them as a valuable resource. Mr. Purple earned his bachelor's degree in Elementary Education and his master's degree in Education Administration. He taught for 6 years as an elementary teacher and spent 18 years as Principal in another K-8 school in New Hampshire before coming to Light School. He has also been an adjunct faculty member at state university for 15 years. At the time of this study Mr. Purple was serving his tenth year as Principal of Light School.

Mr. Purple explained that:

> The big change has been, ten years ago, we came up with guiding beliefs (see *Setting* section above) as a school, and I think if you look at those, and then look at what's happening in the school, whether you're a three-year-old or you're a fourteen-year-old, you can see those same things happening for all kids.

With the mathematics program, Mr. Purple felt that when he started at Light School the teaching was very traditional and students were learning rote mathematics. During his time as an Elementary Teacher and Principal before coming to Light School, Mr. Purple became interested in how students learned mathematics. He explained that he along with the mathematics teachers spent time learning about methods for teaching mathematics that included looking at the instruction used in other countries. From this experience he came to Light School with a "sense of, how do you go from a traditional math program to one that is based upon more current research, in terms of how kids develop as mathematicians". His concern for the mathematics program led to the decision to have the teachers create their own curriculum, hire Mrs. Pink, and start the algebra initiative discussed above.

The short biographies above include summaries of participants' beliefs and thoughts about mathematics, learning, and teaching. These were mentioned to give the reader an overall idea of the teachers and administrators involved with this study. Each of the participants' beliefs and thoughts will be explored in more depth in the following chapter. In the next section, the framework used in the analysis of classroom events and interviews will be explained.

Analytical Framework

The analytic framework for this study is based on the concepts and methods of discourse analysis as described by Gee (2005). This framework views language-in-use as situated in specific contexts and involves not just the words that we use but also the social practices that are acted out in particular situations. Discourses, denoted with a capital D, combine "languages, actions, interactions, ways of thinking, believing, valuing, [and] using various symbols, tools and objects" (Gee, 2005, p. 21). Using a social constructivist lens to view the teacher as a learner in the classroom, who brings in past experiences, ways of thinking, beliefs, values and makes meaning within the situations and events that take place in the classroom, stresses the importance of situated meanings. At the core of discourse analysis is the perspective that "the human mind is a 'pattern-recognizing' device" (p. 53) and that "a situated meaning is an image or pattern that we assemble 'on-the-spot' as we communicate in a given context based on our construal of that context and on our past experiences" (p. 65).

"Essentially, discourse analysis involves asking questions about how language, at a given time and place, is used to construct the aspects of the situation network as realized at that time and place and how aspects of the situation network simultaneously give meaning to that language" (Gee, 2005, p.110). The following questions helped guide the analysis phase of this study. In these questions the word 'situation' is used to describe specific instances that I will be analyzing. These situations may occur in the classroom or in the course of the interview. The situations may be a whole class discussion or event, one conversation between the teachers and a student, an event that involves multiple students (with or without the teacher), or could even span the course of more than one lesson. These questions are largely influenced by, or taken directly from Gee (2005, p. 110-113).

What I asked of my data:

- What are the situated meanings of words such as: 'proofs', 'conjecture', 'justify', 'validate', 'discover', 'define', 'inquire'?
- What situated meanings and values seem to be attached to such things as: students' original ideas, students' justification of answers, time constraints, institutional pressures,?
- What institutions (organized groups who share an affiliation through language-in-use) and/or Discourses (see page 68) related to the use of proofs are being (re-) produced in this situation and how are they being stabilized or transformed in the act?

- What identities (roles, positions) seem to be relevant or under construction in the situation? How do the teacher's beliefs, cultural knowledge, content knowledge, feeling, and values affect this identity?
- What sorts of connections – looking backward and/or forward – are made within large utterances and large stretches of the interaction?
- What sorts of connections are made to previous or future interactions?
- What systems of knowledge (aspect of knowledge about mathematical proofs) and ways of knowing are relevant (or irrelevant) in this situation? How are they made relevant (and irrelevant), and in what ways?

These questions were used during my analysis to help code fieldnotes and interview transcripts; create and investigate episodic threads and thematic connections; and, as a lens for interpreting data and writing the text. This will be discussed more in the data analysis section of this chapter.

Research Procedures

During this study, data collection and analysis were ongoing and interrelated. This section will describe the process used to collect and analyze the data. Some of the descriptions will be participant specific while others will be discussed as study wide.

The teachers participating in this study were told that the purpose was to gain insight in the connections between middle school mathematics teachers' knowledge and their classroom practices. The focus of proofs and the process of proving was not discussed until the final interview and participants were asked to not share the focus of the study with other teachers until data collection was complete.

Data collection and initial analysis began on the second day of school. The eighth and seventh grade classrooms were observed for a total of six weeks. Splitting the day between the two classrooms, at least one, usually two, sections of each grade were observed everyday. For continuity, a particular section from each grade was observed everyday. When schedules allowed, one of the other sections was also observed. Using a laptop, I took notes about the day's activities, including the overall classroom Discourse; individual student-student, student-teacher, and teacher-teacher discourse; and the actions/pedagogical choices made by the teachers. In paying attention to the overall classroom Discourse I focused on details that would allow me to construct a fuller, multilayered understanding of the nature and processes of classroom life beyond simply what was being said or done. For example, I noted when teachers kneeled down to be eye level with students, or when teachers' smiled or showed signs of unhappiness when answering student questions; I noted body language, students' level of participation, noise volume, and group dynamics; I drew pictures and wrote notes about the posters and displays on the classroom walls, along with the messages or values the teachers explained these to portray. These details not only allowed me to analyze a fuller picture of the events, they also helped to clarify the events for teachers during interviews.

Using the concepts of *proofs* and *the process of proving* discussed in Chapter 2, observations and notes were focused on relevant Discourses (see page 68), tasks, activities, and pedagogical choices. Every night the notes taken during observations were written into fieldnotes, and documents collected were categorized (e.g. handouts, student work) and filed. Initial analysis was conducted as often as possible and usually occurred at the end of the week. This initial analysis was used to find events or documentation to discuss with teachers during interviews.

Participant observation during these six weeks included classroom observations, informal conversations, collection of student work including teachers' comments and grading rubrics, and collection of teachers' lesson plans. Data from participant observations was used to inform the first topical question – *How do teachers make decisions about whether or not to include the use of proof in their classrooms? And in what ways?* – by informing such sub-questions as: How does the teacher present mathematical concepts?; How does the teacher answer student questions?; What content does the teacher include in the course?; and, How is the class time structured?.

The first interview with teachers occurred during the fourth week of observations. Interviews were conducted with Mrs. Yellow, Mrs. Green, and Mrs. Red. These interviews were focused on the teachers' past experiences in education, both as students and as teachers; their experiences with mathematics as a discipline; their teacher philosophy; and some discussion about class activities that were observed, students' work that had been collected, and overall class structure and content. Data from the first interviews was used to inform the first topical question – *How do teachers make decisions about whether or not to include the use of proof in their classrooms? And in what way?* – by informing such sub-questions as: What lead her to decide to present mathematical concepts using certain methods?; How did the teacher determine what ways she answers student questions?; What purpose did the teacher allocate to different classroom activities?.

The second interviews occurred at the end of the six weeks of observations. These interviews were focused on the teachers' use of certain pedagogical choices; particular proof related events that had occurred in their classrooms; and students' work related to proofs, as well as the teachers' comments about their work. Mrs. Green participated in two interviews. Because of the content and timing of those interviews her second interview will be categorized as a final interview. Data from the second interviews continued to inform my first topical question.

The final interviews occurred three to five weeks after observations had finished. At least a week before these interviews, teachers were given copies of the NCTM's *Reasoning and Proof Standard* for grades 6 - 8 and *Stylianides' (2007) Definition of Proof* in the context of a mathematics classroom. Each of the teachers read these before the final interviews. The focus of these interviews was on the teachers' reaction to the *Reasoning and Proof Standard*; their reaction to *Stylianides (2007) Definition of Proof*; their beliefs about proofs and the process of proving, particularly in relation to their teaching and student learning. This was set aside for the final interview to help insure that the teachers did not know about the study's focus on proofs and the process of proving. Data from the final interviews was used to inform the second topical question – How do teachers think about or understand proofs and the process of proving? – by informing sub-questions such as: What experiences has the teacher had with mathematical proofs and/or with higher mathematical content?; What experiences has the teacher had related to using proofs and the process of proving in middle school classrooms?; What purpose do teachers allocate to proofs in different contexts (i.e.

college courses, the discipline of mathematics, mathematics education for school age children)?.

Although I have specified which topical question the data from each of the different interviews and from participant observations addresses, there were of course no lines drawn about what questions and information would be used during the entire analysis. The above is an overview of where most of the data from these different sources was focused.

After the first interviews with these teachers the decision was made to stay at Light School and expand my study within the school instead of continuing with teachers in another school. After the six weeks of observations in the seventh and eighth grade classrooms, the school was conducting state testing and three weeks was spent away from the site, analyzing data and preparing for the next set of participants. The procedures followed with the fifth and sixth grade teachers were similar to that explained above. For clarity, the difference will be explained below.

Data collection in the fifth and sixth grade classrooms began during the first week of November and continued for six weeks. Both the fifth and sixth grade classes were beginning new units during this week. Splitting the day between the fifth and sixth grade classes allowed observations to occur during almost all of the time the fifth grade was having "math-time" and at least one section of sixth grade. Unfortunately, the schedules did not allow for the same sixth grade section to be observed everyday. However, a particular section was observed four days a week.

Interviews and participant observation followed the same structure and held the same purposes as those described above. The only difference was in the time of the second and final interviews. Because of the approaching winter break, the second interviews were conducted during the fifth week of observations and the final interviews occurred two weeks after observations were completed. Besides these small differences, the process was the same for the fifth and sixth grade teachers as described above for the seventh and eighth grade teachers.

As a whole, the data collected was used to inform the third topical question – How do teachers form connections between their understanding of proofs and the incorporation of certain teaching methods into their classrooms? – by looking at such sub-questions as: What links/discrepancies are there between teachers' knowledge or beliefs about proof and their classroom practice; and, What makes it easy or hard to consider/implement using proofs in their classrooms?

The next section will more thoroughly explain how data was analyzed throughout this study, and specifically what methods and procedures were used to conduct the final analysis.

Data Analysis

Analysis was conducted in three distinct, yet interrelated phases: coding fieldnotes and interview transcripts using an open coding method, and creating episodic threads, as described by Emerson, Fretz, & Shaw (1995); developing themes at the individual and general levels, and; developing the text. Each of these phases will be described in detail below.

Coding and Episodic Threads

Open coding is characterized by the fact that the researcher does not enter the process of analysis with pre-established codes or categories. "Through coding you *define* what is happening in the data and begin to grapple with what it means" (Charmaz, 2006, p. 46). One technique used when coding data is to be continually asking questions of your fieldnotes and transcripts. Emerson, Fretz, & Shaw (1995) explain that when asking question of data:

> the ethnographer draws on a wide variety of resources, including direct experience of the life and events in the setting; sensitivity toward the concerns and orientations of members; memory of other specific incidents described elsewhere in one's notes; one's own prior experience and insights in other settings; and the concepts and orientation provided by one's own profession of discipline (p. 146).

Included in the recourses used in asking questions of my data were those described by Emerson, Fretz, & Shaw, as well as the theoretical and analytical frameworks of this study. I created short phrases that encompassed the main concepts or one of the main concepts for each of my topical questions; the questions listed as part of my analytical framework; and the theoretical frameworks of proofs, the process of proving, and the functions of proofs and the process of proving. I created codes for these short phrases as a starting place for my coding system. During the coding process I was open to avenues of inquiry outside of those codes and other codes were created when my analysis of the data called for it. To begin, every line of both transcripts and fieldnotes were coded using approximately 45 different codes. These codes can be found in Table 3.1.

Table 3.1
List of Codes

Code	Short Phrase	Description	Source
AQ	Answering Student Questions	Data that would aid in answering: How does the teacher answer student questions?	Topical Question 1
AS	Ascertaining	Data related to teachers and/or students ascertaining as it is described in the Process of Proving Framework.	The Process of Proving Framework- Harel and Sowder (2007)
CA	Classroom Activity	Data that would aid in answering: What purpose did the teacher allocate to different classroom activities?	Topical Question 1
CB	Connections Backward	Data that would aid in answering: What sorts of connections looking backward are made within the large utterances and large stretches of the interaction?	Gee (2005)
CC	Course Content	Data that would aid in answering: What content does the teacher include in the course?	Topical Question 1
CF	Connections Forward	Data that would aid in answering: What sorts of connections looking forward are made within the large utterances and large stretches of the interaction?	Gee (2005)
CM	Communication	Data related to the function of proofs as communication as it is described in the Framework of the Functions of Proof and the Process of Proving.	Framework of the Functions of Proof and the Process of Proving – de Villiers (1999); Hanna & Jahnke (1993); Harel & Sowder (2007); Hersh (1993); and, Yackel & Hanna (2003)
CN	Conjectures	Data related to the function of proofs as Justification, dismissal or modification of a conjecture and the consequences of assumptions as it is described in the Framework of the Functions of Proof and the Process of Proving.	Framework of the Functions of Proof and the Process of Proving – NCTM (2000); Lakatos (1976); Stylianides & Silver (2004); Yackel and Hanna (2003)
CS	Class Structure	Data that would aid in answering: How was the class time structured?	Topical Question 1
D	Discourse	Data that would aid in answering: What Discourses related to the use of proofs are being (re-) produced in this situation and how are they being stabilized or transformed in the act?	Gee (2005)
DC	Differentiating Classroom	Data related to the teacher discussing issues related to differentiating their classroom or observations related to differentiation and proofs.	Data

DS	Discovery	Data related to the function of proofs as discovery as it is described in the Framework of the Functions of Proof and the Process of Proving.	Framework of the Functions of Proof and the Process of Proving – de Villiers (1999); Harel & Sowder (2007); Lakatos (1976); NCTM (2000); and, Yackel & Hanna (2003)
EC	External Conviction Proof	Data related external conviction proofs as they are described in the Proof Framework and engaging in the process of proving using external conviction as it is described in the Process of Proving Framework.	Proof Framework and The Process of Proving Framework – Harel and Sowder (2007)
ED	Exploration of Definitions	Data related to the function of proofs as exploration of definitions as it is described in the Framework of the Functions of Proof and the Process of Proving.	Framework of the Functions of Proof and the Process of Proving – Larson & Zandieh (2005); NCTM (2000); and, Yackel & Hanna (2003)
EI	Empirical- Inductive Proof	Data related empirical-inductive proofs as they are described in the Proof Framework and engaging in the process of proving using empirical-inductive proofs as it is described in the Process of Proving Framework .	Proof Framework and The Process of Proving Framework – Recion and Godino (2001) Harel and Sowder (2007)
EX	Examples	Data related to the teachers or students using examples as a form of argumentation or "proof".	Data and Topical Question I
FI	Future Interactions	Data that would aid in answering: What sorts of connections are made to future interactions?	Gee (2005)
GN	Generality	Data related to teachers and/or students engaging in the process of generality as it is described in the Process of Proving Framework	The Process of Proving Framework- Harel and Sowder (2007)
GZ	Generalization	Data related to the function of proofs as generalization as it is described in the Framework of the Functions of Proof and the Process of Proving.	Framework of the Functions of Proof and the Process of Proving – Ellis (2007); NCTM (2000); Stylianides & Silver (2004); and, Yackel & Hanna (2003)
HY	Hypothesis	Data related to the teachers or students making hypothesis (particularly in the fifth grade when they used "word hypothesis" to work on definitions).	Data
I	Identities	Data that would aid in answering: What Identities (role, positions) seem to be relevant or under construction in the situation? How do teacher's beliefs, cultural knowledge, content knowledge, feelings, and values affect this identity?	Gee (2005)

IA	Informal Arguments	Data related informal arguments as they are described in the Proof Framework and engaging in the process of proving using informal argumentation as it is described in the Process of Proving Framework.	Proof Framework and The Process of Proving Framework – Recion and Godino (2001)
IQ	Inquiry	Data related to the function of proofs as inquiry as it is described in the Framework of the Functions of Proof and the Process of Proving.	Framework of the Functions of Proof and the Process of Proving – NCTM (1991)
LI	Logical Inference	Data related to teachers and/or students engaging in the process of proving using logical inference rules as it is described in the Process of Proving Framework	The Process of Proving Framework- Harel and Sowder (2007)
MA	Main Activity	Data that would aid in answering: What is the main activity going on in the situation? What sub-activities compose this activity? What actions compose these sub-activities? Do these actions, sub-activities, or activities relate to proofs, and how?	Gee (2005)
MS	Multiple Solutions	Data related to the teachers or students discussing the chance of multiple solutions or multiply solutions to a particular problem.	Data
PE	Past Experiences of the Teachers	Data that would aid in answering: What experiences has the teacher had with mathematical proofs, the process of proving, with higher mathematical content?	Topical Question 2
PI	Previous Interactions	Data that would aid in answering: What sorts of connections are made to previous interactions?	Gee (2005)
PP	Purpose of Proof	Data that would aid in answering: What purposes do teachers allocate to proofs in different contexts (i.e. college courses, the discipline of mathematics, mathematics education for school age children)?	Topical Question 2
PS	Persuading	Data related to teachers and/or students persuading as it is described in the Process of Proving Framework	The Process of Proving Framework- Harel and Sowder (2007)
Q	Question that students should ask	Data related to the teachers or student discussing what questions they should ask them selves during investigations or times they were proving answers or solutions to each other.	Data
SD	Stylianides' Definition of Proof	Data related to the teachers responses and discussions about Stylianides' definition of proof.	Proof Framework – Stylianides (2007) Harel and Sowder (2007)

SI	Student Interests	Data related to the teachers using students' interests to generate data or make sense of the mathematics they were learning.	Data
SM	Situated Meaning	Data that would aid in answering: What are the situated meanings of words such as: 'proofs', 'conjecture', 'justify', 'validate', 'discover', 'define', 'inquire'? What situated meanings seem to be attached to such things as: students' original ideas, students' justification of answers, time constraints, institutional pressures?	Gee (2005)
SP	Students Using Proofs	Data related to the students engaging in proofs or the process of proving.	Data
SS	School Structure	Data about the school's structure and the value placed on the school structure by the teachers.	Data
SV	Situated Values	Data that would aid in answering: What situated values seem to be attached to such things as: students' original ideas, students' justification of answers, time constraints, institutional pressures?	Gee (2005)
SW	Show Work	Data related to the teachers asking students to show their work.	Data
TB	Teachers Beliefs	Data about the teacher's beliefs that would aid in answering Topical Question 2, that did not fall into other categories and would be analyzed more specifically at later stages of analysis.	Data and Topical Question 2
TE	Teaching Experience	Data that would aid in answering: What experience does the teacher have related to using proofs and the process of proving in teaching middle school?	Topical Questions 1, 2, and 3.
TK	Teachers Knowledge	Data about the teacher's knowledge that would aid in answering Topical Question 2, that did not fall into other categories and would be analyzed more specifically at later stages of analysis.	Data and Topical Question 2
TP	Teacher Presentation of Material	Data that would aid in answering: How did the teacher determine what forms of content presentation to use? What led her to decided to present mathematical concepts using certain methods?	Topical Questions 1, 2, and 3.

TR	Transformational Proof	Data related transformational proofs as they are described in the Proof Framework and engaging in the process of proving using transformational proof schemes as it is described in the Process of Proving Framework.	Proof Framework and The Process of Proving Framework - Harel and Sowder (2007)
TU	Teachers Using Proofs	Data related to the teachers engaging in proofs or the process of proving.	Data
UB	Using Books	Data related to the students using a textbook.	Data
V	Verification	Data related to the function of proofs as verification as it is described in the Framework of the Functions of Proof and the Process of Proving.	Framework of the Functions of Proof and the Process of Proving – de Villiers (1999); Harel & Sowder (2007); Hersh (1993); and, Yackel & Hanna (2003)

During the coding process tentative linkages began to arise and I moved away from the coding stage and started working with my data to create episodic threads. Episodic threads are a way to organize data, linking related events and/or interviews (Emerson, Fretz, and Shaw, 1995). Episodic threads reflect what Seidman (2006) describes as "making and analyzing thematic connections" (p. 125). Siedman explains that this is a way to organize interview excerpts into categories, which can then be analyzed for connected threads and patterns. As part of my analysis, episodic threads were created using a number of different strategies. The one presented here, as an example, was put together based on connections to a specific classroom event and Mrs. Yellow's discussion about her use and understanding of proofs, related to this event and then broadly. This episodic thread was used during the analysis presented in Part II of the following chapter.

Example of an Episodic Thread:

Episodic Thread – Mrs. Yellow, "Proof" of Distributive Property.

Fieldnotes – 9/15
Mrs. Yellow at a station with six students.

Section 1- group 1
Y-"when we have numbers we have two ways of solving these problems. Distribution Property and Order of Operations. But if we have a variable anywhere we have to use the distributive property"

Y wrote and talked through (with student prompting):

2 a) 3(2 + 3+ 5)

1st way, order of operations: 3(10) = 30
2nd way: 3(2) + 3(3) + 3(5)= 6 + 9 +15= 30

Y: "The second way is going to seem longer but I want **to prove** that you get the same answer.

Y- "so we get the same answer either way, which way looks a little more friendly?"

Answering 2b) $\frac{1}{3}(9+6-3)$

1st way: order of operations
S1: "12"
Y: "okay, tell me what you are thinking"
S1: "so I did what was in the parentheses first so 9+5= 15 the – 3 = 12"
Y: "good, what would you do next?"
S1: "1/3 time 12 which is 4"
Y wrote: 1/3 (12) = 4, "good"

2nd way:
S1 said, while Y wrote: "1/3(9) + 1/3(6) + 1/3(-3)
Y: "Talk me through your thinking"

Section 1 – Group 2
Y went through a similar discussion with this group; she did not tell them she was proving this for them as she did with the first section.

At the end she assigned the group homework and S2 said: "do you want us to do it both ways"
Y: "No you can't, because you can't combine something like 2x + 8 the way we did with the numbers"
Y: "we did the two ways, because this **kind of proves** that the distributive property works"

Section 2 – group 1
Y: "What is the distributive property?"
S2: gave an answer and Y wrote: like a # (numbers), as an interpretation of what S2 was saying.

Y- "give me an example"
S2: 2(3x + 15) = 6x + 30

Y "we are going to do this in two ways just **to show** you that the distributive property will give you the right answer."

S5: "it's the same thing just written differently' – when talking about 2b and 2c.

Y did not mention proof or prove with this group.

Interview 1 pg. 17:
ME: The first way, where they actually combine the numbers first, and the second way, where they didn't, like, how did you decide to do both ways? What led you to that decision?
Y: Because I don't think kids understand why you use the distributive property if you can just combine like terms.
ME: OK.
Y: I think, I mean --
ME: I mean, I'm curious, because you had said to one group, like, I'm doing both ways to show you that the distributive property works. And then, one group, I think you said, I'm doing this to prove it. And then the next group you're like, I'm doing this to kind of prove it.
Y: **I'm not really proving it. I'm sort of proving it. I guess those all meant the same thing in my head.**
ME: Yeah, so, what is that thing, I guess is what, like, what is that, and why is that important for you? Or how do you decide to do that? I'm not asking this right.
Y: Because I don't, no, maybe I'm understanding. Because they, I want them to **buy into that it's important to understand**, and that something that, if I take **something that they already know how to do and try to build on that, it's going to make what they learn easier**. And so, if you can figure, OK, so, they already know how to simplify, with parentheses, order of operations, they already know that. So, if you can, if I can **help make the connection from what they already know to what they don't know**, hopefully it'll make, they'll learn something new and be able to extend it to the next step, I guess, would be why I do it.

Interview 2, pg. 8
I showed Mrs. Yellow the above part of Interview 1.
ME: I was wondering if maybe we could talk about, what is that thing? What is proving, showing, what did you mean when you said, it all means the same thing?
Y: I guess I meant that, **it meant them understanding that the distributive property, how the distributive property works**. So, maybe **not a proof, but how they understand the distributive property**. So that, OK, yes, you can multiply, you know, each term in the parentheses by that first number, OK, I know the procedure. But they also know order of operations, and I have seen kids, in the past, think they see the parentheses in the distributive property and they don't think about combining like terms. They think, parentheses, do that first. Combine [Pause]
ME: And they get stuck here.
Y: And they get stuck.
ME: OK.
Y: So, **the thing was probably understanding** the distributive property.

Interview 2, pg. 9

Y: I rarely, rarely do a **formal proof**. I'll, we may do, when I've taught algebra, that was before school, we would prove the quadratic formula, and we would do some small proofs with properties. And, I have maybe done one, you know, two-column geometry proof with them, that's four steps. But, for the most part, it's really, I have found that **it's a very hard concept for eighth graders**.
ME: OK.
Y: A formal, to do a formal proof. So, I guess, in thinking about it, in loose terms, would be, can, it's almost, **can you show how you got from here to the endpoint. You know, which I guess a formal proof is, with reasoning in there**. And I think they can do that, in very informal ways. You know, they can talk through, and they **can explain why we did this, and this, and this. So, I think, when I say proof to them, that's probably, that's what I mean, is, can you show me how you got from this point to this point? What were your steps, or what was your reasoning? Not that steps are always reasoning, but, you know, what was your thinking? Or your justification.** And that, now, I don't know, now I'm thinking about all these words. You do, I do use them interchangeably, and I don't know if I should be. But then, also, like, how formal do you make it, and then does that, I don't know, I can't get my head around that thought. Like, if you make formal proofs, or you're very strict about, this is what I expect when showing your thinking, are you going to stop kids from showing anything?

The episodic threads pulled together data related to individual teachers or multiple teachers and administrators. The example given above is focused on a single teacher and a single classroom event. Pulling together data from multiple teachers occurred when creating episodic threads related to topics such as school structure. Pieces of interviews and fieldnotes were gathered to present a clearer picture of the events and their meanings. This process encompassed aspects of focused coding and reflection. Emerson, Fretz, & Shaw (1995) explain that focused coding:

> Involves building up and elaborating analytically interesting themes, both by connecting data that initially may not have appeared to go together and by delineating sub-themes and subtopics that distinguish differences and variations within the boarder topic (p. 160).

The reflective aspect of episodic threads is an inductive process where "rather than simply tracing out what the data tell, the fieldworker renders the data meaningful" (Emerson, Fretz, & Shaw, 1995, p. 168). Part of the process of creating episodic threads was creating themes. Although this process is described in the next section, as stated above these phases were interrelated.

Developing Themes

During the process of writing episodic threads, themes were working on two levels. First, particular themes were the focus for each particular episodic thread, and second, themes amongst the threads emerged and became new lines of inquiry. As suggested by (Emerson, Fretz, & Shaw, 1995), "it is useful to sort fieldnotes on the basis of these themes" (p. 159). The significant themes that emerged on both individual and general levels are listed below:

- The teachers incorporate activities and discussions utilizing proofs and the process of proving.
- The teachers' hold dual understandings of proof. One related to their own educational experiences and one related to the role of proofs and the process of proving used in their classrooms.
- The teachers had negative experiences with proofs in their higher-level mathematics course in college.
- When thinking about proofs related to their own educational experience, the teachers demonstrated discomfort, fear, and dislike for proofs and the process of proving.

- When thinking about proofs related to their students' learning of mathematics, the teachers value the use of proofs and the process of proving as part of their classroom practices.
- When thinking about proofs related to their students' learning of mathematics, the teachers believe that proofs and the process of proving relate to their own teaching styles, philosophies, and practices.
- The teachers' knowledge of proofs varies and included aspects of *Informal Argumentation*, *Empirical-inductive Proofs*, and *Transformational Proofs.*
- When thinking about proofs related to their students' learning of mathematics, the teachers believe that *verification, explanation, communication, discovery, exploration of definitions, generalizations, inquiry, and the justification, dismissal or modification of a conjecture* serve as functions of proofs and the process of proving.

The above list generalizes the group of teachers here for the purposes of listing the emerging themes. During analysis each of the teachers were first considered as individuals and then incorporated into the general themes above. The data from each of the teachers were then analyzed according to the general themes in order to authenticate the general themes as well as give perspective to the teachers as individuals. Not every teacher fit perfectly into each of these themes. This will be discussed more thoroughly in the next few chapters. The last stage of data analysis occurred during the development of the text.

Developing the Text

When writing the next few chapters there was a distinct level of analysis occurring. As described by Emerson, Fretz, & Shaw (1995) "the author must *represent* the particular world he has studied (or some slice or quality of it) for readers who lack direct acquaintance with it" (p. 168). There is a considerable amount of analysis needed to form this *representation.* The following chapters describe the findings as a cohesive set of description accounts. These descriptive accounts are the product of analysis and are also analysis at work. Writing up qualitative results, "requires a constant movement back and forth between specific fieldnotes incidents and progressively more focused and precise analysis" (Emerson, Fretz, & Shaw, 1995). In the following chapter the themes listed above are explored through descriptive accounts of both observed events and teachers' interviews. When writing and rewriting this chapter, the final stages of analysis occurred as the *new world,* that of the participants, was translated into meaningful, descriptive accounts.

Validity

How to establish validity in qualitative research has always been challenged and discussed by many in the field (Freeman et. al., 2007). The fact that different accounts of the same data could both be equally valid (Schram, 2006) can cause researchers and readers to feel uncomfortable about the validity or trustworthiness of claims and theories that come from qualitative research. However, there are ways to ensure that a qualitative research study is both credible and trustworthy. In this study I will use the following means of ensuring validity:

(1) *Fieldnote quality* (Emerson, Fretz, & Shaw, 1995) – To ensure the best quality of fieldnotes, I wrote my fieldnotes as soon as possible after the observation. I was almost always able to write up my fieldnotes the same day as the observations occurred. I took breaks from observations every 3 to 4 hours, and I did my best to follow other recommendations described by Emerson, Fretz, & Shaw (1995), such as do not talk to friends or family about your day until fieldnotes were completed.

(2) *Peer review* (Freeman, et. al., 2007) – I worked with fellow graduate students also conducting qualitative research who reviewed parts of my interviews and fieldnotes and helped to validate my coding system.

(3) *Provide detailed descriptions of my decision making process* (Freeman, et. al., 2007) – Beyond a description of my methodology and research design, I kept track of how I went about conducting this study in real time and included that information where appropriate. I noted any problems encountered, decisions made, and most importantly the reasoning I based those decisions on.

(4) *Provide adequate information* (Freeman, et. al., 2007) – Making sure to represent the relationship between my data and my claims or theories with adequate evidence and reasoning is imperative for readers to be able to assess my research and allow them to make decisions about the validity of my claims or theories.

Since validity "can not be defined in advance by a certain procedure but must be attended to at all times as the study shifts and turns" (Freeman, et. al., 2007, pg. 29), these methods of addressing the validity of my research show the wide range of ways in which I attended to validity throughout this study and together work as a means of validating my research.

CHAPTER 4

FINDINGS

This chapter will explore the findings from this study as they relate to the teachers' meaning making. The findings will be presented in three parts. Part I will address the utilization of knowledge by considering the first topical question, *How do teachers make decisions about whether or not to include the use of proof (or the process of proving) in their classrooms? And in what ways?* The findings from this study show that each of the teachers were incorporating activities and discussions utilizing proofs and the process of proving into their classrooms; and the teachers used *verification, explanation, communication, discovery, exploration of definitions, generalizations, inquiry, and the justification, dismissal or modification of a conjecture* as functions of proofs and the process of proving. Examples of activities and classroom discussions using proofs and the process of proving will be presented along with the teachers' and administrators' reactions to these activities.

Part II will address the teachers' *beliefs* and *knowledge* about proofs and the process of proving. It will also start to address how the teachers make sense of past knowledge and their current practices. I will explore answers to the second topical question, *How do teachers think about or understand proofs and the process of proving?* The findings from this study show the teachers' hold dual understandings of proof. One related to their own educational experiences and one related to the role of proofs and the process of proving used in their classrooms. When thinking about proofs related to their own educational experience, the teachers demonstrated discomfort, fear, and dislike for proofs and the process of proving. When thinking about proofs related to their students learning of mathematics, the teachers value the use of proofs and the process of proving as part of their classroom practices and believe that proofs and the process of proving relate to their own teaching styles, philosophies, and practices. The also believe that *verification, explanation, communication, discovery, exploration of definitions, generalizations, inquiry, and the justification, dismissal or modification of a conjecture* serve as functions of proofs and the process of proving. I will present and analyze the beliefs and understanding of proofs shared by teachers during interviews.

Part III of this chapter will continue to address the connections between the teachers' knowledge and practices. I will address this topic by considering the third topical question, *How do teachers form connections between their understanding of proofs (and the process of proving) and the incorporation of certain teaching methods into their classrooms?* I will explore the discontinuity found in the first two parts of this chapter and present alternative resources utilized by teachers in making meaning of the process of proving, outside of their direct understanding or beliefs about proofs. These resources are associated with the school's structure.

PART I: UTILIZATION OF KNOWLEDGE

In order to address the teachers' utilization of knowledge, this part of the chapter will consider the findings related to the first topical question. *How do teachers make decisions about whether or not to include the use of proof (or the process of proving) in their classrooms? And in what ways?* I will illustrate and analyze activities; the aspects and functions of proofs utilized during these activities; and the teachers' and administrators' thoughts regarding these activities.

Although I am using particular examples here to illustrate the use of proofs and the process of proving, there were more instances that could have been used. One example for the fifth grade classroom, and two examples for each of the other grade levels will be presented. The examples were chosen for a number of reasons. The amount of data for different activities varied depending on the detail of my fieldnotes and the teachers' discussions about the activities. The examples below were chosen from the set of activities for which I had the most descriptive details. For each grade, I also included at least one example of an activity that was used by the teachers throughout the time I was observing their classrooms. The examples of reoccurring activities include a specific example(s) and a discussion of how this was a continuous practice. Finally, the examples were chosen to show the teachers using the process of proving related to a variety of the functions of proof.

The teachers' and administrators' reactions to these activities were used to analyze their decision making process. This analysis comes from direct conversations about the use of these particular activities and their discussions regarding different beliefs and values about teaching and learning.

Fifth Grade

Students as Teachers

In the fifth grade classroom students worked through the process of proving with other classmates as they took on the role of "teacher." When a student showed Mrs. White that they had a level of understanding higher than that of other students she would tell them that they were going to "become a teacher" and instruct them to go work with another student who was having difficulty or who had not thought about the problem or concept yet. The process of proving during these interactions is exhibited in the students' level of *communication*, *explanation*, and mathematical arguments (de Villers, 1999; Hanna, 1989; Hanna & Jahnke, 1993, Harel & Sowder, 2007; Hersh, 1993; NCTM, 2000; Stylianidies & Silver, 2004; Yackel & Hanna, 2003). In the fifth grade classroom, Mrs. White uses the practice of having students become teachers in order to help students who have shown a certain level of understanding think more deeply about mathematical concepts. When I asked her how she decided to use this practice she explained, "I think one of the best tests of [knowing] if somebody understands how to do something is if they can explain it to somebody else. And answer questions that they don't necessarily have written down already."

The students were accustomed to becoming a teacher or being taught by another student. They all wanted to be the teacher and would take a very defensive role when another student was teaching them. However, they never seemed to be defensive because they felt belittled or hurt. They were mathematically defensive. They wanted to push their "teacher" to a point where they would become stuck and then they would be able to work on the mathematics together, on an equal level. These pairs of students would continually ask each other "why did you do that?" or "how do you know that will always work?" During these mathematical arguments students were both *ascertaining* and *persuading* and they were engaging in the process of proving through *generality*. The following is an example of when a student was asked to become a teacher and illustrates one of the ways Mrs. White included the process of proving in her classroom.

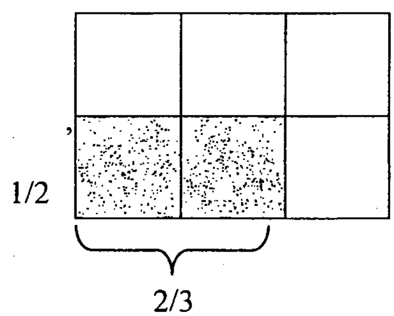
Example

Roger and Sandy are two students in this class who are high achievers in mathematics and were usually working on topics that would normally be taught later in the year or in sixth grade. Two days prior to the following event Mrs. White had assigned Roger problems and activities on multiplying fractions using arrays. He was quickly able to work through the problems but was unable to explain to Mrs. White what he was doing or why he thought it worked. When Mrs. White asked if he had any questions, he told her that he did not and that he understood what he was doing.

Mrs. White gathered Roger and Sandy in one of the classroom's work areas and explained to them what she expected, "Sandy, I am going to have Roger explain how to do this multiplication, he is ready to move on to the division of fractions but I want to make sure he really understands this first. Roger, one of the things you really need to work on is how to communicate your ideas and check in with the person to make sure they are understanding what you are saying. When you are learning a new math skill the way you know it the best is if you can explain it to someone else." Mrs. White walked away and started working with a group of students in the back of the classroom.

Roger sat down next to Sandy, explaining his understanding of what they needed to do. "Okay, so we are going to set up these arrays. What we need to do is to break down the fractions and then multiply like we did when finding the area of a rectangle." The following is an example of the type of work, using an array model to multiply fractions, Roger was explaining to Sandy.

Diagram 4.1:

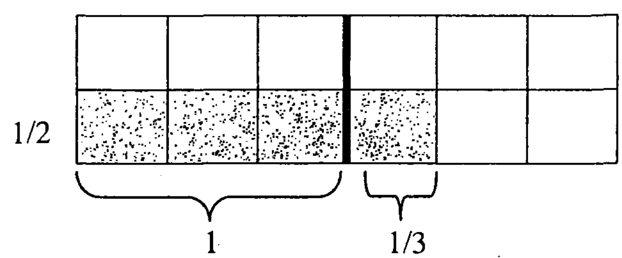


$$\frac{1}{2} \times \frac{2}{3} = \frac{2}{6}$$

Roger and Sandy worked on a few problems together. Sandy kept asking Roger to slow down and explain why he was doing certain things. She also asked him why he thought this worked. They continued to work together with Sandy asking questions and Roger working to explain his thinking. Sometimes Roger would struggle with the "why" questions but he was very comfortable explaining the procedure. If Sandy was not satisfied with his answers she would push him for more information. Once she even quoted Mrs. White saying, "Roger you are not supposed to just show me how to do this you are supposed to be working on how to communicate and explain them to me." It was through this communication of ideas that students were *ascertaining*, working to convince themselves, and *persuading*, working to convince each other. During the processes of *ascertaining* and *persuading* these students demonstrated their abilities to use the process of proving as a function of *communication* and *explanation* of mathematical concepts.

When Roger and Sandy reached a question that included a mixed number they became stuck and went to Mrs. White for help. The following is an example of the type of work, using an array model to multiply a fraction by a mixed number, Roger and Sandy were trying to solve.

Diagram 4.2:



$$\frac{1}{2} \times 1\frac{1}{3} = \frac{4}{6} \; or \; \frac{2}{3}$$

Sandy waited patiently for Mrs. White to finish working with another student and then told her, "Neither of us could get this one," and she pointed to the problem they were stuck on. Roger continued to explain their confusion by asking a more specific question, "Are we supposed to extend this out this way or that way, I am not sure?" Mrs. White was reserved with her response and gave them only enough to continue thinking about the problem on their own, "you need to know how many boxes make a whole still." This seemed to be enough information for Roger who quickly exclaimed, "oh, okay, I get it." Mrs. White's pedagogical choice to answer Roger and Sandy's question without showing them a procedure or directly answering their question is an example of how she used the action of *coaching*. This is considered to be an action of *coaching* because of how Mrs. White coaxed and encouraged the students to develop reasoned arguments on their own, which lead the students to engage in *the process of proving* how and why they could use an array model to multiply fractions (Brown and Renshaw, 2000; Martin et al., 2005).

Roger led Sandy back to the area where they were working and started to explain to her that, "We need to make sure to include the whole number as a group of ones where each of the blocks is one and then we break up those wholes into pieces the same way we did it when we had only one." Sandy seemed satisfied with this explanation and they worked on few more problems together.

Later, right at the end of class, Roger told Mrs. White, "Okay, we have another question." Mrs. White smiled and was pleased that this teaching session had been a source for Roger to formulate questions about a topic he had just an hour ago believed he completely understood, "great I am glad I had you teach this then, isn't it helping?" Roger agreed, "yep." Because it was the end of class they left the question for the next day.

Supporting Evidence

During our first interview Mrs. White explained "It's really important for me that kids not be bored. I don't want to go too slow." I asked her how she was able to differentiate her classroom and she talked about using the practice of having students become teachers as a way to help students like Roger. She said that for Roger "the main thing he needs to work on is making his thinking understandable to others, because he does so much in his head, but he can't possibly explain it...he needs to learn how to use language to help others see his thought process."

I observed this practice of having students become teachers happen multiple times. It was an activity that was used for everyone in the classroom, not just for the students who were working on more advanced math. Another example of using students as teachers involves Jasmine, a student in this class with special needs. Mrs. White had been working with Jasmine on making factor trees. Jasmine was really focused and was able to create factor trees and explain how she was using division to figure out the branches of her tree. Mrs. White describes this example with her own enthusiasm for the moment. "Jasmine was having a lot of math confidence issues. I worked with her one-on-one explaining prime factorization. She got it pretty quickly, and I asked her to teach it to somebody else. I don't think she had ever been asked to teach to somebody else and so she just went up to cloud nine and was like, 'I'm a prime factorization god!' She put her hands up in the air and ran around saying, 'I'm a prime factorization god!' She taught it to another student, she taught it to her dad, she taught it to her mom. It really flipped her whole attitude towards mathematics around. That was really exciting for me to see."

Concluding Remarks

Mrs. White explained that her decision to use this activity with her students came from her belief that you really need to know something to be able to teach it and that sometimes children learn best from other children. "Different kids come up with different questions. I found that, sometimes, kids learn better from other kids, because they were able to see their misunderstandings more clearly…through teaching, sometimes the kids explain the mathematics in a way I wouldn't think of." The value

Mrs. White placed on students teaching and learning from each other demonstrates a belief that the process of proving, particularly as it functions as a way to *communicate* and *explain*, is one that is important to in her students' education.

The students in this class really pushed each other to mathematically communicate their thinking and understanding. During their conversations about the mathematical concepts, students worked to remove doubt about the their ideas and in doing so employed forms of reasoning that modeled both *ascertaining* and *persuading.* Their involvement with the process of proving is illustrated by their engagement in *ascertaining* and *persuading,* during which they demonstrated their abilities to use the process of proving as a function of *communication* and *explanation*. Some questions asked by students during this type of activity were similar to the question of "how do you know this will always work?" This idea of showing that something will work for all such situations demonstrates the process of proving as *generality*. Mrs. White decided to use this activity of students becoming a teacher as a way to create opportunities for her students to *communicate* and *justify* their ideas and ways of reasoning, as well as explore each other's understandings. The example also demonstrates Mrs. White using the action of *coaching* as a means to help students build their own reasoning.

Sixth Grade

<u>If – then Statements</u>

In the sixth grade class students engaged in the process of proving using informal if – then statements by *exploring definitions*, and using them to *justify* mathematical claims (Larson & Zandieh, 2005; Stylianides, 2007; NCTM, 2000; Yackel & Hanna, 2003). The following example shows how if – then statements were explored during a combination of whole class discussions, in small groups, and on homework/worksheets. During this justification process students were engaged in deductive reasoning and exhibited the use of the process of proving through *logical inference.*

Example

To begin class one day Ms. Blue asked students, "Does anyone know what the recursive rule is? I know you did that in 5th grade." Most of the class put their hands up and Ms. Blue called on Kenny who struggled a little to collect his thoughts about the recursive rule and eventually was able to construct a definition, "what you add to the thing before, to get the next one." Ms. Blue reformulated this by writing, "the constant number that is added to the previous element to get the next value." She checked with Kenny to make sure this is what he meant. After he approved she asked the class if they agreed on this definition. Students nodded their heads or said "yes". None of the students challenged this definition, although that was seen during other classes. Mrs. Blue's use of *revoicing* here clarified Kenny's answer while maintaining the value of the idea coming from Kenny and not her or another source of authority. Her action of checking with the students to verify the correctness of this statement shows the students engaging in the process of proving by working with statements accepted by the classroom community, as described in *Stylianides' Definition of Proof.* As stated above students did not challenge this definition and so the validation or acceptance of this statement was quick. However, when students did challenge definitions the process of accepting a given statement relied heavily on the students engaging in *ascertaining* and *persuading.*

After the class had agreed on this definition of the recursive rule, Ms. Blue handed out a worksheet with 9 different tables partially filled in. The input values all started with 0, increased by 1, and ended with 5. The output values where given for the first three inputs and left blank for the last three. The directions on the worksheet were to: 1. Complete each table, 2. Write the rule for each in words and as an equation, 3. Show the change in Y on the side of each table. Ms. Blue gave the following verbal directions, "First figure out the recursive rule, then finish the chart and write the equation in words and then with math symbols. Do a few of these, and then I want to see if you can find a pattern between the recursive rule and what is happening in the equation." Students started working on the worksheet, some worked together and some chose to work individually. Ms. Blue walked around and worked with students who were having difficulties or asking questions. After she had worked with a few students she told the class, "If you are having a hard time figuring out the pattern it is okay to fill out the chart using the recursive rule until you can figure out the pattern." She also suggested that they highlight the change in Y written on the sides of the tables and where they saw this number in their equations. After about 15 minutes Ms. Blue brought their attention back to the overhead so they could, "share what they had discovered." Students expressed their ideas about the connections between the recursive rule and the equation. Students said things like, "So, if the recursive rule is to add 5, then I will have 5 times x in my equation." Eventually, Ms. Blue summarized their findings through the action of *revoicing*, "the recursive rule gives me the number that is in front of the x in the equation." She asked the class if they agreed and they nodded or said "yes." A few did not respond. Again Ms. Blue engaged her students in the process of accepting a given statement. They worked on one of the problems from the worksheet as a class. Ms. Blue asked what they found for the recursive rule for question G. One of the students said, "ten." Ms. Blue wrote on the board:

Recursive Rule: Add ten to the previous element to get the next value.

So,

And she waited for a response. Another student answered, "So, I know that I will have 10 times x in my equation." Ms. Blue asked her to explain why she knew this. "Well, we just figured out that if I have ten as a recursive rule, then I have 10 times x in my equation." Although this example shows the students working procedurally to find the equations of a line, it also shows that they are thinking about if – then statements and what it means to use a mathematical statement that has been agreed upon to connect something they know to the next step in generalizing that knowledge. This demonstrates the students working with the process of proving through *generality* and *logical inferences*. In the example above this can be seen by the students using their understanding of the recursive rule for solving problem G. The students used the recursive rule to conclude that if they are adding 10 to the previous element to get the next value, then they know that they are going to have 10 times x in the equation. The student's explanation as to how she had worked through this conclusion shows her use of the definition of the recursive rule and how that enabled her to formulate the "then" part of her conclusion.

I am using this example to show Ms. Blue's use of definitions and their connection to if – then statements. It was common practice for her to write a known statement on the board followed on the next line by "So," and ask her students to continue the argument as well as explain the connection between the known statement and the students continuation of that statement. Many times this connection relied on a simple definition. Students would answer with, "So, I know ______, because_____ (and they would state a definition)." This activity shows that the students understand the need to justify based on *logical inference* rules and therefore are engaging in an aspect of a *Transformational Proof Scheme*.

Supporting Evidence

Ms. Blue and I discussed her use of these informal if – then statements during our second interview. I explained to her that I had often noticed her stating and writing a fact or observation made by one of her students, following it with "So," and then waiting for the class to complete the thought or sentence. She thought about this, and said:

> I guess I don't do it intentionally, in terms of, I've never really thought about it. But I think, because I, probably, before, have just written, so, we have this, or, this is what we have, and a lot of them are just staring at me like, yeah, I don't care. So I am trying to lead them to their next step, because I mean, that's something that, I think, a lot of them are just getting to the point where they'll look at a table or something, and they're just like, OK, it's a table, and I'm trying to make them see the connection. So, OK, if we have this, what can we do with it? … I want them to, you know, look at what else we would do with it, or what does that mean, and just trying to make those connections. And so that's my way, I guess, of leading them a little bit, saying there is something to connect it with, but not telling them what it is.

Her reference to her students making connections using these if – then statements illustrates how she is working with her students to build on their reasoning abilities. The pedagogical choices to use the action of *revoicing*, engaging her students is the process of creating and accepting mathematical definitions and rules, and her use of informal if – then statements shows Ms. Blue involving her students in aspects of the process of proving. The fact that she viewed this practice as unintentional will be explored in Part II of this chapter.

Concluding Remarks

Ms. Blue engaged her students in the process of proving by utilizing informal if – then statements. Her use of if – then statements had students *exploring definitions*, *justifying* claims, and working with statements accepted by the classroom community. The students worked with deductive reasoning as they relied on *logical inference* rules as a form of justification. Through the use of these *logical inferences* students showed a level of *Transformational Proofs*. Ms. Blue explains that her decision to use this practice was unintentional but that she values the exercise as a way for her students to make connections between what they are learning and their past knowledge.

Building Patterns to Discover Equations

This example illustrates students working with proofs and the process of proving as a function of *generalizing* and *justifying, dismissing and, modifying conjectures* (Ellis 2007; Lakatos 1976; NCTM, 2000; Styianides & Silver, 2004; and Yackel & Hanna, 2003). Students used inductive reasoning to find patterns and create a general equation to fit their data. They were then expected to verify that these equations worked for their data and for further sets of similar data. This activity had students first *ascertain* and then create a convincing argument to *persuade* their teacher. The example explores an assignment called a Problem Solver. Before describing this particular example the structure and purpose of Problem Solvers will be explained.

Problem Solvers

Problem Solvers are a school-wide type of assessment designed specifically for students to think deeply about the mathematics they are learning and to document their thinking. During my interview with the Curriculum Coordinator, I asked Mrs. Pink to talk a little about Problem Solvers and the Grading Rubric that is used for assessing these assignments. There are two different rubrics, one for grades K – 4 and one for grades 5 – 8. Our discussion was focused on the rubric for grades 5 – 8. A copy of this rubric can be found in Appendix A. Mrs. Pink worked on a version of this rubric before working at Light School. She introduced the rubric to the teachers at Light School and they worked together to revise it and "make it their own." The rubric is constantly being modified based on what the teachers learn year to year. I will refer to this rubric later when sharing an episode from the Seventh Grade. However, between the time that I was in the Seventh Grade and the time I was in the Sixth, the rubric had been modified. The differences are slight, but there are two different versions discussed in this dissertation.

During our interview I asked Mrs. Pink if she saw the presence of proofs in the Grading Rubric. Referring to the rubric's criteria for *Understanding the Mathematics*, she explained that:

> Here, in *defending the reasonableness of your answer* no matter what you come up with, the fact that, other than you saying, I did this, and I did this, and therefore it has to be this, that you can also add something else that would show why you believe this is true.

Having students explore why they believe their answers to be true, is fundamental in building an understanding for the process of proving (Knuth & Sutherland, 2004; Stylianides & Silver 2004; Yackel & Hanna, 2003). Mrs. Pink pointed out how the criteria for *Documentation/Organization of Reasoning* are also directly linked to her beliefs and understanding of mathematical proofs:

> In the *Documentation and Organization of Reasoning*, we want the children to be able to communicate such that somebody else could follow their thinking… we're going to teach you how to document. How to communicate… I believe that it is that documentation of thinking that then becomes the tool for thinking itself…When you talk about proof…you can think of it as a documentation of your reasoning, such that it flows, each step makes sense, and you achieve an answer that is logical to the question asked.

The connection between proofs and the documentation of thinking is one that may be lost in a formal and final proof that one sees in a textbook (Hanna, 1989). However, as anyone works through the process of proving, it is essential that this documentation take place (Hersh, 1993).

Example

For the Problem Solver used in this particular example Mrs. Pink worked with Ms. Blue to modify an activity into a suitable question. During the second interview with Ms. Blue we talked about this modification. Copies of the original activity and the Problem Solver can be found in Appendix B. Ms. Blue explained that the original activity was too structured for a Problem Solver, "there were more specific questions on it, as opposed to just kind of leaving it open for interpretation, solving it in different ways." She went on to explain that they reworked the problem to make it more "open-ended" and in doing so created a problem that was good for this activity because the pattern was not obvious, "most of [the students] couldn't look at it and figure out what the equation was. They really had to sort through it." Ms. Blue's pedagogical choice to have her students engaging in an open-ended task where they were expected to *generalize* and *justify, dismiss and, modify conjectures* demonstrates her belief that students should be able to "understand the whys behind the math." As described in the previous chapter, understanding the whys in mathematics is key to being able to engage in the process of proving.

The task of this Problem Solver asked students to build a pattern and discover an equation. The students needed to figure out how to get a combination of adults and children across a river using one boat that could fit up to two children or one adult. Their work was first to *ascertain*, convince themselves that their general equation represented their discovered pattern, and then create a convincing argument to *persuade* their teacher that their general equations were valid. Students approached this in different ways, some created tables, others drew pictures, and others used both. By our third interview, Ms. Blue finished grading the Problem Solvers and we discussed the students' strategies and the connection between this type of activity and the process of proving. Copies of students' work can be found in Appendix C.

One of the students we discussed was Cory. Cory used a picture to build a pattern using the boat to get all of the adults across the river first and back to get all of the children across. He then wrote a generalized statement and an equation to represent his findings. His picture shows four steps. In the first step, he uses the boat to get the two children across the river. In the second step, he uses the boat to bring back one of the children. In the third step he uses the boat to bring one adult across. Then in the forth step he uses the boat to bring the child who had been left on the opposite side back. He then notes: *Repeat the pattern until all of the adults are across. Then repeat step 1.* Cory then gives his solution along with an explanation:

> *The number of one-way trips for 8 adults and 2 children would be 33. It's 4 trips for each adult and then one extra for the two children.*

He goes on to give a mathematical generalization:

> *For any number adults and two children the formula would be 4x (4 times the number of adults) + 1;* and two examples: *For example, with 36 adults, there would be a total of 145 trips. With 6 adults there would be a total of 25 trips [(4x6) +1] required.*

For Cory's *Understanding the Mathematics* grade, Ms. Blue marked that he exceeded the standard because his solution used both a *mathematical rule (generalization) whose derivation is clearly explained or proved another way* and *proves the correctness of the answer by solving the problem a different way.* Ms. Blue explained that:

> He kind of just jumped right to the answer. But it was clear, and I know that's how he thinks, and it's not like, you know, some kids who just got the answer, I mean, you could tell, he knew what he was doing, and he had thought about it, and he understood the problem.

Ms. Blue's assessment of Cory's solution seems to rely not only on his work but also on her understanding of the student. Cory's solution does state a generalized rule, however the derivation of this rule is not clearly explained. A reader of this solution would need to piece together how Cory moved from the picture, to the found pattern, and then to the generalized rule. As stated by Ms. Blue, "he kind of just jumped right to the answer." It was through her understanding of "how he thinks" that she was able to assess that "he knew what he was doing, and he had thought about it, and he understood the problem." Furthermore, Cory's solution does not prove the correctness of the answer by solving the problem in a different way. Reading Cory's solution, it seems more appropriate that his solution would have earned a grade of *Meeting Standard* based on the criteria of *defends the reasonableness of the answer with a clear explanation and/or applies a discovered mathematical rule to at least 2 cases to prove its effectiveness.* Even for this grade, Cory's use of his equation to solve for 36 and 6 adults is not justified using anything other than his equation, and so this grade would be justified because of his defense of the reasonableness of the answer with a clear explanation, which is present in his picture and further explanation. Considering both Ms. Blue's statement and my own assessment of Cory's work based solely on his written solution, it is clear that Ms. Blue's assessment relies on more than Cory's written work, namely, her understanding of the student. I asked her to explain how she decided on this grade and she explained that:

> I think this is a really hard, this is one of the hardest things for the students in a Problem Solver. I have so many say my answer must be right because I tried it again and it worked. And I'm like, well, that doesn't prove.

This statement shows that Ms. Blue is expecting a level of proof in her students' solutions. She went on to explain that she expects her students to go further in "connecting the mathematics or, you know, understanding the mathematics." Although, I have disagreed with Ms. Blue's assessment of Cory's work, it is clear to me that in creating and justifying his conjecture Cory exhibited the process of proving. Moreover, it is clear that Ms. Blue used this activity to engage her students in the process of proving.

Supporting Evidence

When I asked Ms. Blue if she had any reaction to the Stylianides (2007) definition, she referred back to the use of justification in the Problem Solvers. She believed that all of her students could in fact justify their answers, but few of them understand what that means when they are working individually and writing down their justifications rather than sharing them with herself or classmates.

> I think part of it for the Problem Solvers, I don't think it's that they couldn't do it, and they couldn't justify their answer, because if I was interviewing, if I was

> sitting with them, they'd be able to clearly justify it, or be like, oh, that doesn't work. But when they're just writing it down, the one thing, with the Problem Solvers is, it's completely independent. Whereas, that's why I think I see it more in class, when they're working with people, or I'm there, and I can say, well, give me a little bit more, because I think most of them understand why… if I was sitting there and saying, tell me more. Like, what can you prove? I think most of them could kind of push themselves to get there. But when they're sitting there by themselves, it's just like they don't push themselves that way, so it's just like, of course it's right.

The need to justify answers in a way that is beyond that of simple examples, shows one connection between all Problem Solvers and proofs and the process of proving. Ms. Blue also points out here that the students don't always know how to independently develop a full "proof" at this grade level. However, with her and independently, the students are developing ideas about *proofs* and *the process of proving.*

During our third interview I asked Ms. Blue to discuss her expectations for this Problem Solver by explaining how she graded Abby's solution. Abby satisfied the criteria for Meeting Standard for all six criteria, for part one of the problem, which was focused on finding a solution for eight adults and two children. However, she did not formulate an equation, so she earned lower marks for half of the categories. Abby based her solution on a chart she made of all 33 trips. After writing out her chart Abby wrote:

> *Why my concluded answer of 33 one way boat trips is correct is because all of the boat trips have an equal or less weight, which is equivalent to 1 adult or 2 children. But, since 1 child is less than 1 adult or 2 children, that would still be an appropriate move. I have also carefully counted the number f boat trips on my chart, so my answer should be accurate. Also at the end, when I had finished the chart, I was sure I had finished when, in the total people on the other side column, I had has 8a and 2c (or 8 adults and 2 children).*

Concentrating on the two criteria most related to mathematical proofs, for *Understanding the Mathematics* Ms. Blue highlighted that Abby *defended the reasonableness of the answer with a clear explanation.* However, Ms. Blue explained that Abby actually earned a grade of Below Standard, "because [Abby] just told how [she] solved the problem and the weight does not explain why 33 works." We can see here the emphasis Ms. Blue places on *explaining.* Abby's discussion on weight, although not clearly stated, was a justification for allowing one child to make a trip in the boat. However, since the problem states that the boat can hold one child, two children, or one adult, this justification is not needed and as Ms. Blue pointed out, this does not justify her answer of 33 one-way trips.

Abby earned a Meeting Standard grade for *Documentation/Organization of Reasoning.* Ms. Blue commented on Abby's solution that she had an "excellent explanation of [her] thought process." Ms. Blue also wrote this next to a section of Abby's solution where Abby wrote:

> *So I thought ahead again and if I took back a child I could replace that with 2 the next time. So, that is what I did. Then after the next 3 move[s] of repeated steps. I noticed a pattern of down 2 children, back one child, down 2 one adult, and then one child back. I decided to mark this down and from that point on it was just repeated with one more adult each set. Finally at the end when I had 8 adults and it was the last set of the pattern I crossed 2 children and since that was what we needed to get to I was done!*

Abby's explanation of her process explains how she solved this problem. Although, Abby worked though the whole problem without generalizing her pattern, she was able to find a solution using a method that worked for her and she noted a pattern after the first four trips and continued to use this pattern until all the adults were on one side and then finished by making one more trip to retrieve the last child. Ms. Blue's assessment of Abby's solution shows the importance of the generalized equation, and the explanation and justification of the equation was to this activity. Ms. Blue commented about Abby's explanation saying:

> She's never done a Problem Solver for us. She hasn't been in this school before. So, she actually, her answer is really, I mean, she's there, sort of. She didn't get the final answer, but she's really close. She had the right strategy, and she kind of figured it out, but she couldn't, you can just tell, she hasn't had the experience with explaining her work, she didn't get quite there.

She later explained to me how it takes new students time to understand what is expected of them during these types of activities, particularly Problem Solvers.

Concluding Remarks

Besides the demands placed on *justification*, *explanation*, and *discovery* during all Problem Solvers, this particular activity was linked to proofs and the process of proving because of its content. Students were asked to find a pattern and formalize that pattern into an equation that would work for any number of adults and 2 children. The use of *generalization* and *verification* here demonstrate the relationship to the process of proving.

This example illustrates how Ms. Blue included the process of proving through the use of a Problem Solver. This example demonstrates the expectations for students to use the process of proving as a function of *generalization*, *justification*, and *discovery*. The students were involved with *ascertaining*, convincing themselves of the answer through charts, diagrams, etc., and *persuading*, through their written explanations as required by the Problem Solver. The main goal of the Problem Solver was to have students build a pattern in order to discover an equation. This process also reflects a connection to the process of proving since it is based on *generalizations* and *discovery*. Ms. Blue's decision to use this Problem Solver may have come from the school structure, which will be discussed later in this chapter. However, she also talks about her decision to use this type of activity as a way for her students "to be able understand the whys behind the math" they are learning and work with "open-ended problems" that can be "solved using different strategies."

Seventh Grade

Investigation

In the seventh grade classroom Mrs. Red stresses that her students ask questions of their data to find out "what do you know?" and "what do you need to know?" and then work with that information to create conjectures and draw conclusions. Her focus on questioning is present in the classroom in a number of ways. She even has a poster of Einstein with the quote that says, "Even Einstein asked questions." During the second week of the school year the seventh graders started working on their first "investigation." Mrs. Red called certain activities "investigations" based on the level of questioning. As she explained to her students "in an investigation you are trying to discover something." As she continued to explain "investigations" the central role of questions and conjectures was discussed. Mrs. Red asked her students "If you have to discover something what do you have to do?" One of the students responded, "look for clues." An other student responded, "ask questions." Mrs. Red nodded her head and went on "you have to ask questions, right, even ask questions of yourself." She then explained that during an investigation, "I always ask 'what do I know' and 'what do I need to know.' These are the most important questions to ask yourself and so I will always ask these questions, and you should, too." The following example is from the first and second day they worked on this activity. In this example, students engaged in the processes of *discovery*, *exploration of definitions,* and *conjecturing* (deVilliers, 1999; Ellis, 2007; Harel & Sowder, 2007; Lakatos 1976; Larson & Zandieh, 2005; NCTM, 2000 Yakel & Hanna, 2003).

Example

After Mrs. Red discussed that they would be doing an "investigation," and what that meant in her classroom, she gave directions for the activity, "We are discovering about a histogram. It is like you are walking into a crime scene and all you know about a histogram is that it is like a bar graph and it shows how often data fall in different ranges. That's all you know. Then it is like a game, and you are trying to get as much information as you can." The students used their laptops and accessed *NCTM's Illuminations Histogram Tool.* NCTM describes this tool as a way to "create a histogram for analyzing the distribution of a data set using data that you enter or using pre-loaded data that you select." Students were expected to "play" with the pre-loaded data or enter their own data sets. Their first task was to come up with questions about histograms and their second task was to find answers to their questions. In other words, the students were to conjecture about possible solutions to their questions and work to justify or dismiss these conjectures.

Students spent most of the first day "playing" with the *Histogram Tool.* Some students used the pre-loaded data sets. Other students put in their own data. Some students used data sets that were part of their homework from the day before, when they collected statistical data from a newspaper or magazine. The main focus of this class was for students to figure out what they already knew about histograms and to start asking questions about what they did not know. About halfway through the class Mrs. Red explained, "You should be writing down your questions like, what is standard deviation? What is a frequency table? What do you need to find out?" After a student asked her to repeat these questions Mrs. Red said:

> You are supposed to be coming up with your own questions, I was just modeling how you ask questions. You need to be asking questions of your own...Ask questions about what is happening, what does this number mean, what does this word mean, what happens if I change this. Act like you would if you're lost and trying to find your way somewhere.

Later a student, who was new to Light School, explained to Mrs. Red that he had never really done an "investigation" before. She told him, "all you have to do is write down all the questions you have, there is no failure here, it is about your questions and no one can tell you that you are wrong." He started listing possible questions for her and she nodded, telling him he was "asking some great questions" and that tomorrow they would continue by starting to look for answers to these questions.

The following day was focused on creating conjectures and discovering answers to the questions they had raised the day before. Questions that were raised about vocabulary terms were answered using resource books. Changing data sets and playing more with the *Histogram Tool* was used to investigate other questions, such as the one raised by Jon. Jon was working on his computer trying to figure out what happened with a data entry that was the same number as one of the end points of his intervals. He worked on this with a few sets of data and was struggling a little. Eventually, he created the data set to be: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 with an interval size of 2. Later, he explained to Mrs. Red that,

> I changed the data set by taking away 3, and saw that the interval ending in 3 went down but the interval starting with 3 stayed the same. So, I, you know, figured

> that the 3 was part of this interval (pointing to the interval that ended in 3) not that one (pointed to the interval that started with 3).

As he was explaining this he walked through the steps on his computer. Jon conjectured that his question had two possible solutions and through the use of the *Histogram Tool* he found a pattern and was able to conclude that the data point belonged in the lower interval using inductive reasoning. This example shows Jon using questions to create a conjecture and discovering a definition, which he and the class later referred to as a "rule."

When Mrs. Red brought the class together to have a group discussion about what they had learned and what questions they still had, she asked Jon to share his "cool discovery." The class worked through a couple more examples and decided to *generalize* Jon's discovery and make it a "rule." Later Mrs. Red asked them to create a histogram as a class. One of the pieces of data happened to be a number that was an end point of an interval. When she asked the class where that data belonged, one of the students replied, "it should go with the lower interval." Mrs. Red asked her why, and she said, "because of Jon's rule."

This episode shows how Mrs. Red expected her students to discover mathematical definitions and "rules" for themselves, and how she fostered the need for questioning in her classroom. Illustrated above, Jon worked inductively with different sets of data to come to a *generalization* that he shared with his classmates. Mrs. Red then walked the rest of her class through this process by showing them multiple examples of how this generalization works.

Concluding Remarks

Almost every day Mrs. Red would ask her students "What do you know?" and "What do you need to know?" When I asked her about how she decided to use these questions during our second interview, she told me that, "they almost become your mantra; they almost become your philosophy." She explained that having students ask themselves "What do I know?" is important because she expects her students to "recognize what pieces they know, so that they have a basis." She explained the importance of knowing where you need to go, or asking yourself "What do I need to know?" to her students one day. She said, "If you can figure out where you want to end up, what would be desirable, and then walk backwards then you can start to see 'how do I get there'." The connection between her practice of continually asking her students to think about these questions and the process of proving is significant. It focuses on conjectures with both unexpected and expected results, as well as building knowledge using inductive and deductive reasoning.

While making mathematical discoveries students participated in the beginning processes of proving by asking questions, discovering patterns, and creating and testing conjectures. First, the students sought out and answered their own questions using patterns, such as the example of Jon's discovery. Second, they tested conjectures using multiple examples to inductively reason about the validity of the conjectures, such as with the whole class discussion of Jon's "cool discovery." Finally, they explored definitions by looking for "rules" that they would be able to apply to similar circumstances, such as with the example of one student referring to "Jon's rule," or the found definition as a reason for making decisions with other sets of data. Mrs. Red's pedagogical choice to include "investigations" as part of her classroom practices relate to her "philosophy" of engaging students in situations where they ask themselves questions, particularly "what do I know" and "what do I need to know," which is significantly related to the use of proofs and the process of proving.

Argue Your Answer Mathematically

The following example shows the inclusion of the process of proving by having students *justify* and *explain* their answers by using mathematically sound arguments (deVilliers, 1999; Hanna, 1989; Harel & Sowder, 2007; Hersh, 1993; Stylianides, 2007; Stylianides & Silver, 2004;Yackel & Hanna, 2003). These mathematical arguments show the students *persuading* using mathematics to remove doubt about their solutions. Mrs. Red assigned a Problem Solver during the forth week of classes. As with the Problem Solver explained earlier in the *Building Patterns to Discover Equations* section of this chapter, this activity is designed for students to think deeply about the mathematics they are learning. A copy of this rubric can be found in Appendix D.

Example

As part of the current unit students were analyzing data using mean, medium, and mode, as well as different types of graphs. The Problem Solver asked students to determine which of three different golfers was "the best." Students needed to look at the data in multiple ways to make their determination. For example, if students only looked at the mean for each golfers' distances they would find that all of the means were the same and so making a determination about which golfer was "the best" based on this analysis would not give a complete picture of the golfers' abilities. The directions on the Problem Solver stated, "Analyze the results in as many different ways as you know. Present a mathematical argument to back up your decisions about who the winner was and why they won." A copy of this Problem Solver can be found in Appendix E, the source for this problem is unknown.

When Mrs. Red assigned the Problem Solver she told her students, "You need to find the answer and to argue your answer mathematically." She gave them some time in class to start working on the Problem Solver. After a few minutes one of the students, Mike, asked, "can you argue that no one should win?" Mr. Orange, the special education teacher in the room, asked, "Are you arguing mathematically or are you saying there shouldn't be competition and they should just be happy golfing?" Mike laughed and said, "No, I mean mathematically there isn't anything saying one or the other." Another student told Mike to look at the consistency. Mike told her "but there is nothing saying that consistency is better than the person who had the longest distance." In fact, the Problem Solver did give directions to look at the consistency. Mr. Orange continued to work with Mike trying to understand what his position was. "So are you saying that as soon as you choose someone, you can mathematically argue that, no, it should not be them?" Mike agreed, "Right!" Mrs. Red, who had been listening to this conversation, jumped in and told Mike, "I guess you could argue that, as long as your argument is mathematical."

Supporting Evidence

During our first interview, I asked Mrs. Red what she expected her students to learn by doing a Problem Solver. She explained how Problem Solvers, specifically referring to the one described above, are used to help students understand

> how to mathematically argue something, using backup arguments...They can't just say that Rick won the game, and then that's it. They need to say because his numbers were close together, in a range, and use math vocabulary...it's showing them how to use math factual information.

Mrs. Red often referred to this idea of "arguing your answer mathematically" demonstrating her expectation of *persuading* based on mathematical facts. During our second interview I shared the classroom episode described above and asked her, "In general, if you were to say to a student, you need to argue your answer mathematically, what would you mean by that?" She started by explaining what she meant during this specific episode,

> As far as that problem, specifically, I wanted kids to look at the three central tendencies, so, mean, median, and mode. And use those to argue your answer. And that was kind of the trick of it all, because the means all were the same, so they had to figure out the median and the range and use that information to say, who was consistent and not consistent.... And using the graphs that they learned.

She went on to say that, "When I say, a proof, you know, I'm still looking for them to tell me what they've learned, and explain, exactly, mathematically, where they're getting their justification for it." Again here we see the idea of students engaging in the act of *persuading*. Her thoughts and understanding about proofs will be explored in Part II of this chapter.

Before discussing some students' work on the Problem Solver, I will make some connection between Mrs. Red's use of "arguing your answer mathematically" and the process of proving. Probably, the clearest connection is to that of *justification*. Mrs. Red used this phrase with students who lacked discipline in showing the work or writing out their thought process. Many of the times she asked students to "argue your answer mathematically" she would follow with, "you need to justify your answers." Her use of this phrase and the meaning she gives to it through further asking students for justifications shows an expectation of *explanation, communication* and the process of *persuading.*

Additional Examples and Supporting Evidence

The work of two students, Lee and Emma, will be used to analyze Mrs. Red's assessment of this Problem Solver. Their work along with Mrs. Red's comments and grades can be found in Appendix F. Emma's work earned her a grade of "Meeting Standard" for all six criteria on the grading rubric. For the purposes of this analysis, the criteria of *Problem Solving- Connecting the Mathematics* and *Communication – Documentation of Reasoning*, will be explored. As a contrast, Lee earned a grade of "Below Standard" for both *Problem Solving- Connecting the Mathematics* and *Communication – Documentation of Reasoning.* In order to meet the standard for *Connecting the Mathematics* students need to *prove the correctness of the answer by solving the problem in a different way and/or defend the reasonableness of the answered with a clear explanation and/or apply a discovered mathematical rule to at least 2 new, higher classes.* Mrs. Red marked that Emma *proved the correctness of the answer by solving the problem in a different way* and *defended the reasonableness of the answer with a clear explanation.* Emma's answer included the mean, medium, and range for all three golfers as well as individual histograms and box-and-whisker graphs for each golfer. After her histograms, Emma concluded that the golfer named Sarah was the best because she was the most consistent:

I think that Sarah is the most consistent chipper. She has a piece of data in almost every interval. Rick has many spaces and Mike has many skips. Neither is very consistent. Almost all of her bars are touching too.

Emma has argued her answer mathematically by analyzing the data as it was displayed in a histogram. She goes on to create and analyze the data in a box-and-whisker graph. This demonstrates how Emma supported her answer by solving the problem using a different method. After she showed each golfer's data in individual box-and-whisker graphs Emma writes: *Sarah is still the most constant after I did two graphs. Her boxes were really close together and the others were really spread out.* This student followed the expectations outlined in the rubric by solving the problem in a different way.

In order to meet the standard for *Communication – Documentation of Reasoning* the teacher needs to see that:

The documentation of the correct or incorrect solution process clearly shows how the problem was solved and the reasoning used:
Computations used are noted
Presentation is in a logical order
All parts are connected and labeled
Answer(s) is highlighted
Mathematical explanations or arguments are clear.

Emma's work, as noted by Mrs. Red, was very organized, and she met each of the requirements above. Her evidence was not only present but was also explained and connected to her answer. Emma justified her choice for Sarah by concluding that she was the more consistent player. Using her graphs, she presented a clear argument for consistency by referencing the closeness of Sarah's data.

Lee did not perform as well on this Problem Solver. As stated above he earned a grade of "Below Standard" for both *Problem Solving- Connecting the Mathematics* and *Communication – Documentation of Reasoning.* Because his "response was incomplete" he was unable to satisfy any of the requirements for either Approaching Standard or Meeting Standard for the *Connecting the Mathematics* criteria. Although Lee did choose Sarah as the winner and stated that she was the more consistent player, he made no connections to his evidence to support this claim. Lee included each player's mean, medium, mode, and range, as well as both a circle graph and a histogram. Lee's circle graph and histogram displayed each player's range on the same graph. He does not refer to any analysis of these graphs in his answer. He simply states that *Shown in the data Sarah's shot was the best! She was more consistent than Rick or Mike!* This lack of explanation also affected his grade for the *Documentation of Reasoning* criteria. Because his *documentation of the correct or incorrect solution process contain little or no evidence of how the problem was solved of the reasoning used* Lee received a grade of Below Standard for this criteria. Mrs. Red held her students to a high level of *justification* and reasoning as was explored above using two students' solutions to the Problem Solver. She describes her decision to include this activity as one based on her belief that students should learn to "argue their answers mathematically" and be able to "justify where they're getting their justification for it."

Concluding Remarks

Mrs. Red included proofs and the process of proving by engaging her students in this Problem Solver. The expectations for *justifying*, *explaining*, and "arguing answers mathematically" for all Problem Solvers and other activities done in the seventh grade show how Mrs. Red utilized proofs and the process of proving in her classroom activities. The students' mathematical arguments show them engaging the process of *persuading*.

Her decision to include this activity may have been due to the school's structure; this will be discussed later in this chapter.

Eighth Grade

Using Mathematics to Inquire about Mathematics

The following example pulls together episodes that span a three week time period. The central task for all of these episodes was The Locker Problem (given below). Students began working on The Locker Problem the second day of classes and continued to work on the problem and extension exercises related to The Locker Problem into the third week of classes. During that time students were engaged in the functions of *communication, justification, dismissal and modification of conjectures*; *discovery*; and, *inquiry* as they worked with proofs and the process of proving (deVilliers, 1999; Ellis, 2007; Harel & Sowder, 2007; Hersh, 1993; Lakatos, 1976; NCTM, 2000; NCTM, 1991; Stylianides & Silver, 2004; Yackel & Hanna, 2003). Students engaged in both *ascertaining* and *persuading.* They demonstrated *Empirical Proof* through the use of charts and examples and demonstrated *Transformational Proofs* through the use of *generality* and *logical inference*.

Introduction of the Locker Problem – Communication

Mrs. Yellow and Mrs. Green introduced The Locker Problem as an activity about communication and using different strategies to solve problems. They had students get into groups according to their believed strengths. Mrs. Green told students, "If you like to talk go sit at that table", pointing to the table in the back of the room; "If you like to write, go to that table", pointing to a table in the middle of the room; "If you like to act go the that table", pointing to table in the front of the room. The students got themselves into groups and Mrs. Yellow handed out a copy of The Locker Problem to each student:

> *The Locker Problem*
> *At a new junior high school, there are exactly 1000 students and 1000 lockers. The lockers are numbered in order from 1 to 1000. On April Fool's Day the students played the following prank. The first student to enter the building opened every locker. The second student closed every locker that had an even number. The third student changed every third locker, closing those that were open and opening those that were closed. The forth student changed every fourth locker, and so on. After all 1000 students passed through the locker room, which lockers were open?*

The students at the back table were only allowed to talk to each other about the problem; they could not use gestures or write anything down. The students at the middle table were only allowed to write things down, they could not use gestures or talk to each other. The students at the front table were only allowed to act, they could use gestures and any type of acting, but they could not talk or write anything down.

Mrs. Green explained, "after we broke the kids up, we just kind of waited to see what they would do...So, while they were solving it, especially the kids that had to act, they had to do some crazy things, and get up and move around, and I mean, it was fun to watch them trying to solve it using different strategies." The kids who were acting asked if they were allowed to use props, Mrs. Yellow told them they could. The group went out into the hall and used the lockers, acting as if they were the students in the problem. The students who could write did not work with each other very much. Instead they spent the time working on the problem individually. The students who could talk showed difficulty expressing themselves mathematically without being able to write or use gestures. A few times Mrs. Green had to tell them to sit on their hands so that they would not gesture. By the end of class none of the students had made much progress on solving the problem. Before the end of class Mrs. Yellow wrote the following on the board and ask the students to answer them in their notebooks.

1) The benefit of using only talking, writing, or acting was…
2) The challenge to using only talking, writing, or acting was…
3) What strategy (or strategies) worked best for you? Why?

She then explained that they would be talking about the communication aspect of this task the following day.

The following day the teachers and students sat in a circle in the middle of the room and discussed their answers to the above questions. Students said things like:

- "With acting we could use the lockers so we could see it on a smaller scale"
- "With talking you could know or hear what others were thinking"
- "Talking is the most natural way for us to communicate with each other so that was easier"
- "With the writing it went slower so you could think about it more before you shared"
- "When we were writing things down you could go back and look at what you did"
- "With talking you had to keep it all in your head"
- "When we were talking it was like you had to do two things at once, you had to think about what you want to say, and then think about how to communicate it"

To this last comment Mrs. Yellow said, "So, it was like you had two thinking processes, first you had to think about the problem, then you had to think about how to communicate it". Students from all of the different groups explained that this was part of what they struggled with too.

Mrs. Yellow and Mrs. Green had introduced the Locker Problem as a way for students to think about mathematical *communication*. Then, as Mrs. Yellow explained, "it sort of took on a life of its own, and I went, oh, we weren't going to do factoring until probably December. But they were getting it." Mrs. Yellow's pedagogical choice to continue to work on factors was based on her knowledge of the students' abilities to reason about The Locker Problem and other related problems on factoring. As she explained "the enthusiasm and focus of the students, kind of, had us [Mrs. Yellow and Mrs. Green] keep going to continue to work on the Locker Problem and continued to work with factors."

Examples – Empirical Proofs

After the classroom discussion about communication Mrs. Yellow told the students to get into pairs and asked them what they thought they should be doing as they worked together. A few students answered saying, "ask how they solved it," "ask what they were thinking," "ask questions when I don't get something." Mrs. Yellow *revoiced* their responses, " So, my job when listening to someone who is explaining their thinking is to ask questions, learn from them, and also help them." Her *revoicing* focused on *explaining* and *communicating* thinking processes as they are used during the process of proving. Working together to ask and answer questions related to the thinking or reasoning behind a problem engaged these students in the processes of *persuading* and *ascertaining* in relation to their conjectures and solutions to the Locker Problem.

After students had started working in their pairs, Mrs. Yellow and Mrs. Green walked around and talked with students. Here are a few of the conversations Mrs. Yellow had with students, along with some of her reactions to these conversations during interviews.

Tom was working on figuring out whether locker 17 was going to be open or closed. Tom explained to his partner that he thought 17 would be closed because it was a prime number, "since 17 is prime, only the first and 17th person will touch it, so it is closed". The other student nodded her head and Mrs. Yellow bent down next to Tom and said, "great, can you try what you just said with other numbers?" Tom squinted his eyebrows and said, "Yeah, I think so". Mrs. Yellow then asked him, "do you know what a conjecture is?" Tom replied "yes", and Mrs. Yellow said, "Okay, then can you make a conjecture about what you just said about prime numbers?" Tom nodded his head and started writing a general statement; *numbers that have two factors like prime numbers will be closed.*

Later Mrs. Yellow stopped and talked with Randy. Randy had figured out the first few lockers that would end up open (1, 4, 9, and 16). He found an addition pattern, "the number of lockers in between each of the open lockers is the next odd number." Mrs. Yellow asked, "So can you predict what locker will be open next?" Randy looked at his paper and then back up and said, "25." Mrs. Yellow nodded her head, "OK, can you test it?" Randy nodded his head again. Mrs. Yellow said, "OK, can you prove it to me?" Randy explained that, "see I figured out that 1, 4, 9, and 16 would stay open. If you add 3 to 1 you get 4, if you add 5 to 4 your get 9, if you add 7 to 9 you get 16. The next odd number is 9 so if I add 9 to 16, I will get 25". Mrs. Yellow said, "okay you have convinced me that your pattern is working." These students, Tom and Randy demonstrated the use of inductive reasoning to find patterns. They also demonstrated *Empirical Proofs* as they worked to verify their generalized patterns using specific examples. During their conversations with Mrs. Yellow these students were engaging in the process of *persuading* and demonstrated the use of proofs as a function of *explanation* and *communication*.

Supporting Evidence

During our second interview I showed Mrs. Yellow my fieldnotes of her conversation with Randy. I had underlined *predict*, *test*, *prove*, and *convince*. I asked her if she saw any connections between this episode and the work of a mathematician. Mrs. Yellow explained:

> They [mathematicians] start looking for a pattern, and then from patterns, they're going to say, "OK, what do I notice from the pattern? Can they predict what's next?"...That is what they do.

She went on to explain some of her thoughts about her conversation with Randy:

> I'm not surprised by this conversation. I would say it probably happens more often. The thing that surprises me is that I wasn't consciously going through that. That wasn't a conscious decision I was making, it sort of felt natural to me...That just sort of happened, but the conversation doesn't surprise me, asking kids to predict, asking kids to test, asking them to prove something, I feel like we do that all the time.

Mrs. Yellow's belief that this is common practice in her classroom is something I found during my observations. Her statement of this practice being unconscious will be discussed in Part II of this chapter.

Example – Justification, dismissal and modification of conjectures

Randy was not the only student to focus on this addition pattern. After some of the students in one of the groups shared the addition pattern one student, Sarah, who had originally believed she had found the pattern by using perfect squares started to question if the addition pattern would work, too. Sarah had tried out every term up to 1000 and found that this pattern gave all the perfect squares, thus *ascertaining* for herself that this pattern also worked using an *Empirical Proof.* Mrs. Yellow *coached* the group, asking which pattern would be the best to use as a way to determine if a higher numbered locker would be open or closed. This is considered to be an action of *coaching* because of how Mrs. Yellow encouraged the students to develop an argument about the usefulness of their solutions (Brown and Renshaw, 2000; Martin et al., 2005). The students thought about the number 961. As the group's conversation continued, Mrs. Yellow's *coaching* led the students to *prove* that the locker numbered 961 would be open and discovery that the pattern involving factors was more useful. Mrs. Yellow asked the students who had found the addition pattern if this locker was open or closed. They all agreed that it was open. Mrs. Yellow asked them to explain why and Sarah showed her the chart of each term. Mrs. Yellow said, "So I have to know about all the terms before, in order to know about this term?" Sarah nodded her head. One of the students who had found the pattern using perfect squares said, "you don't need that with this pattern, you only need to know about that number." The group agreed it would be easier to figure out whether a number was a perfect square. During this time those students who had found the addition pattern *modified their original conjectures.* Through the use of *persuading* with this one example (locker # 961), the teachers and students engaged in *the process of proving* and led some students to engage in *justifying and modifying* their conjectures as functions of this process of proving.

Example – Transformational Proof

During a whole class discussion at the beginning of the third week of classes Mrs. Yellow asked students to share their thoughts on the locker problem. Matt yelled out, "The lockers that were open were square numbers". Amy said, "yeah, because all the factors of those numbers are prime." Mrs. Yellow asked Amy to explain what she meant. During her explanation Amy talked about square numbers having an odd number of factors. Mrs. Yellow pointed out that she previously said that square numbers had a prime number of factors. John jumped in and said to Amy, "you meant that all square numbers have an odd number of factors." Amy agreed, "Yeah, that's what I meant to say." Amy went on to explain why it mattered that square numbers had an odd number of factors. "Since there are an odd number of factors only that many people will touch the lockers and so it will be open because there is not another factor or person to come back and reclose it." Mrs. Yellow praised her saying, "Good, Amy you are adding the why to the solution." The focus here is not only finding the solution but also *explaining* why this solution works as a function of the process of proving. Though the use of *generality*, *logical inference*, and her level of explanation, Amy exhibited a *Transformational Proof Scheme* in her solution to the Locker Problem.

Example – Ascertaining and discovery

Other activities were going on throughout the three weeks that The Locker Problem was being explored including mental math activities, discussions on how to read mathematical text, playing The Factor Game on laptops (*NCTM Illumination* website), and individual conferences.

As part of their classroom practices, Mrs. Yellow and Mrs. Green have each student meet individually with one of them every two weeks. The following is part of a conversation that took place between Mrs. Green and one of her students, Diane, during an individual conference. Mrs. Green and Diane went through the different activities and tasks that had been assigned over the last few weeks. This conversation picks up when they were looking over one of the assignments that had been due before the whole class discussion presented above. Mrs. Green saw that Diane had not completely solved the problem and asked her, "Do you think you could have figured out The Locker Problem on your own if you had had more time?" Diane explained, "I could have gotten there, I had realized that it had to be odd. And now I know that it has to be a square number for it to have an odd number of factors." Diane flipped through her notebook to one of the extension problems done later in the week. "This is my revelation homework. Doing this I had an epiphany and was like, I totally get this!" She went on to explain that the solution to The Locker Problem was "clear in [her] mind", because she had seen, during a homework problem focused on factor trees why square numbers have an odd number of factors and that no other numbers would. This episode shows Diane working to clarify the mathematical concepts that were used by other students to create and justify conjectures about the Locker Problem. This demonstrates Diane engaging in the process of proving, since she is discovering, for herself, the facts used by others in their justifications. By doing this Diane had now extended her understanding of why the square numbered lockers remain open while the others end up closed, and she is thus engaged in the process of proving through *ascertaining*.

Inquiry

The use of The Locker Problem by Mrs. Yellow and Mrs. Green is a great example of the type of *inquiry* associated with the process of proving. Although the problem is based in a "real world" context, soon after the problem was given the lockers became irrelevant and all of the *inquiry* done by the students was based on mathematical concepts, namely the concepts of factors and perfect squares. The students' conjectures were based on the properties of the numbers and they began proving by playing with the numbers not with the idea of a locker. In this way the students were using mathematics to inquire about mathematics, which, as discussed in chapter 2, is the type of inquiry that is a function of proofs and the process of proving.

Concluding Remarks

The Locker Problem was introduced as a way for students to think about *communicating* their thinking and thus the students talked about one of the major functions of proofs. Throughout the work that students did involving The Locker Problem they were *making conjectures*, *explaining* and *justifying* their conjectures to other students and their teachers, and sometimes *modifying* their original conjecture. During this process students partook in both *ascertaining* and *persuading*. The students demonstrated *Empirical Proofs* through the use of charts and specific examples and deductive proofs that indicate a *Transformational Proof Scheme* through the use of *generality* and *logical inference* as they found and verified generalized patterns. Mrs. Yellow and Mrs. Green's decision to include these types of activities will be discussed further at the end of the *Explaining Reasoning* example.

Explaining Reasoning

The following example is based on the work of two students done on a homework assignment that was completed a few weeks after the completion of the in-class work focused on the Locker Problem. At this point the class had discussed the solution to the Locker Problem and proofs, such as the one given by Amy above, had been presented. In this exercise the students were asked to think back to the Locker Problem and explain if the locker numbered 144 would be open or closed. During our second interview, I asked Mrs. Yellow to discuss each of the two students' solutions to problem number 5, *Thinking about the locker problem, would locker 144 be open or closed. Explain your reasoning*. A copy of the homework sheet and students' work can be found in Appendix G. Asking students to explain their reasoning requested them to engage in the process of proving and particularly the process of *persuading*.

While discussing Theresa's answer,

> *Locker #144 would be open. I know this because all lockers with an odd number of factors are open. Prime factors are closed because they have an even number of factors.*

I asked Mrs. Yellow if she would consider Theresa's answer to be a proof. She explained that,

> Yeah, it's, I would say partly, and it's hard because I am looking also at the student. I would say, it's partly a proof. And why I would say it is because we discussed it, and so this particular student, I think would know, OK, what I'm supposed to remember from this problem is this. And she would put that into her memory. But then, had she factored 144 and then wrote out the prime factorization to figure out that there were an odd number of factors, So, I think she is almost getting it. But I probably would have wanted her to say more why there, why an even number of factors is closed.

I then asked Mrs. Yellow if the problem had been restated to say, "thinking about the locker problem, can you prove the locker number 144 would be open, do you think that this would constitute what would be defined as a proof, in your classroom?" She thought a moment and said, "It would probably be pretty close to a proof, I think." This conversation reflects Mrs. Yellow's expectation for *explanations* to focus on "the whys" and the use of proofs as an *explanation* of reasoning. It also shows how Mrs. Yellow valued the student's knowledge that all she needs is to figure out is if 144 has an odd or even number of factors and her ability to factor out 144 to see its prime factorization. When Mrs. Yellow says that the student "is almost getting it" and that she would like to see "more why" she is acknowledging that this answer misses the fact that the student did not connect 144 with being a perfect square, which is how we know it has an odd number of factors. In Theresa's answer we can see her working to *persuade* her teachers that locker 144 would be open. However, because of an incomplete proof of this solution, Mrs. Yellow was not fully satisfied with Theresa's solution. Further discussion about Mrs. Yellow's uncertainty about what would be considered a proof, her comments of "I think" and "partly a proof," will be discussed in Part II of this chapter.

The next student, Kathryn, states that she is giving two reasons in her answer to number 5:

Locker number 144 would be open. The reason for that is because every locker that's open has/is 2 things. It's a square number and it has an odd number of factors. Another reason is because only three people touched the locker [since] the factors are 1, 12, and 144. So the first student opens the locker, the second student closes it and then the third student opens it again.

I was curious about Mrs. Yellow's reaction to Kathryn's answer and asked what she thought about the student's use of two reasons. She explained that this student is "very much an overachiever, she wants to make sure she has all of her bases covered." I asked Mrs. Yellow if she thought the student covered all of her bases in the first part of her explanation. Mrs. Yellow said:

> I don't know. Possibly, if we had modeled, OK here are questions, what's a sufficient answer for this question. But I also think that our students are always asked to defend and explain, in their thinking and their reasoning, and sometimes, like on the Problem Solver rubric we ask them to try more than one strategy.

I then explained that my curiosity about the extra explanation came from research concerning students not trusting deductive reasoning and that they tended to rely instead on examples (Healy & Hoyles, 2000; Knuth, Slaughter, Choppin, & Sutherland, 2000; Knuth & Sutherland, 2004). Mrs. Yellow connected her students' use of examples with the classroom practices and school expectations.

> That's interesting that you said that, because the other thing is, we've talked about, sometimes it's helpful to give an example to back up what you're saying. But I don't know if so much is, it's another reason, rather than, here's an example that shows it. You know, she probably sees it as another reason, because it's another explanation.

Mrs. Yellow's insight into her student's thought process clarified her use of two explanations as one that was expected of her, not necessarily one she needed to reason about the problem. Moreover, the problem does in fact ask the students to explain why locker 144 would be open or closed, and thus, although Kathryn stated that she was providing two reasons, her "second reason" could also be seen as an connection between her general statement and the particular problem she was being asked to solve. One piece of Kathryn's solution that is missing is a statement about the connection that all square numbers have an odd number of factors. It seems as though she might be stating these as two separate properties of the lockers that will remain open. Of course, more investigation with the student would be needed to know for sure what her thoughts were in regards to this answer. Considering any of the possible reasons for Kathryn's decision to give two reasons in her solution, it is clear that her goal was to in fact prove her solution by giving enough explanation to *persuade* her teacher (or the reader) that locker 144 would be open.

Supporting Evidence

Mrs. Yellow and Mrs. Green both explained during interviews that they included open-ended problems and questions that focused on reasoning because of their desire for students to understand the mathematical concepts they were learning on a deep and meaningful level. During the second interview, when I asked Mrs. Yellow about using extension problems like the one above, she explained that both the teachers and students became invested in the problem and so they looked for extension problems like number 5.

> There's a lot of math here [in the Locker Problem], so then for us [Mrs. Yellow and Mrs. Green], it was sitting down and figuring out "here's all the things that you can get out of it." So we looked back at where we took the problem from and they had these extension problems. We looked and said, "OK, we've got a group of kids who know what the solution is. We have a group of kids who still don't know how to figure out if the locker is open or closed. We need to have opportunities for the kids that get it to push their thinking a little bit, and then we need to get the other kids to at least understand factors.

I asked Mrs. Yellow if the idea of extension problems was similar to the practice of stretching I had seen in earlier grades. She said that it was the same idea only she doesn't "use the word stretching as much as [she] use[s] extension." I asked her if she thought it was important to ask them to do that and she said, "Absolutely… because it sort of goes back to the thinking about thinking" which she had discussed earlier in the interview as

> kids develop[ing] their skills, if they're able to ask questions, if they're able to think about their thinking, it means they're thinking deeply about something, instead of just doing it. And so, if you're thinking about your thinking, or, you know, they're going through a problem and they're actually asking themselves questions, you could be making connections, you could be inferring things.

Mrs. Yellow's connection between extension problems, stretching, and students thinking about their thinking establishes the fact that her pedagogical choice to use this type of problem was as a means to engage students in thinking deeply about the mathematics they were learning.

In the answers provided by Theresa and Kathryn we see that they are *explaining* their reasons and working with *logical inferences* to *communicate* their reasoning. In this way the students demonstrated a use of *Transformational Proofs Schemes.*

During our first interview, Mrs. Green said that one of the important aspects of using an activity as they did with the Locker Problem, was that students would make conjectures. I asked her what that would look like in her classroom, "what would you see in your classroom when students are making conjectures?" Mrs. Green said, "lots of noise. I hope kids are arguing with each other. I love it when they argue about math, you know what I mean? About a new problem, or when they are working together." I asked her what she expected students to get out of that experience and she explained

> learning how to ask questions, I think, is a huge one. Because that can apply to anything. I mean, not only just school. Problem solving, that goes right along with answering questions. Being independent, because they can work together, they can, you know, I guess, independently ask the questions, but they can realize, themselves, I'm not understanding this, and I see that as a form of being independent. Realizing that they need to get help. You know, and all those things apply, across the board, not just school.

Mrs. Green's explanation of making conjectures and engaging students in mathematical arguments to foster questioning, problem solving, and independent learning establishes her pedagogical choice to include activities like those associated with the Locker Problem as one based on her beliefs that students should learn mathematics by engaging in these practices.

Concluding Remarks

This example shows how students are given opportunities to work with deductive reasoning, and are asked to explain their reasoning. Using these types of activities Mrs. Yellow and Mrs. Green include proofs and the process of proving by engaging students in *verification, explanation,* and *justification, dismissal or modification of a conjecture.* Students engaged in the process of proving and specifically the process of *persuading.* The students demonstrated *Transformational Proof Schemes* through the use of *generality* and *logical inference*. Mrs. Yellow and Mrs. Green's pedagogical choice to include these types of activities are similar and rely mainly on their desire for students to learn mathematics on a deep level by fostering students' questioning abilities, problem solving strategies, and their ability to think about their thinking.

The inclusion of proofs and the process of proving

One of the purposes of Part I was to provide the above examples and analysis in order to show what ways the teachers were using proofs and the process of proving in their classrooms. In the example of *Students as Teachers,* from the fifth grade classroom, students were engaging in *verification*, *explanation*, and *communication* through the process of proving. Using mathematical arguments, students were both *persuading* and *ascertaining* in relation to methods for multiplying fractions.

In the example of *If –Then Statements,* from the sixth grade classroom, students *explored definitions*, and used them to *justify* mathematical claims about relationships between recursive and explicit expressions. During this *justification* process students were engaged in deductive reasoning and exhibited the use of *Transformational Proofs Schemes* through *logical inference* and *generality.* Students also engaged in the process of proving by working with statements accepted by the classroom community, as described in *Stylianides' Definition of Proof.* The example of *Building Patterns to Discover Equations* illustrates students making and *justifying* mathematical claims about generalized equations they created based on observed patterns. Students used inductive reasoning to find patterns and then created a general equation to fit their data. The demands placed on *justification*, *explanation*, and *discovery* during all Problem Solvers also shows this activity's relation to the process of proving. Students exhibited their use of *Transformational Proof Schemes* by *generalizing* and *justifying* that an equation would work for any number of adults and 2 children. Their work was first to *ascertain* certainty for themselves and then to create a convincing argument to *persuade* their teacher that their answer and general equations were valid.

In the seventh grade classroom example titled *Investigation*, students engaged in the process of proving through *discovery*, *exploration of definitions*, and deductive reasoning, through an investigation of histograms. Students came up with questions and during the process of answering these questions engaged in the processes of conjecturing and *ascertaining* in relation to their conjectures. One of the students had stated a fact for the other students and through the uses of multiple examples, the teacher *persuaded* the students of it truth. Later in this example another student cites the definition, "Jon's rule," as a way to justify her answer. In the example *Argue Your Answer Mathematically*, the expectation for students to *justify* and *explain* their choice of best golfer by using mathematically sound arguments demonstrates the use of proofs and the process of proving. These mathematical arguments show the teacher's understanding of the importance of *persuading*, using mathematics to remove doubt.

In the eighth grade classroom example titled, *Using Mathematics to Inquire about Mathematics,* The Locker Problem was used to give students the opportunity to make conjectures and to *explain* and *justify* their conjectures to other students and their teachers. The use of this problem is a great example of the type of *inquiry* associated with the process of proving since the *inquiry* is done by thinking about mathematical concepts such as factors and square numbers. The activities had students thinking about *communication*. Students engaged in the process of proving through both *ascertaining* and *persuading* in relation to their conjectures and solutions to the Locker Problem. The students demonstrated *Empirical Proofs* through the use of charts and specific examples and *Transformational Proof Schemes* through the use of *generality* and *logical inference*. The other example associated with the Locker Problem, *Explaining Reasoning*, also demonstrates the actions just described.

Although other examples could have been given, for the purposes of this study I presented a limited, yet demonstrative, number of examples. Since the focus of this study is not solely on the presence of proofs and the process of proving, the time spent presenting such activities was contained to the above material. From these examples and episodes, the case has been made that each teacher, and the participants as a group, have a strong presence of proofs and the process of proving in their classrooms.

Teachers' decisions about including the use of proof or the process of proving

The second purpose of Part I was to explore how teachers decided to include the use of proofs and the process of proving into their classroom practices. Each of the teachers explained and demonstrated different reasons for including the above activities.

Mrs. White had students in her classroom become teachers in order to help them make their "thinking understandable to others." She explained that the decision to use this activity with her students came from her belief that you really need to know something to be able to teach it and that sometimes children learn best from other children.

Ms. Blue explained that her decision to use informal if – then statements was unintentional. However, she also discussed how she used this practice because she values it as a way for her students to make connections between what they are learning and their past knowledge. Although Ms. Blue's decision to use the Problem Solver may come from the school's structure, she also talks about her decision to use this type of activity because she wants her students "to be able to understand the whys behind the math" they are learning and work with "open-ended problems" that can be "solved using different strategies."

Mrs. Red engaged her students in investigations based on her philosophy of having students partake in mathematics by formulating and answering questions, particularly the questions, "what do you know?" and "what do you need to know?" Her decision to include the Problem Solver may have been due to the school's structure. However, she describes her expectation for students to "argue your answer mathematically" as one based on her belief that students should learn to be able to "justify where they're getting their justification."

Mrs. Yellow and Mrs. Green decided to use the Locker Problem to initiate a conversation about communication. One of the reasons these teachers decided to extend the mathematical concepts further into factoring was because of the students' "focus" and "enthusiasm." They included open-ended problems and questions that focused on reasoning because of their desire for students to understand the mathematical concepts they were learning on a deep and meaningful level. For Mrs. Yellow her explanation for this decision was mainly focused on students "explaining the ways." Mrs. Green's explanation was focused mainly on the use of "conjectures" and students' "arguing about math."

Each of these teachers focused on their beliefs about teaching and learning when discussing their reasons for including these activities. In the following part of this chapter the teachers' knowledge and beliefs about proofs and the process of proving will be explored.

PART II: KNOWLEDGE, BELIEFS, AND MAKING SENSE

Part II will address the teachers' *beliefs* and *knowledge* about proofs and the process of proving. It will also begin to address how the teachers make sense of past knowledge and their current practices. I will explore answers to the second topical question, *How do teachers think about or understand proofs and the process of proving?* The focus of *knowledge* and *beliefs* will be related to the discipline of mathematics, their own mathematical education, and their teaching practices, as described in Chapter 2. The data used for this part of the chapter came from the interviews conducted with the teachers. Recall that, as stated in Chapter 3, classroom events and students' work were used during these interviews and before the final interview, each teacher read the NCTM Reasoning and Proof Standard (2000) for their respected grade level as well as *Stylianides' Definition of Proof.* An analysis of each individual teacher will be presented first, followed by an analysis of the group of participants. Where appropriate, the teachers' judgments of students' proofs and the teachers' demonstrated fluencies or difficulties with using proofs during classroom activities will be explored. For each teacher two or three major points of investigation will be presented. These were chosen due to quality of data and the connection to *knowledge* and *beliefs* as framed in Chapter 2. The major themes discussed throughout Part II include:

- The teachers hold dual understandings of proof: one related to their own educational experiences and one related to the role of proofs and the process of proving used in their classrooms.

- The teachers had negative experiences with proofs in their higher-level mathematics courses in college.
- When thinking about proofs related to their own educational experience, the teachers demonstrated discomfort, fear, and dislike for the proofs and the process of proving.
- When thinking about proofs related to their students' learning of mathematics, the teachers value the use of proofs and the process of proving as part of their classroom practices.
- When thinking about proofs related to their students' learning of mathematics, the teachers believe that proofs and the process of proving relate to their own teaching styles, philosophies, and practices.
- The teachers' knowledge of proofs varies and included aspects of *Informal Argumentation*, *Empirical-inductive Proofs*, and *Transformational Proofs.*
- The teachers believe that *verification, explanation, communication, discovery, exploration of definitions, generalizations, inquiry, and the justification, dismissal or modification of a conjecture* serve as functions of proofs and the process of proving.

Mrs. White

In the following analysis of Mrs. White's *knowledge* and *beliefs* about proofs and the process of proving, three topics will be explored. First, Mrs. White's belief that she did not engage in proofs or the process of proving as part of her own education will be discussed. Second, Mrs. White's beliefs concerning the use of examples as a means of proving in her classroom will be analyzed along with a discussion about classroom practices utilized by Mrs. White that engaged her students in the process of proving using examples and reasoning beyond that of using examples. Third, the values Mrs. White placed on proofs and the process of proving during our third interview will be explored.

Mrs. White's experiences as a student do not seem to have afforded her the opportunities to work with proofs or the process of proving. Although she values parts of her education, believing that she was a "good math student," she also recognizes flaws in learning in a "rote way," explaining that she never understood the applications or concepts, particularly those learned in calculus.

Mrs. White believed herself to be "a good math student" despite her lack of conceptual understanding. During our first interview I asked Mrs. White about the practice of using students as teachers. She explained how this practice was different from her own experiences as a mathematics student "when you're just given a formula and told how to use it." She went on to say that

> For me, I always learned math in a rote way, I was a person in Calculus, who did the 17 practice AP tests, so when I got to the real test, I mean, I had no idea what calculus was about, but I knew how to do every single kind of those problems, because somebody taught me how to do the problems...I got a five on the AP test, but I didn't understand any of it.

During our final interview Mrs. White mentioned the need for students to understand the applications about the mathematics they were learning. I asked her if she thought this was a big factor in her choices for using certain teaching strategies. She revisited her own education and explained,

> When I was in school, nobody asked me to think. Well, they just told me the formula, and they said, here, plug in the numbers. I did very well in math, all through high school, and even through the beginning of college, when I took it, and I couldn't tell you how you ever applied it. Calculus was more about plug in

> the numbers and get the derivative, and you do it, there. I didn't know what it was for. I can't say really now what it is for. But I know how to put in the numbers, if somebody shows me the formula.

Mrs. White's accounts of her mathematics education suggest that her *knowledge* about proofs and the process of proving through her own mathematics education is minimal at best. However, her discussion about the difference between her own education and the one she is working to provide for her students suggests that she *believes* that a lack of proofs and the process of proving was detrimental to her understanding of mathematical concepts and their applications.

Mrs. White believes in using examples as a means of proving in her classroom. During our first interview, I asked Mrs. White about a classroom episode where the students were using divisibility rules. Namely, they were figuring out whether a number was divisible by 3 by adding up the digits and figuring out if that number was divisible by 3. I asked Mrs. White if prior to my observations she had shown her students why this rule worked. She explained that they had worked with examples to check the rule:

> Yeah, so, if they had a number that was, like, 363, you would know, all right, you could just say, is this divisible by three? Oh, it should be, because it works. Well, check it. And then you could check it. We've done checking things, and then they've used a calculator, sometimes, to check it... I gave them really big numbers, they were, like, eight- or ten-digit numbers and had them test it. Or, there were bigger numbers, and they were trying to write the prime factorization, but using the rules of divisibility to figure out, what should I start dividing by, kind of thing. And that was our purpose for it. So, to look at bigger numbers, and see if we could find the prime factorization for it, by not having to guess every single number. But it is just a rule that they memorize.

This example demonstrates Mrs. White engaging her students in *Empirical-inductive Proofs*. I asked Mrs. White if she felt it was okay to give students a rule before they understand why it works. She told me,

> Sometimes, I feel, I feel OK. I feel like, some things, you need to just give them the tricks, because, hopefully, they're going to stick with the math, and they're going to get exposed to the proof, later. And, I think that you can do it, on a really simple level, by just giving them example problems, and having them try it.

Mrs. White's mention of her students "get[ting] exposed to the proof later" demonstrates her *knowledge* of examples not constituting a mathematical proof. However, Mrs. White *believes* that her students' abilities to reason about mathematics, at least this level of mathematics, is through the use of examples. She explained that the only way she can work with her students using proofs is by using examples. "I feel like the only way I do it with them for the proofs is to practice it [work through multiple examples]."

During our third interview, Mrs. White discussed classroom activities she believed to illustrate the types of reasoning and proofs described in the NCTM Standard. One of the activities she discussed was the Students as Teachers (analyzed in Part I). The second activity she discussed involved students becoming a court judge.

> Sometimes, we've done problems where we'll have, like, just a multiplication, like a double-digit by a double-digit. And there'll be three different answers, and you [one of the students] have to be the court judge, and you have to say why the one is correct over the others. But you have to prove why that one is correct and why the others are wrong. Like, you can't just say, this one's right and this one's wrong. But you have to be able to explain why.

The Students as Teachers activity and this activity with the acting out of judgments made by a court judge show Mrs. White engaging her students in the process of *persuading* and *ascertaining* beyond that of proof by example, and also highlights her *belief* that proving is associated with "be[ing] able to explain why."

Mrs. White's use of examples as a means to prove certain mathematical facts, and her discussion about the need to use examples as the only way her students can engage in proofs of these facts, suggests that she *believes* that *Empirical-inductive Proofs* should play a role in her students' education. Her discussion about students being "exposed to the proof later" suggests that her own *knowledge* of proofs includes an understanding that examples do not constitute a mathematical proof. Moreover, particular activities utilized in this classroom suggest that she believes that in certain circumstances her students have the ability to engage in the process of proving through *ascertaining* and *persuading*, using forms of reasoning which go beyond that of proof by example. Her association of these activities and proof demonstrate her *belief* that proofs and the process of proving function as a means of *explanation* and *justification.*

During our third interview Mrs. White discussed the use of proofs and the process of proving she was already using in her classroom (e. g. the courtroom judge example above) and how she wanted to incorporate more proofs into her classroom practices. After reading the NCTM grades 3-5 Reasoning and Proof Standard and *Stylianides' Definition of Proof*, Mrs. White explained

> I feel like I wanted it [proofs] to be more stressed. Like, in reading it [NCTM grades 3-5 Reasoning and Proof Standard], I realized, oh, well, sometimes the language is too fed to them, or sometimes there's not enough time for exploration, I guess. And I worry, because I feel like they're [her students] pretty good at math, but when I give them the actual Problem Solvers, they're bad. Like, they get stuck. They don't know where to start. And if you give them a strategy, then they can figure it out. But I'm not sure how to make them, or how to allow them to discover the strategies more on their own, in a given time period. Because you're like, the problem's due. Do the problem. And you kind of know that they know it, but they don't come up with the strategies. So, that's been frustrating to me, there's only a few kids who didn't need [the strategy] fed to them. So, I wrote on there (pointing to the first page of the NCTM grades 3-5 Reasoning and Proof Standard), "with support, with support, with support."

Mrs. White seemed to be focused on the amount of time students would need to work through a problem on their own and her own pedagogical content knowledge. Her comment about her own abilities in using teaching strategies related to students working with the process of proving as a function of *discovery*, "I'm not sure how to make them, or how to allow them to discover the strategies more on their own, in a given time period," suggests that she either does not have or is uncomfortable with utilizing knowledge related to these types of teaching practices. However, she also mentioned that she wanted to start having more proof related events in her classroom a few times during this final interview. Her comments included: "I definitely feel I could do more with the why, and trying to get them [the students] to come through on it"; "I will infiltrate all this proof stuff"; and a comment about her future lesson planning

> I think what I'm taking from this conversation is that that's an area that I could just focus on a little bit, when I'm thinking about planning my lessons, or thinking about how do I introduce the why? Or, where is the why going to come in this span of things that I'm going to do? Where is the why introduced, or how can I have them respond to it in a why way.

These comments suggest that Mrs. White *believes* the use of proofs and the process of proving are useful teaching and learning tools.

In summary, Mrs. White's accounts of her mathematics education suggest that her *knowledge* about proofs and the process of proving through her own mathematics education is minimal at best and that she *believes* that the lack of proofs was detrimental to her understanding of mathematical concepts and their applications. Mrs. White *believes* that using *Empirical-inductive Proofs* is appropriate for her students, while her own *knowledge* of proofs includes an understanding that examples do not constitute a mathematical proof. Mrs. White engages her students in activities that utilize proofs and the process of proving beyond that of proof by example and *believes* that she should incorporate more proofs into her classroom practices.

Ms. Blue

In the following analysis of Ms. Blue's *knowledge* and *beliefs* about proofs and the process of proving, two topics will be explored. First, Ms. Blue's *beliefs* about her mathematics education will be explored along with a discussion about how she *believes* this affects her classroom practices related to proofs and the process of proving. Second, Ms. Blue's *belief* that she uses proof related activities in her class unintentionally will be discussed.

Ms. Blue *believes* that her mathematics education does little to help her in her profession as a sixth grade mathematics teacher. During our first interview, I asked Ms. Blue if there were any educational experiences she felt influenced her teaching practices. She explained

> I really think that the Education classes helped so much more than the math classes. Because even the Math Ed ones, I felt like I got in here, and the things that I wish I had learned, like just the different ways of teaching multiplication…or partial products, I didn't even know, which is so simple, but it would have been great to have learned that before I came here…I felt like there were a lot of those type of things, or, you know, in Middle School Math, it would have been great to see how to multiply and divide fractions using pictures, and that sort of thing… a lot of the advanced math classes, I mean, I can't even tell you the course titles right now, never mind what I learned in them. Like, I understand that they want you to take higher math than your students, which I completely understand, but it just seemed like it was way too much. I didn't think that, for me, personally, those classes weren't helpful.

This discussion shows how little Ms. Blue valued her higher mathematics education, which includes a number of proof based courses. During our second interview I was interested in who she believed to be "they" in her statement that, "I understand that they want you to take higher math than your students." I shared a copy of this part of our first interview and asked her who she thought the "they" was. She explained,

> I think, in society, as a whole, it's a good idea for your teachers, in general, to know more than the specific subject you're teaching, which, I mean, more than

> the specific area you're teaching. Which, I think, in theory, is a good idea, and even practical, like, it's good for me to know what happens, even, in this school, in seventh and eighth grade… I understand that. It's just that, I think that it was too broad, I mean, like, all of middle school and secondary education had to do so much more that I really never really use.

Ms. Blue's response suggests that she values learning about the mathematics taught in grades above the one she is teaching but that learning the mathematics taught in higher level courses has not been useful to her. Ms. Blue continued and returned to the question who is "they."

> I would say, people, like, figuring out the math education system… And it just seems like it's pretty, like, whoever they are, are pretty detached from the math education students, and teachers.

This statement suggests that Ms. Blue *believes* that the *knowledge* she gained in her own education is detached from the *knowledge* she uses as a teacher and the *knowledge* her students need her to have. The statement also continues to suggest how little she values her mathematics education. Thinking about this during our interview I asked Ms. Blue if there was anything she thought was helpful from her experiences of taking mathematics course in college. She thought for a moment and said,

> Looking back right now, no. Like, whether or not there's things that I took from it and I didn't realize, then, maybe, but, like, nothing direct.

I asked her what about the proof course she had mentioned earlier as one of the higher-level mathematics courses she had to take. She said, "That was just a painful class… if you taught it, like, specifically geared toward the younger grades, I think it would be interesting." Ms. Blue's discussions concerning her mathematics education, including proof based courses, demonstrate her *belief*, that her mathematics education does little to help her in her profession. In fact, the only time she discussed how these courses influenced her teaching, she explained that her negative experiences with proofs prevented her from looking at them as a tool for teaching and learning. During our final interview, Ms. Blue talked about how the NCTM Standard linked proofs and conceptual learning. She then explained

> I probably intentionally don't put them [proofs and conceptual learning] together, because I think of proof as the setting it up, and I didn't like that, you know, the formalization of it, whereas the informal, just kind of doing it, the proof, the why's and that sort of thing just seems to work.

I asked Ms. Blue where she got her formal idea of proof from and she said,

> The only time I was, at least knew I was doing proof, when I think of it, was in geometry in high school, with two-column proofs, and then my college classes.

This discussion suggests that Ms. Blue holds a negative view of proofs and relates that negativity to her experiences in high school and college. Moreover, this negative *belief* about proofs prevented her from thinking about proofs as a useful tool for teaching and learning mathematics.

Although Ms. Blue did not consider herself to be using of proofs and the process of proving in her classroom, events such as those described earlier in the *If – then Statements* and *Building Patterns to Discover Equations* sections in Part I of this chapter were observed. When Ms. Blue reflected on these events and others related to proofs she explained that her use of such events in unintentional. As discussed above, during our second interview Ms. Blue explained that her use of informal if – then statements was unintentional, "I guess I don't do it intentionally, in terms of, I've never really thought about it." I asked her if she saw any connections to this practice and the use of proofs. She said, " I guess, I didn't even realize I was doing that." Her discussion about this activity suggests that she was not intentionally using it as form of proof.

During our third interview I asked Ms. Blue if she believed there were any correlations between what she had read in the NCTM Standard and her own teaching practices. She again explained that even though she saw connections to what she had read and her own teaching, that she was not using proofs intentionally in her classroom.

> I definitely saw things that I do. I don't know that I was doing it intentionally. Like, it's not always thought out, oh, I'm trying to teach them proof, or the foundations of proof, but it just makes sense to me that that, of course, they would want to know how. I guess it's just part of the philosophy of the school, too, in just, you're not going to have them just solve a problem and say, OK, that's the answer and that's it. How did you get there? Why does it work? That sort of thing. So, I think I do it, and the school does it, but not necessarily with all of this intent.

The above discussions illustrate Ms. Blue's *belief* that she uses proof related activities in her class, but not for the purpose of engaging her students in proofs or the process of proving. Ms. Blue's discussion about the school's philosophy will be discussed in Part III of this chapter.

In summary, Ms. Blue *believes* that her mathematics education does little to help her in her profession, and, in part, is responsible for her negative view of proofs, which has prevented her from thinking about proofs as a useful tool for teaching and learning mathematics. Ms. Blue *believes* that she uses proof related activities in her class, but not intentionally for the purpose of engaging her students in proofs or the process of proving.

Mrs. Red

In the following analysis of Mrs. Red's *knowledge* and *beliefs* about proofs and the process of proving, three topics will be explored. First, Mrs. Red's *beliefs* about her mathematics education will be discussed. Second, her understanding of proofs as they are used in college and with her students will be analyzed. Third, Mrs. Red's *beliefs* about the connections between proofs and her teaching style and philosophy will be explored.

Mrs. Red *believes* that her mathematics education does little to help her in her profession. During our second interview, I asked Mrs. Red to discuss the impact or helpfulness of the mathematics courses she took in college. Mrs. Blue *believes* that even though the high mathematics course she took built up her knowledge of mathematics, this knowledge has not been helpful to her.

> the other ones, I guess, my other classes were really hard math, which I've never taught, so it just, it helped build my knowledge of math, but that hasn't been that helpful.

As she continues, she explains that in her Geometry course, not only was the content not helpful to her as a teacher, but during the course itself she did not feel as though she was engaging in the process of discovery.

> none of the material have I ever needed, or has it built, like, even algebra, I wouldn't say the geometry class helped me with geometry, per se. Not the Geometry for Teachers, but Geometry in general. Geometry in general, I think, was really hard, and really way advanced than what I do now, and I don't think it helped me discover anything.

This conversation suggests that Mrs. Red *believes* that the mathematics courses have done little to help her in her profession. Moreover, she believes that if she had more opportunities where mathematics and education were linked, then she would have been more prepared to enter the teaching profession.

> I think, even when I came into teaching some of the stuff that I was going to teach, I needed to learn it on my own, because I didn't even learn it in college…I think, as far as education goes, I wish I had more education courses, or linked education and math a ton more.

During our third interview Mrs. Red explained that, "people see it [proof] as what you do in college-based math." Considering her negative beliefs about her experiences with college mathematics and her negative beliefs about proofs, which will be explored below, this connection to proofs and college mathematics suggests that for Mrs. Red, her college mathematics courses are in part responsible for her negative view of proofs.

Mrs. Red has a dual understanding of proofs; one related to her *knowledge* of proofs used on the college level, and another related to her *knowledge* of proofs that she uses with her students. During our first interview Mrs. Red had discussed using proofs with her students during their geometry unit. During our second interview I asked Mrs. Red to revisit this conversation, "In our last interview you mentioned using proofs in your geometry class, can you tell me a little about what you think proof is in your classroom?" Mrs. Red asked, "Did I talk about proof in college, you mean, or proof in my classroom?" I explained and showed her the section of our interview I was referring too. Mrs. Red explained, "because, my proof, the word proof, in college means a very different thing than proof in my classroom." She went on to talk about the kinds of activities she used with her students in geometry. For the purposes of analysis, I will first discuss Mrs. Red's *beliefs* about proofs in college and then come back to her *beliefs* about proofs in her classroom.

After Mrs. Red had explained the activities used in geometry, I asked her if she ever asked them to "prove." She scrunched up her body, holding her fists to her face and shook as she said, "proof, to me, is so, like, when you say proof, I am like ahh, ahh, ahh." After this she relaxed and continued to explain, "Because I think of college, and all of the symbols, and writing it out, and oh, it was so crazy. So, I don't think of it [proof in her classroom] as that type of proof, that on a higher level." This conversation continues to highlight her dual understanding of proofs. Furthermore, it demonstrates that she has negative feelings about the types of proof she engaged in during her college courses.

When Mrs. Red discussed proofs used in her classroom she talked about students "making discoveries," "documenting their thinking process," "explaining what they learned", and "showing their work." During one conversation about proofs in algebra, Mrs. Red explained that *proving* and *explain why* are the same thing. "if they [her students] said y equals 6, and I said prove that to me, they're going to explain why they know. So, it is the same definition." In the discussion about using proofs during their geometry unit mentioned above, Mrs. Red explained that

> During proof activities I have set up a bunch of stations and students would come in and work at a center coming up with theorems and postulates. The kids came in, and they knew they had to just get to work. And so, they had a list of, basically, the grade level expectations, but kid-friendly. And so, then, they had to do activities, and come up with those things.

This suggests that Mrs. Red associated "coming up with theorems and postulates" as part of the process of proving. In other discussions about these stations or centers Mrs. Red stated that the students were working to find and *generalize* patterns. Furthermore, she believed that *discovery* was a function of this process.

> So, when I would pull them back together, after, like, say, four days of them discovering and playing, with pattern blocks, or they had to figure out that certain shapes go together. So, when we did that, when I brought all the kids together, we would have the kids say if they came up with any ideas. And calling them theorems or postulates, do you have any ideas of how to culminate what you have discovered? Did you have any discoveries?

Having students share their discoveries as part of this process suggests that Mrs. Red believes *communication* is a function of proofs and the process of proving. She then had students *explain* their "discoveries" and prove them to her and the rest of the class, suggesting that she believes *explanation* is a function of proofs and the process of proving.

> So, then, they would have, the proof would have to be, they'd have to show how they did it. And, you know, the drawings.

This suggests that she also views *verification* as a function of proofs and the process of proving.

Considering Mrs. Red's discussion about these activities and her *belief* that they were related to proofs, this conversation also suggests that Mrs. Red believed that *inquiry*, as described in Chapter 2, is a function of proofs and the process of proving. This account of the geometry lessons, as well as other instances where Mrs. Red referred to proofs in her classroom as "making discoveries," "documenting their thinking process," "explaining what they learned", and "showing their work," suggest that Mrs. Red *believes* proofs and the process of proving are valuable in her students' education and her understanding of the functions of proofs and the process of proving include: *discovery*, *inquiry*, *explanation*, *communication*, *generalization*, and *verification*.

When thinking about proofs, as she understands them in her classroom, Mrs. Red *believes* that there are connections between proofs and her teaching style and philosophy. During our first interview, Mrs. Red described her teaching style as 'discovery-based" and explained, " I like more discovery, I like kids to get into it more." I asked what she meant by "discovery-based" and she explained that it would be easier to describe what she does for her geometry unit. This was the first time that she had talked about the geometry unit. Her description highlights her belief that her "discovery-based" teaching style is connected to proofs and the process of proving. During this part of the interview Mrs. Red explained that

> the way I did it was, I had out nine different activities or centers that the kids needed to do. And it was based on that I want them to discover that, you know, there's 360 degrees in the circle. So, I may have like a guiding question for them to do, and then they have to play to discover that...the more and more that they played, the more and more they would come up with stuff, and then I'd have them write theorems, and then I'd put them all on a board. Like, so-and-so thinks this, you know? And like, yeah, can we prove it? Can we come up with proofs? And then we work on proofs.

The rest of this conversation was very similar to the one from our second interview discussed above. Here, Mrs. Red's connection to activities that she considers to be discovery-based and her mention of the students presenting and sharing proofs suggests that she connects proofs with her discovery-based teaching style. Moreover, during our second interview she discussed these activities as "proof activities."

In the *Investigation* section of Part I of this chapter, I discussed how Mrs. Red stresses that her students the ask questions "what do you know?" and "what do you need to know?" and how she explained that, "they almost become your mantra; they almost become your philosophy." She referred to these questions as her philosophy a few times during observations and interviews. Mrs. Red connected these questions with the way NCTM describes reasoning and proof. During our third interview, I asked Mrs. Red if she had any reactions to the NCTM Standard. She explained,

> I thought it hit what we do, K through 8, here at this school, as far as helping the kids to develop their own understanding of something that they're learning... I was trying to make the connection to the questions "what do you know?" and "what do you need to know?", and that. And I can see that connection throughout this [the NCTM Reasoning and Proof Standard].

This comment as well as her overall view of both these questions and the NCTM Reasoning and Proof Standard suggests that Mrs. Red *believes* her "philosophy" is connected to proofs and the process of proving.

In summary, Mrs. Red *believes* that her college mathematics courses have done little to help her in her profession and left her under-prepared to start her career. Furthermore, this analysis suggests that Mrs. Red's experiences with college mathematics are in part responsible for her negative views of proofs. Mrs. Red has a dual understanding of proofs; one associated with her college experience, and one associated with the way she uses proofs in her classroom. Her *beliefs* and understanding of proofs used in college level mathematics are negative. Her *beliefs* and understanding of proofs used in her classroom are associated with the functions of *discovery*, *inquiry*, *explanation*, *communication*, *generalization*, and *verification*. She also demonstrates that she values proofs and the process of proving as part of her students' mathematical education. Mrs. Red connects proofs and the process of proving with her "discovery-based" teaching style and the questions of "what do you know?" and "what do you need to know?", which she refers to as "her mantra", "her philosophy."

Mrs. Green

In the following analysis of Mrs. Green's *knowledge* and *beliefs* about proofs and the process of proving, two topics will be explored. First, Mrs. Green's *beliefs* about her own teaching practices and how they relate to proofs and the process of proving will be explored. Second, Mrs. Green's *beliefs* and *knowledge* about what constitutes a proof and her discomfort with the idea of proofs will be analyzed.

Mrs. Green believes that the way students approach problems in her class uses the process of proving. During our final interview, I asked Mrs. Green where she saw proof going on in her classroom. She explained

> Where I see proof going on is when you approach problems the way that we try to, with the questioning, they almost have to sort of go through the process. They're not doing formal proof. So, it's like that type of thing, where they have to sort of do the steps, like a proof, when they are discovering.

This discussion demonstrates how Mrs. Green associates the process of proving with events in her class that are based on discovery and questioning. Her *belief* that her students are not participating in "doing formal proof" will be discussed below as part of her *knowledge* of what constitutes a proof. Mrs. Green goes on to explain that activities such as those where students derive formulas are also related to the process of proving.

> Different activities where they have to kind of derive the formula. And I think, like that, is definitely almost like, you know, the kids are going through that process, and I just think that's a great way for them to think about it and discover it, and remember it. And understanding how it works, how it got there, why the formula exists, helps them remember it and understand it and use it, all better.

This conversation establishes that Mrs. Green values proofs and the process or proving as part of her students' education.

Later in this interview I asked Mrs. Green if she had any reaction to *Stylianides' Definition of Proof*. While she was discussing part (ii), "it employs forms of reasoning that are valid and known to, or with in the conceptual reach of, the classroom community," Mrs. Green explained that her students use counterexamples and if-then statements.

> Even if we know that it's [a student's conjecture] wrong, you know, you've got to write it up there, and you've got to validate it, and accept their thinking, because they're going for it, which is great, and so, and then say, well, what about this situation? And then they can revise. So, I mean, that definitely happens, that counterexample. And so, just coming up with their conjecture, they have to think, if-then. If this situation arises, then does my conjecture work. If put it through my conjecture box, does it come out with what I want? So they absolutely use those forms of reasoning.

This conversation shows that Mrs. Green recognizes different forms of proofs and associates them with forms of reasoning used in her classroom. It also suggests that Mrs. Green understands *justification, dismissal or modification of a conjecture* as a function of proofs and the process of proving.

During our first interview Mrs. Green discussed the use of conjectures in her classroom as part of her teaching philosophy.

> If we are introducing a new topic, we try to kind of throw it out there a little bit and let them ask questions…come up with conjectures…let them explore it a little bit before we get into the 'this is how it works'. And even, you know, sometimes they can come up with the rules, how it works…So, that, we get into some good group discussions that way.

The similarities between Mrs. Green's discussion about her teaching philosophy and how she *believes* proofs are present in her classroom suggest that she considers proofs and the process of proving to be an essential feature of her students' learning of mathematics. Towards the end of our last interview I asked Mrs. Green if she had anything else she would like to discuss about proofs.

Her *belief* that proofs and the process of proving are essential features of her students learning of mathematics became even more clear.

> I think the only thing I would add is that, maybe, I would want to point this out to the kids, as well. That, wow, look, we just went through a proof process, just to put a name to it, so that they could know what they've done, as well. And then, proof doesn't have to be only in geometry with the two-column proofs. It doesn't have to be just that for them.

This highlights Mrs. Green's *belief* that proofs in her classroom do not need to be formal and that she *believes* she was engaging her students in the process of proving, only she was not connecting that process with the process of proving for herself or her students.

Mrs. Green *believes* that mathematical proofs and the forms of proof she associated with her classroom practices are two different things. During our final interview, Mrs. Green discussed the process of proving in her classroom as "the creation process, and the creation process is very valuable." She is again showing a *belief* that proofs are valuable to her students. As she continues she explains that this process is not one she had associated with proofs.

> I never really thought about what we do as proof. Because I remember when, you know, classes I've taken on proof, where it was just like, ugh, I don't want to do this!

Mrs. Green's *belief* that mathematical proofs are different from what her students are engaging in came up again while she discussed telling her students that they were doing proofs. She said,

> You know that [what she and her students were doing in class] is a proof, but, proof, a mathematical proof is something more structural, but we did it [proof] here, in a different form.

This reflects her earlier statement, discussed above, that her students are not participating in "doing formal proof". As she reflected on *Stylianides' Definition of Proof* she discussed her fears about formal or mathematical proofs

> Proof, mathematical proofs is still terrifying for me, but if you think about this definition, with the classroom community, it doesn't have to be so scary.

The fearful and negative *beliefs* Mrs. Green shared during interviews suggests that the disconnection she has with proofs and her classroom practices are related to her prior experiences with proofs.

In summary, Mrs. Green *believes* that proofs and the process of proving are essential features of her students' learning of mathematics. She understands *justification, dismissal or modification of a conjecture* as a function of proofs and the process of proving. And she associates aspects of using proofs and the process of proving with her teaching philosophy. Mrs. Green demonstrated fearful and negative beliefs about formal or "mathematical proofs." She also discussed how she did not associate what was happening in her classroom with formal or "mathematical proofs." The disconnection she has with mathematical proofs and her classroom practices may be related to her prior experiences with proofs.

Mrs. Yellow

In the following analysis of Mrs. Yellow's *knowledge* and *beliefs* about proofs and the process of proving, two topics will be explored. First, Mrs. Yellow's uncertain *knowledge* about what constitutes a proof will be analyzed. Second, Mrs. Yellow's *beliefs* about her past experiences with proofs and the process of proving will be discussed.

Mrs. Yellow demonstrated uncertainty about what would be considered a proof in her classroom. During our second interview I asked Mrs. Yellow about an assignment which asked students to write a conjecture, "can you explain want you would expect from your students as a conjecture?" Mrs. Yellow hesitated and said,

> OK, so, to me, a conjecture is, it's a mathematical, I don't know if law is the right word, but it's been, I don't think it's been formally proven. Now it's like, my God, I don't know that definition...But I feel like, it's something that, you see patterns of something, and you come up with an idea of, you come up with a conclusion based on patterns. That's how I see it.

Mrs. Yellow's hesitation and discussion suggests that she was uncertain about her own understanding of a conjecture. She continued,

> we've asked them [her students] to come up with conjectures, what's that? You know. What do you notice? You know, try different examples. Do you see the

same thing happening over and over, kind of a generalization, based on something? That's how I would define it.

As she related her understanding of conjecture to the way she uses it with her students, her definition seemed to start coming together as an observation that is generalized. As she continued she focused on generalizations and included the idea of translating observations into mathematical language.

> So, but you're finding a pattern and putting it into math language. And I think that's how I explained it. That's how I would see it. But I think it helps them, then, make generalizations, if they're going to come up with formulas or they're going to come up with why.

I asked Mrs. Yellow, "What would you consider to be math language?" She explained

> I could see it being a couple of different things. I could see it being just math, using math vocabulary that we've been talking about. I could also see it being an expression or an equation, something using symbols or notation.

Her idea of "math language" and its role in making a conjecture suggests that she *believes* there is formalness associated with the process of conjecturing. She continued to try and define conjecture:

> I would say it's probably more based on a pattern or, which you can, you can make a conclusion on a pattern. It's like I'm going in circles here, sorry.

Mrs. Yellow's inability to narrow down her understanding of conjecture and her statement about circling further suggest that she in uncertain of her own understanding of this idea. Since conjecturing plays a major role in the process of proving this uncertainty is related to the uncertainty she demonstrates with her knowledge of proofs.

Mrs. Yellow's uncertainty about proofs was demonstrated during a teaching episode and her discussion about this episode during our first and second interviews. During the fourth week of classes the following episodes were observed:

> Mrs. Yellow and Mrs. Green had set up four stations for students to work on throughout the class. The students moved around the stations in groups of six. Mrs. Yellow's station was based on students learning about the distributive property. While she was working with the first group of students Mrs. Yellow explained, "When we have numbers we have two ways of solving these problems. Distribution Property and Order of Operations. But if we have a variable anywhere we have to use the distributive property." She worked through a couple of examples with students prompting what steps she should take. As she worked through the second example, 3(2 + 3+ 5) = ?, she solved the equation two ways. The first way used order of operations, and she wrote: 3(10) = 30. The second used the distributive property, and she wrote: 3(2) + 3(3) + 3(5)= 6 + 9 +15= 30. Mrs. Yellow explained to her students "The second way is going to seem longer but I want to prove that you get the same answer. She then had the students work through other examples.

The important thing to note about this episode is Mrs. Yellow's statement that she was working the problems out in two ways in order to "prove that you get the same answer."

> As Mrs. Yellow worked with the second group of students she went through a similar discussion as the one with the first group. However, she did not mention that she was working the problems out into two ways in order to prove that you would get the same answer. After she had assigned the students their homework one of the students asked, "do you want us to do it both ways?" Mrs. Yellow said, "No you can't, because you can't combine something like 2x + 8 the way we did with the numbers. We did the two ways, because this kind of proves that the distributive property works."

The important thing to note about this episode is Mrs. Yellow's statement that she was working the problems out in two ways in order to "kind of prove" the distributive property.

> During the first class, Mrs. Yellow only had enough time to work with two of the three groups. The next section of students came into class Mrs. Yellow started with her first group by asking, "What is the distributive property?" This session was similar to the others. As she worked with this group she explained "we are going to do this in two ways just to show you that the distributive property will give you the right answer." She never mentioned proofs or proving during this session.

The important thing to note about this episode is Mrs. Yellow's statement that she was working the problems out in two ways "to show you that the distributive property will give you the right answer."

During our first interview I described my observations of these three sessions and explained my curiosity of three different ways she had justified the activity of working the problems out in two different ways, "I mean, I'm curious, because you had said to one group, I'm doing both ways to show you that the distributive property works. And then, one group, I think you said, I'm doing this to prove it. And then the next group you said, I'm doing this to kind of prove it." Mrs. Yellow explained, "I'm not really proving it. I'm sort of proving it. I guess those all meant the same thing in my head." I asked her what that "thing" is, and why it was important for her to include it in her lesson. She explained that:

> I want them to buy into that it's important to understand, if I take something that they already know how to do and try to build on that, it's going to make what they learn easier. And so, if you can figure, OK, so, they already know how to simplify, with parentheses, order of operations, they already know that. So, if you can help make the connection from what they already know to what they don't know, hopefully it'll make, they'll learn something new and be able to extend it to the next step, I guess, would be why I do it.

Mrs. Yellow's explanation about why she included this type of teaching session as part of her classroom activities suggests that she values her students' abilities to build on prior knowledge and form connections to new material. This conversation ended up following a different direction and so I decided to revisit her understanding of what the "thing" was that connected "proof," "show," and "kind-of prove" during our second interview. I showed Mrs. Yellow the section of our first interview where we had discussed these classroom episodes and asked to talk about what the "thing" was, and what she meant by them all meaning the same thing. Mrs. Yellow explained

> I guess I meant that, it meant them understanding that the distributive property, how the distributive property works. So, maybe not a proof, but how they understand the distributive property...So, the thing was probably understanding the distributive property.

Mrs. Yellow's explanation suggests that she was unsure of whether she had actually "proved" the distributive property. It also suggests that she associated her actions during class with students' understanding and that she does not necessarily associate proofs with how students come to understand. As she continued, her *beliefs* about using proofs, at least formal proofs, in her classroom were discussed.

> I rarely, rarely do a formal proof. I'll, we may do, when I've taught algebra, that was before school, we would prove the quadratic formula, and we would do some small proofs with properties. And, I have maybe done one, you know, two-column geometry proof with them, that's four steps. But, for the most part, it's really; I have found that it's a very hard concept for eighth graders.

Mrs. Yellow's association with formal proofs and two-column geometry proofs show one level of her *knowledge* about what constitutes a proof.

> A formal, to do a formal proof. So, I guess, in thinking about it, in loose terms, would be, can, it's almost, can you show how you got from here to the endpoint. You know, which I guess a formal proof is, with reasoning in there.

Mrs. Yellow's comparison between "show how you got from here to the endpoint" and formal proofs, suggests that she believes that in "show[ing] how you got from here to the endpoint" there is a level of reasoning that is not included in formal proofs. She went on to talk about her students' abilities with proofs.

> I think they can do that, in very informal ways. You know, they can talk through, and they can explain why we did this, and this, and this. So, I think, when I say proof to them, that's probably, that's what I mean, is, can you show me how you got from this point to this point? What were your steps, or what was your reasoning? Not that steps are always reasoning, but, you know, what was your

> thinking? Or your justification. And that, now, I don't know, now I'm thinking about all these words. You do, I do use them interchangeably, and I don't know if I should be.

Mrs. Yellow is again showing uncertainty about her use and understanding about proofs. She seems to be exhibiting a level of understanding of proofs as *Informal Argumentations* when she explains that she is using "proof" as a word that is interchangeable. She is also demonstrating an understanding that proofs function as a means of *justification,* that they rely on more than just showing work, that they rely on reasoning as well. Her doubts about using formal proofs in her class seem to come from a *belief* that they do not aid in understanding and may hinder student learning.

> But then, also, like, how formal do you make it, and then does that, I don't know, I can't get my head around that thought. Like, if you make formal proofs, or you're very strict about, this is what I expect when showing your thinking, are you going to stop kids from showing anything?

Mrs. Yellow's concern about using formal proofs may come from her past experiences with proofs which will be discussed later. The above analysis suggests that Mrs. Yellow is uncertain about her understanding and use of proofs. It also highlights some of her *beliefs* about proofs and their place in her classroom. This will be explored more below. However, one more example of Mrs. Yellow's uncertainty will be discussed before moving on.

In Part I of this chapter, Mrs. Yellow's judgment about whether or not a student's solution would be considered a proof was discussed. An analysis of her uncertainty was left for this portion of the chapter. As a reminder let us revisit the student's solution and Mrs. Yellow's comments.

While discussing Theresa's answer,

> *Locker #144 would be open. I know this because all lockers with an odd number of factors are open. Prime factors are closed because they have an even number of factors.*

I ask Mrs. Yellow if she would consider Theresa's answer to be a proof. She explained that,

> Yeah, it's, I would say partly, and it's hard because I am looking, also at the student. I would say, it's partly a proof. And why I would say it is because we discussed it, and so this particular student, I think would know, OK, what I'm supposed to remember from this problem is this. And she would put that into her memory. But then, had she factored 144 and then wrote out the prime factorization to figure out that there were an odd number of factors, So, I think she is almost getting it. But I probably would have wanted her to say more why there, why an even number of factors is closed.

I then asked Mrs. Yellow if the problem had been restated to say, "thinking about the locker problem, can you prove the locker number 144 would be open, do you think that this would constitute what would be defined as a proof, in your classroom?" She thought a moment and said, "It would probably be pretty close to a proof, I think."

Mrs. Yellow's judgment of whether or not Theresa's answer would constitute a proof including "probably" and "I think" demonstrates Mrs. Yellow's uncertainty. Moreover, Mrs. Yellow's *belief* that Theresa's answer is "partly a proof" because it is incomplete and lacking in explanation, while she also believes that this answer is "pretty close to a proof", demonstrates that she is not clear about her understanding of proof.

Through her vagueness about conjectures, her own uses of "proof" with her students, and her uncertainty about judging students' answers as proofs, this analysis suggests that Mrs. Yellow was uncertain about her understanding and use of proofs. However, this analysis also highlights some deep understanding of proofs, even if Mrs. Yellow was unaware of her own knowledge.

Mrs. Yellow demonstrated mixed *beliefs* about her past experiences with proofs and the process of proving. During her experiences in high school she had some mathematics teachers that taught procedurally "I was told what to do. I don't think I ever was asked to think," while in other classes she was "made to explain and justify." Mrs. Yellow explained that she enjoyed the classes that made her explain and justify, more than those where she was just told what to do, "even though they were much harder." She related this to her own teaching, explaining "I don't know how else I would approach it [teaching math], what other way would I go about teaching something, if I didn't ask them to look for patterns, and try to predict, and try to explain and justify." Mrs. Yellow's experiences in high school and as a middle school teacher suggest that she values teaching and learning associated with proofs and the process of proving.

Mrs. Yellow's feelings about proofs and her ability to prove changed from when she was a calculus student to when she took her proofs course in college. During our second interview, I asked Mrs. Yellow about her experiences in college level mathematics courses. She explained

> My proofs course was so hard, and it was so theoretical, that it was just, I could not get my head around it, at all. But I loved proofs that I had done in calculus, I mean, we had done proving in calculus, and that was cool, like, I get this. This makes sense. Proof in geometry in high school, it took me a long time to get, but I liked it. But there's steps that you have to take, in order to get to a higher level of proof, and I feel like I missed something between, you know, my calculus classes and then formal, then the math proof class. There was something, there was some gap, for me, that I missed some step. And I don't know if it was, you know, the professor I had, or just me. I wasn't there, to make that way theoretical leap in proof. I mean, I couldn't even tell you what we did. It was just, I would go in and be like, I don't know what's happening. And I would go in, and I would ask, I would try to ask questions, but it was almost to the point of, like, I don't even know what question to ask. You know, and I did the best I could with what I had, and I ended up doing fine in the class. But as far as, like, that didn't really matter to me, because I, the only thing that I took away from that was, hmm, I really can't do proof.

Mrs. Yellow's account suggests that her uncertainty about proofs discussed above may come from her negative experience with proofs in college. While her "love" for proofs and ability to use proofs as a means of making sense of calculus may contribute the many ways in which she views proofs as valuable for her students and utilized proofs and the process of proving in her classroom.

Similarly to Mrs. Green, Mrs. Yellow *believes* that it would be beneficial for her students to understand when they are working with proofs and the process of proving. During our third interview, I asked Mrs. Yellow if she had any reaction to the NCTM Reasoning and Proof Standard. She explained

> When reading it [the NCTM Reasoning and Proof Standard] I was thinking about, of, oh, do we say to the kids, you're doing a proof. And give them that concrete, wow, I'm doing proof. Like, I know I've done it when I've done more of a formal proof with them. This is a proof, and they're like, whoa, this is hard. You know, when you prove the quadratic. They're like, whoa, wait, and I totally don't get that.

Mrs. Yellow's discussion about relating the processes her students are engaging in with the process of proving, by telling them that "you're doing a proof," highlights her belief that she is engaging her students in proofs and the process of proving, and that helping them to realize that would be valuable to their education.

In summary, Mrs. Yellow is uncertain of her own understanding of conjectures, about her understanding and use of proofs in her classroom, and her own abilities to judge students' work as constituting a proof. However, even if Mrs. Yellow is unaware of her *knowledge,* she exhibited deep and established understanding of proofs. Mrs. Yellow's experiences in high school and as a middle school teacher suggest that she values teaching and learning associated with proofs and the process of proving. Mrs. Yellow's experiences with proofs in college were both negative and positive. Her negative experience with proof may account for her uncertainty about proofs; while her "love" for proofs and ability to use proofs as a means of making sense of calculus may contribute to the many ways in which she views proofs as valuable for her students and utilized proofs and the process of proving in her classroom. Mrs. Yellow believes that her students are engaging in proofs and the process of proving and that it would be beneficial for them to understand when they are working with proofs and the process of proving. Furthermore, she believes that she should make this point explicit.

Analysis of Major Themes

Above, two or three topics for each teacher were explored and analyzed. Through this analysis two general themes were found. In this section those themes will be addressed using the findings and evidence discussed above. First, the teachers' understanding of proofs and the process of proving related to their own educational experiences and the effect of these experiences will be explored. Second, the teachers' understanding of proofs and the process of proving related to their students' education and the connection between their practices and this understanding will be explored. The overarching theme is the difference between these two understandings and the suggestion that these differences were necessary for the teachers to utilize and make sense of proofs as meaningful tools for the teaching and learning of mathematics.

In their discussions about prior experiences with proofs during their own education the teachers explained these experiences to be either negative or non-existent. In Mrs. White's description of her mathematical education she explained that she learned mathematics in a rote and procedural way. She recalled teachers giving her formulas and telling her how to plug in numbers in order to find the solution, and her study habits as consisting of simply practicing a large number of similar problems. In her account of these experiences she contrasted her experiences with the proof related activities she provides for her students. This suggests that Mrs. White's connection between her own experiences and proofs was almost non-existent.

Ms. Blue connected her experiences with college mathematics to proofs and recalled these experiences as negative. She believes that she did not gain anything from her higher level mathematics courses relevant to her profession; that her only experiences with proofs was in high school geometry and college; and that proof is formal, painful, and not connected to conceptual learning. This suggests that Ms. Blue's connection between her own experiences and proofs resulted in negative beliefs about proofs and their usefulness in understanding mathematics.

Mrs. Red also connected her experiences with college mathematics to proofs and recalled these experiences as negative. Similarly to Ms. Blue, Mrs. Red believes that her college mathematics courses have done little to help her in her profession; that her experiences with proofs in college were negative; and that proofs are formal and intimidating. This suggests that Mrs. Red's connection between her own experiences and proofs resulted in negative beliefs about proofs and their usefulness in understanding mathematics.

Mrs. Green explained that her experiences in college involving proofs was terrifying and only allowed her opportunities to engage in formal proofs. She disassociated her believes about proofs and the experiences with conceptual learning she was providing for her students. This suggests that Mrs. Green's connection between her own experiences and proofs resulted in negative beliefs about proofs and their usefulness in understanding mathematics.

Mrs. Yellow explained that her experiences with proofs in college were both positive and negative. She valued the use of proofs in helping her to understand the concepts of calculus but believed the only thing she took away from her proof course was the belief that she could not do proof. Mrs. Yellow also described situations where formal proofs were unhelpful, if not detrimental, to her students' understanding of certain concepts. This suggests that Mrs. Yellow holds both positive and negative beliefs about proofs and their usefulness in aiding in the learning of mathematics. Each of the teachers discussed negative or non-existent experiences with proofs during their education.

The teachers' negative experiences seem to have affected their classroom experiences in two major ways. The teachers thought negatively about proofs and the process of proving outside of the context of their students' learning, and it may be the origin of the teachers' dual understandings of proofs. This dual understanding of proofs allowed the teachers to make sense of proofs as a meaningful way for their students to learn mathematics by not connecting this understanding of proof to their experiences with higher mathematics. The exception to this finding is Mrs. White, who believes the lack of proofs was detrimental to her own education and includes proofs and the process of proving in her classroom as a means to help her students learn mathematics in a way that she was not afforded.

Ms. Blue's discussion about unintentionally using proofs in her teaching practices because she does not associate the proofs with conceptual learning is the best evidence of the teachers not thinking about proofs as a useful tool for teaching and learning. In the analysis of the other teacher's beliefs about using proofs in their classrooms, the evidence of not thinking about proofs as a useful tool for teaching and learning comes mainly from their need to think about proofs in their classrooms as different from the proofs they used in higher-level mathematics courses.

Mrs. Red explicitly discussed her dual understanding of proofs and explained that the proofs she associates with college are different from the proofs she uses in her classroom. Mrs. Green also addressed this dual understanding by explaining that she associates higher-level mathematics as using formal proofs and does not associate what is happening in her classroom with formal proofs. Mrs. Yellow's dual understanding of proofs may be founded in her different experiences with proofs in college. Her understanding of proofs as a means of making sense of calculus may contribute to her beliefs about how she uses proofs in meaningful ways with her students, while her belief that both she and her students struggle with a certain different levels of formal proofs may contribute to her belief that proofs, at least formal proofs, are unhelpful in the teaching and learning of mathematics.

The finding that these teachers were in fact engaging their students in the process of proving, and the value they placed on using proofs and the process of proving in their classrooms, together with their negative beliefs about proofs in their own education, suggests that one of the ways in which these negative experiences affect the teachers' classroom practices is that, in order to make sense of proofs as meaningful tools, the teachers have disconnected their experiences in college with their classroom practices by creating this dual understanding.

By creating this dual understanding, the teachers were able to think about proofs and the process of proving as valuable to their students and as having the functions of *verification, explanation, communication, discovery, exploration of definitions, generalizations, inquiry, and the justification, dismissal or modification of a conjecture.* The evidence above has demonstrated that each of the teachers in this study believe that proofs and the process of proving has the functions of at least some of those listed above. This includes Mrs. White for whom there is no evidence to suggest a dual understanding. Although not all of them discussed each of these functions individually, as a group each of these functions were recognized and all of the teachers believe that when proofs and the process of proving are functioning in these ways the experiences are valuable to their students' education. Furthermore, when thinking about proofs related to their students' education, the teachers related proofs and the process of proving to their teaching style, philosophy, and practices.

In Part I of this chapter Mrs. White's use of proofs and the process of proving was demonstrated in the *Students as Teachers* section. In this episode students were engaging in the processes of *ascertaining* and *persuading* and demonstrated their abilities to use the process of proving as a function of *communication*, *explanation*, and *generality*. Mrs. White values these type of activities and shows a desire for including more proof related activities in her teaching practices.

In the *If – Then Statements* and *Building Patterns to Discover Equations* sections of Part I of this chapter, Ms. Blue engaged her students in proofs and the process of proving as functions of *exploring definitions, justifying*, *generalization*, and *discovery.* Ms. Blue believes that she used proofs and the process of proving during these activities and other classroom practices, however she views the use of proofs as being unintentional yet valuable.

Mrs. Red believes and understands proofs used in her classroom practice are associated with the functions of *discovery, inquiry, explanation, communication, generalization, and verification.* These functions were demonstrated in the sections *Investigation* and *Argue Your Answer Mathematically* sections of Part I of this chapter. She connects proofs and the process of proving with her "discovery-based" teaching style and with her "mantra" or "philosophy" of having students ask themselves, "what do I know?" and "what do I need to know?"

Mrs. Green connects her teaching practices of discovery and questioning with the process of proving. She believes there is a creation process in proving and that this process is valuable for her students to engage in this process. She refers to *justification, dismissal, or modification or conjecture* as an important function of proofs and the process of proving. In *Using Mathematics to Inquire about Mathematics* and *Explaining Reasoning* sections of Part I of this chapter, Mrs. Green and Mrs. Yellow's use of discovery, *verification, explanation, inquiry,* and *justification, dismissal and modification of conjectures* were demonstrated.

Mrs. Yellow associates her own teaching style with proofs as she describes it as asking students to look for patterns, make a conjecture, and then explain and justify their conjectures. She uses proofs and the process of proving often in her classroom and as described above connects *discovery, verification, explanation, inquiry,* and *justification, dismissal and modification of conjectures* as functions of proofs and the process of proving in both her teaching theories and practices.

In summary, this section analyzed the teachers' understanding of proofs and the process of proving related to their own educational experiences. For most of these teachers those experiences and their beliefs about proofs from those experiences were negative. The teachers' understanding of proofs and the process of proving related to their students' education were explored. For all of these teachers, proofs and the process of proving are thought of as valuable and useful in their students' education. The difference between these two understandings as well as the teachers' abilities to utilize proofs in meaningful ways during classroom activities, suggests that these different understandings are necessary and part of how the teachers make sense of their past knowledge and current experiences.

The following part of this chapter will explore alternative resources, outside of their direct knowledge or beliefs about proofs, that relate to the way these teachers utilized and thought about proofs in their classroom practices.

PART III: SCHOOL STRUCTURE

INTERCONNECTION OF PRACTICE AND KNOWLEDGE

This part of the chapter will continue to explore the discontinuity between the first two parts by presenting alternative resources related to the school's structure, utilized by teachers in *making meaning* of proofs and the process of proving, as were understood in the context of their students' education. I will address this topic by considering the third topical question, *How do teachers form connections between their understanding of proofs (and the process of proving) and the incorporation of certain teaching methods into their classrooms*? In the previous part of this chapter the claim was made that some of the teachers hold dual understandings of proof: one related to their own educational experiences and one related to the role of proofs and the process of proving used in their classrooms. Recognizing the dual understandings of proofs leads to further analysis of the data for recourses or opportunities that may have allowed for this dual understanding and connected the teachers' understanding of proofs and process of proving, as were understood in relation to their students' education, to their classroom practices. The teachers discussed these alternative resources as being part of Light School's philosophy and structure.

Individual Instruction

The school's philosophy related to individual instruction allowed the teachers opportunities to think about the learning styles of each individual student and is one of the guiding principles behind the low teacher-to-student ratios discussed in Chapter 3. Although individual instruction was directly linked to the use of proofs and the process of proving, the teachers discussed this as a resource they used when creating and implementing classroom activities that supported the use of proofs and the process of proving. This philosophy is also one of guiding principles behind the decision for the school to create their own curriculum, which will be discussed later as one of the major resources for teachers to think about the use of proofs and the process of proving in their classrooms. Mr. Purple explained that Light School is "very student centered," meaning "whoever the child is, when they enter our classroom here, that's where you pick them up at... We should be able to meet their needs and help them grow, individually." He went on to explain that "individual instruction" is "making it make sense for each child." A reflection of this philosophy is the school's student-to-teacher ratios. When I asked Mr. Purple about class size and using multiple teachers in the classrooms, he explained that "the more we personalize it, the more we connect with individuals, the more effective it is."

In the fifth grade classroom, where there are two teachers during mathematics classes, Ms. Sage explained that a large part of her job is helping to assess "where the kids are at, and planning from there, based on what the kids need." During planning periods Mrs. White and Ms. Sage were continually discussing individual students and what pedagogical choices they needed to make in order to insure that students "moved up their own learning trajectories." During these planning sessions they decided to use activities, such as Students as Teachers, based on their discussions about individual students. Both of these teachers referred to each other as recourses for gathering data about students' understandings and someone they could talk about different pedagogical choices with. This suggests that the school's philosophy related to individual instruction was the basis for two resources used by the teachers in connecting the use of proofs and the process of proving and their classroom practices. First, as a guiding principle behind the decision for them to both teach in the classroom, this philosophy offered the teachers an opportunity to communicate with each other about individual students and pedagogical choices. Second, the use of proof related activities, such as *Students as Teachers*, were based partly on their reflections about individual instruction.

Mrs. Red explained her relationship with the Special Educator in her classroom, Mr. Orange, as "kind of co-teach[ers]." She described how they sometimes planned together and how Mr. Orange helps her "accommodate for kids of all needs." This relationship was not connected to decisions about proof related activities. However, when Mrs. Red discussed proof related activities, such as those used during her geometry unit, she always used the pronoun "we." This suggests that she and Mr. Orange worked together to both create and implement the lessons. Referring to activities she believes fit with the NCTM Reasoning and Proof Standard, Mrs. Red explained that having two teachers in the room "aids" in the ability "to do these types activities. It would be harder to do alone." This suggests that, as a guiding principle behind the decision for them to both teach in the classroom, the school's philosophy related to individualized instruction may have fostered Mrs. Red's decisions to incorporate proof related activities, that she may have been less likely to have used if she were the only teacher in the classroom.

During a conversation about going on maternity leave, Mrs. Yellow explained, "we need a second teacher in the classroom, because of the needs in the classroom.

Which is why [Mrs. Green] and I are together." Having only 18 students in the class and little or no discipline issues, the needs of the classroom are not because of student population, but because of the algebra content being taught, and the teaching methods Mrs. Yellow and Mrs. Green utilize. One of these is conducting individual conferences with each student bi-weekly. During these conferences each student works with the teacher to create and meet an individualized goal.

Mrs. Green also explained how they adjust for the needs of individual students by using stations during class time.

> If we know the kids need more one-on-one time, we'll try to sit with them. I mean, that's why we split into stations and the third station can be split into two groups, one for the kids that were ready to work on a new skill and one for the kids that still need to work on a previous skill.

Working with students at different stations, or splitting one station up to work with different groups at that station is one way which these teachers utilize having two teachers in the classroom in order to pay attention to the individual learning needs of each student. Mrs. Yellow emphasized the focus of individual instruction and how it allows them to be less curriculum driven and more student-centered than she has seen at other schools.

> Here [at Light School] we take the student, and where they are at, and keep pushing them...[at other schools] they are more curriculum driven, and we're more kid driven because of the structure of the school...I am grateful that I'm here and not there.

This suggests that Mrs. Yellow values the school's philosophy and relates her own focus on individual instruction with the school's philosophy and the opportunities that affords her.

As stated above, the focus on individual instruction is not directly linked to the use of proofs and the process of proving in the teachers' classrooms. However, this aspect of the school's philosophy has clearly afforded the teachers with opportunities to work with multiple teachers in their classrooms and student-centered instruction, which in turn, allows them to make pedagogical choices that lend themselves to the use of proofs and the process of proving as was exemplified in Part I of this chapter.

Monitoring for Meaning

Another resource that helped teachers make meaning of their knowledge of proofs and the process of proving being utilized in their classroom practices, was a schoolwide initiative to focus on meta-cognition with the students. The initiative is based on the work done by Ellen Keen. One of the strategies the school was working on during my observations was called *Monitoring for Meaning*. During our second interview Mrs. Yellow referenced a student's questions on the side of a homework assignment and said "this is great, this is some of the stuff we have been doing with the *Monitoring for Meaning*." I asked her to explain what this *Monitoring for Meaning* thing was all about. Mrs. Yellow explained

> *Monitoring for Meaning* is the first strategy that everybody in the school is working on. And, for me, *Monitoring for Meaning* was the hardest strategy for me to get my head around. We've talked a lot, we have middle school meetings once a month, and so we have talked a lot, as a middle school, what does that mean, what does that look like?

This part of Mrs. Yellow's response highlights the collaborative nature of the school and how teachers saw each other as resources for making sense of their knowledge and their pedagogical choices. This was addressed a bit above in relation to the teachers who shared classrooms and will be explored further when discussing the school curriculum.

As Mrs. Yellow went on she explained her understanding of *Monitoring for Meaning.*

> *Monitoring for Meaning*, to me, was abstract… my understanding, that I took away from the meetings and reading her books and talking with her was, it's stopping, and asking yourself, do you understand? If not, what could you do to understand? Or what are you thinking about while you're reading? I'm stuck. What are some different things you can do to help yourself get unstuck? But it's more the recognizing, I'm not understanding what I'm reading, or I'm understanding what I'm reading. I think that's the basic piece, and then the next step would be, OK, so, what are the things you do to help yourself understand?

This part of Mrs. Yellow's response connected the use of *Monitoring for Meaning* with students' abilities to think about their level of understanding. *Monitoring for Meaning* was introduced to the teachers and students at Light School as a reading strategy. During the above conversation Mrs. Yellow referred to understanding of reading but did not connect this to her own teaching or mathematics. After she had finished describing her understanding of *Monitoring for Meaning* I asked Mrs. Yellow if she saw any connection to this and the learning of mathematics. She said,

> I think it's huge, actually. The more that I'm seeing some of the kids develop their skills, they're more able to ask questions, they're more able to think about things, I think it means they're thinking deeply about something, instead of just doing it. And so, if you're thinking about your thinking, they're going through a problem and they're actually asking themselves questions. When we say thinking about your thinking, it could be asking yourself questions, you could be making connections, you could be inferring things? They're doing those things and I've seen it with some of the kids, in their binders, they're now writing questions on the side, and they're writing notes of, OK, so, this is how I would describe the steps to factoring a polynomial. And for me, it's helpful, when I'm looking at their work, because I know how that particular student was thinking…So, I think it has a lot of ramifications in the classroom, and implications, maybe, is a better word.

Mrs. Yellow's connection between *Monitoring for Meaning* and mathematics continues to show her understanding of this strategy to be one focused on the students' thinking about their own understanding and also highlights the value she places on this strategy and its implications in the classroom, at least in her own classroom. It also gave her a way of thinking about the students' meta-cognition and how that affected her own teaching practices, particularly asking her students to think about their thinking.

Mrs. Yellow's experiences with *Monitoring for Meaning* during conversations with her colleagues and with her students seems to be a resource for Mrs. Yellow to make meaning of the use of proofs and the process of proving in her classroom. During our third interview when Mrs. Yellow was explaining her reaction to the NCTM Standards she said, "It all goes back to the thinking about thinking, and the *Monitoring for Meaning* type of thing." This suggests that the schoolwide focus on *Monitoring for Meaning* was a resource for Mrs. Yellow to use in making meaning of her knowledge of proofs and the process of proving and her use of proof related activities as part of her classroom practices.

All of the teachers described *Monitoring for Meaning* similarly to how it was discussed by Mrs. Yellow, either to me, or their students during the time I was observing. Although there was no evidence to support this as a resource utilized by the other teachers for making meaning of proofs and the process of proving, the use of this resource, particularly by Mrs. Yellow, was worth noting.

Curriculum Coordinator

One of the major resources utilized by the teachers was working with the school's curriculum coordinator, Mrs. Pink. Each of the teachers views Mrs. Pink as a major resource for them. Mrs. Pink works with the teachers to create their curriculum, helps them implement certain teaching strategies and activities, and is regarded very highly by both the teachers and Mr. Purple. Before discussing how the teachers view Mrs. Pink as a resource, a short analysis of her *beliefs* and *knowledge* of proofs and the process of proving will be presented.

Mrs. Pink connected proofs with the functions of *communication*, *explanation*, and *verification*. When I asked Mrs. Pink about connections between proofs and the Problem Solver rubric she explained that proofs were present because it was asking students to "document their thinking" and it is about "how to communicate." Her comments suggest that Mrs. Pink *believes* that one of the functions of proofs and the process of proving is *communication* and that she associates proofs with the process of thinking and then documenting that process. She went on to explain how she thinks about proofs.

> When you talk about proof, when I think of proof, you can think of it in two different ways. You can think of it as a documentation of your reasoning, such that it flows, each step makes sense, and you achieve an answer that is logical to the question asked. Or, it can also be, it can also be a secondary measure of, you know, checking.

Mrs. Pink's explanation of thinking about proofs in two different ways is not the same as the dual understanding held by the teachers. Rather, this explanation shows Mrs. Pink thinking about the process of proving as the documentation of thinking and then a proof as the final form of that documentation that can be used later to think about your thinking. This becomes more clear as she continued to explain a proof as a tool.

> I fundamentally believe that it is, it is getting kids to use the tools, getting kids to document their thinking, that then, that documentation can actually become a tool for them. Because when it gets more complex, they can see their thinking at each step. And know, this is where I am. This is what I have to do next. Not because they're following their script, but because that's a step that makes sense.

Mrs. Pink's description of proofs as a tool for students to use in understanding that a certain "step" makes sense demonstrates her belief that a function of proofs is explanation, as to why that "step" makes sense, and as a function of *verification*, when looked at as a way of "checking" that this is true. This again becomes more clear as she explains that proofs are "evidence of why."

> It's showing that evidence of the thinking that I need to follow...in many ways, it sort of gives evidence of why this is the answer, I believe proof should be this, this evidence of why.

Mrs. Pink's *beliefs* about proofs and the process of proving demonstrate that she considers them to aide in student learning and can be used as functions of communication, explanation, and verification. Furthermore, she *believes* that both students and teachers should be engaging with proofs in the classroom activities. She explained that "I want a child to be giving evidence in all aspects of their thinking, like proofs, I also want teachers to be doing proof in their teaching."

Mrs. Pink's *beliefs* about the usefulness of proofs and her desire to have both teachers and students engaging in proofs is a resource for the teachers at Light School that may have contributed to their dual understandings and their making meaning of proofs and the process of proving related to their students' education. This is particularly true since the teachers described Mrs. Pink as a major resource in aiding them with their professional responsibilities.

The teachers believed that Mrs. Pink was a key resource for them when making decisions about the school curriculum, specific activities, or pedagogical strategies. In order to demonstrate the beliefs held by the teachers, a few examples taken from interviews with one teacher from each grade will be supplied. During our first interview I asked Mrs. White, "how do you make methodological decisions about how to present the math?" She said

> I get help from our math coordinator, she is really helpful and I go to her a lot. I get help from my team. I have never taken a math methods course… So, it's kind of been from talking with Mrs. Pink and my colleagues, seeing what they do.

Mrs. White was the exception to the dual understanding of proofs. We see here again that she had little experience with mathematics before becoming a teacher. Her reliance on Mrs. Pink as a resource in making methodological decisions, along with Mrs. Pink's *beliefs* about proofs, suggests that working with Mrs. Pink was a resource used by Mrs. White in *making meaning* of the activities and discourse related to proofs and the process of proving found in her classroom.

Ms. Blue also discussed Mrs. Pink as a resource in making decisions about how to structure units and even how to phrase certain questions. Ms. Blue's use of Mrs. Pink as a resource was discussed in Part I, concerning the modification of the problem used in the episode *Building Patterns to Discover Equations.* During our conversation about this problem I asked Ms. Blue how often she uses Mrs. Pink as a resource. She explained

> Last year and the year before, my first two years, we met once a week, on a regular basis. We had, it was just a weekly time with her, and I would just sit and talk about what I was planning to do, or look at the year. And sometimes, we'd just kind of look at where I was in the year, and sometimes it was more specific questions. She was there for whatever I needed her for, so it wasn't at all a structured, like, OK, show me your plans for next week, or anything. But it was just, what did I need? So, sometimes it was more, look at the whole unit. Sometimes it would be a problem like this, where I'd just kind of say I need help, and so now I just go see her whenever I need her…she's such a huge resource of information, so she definitely helps me to figure it all out…I catch her at all different points, for either a quick question like this or a bigger, like, I want to start this project, or this unit, and I'm not quite sure how to start it, or something like that. So, I use her for all sorts of things.

This conversation highlights Mrs. Pink as a resource for Ms. Blue. Mrs. Red also described Mrs. Pink as a resource. During our second interview Mrs. Red was describing her educational experiences and explained that Mrs. Pink had taught one of the classes she took before Mrs. Red had taken it. She went on to say

> I wish I had, [Mrs. Pink], I took a class from her when I got here [to Light School]. And that was really helpful, because she has such a great way of thinking about things, and she has so many activities...It's great, because, I learned a ton from her, and a ton from my colleagues, here.

Clearly, Mrs. Red thinks of Mrs. Pink as a useful resource. During our third interview, Mrs. Red explained how she attributes Mrs. Pink with bringing in activities related to proofs. Mrs. Red was describing her reactions to the NCTM Reasoning and Proof Standard when she explained that while she was reading it she was thinking, "oh, we do this, I see a connection to this [the NCTM Reasoning and Proof Standard] and what we do at this school." I asked her if she could think of any examples, after she had described a few she said, "I would attribute to [Mrs. Pink] bringing that stuff [the proof related activities] to our school. Where kids were able to come up with things on their own. We do a good job of building it [proof] with our students." Mrs. Red's connection with Mrs. Pink and proof related activities suggests that Mrs. Pink was a major resource for Mrs. Red's dual understanding, and her decisions to use proofs in her classroom.

Mrs. Green directly linked Mrs. Pink with her beliefs about using proofs in the classroom and her ability to lead these activities. During our final interview when Mrs. Green was sharing her reaction to the NCTM Reasoning and Proof Standard, she described a few activities, mainly those linked to the Locker Problem. I asked Mrs. Green how she made decisions about using these types of activities in her classroom. During one of her explanations she described how Mrs. Pink is a resource.

> [Mrs. Pink] is great, with some of the math classes that we've had she has really helped. She's great with different ideas and different ways to do things, and that was probably one of the reasons that we figured out how to use proofs, you know,

how to apply it, or how to think about it.

This explanation suggests that Mrs. Green saw Mrs. Pink not only as a resource, but also as a resource directly linked to her own thoughts and uses of proofs.

Although not all of the teachers directly link Mrs. Pink with their thoughts of uses of proofs, they all described her as a resource that they respected and utilized often. Together, Mrs. Pink's *knowledge* and *beliefs* about proofs and the teachers' beliefs regarding her as a resource they utilize when making decisions about classroom practices, suggest that Mrs. Pink is one of the major resources that allows for the teachers' dual understanding and the connections between the teachers' understanding of proofs and process of proving, as were understood in relation to their students' education, and their classroom practices.

As the curriculum coordinator at a school that does not use published curriculum or certain textbooks, Mrs. Pink was also part of the next resource used by teachers in making meaning of their knowledge of proofs and their classroom practices. The teachers in this school do not have a prescribed curriculum, or even a textbook that they are to use with their students. This process of creating their own curriculum was a major resource for them in making meaning of the use of proofs and the process of proving in their classrooms.

Creation and Implementation of their own Mathematics Curriculum

As described in the Chapter 3, The Light School uses no published curriculum, based on the belief that every student learns in individual and different ways. The faculty and administrators concluded that none of the published mathematics curriculum programs would allow them to facilitate the individual learning needs of their students and so they hired Mrs. Pink to supervise and manage the creation of their own curriculum. Mrs. Pink and the mathematics teachers created their curriculum and continually revise the program. They have based their curriculum on NCTM Principles and Standards (2000) and the New Hampshire Grade Level Expectations. At the heart of this curriculum is the concept of teaching for understanding. The teachers' experiences with both creating and using this program has offered them opportunities to think about their teaching practices and the content.

One of the pieces of the curriculum at Light School is the use of Problem Solvers. These have been described above and have been related to proofs and the process of proving by both Mrs. Pink and the teachers. They are mentioned here because they highlight the consistent nature of the school's curriculum, and demonstrate the presence of proofs and the process of proving in its underlying structure.

As part of the effort to create and continually revise their mathematics curriculum, the teachers collaborate with each other and to view each other as resources for making sense of their knowledge and their pedagogical choices. During our first interview I asked Mrs. Yellow to describe the curriculum and how she decided what content she was would teach and how. She explained

> OK, so, in our school, we've worked this out with, sixth, seventh and eighth grade math teachers and [Mrs. Pink]. We've taken the GLE's, and gone through them in a lot of detail, and sort of divided them up, knowing that we wanted all kids to have completed Algebra I by the time they graduate from [Light School]...So, we've gone through the GLE's, broken things apart, figured out what they mean, and from there, that's how we've decided, OK, here's our agenda. So, working off the GLE's is how we decide what our curriculum is, and what goes together, and how we want to teach it.

Mrs. Yellow's explanation shows a collaborative effort involving the teachers in grades 6 – 8, along with Mrs. Pink. She also explains that this process involved not only decisions about content but also about pedagogical choices, about "how [they] want to teach it." Working together in this way offered the teachers opportunities to connect the mathematics across the grade levels. As describe by Mr. Red,

> I think our school builds a lot, like, we build on each year, you know, in the school. But, I think that, like, if you observed the other class, like, younger grades, too, you would see the build up, on everything. And the structure, it's just very different than other schools. I'm very connected to every class, like, each of our classes, trying to make that connection, from year to year to year. And from, like, even eighth grade to first grade, and looking at how we're going to help each other to build.

Mrs. Red's description of the connections made between grade levels and the teachers thinking about helping each other build on what had been previously taught, demonstrates one effect that creating a curriculum together may have had on the teachers' practices. Ms. Blue also commented on the coordination between teachers and how that influenced her teaching.

> I think the collaboration between grade levels and working with a group of teachers is huge, because I think it's such an advantage to know that, like, 90% of my kids come from, probably more than that, probably 95, have come from the fifth grade here and I know what they've taught. So, it's not a big deal, if they don't learn a specific skill in fifth grade, because I know most of my class hasn't learned it, I can just teach it, and there's probably something that they covered last year that, so I don't have to teach it this year.

During our final interview, Ms. Blue connected her desire to incorporate more proofs in her class with her ability to collaborate with the other teachers in the school. She explained that

> It [proofs] would be a good discussion, I think, between the whole middle school, on what we've all been doing, and where we could be going with it. There are some where, or maybe just to keep the conversation flow, even if it's just to make sure that I'm talking about it in sixth, [Mrs. Red]'s talking about it in seventh, and

> [Mrs. Yellow]'s talking about it in eighth. Because that's one thing, I don't, I can't remember talking about proofs in particular.

Ms. Blue's explanation highlights some of the comments discussed above, that the teachers wanted to make it more explicit to their students that they were working with proofs or the process of proving. Here, Ms. Blue's discussion is about the teachers making that explicit for each other during planning and other conversations. This suggests that even if the teachers' experiences with creating and revising the curriculum did not directly become a resource for making sense of proofs and the process of proving already, the finding that this is a resource for which the teachers could make sense of proofs and the process of proving is well established.

One of the inconsistencies found during this study was the unfamiliarity with the NCTM Reasoning and Proof Standard demonstrated by the teachers during final interviews and their claims to have based their curriculum on the Grade Level Expectations and the NCTM Standards. During my analysis I found no evidence that allowed me to make sense of this inconsistency, and I would offer that it is a point of interest for future work with these teachers.

Respect within and for the Schools' Structure

The resources described above were all related to the school's structure. These aspects of the school's structure would not have become resources for the teachers if there was not a shared respect between all of the teachers and administrators, or if these aspects were not as highly regarded and followed through on by all of the members of the school's faculty.

Mr. Purple explained some of the issues with using a curriculum that was created by the teachers. I asked Mr. Purple if he had faced any problems with the school board or parents when the school moved away from a traditional mathematics curriculum. He explained

> At first, yes, not just in mathematics, in everything, my first three years was hell. You know, because it was almost like the emperor's clothes, you know, where a grade school where the kids are coming from well-educated families, so they're going to look OK. But in reality, when you actually got in and looked in the classroom, what was happening, in practice, it wasn't very good. And so, and so a lot of parents we had, had gone through traditional educations. And they feel like it worked for them. They thought it was great, because they became CEOs, and all of those other things, so they thought, why? Why would you want to change it? And so, it took a while.

Mr. Purple's discussion about coming up against parents who did not want the school practices to change shows how determined he was to make the changes he did and implement programs that reflected the Guiding Principles. He went on to say that he meets less resistance from parents now, and attributes this to the success of the teachers.

> But I think the teachers have done an amazing job, in terms of using kids to show them [the parents], and when they start seeing, now, that their kids are secure and doing things as a six-year-old, they [the parents] could never do, that's, like, wow. And so, I see and hear less of that, today, than I've ever heard before.

Mr. Purple appreciates what the teachers have accomplished and is excited to see the results of their hard work. He explained

> I am excited, to what's happening. When I see kids doing stuff, and talk about how they're solving problems, or trying different strategies. I mean, to me, that's exciting. I'm really excited.

As the other administrator in this study, Mrs. Pink's respect for the teachers helps to demonstrate the respect of the administration for both the teachers and the ways in which the teachers are working with the students. She explained, " I think we have a wonderful group of teachers." The respect for Mrs. Pink from both Mr. Purple and the mathematics teachers in this study was discussed above. The teachers also shared their respect for Mr.

Purple and the school's philosophy. Ms. Sage explained that

> The philosophy of the school is why I want to be here, but I think that's why most of the teachers want to be here. Not for the pay or anything else, but just because, to be able to teach on the kind of level we are able to teach at is a fantastic thing.

When I asked Mrs. Yellow about the how the school's structure affects her ability to teach mathematics, she simply replied, "I think it lets me teach math." I asked her, "Yeah? Just totally?" And she said "Yep, totally." She then went on to describe meetings and support from administrators that she felt was a large part of the school's structure. But the most informative part of this conversation was the simple answer of, "it lets me teach math." Ms. Blue commented on the support from Mr. Purple, explaining that, "our principal is very flexible. He lets us do what we need to do, to make it meaningful for the kids, which is what we're here for, the kids. So, it's awesome." The mutual respect throughout the administration and teachers at Light School has afforded resources for the teachers to utilize in making meaning of proofs and the process of proving in their classrooms, and may even be a resource itself.

In this part of the chapter different resources that the teachers used in making meaning of proofs and the process of proving were presented. Although these alternative resources do not give a full picture to the teachers' *meaning making*, their presence supports the finding that resources or opportunities, outside of the teachers' direct beliefs or knowledge of proofs and the process of proving, allowed for a dual understanding and connected the teachers' understanding of proofs and process of proving, as were understood in relation to their students' education, to their classroom practices.

CHAPTER 5

CONCLUSION

In the previous chapter findings related to the different aspects of *making meaning:* utilization of knowledge, knowledge, beliefs, making sense of past knowledge and current experiences, and interconnection of practice and knowledge were presented. This chapter will reflect on these findings as a whole and address my main research question, *How do middle school mathematics teachers make meaning of their knowledge of proofs and the process of proving in the context of their classroom?* I will then discuss possible implications of this study related to the education and professional development of middle school mathematics. The limitations of this study will be addressed, and directions for future research will be explored.

Reflection on Major Findings

The most significant finding in this study is found in answering, *How do middle school mathematics teachers make meaning of their knowledge of proofs and the process of proving in the context of their classroom practices?*, for the group of teachers who had negative experiences with proofs and the process of proving prior to becoming teachers. For these teachers the alternative resources discussed earlier, and probably others that were not found during this study, allow the teachers to hold dual understandings of proofs and the process of proving. When *making meaning* of their knowledge of proofs and the process of proving in the context of their classrooms, the findings suggest that the teachers rely on an understanding of proofs and the process of proving that is different and disconnected from their understanding of proofs and the process of proving related to their experiences prior to becoming teachers. The teachers' negative and sometimes fearful *beliefs* and feelings about proofs in higher level mathematics courses show that the teachers did not consider proofs to have functioned in the many ways described in Chapters 2 in relation to their own education. However, the findings presented in Part I of Chapter 4 demonstrate how the teachers were utilizing proofs and the process of proving as functions of *verification, explanation, communication, discovery, exploration of definitions, generalization, inquiry, and justification, dismissal, or modification of a conjecture*. Moreover, the findings in Part II of Chapter 4 suggest that the teachers understand these functions of proofs and the process of proving in relation to their students' education and that they value the use of proofs and the process of proving as tools for the teaching and learning of mathematics.

When *making meaning* of their knowledge of proofs and the process of proving in the context of their classrooms, the findings suggest that the teachers rely on an understanding of proofs and the process of proving that is different and disconnected from their understanding of proofs and the process of proving related to their experiences prior to becoming teachers. Although I have no evidence to suggest that the teachers would not use proofs and the process of proving without this dual understanding, it is reasonable to assume that the teachers would have not incorporated proofs if they were relying on their understanding of proofs in relation to their educational experiences during the process of *making meaning* of proofs in their classrooms. Because of their negative and fearful beliefs about proofs in their own education, the teachers deemed them as unuseful in their own learning, and thus, it is reasonable to assume that without this dual understanding, they would have deemed proofs as unuseful in their students' learning.

In conclusion, during the process of *making meaning* of proofs and the process of proving in the context of their classroom practices, these teachers rely on their *knowledge* and *beliefs* about proofs and the process of proving in relation to their students' learning. They *make sense of their past knowledge and current experiences* by focusing only on their *knowledge* and *beliefs* about proofs and the process of proving in relation to their students' learning, leaving the *knowledge* and *beliefs* associated with their experiences with proofs in higher level mathematics courses as disjointed from this process. In other words, the past knowledge utilized in this aspect of *making meaning* includes their past knowledge related to their understanding of proofs and the process of proving in relation to their students' learning, and not their past knowledge of proofs and the process of proving related to their own educational experiences. The school's structure and philosophy influenced the teachers' professional practices and created situations for them to think about proofs and the process of proving during the creation and implementation of their own curriculum, providing professional development around areas like *Monitoring of Meaning*, and focusing classroom practices on individual instruction. In negotiating their *knowledge* and *beliefs* as part of their professional practices, the teachers utilized these opportunities in creating their dual understanding of proofs and the process of proving. Thus, the aspect of *making meaning* related to *the interconnections of practice and knowledge* was the foundation for creating their dual understanding.

For the two teachers in this study who did not have negative experiences with proofs during their own education, the above analysis needs only to be modified by removing the dual understanding. During the process of *making meaning* of proofs and the process of proving in the context of their classroom practices, these teachers also relied on their *knowledge* and *beliefs* about proofs and the process of proving in relation to their students' learning. The past knowledge utilized in making sense of past knowledge and current experiences was related to their understanding of proofs and the process of proving in relation to their students' learning, particularly for Mrs. White who expressed that she had little to no experience with proofs and the process of proving prior to becoming a teacher. The school's structure and philosophy influenced the teachers' professional practices and in negotiating their *knowledge* and *beliefs* as part of their professional practices, the teachers utilized these opportunities to think about the use of proofs and the process of proving. Thus, the aspect of *making meaning* related to the *interconnections of practice and knowledge* was also the foundation for these teachers to think about the use of proofs and the process of proving in their classroom practices.

Implications

The finding that the teachers hold dual understandings of proofs and the process of proving, and that their experiences with proofs during their college education were not considered during the process of *making meaning* of proofs in the context of their classrooms, suggests that the teachers' experiences in higher level mathematics courses were not beneficial in relation to their use or understanding of proofs and the process of proving. In order to address issues such as these it is imperative for teacher educators to focus more on how the teachers are learning mathematics, and not just the mathematical content of their college courses. This finding is supported by the Mathematical Science Education Board (1996) who suggested that

> It is not just the mathematics. Knowing mathematics does not ensure the effectiveness of prospective teachers. How they come to know their mathematics matters as well (p. 12).

It is also supported by the Conference Board of the Mathematical Sciences (2001) who recommend that:

> Middle grades teachers need to have opportunities to come to understand the types of reasoning middle grade students are able to undertake, and then be able to challenge their students in ways that will lead them to reason and make sense of mathematics. They need to provide their students with opportunities to explore, conjecture, provide counterexamples, and justify (p. 99).

CBMS's recommendation is clearly related to proofs.

The necessity for teacher educators to provide opportunities for teachers to engage in the process of proving in positive and meaningful ways is made clear by Harel and Sowder (2007)

> Many teachers are unlikely to teach proof well, since their own grasp of proof is limited. It is important to determine better the extent to which teachers are equipped to deliver a curriculum in which proof is central (p. 836-837).

As discussed in chapter 2, research has shown us that both preservice teachers and inservice teachers possess a minimal understanding about proofs and the process of proving. As explained by Stylianides and Ball (2008) this suggests an "inadequate preparation of many teachers to effectively cultivate proving in their classrooms" (p. 329). The findings of this study support that without the alternative resources utilized by the teachers, their own preparation would not have been enough for them to cultivate proving in their classrooms as efficiently and positively as was shown in Part I of Chapter 4.

The findings from this study suggest that the teachers were able to overcome their negative experiences with proofs through their professional practices and development as inservice teachers. This suggests that when teachers are afforded opportunities to think about proofs and the process of proving in relation to their students' education, they are able to think about and use proofs efficiently and positively in their classrooms. This finding suggests that school districts and universities should make it a priority to offer teachers opportunities such as those described in Part III of Chapter 4. This recommendation is supported by Stylianides (2007) who suggests that teacher education programs, both preservice and inservice,

> offer teachers the necessary guidance and equip them with the necessary resources so that they can effectively cultivate proof and proving among their students (p. 318).

Moreover, the teachers' use of professional practices and development provided to them through the school's structure and philosophy suggests that we need to pay closer attention to how school structure affects teachers' teaching practices.

Limitations and Direction for Future Research

One of the limitations of this study is based on both time constraints and lack of a framework for alternative resources. Since the finding of alternative resources came from the analysis of this study, I was unable to focus interviews and observation on this concept. I also did not enter the study with a framework for viewing alternative resources as a means of *making meaning*. I propose that one area of future research would be to create a framework for viewing the alternative resources used by teachers *making meaning* of the use of proofs and the process of proving in their classrooms. With more time this would have been a useful tool for further investigation with the teacher participants in this study. Using this framework, research about the process of *making meaning* could further our understanding in the knowledge for teaching proofs and using proofs as a tool for teaching and learning.

Further analysis of the usefulness of the framework of *making meaning,* presented as part of this study, is needed. More studies centered on the process of teachers *making meaning* would be useful to the field of mathematics education as we move forward in our attempt to understand teachers' knowledge. Future research associated with *making meaning* would help to clarify and validate the framework, as well as the need to look at the multifaceted knowledge base used by teachers. I propose that one area of future research would be to validate this framework, and that other research on teachers' knowledge should pay close attention to the knowledge used by teachers in their profession.

This study was conducted in an affluent community, in a school whose principal was strongly committed to students' learning of mathematics through the use of multiple strategies, including proofs and the process of proving. One of the limitations of this study relates to the fact that all of the data came from this school, and that this school's structure seems to be unique. The teachers' use of professional practices and development as inservice teachers that were provided to them through the school's structure and philosophy suggests that we need to pay closer attention to how school structure affects teachers' teaching practices. One area of future research would be focusing on school structure in relation to teachers' teaching practices. This research could aide in our understanding of teachers' pedagogical choices in relation to school structure and may help to frame alternative resources used by teachers working in different settings.

Finally, one possible limitation of this study is related to the teachers' past experiences with proofs. None of the teachers in this study reported having positive experiences with proofs in upper level mathematics course. Mrs. Yellow was partly an exception to this generalization. Recall however, that her "good" experiences with proofs were during calculus, while her "bad" experiences with proofs were during mathematics courses that are taught after calculus. Mrs. White reported having no higher-level mathematics courses and so her experiences with proofs were limited. This may have created a limitation, because the teacher populations who have had "good" experiences with proofs in college were not represented in the participant population. I propose that this was not a problem with the study, but that future research conducted on teachers *making meaning* and alternative resources with teachers who report having "good" experiences with proofs in higher level mathematics courses would be beneficial to the field. Moreover, research concerning what types of experiences with proofs during higher level mathematics courses positively or negatively affect teachers' *knowledge* and *beliefs* about proofs is suggested.

APPENDIX A

PROBLEM SOLVING RUBRIC FOR *BUILDING PATTERNS TO DISCOVER EQUATIONS*

PROBLEM SOLVING AND COMMUNICATION CRITERIA - Grades 5-8

10/23/08

PROBLEM SOLVING - UNDERSTANDING OF TASK AND USE OF APPROPRIATE STRATEGY

Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
Strategy(ies) or procedure(s) used would not work to solve the given problem or There is no evidence of the strategy(ies) or procedure(s) used to solve the problem	Strategy(ies) or procedure(s) used would only work to solve part of the problem or Strategy(ies) or procedure(s) used would work, but is not executed accurately or Strategy(ies) used are not grade level appropriate	Strategy(ies) or procedure(s) used is grade level appropriate and used accurately	Strategy(ies) or procedure(s) used is sophisticated for the expectations of the grade level and leads directly to a full solution (See grade expectations)

PROBLEM SOLVING - ACCURACY OF ANSWER

Below Standard	Approaching Standard	Meeting Standard	
There is no answer to the problem or No part of the answer is correct	The answer(s) is only partially correct	The answer(s) is correct for all parts of the problem	

PROBLEM SOLVING – UNDERSTANDING THE MATHEMATICS

Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
The response only solves the problem or The response is incomplete or There is no explanation or derivation of formulas that appear as an answer	The response does one or more of the following: - makes an observation about the important mathematics in the problem and/or - states an important pattern or rule	The response does one or more of the following: - defends the reasonableness of the answer with a clear explanation and/or - applies a discovered mathematical rule to at least 2 cases to prove its effectiveness	The response does one or more of the following: - a mathematical rule (generalization) whose derivation is clearly explained or proved another way and/or - proves the correctness of the answer by solving the problem a different way

COMMUNICATION - MATHEMATICAL VOCABULARY AND SYMBOLIC NOTATION

Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
Relevant content vocabulary or symbolic notation is absent and/or There is a significant error in use of content vocabulary or symbolic notation	There is some (limited) use of relevant content vocabulary or symbolic notation and/or There is a minor error in the content vocabulary or symbolic notation used	There is an appropriate and accurate use of mathematical content vocabulary for the grade level and/or There is an appropriate and accurate use of symbolic notation for the grade level (see grade expectations)	Mathematical content vocabulary used is sophisticated for the grade level and/or Symbolic notation used is sophisticated for the grade level (see grade expectations)

Symbolic Notation: Mathematical signs and symbols (e.g., %, ≥, Σ, ≐, →, ⊥, $f()$, !, {}, ≐, ...)

COMMUNICATION - MATHEMATICAL REPRESENTATION

Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
No representation is used to solve or communicate any aspect of the problem or An inappropriate representation is used	An appropriate representation is attempted, but it may be incomplete or have a minor flaw	There is an appropriate and accurate use of mathematical representation(s) for the grade level (see grade expectations)	The representation(s) used are sophisticated for the grade level and/or The representations are linked to equations, models or other representations (see grade expectations)

Mathematical Representations: Graphs, plots, charts, tables, models, diagrams, keys

COMMUNICATION – DOCUMENTATION/ORGANIZATION OF REASONING

Below Standard	Approaching Standard	Meeting Standard
The documentation of the correct or incorrect solution process contains little or no evidence of how the problem was solved or the reasoning used	The documentation of the correct or incorrect solution process contains some evidence of how the problems was solved and the reasoning used but there are some gaps or unclear parts	The documentation of the correct or incorrect solution process clearly shows how the problem was solved and the reasoning used: Computations used are noted Presentation is in a logical order All parts are connected and labeled Answer(s) is highlighted Mathematical explanations or arguments are clear

FRAMEWORK FOR PROBLEM SOLVING

TITLE: Write the title of the task (if there is one) at the top of your paper.

QUESTION: Write the question you are trying to answer.

INFORMATION:

First list the information provided that will help you solve the problem

Next, identify any information you need to research to help you solve the problem. (this may not be needed)

Document the information you obtained from your research

STRATEGY: Choose a strategy (or combination of strategies) that you want to use to try to solve the problem. The strategy might be:

- Making a quick sketch of the problem (or a complete diagram)
- Using manipulative materials to model the problem
- Acting out the problem with people or things
- Making a table of information
- Working backwards
- Solving a simpler problem first and then successively bigger ones
- Finding and using a pattern
- Successive Approximation (Guess and Check)

SOLUTION: Document all of the thinking and work you did to solve the problem

ANSWER: Pinpoint your answer(s) to the to the question(s) asked by highlighting it or boxing it in...or using some other means

SHOWING UNDERSTANDING OF THE MATHEMATICS - STRETCHING

Justification: Explain logically why your solution must be correct

Verification: Solve the problem a different way to check your answer

Find an important pattern or rule in the problem

Application: Apply your pattern or rule to at least 2 cases of the situation to prove it works

Generalization: Make a rule for finding the solution to any version of the problem (when the numbers in the problem change, but the situation is the same)

COMMUNICATION CHECK:

VOCABULARY: Did you use appropriate content vocabulary?

NOTATION: Did you use appropriate mathematical notation?

REPRESENTATION: Did you use appropriate and accurate representations?

PRESENTATION: Is your work organized and clear to a reader?

APPENDIX B

COPIES OF THE ORIGINAL ACTIVITY AND THE PROBLEM SOLVER FOR

BUILDING PATTERNS TO DISCOVER EQUATIONS

Original Activity:

3 Crossing the River

USING TABLES TO DESCRIBE AND PREDICT PATTERNS

Examining a pattern can help you develop a general rule that applies to any stage of the pattern. In this investigation you will look for a pattern to solve the problem of getting a group of hikers across a river using one small boat.

Find a Rule for Any Number

How can finding a pattern help solve for all cases?

Think carefully about how the hikers could cross the river using just one boat. It may be helpful to act it out or use a diagram to solve the problem. As you work, make a table showing how many trips it takes for 1 to 5 adults and 2 children to cross. Look for the pattern, then use it to find how many trips are required for the other groups to cross to the other side.

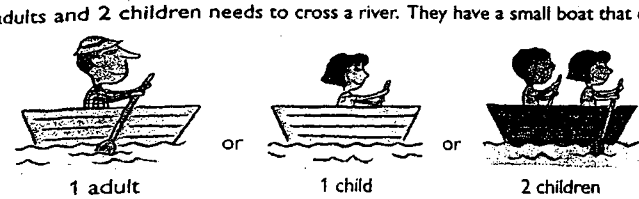
1. How many one-way trips does it take for the entire group of 8 adults and 2 children to cross the river? Tell how you found your answer.
2. How many trips in all for 6 adults and 2 children?
3. 15 adults and 2 children?
4. 23 adults and 2 children?
5. 100 adults and 2 children?

Tell how you would find the number of one-way trips needed for any number of adults and two children to cross the river.

(Everyone can row the boat.)

Ten Hikers—One Boat

A group of 8 adults and 2 children needs to cross a river. They have a small boat that can hold either:



Original Activity (continued):

Use Your Method in Another Way

Use the pattern to find the number of adults who need to cross the river for each case.

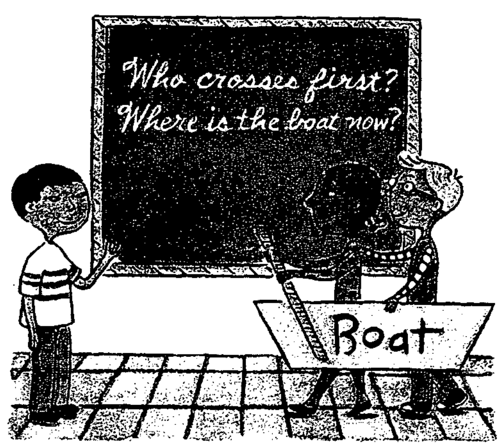
1. It takes 13 trips to get all of the adults and the 2 children across the river.
2. It takes 41 trips to get all of the adults and the 2 children across the river.
3. It takes 57 trips to get all of the adults and the 2 children across the river.

How can you work backward from what you know?

Tell How You Look for Patterns

Write a friend a letter telling how you look for patterns. Give examples from the patterns you have investigated so far. Answers to the following questions will help you write your letter.

- How can a table help you discover and describe a pattern?
- What other tools are helpful?
- How does finding a pattern help you solve problems?



hot Words: pattern, table

HW Homework
page 356

Problem Solver:

Crossing the River

Name:_____________ __________ Date:___________ Class:______

Ten Hikers—One Boat (Everyone can row the boat.)

A group of 8 adults and 2 children needs to cross a river. They have a small boat that can hold either:

1 adult or 1 child or 2 children

Think carefully about how the hikers could cross the river using just one boat. You may use any method you would like to solve the problem, just be sure to document your work.

1. How many one – way trips does it take for the entire group of 8 adults and 2 children to cross the river? (Explain how you found your answer)

2. Tell how you would find the number of one-way trips needed for any number of adults and two children to cross the river. (Everyone can row the boat)

Use the framework to set up your problem solver.
Self-evaluate your performance on the rubric provided.

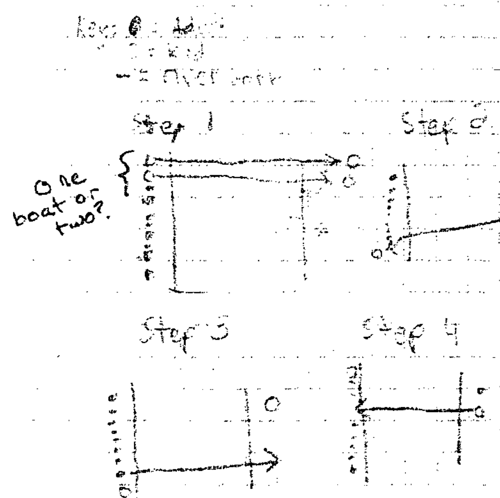
APPENDIX C

STUDENT WORK FOR *BUILDING PATTERNS TO DISCOVER EQUATIONS*

CORY'S WORK:

Crossing The River

How many one-way trips does it take a group of 8 adults and 2 children to cross a river in small boat? How would you find the number of one way trips for X adults & 2 kids? One adult can fit in the boat. One child can fit in the boat. Two children can fit in the boat.



Repeat the pattern until all of the adults are across. Then repeat step 1.

The number of one-way trips for 8 adults and 2 children would be 33. It's 4 trips for each adult and then one extra for the two children. For any number of adults and 2 children the formula would be 4x (4 times the number of adults) +1. For example, with 36 adults there would be a total of 145 trips. With 6 adults there would be a total of 25 trips. [4x6+1] required.

So the equation is $4X+1=T$

X is # of adults
T is # of trips

CORY'S GRADING RUBRIC:

PROBLEM SOLVING AND COMMUNICATION CRITERIA - Grades 5-8

10/23/08

PROBLEM SOLVING - UNDERSTANDING OF TASK AND USE OF APPROPRIATE STRATEGY

3+

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PROBLEM SOLVING - ACCURACY OF ANSWER

3

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PROBLEM SOLVING – UNDERSTANDING THE MATHEMATICS

4+

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Mathematical Representations: Graphs, plots, charts, tables, models, diagrams, keys

COMMUNICATION – DOCUMENTATION/ORGANIZATION OF REASONING

(3)

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ABBY'S WORK:

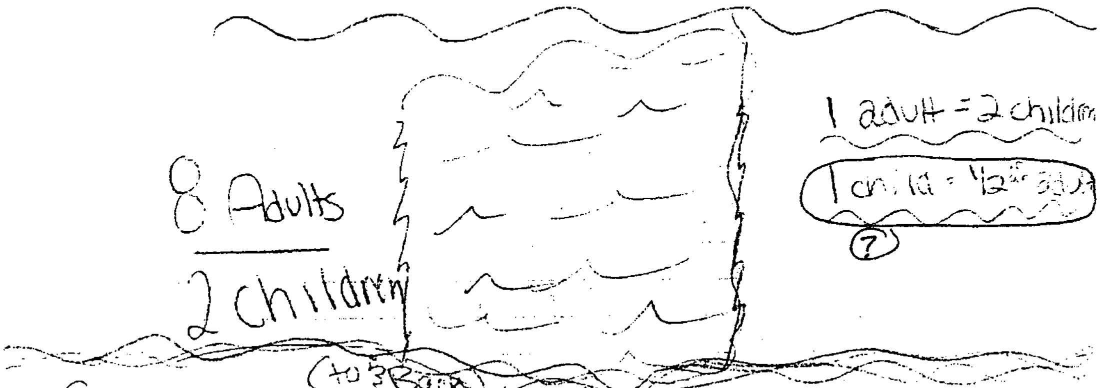
Crossing the river:

1.) How many one way trips would it take to cross the river (every body getting across)? ✓

2.) Tell how you found that number. ✓

Part I - boat can hold 1 adult, 1 child or, 2 children ✓

Part II - one person will have to stay in the boat at all times ✓



CHART

PART I:

Might also help to have a column with who is on the side

# of - boat trips (total at ends) (to & Back)	# of - adults / children	# of - Total people on opposite side
1 T	2c	2c
1 B	1c	1c
1 T	1a	1a, 1c
1 B	1c	1a
1 T	2c	1a, 2c
1 B	1c	1a, 1c
1 T	1a	2a, 1c
1 B	1c	2a
1 T	2c	2a, 2c
1 B	1c	2a, 1c
1 T	1a	3a, 1c
1 B	1c	3a
1 T	2c	3a, 2c
1 B	1c	3a, 1c

pattern

Key
T = To
B = Back
a = adult
C = children

1T	1a	4a, 1c
1B	1c	4a
1T	2c	4a, 2c
1B	1c	4a, 1c
1T	1a	5a, 1c
1B	1c	5a
1T	2c	5a, 2c
1B	1c	5a, 1c
1T	1a	6a, 1c
1B	1c	6a
1T	2c	6a, 2c
1B	1c	6a, 1c
1T	1a	7a, 1c
1B	1c	7a
1T	2c	7a, 2c
1B	1c	7a, 1c
1T	2a	8a, 1c
1B	1c	8a
1T	2c	8a, 2c

[redacted]

I'm not sure what you're talking about here with the weight.

Why my ~~concluded~~ answer of 33 one way boat trips is correct is because all of the boat trips have an equal or less weight, which is equivilant to 1 adult or 2 children. But, since 1 child is less than 1 adult or 2 children, that would still be an approprate move. I have also carefully counted the number of boat trips on my chart, so my answer should be accurate. Also at the end, when I had finished the chart, I was sure I had finished when, in the total people on other side colum, I had 8a and 2c (or 8 adults and 2 children).

PART II:

How I got the answer of 33 one way boat trips, is that I first decided that the most logical way to begin this problem would to cross the largest possible amount of people, because one person always has to be in the boat while it crosses, so it can get across. Thats why if they are 2 people at least one person will get to stay on the opposite side. Otherwise yes! it really would be pointless. Next I really had to send back a child, so that what I did. After that I chose an adult because if I did another child the process would yes! just repeat itself. Then I had to do the next step in my chart, so I thought ahead again and if I took back a child I could replace that with 2 the next time. So, that's what I did. Then after the next 3 move of repeated steps, I noticed a pattern of down 2 children, back one child, down one adult, and then on child back. I decided to mark the down and from that point on it was just repeated with one more adult each set. Finally at the end when I had 8 adults and it was the last set of the pattern I crossed 2 children and since that was what we needed to get to I was done. Then I counted up all of the boat trips and got 33, which concludes to my answer, 33 one way boat trips. ✓ But this is part 1 - How would you do it for any number?

excellent explaination of your thought process!

ABBY'S GRADING RUBRIC:

PROBLEM SOLVING AND COMMUNICATION CRITERIA - Grades 5-8

10/23/08

(3)

PROBLEM SOLVING - UNDERSTANDING OF TASK AND USE OF APPROPRIATE STRATEGY			
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(2)

PROBLEM SOLVING - ACCURACY OF ANSWER			
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(1)

PROBLEM SOLVING – UNDERSTANDING THE MATHEMATICS			
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← Because you just told how ya solved the problem and the weight doesn't explain why 33 works

(2)

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Symbolic Notation: Mathematical signs and symbols (e.g., %, ≥, Σ, ∈, →, ⊥, $f()$, !, { }, ≈, ...)

(3)

COMMUNICATION - MATHEMATICAL REPRESENTATION			
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Mathematical Representations: Graphs, plots, charts, tables, models, diagrams, keys

(3)

COMMUNICATION – DOCUMENTATION/ORGANIZATION OF REASONING		
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APPENDIX D

PROBLEM SOLVING RUBRIC FOR *ARGUE YOUR ANSWER MATHEMATICALLY*

PROBLEM SOLVING AND COMMUNICATION CRITERIA - Grades 5-8

1/6/06

PROBLEM SOLVING - UNDERSTANDING OF TASK AND USE OF APPROPRIATE STRATEGY			
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PROBLEM SOLVING - ACCURACY OF ANSWER			
Below Standard	Approaching Standard	Meeting Standard	
There is no answer to the problem or The answer is not correct or No work is present to support the final correct answer(s)	The final correct answer(s) is only partially supported by the work presented	The final correct answer(s) is fully supported by the work presented	

PROBLEM SOLVING - CONNECTING THE MATHEMATICS			
Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
The response only solves the problem or The response is incomplete	The response goes beyond the solution because it includes one or more of the following: - a mathematically relevant observation about the problem - a connection of the underlying mathematical concept(s) in this problem to a similar problem or a real world application - an important pattern or rule	The response - proves the correctness of the answer by solving the problem a different way and/or - defends the reasonableness of the answer with a clear explanation and/or - applies a discovered mathematical rule to at least 2 new, higher cases	The response includes - a mathematical rule (generalization) whose derivation is clearly explained or proved another way and/or - the creation and solution of a more challenging version of the problem

COMMUNICATION - MATHEMATICAL VOCABULARY AND SYMBOLIC NOTATION			
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APPENDIX E

PROBLEM SOLVER FOR *ARGUE YOUR ANSWER MATHEMATICALLY*

Who is the Best?

Rick, Mike, and Sarah are all on their school's golf team. They have been practicing their chipping. Each player thinks they are the best chipper on the team. To decide who is right, they have a contest. Each player chips 10 balls onto the same green. The balls are different colors so they can tell them apart. When their finish, they measure the distance from each ball to the cup in inches. Here are the results:

Rick: 40, 46, 60, 95,1 00, 105, 120, 152, 312, 320
Mike: 52, 60, 64, 76, 88, 120, 184, 188, 230, 288
Sarah: 84, 99, 120, 129, 130, 135, 136, 152, 165, 200

When the contest was over, the kids still couldn't decide on the winner. The balls were all spread out. No one was close every time. They ask the coach for advice. He said, "In the game of golf, getting close and being consistent are important. So, you should consider who is closest and most consistent. Don't just consider who had the best shot. You're math whizzes – I'm sure you can figure it out."

Help thekids decide who won. Analyze the results in as many different ways as you know. Present a mathematical argument to back up your decision about who the winner was and why they won.

APPENDIX F

STUDENT WORK FOR *ARGUE YOUR ANSWER MATHEMATICALLY*

EMMA'S WORK:

Who Is The Best?

- Who is the most consistent chipper?

Mike: 52, 60, 64, 76, 88, 120, 184, 188, 230, 288
Mean: 135
Median: 104
Mode: No Mode
Range: 236

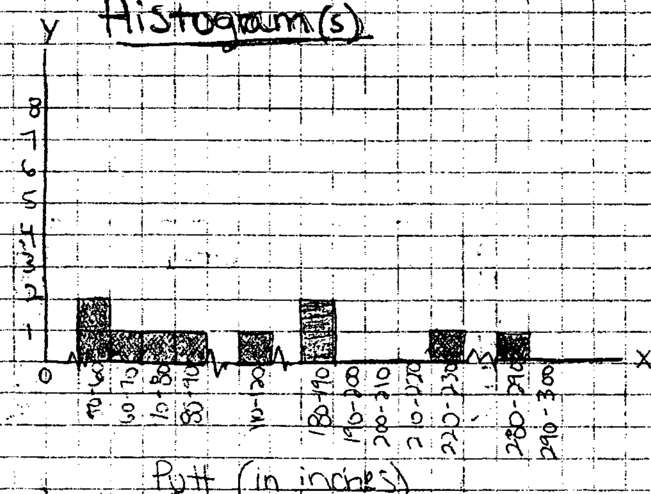
Rick: 40, 46, 60, 95, 100, 105, 120, 152, 312, 320
Mean: 135
Median: 102.5
Mode: No Mode
Range: 280

Sarah: 84, 99, 120, 129, 130, 135, 136, 152, 165, 200
Mean: 135
Median: 132.5
Mode: No Mode
Range: 216

Histogram(s)

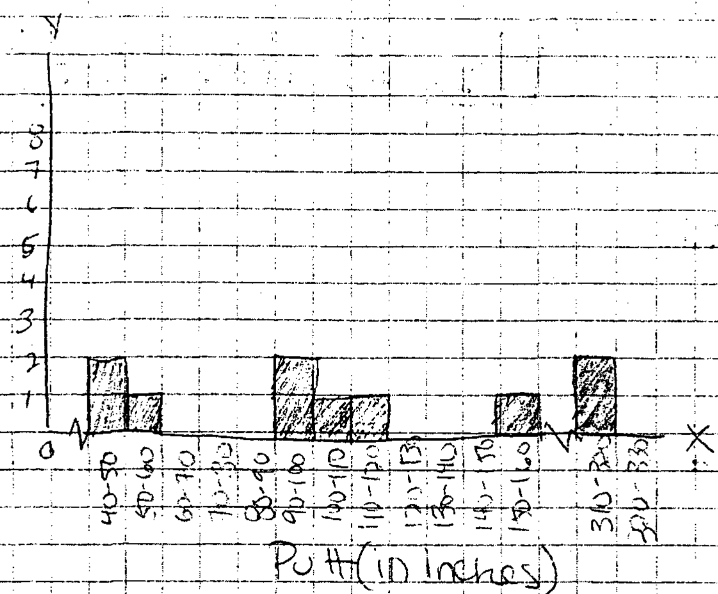
Mike:

	Frequency	
50-60	II	2
60-70	I	1
70-80	I	1
80-90	I	1
110-120	I	1
180-190	II	2
190-200		0
200-210		0
210-220		0
220-230	I	1
280-290	I	1
290-300		0



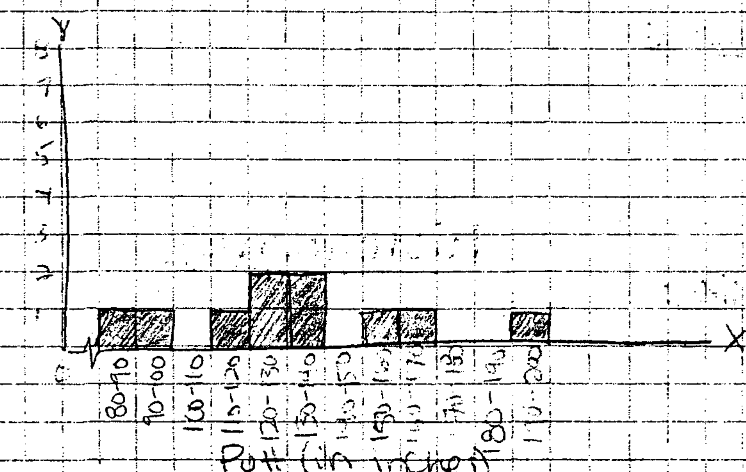
Rick:

40-50	ll	2
50-60	l	1
60-70		0
70-80		0
80-90		0
90-100	lll	2
100-110	l	1
110-120	l	1
120-130		0
130-140		0
140-150		0
150-160	l	1
310-320	ll	2
320-330		0



Sarah:

80-90	l	ll
90-100	l	ll
100-110		0
110-120	l	ll
120-130	ll	2
[illegible]	lll	2
[illegible]		0
[illegible]	l	1
[illegible]-170	l	1
170-180		0
180-190		0
190-200	l	1



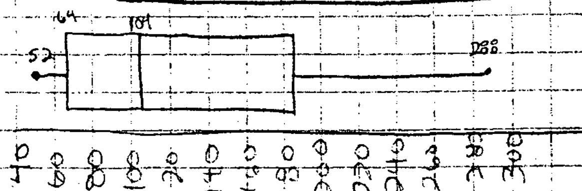
ANSWER

I think that **Sarah** is the most consistent chipper. She has a piece of data in almost every interval. Rick has many spaces, and Mike has many skips. Neither is very consistent. Almost all of her bars are touching [illegible].

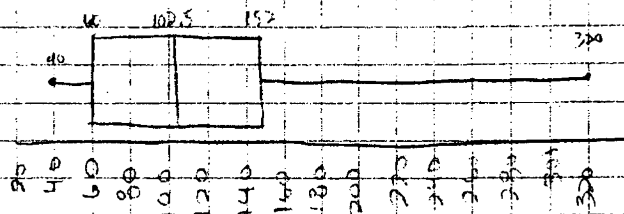
ANSWER

Box-and-Whisker(s)

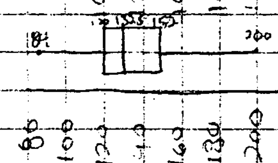
Mike:



Rick:



Sarah:



ANSWER

Sarah is still the most consiste[nt] after I did two graphs. Her boxes were really close together and the others were really spread out.

FINAL ANSWER

I think Sarah's the most consistent chipper, because she has the smallest range and you can see that in every graph I made her shots were very close together and she kept them even throughout the whole range of all 3 graphs.

EMMA' GRADING RUBRIC:

PROBLEM SOLVING AND COMMUNICATION CRITERIA - Grades 5-8

PROBLEM SOLVING - UNDERSTANDING OF TASK AND USE OF APPROPRIATE STRATEGY			
Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
Strategy(ies) or procedure(s) used would not work to solve the given problem or There is no evidence of the strategy(ies) or procedure(s) used to solve the problem	Strategy(ies) or procedure(s) used would only work to solve part of the problem or Strategy(ies) or procedure(s) would work but it is not carried through to achieve a final answer	Strategy(ies) or procedure(s) used leads to a full solution ☐ Efficient	Strategy(ies) or procedure(s) used is sophisticated for the expectations of the grade level, and leads directly to a full solution ☐ Efficient (See grade expectations)

PROBLEM SOLVING - ACCURACY OF ANSWER			
Below Standard	Approaching Standard	Meeting Standard	
There is no answer to the problem or The answer is not correct or No work is present to support the final correct answer(s)	The final correct answer(s) is only partially supported by the work presented	The final correct answer(s) is fully supported by the work presented Think about who is Closest.	

PROBLEM SOLVING - CONNECTING THE MATHEMATICS			
Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
The response only solves the problem or The response is incomplete	The response goes beyond the solution because it includes one or more of the following: - a mathematically relevant observation about the problem - a connection of the underlying mathematical concept(s) in this problem to a similar problem or a real world application - an important pattern or rule	The response - proves the correctness of the answer by solving the problem a different way and/or - defends the reasonableness of the answer with a clear explanation and/or - applies a discovered mathematical rule to at least 2 new, higher cases	The response includes - a mathematical rule (generalization) whose derivation is clearly explained or proved another way and/or - the creation and solution of a more challenging version of the problem

COMMUNICATION - MATHEMATICAL VOCABULARY AND SYMBOLIC NOTATION			
Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
Relevant content vocabulary or symbolic notation is absent and/or There is a significant error in use of content vocabulary or symbolic notation and/or There is no explanation or derivation of formulas that appear as an answer	There is some (limited) use of relevant content vocabulary or symbolic notation and/or There is a minor error in the content vocabulary or symbolic notation used	There is an appropriate and accurate use of mathematical content vocabulary for the grade level and/or There is an appropriate and accurate use of symbolic notation for the grade level (see grade expectations)	Mathematical content vocabulary used is sophisticated for the grade level and/or Symbolic notation used is sophisticated for the grade level (see grade expectations)

Symbolic Notation: Mathematical signs and symbols (e.g., %, ≥, Σ, ●, →, ⊥, f(), √, {}, ≠,...)

COMMUNICATION - MATHEMATICAL REPRESENTATION			
Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
No representation is used to solve or communicate any aspect of the problem or An inappropriate representation is used	An appropriate representation is attempted, but it may be incomplete or have a minor flaw	There is an appropriate and accurate use of mathematical representation(s) for the grade level (see grade expectations)	The representation(s) used are sophisticated for the grade level and/or The representations are linked to equations, models or other representations (see grade expectations)

Mathematical Representations: Graphs, plots, charts, tables, models, diagrams, keys

COMMUNICATION - DOCUMENTATION OF REASONING		
Below Standard	Approaching Standard	Meeting Standard
The documentation of the correct or incorrect solution process contains little or no evidence of how the problem was solved or the reasoning used	The documentation of the correct or incorrect solution process contains some evidence of how the problem was solved and the reasoning used but there are some gaps or unclear parts	The documentation of the correct or incorrect solution process clearly shows how the problem was solved and the reasoning used: Computations used are noted Presentation is in a logical order All parts are connected and labeled Answer(s) is highlighted Mathematical explanations or arguments are clear

Your organization is Commendable!

LEE'S WORK:

- Who is the best?

Rick, mike and Sarah each took 10 shots (chips) on to the same green whos was best?

I think that Rick will do the best since he started off really low and finnished high and it looked from the data he had the best shot.

Mean, median and mode, cirle graph and Histogram chart.

Rick: Mean: 1350 ÷ 10 = (135)

Median: (62.5)

Mode: N/A

Range: (280)

Mike: Mean: 1350 ÷ 10 = (135)

Median: 88, 120 ÷ 2 = (104) Range: (236)

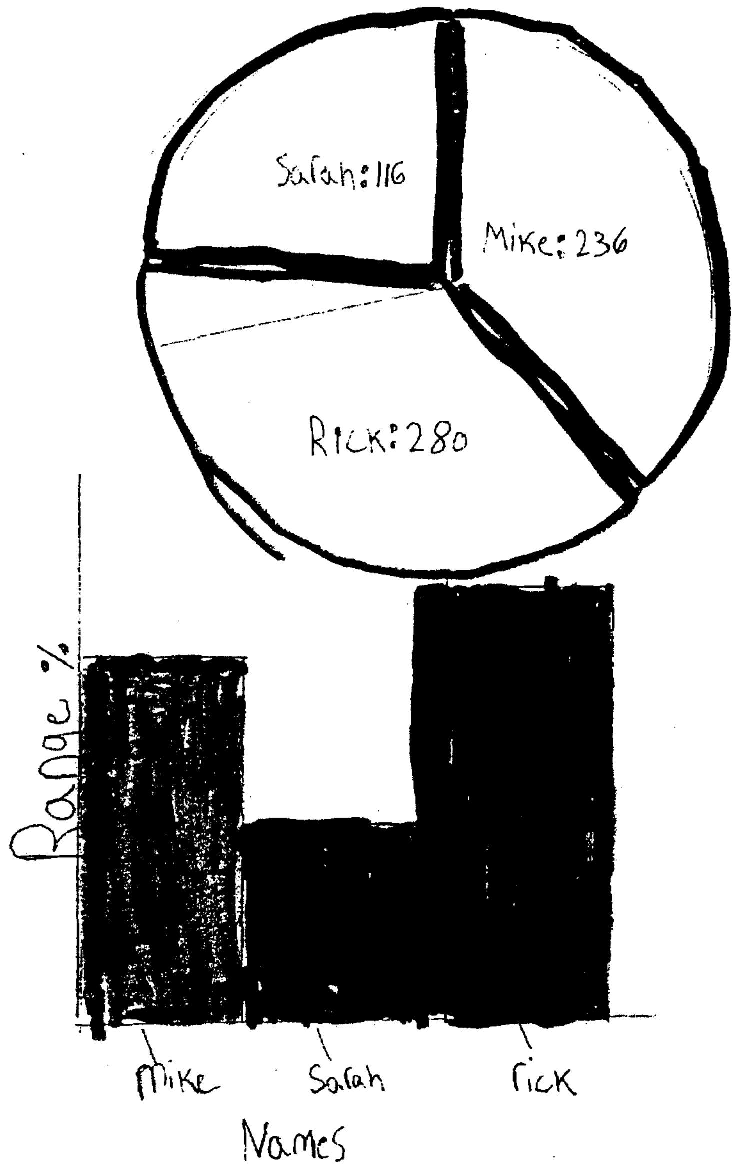
Mode: N/A

Sarah: Mean: 1350 ÷ 10 = (135)

Median: 130, 135 = (132.5) Range: (116)

Mode: N/A

Shown in the data Sarahs shot was the best! She was more consistent than rick or mike!



Sarah:116
Mike:236
Rick:280
Range %
Mike
Sarah
rick
Names

LEE'S GRADING RUBRIC:

PROBLEM SOLVING AND COMMUNICATION CRITERIA - Grades 5-8

PROBLEM SOLVING - UNDERSTANDING OF TASK AND USE OF APPROPRIATE STRATEGY

Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
Strategy(ies) or procedure(s) used would not work to solve the given problem or There is no evidence of the strategy(ies) or procedure(s) used to solve the problem	Strategy(ies) or procedure(s) used would only work to solve part of the problem or Strategy(ies) or procedure(s) would work but it is not carried through to achieve a final answer	Strategy(ies) or procedure(s) used leads to a full solution ☐ Efficient	Strategy(ies) or procedure(s) used is sophisticated for the expectations of the grade level and leads directly to a full solution ☐ Efficient (See grade expectations)

PROBLEM SOLVING - ACCURACY OF ANSWER

Below Standard	Approaching Standard	Meeting Standard	
There is no answer to the problem or The answer is not correct or No work is present to support the final, correct answer(s)	The final correct answer(s) is only partially supported by the work presented	The final correct answer(s) is fully supported by the work presented	

you need to combine strategies to make your arguement strong.

PROBLEM SOLVING - CONNECTING THE MATHEMATICS

Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
The response only solves the problem or The response is incomplete	The response goes beyond the solution because it includes one or more of the following: - a mathematically relevant observation about the problem - a connection of the underlying mathematical concept(s) in this problem to a similar problem or a real world application - an important pattern or rule	The response - proves the correctness of the answer by solving the problem a different way and/or - defends the reasonableness of the answer with a clear explanation and/or - applies a discovered mathematical rule to at least 2 new, higher cases	The response includes - a mathematical rule (generalization) whose derivation is clearly explained or proved another way and/or - the creation and solution of a more challenging version of the problem

COMMUNICATION - MATHEMATICAL VOCABULARY AND SYMBOLIC NOTATION

Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
Relevant content vocabulary or symbolic notation is absent and/or There is a significant error in use of content vocabulary or symbolic notation and/or There is no explanation or derivation of formulas that appear as an answer	There is some (limited) use of relevant content vocabulary or symbolic notation and/or There is a minor error in the content vocabulary or symbolic notation used	There is an appropriate and accurate use of mathematical content vocabulary for the grade level and/or There is an appropriate and accurate use of symbolic notation for the grade level (see grade expectations)	Mathematical content vocabulary used is sophisticated for the grade level and/or Symbolic notation used is sophisticated for the grade level (see grade expectations)

Symbolic Notation: Mathematical signs and symbols (e.g., %, ≥, Σ, π, →, ⊥, $f()$, !, { }, ≈, ...)

COMMUNICATION - MATHEMATICAL REPRESENTATION

Below Standard	Approaching Standard	Meeting Standard	Exceeding Standard
No representation is used to solve or communicate any aspect of the problem or An inappropriate representation is used	An appropriate representation is attempted, but it may be incomplete or have a minor flaw	There is an appropriate and accurate use of mathematical representation(s) for the grade level (see grade expectations)	The representation(s) used are sophisticated for the grade level and/or The representations are linked to equations, models or other representations (see grade expectations)

your representations are not accurate.

Mathematical Representations: Graphs, plots, charts, tables, models, diagrams, keys

COMMUNICATION - DOCUMENTATION OF REASONING

Below Standard	Approaching Standard	Meeting Standard
The documentation of the correct or incorrect solution process contains little or no evidence of how the problem was solved or the reasoning used	The documentation of the correct or incorrect solution process contains some evidence of how the problem was solved and the reasoning used but there are some gaps or unclear parts	The documentation of the correct or incorrect solution process clearly shows how the problem was solved and the reasoning used Computations used are noted Presentation is in a logical order All parts are connected and labeled Answer(s) is highlighted Mathematical explanations or arguments are clear

Why do you believe Sarah is the most consistent?

APPENDIX G

STUDENT WORK FOR *EXPLAINING REASONING*

THERESA'S WORK:

Name
Date 4/12/00

Factors and the Locker Problem

1) Define a prime number and give an example.

A prime number is a number with only 2 factors. Its two factors are 1 and itself.
Example: 2

2 = 2 only has 2 factors

2) Find the prime factorization of 50.

$2^1 \cdot 5^2$

3) How many factors of 50?

$2^1 \cdot 5^2$

$2 \cdot 3 = 6$

6 factors of 50

4) What are the factors of 50?

$2^0 \times 5^0 = 1$
$2^1 \times 5^0 = 2$
$2^0 \times 5^1 = 5$
$2^0 \times 5^2 = 25$
$2^1 \times 5^1 = 10$
$2^1 \times 5^2 = 50$

5) Thinking about the locker problem, would locker number 144 be open or closed? Explain your reasoning.

144 = 72 · 2, 72 = 36 · 2, 36 = 18 · 2 ...

$2^4 \times 3^2$

$5 \times 3 = 15$ (odd)

Locker #144 would be open. I know this because all lockers with an odd number of factors are open. Prime factors are closed because they have an even number of factors.

Great!

6) Factor $33a^3b^2$

Good start

$3 \cdot 11 \cdot a \cdot a \cdot a \cdot b \cdot b$

$33a^3b^2$ → 11, a^3b^2 → a, a^2b^2 → a, ab^2 → a ...

7) Factor -15

-15 → 5, 3

KATHRYN'S WORK:

Name ____
Date 9/12/08

Factors and the Locker Problem

1) Define a prime number and give an example.
A prime number is a number that only has 2 factors. 1 and itself. For example, 7 is a prime number because the only factors it has is 1 and 7.

2) Find the prime factorization of 50.
On the back of the page

3) How many factors of 50?
On the back of the page.

4) What are the factors of 50?
1, 2, 5, 10, 25, 50

On the back of the page (work)

5) Thinking about the locker problem, would locker number 144 be open or closed? Explain your reasoning. Locker number 144 would be open. The reason for that is because every locker that's open has/is 2 things. It's a square number and it has an odd number of factors. Another reason is because only three people touch the locker sense the factors are 1, 12 and 144 so the first student opens the locker, 2nd student closes it and then, the 3rd student opens it again.

You are right that 144 has an odd # of factors, but it has more than 3 factors.

6) Factor $33a^3b^2$
On the back

7) Factor -15
On the back

APPENDIX H

INSTITUTATIONAL REVIEW BOARD APPROVAL

University *of* New Hampshire

Research Conduct and Compliance Services, Office of Sponsored Research
Service Building, 51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

29-Feb-2008

Paddack, Megan
Mathematics & Statistics, Kingsbury Hall
2 Dock Road
Gilford, NH 03249

IRB #: 4110
Study: Middle school mathematics teachers' meaning of proof
Approval Date: 29-Feb-2008

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Exempt as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 101(b) with the following comment(s):

The researcher is approved to conduct her study at Massabesic Middle School. Before conducting her study in other schools, the researcher needs to forward to the IRB a letter in support of the study from the principal.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* (This document is also available at http://www.unh.edu/osr/compliance/irb.html.) Please read this document carefully before commencing your work involving human subjects.

Upon completion of your study, please complete the enclosed pink Exempt Study Final Report form and return it to this office along with a report of your findings.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Manager

cc: File
Graham, Karen

University *of* New Hampshire

Research Conduct and Compliance Services, Office of Sponsored Research
Service Building, 51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

28-Aug-2008

Paddack, Megan
Mathematics & Statistics
Kingsbury Hall
2 Dock Road
Gilford, NH 03249

IRB #: 4110
Study: Middle school mathematics teachers' meaning of proof
Approval Expiration Date: 27-Aug-2009
Modification Approval Date: 27-Aug-2008
Modification: Addition of children as subjects

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved your modification to this study, as indicated above. Further changes in your study must be submitted to the IRB for review and approval prior to implementation.

Approval for this protocol expires on the date indicated above. At the end of the approval period you will be asked to submit a report with regard to the involvement of human subjects in this study. If your study is still active, you may request an extension of IRB approval.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* This document is available at http://www.unh.edu/osr/compliance/irb.html or from me.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Manager

cc: File
Graham, Karen

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