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Morita equivalence for group -theoretical categories

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MORITA EQUIVALENCE FOR
GROUP-THEORETICAL CATEGORIES

BY

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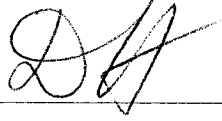
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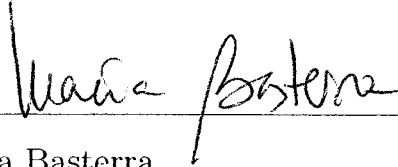
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ABSTRACT

MORITA EQUIVALENCE FOR GROUP-THEORETICAL CATEGORIES

by

Deepak Naidu

University of New Hampshire, May, 2007
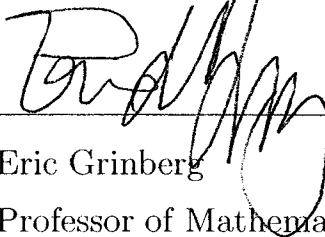
We give necessary and sufficient conditions for two pointed categories to be dual to each other with respect to a module category. Whenever the dual of a pointed category with respect to a module category is pointed, we give explicit formulas for the Grothendieck ring and for the associator of the dual. This leads to the definition of *categorical Morita equivalence* on the set of all finite groups and on the set of all pairs (G, ω), where G is a finite group and $\omega \in H^3(G, k^\times)$. A group-theoretical and cohomological interpretation of this relation is given. As an application, we give a series of concrete examples of pairs of groups that are categorically Morita equivalent but have non-isomorphic Grothendieck rings. In particular, the representation categories of the Drinfeld doubles of the groups in each example are equivalent as braided tensor categories and hence these groups define the same modular data.

The notion of a nilpotent fusion category, which categorically extends the notion of a nilpotent group, was introduced by Gelaki and Nikshych. We give sufficient conditions for a group-theoretical category to be nilpotent.

We classify Lagrangian subcategories of the representation category of a twisted quantum double $D^\omega(G)$, where G is a finite group and ω is a 3-cocycle on it. This gives a description of all braided tensor equivalences between twisted quantum doubles of fi-

nite groups. We also establish a canonical bijection between Lagrangian subcategories of $\text{Rep}(D^\omega(G))$ and module categories over the category Vec_G^ω of twisted G-graded vector spaces such that the dual fusion category is pointed. As a consequence, we establish that two group-theoretical fusion categories are weakly Morita equivalent if and only if their centers are equivalent as braided tensor categories.

CHAPTER 1

INTRODUCTION

Fusion categories arise in several areas of mathematics such as representation theory, quantum groups, and operator algebras. Several results concerning the structure and classification of fusion categories have appeared in literature (see [ENO] and references therein). There is an important class of fusion categories called *group-theoretical*. As the name suggests, these are fusion categories that come from a group-theoretical datum. One of the reasons for the importance of group-theoretical categories is that it is not known, at the time of writing, whether there exists a finite-dimensional Hopf algebra whose representation category is not group theoretical. In this work we study an equivalence relation called *weak Morita equivalence* on the class of group-theoretical categories.

Throughout this work we will work over an algebraically closed field k of characteristic 0. Unless otherwise stated all cocycles appearing in this work will have coefficients in the trivial module $k^\times$. A right module category over a tensor category $\mathcal{C}$ is a category $\mathcal{M}$ together with a functor $\mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and certain associativity and unit constraints satisfying some natural axioms (see [O1] and references therein). The dual of a tensor category $\mathcal{C}$ with respect to a module category is the category $\mathcal{C}^*_{\mathcal{M}} := \mathrm{Func}(\mathcal{M}, \mathcal{M})$ whose objects are $\mathcal{C}$-module functors from $\mathcal{M}$ to itself and mor-

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phisms are natural module transformations. The category $\mathcal{C}^*_\mathcal{M}$ is a tensor category with tensor product being composition of module functors.

A *fusion category* over k is a k-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects and finite-dimensional Hom-spaces such that the neutral object in simple (see [ENO]). If $\mathcal{C}$ is a fusion category and $\mathcal{M}$ is a semisimple indecomposable module category over $\mathcal{C}$, then it is known that the dual category $\mathcal{C}^*_\mathcal{M}$ is a fusion category. The duality of fusion categories is known to be an equivalence relation [Mul].

A fusion category is said to be *pointed* if all its simple object are invertible. Every pointed category is equivalent to a fusion category Vec^ω_G whose objects are vector spaces graded by the finite group G and whose associativity constraint is given by the 3-cocycle $\omega \in Z^3(G, k^\times)$. Let us denote $\mathrm{Vec}_G := \mathrm{Vec}^1_G$. A fusion category is called *group-theoretical* if it is equivalent to the dual of a pointed category with respect to some semisimple indecomposable module category.

We use the notion of *weak Morita equivalence* [Mul] of fusion categories to study an equivalence relation called *categorical Morita equivalence* on the set of all finite groups and on the set of all pairs (G, ω), where G is a finite group and $\omega \in H^3(G, k^\times)$. Namely, we say that two groups G and G' (respectively, two pairs (G, ω) and (G', ω')) are categorically Morita equivalent if Vec_G is dual to $\mathrm{Vec}_{G'}$ (respectively, Vec^ω_G is dual to $\mathrm{Vec}^{\omega'}_{G'}$) with respect to some semisimple indecomposable module category. This equivalence relation extends the notion of *isocategorical groups*, i.e., groups with equivalent tensor categories of representations, studied in [Da] and [EG]. Our motiva-

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tion to study categorical Morita equivalence comes from the question about existence of finite-dimensional semisimple Hopf algebras with non group-theoretical representation categories asked in [ENO, Question 8.45]. We think that understanding equivalence classes of categorically Morita equivalent groups is a natural step towards answering this question.

The notion of a nilpotent fusion category, which categorically extends the notion of a nilpotent group, was introduced in [GN]. We give sufficient conditions for a group-theoretical category to be nilpotent.

Let G be a finite group and ω be a 3-cocycle on G. In [DPR1, DPR2] R. Dijkgraaf, V. Pasquier, and P. Roche introduced a quasi-triangular quasi-Hopf algebra $D^\omega(G)$. When $\omega = 1$ this quasi-Hopf algebra coincides with the Drinfeld double $D(G)$ of G and so $D^\omega(G)$ is often called a *twisted quantum double* of G. It is well known that the representation category $\mathrm{Rep}(D^\omega(G))$ of $D^\omega(G)$ is a modular category [BK, T] and is braided equivalent to the center [K] of the fusion category Vec_G^ω.

In [DGNO] a criterion for a modular category $\mathcal{C}$ to be braided equivalent to the center of a category of the form Vec_G^ω for some finite group G and $\omega \in Z^3(G, k^\times)$ is given. Namely, such a braided equivalence exists if and only if $\mathcal{C}$ contains a Lagrangian subcategory, i.e., a maximal isotropic subcategory of dimension $\sqrt{\dim(\mathcal{C})}$. More precisely, Lagrangian subcategories of $\mathcal{C}$ parametrize the classes of braided equivalences between $\mathcal{C}$ and centers of pointed categories, see [DGNO, Section 4].

This means that a description of Lagrangian subcategories of $\mathrm{Rep}(D^\omega(G))$ for all groups G and 3-cocycles ω is equivalent to a description of all braided equivalences

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between representation categories of twisted group doubles. Such equivalences for elementary Abelian and extra special groups were studied in [MN] and [GMN].

We classify Lagrangian subcategories of $\text{Rep}(D^\omega(G))$. In view of the above remarks this gives a description of all braided tensor equivalences between twisted quantum doubles of finite groups. We also establish a canonical bijection between Lagrangian subcategories of $\text{Rep}(D^\omega(G))$ and module categories over the fusion category Vec_G^ω such that the dual fusion category is pointed. As a consequence, we obtain that two group-theoretical fusion categories are weakly Morita equivalent if and only if their centers are equivalent as braided tensor categories.

The main results of this work are:

(1) Computation of the dual of Vec_G^ω with respect to a semisimple indecomposable module category when the dual is pointed, including explicit formulas for the Grothendieck ring and the associated 3-cocycle.

(2) Necessary and sufficient conditions for two pointed categories to be dual to each other with respect to a module category.

(3) A series of concrete examples of pairs of groups (G_1, G_2) that are categorically Morita equivalent but have non-isomorphic Grothendieck rings (and hence, inequivalent representation categories). A consequence of the categorical Morita equivalence of these groups is that the representation categories $\text{Rep}(D(G_1))$ and $\text{Rep}(D(G_2))$ of their Drinfeld doubles are equivalent as braided tensor categories and so, in particular, these groups define the same modular data. To the best of our knowledge these

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are first examples of finite groups with this property, cf. a discussion of a finite group modular data in [CGR].

(4) Sufficient conditions for a group-theoretical category to be nilpotent.

(5) Classification of Lagrangian subcategories of $\text{Rep}(D^\omega(G))$. As a consequence, we obtain that two group-theoretical fusion categories are weakly Morita equivalent if and only if their centers are equivalent as braided tensor categories.

Below we give a brief description of the contents of each Chapter.

Chapter 1 is this Introduction.

In Chapter 2, we recall necessary definitions and results from cohomology of groups and projective representations. We also recall some definition and results on fusion categories, module categories, duals of fusion categories, graded fusion categories, nilpotent fusion categories, and modular categories.

In Chapter 3, we give necessary and sufficient conditions for the dual of a pointed category with respect to a module category to be pointed. We show that the Grothendieck ring of the dual of a pointed category with respect to a module category when the dual is pointed is the group ring of a certain crossed product of groups. We also find an explicit formula for the 3-cocycle associated to the dual category. We introduce the notion of *categorical Morita equivalence* on the set of all finite groups and on the set of all pairs (G, ω), where G is a finite group and $\omega \in H^3(G, k^\times)$. We give a group-theoretical and cohomological interpretation of this relation. Finally, as an application we give a series of examples of pairs of groups that are categorically Morita equivalent but have non-isomorphic Grothendieck rings.

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In Chapter 4, we give sufficient conditions for a group-theoretical category to be nilpotent.

In the final Chapter, Chapter 5, we classify Lagrangian subcategories of $\mathrm{Rep}(D^\omega(G))$. As a consequence, we obtain that two group-theoretical categories are weakly Morita equivalent if and only if their centers are equivalent as braided tensor categories.

All categories considered in this work are assumed to be small. All k-linear abelian categories considered in this work are assumed to have finite dimensional Hom-spaces and finite number of isomorphism classes of simple objects. All functors between k-linear categories are assumed to be additive and k-linear on the space of morphisms.

CHAPTER 2

PRELIMINARIES

In this Chapter we recall necessary definitions and results from cohomology of groups and projective representations. We also recall some definitions and results from fusion categories, module categories, duals of fusion categories, graded fusion categories, nilpotent fusion categories, and modular categories.

2.1 Cohomology of groups and Shapiro's Lemma

Throughout this section G will denote a finite group. Let M be a left G-module with action denoted by $(g, m) \mapsto g \triangleright m$, for $g \in G$, $m \in M$. We define a cochain complex $C(G, M) = (C^n(G, M))_{n \geq 0}$ of G with coefficients in M as follows. Let $G^n = G \times \cdots \times G$ (n factors) and $C^n(G, M) = \text{Fun}(G^n, M)$ be the set of all n-cochains. By convention, $G^0(G, M) = M$. A n-cochain f is said to be normalized if $f(g_1, g_2, \ldots, g_n) = 0_M$ whenever $g_i = 1_G$ for some $i \in \{1, 2, \ldots, n\}$. All n-cochains are assumed to be normalized. Let $\delta^n : C^n(G, M) \to C^{n+1}(G, M)$ be the coboundary operator given by

$$
\begin{aligned}
(\delta^n f)(g_1, \ldots, g_{n+1}) &= g_1 \triangleright f(g_2, \ldots, g_{n+1}) \\
&+ \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) \\
&+ (-1)^{n+1} f(g_1, \ldots, g_n),
\end{aligned}
$$

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for all $f \in C^n(G, M)$.

If M is a right G-module, we denote the action by $(m, g) \mapsto m \lhd g$, for $g \in G$, $m \in M$. Also, define $\underline{\delta}^n : C^n(G, M) \to C^{n+1}(G, M)$ by

$$
\begin{aligned}
(\underline{\delta}^n f)(g_1, \ldots, g_{n+1}) \;=\; & f(g_2, \ldots, g_{n+1}) \\
& + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) \\
& + (-1)^{n+1} (f(g_1, \ldots, g_n) \lhd g_{n+1}),
\end{aligned}
$$

for all $f \in C^n(G, M)$.

Let $Z^n(G, M) = Ker(\delta^n)$ be the set of n-cocycles and $B^n(G, M) = Im(\delta^{n-1})$ be the space of n-coboundaries. Similarly, let $\underline{Z}^n(G, M) = Ker(\underline{\delta}^n)$ and $\underline{B}^n(G, M) = Im(\underline{\delta}^{n-1})$. The n-th cohomology group $H^n(G, M)$ of G with coefficients in M is the quotient $Z^n(G, M)/B^n(G, M)$, $(n \geq 1)$. Also, let $\underline{H}^n(G, M) = \underline{Z}^n(G, M)/\underline{B}^n(G, M)$.

Let M be a left module over two finite groups K and K'. Any homomorphism $a : K' \to K$ induces a homomorphism between the cohomology groups:

$$ H^n(K, M) \to H^n(K', M) : \overline{\omega} \mapsto \overline{\omega^a} := \overline{\omega \circ a^{\times n}}. \tag{2.1} $$

Let H be a subgroup of G. Let $p : G \to H \backslash G$ be the usual surjection, i.e., $p(g) := Hg$, for all $g \in G$. We will denote $p(1_G)$ by 1. For each $x \in H \backslash G$ choose a representative $u(x)$ in G; i.e., an element $u(x)$ with $pu(x) = x$. In particular, choose $u(1) = 1_G$. The set $H \backslash G$ is a right G-set with the obvious action: $x \lhd g := p(u(x)g)$, $x \in H \backslash G$ and $g \in G$.

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Also, the set $\{u(x) \mid x \in H\backslash G\}$ is a right G-set: $u(x) \triangleleft g = u(x \triangleleft g)$, $x \in H\backslash G$ and $g \in G$. The elements $u(x)g$ and $u(x \triangleleft g)$ differ by an element $\kappa_{x,g}$ of H, for all $x \in H\backslash G$ and $g \in G$:

$$u(x)g = \kappa_{x,g}u(x \triangleleft g). \tag{2.2}$$

The following relation holds:

$$\kappa_{x,g_1g_2} = \kappa_{x,g_1}\kappa_{x\triangleleft g_1,g_2}, \tag{2.3}$$

for all $x \in H\backslash G$ and $g_1, g_2 \in G$

The abelian group $\mathrm{Fun}(H\backslash G, k^\times)$ of functions from $H\backslash G$ to $k^\times$ is a left G-module: $(g \triangleright f)(x) = f(x \triangleleft g)$, $x \in H\backslash G$ and $g \in G$. Let us regard $k^\times$ as a trivial left H-module. It is easy to see that $\mathrm{Fun}(H\backslash G, k^\times)$ is isomorphic to the coinduced module $\mathrm{Coind}_H^G k^\times = \mathrm{Hom}_H(G, k^\times)$. Throughout this work we will identify the coinduced module $\mathrm{Coind}_H^G k^\times$ with $\mathrm{Fun}(H\backslash G, k^\times)$.

Let $C := \mathrm{Coind}_H^G k^\times = \mathrm{Fun}(H\backslash G, k^\times)$ and $K := H\backslash G$. The action of G on K restricts to an action of H on K. Let K^H denote the set of elements of K that are stable under the action of H. Note that K^H forms a group that is isomorphic to $H\backslash N_G(H)$, where $N_G(H)$ is the normalizer of H in G. Denote by $\widehat{H}$ the group $\mathrm{Hom}(H, k^\times)$.

By Shapiro's Lemma there is an isomorphism between $H^n(G, C)$ and $H^n(H, k^\times)$ for each $n \in \mathbb{N}$. It is well known that the restriction maps induces this isomorphism.

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We will need the explicit form of the inverse of the restriction map when $n = 1, 2$.

Lemmas 2.1.1 and 2.1.2 provide these inverse maps.

Lemma 2.1.1. The following map induces an isomorphism between $H^1(H, k^\times) = \widehat{H}$ and $H^1(G, C)$:

$$\varphi_1 : Z^1(H, k^\times) \to Z^1(G, C), \qquad (\varphi_1(\rho)(g))(x) = \rho(\kappa_{x,g}), \qquad (2.4)$$

for all $\rho \in Z^1(H, k^\times)$, $g \in G$, $x \in K$.

Proof. We will first show that $\varphi_1(\rho) \in Z^1(G, C)$, for all $\rho \in Z^1(H, C^\times)$. We need to show that $\varphi_1(\rho)$ satisfies the equation:

$$(\varphi_1(\rho)(g_1))(x) \, (\varphi_1(\rho)(g_2))(x \triangleleft g_1) = (\varphi_1(\rho)(g_1 g_2))(x)$$

$$\Leftrightarrow \rho(\kappa_{x, g_1}) \rho(\kappa_{x \triangleleft g_1, g_2}) = \rho(\kappa_{x, g_1 g_2}),$$

for all $x \in K$, $g_1, g_2 \in G$.

The 1-cocycle condition on ρ is:

$$\rho(h_1) \rho(h_2) = \rho(h_1 h_2).$$

Put $h_1 = \kappa_{x\, g_1}$ and $h_2 = \kappa_{x \triangleleft g_1, g_2}$ in the above equation and use (2.3) to obtain the desired equation.

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The map φ_1 induces a homomorphism:

$$\widetilde{\varphi_1} : H^1(H, k^\times) \to H^1(G, C).$$

Let ψ_1 denote the restriction homomorphism:

$$\psi_1 : Z^1(G, C) \to Z^1(H, k^\times), \qquad \psi_1(\gamma)(h) = \gamma(h)(1), \tag{2.5}$$

for all $\gamma \in Z^1(G, C)$ and $h \in H$. Let $\widetilde{\psi_1}$ denote the induced homomorphism:

$$\widetilde{\psi_1} : H^1(G, C) \to H^1(H, k^\times). \tag{2.6}$$

It remains to show that the homomorphisms $\widetilde{\varphi_1}$ and $\widetilde{\psi_1}$ are inverse to each other. It suffice to show that $\psi_1 \circ \varphi_1 = Id_{Z^1(H, k^\times)}$. Pick any $\rho \in Z^1(H, k^\times)$. Then $\psi_1(\varphi_1(\rho))(h) = (\varphi_1(\rho)(h))(1) = \rho(\kappa_{1,h}) = \rho(h)$, for all $h \in H$. So $\psi_1 \circ \varphi_1 = Id_{Z^1(H, k^\times)}$ and the Lemma is proved. ∎

Lemma 2.1.2. The following map induces an isomorphism between $H^2(H, k^\times)$ and $H^2(G, C)$:

$$\varphi : Z^2(H, k^\times) \to Z^2(G, C), \qquad (\varphi(\mu)(g_1, g_2))(x) = \mu(\kappa_{x,g_1}, \kappa_{x \lhd g_1, g_2}), \tag{2.7}$$

for all $\mu \in Z^2(H, k^\times)$, $g_1, g_2 \in G$, and $x \in K$.

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Proof. We will first show that $\varphi(\mu) \in Z^2(G, C)$, for all $\mu \in Z^2(H, k^\times)$. We need to show that $\varphi(\mu)$ satisfies the following equation for all $g_1, g_2, g_3 \in G$ and $x \in K$:

$$(\varphi(\mu)(g_2, g_3))(x \lhd g_1) \; (\varphi(\mu)(g_1, g_2 g_3))(x) = (\varphi(\mu)(g_1 g_2, g_3))(x) \; (\varphi(\mu)(g_1, g_2))(x)$$

$$\Leftrightarrow \mu(\kappa_{x \lhd g_1, g_2}, \kappa_{x \lhd g_1 g_2, g_3}) \, \mu(\kappa_{x, g_1}, \kappa_{x \lhd g_1, g_2 g_3}) = \mu(\kappa_{x, g_1 g_2}, \kappa_{x \lhd g_1 g_2, g_3}) \, \mu(\kappa_{x, g_1}, \kappa_{x \lhd g_1, g_2}).$$

The 2-cocycle condition on μ is:

$$\mu(h_2, h_3) \, \mu(h_1, h_2 h_3) = \mu(h_1 h_2, h_3) \, \mu(h_1, h_2),$$

for all $h_1, h_2, h_3 \in H$. Put $h_1 = \kappa_{x, g_1}$, $h_2 = \kappa_{x \lhd g_1, g_2}$ and $h_3 = \kappa_{x \lhd g_1 g_2, g_3}$ in the above equation and use (2.3) to obtain the desired equation.

Let us now show that φ preserves coboundaries. Let $\alpha : H \to k^\times$ be any function. Then $\varphi(\delta^1 \alpha)$ is a coboundary. Indeed, define a function $\widetilde{\alpha} : G \to C$ by $(\widetilde{\alpha}(g))(x) := \alpha(\kappa_{x, g})$. Now, $(\varphi(\delta^1 \alpha)(g_1, g_2))(x) = (\delta^1 \alpha)(\kappa_{x, g_1}, \kappa_{x \lhd g_1, g_2}) = \dfrac{\alpha(\kappa_{x \lhd g_1, g_2}) \alpha(\kappa_{x, g_1})}{\alpha(\kappa_{x, g_1} \kappa_{x \lhd g_1, g_2})}$

$= \dfrac{\alpha(\kappa_{x \lhd g_1, g_2}) \alpha(\kappa_{x, g_1})}{\alpha(\kappa_{x, g_1 g_2})} = ((\delta^1 \widetilde{\alpha})(g_1, g_2))(x)$. So φ preserves coboundaries and hence it induces a map:

$$\widetilde{\varphi} : H^2(H, k^\times) \to H^2(G, C). \tag{2.8}$$

Let ψ denote the restriction map:

$$\psi : Z^2(G, C) \to Z^2(H, k^\times), \qquad \psi(\gamma)(h_1, h_2) = \gamma(h_1, h_2)(1), \tag{2.9}$$

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for all $\gamma \in Z^2(G, C)$ and $h_1, h_2 \in H$. Let $\widetilde{\psi}$ denote the induced map:

$$\widetilde{\psi} : H^2(G, C) \to H^2(H, k^\times).\qquad(2.10)$$

It remains to show that the maps $\widetilde{\varphi}$ and $\widetilde{\psi}$ are inverse to each other. It suffice to show that $\psi \circ \varphi = Id_{Z^2(H, k^\times)}$. Pick any $\mu \in Z^2(H, k^\times)$. Then $\psi(\varphi(\mu))(h_1, h_2) = (\varphi(\mu)(h_1, h_2))(1) = \mu(\kappa_{1, h_1}, \kappa_{1 \lhd h_1, h_2}) = \mu(h_1, h_2)$, for all $h_1, h_2 \in H$. So $\psi \circ \varphi = Id_{Z^2(H, k^\times)}$ and the Lemma is proved. ∎

There is a right action of K^H on $C^n(G, C)$:

$$(\gamma, x) \mapsto {}^x\gamma, \qquad {}^x\gamma(g_1, \ldots, g_n)(y) := \gamma(g_1, \ldots, g_n)(p(u(x)u(y))),$$

for all $\gamma \in C^n(G, C)$, $g_1, \ldots, g_n \in G$, $x \in K^H$, and $y \in K$.

It is routine to check that the above action is independent of the function $u : K \to G$. This induces a right action of K^H on $Z^n(G, C)$ and $H^n(G, C)$. If H is normal in G, then $K^H = K$ and

$$ {}^x\gamma(g_1, \ldots, g_n)(y) := \gamma(g_1, \ldots, g_n)(xy),$$

for all $\gamma \in C^n(G, C)$, $g_1, \ldots, g_n \in G$, and $x, y \in K$.

If H is normal in G, we can also define a right action of G on $C^n(G, C)$:

$$ {}^g\gamma(g_1, \ldots, g_n)(y) := \gamma(g_1, \ldots, g_n)(p(g)y),\qquad(2.11)$$

for all $\gamma \in C^n(G, C), g, g_1, \ldots, g_n \in G, y \in K$.

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Also, if H is normal in G, then $Z^n(H, k^\times)$ is a right G-module:

$$(\mu, g) \mapsto \mu^g, \qquad \mu^g(h_1, \ldots, h_n) = \mu(gh_1g^{-1}, \ldots, gh_ng^{-1}),$$

for $\mu \in Z^n(H, k^\times)$, $g \in G$ and $h_1, \ldots, h_n \in H$.

If H is abelian and normal in G, then $Z^n(H, k^\times)$ becomes a right K-module:

$$(\mu, x) \mapsto \mu^{u(x)},$$

for $\mu \in Z^n(H, k^\times)$ and $x \in K$. This induces an action of K on $H^n(H, k^\times)$.

Lemma 2.1.3. If H is abelian and normal in G, then the map ψ_1 defined in (2.5) is a K-module map.

Proof. Pick any $\gamma \in Z^1(G, C)$ and $x \in K$. We have $\psi_1(^x\gamma)(h) = (^x\gamma)(h)(1) = \gamma(h)(x)$ and $(\psi_1(\gamma)^x)(h) = \psi_1(\gamma)(u(x)hu(x)^{-1}) = \gamma(u(x)hu(x)^{-1})(1)$. By Lemma 2.1.1 we know that $\gamma = (\delta^1\alpha)\,\varphi_1(\rho)$, for some $\alpha \in C$ and $\rho \in \widehat{H}$. We have,

$$\gamma(h)(x) = ((\delta^1\alpha)\,\varphi_1(\rho))(h)(x)$$
$$= \frac{\alpha(x \lhd h)}{\alpha(x)}\,\rho(\kappa_{x,h})$$
$$= \rho(u(x)hu(x)^{-1})$$

and

$$\gamma(u(x)hu(x)^{-1})(1) = ((\delta^1\alpha)\,\varphi_1(\rho))(u(x)hu(x)^{-1})(1)$$
$$= \frac{\alpha(1 \lhd u(x)hu(x)^{-1})}{\alpha(1)}\,\rho(\kappa_{1,\,u(x)hu(x)^{-1}})$$
$$= \rho(u(x)hu(x)^{-1}).$$

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It follows that ψ_1 is K-linear and the Lemma is proved. ∎

Lemma 2.1.4. If H is abelian and normal in G, then the map $\widetilde{\varphi}$ defined in (2.8) is a K-module map.

Proof. Pick any $\mu \in Z^2(H, k^\times)$. It suffices to show that $\psi({}^x\varphi(\mu))$ is cohomologous to $\psi(\varphi(\mu^x)) = \mu^x$ in $H^2(H, k^\times)$, for all $x \in K$. We will actually show that $\psi({}^x\varphi(\mu)) = \mu^x$, for all $x \in K$. We have,

$$
\begin{aligned}
\psi({}^x\varphi(\mu))(h_1, h_2) &= {}^x\varphi(\mu)(h_1, h_2)(1) \\
&= \varphi(\mu)(h_1, h_2)(x) \\
&= \mu(\kappa_{x, h_1}, \kappa_{x \triangleleft h_1, h_2}) \\
&= \mu(\kappa_{x, h_1}, \kappa_{x, h_2}) \\
&= \mu(u(x)h_1 u(x)^{-1}, u(x)h_2 u(x)^{-1}) \\
&= \mu^x(h_1, h_2)
\end{aligned}
$$

for all $h_1, h_2 \in H$ and $x \in K$. So $\psi({}^x\varphi(\mu)) = \mu^x$, for all $x \in K$ and the Lemma is proved. ∎

Pick any $\mu \in Z^2(G, \operatorname{Coind}_H^G k^\times)$. The 2-cocycles μ and $\varphi(\psi(\mu))$ are cohomologous. So, there exists $\eta \in C^1(G, \operatorname{Coind}_H^G k^\times)$ which satisfies:

$$
\mu = (\delta^1 \eta)\ \varphi(\psi(\mu)). \tag{2.12}
$$

Lemma 2.1.5. The restriction $\operatorname{res}(\eta)$ of the map η is an element of $\widehat{H}$.

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Proof. Equation (2.12) means that

$$\mu(g_1, g_2)(x) = (\delta^1 \eta)(g_1, g_2)(x) \; \varphi(\psi(\mu))(g_1, g_2)(x)$$

$$= \frac{\eta(g_2)(x) \; \eta(g_1)(x)}{\eta(g_1 g_2)(x)} \; \mu(\kappa_{x, g_1}, \kappa_{x \triangleleft g_1, g_2})(1),$$

for all $x \in K$, and $g_1, g_2 \in G$.

Put $x = 1$, $g_1 = h_1$, $g_2 = h_2 \in H$, and use $u(1) = 1_G$ to get

$$\eta(h_1)(1) \; \eta(h_2)(1) = \eta(h_1 h_2)(1),$$

i.e., $res(\eta) \in \widehat{H}$ and the Lemma is proved. ∎

2.2 The Schur multiplier of an abelian group.

Let H be a finite abelian group. Let $\Lambda^2 H$ denote the abelian group of alternating bicharacters on H, i.e.,

$$\Lambda^2 H := \left\{ B : H \times H \to k^\times \; \middle| \; \begin{array}{l} B(h_1 h_2, h) = B(h_1, h) B(h_2, h), \\[4pt] B(h, h_1 h_2) = B(h, h_1) B(h, h_2), \text{ and} \\[4pt] B(h, h) = 1, \text{ for all } h, h_1, h_2 \in H \end{array} \right\}.$$

Note 2.2.1. Let $B \in \Lambda^2 H$. The condition $B(h, h) = 1$, for all $h \in H$ implies that $B(h_1, h_2) B(h_2, h_1) = 1$, for all $h_1, h_2 \in H$. Indeed, we have $B(h_1, h_2) B(h_2, h_1) = B(h_1, h_2) B(h_1, h_1) B(h_2, h_1) B(h_2, h_2) = B(h_1, h_2 h_1) B(h_2, h_1 h_2) = B(h_1 h_2, h_1 h_2) = 1$, for all $h_1, h_2 \in H$.

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Define a homomorphism $alt : Z^2(H, k^\times) \to \Lambda^2 H : \mu \to alt(\mu)$, by

$$alt(\mu)(h_1, h_2) := \frac{\mu(h_2, h_1)}{\mu(h_1, h_2)}, \quad h_1, h_2 \in H.$$

Lemma 2.2.2. The homomorphism alt induces an isomorphism between $H^2(H, k^\times)$ and $\Lambda^2 H$.

Proof. It is evident that $alt(B^2(H, k^\times)) = \{1\}$. So alt induces a homomorphism which, by abuse of notation, we also denote by alt:

$$alt : H^2(H, k^\times) \to \Lambda^2 H : \overline{\mu} \mapsto alt(\mu). \tag{2.13}$$

The homomorphism alt is injective. Indeed, let $\mu \in Z^2(H, k^\times)$ and suppose $alt(\mu) = 1$. So μ is symmetric. Recall that there is a bijective correspondence between symmetric classes in $H^2(H, k^\times)$ and equivalences classes of abelian central extensions of $k^\times$ by H. Since μ is symmetric, the central extension $1 \to k^\times \to E_\mu \to H \to 1$ that μ determines is abelian. Since $k^\times$ is algebraically closed, it is a divisible group. So every abelian extension of $k^\times$ by H splits. So, in particular, the previous exact sequence splits. Therefore, $\mu \in B^2(H, k^\times)$. It follows that alt is injective.

To see that alt is surjective, pick any $B \in \Lambda^2 H$. Since H is a finite abelian group, we can write $H = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_l\mathbb{Z}$. Let c_i be a generator of $\mathbb{Z}/n_i\mathbb{Z}$ (written multiplicatively), $i = 1, 2, \cdots, l$. Let $\varphi_i : \mathbb{Z}/n_i\mathbb{Z} \hookrightarrow H$, $i = 1, 2, \cdots, l$ be the usual inclusions. Define $\lambda_{rs} := B(\varphi_r(c_r), \varphi_s(c_s))$, $r, s \in \{1, 2, \cdots, l\}$. Since B is alternating,

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the relation $\lambda_{rs}\lambda_{sr} = 1$ holds for all $r, s \in \{1, 2, \cdots, l\}$. Also, note that $\lambda_{rr} = 1$, for all $r \in \{1, 2, \cdots, l\}$. We have

$$B(\oplus_{r=1}^{l} c_r^{p_r},\ \oplus_{s=1}^{l} c_s^{q_s}) = B(\Pi_{r=1}^{l} \varphi_r(c_r^{p_r}),\ \Pi_{s=1}^{l} \varphi_s(c_s^{q_s}))$$

$$= \Pi_{r=1}^{l}\Pi_{s=1}^{l} B(\varphi_r(c_r)^{p_r},\ \varphi_s(c_s)^{q_s})$$

$$= \Pi_{r=1}^{l}\Pi_{s=1}^{l} \lambda_{rs}^{p_r q_s}$$

$$= \Pi_{r=1}^{l}\Pi_{s=1, s<r}^{l} \lambda_{rs}^{(p_r q_s - p_s q_r)}.$$

In the last equality used the relations $\lambda_{rr} = 1$ and $\lambda_{rs}\lambda_{sr} = 1$, $r, s \in \{1, 2, \cdots, l\}$. Now, define a map $\mu : H \times H \to k^{\times}$ by $\mu(\oplus_{r=1}^{l} c_r^{p_r},\ \oplus_{s=1}^{l} c_s^{q_s}) := \Pi_{r=1}^{l}\Pi_{s=1, s<r}^{l} \lambda_{rs}^{-p_s q_r}$. It is evident that μ is a bicharacter on H. So $\mu \in Z^2(H, k^{\times})$. It is also evident that $alt(\mu) = B$ and the Lemma is proved. ∎

Remark 2.2.3. Suppose H is a normal abelian subgroup of a finite group G. The abelian group $\Lambda^2 H$ is a right G-module:

$(B, g) \mapsto B^g$, $B^g(h_1, h_2) := B(gh_1g^{-1}, gh_2g^{-1})$, $g \in G$, $h_1, h_2 \in H$. It is evident that the map alt is G-linear. So $H^2(H, k^{\times})$ and $\Lambda^2 H$ are isomorphic as G-modules.

2.3   Projective representations

Definition 2.3.1. Let V be a finite-dimensional vector space over k. A mapping $\rho : G \to GL(V)$ is called a *projective representation* of the finite group G with 2-cocycle $\alpha : G \times G \to k^{\times}$ if it satisfies $\rho(1_G) = \mathrm{id}_V$, and $\rho(g_1)\rho(g_2) = \alpha(g_1, g_2)\, \rho(g_1 g_2)$, for all $g_1, g_2 \in G$.

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Remark 2.3.2. (i) There are obvious notions of irreducible projective representations and direct sum of projective representations.

(ii) Let ρ_1 and ρ_2 be projective representations of a finite group G with 2-cocycles α and β, respectively. Then their tensor product is a projective representation of G with 2-cocycle $\alpha\beta$.

(iii) Let ρ be projective representation of a finite group G with 2-cocycles α. Then there is a notion of a dual representation ρ^* which is a projective representation of G with 2-cocycle α^{-1}.

Lemma 2.3.3. Let $\rho : G \to GL(V)$ be a projective representation with 2-cocycle α. Then $V \otimes V^*$ becomes a G-module and its decomposition into irreducible modules of G contains a copy of the trivial module.

Proof. That $V \otimes V^*$ is a G-module follows from the above remarks. Define $g \triangleright T := \rho(g) \circ T \circ \rho(g)^{-1}$, for all $g \in G$, and $T \in \mathrm{End}(V)$. With this action, $\mathrm{End}(V)$ becomes a G-module. It can be shown that the usual vector space isomorphism between $V \otimes V^*$ and $\mathrm{End}(V)$ is G-linear. Now, $\{\lambda \, \mathrm{id}_V \mid \lambda \in k\}$ is a submodule of $\mathrm{End}(V)$ on which G acts trivially and the Lemma is proved. ∎

Let $\rho : G \to GL(V)$ be a projective representation with 2-cocycle α. If we identify $GL(V)$ with $GL(n, k)$ where $n = \dim_k(V)$, then the resulting map is called a *projective matrix representation*. In this work, by projective representation we will always mean a projective matrix representation. The dual representation ρ^* of a

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projective representation ρ of G is defined by $\rho^*(g) := (\rho(g)^T)^{-1}$, for all $g \in G$, where the superscript ^{T} stands for the transpose of a matrix.

Definition 2.3.4. Two projective representations $\rho_1 : G \to GL(n, k)$ and $\rho_2 : G \to GL(n, k)$ with 2-cocycle α are isomorphic if there is a matrix $A \in GL(n, k)$ such that $\rho_2(g) = A\rho_1(g)A^{-1}$, for all $g \in G$.

2.4 Abelian categories

References for this Section are [BK] and [Mac].

Definition 2.4.1. An *additive category* $\mathcal{C}$ is a category satisfying the following axioms.

(A1) $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group, for all $X, Y \in \operatorname{Obj}(\mathcal{C})$

(A2) There exists a zero object $\mathbf{0} \in \mathcal{C}$: $\operatorname{Hom}_{\mathcal{C}}(X, \mathbf{0}) = \operatorname{Hom}_{\mathcal{C}}(\mathbf{0}, X) = \{0\}$, for all $X \in \operatorname{Obj}(\mathcal{C})$

(A3) Finite direct sums exist, i.e., for all $X_1, X_2 \in \operatorname{Obj}(\mathcal{C})$ there exists $Y \in \operatorname{Obj}(\mathcal{C})$ and morphisms $i_1 : X_1 \to Y$, $i_2 : X_2 \to Y$, $p_1 : Y \to X_1$, and $p_2 : Y \to X_2$ such that $p_1 \circ i_1 = \operatorname{id}_{X_1}$, $p_2 \circ i_2 = \operatorname{id}_{X_2}$, and $i_1 \circ p_1 + i_2 \circ p_2 = \operatorname{id}_Y$.

Example 2.4.2. Let R be a ring and let $\mathcal{C} :=$ category of left R-modules. Then $\mathcal{C}$ is an additive category.

Definition 2.4.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between additive categories. We say that F is *additive* if $\operatorname{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F} \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a group homomorphism, for all $X, Y \in \operatorname{Obj}(\mathcal{C})$.

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Remark 2.4.4. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. Then $F(X) \oplus F(Y) \cong F(X \oplus Y)$, for all $X, Y \in \mathrm{Obj}(\mathcal{C})$.

Definition 2.4.5. Let $\mathbb{F}$ be a field. An additive category $\mathcal{C}$ is $\mathbb{F}$-*linear* if for all $X, Y, Z \in \mathrm{Obj}(C)$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\mathbb{F}$-vector space and the composition $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is a $\mathbb{F}$-bilinear map.

Definition 2.4.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between $\mathbb{F}$-linear categories. We say that F is $\mathbb{F}$-*linear* if $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is $\mathbb{F}$-linear, for all $X, Y \in \mathrm{Obj}(\mathcal{C})$.

Definition 2.4.7. Let $\mathcal{C}$ be an additive category. Let $f : X \to Y$ be a morphism in $\mathcal{C}$.

The *kernel* of f is an object $K \in \mathrm{Obj}(\mathcal{C})$ together with a morphism $i : K \to X$ such that $f \circ i = 0$ and for all morphisms $i' : K' \to X$ such that $f \circ i' = 0$ there is a unique morphism $k : K' \to K$ such that $i' = i \circ k$.

The *cokernel* of f is an object $C \in \mathrm{Obj}(\mathcal{C})$ together with a morphism $j : Y \to C$ such that $j \circ f = 0$ and for all morphisms $j' : Y \to C'$ such that $j' \circ f = 0$ there is a unique morphism $k : C \to C'$ such that $j' = k \circ j$.

Remark 2.4.8. If kernel exists, it is unique up to a unique isomorphism. The same is true for cokernel.

Example 2.4.9. Let $\mathcal{C} :=$ the category of abelian groups. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then, $coker(f) \cong Y/\mathrm{Im}(f)$.

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Definition 2.4.10. An *abelian category* is an additive category $\mathcal{C}$ in which every morphism $f : X \to Y$ admits the following decomposition $K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$ such that

(i) $j \circ i = f$,

(ii) $K \xrightarrow{k} X = ker(f)$, $Y \xrightarrow{c} C = coker(f)$ (kernel and cokernel exist), and

(iii) $I = coker(k) = ker(c)$.

Example 2.4.11. Let R be a ring and let $\mathcal{C} :=$ category of left R-modules. Then $\mathcal{C}$ is an abelian category.

Definition 2.4.12. Let $\mathcal{C}$ an abelian category and let $X \in \mathrm{Obj}(\mathcal{C})$. The object X is said to be *simple* if it is non-zero and has no subobjects other than the zero object and X. The object X is said to be semisimple if it is isomorphic to a finite direct sum of simple objects. The abelian category $\mathcal{C}$ is said to be *semisimple* if every object in $\mathcal{C}$ is semisimple.

Remark 2.4.13. Let $\mathbb{F}$ be an algebraically closed field. Let $\mathcal{C}$ be a $\mathbb{F}$-linear abelian category. Schur's Lemma holds for $\mathcal{C}$, i.e., for any two simple objects X and Y of $\mathcal{C}$, the following statements hold.

(i) For any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, either $f = 0$ or f is an isomorphism.

(ii) $\mathrm{Hom}_{\mathcal{C}}(X, X) = \mathbb{F} \cdot \mathrm{id}_X$.

(iii) $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathbb{F}$, if $X \cong Y$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \{0\}$, if $X \ncong Y$.

Proof: The first statement follows from the fact that the kernel and cokernel of a morphism define subobjects. For the second statement, pick any $f \in \mathrm{Hom}_{\mathcal{C}}(X, X)$.

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Then f defines a linear transformation: $\mathrm{Hom}_{\mathcal{C}}(X, X) \to \mathrm{Hom}_{\mathcal{C}}(X, X) : g \mapsto f \circ g$. Since $\mathbb{F}$ is algebraically closed, there exists a scalar $\lambda \in \mathbb{F}$ and a nonzero morphism $g \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that $f \circ g = \lambda g$. Note that by (i), g is an isomorphism. We have $f \circ g = \lambda g = (\lambda \mathrm{id}_X \circ g) \Leftrightarrow f = \lambda \mathrm{id}_X$. So $\mathrm{Hom}_{\mathcal{C}}(X, X) = \mathbb{F} \cdot \mathrm{id}_X$. For the last statement, suppose $X \cong Y$ and fix an isomorphism $l : X \xrightarrow{\sim} Y$. Pick any $h \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Then, $l^{-1} \circ h \in \mathrm{Hom}_{\mathcal{C}}(X, X)$. By (ii), $l^{-1} \circ h = \lambda \mathrm{id}_X$, for some $\lambda \in \mathbb{F}$. So $h = \lambda l$, and it follows that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathbb{F}$.

2.5   Fusion categories

References for this Section are [BK], [ENO], and [Mac].

Definition 2.5.1. A *tensor category* $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ is a category $\mathcal{C}$ along with a *tensor product* bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, *the unit object* 1, and natural isomorphisms

$$\alpha : \otimes(\otimes \times \mathrm{id}) \xrightarrow{\sim} \otimes(\mathrm{id} \times \otimes)$$

$$\lambda : \otimes(1 \times \mathrm{id}) \xrightarrow{\sim} \mathrm{id}$$

$$\rho : \otimes(\mathrm{id} \times 1) \xrightarrow{\sim} \mathrm{id}$$

called *associativity constraint* and *right* and *left unit constraints*, respectively, satisfying the following commutative diagrams called *pentagon* and *triangle* axioms.

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$$((X \otimes Y) \otimes Z) \otimes W$$

$$\alpha_{X,Y,Z} \otimes \mathrm{id}_W$$

$$\alpha_{X \otimes Y, Z, W}$$

$$(X \otimes (Y \otimes Z)) \otimes W \qquad\qquad (X \otimes Y) \otimes (Z \otimes W)$$

$$\alpha_{X, Y \otimes Z, W} \Big\downarrow \qquad\qquad\qquad\qquad\qquad \Big\downarrow \alpha_{X, Y, Z \otimes W}$$

$$X \otimes ((Y \otimes Z) \otimes W) \xrightarrow{\ \mathrm{id}_X \otimes \alpha_{Y,Z,W}\ } X \otimes (Y \otimes (Z \otimes W))$$

$$(X \otimes 1) \otimes Y \xrightarrow{\ \alpha_{X,1,Y}\ } X \otimes (1 \otimes Y)$$

$$\rho_X \otimes \mathrm{id}_Y \searrow \qquad\qquad \swarrow \mathrm{id}_X \otimes \lambda_Y$$

$$X \otimes Y$$

commute for all objects X, Y, Z, W in $\mathcal{C}$.

Note 2.5.2. In the previous definition, if $\mathcal{C}$ is an abelian tensor category, then we will additionally require that $\otimes$ is biadditive on the space of morphisms. If $\mathcal{C}$ a $\mathbb{F}$-linear tensor category for some field $\mathbb{F}$, then we additionally require that $\otimes$ is $\mathbb{F}$-bilinear on the space of morphisms.

Definition 2.5.3. Let $\mathcal{C} = (\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ and $\mathcal{C}' = (\mathcal{C}', \otimes, 1', \alpha, \lambda, \rho)$ be tensor categories. A *tensor functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ together with a natural isomorphism

$$\varphi : \otimes(F \times F) \xrightarrow{\sim} F\otimes$$

and an isomorphism $\nu : 1' \xrightarrow{\sim} F(1)$ such that the diagrams

$$(F(X) \otimes F(Y)) \otimes F(W) \xrightarrow{\ \alpha_{F(X),F(Y),F(Z)}\ } F(X) \otimes (F(Y) \otimes F(Z))$$

$$\varphi_{X,Y} \otimes \mathrm{id}_{F(Z)} \Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow \mathrm{id}_{F(X)} \otimes \varphi_{Y,Z}$$

$$F(X \otimes Y) \otimes F(Z) \qquad\qquad\qquad F(X) \otimes F(Y \otimes Z)$$

$$\varphi_{X \otimes Y, Z} \Big\downarrow \qquad\qquad\qquad\qquad\qquad \Big\downarrow \varphi_{X, Y \otimes W}$$

$$F((X \otimes Y) \otimes Z) \xrightarrow{\ F(\alpha_{X,Y,Z})\ } F(X \otimes (Y \otimes Z))$$

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$$1' \otimes F(X) \xrightarrow{\lambda_{F(X)}} F(X)$$

with vertical maps $\nu \otimes \mathrm{id}_{F(X)}$ (left, downward) and $F(\lambda_X)$ (right, upward), and bottom row

$$F(1) \otimes F(X) \xrightarrow{\varphi_{1,X}} F(1 \otimes X)$$

and

$$F(X) \otimes 1' \xrightarrow{\rho_{F(X)}} F(X)$$

with vertical maps $\mathrm{id}_{F(X)} \otimes \nu$ (left, downward) and $F(\rho_X)$ (right, upward), and bottom row

$$F(X) \otimes F(1) \xrightarrow{\varphi_{X,1}} F(X \otimes 1)$$

commute for all objects X, Y, Z in $\mathcal{C}$.

Definition 2.5.4. Let $\mathcal{C} = (\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ be a tensor category and let X be an object in $\mathcal{C}$. A *right dual* to X is an object X^* such that there exist morphisms

$$e_X : X^* \otimes X \to 1,$$

$$c_X : 1 \to X \otimes X^*,$$

called *evaluation* and *coevaluation* morphisms, such that the following compositions

$$X \xrightarrow{\lambda_X^{-1}} 1 \otimes X \xrightarrow{c_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes e_X} X \otimes 1 \xrightarrow{\rho_X} X$$

$$X^* \xrightarrow{\rho_{X^*}^{-1}} X^* \otimes 1 \xrightarrow{\mathrm{id}_{X^*} \otimes c_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha_{X^*,X,X^*}^{-1}} (X^* \otimes X) \otimes X^* \xrightarrow{e_X \otimes \mathrm{id}_{X^*}} 1 \otimes X^* \xrightarrow{\lambda_{X^*}} X^*$$

are equal to the identity isomorphisms id_X and id_{X^*}, respectively.

A *left dual* to X is an object $^*X$ such that there exist morphisms

$$e'_X : X \otimes {}^*X \to 1,$$

$$c'_X : 1 \to {}^*X \otimes X,$$

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such that the following compositions

$$X \xrightarrow{\rho_X^{-1}} X \otimes 1 \xrightarrow{\mathrm{id}_X \otimes c'_X} X \otimes (^*X \otimes X) \xrightarrow{\alpha_{X,^*X,X}^{-1}} (X \otimes {}^*X) \otimes X \xrightarrow{e'_x \otimes \mathrm{id}_X} 1 \otimes X \xrightarrow{\lambda_X} X$$

$$^*X \xrightarrow{\lambda_{^*X}^{-1}} 1 \otimes {}^*X \xrightarrow{c'_X \otimes \mathrm{id}_{^*X}} (^*X \otimes X) \otimes {}^*X \xrightarrow{\alpha_{^*X,X,^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathrm{id}_{^*X} \otimes e'_X} {}^*X \otimes 1 \xrightarrow{\rho_{^*X}} {}^*X$$

are equal to the identity isomorphisms id_X and $\mathrm{id}_{^*X}$ respectively.

Definition 2.5.5. A tensor category $\mathcal{C}$ is called *rigid* if every object in $\mathcal{C}$ has right and left duals.

Example 2.5.6. In the tensor category Vec of finite-dimensional vector spaces over a field $\mathbb{F}$ every object V has a (left or right) dual $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$. The evaluation and coevaluation morphisms are: $ev_V : V^* \otimes V \to \mathbb{F} : f \otimes v \mapsto f(v)$ and

$coev_V : \mathbb{F} \to V \otimes V^* : \alpha \mapsto \alpha \sum_i v_i \otimes f^i$, where $\{v_i\}$, $\{f^i\}$ are dual bases.

Definition 2.5.7. A *fusion category* over an algebraically closed field $\mathbb{F}$ is a $\mathbb{F}$-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects and finite-dimensional Hom-spaces such that the neutral object in simple.

Remark 2.5.8. A fusion category can be thought of as the "categorification" of the notion of a ring.

Example 2.5.9. Let $\mathbb{F}$ be an algebraically closed field.

(i) The category Vec of finite dimensional vector spaces over $\mathbb{F}$ is a fusion category.

(ii) Let G be a finite group such that $|G|$ is invertible in $\mathbb{F}$. Then the category $\mathrm{Rep}(G)$ of finite-dimensional representations over $\mathbb{F}$ of G is a fusion category.

(iii) More generally, let H be a finite-dimensional semisimple Hopf algebra over $\mathbb{F}$. Then the category $\operatorname{Rep}(H)$ of finite-dimensional representations over $\mathbb{F}$ of H is a fusion category. Tensor products and dual objects are defined using the comultiplication and antipode, respectively.

Definition 2.5.10. An object X in a fusion category is said to be *invertible* if the evaluation and coevaluation morphisms $e_X : X^* \otimes X \to 1$ and $c_X : 1 \to X \otimes X^*$ are isomorphisms.

Definition 2.5.11. A fusion category is said to be *pointed* if all its simple objects are invertible.

Example 2.5.12. Let G be a finite group and $\omega \in Z^3(G, k^\times)$. Let $\operatorname{Vec}_G^\omega$ be the category of finite-dimensional vector spaces over k graded by the group G with morphisms being linear transformations that respect the grading. Then $\operatorname{Vec}_G^\omega$ becomes a pointed category with tensor product given by:

$$(V \otimes W)_g := \oplus_{x,y \in G : xy=g} V_x \otimes_k W_y,$$

for all $V, W \in \operatorname{Obj}(\operatorname{Vec}_G^\omega)$, and associativity constraint given by:

$$(U_{g_1} \otimes V_{g_2}) \otimes W_{g_3} \to U_{g_1} \otimes (V_{g_2} \otimes W_{g_3}) : (u \otimes v) \otimes w \mapsto \omega(g_1, g_2, g_3) \cdot (u \otimes (v \otimes w)).$$

A category is called *skeletal* if all isomorphic objects in the category are actually equal. Every category is equivalent to a skeletal category. It is convenient to work

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with a skeletal category $\mathcal{V}_G^\omega$ equivalent to Vec_G^ω. Let $\mathcal{V}_G^\omega$ be the fusion category with simple objects g, $g \in G$. The tensor product is defined by $g_1 \otimes g_2 = g_1 g_2$, and the associativity isomorphisms are $\omega(g_1, g_2, g_3) id_{g_1 g_2 g_3}$. The unit object is 1_G. The left and right unit isomorphisms are $\omega(1_G, 1_G, g) id_g$ and $\omega(g, 1_G, 1_G) id_g$, respectively. The previous statement follows from the triangle axiom for tensor categories. Since we can assume that all cocycles are normalized, the left and right unit isomorphisms are the identity morphisms. The left and right dual objects of g are $g^* = {}^*g = g^{-1}$. If G' is another group and $\omega' \in Z^3(G', k^\times)$, then $\mathcal{V}_G^\omega \cong \mathcal{V}_{G'}^{\omega'}$ if and only if there is an isomorphism $a : G \to G'$ such that ω' and ω^a are cohomologous.

Remark 2.5.13. Every pointed category is equivalent to Vec_G^ω for some finite group G and 3-cocycle $\omega \in Z^3(G, k^\times)$.

Sketch of proof: Let $\mathcal{C} = (\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ be a pointed category. A *skeleton* of a category $\mathcal{D}$ is any full subcategory $\overline{\mathcal{D}}$ such that each object of $\mathcal{D}$ is isomorphic (in $\mathcal{D}$) to exactly one object of $\overline{\mathcal{D}}$. Every category is equivalent to any of its skeletons. Let us constructs a skeleton $\overline{\mathcal{C}}$ of $\mathcal{C}$: choose one object from each isomorphism class of objects in $\mathcal{C}$. Let $\text{Obj}(\overline{\mathcal{C}})$ be the set of all objects chosen above. For any $X \in \text{Obj}(\mathcal{C})$, by $\overline{X}$ we mean the object in $\overline{\mathcal{C}}$ that represents the object X. Define $\text{Hom}_{\overline{\mathcal{C}}}(X, Y) :=$ $\text{Hom}_{\mathcal{C}}(X, Y)$. Define tensor product $\odot$ in $\overline{\mathcal{C}}$: $X \odot Y := \overline{X \otimes Y}$, for all $X, Y \in \text{Obj}(\overline{\mathcal{C}})$. Fix isomorphisms $\beta(X, Y) : X \odot Y \xrightarrow{\sim} X \otimes Y$ in $\mathcal{C}$, for all $X, Y \in \text{Obj}(\overline{\mathcal{C}})$. We now define associativity constraint $\overline{\alpha}$ in $\overline{\mathcal{C}}$. For any $X, Y, Z \in \text{Obj}(\overline{\mathcal{C}})$ define $\overline{\alpha}_{X,Y,Z}$ to be the following composition.

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$$(X \odot Y) \odot Z \xrightarrow{\beta(X \odot Y, Z)} (X \odot Y) \otimes Z \xrightarrow{\beta(X, Y) \otimes id_Z} (X \otimes Y) \otimes Z \xrightarrow{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z)$$

$$\xrightarrow{(id_X \otimes \beta(Y, Z))^{-1}} X \otimes (Y \odot Z) \xrightarrow{\beta(X, Y \odot Z)^{-1}} X \odot (Y \odot Z).$$

Left and right unit constraints are defined in the obvious way. It can be shown that the necessary axioms (pentagon, triangle) are satisfied. Then $\overline{C}$ is a fusion category that is equivalent to C. Since C is pointed, the simple objects of $\overline{C}$ form a finite group G and the associativity constraint $\overline{\alpha}$ in $\overline{D}$ gives rise to a 3-cocycle $\omega \in Z^3(G, k^\times)$. The cohomology class of this 3-cocycle does not depend on the choices made in the construction of $\overline{C}$. Then $C \cong \overline{C} \cong \mathrm{Vec}_G^\omega$ as fusion categories.

Definition 2.5.14. The *Grothendieck ring* $\mathcal{K}_0(C)$ of a fusion category C is the free $\mathbb{Z}$-module generated by the isomorphism classes of simple objects of C with the multiplication coming from the tensor product in C.

Remark 2.5.15. The Grothendieck ring of a fusion category is a based unital ring (see [O1]).

Example 2.5.16. The Grothendieck ring of the fusion category Vec_G^ω, where G is a finite group and ω is a 3-cocycle on G, is isomorphic to the group ring $\mathbb{Z}[G]$.

Definition 2.5.17. Let C be a fusion category. Let $\mathcal{K}_0(C)$ be the Grothendieck ring of C. For any object $X \in \mathrm{Obj}(C)$, define the *Frobenius-Perron dimension* of X, $\mathrm{FPdim}(X)$, to be the largest positive eigenvalue (which exists by the Frobenius-Perron theorem, see [Ga]) of the matrix $[X]$ of multiplication by $\overline{X}$ in $\mathcal{K}_0(C)$, where $\overline{X}$ is the image of X in $\mathcal{K}_0(C)$. The Frobenius-Perron dimension of C, $\mathrm{FPdim}(C)$, is

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the sum of squares of the Frobenius-Perron dimension of the objects in any complete set of representatives of simple objects of $\mathcal{C}$.

Example 2.5.18. Consider the fusion category $\mathrm{Rep}(H)$, where H is a semisimple finite-dimensional Hopf algebra over an algebraically closed field $\mathbb{F}$. Then $\mathrm{FPdim}(V) = \dim_{\mathbb{F}}(V)$, for all $V \in \mathrm{Obj}(\mathrm{Rep}(H))$.

2.6 Module categories

A module category over a tensor category can be thought of as the "categorification" of the notion of a module over a ring. Recall some definitions from [O1]:

Definition 2.6.1. A right *module category* over a tensor category $(\mathcal{C}, \otimes, 1_{\mathcal{C}}, \alpha, \lambda, \rho)$ is a category $\mathcal{M}$ together with a bifunctor $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and functorial associativity and unit isomorphisms: $\mu_{M,X,Y} : M \otimes (X \otimes Y) \to (M \otimes X) \otimes Y$, $\tau_M : M \otimes 1_{\mathcal{C}} \to M$ for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, $M \in \mathrm{Obj}(\mathcal{M})$ such that the diagrams

$$M \otimes (X \otimes Y) \otimes Z \qquad (2.14)$$

$$\mathrm{id}_M \otimes \alpha_{X,Y,Z} \qquad \mu_{M, X \otimes Y, Z}$$

$$M \otimes (X \otimes (Y \otimes Z)) \qquad (M \otimes (X \otimes Y)) \otimes Z$$

$$\mu_{M, X, Y \otimes Z} \qquad \mu_{M, X, Y} \otimes \mathrm{id}_Z$$

$$(M \otimes X) \otimes (Y \otimes Z) \xrightarrow{\mu_{M \otimes X, Y, Z}} ((M \otimes X) \otimes Y) \otimes Z$$

$$M \otimes (1_{\mathcal{C}} \otimes Y) \xrightarrow{\mu_{M, 1_{\mathcal{C}}, Y}} (M \otimes 1_{\mathcal{C}}) \otimes Y \qquad (2.15)$$

$$\mathrm{id}_M \otimes \lambda_Y \qquad \tau_M \otimes \mathrm{id}_Y$$

$$M \otimes Y$$

commute for all $M \in \mathrm{Obj}(\mathcal{M})$, $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$.

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Note 2.6.2. In the previous definition, if $\mathcal{C}$ is an abelian tensor category, then we will additionally require $\mathcal{M}$ to be an abelian category and $\otimes$ to be additive on the space of morphisms. If $\mathcal{C}$ is a $\mathbb{F}$-linear category for some field $\mathbb{F}$, then we will additionally require $\mathcal{M}$ to be a $\mathbb{F}$-linear category and $\otimes$ to be $\mathbb{F}$-bilinear on the space of morphisms.

Definition 2.6.3. Let $(\mathcal{M}_1, \mu^1, \tau^1)$ and $(\mathcal{M}_2, \mu^2, \tau^2)$ be two right module categories over a tensor category $\mathcal{C}$. A *module functor* from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ together with functorial isomorphisms $\gamma_{M,X} : F(M \otimes X) \to F(M) \otimes X$ for all $X \in \mathrm{Obj}(\mathcal{C})$, $M \in \mathrm{Obj}(\mathcal{M}_1)$ such that the diagrams

$$
\begin{array}{ccc}
& F(M \otimes (X \otimes Y)) & \\
\overset{F(\mu^1_{M,X,Y})}{\swarrow} & & \overset{\gamma_{M,X \otimes Y}}{\searrow} \\
F((M \otimes X) \otimes Y) & & F(M) \otimes (X \otimes Y) \\
\Big\downarrow {\scriptstyle \gamma_{M \otimes X, Y}} & & \Big\downarrow {\scriptstyle \mu^2_{F(M),X,Y}} \\
F(M \otimes X) \otimes Y & \xrightarrow{\;\;\gamma_{M,X} \otimes \mathrm{id}_Y\;\;} & (F(M) \otimes X) \otimes Y
\end{array}
\tag{2.16}
$$

$$
\begin{array}{ccc}
F(M \otimes 1_{\mathcal{C}}) & \xrightarrow{\;\;F(\tau^1_M)\;\;} & F(M) \\
{\scriptstyle \gamma_{M,1_{\mathcal{C}}}} \searrow & & \nearrow {\scriptstyle \tau^1_{F(M)}} \\
& F(M) \otimes 1_{\mathcal{C}} &
\end{array}
\tag{2.17}
$$

commute for all $M \in \mathrm{Obj}(\mathcal{M}_1), X, Y \in \mathrm{Obj}(\mathcal{C})$.

Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there exists a module functor from $\mathcal{M}_1$ to $\mathcal{M}_2$ which is an equivalence of categories. For two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over a tensor category $\mathcal{C}$ their *direct sum* is the category $\mathcal{M}_1 \oplus \mathcal{M}_2$ with the obvious module category structure. A module category is *indecomposable* if it is not equivalent to a direct sum of two non-trivial module categories.

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Definition 2.6.4. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two right module categories over a tensor category $\mathcal{C}$. Let (F^1, γ^1) and (F^2, γ^2) be module functors from $\mathcal{M}_1$ to $\mathcal{M}_2$. A *natural module transformation* from (F^1, γ^1) to (F^2, γ^2) is a natural transformation $\eta : F^1 \to F^2$ such that the square

$$
\begin{array}{ccc}
F^1(M \otimes X) & \xrightarrow{\eta_{M \otimes X}} & F^2(M \otimes X) \\
{\scriptstyle \gamma^1_{M,X}} \downarrow & & \downarrow {\scriptstyle \gamma^2_{M,X}} \\
F^1(M) \otimes X & \xrightarrow[\eta_M \otimes id_X]{} & F^2(M) \otimes X
\end{array}
\tag{2.18}
$$

commutes for all $M \in \mathrm{Obj}(\mathcal{M})$, $X \in \mathrm{Obj}(\mathcal{C})$.

Example 2.6.5. Let us recall a description of semisimple indecomposable module categories over $\mathcal{V}_G^\omega$ (the skeletal fusion category defined in Example 2.5.12) given in [O2]. Let $\mathcal{M}$ be a semisimple indecomposable right module category over $\mathcal{V}_G^\omega$ with module category structure μ. Without loss of generality we may assume that $\mathcal{M}$ is skeletal. The set of simple objects of $\mathcal{M}$ is a transitive right G-set and hence can be identified with the set of right cosets $H \backslash G$ for some subgroup H of G. So the set of all simple objects of $\mathcal{M}$, $\mathrm{Irr}(\mathcal{M}) = H \backslash G$. All the isomorphisms μ_{x, g_1, g_2}, $x \in H \backslash G$, $g_1, g_2 \in G$ are given by scalars. So we can regard μ as an element of $C^2(G, \mathrm{Coind}_H^G k^\times)$:

$$
\mu(g_1, g_2)(x) := \mu_{x, g_1, g_2}, \qquad x \in H \backslash G, g_1, g_2 \in G.
$$

We may assume that the 2-cochain μ is normalized. Since the unit constraint in $\mathcal{V}_G^\omega$ is trivial, the commutativity of Triangle 2.15 implies that the unit constraint in $\mathcal{M}$

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is trivial. Let us regard ω as an element of $Z^3(G, \mathrm{Coind}_H^G k^\times) \subset C^3(G, \mathrm{Coind}_H^G k^\times)$ by treating $\omega(g_1, g_2, g_3)$ as a constant function on $H\backslash G$, for all $g_1, g_2, g_3 \in G$. The commutativity of the Pentagon 2.14 implies that

$$\delta^2 \mu = \omega \qquad (2.19)$$

i.e.,

$$\mu(g_2, g_3)(x \triangleleft g_1)\ \mu(g_1 g_2, g_3)(x)^{-1}\ \mu(g_1, g_2 g_3)(x)\ \mu(g_1, g_2)(x)^{-1} = \omega(g_1, g_2, g_3),$$

$$(2.20)$$

for all $g_1, g_2, g_3 \in G, x \in H\backslash G$.

The previous equation, in particular, means that ω restricted to $H \times H \times H$ represents the trivial class in $H^3(H, k^\times)$. So semisimple indecomposable right module categories over $\mathcal{V}_G^\omega$ are given by pairs (H, μ), where H is a subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $\mu \in C^2(G, \mathrm{Coind}_H^G k^\times)$ is a 2-cochain satisfying $\delta^2 \mu = \omega$, where ω is regarded as an element of $Z^3(G, \mathrm{Coind}_H^G k^\times)$.

Let H be a subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial. Let $\Lambda_{H,\omega} := \{\mu \in C^2(G, \mathrm{Coind}_H^G k^\times) \mid \delta^2 \mu = \omega\}$. Two elements in $\Lambda_{H,\omega}$ give rise to equivalent module categories if they differ by some element in $B^2(G, \mathrm{Coind}_H^G k^\times)$; we will say that such elements are equivalent. Let $\overline{\Lambda}_{H,\omega}$ denote the equivalence classes of $\Lambda_{H,\omega}$. There is a (in general non-canonical) bijection between $\overline{\Lambda}_{H,\omega}$ and $H^2(H, k^\times)$. Note that $\overline{\Lambda}_{H,1} = H^2(G, \mathrm{Coind}_H^G k^\times) \cong H^2(H, k^\times)$.

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2.7 The dual category

Let $\mathcal{C}$ be a tensor category and let $\mathcal{M}$ be a right module category over $\mathcal{C}$. The notion of the dual of $\mathcal{C}$ with respect to $\mathcal{M}$ is analogous to the notion of the dual of a ring R with respect to an R-module M, which is defined to be the endomorphism ring $\text{End}_R(M)$.

Definition 2.7.1. The dual category of $\mathcal{C}$ with respect to $\mathcal{M}$ is the category $\mathcal{C}_{\mathcal{M}}^* := \text{Func}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ whose objects are $\mathcal{C}$-module functors from $\mathcal{M}$ to itself and morphisms are natural module transformations.

The category $\mathcal{C}_{\mathcal{M}}^*$ is a tensor category with tensor product being composition of module functors. Let (γ^1, F^1), $(\gamma^2, F^2) \in Obj(\mathcal{C}_{\mathcal{M}}^*)$, where γ^1, γ^2 represent the module functor structure on the functors F^1 and F^2, respectively. Then, $(\gamma^1, F^1) \otimes (\gamma^2, F^2) = (\gamma, F^1 \circ F^2)$, where γ is defined by: $\gamma_{M,X} := \gamma^1_{F^2(M),X} \circ F^1(\gamma^2_{M,X})$ for all $M \in \mathcal{M}$, $X \in \mathcal{C}$. Let $\eta : (\gamma^1, F^1) \to (\gamma^2, F^2)$ and $\eta' : (\gamma^3, F^3) \to (\gamma^4, F^4)$ be morphisms in $\mathcal{C}_{\mathcal{M}}^*$, i.e., natural module transformations. Then their tensor product $\eta \otimes \eta'$ is defined by: $(\eta \otimes \eta')(M) := \eta_{F^4(M)} \circ F^1(\eta'_M)$. Denote by $(1, id_{\mathcal{M}})$ the obvious unit object.

Remark 2.7.2. It is known that if $\mathcal{C}$ is a fusion category and $\mathcal{M}$ is a semisimple indecomposable module category over $\mathcal{C}$, then $\mathcal{C}_{\mathcal{M}}^*$ is a fusion category. In this case, it is known that $\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{C}_{\mathcal{M}}^*)$ (see [ENO]).

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Definition 2.7.3. Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *weakly Morita equivalent* if there exists an indecomposable semisimple right module category $\mathcal{M}$ over $\mathcal{C}$ such that the categories $\mathcal{C}_{\mathcal{M}}^*$ and $\mathcal{D}$ are equivalent as fusion categories.

Remark 2.7.4. It was shown by Müger that the above relation is indeed an equivalence relation.

Definition 2.7.5. A fusion category $\mathcal{C}$ is said to be *group theoretical* if it is weakly Morita equivalent to a pointed category.

Remark 2.7.6. A fusion category is group-theoretical if and only if it is equivalent to the fusion category $(\mathrm{Vec}_G^\omega)_{\mathcal{M}}^*$ for some finite group G and $\omega \in Z^3(G, k^\times)$ and some semisimple indecomposable module category $\mathcal{M}$ over Vec_G^ω. A classification of semisimple indecomposable module categories over Vec_G^ω was given in Example 2.6.5. A finite-dimensional semisimple Hopf algebra H is said to be *group-theoretical* if its representation category $\mathrm{Rep}(H)$ is group-theoretical. It is not known to the author, at the time of writing, if there exists a finite-dimensional semisimple Hopf algebra that is not group-theoretical.

2.8 Graded and nilpotent fusion categories

The following is taken from [GN].

Let (R, B) denote a based ring R with basis B and let $\mathcal{C}$ be a fusion category.

Definition 2.8.1. (R, B) is said to be *graded* by a finite group G if there is a partition $B = \cup_{g \in G} B_g$, such that $R = \oplus_{g \in G} R_g$, where R_g is a $\mathbb{Z}$-submodule of R generated by B_g and $R_{g_1} R_{g_2} \subset R_{g_1 g_2}$, $R_g^* = R_{g^{-1}}$, for all $g, g_1, g_2 \in G$.

Definition 2.8.2. $\mathcal{C}$ is said to be *graded* by a finite group G if $\mathcal{C}$ decomposes into a direct sum of full abelian subcategories $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ such that $\mathcal{C}_g \neq 0$, $\mathcal{C}_g^* = \mathcal{C}_{g^{-1}}$ and the tensor product maps $\mathcal{C}_{g_1} \times \mathcal{C}_{g_2}$ to $\mathcal{C}_{g_1 g_2}$, for all $g, g_1, g_2 \in G$.

Let R_{ad} denote the based subring of R generated by all basic elements of R contained in $X X^*$, $X \in B$. Let $R^{(0)} := R$, $R^{(1)} := R_{ad}$, and $R^{(n)} := (R^{(n-1)})_{ad}$, for every positive integer n. Similarly, let $\mathcal{C}_{ad}$ denote the full tensor subcategory of $\mathcal{C}$ generated by all simple subobjects of $X \otimes X^*$, X simple object of $\mathcal{C}$. Let $\mathcal{C}^{(0)} := \mathcal{C}$, $\mathcal{C}^{(1)} := \mathcal{C}_{ad}$, and $\mathcal{C}^{(n)} := (\mathcal{C}^{(n-1)})_{ad}$, for every positive integer n.

Definition 2.8.3. R is said to be *nilpotent* if $R^{(n)} = \mathbb{Z}1$, for some n. The smallest n for which this happens is called the *nilpotency class* of R.

Definition 2.8.4. $\mathcal{C}$ is said to be *nilpotent* if $\mathcal{C}^{(n)} = \mathrm{Vec}$, for some n. The smallest n for which this happens is called the *nilpotency class* of $\mathcal{C}$.

Note 2.8.5. Note that a fusion category is nilpotent if and only if its Grothendieck ring is nilpotent.

Example 2.8.6. (1) The fusion category Vec_G^ω, where G is a finite group and ω is a 3-cocycle on G, is nilpotent. Its nilpotency class is equal to 1.

(2) Let G be a finite group and $\mathcal{C} := \mathrm{Rep}(G)$. Then $\mathcal{C}_{ad} \cong \mathrm{Rep}(G/Z(G))$, where $Z(G)$

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is the center of G. Furthermore, $\mathcal{C}$ is nilpotent if and only if G is nilpotent.

(3) The Tambara-Yamagami categories [TY] are nilpotent with nilpotency class equal to 2.

2.9 Braided tensor categories, ribbon categories, and modular categories

Definition 2.9.1. A *braided tensor category* $\mathcal{C}$ is a tensor category $(\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ equipped with a natural isomorphism called *braiding*: $\sigma : \otimes \to \otimes \tau$ (where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C} : (X, Y) \mapsto (Y, X)$ is the flip functor) satisfying the following commutative diagrams called *hexagon axiom*:

$$
\begin{array}{ccccc}
 & & X \otimes (Y \otimes Z) \xrightarrow{\sigma_{X,Y \otimes Z}} (Y \otimes Z) \otimes X & & \\
 & \overset{\alpha_{X,Y,Z}}{\nearrow} & & \overset{\alpha_{Y,Z,X}}{\searrow} & \\
(X \otimes Y) \otimes Z & & & & Y \otimes (Z \otimes X) \\
 & \overset{\sigma_{X,Y} \otimes \mathrm{id}_Z}{\searrow} & & \overset{\mathrm{id}_Y \otimes \sigma_{X,Z}}{\nearrow} & \\
 & & (Y \otimes X) \otimes Z \xrightarrow{\alpha_{Y,X,Z}} Y \otimes (X \otimes Z) & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & (X \otimes Y) \otimes Z \xrightarrow{\sigma_{X \otimes Y, Z}} Z \otimes (X \otimes Y) & & \\
 & \overset{\alpha_{X,Y,Z}^{-1}}{\nearrow} & & \overset{\alpha_{Z,X,Y}^{-1}}{\searrow} & \\
X \otimes (Y \otimes Z) & & & & (Z \otimes X) \otimes Y \\
 & \overset{\mathrm{id}_X \otimes \sigma_{Y,Z}}{\searrow} & & \overset{\sigma_{X,Z} \otimes \mathrm{id}_Y}{\nearrow} & \\
 & & X \otimes (Z \otimes Y) \xrightarrow{\alpha_{X,Z,Y}^{-1}} (X \otimes Z) \otimes Y & &
\end{array}
$$

for all objects X, Y, Z in $\mathcal{C}$.

Definition 2.9.2. A braided tensor category $\mathcal{C}$ with braiding σ is called *symmetric* if $\sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X \otimes X}$, for all objects X and Y in $\mathcal{C}$.

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Definition 2.9.3. A tensor functor (F, φ, ν) from a braided tensor category $\mathcal{C}$ (with braiding σ) to a braided tensor category $\mathcal{C}'$ (with braiding σ') is *braided*, if for any pair X, Y of objects in $\mathcal{C}$, the square

$$
\begin{array}{ccc}
F(X) \otimes F(Y) & \xrightarrow{\varphi_{X,Y}} & F(X \otimes Y) \\
{\scriptstyle \sigma'_{F(X),F(Y)}} \downarrow & & \downarrow {\scriptstyle F(\sigma_{X,Y})} \\
F(Y) \otimes F(X) & \xrightarrow[\varphi_{Y,X}]{} & F(Y \otimes X)
\end{array}
$$

commutes.

Definition 2.9.4. Let $\mathcal{C}$ be a tensor category with associativity constraint α. The *center* $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is the category whose objects are pairs $(V, \sigma_{-,V})$, where $V \in \mathrm{Obj}(\mathcal{C})$ and $\sigma_{-,V}$ is a family of natural isomorphisms $\sigma_{X,V} : X \otimes V \xrightarrow{\sim} V \otimes X$ defined for all objects $X \in \mathrm{Obj}(\mathcal{C})$ such that for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ the following diagram commutes.

$$
\begin{array}{ccccc}
& & (X \otimes Y) \otimes V & & \\
& {\scriptstyle \alpha_{X,Y,V}} \swarrow & & \searrow {\scriptstyle \sigma_{X \otimes Y,V}} & \\
X \otimes (Y \otimes V) & & & & V \otimes (X \otimes Y) \\
{\scriptstyle \mathrm{id}_X \otimes \sigma_{Y,V}} \downarrow & & & & \downarrow {\scriptstyle \alpha^{-1}_{V,X,Y}} \\
X \otimes (V \otimes Y) \xrightarrow{\alpha^{-1}_{X,V,Y}} & (X \otimes V) \otimes Y & \xrightarrow{\sigma_{X,V} \otimes \mathrm{id}_Y} & (V \otimes X) \otimes Y
\end{array}
$$

A morphism from $(V, \sigma_{-,V})$ to $(W, \sigma_{-,W})$ is a morpshism $f : V \to W$ in $\mathcal{C}$ such that for each $X \in \mathrm{Obj}(\mathcal{C})$ we have $(f \otimes \mathrm{id}_X)\sigma_{X,V} = \sigma_{X,W}(\mathrm{id}_X \otimes f)$.

Remark 2.9.5. The center $\mathcal{Z}(\mathcal{C})$ of a tensor category $\mathcal{C}$ has a canonical structure of a braided tensor category.

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Definition 2.9.6. Let $\mathcal{C} = (\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ be a rigid tensor category. For any morphism $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, the *dual* of f is a morphism f^* in $\text{Hom}_{\mathcal{C}}(Y^*, X^*)$ defined by: $f^* := Y^* \xrightarrow{\rho_{Y^*}^{-1}} Y^* \otimes 1 \xrightarrow{\text{id}_{Y^*} \otimes c_X} Y^* \otimes (X \otimes X^*) \xrightarrow{\text{id}_{Y^*} \otimes (f \otimes \text{id}_{X^*})} Y^* \otimes (Y \otimes X^*) \xrightarrow{\alpha_{Y^*,Y,X^*}^{-1}}$ $(Y^* \otimes Y) \otimes X^* \xrightarrow{e_Y \otimes \text{id}_{X^*}} 1 \otimes X^* \xrightarrow{\lambda_{X^*}} X^*$, where e and c are the evaluation and coevaluation morphisms, respectively.

Definition 2.9.7. A rigid braided tensor category $\mathcal{C}$ (with braiding σ) is said to be *ribbon* if it is equipped with a natural isomorphism called *twist*: $\theta : \text{id}_{\mathcal{C}} \to \text{id}_{\mathcal{C}}$, satisfying $\theta_{X \otimes Y} = \sigma_{Y,X} \circ \sigma_{X,Y} \circ (\theta_X \otimes \theta_Y)$, $\theta_1 = \text{id}_1$, and $\theta_{X^*} = (\theta_X)^*$, for all $X, Y \in \text{Obj}(\mathcal{C})$.

Definition 2.9.8. Let $\mathcal{C}$ be a $\mathbb{F}$-linear ribbon category with braiding σ, twist θ, and neutral object 1 such that $\text{Hom}_{\mathcal{C}}(1, 1) \cong \mathbb{F}$. For any endomorphism $f \in \text{Hom}_{\mathcal{C}}(X, X)$, define its *trace*, $tr(f) := 1 \xrightarrow{c_X} X \otimes X^* \xrightarrow{(\theta_X \circ f) \otimes \text{id}_{X^*}} X \otimes X^* \xrightarrow{\sigma_{X,X^*}}$ $X^* \otimes X \xrightarrow{e_X} 1 \in \mathbb{F} \cong \text{Hom}_{\mathcal{C}}(1, 1)$, where e and c are the evaluation and coevaluation morphisms, respectively. For any object $X \in \text{Obj}(\mathcal{C})$, the value $tr(\text{id}_X)$ is called the *dimension* of X and denoted $d(X)$.

Remark 2.9.9. In the previous definition, the identification of $\text{Hom}_{\mathcal{C}}(1, 1)$ with $\mathbb{F}$ is by the isomorphism $\mathbb{F} \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(1, 1) : \lambda \mapsto \lambda \cdot \text{id}_1$.

Definition 2.9.10. A ribbon fusion category $\mathcal{C}$ with isomorphism classes of simple objects enumerated as $\{X_1 = 1, X_2, \cdots, X_n\}$ is said to be *modular* if the *S-matrix* with entries $S_{i,j} := tr(\sigma_{X_j,X_i} \circ \sigma_{X_i,X_j})$ is invertible.

Remark 2.9.11. The name "modular" comes from the fact that modular categories give rise to projective representations of the modular group $SL(2, \mathbb{Z})$.

Example 2.9.12. Let G be a finite group. Its group algebra $k[G]$ over k is a Hopf algebra with k-basis $\{x \mid x \in G\}$ and

$$
\begin{array}{lll}
\textit{multiplication} & x \otimes y \mapsto xy, & x, y \in G, \\[2mm]
\textit{unit} & 1_G, & \\[2mm]
\textit{comultiplication} & \Delta(x) = x \otimes x, & x \in G, \\[2mm]
\textit{counit} & \epsilon(x) = 1, & x \in G, \\[2mm]
\textit{antipode} & \gamma(x) = x^{-1} & x \in G.
\end{array}
$$

The Hopf algebra dual to $k[G]$ is isomorphic to the function algebra $F(G)$ of the group G. It has k-basis $\{\delta_g \mid g \in G\}$ where

$$
\delta_g(x) = \delta_{g,x} = \begin{cases} 1 & \text{for } g = x, \\[2mm] 0 & \text{for } g \neq x. \end{cases}
$$

It has

$$
\begin{array}{lll}
\textit{multiplication} & \delta_g \delta_h = \delta_{g,h}\delta_g, & g, h \in G, \\[2mm]
\textit{unit} & 1 = \displaystyle\sum_{g \in G} \delta_g, & \\[4mm]
\textit{comultiplication} & \Delta(\delta_g) = \displaystyle\sum_{g_1 g_2 = g} \delta_{g_1} \otimes \delta_{g_2}, & g \in G, \\[4mm]
\textit{counit} & \epsilon(\delta_g) = \delta_{g,1_G}, & g \in G, \\[2mm]
\textit{antipode} & \gamma(\delta_g) = \delta_{g^{-1}} & x, g \in G.
\end{array}
$$

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As a vector space the *Drinfeld double* $D(G)$ of G is $F(G) \otimes_k k[G]$. $D(G)$ is a Hopf

algebra with

$$multiplication \qquad (\delta_g \otimes x)(\delta_h \otimes y) = \delta_{g,xhx^{-1}}(\delta_g \otimes xy), \qquad x,y,g,h \in G,$$

$$unit \qquad 1 = \sum_{g \in G} \delta_g \otimes 1_G,$$

$$comultiplication \qquad \Delta(\delta_g \otimes x) = \sum_{g_1 g_2 = g} (\delta_{g_1} \otimes x) \otimes (\delta g_2 \otimes x) \qquad x,g \in G,$$

$$counit \qquad \epsilon(\delta_g \otimes x) = \delta_{g,1_G}, \qquad x,g \in G,$$

$$antipode \qquad \gamma(\delta_g \otimes x) = \delta_{x^{-1}g^{-1}x} \otimes x^{-1} \qquad x,y \in G.$$

The category $\mathrm{Rep}(D(G))$ of finite-dimensional representations of $D(G)$ as a k-algebra

is a modular category. A description of simple object and explicit formulas for the

S-matrix and twist of $\mathrm{Rep}(D(G))$ are mentioned in Chapter 5.

Remark 2.9.13. It is known that for any finite group G, the categories $\mathcal{Z}(\mathrm{Vec}_G)$

and $\mathrm{Rep}(D(G))$ are equivalent as braided tensor categories. A twisted version (called

twisted quantum double of G) $D^\omega(G)$, where ω is a 3-cocycle on the finite group G was

introduced in [DPR1]. Note that $D^\omega(G)$ is a quasi-triangular quasi-Hopf algebra. It

is known that $\mathrm{Rep}(D^\omega(G))$ is a modular category. It is also known that $\mathrm{Rep}(D^\omega(G))$

is equivalent to $\mathcal{Z}(\mathrm{Vec}_G^\omega)$ as a braided tensor category.

2.10 Centralizers in modular categories

Let $\mathcal{C}$ be a modular category with braiding σ, twist θ, and S-matrix S (see [BK]).

Let $\mathcal{D}$ be a full (not necessarily tensor) subcategory of $\mathcal{C}$. Its dimension is defined by

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$\dim(\mathcal{D}) := \sum_{X \in \mathrm{Irr}(\mathcal{D})} d(X)^2$, where $\mathrm{Irr}(\mathcal{D})$ is the set of isomorphism classes of simple objects in $\mathcal{D}$. In [Mu2], Müger introduced the notion of the centralizer $\mathcal{D}' = C_{\mathcal{C}}(\mathcal{D})$ of $\mathcal{D}$ in $\mathcal{C}$ as the full subcategory defined by

$$\mathrm{Obj}(\mathcal{D}') := \{X \in \mathcal{C} \mid \sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X \otimes Y}, \text{ for all } Y \in \mathcal{D}\}.$$

It was shown that $\mathcal{D}'$ is a fusion subcategory of $\mathcal{C}$ and that

$$\dim(\mathcal{D}) \cdot \dim(\mathcal{D}') = \dim(\mathcal{C}). \tag{2.21}$$

Following M. Müger, we will say that two objects $X, Y \in \mathcal{C}$ *centralize* each other if $\sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X \otimes Y}$. For simple X and Y this condition is equivalent to $S(X,Y) = d(X)d(Y)$ [Mu2, Corollary 2.14].

Remark 2.10.1. If $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ such that all objects in $\mathcal{D}$ centralize each other, i.e., $\mathcal{D} \subseteq \mathcal{D}'$ then $(\dim(\mathcal{D}))^2 \leq \dim(\mathcal{C})$. Indeed, we have $\dim(\mathcal{D}) \leq \dim(D')$ and so it follows from (2.21) that $(\dim(\mathcal{D}))^2 \leq \dim(\mathcal{C})$. In particular, if $\mathcal{D}$ is a symmetric fusion subcategory of $\mathcal{C}$, then $(\dim(\mathcal{D}))^2 \leq \dim(\mathcal{C})$.

Lemma 2.10.2. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ (which is not apriori assumed to be closed under the tensor product or duality) such that $\mathcal{D} \subseteq \mathcal{D}'$. Then the fusion subcategory $\tilde{\mathcal{D}} \subseteq \mathcal{C}$ generated by $\mathcal{D}$ is symmetric.

Proof. We may assume that $\mathcal{D}$ is closed under taking duals. Indeed, it follows from [ENO, Proposition 2.12] that X centralizes Y if and only if X centralizes Y^* for any two simple objects X, Y in $\mathcal{C}$.

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Let Z_1, Z_2 be simple objects in $\tilde{\mathcal{D}}$. There exist simple objects X_1, X_2, Y_1, Y_2 in $\mathcal{D}$ such that Z_1 is contained in $X_1 \otimes Y_1$ and Z_2 is contained in $X_2 \otimes Y_2$. By [Mu2, Lemma 2.4 (i)], it follows that Z_1 centralizes $X_2 \otimes Y_2$, and hence Z_1, Z_2 centralize each other. ∎

Corollary 2.10.3. *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ such that $\mathcal{D} \subseteq \mathcal{D}'$ and $dim(\mathcal{D})^2 = dim(\mathcal{C})$. Then $\mathcal{D}$ is a symmetric fusion subcategory.*

2.11 Lagrangian subcategories and braided equivalences of twisted group doubles

Let $\mathcal{C}$ be a modular category. Let us assume that $\mathcal{C}$ has integral Frobenius-Perron dimensions of simple objects. It was shown in [ENO] that any such category is equivalent to the representation category of a semisimple quasi-Hopf algebra and has a canonical spherical structure with respect to which the categorical dimension of any object is equal to its Frobenius-Perron dimension. In particular, all categorical dimensions are positive integers. Let us recall some definitions and results from [DGNO].

Definition 2.11.1. A fusion subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be *isotropic* if the twist θ of $\mathcal{C}$ restricts to identity on $\mathcal{D}$.

Definition 2.11.2. A fusion subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be *Lagrangian* if it is isotropic and $(dim(\mathcal{D}))^2 = dim(\mathcal{C})$.

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Remark 2.11.3. (i) The definitions of isotropic and Lagrangian subcategories above are motivated by the example below.

(ii) Isotropic subcategories are necessarily symmetric.

Example 2.11.4 ([DGNO]). Let G be a finite abelian group. Let $q : G \to k^\times$ be a quadratic form, i.e., for all $g \in G$, $q(g^{-1}) = q(g)$ and the map

$$b : G \times G \to k^\times : (g_1, \, g_2) \mapsto \frac{q(g_1 g_2)}{q(g_1)q(g_2)}$$

is a symmetric bicharacter. The pair (G, q) is known in literature as a *metric group*. A subgroup H of G is called *isotropic* is $q|_H = 1$. An isotropic subgroup H of G is called *Lagrangian* if $H^\perp = H$. Let $\mathcal{C}(G, q) := \mathcal{V}_G^1$ (the skeletal pointed category defined in Example 2.5.12). The quadratic form q gives $\mathcal{C}(G, q)$ the structure of a braided category (see [Q]):

$$k \cong \mathrm{Hom}_{\mathcal{C}(G, q)}(g_1 g_2, \, g_1 g_2) \ni g_1 \otimes g_2 \xrightarrow{\sim} g_2 \otimes g_1 := q(g_1 g_2)\mathrm{id}_{g_1 g_2},$$

for all $g_1, g_2 \in G$. The category $\mathcal{C}(G, q)$ is modular if and only if the symmetric bicharacter b associated to q is non-degenerate. It is also know that isotropic and Lagrangian subcategories of $\mathcal{C}(G, q)$ correspondence to isotropic and Lagrangian subgroups of (G, q), respectively.

Consider the set of all braided tensor equivalences $F : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathcal{P})$, where $\mathcal{P}$ is a pointed fusion category. There is an equivalence relation on this set defined as follows.

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We say that $F_1 : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathcal{P}_1)$ and $F_2 : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathcal{P}_2)$ are equivalent if there exits a tensor

equivalence $\iota : \mathcal{P}_1 \xrightarrow{\sim} \mathcal{P}_2$ such that $\mathcal{F}_2 \circ F_2 = \iota \circ \mathcal{F}_1 \circ F_1$, where $\mathcal{F}_i : \mathcal{Z}(\mathcal{P}_i) \to \mathcal{P}_i$, $i = 1, 2$,

are the canonical forgetful functors. Let $\mathrm{E}(\mathcal{C})$ be the collection of equivalence classes

of such equivalences. Informally, $\mathrm{E}(\mathcal{C})$ is the set of all "different" braided equivalences

between $\mathcal{C}$ and representation categories of twisted group doubles.

Let $\mathrm{Lagr}(\mathcal{C})$ be the set of all Lagrangian subcategories of $\mathcal{C}$.

In [DGNO, Theorem 4.5] it was proved that there is a bijection

$$f : \mathrm{E}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Lagr}(\mathcal{C}) \qquad\qquad (2.22)$$

defined as follows. Note that each braided tensor equivalence $F : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathcal{P})$ gives rise

to the Lagrangian subcategory $f(F)$ of $\mathcal{C}$ formed by all objects sent to multiples of

the unit object $\mathbf{1}$ under the forgetful functor $\mathcal{Z}(\mathcal{P}) \to \mathcal{P}$. This subcategory is clearly

the same for all equivalent choices of F.

In particular, the center of a fusion category $\mathcal{D}$ contains a Lagrangian subcategory

if and only if $\mathcal{D}$ is group-theoretical [DGNO].

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CHAPTER 3

CATEGORICAL MORITA EQUIVALENCE FOR GROUP-THEORETICAL CATEGORIES

The results presented in this Chapter are based on [N]. The organization of this Chapter is as follows. In Section 3.1 we give necessary and sufficient conditions for the dual of a pointed category with respect to an indecomposable module category to be pointed. In Section 3.2 we show that the Grothendieck ring of the dual of a pointed category with respect to an indecomposable module category when the dual is pointed is the group ring of a certain crossed product of groups. We also find an explicit formula for the 3-cocycle associated to the dual category. In Section 3.3 we introduce the notion of *categorical Morita equivalence* on the set of all finite groups and on the set of all pairs (G, ω), where G is a finite group and $\omega \in H^3(G, k^\times)$. We give a group-theoretical and cohomological interpretation of these relations. In the final section, Section 3.4, we give a series of examples of pairs of groups that are categorically Morita equivalent but have non-isomorphic Grothendieck rings.

3.1   Necessary and sufficient condition for the dual of a pointed category to be pointed

We fix the following notation for this and the next Section. Let G be a finite group and $\omega \in Z^3(G, k^\times)$. Let H be a subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically

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trivial. Let $K := H\backslash G$ and $C := \text{Coind}_H^G k^\times$. Let $u : K \to G$ be a function satisfying $p \circ u = id_K$ and $u(p(1_G)) = 1_G$, where $p : G \to K$ is the usual surjection. Denote $p(1_G)$ by 1. Let $\kappa : K \times G \to H$ be the function satisfying (2.2). Let $\mathcal{C} := \mathcal{V}_G^\omega$ and let $\mathcal{M} = \mathcal{M}(H, \mu)$ denote the right module category constructed from the pair (H, μ) (see Examples 2.5.12 and 2.6.5), where $\mu \in C^2(G, C)$ is a 2-cochain satisfying $\delta^2 \mu = \omega$. In the previous equation we regarded ω as an element of $Z^3(G, C)$ by treating $\omega(g_1, g_2, g_3)$ as a constant function on K, for all $g_1, g_2, g_3 \in G$. The module category structure of $\mathcal{M}$ is given by μ. If $\omega \equiv 1$, then we will assume that μ belongs to $Z^2(H, k^\times)$ and that the module category structure of $\mathcal{M}(H, \mu)$ is given by $\varphi(\mu)$ (see (2.7)).

Lemma 3.1.1. For each $x \in K^H$, $\frac{{}^x\mu}{\mu}$ is an element of $Z^2(G, \text{Coind}_H^G k^\times)$.

Proof. We have $\delta^2 \mu = \omega$, where ω is regarded as an element of $Z^3(G, \text{Coind}_H^G k^\times)$. It suffices to show that $\delta^2({}^x\mu) = \omega$, for all $x \in K^H$. This follows from the fact that $p(u(x)u(y)) \triangleleft g = p(u(x)u(y \triangleleft g))$, for all $x \in K^H, y \in K$, and $g \in G$. Indeed,

$$(\delta^2({}^x\mu))(g_1, g_2, g_3)(y)$$

$$= {}^x\mu(g_2, g_3)(y \triangleleft g_1) \, {}^x\mu(g_1 g_2, g_3)(y)^{-1} \, {}^x\mu(g_1, g_2 g_3)(y) \, {}^x\mu(g_1, g_2)(y)^{-1}$$

$$= \mu(g_2, g_3)(p(u(x)u(y \triangleleft g_1))) \, \mu(g_1 g_2, g_3)(p(u(x)u(y)))^{-1}$$

$$\times \mu(g_1, g_2 g_3)(p(u(x)u(y))) \, \mu(g_1, g_2)(p(u(x)u(y)))^{-1}$$

$$= \mu(g_2, g_3)(p(u(x)u(y)) \triangleleft g) \, \mu(g_1 g_2, g_3)(p(u(x)u(y)))^{-1}$$

$$\times \mu(g_1, g_2 g_3)(p(u(x)u(y))) \, \mu(g_1, g_2)(p(u(x)u(y)))^{-1}$$

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$$= (\delta^2\mu)(g_1,\ g_2,\ g_3)(p(u(x)u(y)))$$

$$= \omega(g_1,\ g_2,\ g_3)(p(u(x)u(y)))$$

$$= \omega(g_1,\ g_2,\ g_3).$$

So $\delta^2({}^x\mu) = \omega$, for all $x \in K^H$ and the Lemma is proved. ∎

Definition 3.1.2. For each $x \in K^H$, define the set $\mathrm{Fun}_x = \mathrm{Fun}_x(G,\ C)$:

$$\mathrm{Fun}_x := \left\{ \gamma \in C^1(G,\ C) \,\middle|\, \delta^1\gamma = \frac{{}^x\mu}{\mu} \right\}. \qquad (3.1)$$

Lemma 3.1.3. Invertible objects in $\mathcal{C}^*_{\mathcal{M}}$ are given by pairs $(\gamma,\ x)$, where $x \in K^H$ and $\gamma \in \mathrm{Fun}_x$.

Proof. We associate an invertible objects in $\mathcal{C}^*_{\mathcal{M}}$ to each pair $(\gamma,\ x)$, where $x \in K^H$ and $\gamma \in \mathrm{Fun}_x$ as follows: define a map $f_x : K \to K$ by $f_x(y) = p(u(x)u(y))$ for all $y \in K$. Extend the map f_x to a functor $F_x : \mathcal{M} \to \mathcal{M}$. The module functor structure on F_x, which is also denoted by γ, is: $\gamma_{y,g} := \gamma(g)(y)\ id_{(p(u(x)u(y)))\triangleleft g}$ for all $g \in G$ and $y \in K$. The pentagon axiom for a module functor (2.16) is:

$${}^x\mu(g_1,\ g_2)(y)\ \gamma(g_1 g_2)(y) = \gamma(g_1)(y)\ \gamma(g_2)(y \triangleleft g_1)\ \mu(g_1,\ g_2)(y),$$

for all $g_1,\ g_2 \in G$ and $y \in K$.

This condition is satisfied because $\gamma \in \mathrm{Fun}_x$. The inverse of $(\gamma,\ F_x)$ is the module functor associated to the pair $(({}^{p(u(x)^{-1})}\gamma)^{-1},\ p(u(x)^{-1}))$. All invertible objects in $\mathcal{C}^*_{\mathcal{M}}$ arise in this way and the Lemma is proved. ∎

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Two invertible $\mathcal{C}$-module functors (γ^1, x_1) and (γ^2, x_2) are isomorphic in $\mathcal{C}_{\mathcal{M}}^*$ if and only if $x_1 = x_2$ and there exists an element $\alpha \in C$ such that $\gamma^1(g)(y) = \frac{\alpha(y \triangleleft g)}{\alpha(y)} \gamma^2(g)(y)$ for all $g \in G$ and $y \in K$.

This motivates us to define an equivalence relation on the set Fun_x: we define two elements $\gamma^1, \gamma^2 \in \mathrm{Fun}_x$ to be equivalent if there exists an $\alpha \in C$ such that

$$\gamma^1(g)(y) = \frac{\alpha(y \triangleleft g)}{\alpha(y)} \gamma^2(g)(y),$$

for all $g \in G$ and $y \in K$.

Let $\overline{\mathrm{Fun}}_x$ denote the set of equivalence classes of Fun_x under the aforementioned equivalence relation.

Lemma 3.1.4. For each $x \in K^H$, if $\overline{\mathrm{Fun}}_x \neq \emptyset$, then there is a bijection between the sets $\overline{\mathrm{Fun}}_x$ and $H^1(G, C)$ and hence there is a bijection between the sets $\overline{\mathrm{Fun}}_x$ and $\widehat{H}$.

Proof. Suppose $\overline{\mathrm{Fun}}_x \neq \emptyset$, $x \in K^H$. Fix some $\eta \in \mathrm{Fun}_x$. Then the maps

$$\mathrm{Fun}_x \to Z^1(G, C) : \beta \mapsto \frac{\beta}{\eta}$$

and

$$Z^1(G, C) \to \mathrm{Fun}_x : \gamma \mapsto \eta\gamma$$

are inverse to each other. These maps induce a bijections between the sets $\overline{\mathrm{Fun}}_x$ and $H^1(G, C)$. The second statement of the Lemma follows from Shapiro's Lemma. ■

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Theorem 3.1.5. *The fusion category $\mathcal{C}^*_{\mathcal{M}}$ (where $\mathcal{C} = \mathcal{V}^\omega_G$ and $\mathcal{M} = \mathcal{M}(H, \mu)$) is pointed if and only if the following three conditions hold:*

1. H is abelian,

2. H is normal in G and

3. the restriction $\psi({}^g\mu/\mu)$ is trivial in $H^2(H, k^\times)$, for all $g \in G$,

where ψ is the restriction map defined in (2.9). If $\omega \equiv 1$, then we assume that μ belongs to $Z^2(H, k^\times)$ and the module category structure on $\mathcal{M}$ is given by $\varphi(\mu)$ (see (2.7)). The third condition above is then replaced with:

3.' μ represents a G-invariant class in $H^2(H, k^\times)$.

Proof. Suppose that $\mathcal{C}^*_{\mathcal{M}}$ is pointed and let $\mathcal{S} = K^H$, where $K = H\backslash G$. The set of isomorphism classes of simple objects in $\mathcal{C}^*_{\mathcal{M}}$ is given by the set $\bigcup_{s \in \mathcal{S}} \left(\overline{\mathrm{Fun}_s} \times \{s\} \right)$. By the previous Lemma, we have $\mathrm{FPdim}(\mathcal{C}^*_{\mathcal{M}}) \leq |\widehat{H}| \, |\mathcal{S}|$. Note that $|\widehat{H}| \leq |H|$ and $|\mathcal{S}| \leq |K| = \frac{|G|}{|H|}$. By Remark 2.7.2, $\mathrm{FPdim}(\mathcal{C}^*_{\mathcal{M}}) = \mathrm{FPdim}(\mathcal{C}) = |G|$. It follows that we must have $\overline{\mathrm{Fun}_x} \neq \emptyset$ for all $x \in K$, $|\widehat{H}| = |H|$ and $\mathcal{S} = K$. The second condition in the previous sentence means that H is abelian. The third condition means that H is normal in G. The first condition is equivalent to saying that $\frac{{}^x\mu}{\mu}$ is trivial in $H^2(G, C)$, for all $x \in K$. This is equivalent to saying that the restriction $\psi\left(\frac{{}^g\mu}{\mu} \right)$ is trivial in $H^2(H, k^\times)$, for all $g \in G$.

Conversely, suppose that H is abelian and normal in G and that $\psi\left(\frac{{}^g\mu}{\mu} \right)$ is trivial in $H^2(H, k^\times)$, for all $g \in G$. Let $\mathcal{C}'$ denote the full fusion subcategory generated by

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invertible objects of $\mathcal{C}^*_{\mathcal{M}}$. The isomorphism classes of simple objects in $\mathcal{C}'$ are given by elements of the set $\bigcup_{x \in K} \left(\overline{\mathrm{Fun}_x} \times \{x\} \right)$. The size of each set in the previous union is $|H|$. So $\mathrm{FPdim}(\mathcal{C}') = |G|$. It follows that $\mathcal{C}^*_{\mathcal{M}} = \mathcal{C}'$. In other words, every simple object in $\mathcal{C}^*_{\mathcal{M}}$ is invertible, that is, the category $\mathcal{C}^*_{\mathcal{M}}$ is pointed.

The last statement of the theorem follows from Lemma 2.1.3. ∎

Remark 3.1.6. The 2-cocycle $\psi(^g\mu/\mu)$ that appears in the previous Theorem is cohomologous to a 2-cocycle (defined in the Lemma below) that appears in several places in literature, in particular in [DPR1].

For each $x \in G$, define $\Upsilon_x : G \times G \to k^\times$ by

$$\Upsilon_x(g_1, g_2) := \frac{\omega(xg_1x^{-1}, xg_2x^{-1}, x)\omega(x, g_1, g_2)}{\omega(xg_1x^{-1}, x, g_2)}, \qquad \text{for all } g_1, g_2 \in G. \tag{3.2}$$

It is straightforward to verify that $\delta^2 \Upsilon_x = \frac{\omega}{\omega^x}$, for all $x \in G$, where $\omega^x(g_1, g_2, g_3) = \omega(xg_1x^{-1}, xg_2x^{-1}, xg_3x^{-1})$, for all $g_1, g_2, g_3 \in G$.

Lemma 3.1.7. Let H be a normal subgroup of G and let $\mu \in C^2(G, \mathrm{Coind}^G_H k^\times)$ be a 2-cochain that satisfies $\delta^2\mu = \omega$, where ω is regarded as an element of $Z^3(G, \mathrm{Coind}^G_H k^\times)$. The 2-cocycles $\psi\left(\frac{^x\mu}{\mu}\right)$ and $\left(\frac{\psi(\mu)^x}{\psi(\mu)} \times \Upsilon_x|_{H \times H}\right)$ define the same class in $H^2(H, k^\times)$.

Proof. We have

$$\left(\frac{\psi(\mu)^x}{\psi(\mu)} \times \Upsilon_x\right)(h_1, h_2)$$

$$= \frac{\psi(\mu)(xh_1x^{-1}, xh_2x^{-1})}{\psi(\mu)(h_1, h_2)} \times \frac{\omega(xh_1x^{-1}, xh_2x^{-1}, x)\omega(x, h_1, h_2)}{\omega(xh_1x^{-1}, x, h_2)}$$

$$= \frac{\psi(^x\mu)(h_1, h_2)}{\psi(\mu)(h_1, h_2)} \times \frac{\psi(\mu)(x, h_1h_2)}{\psi(\mu)(x, h_1)} \frac{1}{\psi(\mu)(x, h_2)} \times \frac{\psi(\mu)(xh_1x^{-1}, x)}{\psi(\mu)(xh_1h_2x^{-1}, x)},$$

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for all $x \in G, h_1, h_2 \in H$. In the second equality above we used (2.20) with

$$(g_1, g_2, g_3, x) = (x, h_1, h_2, 1_{H\backslash G}), (xh_1x^{-1}, x, h_2, 1_{H\backslash G}), (xh_1x^{-1}, xh_2x^{-1}, x, 1_{H\backslash G})$$

and canceled some factors. Observe that the second and third factors in the last expression in the equalities above define coboundaries and the Lemma is proved. ■

Example 3.1.8. If G is a finite cyclic group, then $H^2(H, k^\times) = \{1\}$ for any subgroup H of G. Hence the dual of Vec_G^ω with respect to any indecomposable module category for any 3-cocycle ω on G is pointed. Also, if G is finite abelian group, then the dual of Vec_G with respect to any indecomposable module category is pointed. The previous statement is not true for Vec_G^ω if ω is a non-trivial 3-cocycle on the abelian group G. Indeed, consider the dihedral group $D_8 = \{r, s \mid r^4 = s^2 = 1, rs = sr^{-1}\}$ and the subgroup $< r^2 >$ of it. It can be shown that $(\text{Vec}_{D_8})^*_{\mathcal{M}(<r^2>, 1)} \cong \text{Vec}_{\mathbb{Z}/2\mathbb{Z})^3}^\omega$, where ω is a certain non-trivial 3-cocycle on $(\mathbb{Z}/2\mathbb{Z})^3$. Now, we know that Vec_{D_8} is dual to the representation category $\text{Rep}(D_8)$. Hence, there must exist an indecomposable module category over $\text{Vec}_{\mathbb{Z}/2\mathbb{Z})^3}^\omega$ with respect to which the dual of $\text{Vec}_{\mathbb{Z}/2\mathbb{Z})^3}^\omega$ is equivalent to the non-pointed fusion category $\text{Rep}(D_8)$. We refer the reader to [CGR] and [GMN] for similar results.

3.2 The dual of a pointed category (when it is pointed)

In this Section, we follow the notation fixed at the beginning of Section 3.1. We will assume that H is abelian and normal in G and that $\frac{^x\mu}{\mu}$ is trivial in $H^2(G, C)$, for all $x \in K$, i.e., we will assume the the conditions of Theorem 3.1.5 hold.

3.2.1 Tensor product and composition of morphisms

It suffices to restrict ourselves to simple objects in $\mathcal{C}^*_\mathcal{M}$. Recall that simple objects in $\mathcal{C}^*_\mathcal{M}$ are given by pairs (γ, x), where $\gamma \in \mathrm{Fun}_x$ (the set defined in (3.1)) and $x \in K$. The element $x \in K$ determines a $\mathcal{C}$-module functor $F_x : \mathcal{M} \to \mathcal{M}$ given by $F_x(y) = xy$, for all $y \in K$. The $\mathcal{C}$-module functor structure on F_x is given by γ. Tensor product (=composition of module functors) in $\mathcal{C}^*_\mathcal{M}$: for any two simple objects (γ^1, x_1) and (γ^2, x_2), $(\gamma^1, x_1) \otimes (\gamma^2, x_2) = ({}^{x_2}\gamma^1 \gamma^2, x_1 x_2)$. It is straightforward to check that ${}^{x_2}\gamma^1 \gamma^2$ is an element of the set $\mathrm{Fun}_{x_1 x_2}$.

Now let us look at morphisms in $\mathcal{C}^*_\mathcal{M}$. It suffices to restrict ourselves to isomorphisms between simple objects. Recall that an isomorphism between two simple objects (γ^1, x) and (γ^2, x) (note that the second coordinates have to be equal for an isomorphism to exist) in $\mathcal{C}^*_\mathcal{M}$ is given by an element $\alpha \in C$ which satisfies:

$\gamma^1(g)(y) = \frac{\alpha(y \triangleleft g)}{\alpha(y)} \gamma^2(g)(y)$, for all $g \in G$ and $y \in K$.

Note 3.2.1. An isomorphism $\alpha : (\gamma^1, x) \to (\gamma^2, x)$ is completely determined by $\alpha(1)$. If α is an automorphism, then $\alpha(y) = \alpha(1)$ for all $y \in K$. Indeed, we have $\gamma^1(g)(y) = \frac{\alpha(y \triangleleft g)}{\alpha(y)} \gamma^2(g)(y)$, for all $g \in G$ and $y \in K$. So we have $\alpha(y \triangleleft g) = \frac{\gamma^1(g)(y)}{\gamma^2(g)(y)} \alpha(y)$. Now let $y = 1$ and $g = u(y)$. Then, $\alpha(y) = \frac{\gamma^1(u(y))(1)}{\gamma^2(u(y))(1)} \alpha(1)$. It is easy to show that the above equality is independent of the choice of the function $u : K \to G$.

Let $\alpha : (\gamma^1, x_1) \to (\gamma^2, x_1)$ and $\beta : (\gamma^3, x_2) \to (\gamma^4, x_2)$ be any two isomorphisms between simple objects in $\mathcal{C}^*_\mathcal{M}$. The tensor product of α and β:

$$(\alpha \otimes \beta)(x) = ({}^{x_2}\alpha \, \beta)(x) = \alpha(x_2 x)\beta(x),$$

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for all $x \in K$. If $\gamma^2 = \gamma^3$, then the composition of β and α is given by $(\beta \circ \alpha)(x) = \beta(x)\alpha(x)$, for all $x \in K$.

3.2.2 The Grothendieck ring

The set of isomorphism classes of simple objects in $\mathcal{C}_{\mathcal{M}}^*$ forms a group Λ:

$$\Lambda = \bigcup_{x \in K} \left(\overline{\mathrm{Fun}}_x \times \{x\} \right) \qquad (\overline{\gamma^1}, x_1) \star (\overline{\gamma^2}, x_2) = (\overline{^{x_2}\gamma^1\,\gamma^2}, x_1 x_2), \qquad (3.3)$$

where for any $\gamma \in \mathrm{Fun}_x$, by $\overline{\gamma}$ we mean the equivalence class of γ in $\overline{\mathrm{Fun}}_x$. The inverse of any $(\overline{\gamma}, x) \in \Lambda$ is $\left(\overline{^{x^{-1}}\gamma^{-1}}, x^{-1} \right)$. The Grothendieck ring $\mathcal{K}_0(\mathcal{C}_{\mathcal{M}}^*)$ is isomorphic to the group ring $\mathbb{Z}[\Lambda]$.

The rest of this Section is devoted to showing that Λ is isomorphic to a certain crossed product of the groups $\widehat{H}$ and K.

Since $\frac{^x\mu}{\mu}$ is trivial in $H^2(G, C)$, for each $x \in K$ we have a maps $\eta_x \in C^1(G, C)$, $x \in K$ such that:

$$\delta^1 \eta_x = \frac{^x\mu}{\mu}. \qquad (3.4)$$

Define a function

$$\tilde{\nu} : K \times K \to C^1(G, C), \qquad \tilde{\nu}(x_1, x_2) = \frac{^{x_2}\eta_{x_1}\,\eta_{x_2}}{\eta_{x_1 x_2}}. \qquad (3.5)$$

Lemma 3.2.2. The function $\tilde{\nu}$ defines an element in $\underline{H}^2(K, H^1(G, C))$.

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Proof. Let us first show that $\tilde{\nu}(x_1, x_2) \in Z^1(G, C)$, for all $x_1, x_2 \in K$. We have

$$\delta^1\eta_{x_1x_2} = \frac{x_1x_2\mu}{\mu} = \frac{x_2(x_1\mu)}{\mu} = \frac{x_2(\delta^1\eta_{x_1}\mu)}{\mu} = \frac{x_2(\delta^1\eta_{x_1})^{x_2}\mu}{\mu} = \delta^1(x_2\eta_{x_1})\,\delta^1\eta_{x_2} = \delta^1(x_2\eta_{x_1}\,\eta_{x_2}).$$

So $\tilde{\nu}(x_1, x_2) \in Z^1(G, C)$, for all $x_1, x_2 \in K$. Now let us show that $\underline{\delta}^2\tilde{\nu} \equiv 1$. We have

$$(\underline{\delta}^2\tilde{\nu})(x_1, x_2, x_3) = \tilde{\nu}(x_2, x_3)\tilde{\nu}(x_1x_2, x_3)^{-1}\tilde{\nu}(x_1, x_2x_3)(^{x_3}\tilde{\nu}(x_1, x_2))^{-1}$$

$$= \frac{^{x_3}\eta_{x_2}\,\eta_{x_3}}{\eta_{x_2x_3}} \times \frac{\eta_{x_1x_2x_3}}{^{x_3}\eta_{x_1x_2}\,\eta_{x_3}} \times \frac{^{x_2x_3}\eta_{x_1}\,\eta_{x_2x_3}}{\eta_{x_1x_2x_3}} \times \frac{^{x_3}\eta_{x_1x_2}}{^{x_3}(^{x_2}\eta_{x_1})\,^{x_3}\eta_{x_2}}$$

$$\equiv 1.$$

The cohomology class of $\tilde{\nu}$ does not depend on the choice of the family of maps $\{\eta_x \mid x \in K\}$. Indeed, let $\{\eta'_x \mid x \in K\}$ be another family of maps satisfying $(\delta^1\eta'_x) = \frac{x_\mu}{\mu}$, for all $x \in K$. We contend that $\tilde{\nu}(x_1, x_2) = \frac{x_2\eta_{x_1}\,\eta_{x_2}}{\eta_{x_1x_2}}$ and $\tilde{\nu}'(x_1, x_2) = \frac{x_2\eta'_{x_1}\,\eta'_{x_2}}{\eta'_{x_1x_2}}$ define the same class in $\underline{H}^2(K, Z^1(G, C))$. We have, $\delta^1\left(\frac{\eta_x}{\eta'_x}\right) = 1$, i.e. $\frac{\eta_x}{\eta'_x} \in Z^1(G, C)$, for all $x \in K$. Define $\beta : K \to Z^1(G, C)$ by $\beta(x) := \frac{\eta_x}{\eta'_x}$, for all $x \in K$. Then,

$$\tilde{\nu}(x_1, x_2) = \frac{x_2\eta_{x_1}\,\eta_{x_2}}{\eta_{x_1x_2}} = \frac{x_2\beta(x_1)\,^{x_2}\eta'_{x_1}\,\beta(x_2)\,\eta'_{x_2}}{\beta(x_1x_2)\eta'_{x_1x_2}} = (\delta^1\beta)(x_1, x_2)\,\tilde{\nu}'(x_1, x_2)$$ and the Lemma

is proved. ∎

Corollary 3.2.3. *The function $\nu = \psi_1 \circ \tilde{\nu}$ defines an element in $\underline{H}^2(K, \widehat{H})$ (where ψ_1 is defined in (2.5)) .*

Proof. This follows immediately from Lemmas 2.1.3 and 3.2.2. ∎

Remark 3.2.4. If $\omega \equiv 1$, then the element ν in the previous Corollary is the image of μ under the following composition.

$$H^2(H, k^\times)^K \longrightarrow H^2(G, C)^K \longrightarrow \underline{H}^2(K, H^1(G, C)) \longrightarrow \underline{H}^2(K, \widehat{H}). \qquad (3.6)$$

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The first map in the above composition comes from φ (2.7), the second from (3.5) and third is induced from the map ψ_1 (2.5). Maps similar to the one in (3.6) appears in [Da] and [EG].

Let us put a group structure on the set $\widehat{H} \times K$. For any two pairs $(\rho_1,\, x_1)$, $(\rho_2,\, x_2)$ define their product by:

$$(\rho_1,\, x_1)(\rho_2,\, x_2) = (\nu(x_1,\, x_2)\, \rho_1^{x_2}\rho_2,\, x_1 x_2). \tag{3.7}$$

Associativity follows from Corollary 3.2.3. We denote this group by $\widehat{H} \rtimes_\nu K$.

As mentioned in Lemma 3.1.4, the set $\overline{\mathrm{Fun}_x}$ and $\widehat{H}$ are in bijection for each $x \in K$. The following maps induce this bijection.

$$\zeta_x : \widehat{H} \to \mathrm{Fun}_x, \qquad \zeta_x(\rho) := \eta_x\, \varphi_1(\rho), \tag{3.8}$$

$$\theta_x : \mathrm{Fun}_x \to \widehat{H}, \qquad \theta_x(\gamma) := \psi_1(\gamma/\eta_x),$$

where the maps φ_1 and ψ_1 are defined in (2.4) and (2.5), respectively.

Theorem 3.2.5. *The Grothendieck ring $\mathcal{K}_0(\mathcal{C}_\mathcal{M}^*) = \mathbb{Z}[\Lambda]$ is isomorphic to the group ring $\mathbb{Z}[\widehat{H} \rtimes_\nu K]$.*

Proof. Suffices to show that the groups Λ and $\widehat{H} \rtimes_\nu K$ are isomorphic. Define a map $T : \widehat{H} \rtimes_\nu K \to \Lambda$ by $T((\rho,\, x)) := (\overline{\zeta_x(\rho)},\, x)$. For all $(\rho_1,\, x_1)$, $(\rho_2,\, x_2) \in \widehat{H} \rtimes_\nu K$, we have

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$$T((\rho_1, x_1)(\rho_2, x_2)) = T((\nu(x_1, x_2)\, \rho_1^{x_2}\rho_2,\ x_1 x_2))$$

$$= (\overline{\zeta_{x_1 x_2}(\nu(x_1, x_2)\, \rho_1^{x_2}\rho_2)},\ x_1 x_2)$$

and

$$T((\rho_1, x_1)) \star T((\rho_2, x_2)) = (\overline{\zeta_{x_1}(\rho_1)},\ x_1) \star (\overline{\zeta_{x_2}(\rho_2)},\ x_2)$$

$$= (\overline{{}^{x_2}(\zeta_{x_1}(\rho_1))\, \zeta_{x_2}(\rho_2)},\ x_1 x_2)$$

We contend that $\theta_{x_1 x_2}({}^{x_2}\zeta_{x_1}(\rho_1)\, \zeta_{x_2}(\rho_2)) = \nu(x_1, x_2)\, \rho_1^{x_2}\rho_2$. Indeed, for all $h \in H$, we have

$$\begin{aligned}
\theta_{x_1 x_2}({}^{x_2}(\zeta_{x_1}(\rho_1))\, \zeta_{x_2}(\rho_2))(h) &= \frac{({}^{x_2}(\zeta_{x_1}(\rho_1))\, \zeta_{x_2}(\rho_2))(h)(1)}{\eta_{x_1 x_2}(h)(1)} \\
&= \frac{(\zeta_{x_1}(\rho_1)(h))(x_2)\ (\zeta_{x_2}(\rho_2)(h))(1)}{\eta_{x_1 x_2}(h)(1)} \\
&= \frac{(\varphi_1(\rho_1)(h))(x_2)\ \eta_{x_1}(h)(x_2)\ (\varphi_1(\rho_2)(h))(1)\ \eta_{x_2}(h)(1)}{\eta_{x_1 x_2}(h)(1)} \\
&= (\nu(x_1, x_2)\, \rho_1^{x_2}\, \rho_2)(h)
\end{aligned}$$

Hence, $\overline{\zeta_{x_1 x_2}(\nu(x_1, x_2)\, \rho_1^{x_2}\rho_2)} = \overline{{}^{x_2}(\zeta_{x_1}(\rho_1))\, \zeta_{x_2}(\rho_2)}$. This shows that T is a group homomorphism. It is evident that T is a bijection and the Theorem is proved. ∎

Example 3.2.6. Suppose the order of H is relatively prime to the order of the group K and suppose $\psi({}^{x}\mu/\mu)$ is trivial in $H^2(H, k^{\times})$, for all $x \in K$. Then the Grothendieck ring of $\mathcal{C}_{\mathcal{M}}^{*}$ is isomorphic to $\mathbb{Z}[\widehat{H} \rtimes K]$. Indeed, since $|H|$ and $|K|$ are relatively prime we have $H^2(K, \widehat{H}) = \{1\}$ which implies that ν is trivial in $H^2(K, \widehat{H})$.

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3.2.3 A Skeleton

A *skeleton* of a category $\mathcal{D}$ is any full subcategory $\overline{\mathcal{D}}$ such that each object of $\mathcal{D}$ is isomorphic (in $\mathcal{D}$) to exactly one object of $\overline{\mathcal{D}}$. Every category is equivalent to any of its skeletons. Let us recall how one constructs a skeleton $\overline{\mathcal{D}}$ of any tensor category $\mathcal{D}$ with associativity constraint a and tensor product $\otimes$. The construction is as follows: choose one object from each isomorphism class of objects in $\mathcal{D}$. Let $\mathrm{Obj}(\overline{\mathcal{D}})$ be the set of all objects chosen above. For any $X \in \mathrm{Obj}(\mathcal{D})$, by $\overline{X}$ we mean the object in $\overline{\mathcal{D}}$ that represents the object X.

Define $\mathrm{Hom}_{\overline{\mathcal{D}}}(X, Y) := \mathrm{Hom}_{\mathcal{D}}(X, Y)$. Define tensor product $\odot$ in $\overline{\mathcal{D}}$: $X \odot Y := \overline{X \otimes Y}$, for all $X, Y \in \mathrm{Obj}(\overline{\mathcal{D}})$. Fix isomorphisms $\beta(X, Y) : X \odot Y \xrightarrow{\sim} X \otimes Y$ in $\mathcal{D}$, for all $X, Y \in \mathrm{Obj}(\overline{\mathcal{D}})$. For any $f \in \mathrm{Hom}_{\overline{\mathcal{D}}}(X, Y)$ and $g \in \mathrm{Hom}_{\overline{\mathcal{D}}}(X', Y')$ define its tensor product: $f \odot g = \beta(X', Y')^{-1} \circ (f \otimes g) \circ \beta(X, Y)$.

We now define associativity constraint $\overline{a}$ in $\overline{\mathcal{D}}$. For any $X, Y, Z \in \mathrm{Obj}(\overline{\mathcal{D}})$ define $\overline{a}_{X,Y,Z}$ to be the following composition:

$$(X \odot Y) \odot Z \xrightarrow{\beta(X \odot Y, Z)} (X \odot Y) \otimes Z \xrightarrow{\beta(X,Y) \otimes id_Z} (X \otimes Y) \otimes Z \xrightarrow{a_{X,Y,Z}} X \otimes (Y \otimes Z)$$

$$\xrightarrow{(id_X \otimes \beta(Y, Z))^{-1}} X \otimes (Y \odot Z) \xrightarrow{\beta(X, Y \odot Z)^{-1}} X \odot (Y \odot Z).$$

Left and right unit constraints are defined in the obvious way. It can be shown that the necessary axioms (pentagon, triangle) are satisfied. Hence $\overline{\mathcal{D}}$ is a tensor category. The tensor categories $\mathcal{D}$ and $\overline{\mathcal{D}}$ are equivalent as tensor categories. Indeed, define a functor $F : \overline{\mathcal{D}} \to \mathcal{D}$ by $F(X) = X$ and $F(f) = f$ for any object X and

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any morphism f in $\overline{\mathcal{D}}$. It is evident that this functor is essentially surjective and fully faithful. Thus F establishes an equivalence of the categories $\overline{\mathcal{D}}$ and $\mathcal{D}$. Let us put a tensor functor structure on F. We need natural isomorphisms $J(X, Y) :$ $F(X \odot Y) = X \odot Y \xrightarrow{\sim} X \otimes Y = F(X) \otimes F(Y)$. Let $J(X, Y) = \beta(X, Y)$. Then it is straightforward to show that J is a natural isomorphism and that all the necessary axioms are satisfied.

Remark 3.2.7. If $\mathcal{D}$ is a pointed fusion category, then the simple objects of a skeleton $\overline{\mathcal{D}}$ of $\mathcal{D}$ form a group and the associativity constraint in $\overline{\mathcal{D}}$ gives rise to a 3-cocycle. The cohomology class of this 3-cocycle does not depend on the choices made in the construction of $\overline{\mathcal{D}}$.

The function κ defines an element in $Z^2(K, H)$:

$$\kappa(x_1, x_2) := \kappa_{x_1, u(x_2)}. \tag{3.9}$$

Note that the cohomology class of κ is independent on the choice of the function u. Also note that the cohomology class that κ defines in $H^2(K, H)$ is equal to the cohomology class associated to the the exact sequence $1 \to H \to G \to K \to 1$.

Define a 3-cocycle ϖ on the group $\widehat{H} \rtimes_\nu K$ (see (3.7) and (3.5)):

$$\varpi((\rho_1, x_1), (\rho_2, x_2), (\rho_3, x_3)) := (\tilde{\nu}(x_1, x_2)(u(x_3)))(1)\, \rho_1(\kappa(x_2, x_3)), \tag{3.10}$$

for all $(\rho_1, x_1), (\rho_2, x_2), (\rho_3, x_3) \in \widehat{H} \rtimes_\nu K$.

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Remark 3.2.8. (i) It is routine to check that ϖ does indeed define a 3-cocycle and that its cohomology class does not depend on the choice of the function $u : K \to G$.

(ii) A special case, with $\tilde{\nu} \equiv 1$, of the formula in (3.10) appeared in [GMN].

Theorem 3.2.9. *The fusion categories $\mathcal{C}_{\mathcal{M}}^*$ and $Vec_{\widehat{H} \rtimes_\nu K}^{\varpi}$ are equivalent.*

Proof. Let us construct a skeleton $\overline{\mathcal{C}_{\mathcal{M}}^*}$ of the category $\mathcal{C}_{\mathcal{M}}^*$.

Let $\overline{\Lambda} = \bigcup_{x \in K} \{ (\zeta_x(\rho), x) \mid \rho \in \widehat{H} \}$ denote the set of all simple objects of $\overline{\mathcal{C}_{\mathcal{M}}^*}$ (see (3.8) for definition of ζ_x). Tensor product $\odot$ in $\overline{\mathcal{C}_{\mathcal{M}}^*}$: $(\zeta_{x_1}(\rho_1), x_1) \odot (\zeta_{x_2}(\rho_2), x_2) =$

$$\overline{(\zeta_{x_1}(\rho_1),\, x_1) \otimes (\zeta_{x_2}(\rho_2),\, x_2)} = \overline{(^{x_2}(\zeta_{x_1}(\rho_1))\, \zeta_{x_2}(\rho_2),\, x_1 x_2)} = (\zeta_{x_1 x_2}(\nu(x_1,\, x_2)\rho_1^{x_2}\rho_2),\, x_1 x_2).$$

Note that $\overline{\Lambda}$ forms a group (multiplication coming from $\odot$) that is isomorphic to $\widehat{H} \rtimes_\nu K$.

Fix isomorphisms in $\mathcal{C}_{\mathcal{M}}^*$: $C \ni f((\zeta_{x_1}(\rho_1), x_1), (\zeta_{x_2}(\rho_2), x_2)) : (\zeta_{x_1}(\rho_1), x_1) \odot$

$(\zeta_{x_2}(\rho_2), x_2) = (\zeta_{x_1 x_2}(\nu(x_1, x_2)\rho_1^{x_2}\rho_2), x_1 x_2) \xrightarrow{\sim} (^{x_2}(\zeta_{x_1}(\rho_1))\, \zeta_{x_2}(\rho_2), x_1 x_2) =$

$(\zeta_{x_1}(\rho_1), x_1) \otimes (\zeta_{x_2}(\rho_2), x_2)$, for all $(\zeta_{x_1}(\rho_1), x_1)$, $(\zeta_{x_2}(\rho_2), x_2) \in \overline{\Lambda}$. The following equality holds:

$$^{x_2}(\zeta_{x_1}(\rho_1))\, \zeta_{x_2}(\rho_2))(g)(y) = \frac{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2))(y \triangleleft g)}{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2))(y)}$$
$$\times\, \zeta_{x_1 x_2}(\nu(x_1,\, x_2)\rho_1^{x_2}\rho_2)(g)(y),$$

for all $g \in G$, $y \in K$. After using the definition of ζ_{x_1}, ζ_{x_2}, $\zeta_{x_1 x_2}$ and $\tilde{\nu}$, canceling and rearranging, the above equality becomes:

$$\frac{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2))(y \triangleleft g)}{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2))(y)} = \frac{\tilde{\nu}(x_1,\, x_2)(g)(y)\, \rho_1(\kappa_{x_2 y, g})}{\nu(x_1,\, x_2)(\kappa_{y, g})\, \rho_1^{x_2}(\kappa_{y, g})}.$$

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Putting $y = 1$ and $g = u(y)$ in the above relation and canceling, we obtain:

$$f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2))(y) = \frac{\tilde{\nu}(x_1,\, x_2)(u(y))(1)\; \rho_1(\kappa_{x_2,\, u(y)})}{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2))(1)}. \tag{3.11}$$

Remains to calculate the associativity constraint in $\overline{\mathcal{C}_{\mathcal{M}}^*}$ which we denote by ϖ'.

For any $(\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3) \in \overline{\Lambda}$, ϖ' is defined by:

$$\varpi'((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))$$

$$= \frac{(f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2))) \otimes Id_{(\zeta_{x_3}(\rho_3),\, x_3)}}{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2) \odot (\zeta_{x_3}(\rho_3),\, x_3))}$$
$$\times \frac{f((\zeta_{x_1}(\rho_1),\, x_1) \odot (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))}{(Id_{(\zeta_{x_1}(\rho_1),\, x_1)} \otimes f((\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3)))}$$
$$= \frac{{}^{x_3}(f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2)))}{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2) \odot (\zeta_{x_3}(\rho_3),\, x_3))}$$
$$\times \frac{f((\zeta_{x_1}(\rho_1),\, x_1) \odot (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))}{f((\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))}.$$

Note that $\varpi'((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))$ is an automorphism of a simple object in $\mathcal{C}_{\mathcal{M}}^*$. By Note 3.2.1, $\varpi'((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))(y)$ is constant for all $y \in K$. Thus, it suffices to calculate $\varpi'((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))(1)$.

We have,

$$\varpi'((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))(1)$$

$$= \frac{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2))(x_3)}{f((\zeta_{x_1}(\rho_1),\, x_1),\, (\zeta_{x_2}(\rho_2),\, x_2) \odot (\zeta_{x_3}(\rho_3),\, x_3))(1)}$$
$$\times \frac{f((\zeta_{x_1}(\rho_1),\, x_1) \odot (\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))(1)}{f((\zeta_{x_2}(\rho_2),\, x_2),\, (\zeta_{x_3}(\rho_3),\, x_3))(1)}.$$

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$$= \frac{f(((\zeta_{x_1}(\rho_1),\ x_1),\ (\zeta_{x_2}(\rho_2),\ x_2))(1)}{f(((\zeta_{x_1}(\rho_1),\ x_1),\ (\zeta_{x_2}(\rho_2),\ x_2) \odot (\zeta_{x_3}(\rho_3),\ x_3))(1)}$$

$$\times \frac{f(((\zeta_{x_1}(\rho_1),\ x_1) \odot (\zeta_{x_2}(\rho_2),\ x_2),\ (\zeta_{x_3}(\rho_3),\ x_3))(1)}{f(((\zeta_{x_2}(\rho_2),\ x_2),\ (\zeta_{x_3}(\rho_3))),\ x_3)(1)}$$

$$\times \tilde{\nu}(x_1,\ x_2)(u(x_3))(1)\ \rho_1(\kappa_{x_2,\,u(x_3)}).$$

We used (3.11) to obtain the last equality.

Since the cohomology class of ϖ' does not depend on the choice of the isomorphisms $f(\cdot,\ \cdot)$, we can assume that $f(\cdot,\ \cdot)(1) = 1$. Also, regard ϖ' as a 3-cocycle on $\widehat{H} \rtimes_\nu K$. Then we get:

$$\varpi'((\rho_1,\ x_1),\ (\rho_2,\ x_2),\ (\rho_3,\ x_3)) = \tilde{\nu}(x_1,\ x_2)(u(x_3))(1)\ \rho_1(\kappa(x_2,\ x_3)),$$

for all $(\rho_1,\ x_1),\ (\rho_2,\ x_2),\ (\rho_3,\ x_3) \in \widehat{H} \rtimes_\nu K$. That is, $\varpi' = \varpi$ and the Theorem is proved. ∎

Example 3.2.10. Let $G = \mathbb{Z}/4\mathbb{Z} = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$, $\omega = 1$, $H = \{\bar{0},\bar{2}\}$, and $\mu \equiv 1$. Since $\mu \equiv 1$ we can assume that $\tilde{\nu} \equiv 1$ (see (3.5)) and $\nu \equiv 1$ (see Corollary 3.2.3). By Theorem 3.2.5 it follows that $\mathcal{K}_0(\mathcal{C}^*_\mathcal{M}) \cong \mathbb{Z}[\widehat{\mathbb{Z}/2\mathbb{Z}} \times \mathbb{Z}/2\mathbb{Z}]$. Let $\widehat{\mathbb{Z}/2\mathbb{Z}} = \{\rho_0,\ \rho_1\}$, where ρ_1 represents the non-trivial homomorphism. We have, $K = \{H+\bar{0},\ H+\bar{1}\}$. We claim that the associativity constraint ϖ in $\mathcal{C}^*_\mathcal{M}$ is non-trivial. It suffices to show that the restriction of ϖ to some non-trivial subgroup of $\widehat{\mathbb{Z}/2\mathbb{Z}} \times \mathbb{Z}/2\mathbb{Z}$ is non-trivial. Consider the restriction of ϖ to the subgroup $K = \{(\rho_0,\ H + \bar{0}),(\rho_1,\ H + \bar{1})\}$. It suffices to show that there exists a triple of elements in this subgroup such that ϖ evaluated at this triple is not equal to 1. Define the function $u : K \rightarrow G$ by $u(H + \bar{0}) = \bar{0}$ and $u(H + \bar{1}) = \bar{1}$. Since $\tilde{\nu} \equiv 1$, the first factor in the definition of ϖ vanishes. We

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have, $\varpi((\rho_1, H + \bar{1}), (\rho_1, H + \bar{1}), (\rho_1, H + \bar{1})) = \rho_1(\kappa(H + \bar{1}, H + \bar{1})) = \rho_1(\bar{2}) = -1$.

Thus, the 3-cocycle ϖ is non-trivial. In particular, the fusion categories $\mathrm{Vec}^1_{\mathbb{Z}/4\mathbb{Z}}$ and $\mathrm{Vec}^\varpi_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ are dual to each other.

3.3   Categorical Morita equivalence

Using the notion of weak Morita equivalence for fusion categories we put an equivalence relation on the set of all pairs (G, ω), where G is a finite group and $\omega \in H^3(G, k^\times)$:

Definition 3.3.1. We say that two pairs $(G, \bar{\omega})$ and $(G', \bar{\omega}')$ are *categorically Morita equivalent* and write $(G, \bar{\omega}) \approx (G', \bar{\omega}')$ if the fusion categories Vec^ω_G and $\mathrm{Vec}^{\omega'}_{G'}$, are weakly Morita equivalent.

Remark 3.3.2. Note that finding categorically Morita equivalence classes of the set of all pairs (G, ω), where G is a finite group and $\omega \in H^3(G, k^\times)$ amounts to finding weakly Morita equivalence classes of the set of all group-theoretical categories.

We also define an equivalence relation on the set of all groups:

Definition 3.3.3. We say that two groups G and G' are *categorically Morita equivalent* and write $G \approx G'$ if the pairs $(G, 1)$ and $(G', 1)$ are categorically Morita equivalent.

Remark 3.3.4. Two finite groups G and G' are called *isocategorical* if their representation categories $\mathrm{Rep}(G)$ and $\mathrm{Rep}(G')$ are tensor equivalent [EG]. If two groups G and G' are isocategorical, then they are categorically Morita equivalent (this follows

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from the fact that for any group G the categories $\text{Rep}(G)$ and $(\text{Vec}_G^1)^*_{\mathcal{M}(G,1)}$ are tensor equivalent). We show in Section 3.4 that the converse is not true, that is, there do exist groups that are categorically Morita equivalent but not isocategorical.

Remark 3.3.5. It was shown in [O2] that if two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent, then their centers are equivalent as braided tensor categories. It follows that if two groups are categorically Morita equivalent, then the centers of their representation categories are equivalent as braided tensor categories.

Definition 3.3.6. We say that a group G is *categorically Morita rigid* if any group that is categorically Morita equivalent to G is actually isomorphic to G.

Remark 3.3.7. By remark 3.3.5 it follows that abelian groups are categorically Morita rigid. In particular, an abelian group can not be categorically Morita equivalent to a non-abelian group.

The next theorem gives a group-theoretical and cohomological interpretation of categorical Morita equivalence.

Theorem 3.3.8. *Two pairs $(G, \overline{\omega})$ and $(G', \overline{\omega}')$ are categorically Morita equivalent if and only if the following conditions hold:*

1. *G contains a normal abelian subgroup H such that $\omega|_{H \times H \times H}$ is trivial in $H^3(H, k^\times)$,*

2. *there is a 2-cochain $\mu \in C^2(G, \text{Coind}_H^G k^\times)$ such that $\delta^2 \mu = \omega$ and $\psi(^x\mu/\mu)$ is trivial in $H^2(H, k^\times)$, for all $x \in H \backslash G$ and there is an isomorphism*

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$a: G' \xrightarrow{\sim} \widehat{H} \rtimes_\nu (H \backslash G)$ (where ω is regarded as an element of $Z^3(G, \text{Coind}_H^G k^\times)$,

ψ and $\widehat{H} \rtimes_\nu (H \backslash G)$ are defined in (2.9) and (3.7), respectively) and

3. the 3-cocycle $\frac{\varpi^a}{\varpi'}$ is trivial in $H^3(G', k^\times)$ (where ϖ is the 3-cocycle associated to the dual $(Vec_G^\omega)^*_{\mathcal{M}(H,\mu)}$, defined in (3.10)).

Proof. Suppose the pairs (G, ϖ) and (G', ϖ') are categorically Morita equivalent. Then there exists an indecomposable right module category $\mathcal{M}$ over Vec_G^ω such that the categories $(Vec_G^\omega)^*_{\mathcal{M}}$ and $Vec_{G'}^{\omega'}$ are tensor equivalent. So there exists a subgroup H of G such that $\omega|_{H \times H \times H}$ represents the trivial class in $H^3(H, K^\times)$ and 2-cochain $\mu \in C^2(G, \text{Coind}_H^G k^\times)$ (satisfying $\delta^2 \mu = \omega$) which together produce the module category $\mathcal{M}$. Note that $(Vec_G^\omega)^*_{\mathcal{M}}$ must be pointed. By Theorem 3.1.5, it follows that H is abelian and normal in G and that $\psi(^x\mu/\mu)$ is trivial in $H^2(H, k^\times)$, for all $x \in H \backslash G$. Theorem 3.2.9 says that $(Vec_G^\omega)^*_{\mathcal{M}} \cong Vec_{\widehat{H} \rtimes_\nu (H \backslash G)}^\varpi$. It now follows that there must exist an isomorphism $a : G' \to \widehat{H} \rtimes_\nu (H \backslash G)$ such that ϖ^a is cohomologous to ω'. The converse is evident and the Theorem is proved. ∎

Corollary 3.3.9. *Two groups G and G' are categorically Morita equivalent if and only if the following conditions hold:*

1. *G contains a normal abelian subgroup H,*

2. *there exists a G-invariant $\mu \in H^2(H, k^\times)$ such that the groups G' and $\widehat{H} \rtimes_\nu (H \backslash G)$ are isomorphic (where $\widehat{H} \rtimes_\nu (H \backslash G)$ is defined in (3.7)) and*

3. *the 3-cocycle ϖ associated to the dual $(Vec_G)^*_{\mathcal{M}(H,\mu)}$, defined in (3.10), is trivial.*

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3.4 Examples of categorically Morita equivalent groups with non-isomorphic Grothendieck rings

Let p and q be odd primes such that $p - 1$ is divisible by q. Then there exists a unique upto isomorphism non-trivial semidirect product of the groups $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$. Let a and b be generators of the groups $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$, respectively. Let us fix an action of $\mathbb{Z}/q\mathbb{Z}$ on $\mathbb{Z}/p\mathbb{Z}$: fix a $t \in \mathbb{Z}$ ($t \not\equiv 1 \mod p$) such that $t^q - 1$ is divisible by p. Such a t of course exists because $p - 1$ is divisible by q. Then the action of $\mathbb{Z}/q\mathbb{Z}$ on $\mathbb{Z}/p\mathbb{Z}$ is defined by: $a \triangleleft b := a^t$. Let ρ be a generator of the groups $\widehat{\mathbb{Z}/p\mathbb{Z}}$. Then the induced action of $\mathbb{Z}/q\mathbb{Z}$ on $\widehat{\mathbb{Z}/p\mathbb{Z}}$ is given by: $(\rho \triangleleft b)(a) := \rho(a \triangleleft b^{-1})$. But $b^{-1} = b^{q-1}$. So $\rho \triangleleft b = \rho^{t^{q-1}}$.

The subgroup $\mathbb{Z}/p\mathbb{Z}$ (identified with $\mathbb{Z}/p\mathbb{Z} \times \{1\}$) of $\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ can be considered as a right $(\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z})$-module where the action is via conjugation. The dual group $\widehat{\mathbb{Z}/p\mathbb{Z}}$ is also a right $(\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z})$-module with the action being induced from the action of $\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ on $\mathbb{Z}/p\mathbb{Z}$. Let $G := \mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z})$ and $G' := \widehat{\mathbb{Z}/p\mathbb{Z}} \rtimes (\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z})$.

Lemma 3.4.1. The groups G and G' have different number of normal subgroups of order p.

Proof. Note that both groups have the same number of subgroups of order p. We claim that all subgroups of order p in G are normal whereas there exists a non-normal subgroup of order p in G'. The generator of any subgroup of G of order p is of the form $(a^l, (a^m, 1))$, where $l, m \in \{1, \ldots, p\}$ with l and m not simultaneously equal to p.

The elements $(a, (1, 1))$, $(1, (a, 1))$, and $(1, (1, b))$ generate the group G. Note that the element $(a^l, (a^m, 1))$ is stable under conjugation by the first two generators of G. While conjugation by the third generator gives: $(1, (1, b))^{-1}(a^l, (a^m, 1))(1, (1, b)) = (1, (1, b^{-1}))(a^{lt}, (a^{mt}, b)) = (a^{lt}, (a^{mt}, 1)) = (a^l, (a^m, 1))^t$. This shows that all subgroups of order p in G are normal. Consider the subgroup of G' of order p generated by the element $(\rho, (a, 1))$. We have

$(1, (1, b))^{-1}(\rho, (a, 1))(1, (1, b)) = (1, (1, b^{-1}))(\rho^{t^{q-1}}, (a^t, b)) = (\rho^{t^{q-1}}, (a^t, 1))$. Note that the element $(\rho^{t^{q-1}}, (a^t, 1))$ is not a power of $(\rho, (a, 1))$ because $t^{q-1} \not\equiv t \mod p$. This shows that the subgroup of G' of order p generated by the element $(\rho, (a, 1))$ is not normal and the Lemma is proved. ∎

Corollary 3.4.2. *The groups G and G' are categorically Morita equivalent but have non-isomorphic Grothendieck rings.*

Proof. To see that these two groups G and G' satisfy the conditions in Corollary 3.3.9, take H to be the subgroup $\mathbb{Z}/p\mathbb{Z}$ of G and take $\mu \equiv 1$. Observe that the groups $\widehat{H} \rtimes (H \backslash G)$ and G' are isomorphic. Since the exact sequence $1 \to H \to G \to H \backslash G \to 1$ splits and $\mu \equiv 1$, it follows that the 3-cocycle associated to the dual $(\text{Vec}_G)^*_{\mathcal{M}(H, \mu)}$, defined in (3.10), is trivial. So the conditions in Corollary 3.3.9 hold and it follows that the groups G and G' are categorically Morita equivalent. To see that these groups have non-isomorphic Grothendieck rings note that the previous Lemma implies that these groups have different number of quotient groups of order pq. By [Nik, Proposition

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3.11] it follows that the Grothendieck rings $K_0(\text{Rep}(G))$ and $K_0(\text{Rep}(G'))$ are not isomorphic. ∎

Corollary 3.4.3. *The groups G and G' are not isocategorical.*

Proof. This follows immediately from the above Corollary. ∎

Remark 3.4.4. (i) By Remark 3.3.5 the representation categories $\text{Rep}(D(G))$ and $\text{Rep}(D(G'))$ of the Drinfeld doubles of the groups G and G' are equivalent as braided tensor categories and hence these groups define the same modular data.

(ii) Equivalence of certain twisted doubles of groups was investigated in [GMN].

(iii) The above examples of categorically Morita equivalent groups come from a more general construction: start with any finite group G and a finite right G-module H. Consider the semidirect product $H \rtimes G$. We can regard $\widehat{H}$ as a right G-module with the action being induced from the action of G on H. Then the groups $H \rtimes G$ and $\widehat{H} \rtimes G$ are categorically Morita equivalent. Note however that these two groups are not always non-isomorphic.

(iv) By Ito's theorem [Gr, Theorem 6.3.9] it follows that the possible dimensions of irreducible representations of the groups G and G' are 1 and q. It can be shown that the order of the commutator subgroup is p^2 for both groups. Therefore, the order of the abelianization (equal to the number of 1-dimensional representations) of both groups is q. So the group algebras $k[G]$ and $k[G']$ are both isomorphic to

$$\underbrace{k \oplus k \oplus \cdots \oplus k}_{q\ copies} \oplus \underbrace{M_q(k) \oplus M_q(k) \oplus \cdots \oplus M_q(k)}_{(p^2-1)/q\ copies}.$$

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(v) It follows from Corollary 3.4.2 that the groups G and G' have different character tables. This provides a counter-example to the hunch, mentioned in [CGR], that groups defining the same modular data will have the same character table.

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CHAPTER 4

NILPOTENCY AND GRADING OF
GROUP-THEORETICAL CATEGORIES

The main result of this Chapter is Proposition 4.1.9 in which we give sufficient conditions for a group-theoretical category to be nilpotent.

We fix the following notation for this Chapter. Let $\mathcal{C} := \mathcal{V}_G^\omega$, where G is a finite group and $\omega \in Z^3(G, k^\times)$ and let $\mathcal{M} := \mathcal{M}(H, \mu)$ denote the right module category over $\mathcal{C}$ constructed from the pair (H, μ) (see Examples 2.5.12 and 2.6.5), where H is a subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $\mu \in C^2(G, \mathrm{Coind}_H^G k^\times)$ is a 2-cochain satisfying $\delta^2 \mu = \omega$. In the previous equation we regarded ω as an element of $Z^3(G, \mathrm{Coind}_H^G k^\times)$ by treating $\omega(g_1, g_2, g_3)$ as a constant function on $H \backslash G$, for all $g_1, g_2, g_3 \in G$. Let $K := H \backslash G$. Then $\mathrm{Irr}(\mathcal{M}) = K$ and $\mathrm{Obj}(\mathcal{M}) = \{\oplus_{x \in K} n_x x \mid n_x \text{ is a non-negative integer}\}$, where $n_x x = \oplus_{i=1}^{n_x} x$. $\mathrm{Hom}(\oplus_{x \in K} n_x x, \oplus_{y \in K} m_y y) := \oplus_{x \in K} \oplus_{y \in K} \mathrm{Hom}(n_x x, m_y y)$ and

$$\mathrm{Hom}(n_x x, m_y y) := \begin{cases} \{0\}, & \text{if } x \neq y \\ M_{n_x, n_y}(k), & \text{if } x = y \end{cases}$$

where M_{n_x, n_y} is the space of $n_x \times n_y$ matrices with entries from k. Recall that the action of $\mathcal{C}$ on $\mathcal{M}$ is given by the right action (denoted by $\lhd$) of G on K and the module category structure is given by μ.

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Let $u : K \to G$ be a function satisfying $p \circ u = id_K$ and $u(p(1_G)) = 1_G$, where $p : G \to K$ is the usual surjection. We denote by 1_K the object $p(1_G)$ in the category $\mathcal{M}$. Let $\kappa : K \times G \to H$ be the function satisfying (2.2). We will denote by $\boxtimes$ the Kronecker tensor product of matrices. I_n will denote the $n \times n$ identity matrix.

If H is normal in G, define R_x, for each $x \in K$, to be the set of all projective matrix representations of H with 2-cocycle $\psi \left(\frac{x_\mu}{\mu} \right)$ (where ψ is the map defined in (2.9)). Note that if H is normal in G, by Lemma 3.1.1, $\psi \left(\frac{x_\mu}{\mu} \right)$ is indeed a 2-cocycle on H.

$$R_x := \left\{ \rho \,\middle|\, \rho \text{ is a projective matrix representation of } H \text{ with 2-cocycle } \psi \left(\frac{x_\mu}{\mu} \right) \right\}$$
(4.1)

Also, let

$$\overline{R}_x = \text{the set of isomorphism classes of } R_x$$

$$\text{Irr}(R_x) = \text{the set of irreducible elements of } R_x$$

$$\overline{\text{Irr}}(R_x) = \text{the set of isomorphism classes of } \text{Irr}(R_x)$$

$$\overline{\text{Obj}}(\mathcal{C}_\mathcal{M}^*) = \text{the set of isomorphism classes of } \text{Obj}(\mathcal{C}_\mathcal{M}^*)$$
(4.2)

$$\text{Irr}(\mathcal{C}_\mathcal{M}^*) = \text{the set of irreducible objects of } \mathcal{C}_\mathcal{M}^*$$

$$\overline{\text{Irr}}(\mathcal{C}_\mathcal{M}^*) = \text{the set of isomorphism classes of } \text{Irr}(\mathcal{C}_\mathcal{M}^*)$$

Let $\mathcal{D}$ be a semisimple category having finitely many isomorphism classes of simple objects. For any object X of $\mathcal{D}$ let us denote by $\#X$ the number of simple objects (counting multiplicities) in the decomposition of X.

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Lemma 4.1.1. The FP-dimension of any object $(F, \gamma) \in \mathrm{Obj}(\mathcal{C}_{\mathcal{M}}^*)$ is given by $\#(F(s))$, where s is any simple object in $\mathcal{M}$.

Proof. First, observe that $\#(F(s)) = \#(F(1_K))$ for any simple object s in $\mathcal{M}$. Indeed, $\#(F(s)) = \#(F(1_K \lhd u(s))) = \#(F(1_K) \lhd u(s)) = \#(F(1_K))$. The second equality above is because there is an isomorphism $\gamma_{1_K, u(s)}$ between $F(1_K \lhd u(s))$ and $F(1_K) \lhd u(s)$. It is clear that $\#$ extends to a ring homomorphism from the Grothendieck ring $\mathcal{K}_0(\mathcal{C}_{\mathcal{M}}^*)$ of $\mathcal{C}_{\mathcal{M}}^*$ to $\mathbb{Z}$. The statement of the Lemma now follows from [ENO, Lemma 8.3]. $\blacksquare$

Suppose H is normal in G. Then the 2-cocycles $\frac{^x\mu}{\mu}$ and $\varphi\left(\psi\left(\frac{^x\mu}{\mu}\right)\right)$ (where φ and ψ are defined in (2.7) and (2.9), respectively) are cohomologous, for all $x \in K$. So for each $x \in K$, there exists $\eta_x \in C^1(G, \mathrm{Coind}_H^G k^\times)$ which satisfies:

$$\frac{^x\mu}{\mu} = (\delta^1 \eta_x) \, \varphi\left(\psi\left(\frac{^x\mu}{\mu}\right)\right). \tag{4.3}$$

We assume that $\eta_{1_K} \equiv 1$.

Lemma 4.1.2. The following map is well defined.

$$\zeta_x : R_x \to \mathrm{Obj}(\mathcal{C}_{\mathcal{M}}^*) : \rho \mapsto \zeta_x(\rho) = (F, \gamma), \tag{4.4}$$

where (F, γ) is defined by: $F(s) := \dim(\rho) \, xs = \underbrace{xs \oplus xs \oplus \cdots \oplus xs}_{dim(\rho) \; summands}$, for all $s \in K = \mathrm{Irr}(\mathcal{M})$ and $\gamma_{s,g} := \eta_x(g)(s) \, \rho(\kappa_{s,g})$, for all $s \in K$, $g \in G = \mathrm{Irr}(\mathcal{C})$. The Frobenius-Perron dimension of $\zeta_x(\rho)$ is equal to $\dim(\rho)$, for all $\rho \in R_x$.

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Proof. To show that γ is a module functor structure on F, it suffices to show that equation below holds for all $g_1, g_2 \in G$, $s \in K$.

$$\frac{{}^x\mu(g_1, g_2)(s)}{\mu(g_1, g_2)(s)} \; \gamma_{s, g_1 g_2} = \gamma_{s, g_1} \; \gamma_{s \triangleleft g_1, g_2}. \tag{4.5}$$

The right hand side of the above equation is equal to

$$\eta_x(g_1)(s) \; \rho(\kappa_{s, g_1}) \; \eta_x(g_2)(s \triangleleft g_1) \; \rho(\kappa_{s \triangleleft g_1, g_2})$$

$$= \eta_x(g_1)(s) \; \eta_x(g_2)(s \triangleleft g_1) \; \psi\left(\frac{{}^x\mu}{\mu}\right)(\kappa_{s, g_1}, \; \kappa_{s \triangleleft g_1, g_2}) \; \rho(\kappa_{s, g_1 g_2})$$

$$= \eta_x(g_1 g_2)(s) \; (\delta^1 \eta_x)(g_1, g_2)(s) \; \varphi\left(\psi\left(\frac{{}^x\mu}{\mu}\right)\right)(g_1, g_2)(s) \; \rho(\kappa_{s, g_1 g_2})$$

$$= \left(\frac{{}^x\mu}{\mu}\right)(g_1, g_2)(s) \; \eta_x(g_1 g_2)(s) \; \rho(\kappa_{s, g_1 g_2}),$$

which is the left hand side of (4.5). We used (2.3) in the second equality above. The last statement of the Lemma follows from Lemma 4.1.1. $\blacksquare$

Lemma 4.1.3. Let ζ_x, $x \in K$ be the maps defined in (4.4). The following statements hold.

(i) If $\rho_1, \rho_2 \in R_x$ are isomorphic, then $\zeta_x(\rho_1)$ is isomorphic to $\zeta_x(\rho_2)$.

(ii) If $\zeta_x(\rho_1)$ is isomorphic to $\zeta_x(\rho_2)$ for any $\rho_1, \rho_2 \in R_x$, then ρ_1 is isomorphic to ρ_2.

(iii) If $\rho \in R_x$ is irreducible, then $\zeta_x(\rho)$ is simple.

Proof. (i) Suppose $\rho_1, \rho_2 \in R_x$ are isomorphic and let $\zeta(\rho_1) := (F^1, \gamma^1)$ and $\zeta(\rho_2) := (F^2, \gamma^2)$. Let $n = \dim(\rho_1) = \dim(\rho_2)$. To see that (F^1, γ^1) is isomorphic to (F^2, γ^2)

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it suffices to show that these exists a $\beta \in \mathrm{Fun}(K, GL(n, k))$ such that the equation below holds for all $s \in K, g \in G$.

$$\gamma^2_{s,g} = \beta(s)\, \gamma^1_{s,g}\, \beta(s \lhd g)^{-1}.$$

Since ρ_1 is isomorphic to ρ_2, there exists $A \in GL(n, k)$ such that $\rho_2(h) = A\, \rho_1(h)\, A^{-1}$, for all $h \in H$. Put $\beta(s) := A$, for all $s \in K$. Then,

$$\begin{aligned}
\gamma^2_{s,g} &= \eta_x(g)(s)\, \rho_2(\kappa_{s,g}) \\
&= \eta_x(g)(s)\, A\, \rho_1(\kappa_{s,g})\, A^{-1} \\
&= A\, \eta_x(g)(s)\, \rho_1(\kappa_{s,g})\, A^{-1} \\
&= \beta(s)\, \gamma^1_{s,g}\, \beta(s \lhd g)^{-1}.
\end{aligned}$$

It follows that (F^1, γ^1) is isomorphic to (F^2, γ^2).

(ii) Suppose $\zeta_x(\rho_1)$ and $\zeta_x(\rho_2)$ are isomorphic, $\rho_1, \rho_2 \in R_x$. Let $\zeta(\rho_1) := (F^1, \gamma^1)$ and $\zeta(\rho_2) := (F^2, \gamma^2)$. Since (F^1, γ^1) is isomorphic to (F^2, γ^2), there exists $\beta \in \mathrm{Fun}(K, GL(n, k))$ such that the equation $\gamma^2_{s,g} = \beta(s)\, \gamma^1_{s,g}\, \beta(s \lhd g)^{-1}$ holds for all $s \in K, g \in G$. Put $s := 1_K, g = h \in H$ in the previous equation to get

$$\begin{aligned}
\rho_2(h) &= (\eta_x(h)(1_K))^{-1}\, \gamma^2_{1_K, h} \\
&= (\eta_x(h)(1_K))^{-1}\, \beta(1_K)\, \gamma^1_{1_K, h}\, \beta(1_K)^{-1} \\
&= \beta(1_K)\, \rho_1(h)\, \beta(1_K)^{-1}.
\end{aligned}$$

It follows that ρ_1 is isomorphic to ρ_2.

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(iii) Suppose $\rho \in R_x$ is irreducible and let $\zeta_x(\rho) := (F, \gamma)$. Suppose, in order to obtain a contradiction, that (F, γ) is not simple. Then, there exists $(F^1, \gamma^1), (F^2, \gamma^2) \in$ $\mathrm{Obj}(\mathcal{C}^*_\mathcal{M})$ such that $F = F^1 \oplus F^2$, $\gamma = \gamma^1 \oplus \gamma^2$ where $F^1(s) = n_1 xs$, $F^2(s) = n_2 xs$ for all $s \in K$ and n_1, n_2 are positive integers satisfying $n_1 + n_2 = \dim(\rho)$. We have, $\rho(h) = (\eta_x(h)(1_K))^{-1} \gamma_{1_K, h} = (\eta_x(h)(1_K))^{-1} \gamma^1_{1_K, h} \oplus (\eta_x(h)(1_K))^{-1} \gamma^2_{1_K, h}$, for all $h \in H$. Put $\rho_1(h) := (\eta_x(h)(1_K))^{-1} \gamma^1_{1_K, h}$ and $\rho_2(h) := (\eta_x(h)(1_K))^{-1} \gamma^2_{1_K, h}$, for all $h \in H$. It follows from Lemma 2.1.5 and (4.5) that ρ_1, ρ_2 are elements of R_x. So $\rho = \rho_1 \oplus \rho_2$. This contradicts our supposition that ρ is irreducible. Hence, $\zeta_x(\rho)$ is simple whenever $\rho \in R_x$ is irreducible. ∎

Lemma 4.1.4. There is a bijection between the sets $\bigcup_{x \in K} \overline{\mathrm{Irr}}(R_x)$ and $\overline{\mathrm{Irr}}(\mathcal{C}^*_\mathcal{M})$.

Proof. It follows from Lemma 4.1.3 that the maps ζ_x, $x \in K$ in (4.4) induces injections

$$\overline{\zeta}_x : \overline{\mathrm{Irr}}(R_x) \hookrightarrow \overline{\mathrm{Irr}}(\mathcal{C}^*_\mathcal{M}), \tag{4.6}$$

for all $x \in K$, defined by: $\overline{\zeta}_x(\overline{\rho}) := \overline{\zeta_x(\rho)}$ where $\overline{\rho}$ is the isomorphism class represented by ρ and $\overline{\zeta_x(\rho)}$ is the isomorphism class represented by $\zeta_x(\rho)$. Now, the sum of squares of the Frobenius-Perron dimensions of the objects of the set $\bigcup_{x \in K} \mathrm{Im}(\overline{\zeta}_x)$ is equal to

$$\sum_{x \in K} \sum_{\overline{\rho} \in \overline{\mathrm{Irr}}(R_x)} (\dim(\overline{\rho}))^2 = \sum_{x \in K} |H| = |K||H| = |G|.$$

The Lemma now follows from the fact that $\mathrm{FPdim}(\mathcal{C}^*_\mathcal{M}) = \mathrm{FPdim}(\mathcal{C}) = |G|$. ∎

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For each $x \in K$, let $(\mathcal{C}_\mathcal{M}^*)_x$ denote the full abelian subcategory of $\mathcal{C}_\mathcal{M}^*$ with objects given by the set

$$\{(F, \gamma) \mid (F, \gamma) \text{ is isomorphic to a direct sum of objects of the set } \zeta_x(\mathrm{Irr}(R_x))\}$$

Remark 4.1.5. (i) Let $(F, \gamma) \in \mathcal{C}_\mathcal{M}^*$. Then, $(F, \gamma) \in (\mathcal{C}_\mathcal{M}^*)_x$ if an only if there is a non-negative integer n such that $F(s) = nxs$, for all $s \in K$.

(ii) $(\mathcal{C}_\mathcal{M}^*)_{1_K}$ forms a semisimple tensor category with finitely many isomorphism classes of simple objects. The unit object $(1, \mathrm{id}_\mathcal{M})$ of $\mathcal{C}_\mathcal{M}^*$ is contained in $(\mathcal{C}_\mathcal{M}^*)_{1_K}$.

(iii) $\overline{R}_{1_K}$ is a semiring with basis $\overline{\mathrm{Irr}}(R_{1_K})$. The multiplicative structure is given by the tensor product of representations.

(iv) The set $\overline{\mathrm{Obj}}((\mathcal{C}_\mathcal{M}^*)_{1_K})$ of isomorphism classes of $\mathrm{Obj}((\mathcal{C}_\mathcal{M}^*)_{1_K})$ forms a semiring with basis $\mathrm{Im}(\overline{\zeta}_{1_K})$.

Lemma 4.1.6. The map $\overline{\zeta}_{1_K}$ (defined in (4.6)) induces a unit preserving semiring isomorphism between $\overline{R}_{1_K}$ and $\overline{\mathrm{Obj}}((\mathcal{C}_\mathcal{M}^*)_{1_K})$.

Proof. It follows from Lemma 4.1.4 and Remark 4.1.5 that the map defined below is a bijection.

$$\Upsilon : \overline{R}_{1_K} \to \overline{\mathrm{Obj}}((\mathcal{C}_\mathcal{M}^*)_{1_K}) : \overline{\rho} \mapsto \overline{\zeta_{1_K}(\rho)} \tag{4.7}$$

Note that Υ preserves the unit. Let $(F, \gamma) \in \mathrm{Obj}((\mathcal{C}_\mathcal{M}^*)_{1_K})$. Then, γ satisfies the equation $\gamma_{s, g_1 g_2} = \gamma_{s, g_1} \gamma_{s \triangleleft g_1, g_2}$, for all $s \in K$, $g_1, g_2 \in G$. Put $s = 1_K$ and $g_1 = h_1, g_2 = h_2 \in H$ in the previous equation to get the equation $\gamma_{1_K, h_1 h_2} = \gamma_{1_K, h_1} \gamma_{1_K, h_2}$. Define

$$\Upsilon' : \mathrm{Obj}((\mathcal{C}_\mathcal{M}^*)_{1_K}) \to R_{1_K} : (F, \gamma) \mapsto \Upsilon'((F, \gamma)) \tag{4.8}$$

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where $(\Upsilon'((F, \gamma)))(h) := \gamma_{1_K, h}$, for all $h \in H$. We have $\Upsilon' \circ \zeta_{1_K} = \mathrm{id}_{R_{1_K}}$. So Υ' induces a map $\overline{\mathrm{Obj}}((\mathcal{C}^*_\mathcal{M})_{1_K}) \to \overline{R}_{1_K}$ which is inverse to the map Υ defined in (4.7). Let us show that Υ is a semiring isomorphism. It is clear that Υ preserves the additive structure. To see that it preserves the multiplicative structure note that is suffices to show that $\Upsilon'(\zeta_{1_K}(\rho_1) \circ \zeta_{1_K}(\rho_2)) \cong \rho_1 \otimes \rho_2$ (= tensor product of representations ρ_1 and ρ_2), for all $\rho_1, \rho_2 \in R_{1_K}$. Let $\zeta_{1_K}(\rho_1) := (F^1, \gamma^1)$, $\zeta_{1_K}(\rho_2) := (F^2, \gamma^2)$, and $(F^1, \gamma^1) \circ (F^2, \gamma^2) := (F, \gamma)$ for $\rho_1, \rho_2 \in R_{1_K}$. We have,

$$\Upsilon'(\zeta_{1_K}(\rho_1) \circ \zeta_{1_K}(\rho_2))(h) = \gamma_{1_K, h}$$

$$= \gamma^1_{F^2(1_K), h} \circ F^1(\gamma^2_{1_k, h})$$

$$= \gamma^1_{dim(\rho_2)1_K, h} \circ (\gamma^2_{1_K, h} \boxtimes \mathrm{I}_{dim(\rho_1)})$$

$$= (\mathrm{I}_{dim(\rho_2)} \boxtimes \gamma^1_{1_K, h})(\gamma^2_{1_K, h} \boxtimes \mathrm{I}_{dim(\rho_1)})$$

$$= (\gamma^2_{1_K, h} \boxtimes \gamma^1_{1_K, h})$$

$$= \rho_2(\kappa_{1_K, h}) \boxtimes \rho_1(\kappa_{1_K, h})$$

$$= \rho_2(h) \boxtimes \rho_1(h)$$

for all $h \in H$. This shows that $\Upsilon'(\zeta_{1_K}(\rho_1) \circ \zeta_{1_K}(\rho_2)) = \rho_2 \otimes \rho_1 \cong \rho_1 \otimes \rho_2$ and the Lemma is proved. $\blacksquare$

Proposition 4.1.7. *If H is normal in G, then the fusion category $\mathcal{C}^*_\mathcal{M}$ (where $\mathcal{C} = \mathcal{V}^\omega_G$ and $\mathcal{M} = \mathcal{M}(H, \mu)$) is graded by the group $K = H \backslash G$.*

Proof. From Lemma 4.1.4, it is clear that $\mathcal{C}^*_\mathcal{M} = \oplus_{x \in K}(\mathcal{C}^*_\mathcal{M})_x$. Pick any $(F, \gamma) \in (\mathcal{C}^*_\mathcal{M})_x$, $(F^1, \gamma^1) \in (\mathcal{C}^*_\mathcal{M})_{x_1}$, $(F^2, \gamma^2) \in (\mathcal{C}^*_\mathcal{M})_{x_2}$. We need to show that

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(i) $(F, \gamma)^* \in (\mathcal{C}^*_{\mathcal{M}})_{x^{-1}}$ where $(F, \gamma)^*$ is the dual of (F, γ).

(ii) $(F^1, \gamma^1) \circ (F^2, \gamma^2) \in (\mathcal{C}^*_{\mathcal{M}})_{x_1 x_2}$.

There exists non-negative integers n, n_1, n_2 such that $F(s) = nxs$, $F^1(s) = n_1 x_1 s$, $F^2(s) = n_2 x_2 s$, for all $s \in K$. Statement (ii) above follows from Remark 4.1.5 and the fact that $(F^1 \circ F^2)(s) = n_1 n_2 x_1 x_2 s$, for all $s \in K$. To see that Statement (i) above holds, define an object $(F', \gamma') \in (\mathcal{C}^*_{\mathcal{M}})_{x^{-1}}$ as follows: let $F'(s) := nx^{-1}s$, for all $s \in K$, and let $\gamma'_{s,g} := ((\gamma_{x^{-1}s,g})^T)^{-1}$, for all $s \in K, g \in G$. Then γ' defines a module functor structure on F' if and only if the equation below is satisfied for all $s \in K, g_1, g_2 \in G$.

$$\frac{x^{-1}\mu(g_1, g_2)(s)}{\mu(g_1, g_2)(s)} ((\gamma_{x^{-1}s, g_1 g_2})^T)^{-1} = ((\gamma_{x^{-1}s, g_1})^T)^{-1} ((\gamma_{(x^{-1}s) \triangleleft g_1, g_2})^T)^{-1}.$$

Now, γ satisfies (4.5) for all $s \in K, g_1, g_2 \in G$. Replace s with $x^{-1}s$ in (4.5) to get

$$\frac{x\mu(g_1, g_2)(x^{-1}s)}{\mu(g_1, g_2)(x^{-1}s)} \gamma_{x^{-1}s, g_1 g_2} = \gamma_{x^{-1}s, g_1} \gamma_{(x^{-1}s) \triangleleft g_1, g_2}$$

$$\Longleftrightarrow \frac{\mu(g_1, g_2)(s)}{x^{-1}\mu(g_1, g_2)(s)} \gamma_{x^{-1}s, g_1 g_2} = \gamma_{x^{-1}s, g_1} \gamma_{(x^{-1}s) \triangleleft g_1, g_2}.$$

Taking the transpose and inverse of both sides of the above equation, we get

$$\frac{x^{-1}\mu(g_1, g_2)(s)}{\mu(g_1, g_2)(s)} ((\gamma_{x^{-1}s, g_1 g_2})^T)^{-1} = ((\gamma_{x^{-1}s, g_1})^T)^{-1} ((\gamma_{(x^{-1}s) \triangleleft g_1, g_2})^T)^{-1},$$

which is precisely what we need. Now, we claim that (F', γ') is the right dual of (F, γ), i.e., $(F, \gamma)^* = (F', \gamma')$. Note that if (F, γ) is simple, then so is (F', γ'). So it suffices to

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show that the unit object of $\mathcal{C}_{\mathcal{M}}^*$ is contained in the decomposition of $(F, \gamma) \circ (F', \gamma')$ into simple objects. In view of Lemma 4.1.6, it suffices to show that the identity element (=trivial representation of H) of R_{1_K} is contained in $\Upsilon'((F, \gamma) \circ (F', \gamma'))$, where Υ' is defined in (4.8). Let $(F, \gamma) \circ (F', \gamma') = (F'', \gamma'')$. Then,

$$\Upsilon'((F'', \gamma''))(h) = \gamma''_{1_K, h}$$

$$= \gamma_{F'(1_K), h} \circ F(\gamma'_{1_K, h})$$

$$= \gamma_{nx^{-1}, h} \circ (\gamma'_{1_K, h} \boxtimes \mathrm{I}_n)$$

$$= (\mathrm{I}_n \boxtimes \gamma_{x^{-1}, h})(((\gamma_{x^{-1}, h})^T)^{-1} \boxtimes \mathrm{I}_n)$$

$$= ((\gamma_{x^{-1}, h})^T)^{-1} \boxtimes \gamma_{x^{-1}, h}$$

for all $h \in H$. Define a map $\rho : H \to GL(n, k)$ by $\rho(h) := \gamma_{x^{-1}, h}$. Then ρ is a projective representation of H with 2-coycle $\psi\left(\frac{\mu}{x^{-1}\mu}\right)$. So ρ^* is a projective representation of H with 2-cocycle $\psi\left(\frac{\mu}{x^{-1}\mu}\right)^{-1}$. So, we have $\Upsilon((F'', \gamma'')) = \rho^* \otimes \rho$. By Lemma 2.3.3, it follows that the decomposition of $\rho^* \otimes \rho$ into irreducible representations of H contains the trivial representation and the Proposition is proved. ∎

Note that $\overline{\mathrm{Irr}}(R_{1_K})$ generates a based ring that is isomorphic to the Grothendieck ring $\mathcal{K}_0(\mathrm{Rep}(H))$ of the representation category $\mathrm{Rep}(H)$ of H. Also, note that $(\mathcal{C}_{\mathcal{M}}^*)_{1_K}$ is a fusion sub-category of $\mathcal{C}_{\mathcal{M}}^*$.

Corollary 4.1.8. *The based rings* $\mathcal{K}_0(\mathrm{Rep}(H))$ *and* $\mathcal{K}_0((\mathcal{C}_{\mathcal{M}}^*)_{1_K})$ *are isomorphic.*

Proof. In view of Lemma 4.1.6 and Proposition 4.1.7 we only need to show that the map Υ defined in (4.7) preserves duals. But this follows immediately by noting that $\Upsilon'(\zeta_{1_K}(\rho)^*) = \rho^*$, for all $\rho \in R_{1_K}$ where Υ' is the map defined in (4.8). ∎

Proposition 4.1.9. *If H is normal in G and H is nilpotent, then the fusion category $\mathcal{C}^*_{\mathcal{M}}$ (where $\mathcal{C} = \mathcal{V}^\omega_G$ and $\mathcal{M} = \mathcal{M}(H, \mu)$) is nilpotent.*

Proof. First note that $\mathcal{K}_0(\mathrm{Rep}(H))$ is nilpotent because H is nilpotent. Also, note that $(\mathcal{C}^*_{\mathcal{M}})_{1_K}$ is a nilpotent fusion category since by the previous corollary $\mathcal{K}_0((\mathcal{C}^*_{\mathcal{M}})_{1_K}) \cong \mathcal{K}_0(\mathrm{Rep}(H))$. Now observe that $(\mathcal{C}^*_{\mathcal{M}})_{ad}$ is a fusion sub-category of $(\mathcal{C}^*_{\mathcal{M}})_{1_K}$. By [GN, Proposition 4.6] it follows that $(\mathcal{C}^*_{\mathcal{M}})_{ad}$ is nilpotent and hence $\mathcal{C}^*_{\mathcal{M}}$ is nilpotent. ∎

Remark 4.1.10. (i) If the conditions of Proposition 4.1.9 hold, then the nilpotency class of $\mathcal{C}^*_{\mathcal{M}}$ is less than of equal to $1+$ (nilpotency class of H).

(ii) If $\mathcal{C}^*_{\mathcal{M}}$ (where $\mathcal{C} = \mathcal{V}^\omega_G$ and $\mathcal{M} = \mathcal{M}(H, \mu)$) is nilpotent, then H is nilpotent. This follows from the fact that $\mathcal{K}_0(\mathcal{C}^*_{\mathcal{M}})$ contains $\mathcal{K}_0(\mathrm{Rep}(H))$ as a subring.

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CHAPTER 5

LAGRANGIAN SUBCATEGORIES AND BRAIDED TENSOR EQUIVALENCES OF TWISTED QUANTUM DOUBLES OF FINITE GROUPS

The results presented in this Chapter are based on [NN].

In Section 5.1 (respectively, Section 5.2) we classify Lagrangian categories of the representation category of the Drinfeld double (respectively, twisted double) of a finite group. The reason we prefer to treat untwisted and twisted cases separately is because our constructions in the former case do not involve rather technical cohomological computations present in the latter. We feel that a reader might get a better understanding of our results by exploring the untwisted case first. Of course when $\omega = 1$ the results of Section 5.2 reduce to those of Section 5.1.

5.1 Lagrangian subcategories in the untwisted case

We fix notation for this Section. Let G be a finite group. For any $g \in G$, let K_g denote the conjugacy class of G containing g. Let R denote a complete set of representatives of conjugacy classes of G. Let $\mathcal{C}$ denote the representation category $\mathrm{Rep}(D(G))$ of the Drinfeld double of the group G:

$$\mathcal{C} := \mathrm{Rep}(D(G)).$$

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The category $\mathcal{C}$ is equivalent to $\mathcal{Z}(\mathrm{Vec}_G)$, the center of Vec_G. It is well known that $\mathcal{C}$ is a modular category. Let Γ denote a complete set of representatives of simple objects of $\mathcal{C}$. The set Γ is in bijection with the set $\{(a, \chi) \mid a \in R \text{ and } \chi \text{ is an irreducible}$ character of $C_G(a)\}$, where $C_G(a)$ is the centralizer of a in G (see [CGR]). In what follows we will identify Γ with the previous set.

$$\Gamma := \{(a,\, \chi) \mid a \in R \text{ and } \chi \text{ is an irreducible character of } C_G(a)\}. \qquad (5.1)$$

Let S and θ be (see, e.g. [BK], [CGR]) the S-matrix and twist, respectively, of $\mathcal{C}$. Recall that we take the canonical twist. It is known that the entries of the S-matrix lie in a cyclotomic field. Also, the values of characters of a finite group are sums of roots of unity, so they are algebraic numbers. So we may assume that all scalars appearing herein are complex numbers; in particular, complex conjugation and absolute values make sense. We have the following formulas for the S-matrix, twist and dimensions:

$$S((a,\, \chi),\, (b,\, \chi')) = \frac{|G|}{|C_G(a)||C_G(b)|} \sum_{g \in G(a,b)} \overline{\chi}(gbg^{-1})\, \overline{\chi}'(g^{-1}ag),$$

$$\theta(a,\, \chi) = \frac{\chi(a)}{\deg \chi},$$

$$d((a,\, \chi)) = |K_a| \deg \chi = \frac{|G|}{|C_G(a)|} \deg \chi,$$

for all $(a,\, \chi), (b,\, \chi') \in \Gamma$, where $G(a, b) = \{g \in G \mid agbg^{-1} = gbg^{-1}a\}$.

5.1.1 Classification of Lagrangian subcategories of $\mathrm{Rep}(D(G))$

Lemma 5.1.1. Two objects $(a, \chi), (b, \chi') \in \Gamma$ centralize each other if and only if the following conditions hold:

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(i) The conjugacy classes K_a, K_b commute element-wise,

(ii) $\chi(gbg^{-1}) \chi'(g^{-1}ag) = \deg \chi \deg \chi'$, for all $g \in G$.

Proof. By [Mu2, Corollary 2.14] two objects $(a, \chi), (b, \chi') \in \Gamma$ centralize each other if and only if

$$S((a, \chi), (b, \chi')) = \deg \chi \deg \chi'.$$

This is equivalent to the equation

$$\sum_{g \in G(a, b)} \chi(gbg^{-1}) \chi'(g^{-1}ag) = |G| \deg \chi \deg \chi', \qquad (5.2)$$

where $G(a, b) = \{g \in G \mid agbg^{-1} = gbg^{-1}a\}$. It is clear that if the two conditions of the Lemma hold, then (5.2) holds since $G(a, b) = G$.

Now suppose that (5.2) holds. We will show that this implies the two conditions in the statement of the Lemma. We have

$$|G| \deg \chi \deg \chi' = | \sum_{g \in G(a, b)} \chi(gbg^{-1}) \chi'(g^{-1}ag)|$$

$$\leq \sum_{g \in G(a, b)} |\chi(gbg^{-1})| |\chi'(g^{-1}ag)|$$

$$\leq |G| \deg \chi \deg \chi'.$$

So $\sum_{g \in G(a, b)} |\chi(gbg^{-1})| |\chi'(g^{-1}ag)| = |G| \deg \chi \deg \chi'$. Since

$$|G(a, b)| \leq |G|, \quad |\chi(gbg^{-1})| \leq \deg \chi, \quad \text{and} \quad |\chi'(g^{-1}ag)| \leq \deg \chi',$$

we must have $G(a, b) = G$, $|\chi(gbg^{-1})| = \deg \chi$, and $|\chi'(g^{-1}ag)| = \deg \chi'$. The equality $G(a, b) = G$ implies that the conjugacy classes K_a, K_b commute element-

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wise, which is Condition (i) in the statement of the Lemma. Since $|\chi(gbg^{-1})| = \deg \chi$,

and $|\chi'(g^{-1}ag)| = \deg \chi'$, there exist roots of unity α_g and β_g such that $\chi(gbg^{-1}) = \alpha_g \deg \chi$, and $\chi'(g^{-1}ag) = \beta_g \deg \chi'$, for all $g \in G$. Put this in (5.2) to get the

equation

$$\sum_{g \in G} \alpha_g \beta_g = |G|. \tag{5.3}$$

Note that (5.3) holds if and only if $\alpha_g \beta_g = 1$, for all $g \in G$. This is equivalent to saying

that $\chi(gbg^{-1}) \chi'(g^{-1}ag) = \deg \chi \deg \chi'$, for all $g \in G$ and the Lemma is proved. ∎

Lemma 5.1.2. Let E be a normal subgroup of a finite group K. Let $\mathrm{Irr}(K)$ denote

the set of irreducible characters of K. Let ρ be a K-invariant character of E of degree

one. Then

$$\sum_{\chi \in \mathrm{Irr}(K): \chi|_E = (\deg \chi)\rho} (\deg \chi)^2 = \frac{|K|}{|E|}.$$

Proof. Suppose χ is any irreducible character of K. Since ρ is K-invariant, by Clifford's Theorem, if ρ is an irreducible constituent of $\chi|_E$, then

$$\chi|_E = (\deg \chi)\rho. \tag{5.4}$$

By Frobenius reciprocity, the multiplicity of any irreducible χ in $\mathrm{Ind}_E^K \rho$ is equal to

the multiplicity of ρ in $\chi|_E$. The latter is equal to $\deg \chi$ if χ satisfies (5.4) and 0

otherwise. Therefore,

$$\sum_{\chi \in \mathrm{Irr}(K): \chi|_E = (\deg \chi)\rho} (\deg \chi)^2 = \deg \mathrm{Ind}_E^K \rho = \frac{|K|}{|E|},$$

as required. ∎

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Let H be a normal Abelian subgroup of G and let B be a G-invariant alternating bicharacter on H. Then $H = \bigcup_{a \in H \cap R} K_a$. Let

$$\mathcal{L}_{(H,B)} := \text{full Abelian subcategory of } \mathcal{C} \text{ generated by}$$

$$\left\{ (a,\chi) \in \Gamma \;\middle|\; \begin{array}{l} a \in H \cap R \text{ and } \chi \text{ is an irreducible character of } C_G(a) \\[2mm] \text{such that } \chi(h) = B(a,h) \deg \chi, \text{ for all } h \in H \end{array} \right\}.$$

$$(5.5)$$

Proposition 5.1.3. *The subcategory $\mathcal{L}_{(H,B)} \subseteq Rep(D(G))$ is Lagrangian.*

Proof. We have

$$\chi(gbg^{-1}) \chi'(g^{-1}ag) = B(a, gbg^{-1}) \deg \chi \, B(b, g^{-1}ag) \deg \chi'$$

$$= B(a, gbg^{-1}) B(gbg^{-1}, a) \deg \chi \deg \chi'$$

$$= \deg \chi \deg \chi',$$

for all $(a,\chi), (b,\chi') \in \mathcal{L}_{(H,B)} \cap \Gamma, g \in G$. The second equality above is due to G-invariance of B and the third equality holds since B is alternating. By Lemma 5.1.1, it follows that objects in $\mathcal{L}_{(H,B)}$ centralize each other.

Also, we have $\theta_{(a,\chi)} = \frac{\chi(a)}{\deg \chi} = \frac{B(a,a)}{\deg \chi} \deg \chi = 1$, for all $(a,\chi) \in \mathcal{L}_{(H,B)} \cap \Gamma$. Therefore, $\theta|_{\mathcal{L}_{(H,B)}} = \text{id}$.

The dimension of $\mathcal{L}_{(H,B)}$ is equal to $|G|$. Indeed,

$$\dim(\mathcal{L}_{(H,B)}) = \sum_{(a,\chi) \in \mathcal{L}_{(H,B)} \cap \Gamma} d(a,\chi)^2$$

$$= \sum_{(a,\chi) \in \mathcal{L}_{(H,B)} \cap \Gamma} |K_a|^2 (\deg \chi)^2$$

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$$= \sum_{a \in H \cap R} |K_a|^2 \sum_{\chi:(a,\chi) \in \mathcal{L}_{(H,B)} \cap \Gamma} (\deg \chi)^2$$

$$= \sum_{a \in H \cap R} |K_a|^2 \frac{|C_G(a)|}{|H|}$$

$$= \frac{|G|}{|H|} \sum_{a \in H \cap R} |K_a|$$

$$= |G|.$$

The fourth equality above is explained as follows. Fix $a \in H \cap R$. Define $\rho : H \to k^\times$ by $\rho(h) := B(a, h)$. Observe that ρ is a $C_G(a)$-invariant character of H of degree 1 and then apply Lemma 5.1.2.

It follows from Lemma 2.10.2 that $\mathcal{L}_{(H,B)}$ is a Lagrangian subcategory of $\mathrm{Rep}(D(G))$ and the Proposition is proved. ∎

Now, let $\mathcal{L}$ be a Lagrangian subcategory of $\mathcal{C}$. So, in particular, the two conditions in Lemma 5.1.1 hold for all simple objects in $\mathcal{L}$. Define

$$H_\mathcal{L} := \bigcup_{a \in R:(a,\chi) \in \mathcal{L} \text{ for some } \chi} K_a. \tag{5.6}$$

Note that $H_\mathcal{L}$ is a normal Abelian subgroup of G. Indeed, that $H_\mathcal{L}$ is a subgroup follows from the fact that $\mathcal{L}$ contains the unit object and is closed under tensor products. The subgroup $H_\mathcal{L}$ is normal in G because it is a union of conjugacy classes of G. Finally, that $H_\mathcal{L}$ is Abelian follows by Condition (i) of Lemma 5.1.1.

For each $a \in H \cap R$, define $\xi_a : H_\mathcal{L} \to k^\times$ by

$$\xi_a(h) := \frac{\chi(h)}{\deg \chi},$$

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for $h \in H_{\mathcal{L}}$, where χ is any irreducible character of $C_G(a)$ such that $(a, \chi) \in \mathcal{L} \cap \Gamma$. To see that this definition does not depend on the choice of χ, let $(a, \chi), (a, \chi'), (b, \chi'') \in \mathcal{L} \cap \Gamma$ and apply Condition (ii) of Lemma 5.1.1 to pairs $(a, \chi), (b, \chi'')$ and $(a, \chi'), (b, \chi'')$ to get

$$\frac{\chi(gbg^{-1})}{\deg \chi} = \left(\frac{\chi''(g^{-1}ag)}{\deg \chi''} \right)^{-1} \text{ and } \frac{\chi'(gbg^{-1})}{\deg \chi'} = \left(\frac{\chi''(g^{-1}ag)}{\deg \chi''} \right)^{-1},$$

for all $g \in G$. This implies that $\frac{\chi|_{H_{\mathcal{L}}}}{\deg \chi} = \frac{\chi'|_{H_{\mathcal{L}}}}{\deg \chi'}$, for any two pairs $(a, \chi), (a, \chi') \in \mathcal{L} \cap \Gamma$.

For any $a, b \in H_{\mathcal{L}} \cap R$, by Condition (ii) of Lemma 5.1.1, ξ_a and ξ_b satisfy the equation:

$$\xi_a(gbg^{-1}) = \xi_b(g^{-1}ag)^{-1}, \quad \text{for all } g \in G. \tag{5.7}$$

Define a map $B_{\mathcal{L}} : H_{\mathcal{L}} \times H_{\mathcal{L}} \to k^{\times}$ by

$$B_{\mathcal{L}}(h_1, h_2) := \xi_a(g^{-1}h_2 g), \tag{5.8}$$

where $h_1 = gag^{-1}, g \in G, a \in H_{\mathcal{L}} \cap R$.

Proposition 5.1.4. *$B_{\mathcal{L}}$ is a well-defined G-invariant alternating bicharacter on $H_{\mathcal{L}}$.*

Proof. First, let us show that $B_{\mathcal{L}}$ is well-defined. Suppose $gag^{-1} = kak^{-1}$, where $a \in H_{\mathcal{L}} \cap R, g, k \in G$. Then

$$B_{\mathcal{L}}(gag^{-1}, lbl^{-1}) = \xi_a((g^{-1}l)b(g^{-1}l)^{-1})$$

$$= \xi_b((g^{-1}l)^{-1}a(g^{-1}l))^{-1}$$

$$= \xi_b(l^{-1}(gag^{-1})l)^{-1}$$

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$$= \xi_b(l^{-1}(kak^{-1})l)^{-1}$$

$$= \xi_a((l^{-1}k)^{-1}b(l^{-1}k))$$

$$= \xi_a(k^{-1}(lbl^{-1})k)$$

$$= B_{\mathcal{L}}(kak^{-1}, \, lbl^{-1}),$$

for all $b \in H_{\mathcal{L}} \cap R, l \in G$. The second and the fifth equalities above are due to (5.7).

Let $h_1 = kak^{-1}, h_2 \in H_{\mathcal{L}}, g \in G$, where $a \in H_{\mathcal{L}} \cap R, k \in G$. Then

$$B_{\mathcal{L}}(gh_1g^{-1}, \, gh_2g^{-1}) = B_{\mathcal{L}}(gkak^{-1}g^{-1}, \, gh_2g^{-1})$$

$$= \xi_a((gk)^{-1}(gh_2g^{-1})(gk))$$

$$= \xi_a(k^{-1}h_2k)$$

$$= B_{\mathcal{L}}(kak^{-1}, \, h_2)$$

$$= B_{\mathcal{L}}(h_1, \, h_2).$$

So $B_{\mathcal{L}}$ is G-invariant.

Now,

$$B_{\mathcal{L}}(gag^{-1}, \, gag^{-1}) = B_{\mathcal{L}}(a, \, a)$$

$$= \xi_a(a)$$

$$= \frac{\chi(a)}{\deg \chi}$$

$$= \theta_{(a, \chi)}$$

$$= 1,$$

for all $a \in H_{\mathcal{L}} \cap R, g \in G$. The first equality above is due to the G-invariance of $B_{\mathcal{L}}$.

So $B_{\mathcal{L}}(h, \, h) = 1$, for all $h \in H_{\mathcal{L}}$.

Also, $B_{\mathcal{L}}(g_1 a g_1^{-1}, g_2 b g_2^{-1}) B_{\mathcal{L}}(g_2 b g_2^{-1}, g_1 a g_1^{-1}) = \xi_a(g_1^{-1} g_2 b g_2^{-1} g_1) \xi_b(g_2^{-1} g_1 a g_1^{-1} g_2) = 1$, for all $g_1, g_2 \in G, a, b \in H \cap R$. We used (5.7) in the last equality.

To see that $B_{\mathcal{L}}$ is a bicharacter, observe first that ξ_a is a homomorphism, for all $a \in H_{\mathcal{L}} \cap R$. We have

$$B_{\mathcal{L}}(gag^{-1}, h_1) \, B_{\mathcal{L}}(gag^{-1}, h_2) = \xi_a(g^{-1} h_1 g) \, \xi_a(g^{-1} h_2 g)$$

$$= \xi_a(g^{-1} h_1 h_2 g)$$

$$= B_{\mathcal{L}}(gag^{-1}, h_1 h_2),$$

for all $a \in H_{\mathcal{L}} \cap R, g \in G, h_1, h_2 \in H_{\mathcal{L}}$. We conclude that $B_{\mathcal{L}}$ is a G-invariant alternating bicharacter on $H_{\mathcal{L}}$ and the Proposition is proved. ∎

Recall that $\mathrm{Lagr}(\mathcal{C})$ denotes the set of Lagrangian subcategories of a modular category $\mathcal{C}$.

Theorem 5.1.5. *Lagrangian subcategories of the representation category of the Drinfeld double $D(G)$ are classified by pairs (H, B), where H is a normal Abelian subgroup of G and B is an alternating G-invariant bicharacter on H.*

Proof. Let $\mathcal{E} := \{(H, B) \mid H$ is a normal Abelian subgroup of G and $B \in (\Lambda^2 H)^G\}$. Define a map $\Psi : \mathcal{E} \to \mathrm{Lagr}(\mathcal{C}) : (H, B) \mapsto \mathcal{L}_{(H, B)}$, where $\mathcal{C} = \mathrm{Rep}(D(G))$ and $\mathcal{L}_{(H, B)}$ is defined in (5.5). It was shown in Proposition 5.1.3 that $\mathcal{L}_{(H, B)}$ is a Lagrangian subcategory.

To see that Ψ is injective pick any $(H, B), (H', B') \in \mathcal{E}$ and assume that $\Psi((H, B)) = \Psi((H', B'))$. So in particular we will have $\mathcal{L}_{(H, B)} \cap \Gamma = \mathcal{L}_{(H', B')} \cap \Gamma$. Note that

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$H = \cup_{(a,\chi) \in \mathcal{L}_{(H,B)} \cap \Gamma} K_a$ and $H' = \cup_{(a,\chi) \in \mathcal{L}_{(H',B')} \cap \Gamma} K_a$. Since $\mathcal{L}_{(H,B)} \cap \Gamma = \mathcal{L}_{(H',B')} \cap \Gamma$,

it follows that $H = H'$. Also note that for any $(a, \chi) \in \mathcal{L}_{(H,B)} \cap \Gamma = \mathcal{L}_{(H',B')} \cap \Gamma$,

we have $\chi(h) = B(a, h) \deg \chi = B'(a, h) \deg \chi$, for all $h \in H = H'$. Since B, B' are

G-invariant, it follows that $B = B'$. So Ψ is injective.

To see that Ψ is surjective pick any $\mathcal{L} \in \mathrm{Lagr}(\mathcal{C})$. Consider the pair $(H_{\mathcal{L}}, B_{\mathcal{L}})$,

where $H_{\mathcal{L}}$ and $B_{\mathcal{L}}$ are defined in (5.6) and (5.8), respectively. Proposition 5.1.4 showed

that $(H_{\mathcal{L}}, B_{\mathcal{L}})$ belongs to the set $\mathcal{E}$. We contend that $\Psi((H_{\mathcal{L}}, B_{\mathcal{L}})) = \mathcal{L}$. It suffices

to show that $\mathcal{L} \cap \Gamma \subseteq \mathcal{L}_{(H_{\mathcal{L}}, B_{\mathcal{L}})}$. But this hold by definition of $\mathcal{L}_{(H_{\mathcal{L}}, B_{\mathcal{L}})}$ and the

observation that $\frac{\chi|_{H_{\mathcal{L}}}}{\deg \chi} = \frac{\chi'|_{H_{\mathcal{L}}}}{\deg \chi'}$, for any two pairs $(a, \chi), (a, \chi') \in \mathcal{L} \cap \Gamma$, $a \in H_{\mathcal{L}} \cap R$.

So Ψ is surjective and the Theorem is proved. ∎

5.1.2 Bijective correspondence between Lagrangian subcategories and module categories with pointed duals

Let $\mathcal{D}$ be a fusion category and let $\mathcal{M}$ be an indecomposable $\mathcal{D}$-module category.

There is a canonical braided tensor equivalence [EO]

$$\iota_{\mathcal{M}} : \mathcal{Z}(\mathcal{D}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{D}_{\mathcal{M}}^*) \tag{5.9}$$

defined by identifying both centers with the category of $\mathcal{D} \boxtimes (\mathcal{D}_{\mathcal{M}}^*)^{\mathrm{rev}}$-module endo-

functors of $\mathcal{M}$.

Let $f : E(\mathcal{C}) \xrightarrow{\sim} \mathrm{Lagr}(\mathcal{C})$ be the bijection between the set of (equivalence classes

of) braided tensor equivalences between $\mathcal{C}$ and centers of pointed fusion categories

and the set of Lagrangian subcategories of $\mathcal{C}$ defined in [DGNO], see (2.22).

Theorem 5.1.6. *The assignment $\mathcal{M} \mapsto \iota_{\mathcal{M}}$ restricts to a bijection between the set of equivalence classes of indecomposable Vec_G-module categories $\mathcal{M}$ with respect to which the dual fusion category $(Vec_G)^*_{\mathcal{M}}$ is pointed and $E(Rep(D(G)))$.*

Proof. By Theorem 3.1.5, Theorem 5.1.5 and taking into account that the isomorphism $alt : H^2(H, k^\times) \xrightarrow{\sim} (\Lambda^2 H)$ is G-linear, we see that the two sets in question have the same cardinality. Thus, to prove the theorem it suffices to check that for $\mathcal{M} := \mathcal{M}(H, \mu)$ one has $f(\iota_{\mathcal{M}}) \subseteq \mathcal{L}_{(H, alt(\mu))}$, where $\mathcal{L}_{(H, alt(\mu))}$ is the Lagrangian subcategory defined in (5.5).

By definition, $f(\iota_{\mathcal{M}})$ consists of all objects Z in $\mathcal{C} = \mathcal{Z}(Vec_G)$ (identified with $Rep(D(G))$) such that the Vec_G-module endofunctor $F_Z : \mathcal{M} \to \mathcal{M} : M \mapsto M \otimes Z$ is isomorphic to a multiple of $id_{\mathcal{M}}$. Note that here we abuse notation and write Z for both object of the center and its forgetful image.

Let us recall the parametrization of simple objects of $\mathcal{Z}(Vec_G)$ in (5.1). Suppose that a simple Z corresponds to the conjugacy class K_a represented by $a \in R$ and the character afforded by the irreducible representation $\pi : C_G(a) \to GL(V_\pi)$. Then as a G-graded vector space $Z = \oplus_{x \in K_a} V_\pi^x$ and the permutation isomorphism

$$c_{g,Z} : g \otimes Z \xrightarrow{\sim} Z \otimes g$$

is induced from π, where we identify simple objects of Vec_G with the elements of the group G.

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It is clear that F_Z is isomorphic to a multiple of $\mathrm{id}_{\mathcal{M}}$ as an ordinary functor if and only if $K_a \subseteq H$. Note that this implies that $H \subseteq C_G(a)$. Note that for every Vec_G-module functor $F : \mathcal{M} \to \mathcal{M}$ the module functor structure on F is completely determined by the collection of isomorphisms $F(H1 \otimes h) \xrightarrow{\sim} F(H1) \otimes h$, $h \in H$, where $H1$ denotes the trivial coset in $H \backslash G = \mathrm{Irr}(\mathcal{M})$.

For $F = F_Z$ the latter isomorphism is given by the composition

$$(H1 \otimes h) \otimes Z \xrightarrow{\oplus_x \mu(h,x)^{-1} \mathrm{id}_{V_\pi^x}} H1 \otimes (h \otimes Z) \xrightarrow{\mathrm{id}_{H1} \otimes c_{h,Z}} H1 \otimes (Z \otimes h) \xrightarrow{\oplus_x \mu(x,h) \mathrm{id}_{V_\pi^x}} (H1 \otimes Z) \otimes h.$$

Note that the restriction of $c_{h,Z}$ to $h \otimes V_\pi^a$ is given by $\pi(h)$, for all $h \in C_G(a)$. If the above composition equals identity, then $\pi(h) = alt(\mu)(a, h) \, \mathrm{id}_{V_\pi}$, for all $h \in H$. So $Z \in \mathcal{L}_{(H, alt(\mu))}$ and, therefore, $f(\iota_{\mathcal{M}}) \subseteq \mathcal{L}_{(H, alt(\mu))}$, as required. $\blacksquare$

5.2   Lagrangian subcategories in the twisted case

In this Section we extend the constructions in the previous Section when the associativity is given by a 3-cocycle $\omega \in Z^3(G, k^\times)$. Note that the results of this Section reduce to the results in the previous Section when $\omega \equiv 1$.

For this Section we follow the notation fixed at the beginning of the previous section. Let ω be a normalized 3-cocycle on G, i.e., ω is a map from $G \times G \times G$ to $k^\times$ satisfying:

$$\omega(g_2, g_3, g_4) \omega(g_1, g_2 g_3, g_4) \omega(g_1, g_2, g_3) = \omega(g_1 g_2, g_3, g_4) \omega(g_1, g_2, g_3 g_4), \qquad (5.10)$$

$$\omega(g, 1_G, l) = 1,$$

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for all $g, l, g_1, g_2, g_3, g_4 \in G$.

Let $\mathcal{C}$ denote the representation category $\text{Rep}(D^\omega(G))$ of the twisted quantum double of the group G [DPR1, DPR2]:

$$\mathcal{C} := \text{Rep}(D^\omega(G)).$$

The category $\mathcal{C}$ is equivalent to $\mathcal{Z}(\text{Vec}_G^\omega)$. It is well known that $\mathcal{C}$ is a modular category. Replacing ω by a cohomologous 3-cocycle we may assume that the values of ω are roots of unity.

For all $a, g, h \in G$, define

$$\beta_a(h, g) := \omega(a, h, g)\omega(h, h^{-1}ah, g)^{-1}\omega(h, g, (hg)^{-1}ahg). \qquad (5.11)$$

The β_a's satisfy the following equation:

$$\beta_a(x, y)\beta_a(xy, z) = \beta_a(x, yz)\beta_{x^{-1}ax}(y, z), \qquad \text{for all } x, y, z \in G. \qquad (5.12)$$

Observe that the restriction of each β_a to the centralizer $C_G(a)$ of a in G is a normalized 2-cocycle. Let Γ denote a complete set of representatives of simple objects of $\mathcal{C}$. The set Γ is in bijection with the set
$\{(a, \chi) \mid a \in R \text{ and } \chi \text{ is an irreducible } \beta_a\text{-character of } C_G(a)\}$. In what follows we will identify Γ with the previous set:

$$\Gamma := \{(a, \chi) \mid a \in R \text{ and } \chi \text{ is an irreducible } \beta_a\text{-character of } C_G(a)\}. \qquad (5.13)$$

Let S and θ be the S-matrix and twist, respectively, of $\mathcal{C}$. It is known that the entries of the S-matrix lie in a cyclotomic field. Also, the values of α-characters of a finite group are sums of roots of unity, so they are algebraic numbers, where α is any 2-cocycle whose values are roots of unity. So we may assume that all scalars appearing herein are complex numbers; in particular, complex conjugation and absolute values make sense. We have the following formulas for the S-matrix, twist, and dimensions (see [CGR]):

$$S((a,\,\chi),\,(b,\,\chi'))$$

$$= \sum_{g \in K_a, g' \in K_b \cap C_G(g)} \overline{\left(\frac{\beta_a(x,\,g')\beta_a(xg',\,x^{-1})\beta_b(y,\,g)\beta_b(yg,\,y^{-1})}{\beta_a(x,\,x^{-1})\beta_b(y,\,y^{-1})}\right)} \overline{\chi}(xg'x^{-1})\,\overline{\chi}'(ygy^{-1}),$$

$$\theta(a,\,\chi) = \frac{\chi(a)}{\deg\chi},$$

$$d((a,\,\chi)) = |K_a|\,\deg\chi = \frac{|G|}{|C_G(a)|}\,\deg\chi,$$

for all $(a,\,\chi),\,(b,\,\chi') \in \Gamma$, where $g = x^{-1}ax,\,g' = y^{-1}by$.

5.2.1 Classification of Lagrangian subcategories of Rep($D^\omega(G)$)

Remark 5.2.1. Let $\rho : K \to GL(V)$ be a finite-dimensional projective representation with 2-cocycle α on the finite group K. Let χ be the projective character afforded by ρ, i.e., $\chi(x) = \mathrm{Trace}(\rho(x))$, for all $x \in K$. Suppose that the values of α are roots of unity. Then $|\chi(x)| \le \deg\chi$, for all $x \in K$ and we have equality if and only if $\rho(x) \in k^\times \cdot \mathrm{id}_V$.

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Lemma 5.2.2. Two objects $(a, \chi), (b, \chi') \in \Gamma$ centralize each other if and only if the following conditions hold:

(i) The conjugacy classes K_a, K_b commute element-wise,

(ii) $\left(\frac{\beta_a(x, y^{-1}by)\beta_a(xy^{-1}by, x^{-1})\beta_b(y, x^{-1}ax)\beta_b(yx^{-1}ax, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \right) \chi(xy^{-1}byx^{-1}) \chi'(yx^{-1}axy^{-1})$

$= \deg \chi \, \deg \chi'$, for all $x, y \in G$.

Proof. Two objects $(a, \chi), (b, \chi') \in \Gamma$ centralize each other if and only if

$$S((a, \chi), (b, \chi')) = \deg \chi \, \deg \chi'.$$

This is equivalent to the equation:

$$\sum_{g \in K_a, g' \in K_b \cap C_G(g)} \left(\frac{\beta_a(x, g')\beta_a(xg', x^{-1})\beta_b(y, g)\beta_b(yg, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \right) \chi(xg'x^{-1}) \chi'(ygy^{-1})$$

$$= |K_a||K_b| \deg \chi \, \deg \chi',$$

$$(5.14)$$

where $g = x^{-1}ax, g' = y^{-1}by$. It is clear that if the two conditions of the Lemma hold, then (5.14) holds since the set over which the above sum is taken is equal to $K_a \times K_b$.

Now suppose that (5.14) holds. We will show that this implies the two conditions in the statement of the Lemma. We have

$|K_a||K_b| \deg \chi \, \deg \chi'$

$$= \left| \sum_{g \in K_a, g' \in K_b \cap C_G(g)} \left(\frac{\beta_a(x, g')\beta_a(xg', x^{-1})\beta_b(y, g)\beta_b(yg, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \right) \chi(xg'x^{-1}) \chi'(ygy^{-1}) \right|$$

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$$\leq \sum_{g \in K_a, g' \in K_b \cap C_G(g)} \left| \left(\frac{\beta_a(x, g')\beta_a(xg', x^{-1})\beta_b(y, g)\beta_b(yg, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \right) \right| |\chi(xg'x^{-1})| \, |\chi'(ygy^{-1})|$$

$$= \sum_{g \in K_a, g' \in K_b \cap C_G(g)} |\chi(xg'x^{-1})| \, |\chi'(ygy^{-1})|$$

$$\leq |K_a||K_b| \deg \chi \, \deg \chi'$$

So

$$\sum_{g \in K_a, g' \in K_b \cap C_G(g)} |\chi(xg'x^{-1})| \, |\chi'(ygy^{-1})| = |K_a||K_b| \deg \chi \, \deg \chi'.$$

Since $|\{(g, g') \mid g \in K_a, g' \in K_b \cap C_G(g)\}| \leq |K_a||K_b|$, $|\chi(xg'x^{-1})| \leq \deg \chi$, and

$|\chi'(ygy^{-1})| \leq \deg \chi'$, we must have $|\{(g, g') \mid g \in K_a, g' \in K_b \cap C_G(g)\}| = |K_a||K_b|$,

i.e. $\{(g, g') \mid g \in K_a, g' \in K_b \cap C_G(g)\} = K_a \times K_b$, $|\chi(xg'x^{-1})| = \deg \chi$, and

$|\chi'(ygy^{-1})| = \deg \chi'$. The equality $\{(g, g') \mid g \in K_a, g' \in K_b \cap C_G(g)\} = K_a \times K_b$

implies that $K_b \subseteq C_G(g)$, for all $g \in K_a$. This is equivalent to the condition that

K_a, K_b commute element-wise which is Condition (i) in the statement of the Lemma.

Now, (5.14) becomes:

$$\sum_{(g, g') \in K_a \times K_b} \left(\frac{\beta_a(x, g')\beta_a(xg', x^{-1})\beta_b(y, g)\beta_b(yg, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \right) \frac{\chi(xg'x^{-1})}{\deg \chi} \frac{\chi'(ygy^{-1})}{\deg \chi'} = |K_a||K_b|,$$

$$(5.15)$$

where $g = x^{-1}ax$, $g' = y^{-1}by$. Since $|\chi(xg'x^{-1})| = \deg \chi$, and $|\chi'(ygy^{-1})| = \deg \chi'$, by

Remark 5.2.1, $\frac{\chi(xg'x^{-1})}{\deg \chi}$ and $\frac{\chi'(ygy^{-1})}{\deg \chi'}$ are roots of unity. Note that (5.15) holds if and

only if

$$\left(\frac{\beta_a(x, g')\beta_a(xg', x^{-1})\beta_b(y, g)\beta_b(yg, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \right) \chi(xg'x^{-1}) \chi'(ygy^{-1}) = \deg \chi \, \deg \chi',$$

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for all $g \in K_a, g' \in K_b$, where $g = x^{-1}ax, g' = y^{-1}by$. This is equivalent to Condition

(ii) in the statement of the Lemma. ∎

Note 5.2.3. Let E be a subgroup of a finite group K. Let α be a 2-cocycle on K.

Let χ be a projective α-character of E. For any $x \in K$, define χ^x by

$$\chi^x(l) := \alpha(lx, x^{-1})^{-1}\alpha(x, x^{-1}lx)^{-1}\alpha(x, x^{-1})\chi(x^{-1}lx),$$

for all $l \in E$. Then χ^x is a projective α-character of xEx^{-1}. Suppose E is normal in

K. Then χ is said to be K-*invariant* if $\chi^x = \chi$, for all $x \in K$.

Lemma 5.2.4. Let E be a normal subgroup of a finite group K. Let α be a 2-cocycle

on K. Let $\mathrm{Irr}(K)$ denote the set of irreducible projective α-characters of K. Let ρ

be a K-invariant projective $\alpha|_{E \times E}$-character of E of degree 1. Then

$$\sum_{\chi \in \mathrm{Irr}(K) : \chi|_E = (\deg \chi)\rho} (\deg \chi)^2 = \frac{|K|}{|E|}.$$

Proof. The proof is completely similar to the one given in Lemma 5.1.2 except in this

case we apply Clifford's Theorem [Ka, Theorem 8.1] and Frobenius reciprocity [Ka,

Proposition 4.8] for projective characters. ∎

Let H be a normal Abelian subgroup of G.

Recall that $\omega \in Z^3(G, k^\times)$ gives rise to a collection (5.12) of 2-cochains β_a, $a \in G$.

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Definition 5.2.5. We will say that a map $B : H \times H \to k^\times$ is an *alternating ω-bicharacter on H* if it satisfies the following three conditions:

$$B(h_1, h_2) = B(h_2, h_1)^{-1}, \tag{5.16}$$

$$B(h, h) = 1, \tag{5.17}$$

$$\delta^1 B_h = \beta_h|_{H \times H}, \tag{5.18}$$

for all $h, h_1, h_2 \in H$, where the map $B_h : H \to k^\times$ is defined by $B_h(h_1) := B(h, h_1)$, for all $h, h_1 \in H$.

Definition 5.2.6. We will say that an alternating ω-bicharacter $B : H \times H \to k^\times$ on H is *G-invariant* if it satisfies the following condition:

$$B(x^{-1}ax, h) = \frac{\beta_a(x, h)\beta_a(xh, x^{-1})}{\beta_a(x, x^{-1})} B(a, xhx^{-1}), \quad \text{for all } x \in G, a \in H \cap R, h \in H. \tag{5.19}$$

Define

$$\Lambda_\omega^2 H := \{B : H \times H \to k^\times \mid B \text{ is an alternating } \omega - \text{bicharacter on } H\}, \tag{5.20}$$

and

$$(\Lambda_\omega^2 H)^G := \{B \in \Lambda_\omega^2 H \mid B \text{ is } G\text{-invariant}\}. \tag{5.21}$$

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Remark 5.2.7. If $\omega \equiv 1$, then $(\Lambda_\omega^2 H)^G$ is the Abelian group of G-invariant alternating bicharacters on H.

Remark 5.2.8. If B is an alternating ω-bicharacter on H, then the restriction $\omega|_{H \times H \times H}$ must be cohomologically trivial. Indeed, let $\omega_H := \omega|_{H \times H \times H}$. Then B defines a braiding on the fusion category $\mathrm{Vec}_H^{\omega_H}$. The isomorphism $h_1 \otimes h_2 \xrightarrow{\sim} h_2 \otimes h_1$ is given by $B(h_1, h_2)$, for all $h_1, h_2 \in H$, where we identify simple objects of $\mathrm{Vec}_H^{\omega_H}$ with elements of H. It is known (see, e.g., [Q], [FRS]) that in this case ω_H is an *Abelian* 3-cocycle on H. By a classical result of Eilenberg and MacLane [EM] the third Abelian cohomology group of H is isomorphic to the (multiplicative) group of quadratic forms on H. The value of the corresponding quadratic form q on $h \in H$ is given by $q(h) = B(h, h)$. Since B is alternating we have $q \equiv 1$ and so ω_H must be cohomologically trivial.

Let $B \in (\Lambda_\omega^2 H)^G$ and define:

$\mathcal{L}_{(H,B)} :=$ full Abelian subcategory of $\mathcal{C}$ generated by

$$
\left\{ (a, \chi) \in \Gamma \;\middle|\; \begin{array}{l} a \in H \cap R \text{ and } \chi \text{ is an irreducible } \beta_a\text{-character of } C_G(a) \\ \text{such that } \chi(h) = B(a, h) \deg\chi, \text{ for all } h \in H \end{array} \right\}
$$

(5.22)

Proposition 5.2.9. *The subcategory $\mathcal{L}_{(H,B)} \subseteq Rep(D^\omega(G))$ is Lagrangian.*

Proof. Pick any $(a, \chi), (b, \chi') \in \mathcal{L}_{(H,B)} \cap \Gamma$. We have

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$$\left(\frac{\beta_a(x, y^{-1}by)\beta_a(xy^{-1}by, x^{-1})\beta_b(y, x^{-1}ax)\beta_b(yx^{-1}ax, y^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \right)$$

$$\chi(xy^{-1}byx^{-1})\,\chi'(yx^{-1}axy^{-1})$$

$$= \frac{\beta_a(x, y^{-1}by)\beta_a(xy^{-1}by, x^{-1})}{\beta_a(x, x^{-1})}B(a, xy^{-1}byx^{-1})$$

$$\times \frac{\beta_b(y, x^{-1}ax)\beta_b(yx^{-1}ax, y^{-1})}{\beta_b(y, y^{-1})}B(b, yx^{-1}axy^{-1}) \times \deg\chi \deg\chi'$$

$$= B(x^{-1}ax, y^{-1}by)\, B(y^{-1}by, x^{-1}ax)\, \deg\chi \deg\chi'$$

$$= \deg\chi \deg\chi',$$

for all $x, y \in G$. The second equality above is due to (5.19) while the third equality is due to (5.16). Note that K_a, K_b commute element-wise since H is Abelian. By Lemma 5.2.2, it follows that objects in $\mathcal{L}_{(H, B)}$ centralize each other.

Also, $\theta|_{\mathcal{L}_{(H, B)}} = \mathrm{id}$. The proof of this assertion is exactly the one given in Proposition 5.1.3.

Now, fix $a \in H \cap R$ and observe that B_a defines a $C_G(a)$-invariant β_a-character of H of degree 1. Indeed,

$$(B_a)^x(h) = \frac{\beta_a(x, x^{-1})}{\beta_a(hx, x^{-1})\beta_a(x, x^{-1}hx)}B(a, x^{-1}hx)$$

$$= B(x^{-1}ax, x^{-1}hx)^{-1}\, B(a, h)\, B(a, x^{-1}hx)$$

$$= B(a, h),$$

for all $x \in C_G(a), h \in H$. The second equality above is due to (5.19).

The dimension of $\mathcal{L}_{(H, B)}$ is equal to $|G|$. The proof of this assertion is exactly the one given in Proposition 5.1.3 except we appeal to Lemma 5.2.4 in this case.

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It follows from Lemma 2.10.2 that $\mathcal{L}_{(H, B)}$ is a Lagrangian subcategory of $\text{Rep}(D^\omega(G))$ and the Proposition is proved. ■

Lemma 5.2.10. Let H be a normal Abelian subgroup of G. Let $B : H \times H \to k^\times$ be a map satisfying (5.16),(5.17), and (5.19). Suppose $\delta^1 B_a = \beta_a|_{H \times H}$, for all $a \in H \cap R$. Then $B \in (\Lambda_\omega^2 H)^G$.

Proof. We only need to verify that (5.18) holds. We have

$$(\delta^1 B_{x^{-1}ax})(h_1, h_2)$$

$$= \frac{B(x^{-1}ax, h_1)B(x^{-1}ax, h_2)}{B(x^{-1}ax, h_1 h_2)}$$

$$= \left(\frac{\beta_a(x, h_1)\beta_a(xh_1, x^{-1})}{\beta_a(x, x^{-1})} \right) B(a, xh_1 x^{-1}) \times \left(\frac{\beta_a(x, h_2)\beta_a(xh_2, x^{-1})}{\beta_a(x, x^{-1})} \right) B(a, xh_2 x^{-1})$$

$$\times \left(\frac{\beta_a(x, x^{-1})}{\beta_a(x, h_1 h_2)\beta_a(xh_1 h_2, x^{-1})} \right) B(a, xh_1 h_2 x^{-1})^{-1}$$

$$= \frac{\beta_a(x, h_1)\beta_a(xh_1, x^{-1})\beta_a(x, h_2)\beta_a(xh_2, x^{-1})\beta_a(xh_1 x^{-1}, xh_2 x^{-1})}{\beta_a(x, x^{-1})\beta_a(x, h_1 h_2)\beta_a(xh_1 h_2, x^{-1})}$$

$$= \frac{\beta_{x^{-1}ax}(h_1, h_2)\beta_a(xh_1, x^{-1})\beta_a(x, h_2)\beta_a(xh_2, x^{-1})\beta_a(xh_1 x^{-1}, xh_2 x^{-1})}{\beta_a(xh_1, h_2)\beta_a(x, x^{-1})\beta_a(xh_1 h_2, x^{-1})}$$

$$= \frac{\beta_{x^{-1}ax}(h_1, h_2)\beta_a(xh_1, h_2 x^{-1})\beta_{x^{-1}ax}(x^{-1}, xh_2 x^{-1})\beta_a(x, h_2 x^{-1})\beta_{x^{-1}ax}(h_2, x^{-1})}{\beta_a(xh_1, h_2)\beta_a(x, x^{-1})\beta_a(xh_1 h_2, x^{-1})}$$

$$= \beta_{x^{-1}ax}(h_1, h_2),$$

for all $x \in G, a \in H \cap R, h_1, h_2 \in H$. In the second equality above, we used (5.19). In the third equality we used $\delta^1 B_a = \beta_a|_{H \times H}$. and canceled some factors. In the fourth equality we used (5.12) with $(x, y, z) = (x, h_1, h_2)$. In the fifth equality we used (5.12) twice with $(x, y, z) = (x, h_2, x^{-1}), (xh_1, x^{-1}, xh_2 x^{-1})$. In the last equality we used (5.12) twice with $(x, y, z) = (xh_1, h_2, x^{-1}), (x, x^{-1}, xh_2 x^{-1})$. ■

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Now, let $\mathcal{L}$ be a Lagrangian subcategory of $\mathcal{C}$. So, in particular, the two conditions in Lemma 5.2.2 hold for all objects in $\mathcal{L} \cap \Gamma$. Define

$$H_{\mathcal{L}} := \bigcup_{a \in R:(a,\chi) \in \mathcal{L} \cap \Gamma \text{ for some } \chi} K_a. \tag{5.23}$$

Note that $H_{\mathcal{L}}$ is a normal Abelian subgroup of G.

Define a map $B_{\mathcal{L}} : H_{\mathcal{L}} \times H_{\mathcal{L}} \to k^{\times}$ by

$$B_{\mathcal{L}}(h_1, h_2) := \frac{\beta_a(x, h_2)\beta_a(xh_2, x^{-1})}{\beta_a(x, x^{-1})} \times \frac{\chi(xh_2x^{-1})}{\deg \chi}, \tag{5.24}$$

where $h_1 = x^{-1}ax, x \in G, a \in H_{\mathcal{L}} \cap R$ and χ is any β_a-character of $C_G(a)$ such that $(a, \chi) \in \mathcal{L} \cap \Gamma$. The above definition does not depend on the choice of χ. The proof of this assertion is similar to the proof given for the corresponding assertion in the untwisted case.

Proposition 5.2.11. *The map $B_{\mathcal{L}}$ defined in (5.24) is an element of $(\Lambda_{\omega}^2 H)^G$.*

Proof. First, let us show that $B_{\mathcal{L}}$ is well-defined. Suppose $x^{-1}ax = z^{-1}az$, where $a \in H_{\mathcal{L}} \cap R, x, z \in G$. Then

$$
\begin{aligned}
B_{\mathcal{L}}(x^{-1}ax, y^{-1}by) &= \frac{\beta_a(x, y^{-1}by)\beta_a(xy^{-1}by, x^{-1})}{\beta_a(x, x^{-1})} \times \frac{\chi(xy^{-1}byx^{-1})}{\deg \chi} \\
&= \left(\frac{\beta_b(y, x^{-1}ax)\beta_b(yx^{-1}ax, y^{-1})}{\beta_b(y, y^{-1})} \right)^{-1} \left(\frac{\chi'(yx^{-1}axy^{-1})}{\deg \chi'} \right)^{-1} \\
&= \left(\frac{\beta_b(y, z^{-1}az)\beta_b(yz^{-1}az, y^{-1})}{\beta_b(y, y^{-1})} \right)^{-1} \left(\frac{\chi'(yz^{-1}azy^{-1})}{\deg \chi'} \right)^{-1} \\
&= \frac{\beta_a(z, y^{-1}by)\beta_a(zy^{-1}by, z^{-1})}{\beta_a(z, z^{-1})} \times \frac{\chi(zy^{-1}byz^{-1})}{\deg \chi} \\
&= B_{\mathcal{L}}(z^{-1}az, y^{-1}by),
\end{aligned}
$$

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for all $b \in H_{\mathcal{L}} \cap R, y \in G$, where χ' is any irreducible β_b-character of $C_G(b)$ such that $(b, \chi') \in \mathcal{L} \cap \Gamma$. The second and the fourth equalities above are due to Condition (ii) of Lemma 5.2.2.

The map $B_{\mathcal{L}}$ satisfies (5.16) because Condition (ii) of Lemma 5.2.2 holds. Let us show that (5.17) holds for $B_{\mathcal{L}}$:

$$B_{\mathcal{L}}(x^{-1}ax, \, x^{-1}ax)$$
$$= \frac{\beta_a(x, \, x^{-1}ax)\beta_a(ax, \, x^{-1})}{\beta_a(x, \, x^{-1})} \times \frac{\chi(a)}{\deg \chi}$$
$$= \omega(a, \, x, \, x^{-1}ax) \times \frac{\omega(a, \, ax, x^{-1})\omega(ax, \, x^{-1}, \, a)}{\omega(ax, \, x^{-1}ax, x^{-1})} \times \frac{\omega(x, \, x^{-1}ax, \, x^{-1})}{\omega(a, \, x, \, x^{-1})\omega(x, \, x^{-1}, \, a)} \times \theta_{(a, \chi)}$$
$$= \frac{\omega(a, \, x, \, x^{-1}a)\omega(ax, \, x^{-1}, \, a)}{\omega(a, \, x, \, x^{-1})\omega(x, \, x^{-1}, \, a)}$$
$$= 1,$$

for all $x \in G, a \in H_{\mathcal{L}} \cap R$. In the second equality we used the definition of β_a. In the third equality we used (5.10) with $(g_1, \, g_2, \, g_3, \, g_4) = (a, \, x, \, x^{-1}ax, \, x^{-1})$ and used the fact that $\theta_{(a, \chi)} = 1$. In the fourth equality we used (5.10) with $(g_1, \, g_2, \, g_3, \, g_4) = (a, \, x, \, x^{-1}, \, a)$.

The map $B_{\mathcal{L}}$ satisfies (5.19) because $B_{\mathcal{L}}(a, xhx^{-1}) = \frac{\chi(xhx^{-1})}{\deg \chi}$, for all $a \in H \cap R, x \in G, h \in H$. We have $B_{\mathcal{L}}(a, h_1)B_{\mathcal{L}}(a, h_2) = \frac{\chi(h_1)}{\deg \chi} \frac{\chi(h_2)}{\deg \chi} = \beta_a(h_1, h_2)\frac{\chi(h_1 h_2)}{\deg \chi} = \beta_a(h_1, h_2)B_{\mathcal{L}}(a, h_1 h_2)$, for all $a \in H \cap R, h_1, h_2 \in H$. The second last equality above is because H acts as scalars on the projective β_a-representation of $C_G(a)$ whose projective character is χ. By Lemma 5.2.10 it follows that $B_{\mathcal{L}} \in \Lambda_\omega^2 H$ and the Proposition is proved. $\blacksquare$

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Theorem 5.2.12. *Lagrangian subcategories of the representation category of the twisted double $D^\omega(G)$ are classified by pairs (H, B), where H is a normal Abelian subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $B : H \times H \to k^\times$ is a G-invariant alternating ω-bicharacter in the sense of Definition 5.2.6.*

Proof. The proof is completely similar to the one given in Theorem 5.1.5. ∎

5.2.2 Bijective correspondence between Lagrangian subcategories and module categories with pointed duals

Let H be a subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial. Consider the set $\{\mu \in C^2(H, k^\times) \mid \delta^2\mu = \omega|_{H \times H \times H}\}$. An element μ of the previous set satisfies:

$$\mu(h_2, h_3)\, \mu(h_1 h_2, h_3)^{-1}\, \mu(h_1, h_2 h_3)\, \mu(h_1, h_2)^{-1} = \omega(h_1, h_2, h_3). \tag{5.25}$$

for all $h_1, h_2, h_3 \in H$.

We will say that two elements of $\{\mu \in C^2(H, k^\times) \mid \delta^2\mu = \omega|_{H \times H \times H}\}$ are equivalent if they differ by a coboundary. Let

$$\Omega_{H,\omega} := \text{equivalence classes of } \left\{\mu \in C^2(H, k^\times) \mid \delta^2\mu = \omega|_{H \times H \times H}\right\}. \tag{5.26}$$

For each $x \in G$, define $\nu_x : G \times G \to k^\times$ by

$$\nu_x(g_1, g_2) := \frac{\omega(g_1, g_2, x)\omega(g_1 g_2 x g_2^{-1} g_1^{-1}, g_1, g_2)}{\omega(g_1, g_2 x g_2^{-1}, g_2)}, \qquad \text{for all } g_1, g_2 \in G.$$

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It is easy to verify that the following relation holds:

$$\frac{\Upsilon_{x_1 x_2}(g_1, g_2)}{\Upsilon_{x_1}(x_2 g_1 x_2^{-1}, x_2 g_2 x_2^{-1}) \Upsilon_{x_2}(g_1, g_2)} = \frac{\nu_{g_1}(x_1, x_2) \nu_{g_2}(x_1, x_2)}{\nu_{g_1 g_2}(x_1, x_2)}, \quad \text{for all } x_1, x_2, g_1, g_2 \in G,$$

$$(5.27)$$

where Υ is defined in (3.2).

Suppose that H is normal in G. For any $x \in G$ and $\mu \in C^2(H, k^\times)$ such that $\delta^2 \mu = \omega|_{H \times H \times H}$, define $\mu \vartriangleleft x := \mu^x \times \Upsilon_x|_{H \times H}$. It is easy to verify that $\delta^2(\mu \vartriangleleft x) = \omega|_{H \times H \times H}$. This induces an action of G on $\Omega_{H,\omega}$ (defined in (5.26)). Indeed, that this is an action follows from (5.27). Let $(\Omega_{H,\omega})^G$ denote the set of G-invariant elements of $\Omega_{H,\omega}$, i.e.,

$$(\Omega_{H,\omega})^G := \left\{ \mu \in \Omega_{H,\omega} \ \middle| \ \frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H} \text{ is trivial in } H^2(H, k^\times), \text{ for all } x \in G \right\}.$$

$$(5.28)$$

It can be deduced from Theorem 3.1.5 that the set of equivalence classes of indecomposable module categories over Vec_G^ω such that the dual is pointed is in bijection with the set of all pairs (H, μ), where H is a normal Abelian subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $\mu \in (\Omega_{H,\omega})^G$.

Theorem 5.2.12 showed that the set of Lagrangian subcategories of $\mathrm{Rep}(D^\omega(G))$ is in bijection with the set of pairs (H, B), where H be a normal Abelian subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial and $B \in (\Lambda_\omega^2 H)^G$ (defined in (5.20)).

In this Subsection we will first show that the set of equivalence classes of indecomposable module categories over Vec_G^ω such the dual is pointed is in bijection with the set of Lagrangian subcategories of $\mathrm{Rep}(D^\omega(G))$. We will establish the aforementioned

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bijection by showing that there is a bijection between $\Omega_{H,\omega}$ (defined in (5.26)) and $\Lambda_\omega^2 H$ (defined in (5.20)) that restricts to a bijection between $(\Omega_{H,\omega})^G$ and $(\Lambda_\omega^2 H)^G$, where H is any normal Abelian subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial.

Let H be a normal Abelian subgroup of G such that $\omega|_{H \times H \times H}$ is cohomologically trivial. Let $\mu \in C^2(H, k^\times)$ be a 2-cochain satisfying $\delta^2\mu = \omega|_{H \times H \times H}$. Define $alt'(\mu)$ by

$$alt'(\mu)(h_1, h_2) := \frac{\mu(h_2, h_1)}{\mu(h_1, h_2)}.$$

Lemma 5.2.13. The map $alt'(\mu) : H \times H \to k^\times$ defined above is an element of $\Lambda_\omega^2 H$.

Proof. Clearly $alt'(\mu)(h_1, h_2) = alt'(\mu)(h_2, h_1)^{-1}$ and $alt'(\mu)(h, h) = 1$, for all $h, h_1, h_2 \in H$. We have

$$
\begin{aligned}
\frac{alt'(\mu)(h, h_1)\ alt'(\mu)(h, h_2)}{alt'(\mu)(h, h_1 h_2)} &= \frac{\mu(h_1, h)}{\mu(h, h_1)} \times \frac{\mu(h_2, h)}{\mu(h, h_2)} \times \frac{\mu(h, h_1 h_2)}{\mu(h_1 h_2, h)} \\
&= \frac{\mu(h_1, h)\mu(h_2, h)\mu(hh_1, h_2)}{\mu(h, h_2)\mu(h_1 h_2, h)\mu(h_1, h_2)} \times \omega(h, h_1, h_2) \\
&= \frac{\mu(h_1, h)\mu(hh_1, h_2)}{\mu(h, h_2)\mu(h_1, hh_2)} \times \omega(h, h_1, h_2) \times \omega(h_1, h_2, h) \\
&= \frac{\omega(h, h_1, h_2)\omega(h_1, h_2, h)}{\omega(h_1, h, h_2)} \\
&= \beta_h(h_1, h_2),
\end{aligned}
$$

for all $h, h_1, h_2 \in H$. In the second, third, and fourth equalities above we used (5.25) with $(h_1, h_2, h_3) = (h, h_1, h_2), (h_1, h_2, h), (h_1, h, h_2)$, respectively. ∎

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The map alt' induces a map between $\Omega_{H,\omega}$ and $\Lambda^2_\omega H$. By abuse of notation we denote this map also by alt':

$$alt' : \Omega_{H,\omega} \to \Lambda^2_\omega H : \mu \mapsto alt'(\mu). \tag{5.29}$$

Lemma 5.2.14. The map alt' defined above is a bijection.

Proof. First note that alt' is well-defined. Fix $\mu_0 \in C^2(H, k^\times)$ satisfying $\delta^2 \mu_0 = \omega|_{H \times H \times H}$. Let $B_0 := alt'(\mu_0)$. Define bijections $f_1 : \Lambda^2_\omega H \xrightarrow{\sim} \Lambda^2 H : B \mapsto \frac{B}{B_0}$ and $f_2 : \Omega_{H,\omega} \xrightarrow{\sim} H^2(H, k^\times) : \mu \mapsto \left(\frac{\mu}{\mu_0}\right)$. Note that the cardinality of the two sets $\Omega_{H,\omega}$ and $\Lambda^2_\omega H$ are equal. Injectivity, and hence bijectivity, of alt' follows from the equality $f_1 \circ alt' = alt \circ f_2$.

∎

Lemma 5.2.15. The following relation holds:

$$\frac{\Upsilon_x(h_2, h_1)}{\Upsilon_x(h_1, h_2)} = \frac{\beta_{xh_1x^{-1}}(x, h_2)\beta_{xh_1x^{-1}}(xh_2, x^{-1})}{\beta_{xh_1x^{-1}}(x, x^{-1})}, \text{ for all } x \in G, h_1, h_2 \in H.$$

Proof. We have

$$\frac{\Upsilon_x(h_2, h_1)}{\Upsilon_x(h_1, h_2)} \times \frac{\beta_{xh_1x^{-1}}(x, x^{-1})}{\beta_{xh_1x^{-1}}(x, h_2)\beta_{xh_1x^{-1}}(xh_2, x^{-1})}$$

$$= \frac{\omega(xh_2x^{-1}, xh_1x^{-1}, x)}{\omega(xh_2x^{-1}, x, h_1)\omega(xh_1x^{-1}, xh_2x^{-1}, x)} \times \frac{\omega(xh_1x^{-1}, x, x^{-1})\omega(x, x^{-1}, xh_1x^{-1})}{\omega(x, h_1, x^{-1})}$$

$$\times \frac{\omega(xh_2, h_1, x^{-1})}{\omega(xh_1x^{-1}, xh_2, x^{-1})\omega(xh_2, x^{-1}, xh_1x^{-1})}$$

$$= \frac{\omega(xh_2x^{-1}, xh_1x^{-1}, x)\omega(xh_1x^{-1}, x, x^{-1})\omega(x, x^{-1}, xh_1x^{-1})\omega(xh_2x^{-1}, xh_1, x^{-1})}{\omega(xh_1x^{-1}, xh_2x^{-1}, x)\omega(xh_1x^{-1}, xh_2, x^{-1})\omega(xh_2, x^{-1}, xh_1x^{-1})\omega(xh_2x^{-1}, x, h_1x^{-1})}$$

$$= \frac{\omega(x, x^{-1}, xh_1x^{-1})\omega(xh_1h_2x^{-1}, x, x^{-1})}{\omega(xh_1x^{-1}, xh_2x^{-1}, x)\omega(xh_1x^{-1}, xh_2, x^{-1})\omega(xh_2, x^{-1}, xh_1x^{-1})\omega(xh_2x^{-1}, x, h_1x^{-1})}$$

$$= \frac{\omega(x,\, x^{-1},\, xh_1x^{-1})\omega(xh_2x^{-1},\, x,\, x^{-1})}{\omega(xh_2,\, x^{-1},\, xh_1x^{-1})\omega(xh_2x^{-1},\, x,\, h_1x^{-1})}$$

$$= 1,$$

for all $x \in G, h_1, h_2 \in H$. In the first equality above we used the definition of Υ and

β and canceled some factors. In the second, third, fourth, and fifth equalities we used

(5.10) with $(g_1, g_2, g_3, g_4) = (xh_2x^{-1},\, x,\, h_1,\, x^{-1})$, $(xh_2x^{-1},\, xh_1x^{-1},\, x,\, x^{-1})$,

$(xh_1x^{-1},\, xh_2x^{-1},\, x,\, x^{-1})$, and $(xh_2x^{-1},\, x,\, x^{-1},\, xh_1x^{-1})$, respectively. ∎

Lemma 5.2.16. The map alt' defined in (5.29) restricts to a bijection between

$(\Omega_{H,\omega})^G$ and $(\Lambda_\omega^2 H)^G$.

Proof. Let us first show that $alt'((\Omega_{H,\omega})^G) \subseteq (\Lambda_\omega^2 H)^G$. Pick any $\mu \in (\Omega_{H,\omega})^G$. So

$alt\left(\frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H}\right) = 1$, for all $x \in G$. We have

$alt'(\mu)(x^{-1}ax,\, h) \times alt'(\mu)(a,\, xhx^{-1})^{-1}$

$$= \frac{\mu(h,\, x^{-1}ax)}{\mu(x^{-1}ax,\, h)} \times \frac{\mu(a,\, xhx^{-1})}{\mu(xhx^{-1},\, a)}$$

$$= \frac{\mu^x(x^{-1}ax,\, h)}{\mu(x^{-1}ax,\, h)} \times \frac{\mu(h,\, x^{-1}ax)}{\mu^x(h,\, x^{-1}ax)}$$

$$= alt\left(\frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H}\right)(h,\, x^{-1}ax) \times \frac{\Upsilon_x(h,\, x^{-1}ax)}{\Upsilon_x(x^{-1}ax,\, h)}$$

$$= \frac{\Upsilon_x(h,\, x^{-1}ax)}{\Upsilon_x(x^{-1}ax,\, h)}$$

$$= \frac{\beta_a(x,\, h)\beta_a(xh,\, x^{-1})}{\beta_a(x,\, x^{-1})},$$

for all $x \in G, a \in H \cap R, h \in H$. In the fourth equality above we used the fact

that $alt\left(\frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H}\right) = 1$ and in the fifth equality we used Lemma 5.2.15. So

$alt'((\Omega_{H,\omega})^G) \subseteq (\Lambda_\omega^2 H)^G$, as desired.

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Now let us show that $(\Lambda_\omega^2 H)^G \subseteq alt'((\Omega_{H,\omega})^G)$. Pick any $\mu \in \Omega_{H,\omega}$ and suppose that $alt'(\mu) \in (\Lambda_\omega^2 H)^G$. Suffices to show that $alt\left(\frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H}\right) = 1$, for all $x \in G$. Let $B := alt'(\mu)$. We have

$$alt\left(\frac{\mu^x}{\mu} \times \Upsilon_x|_{H \times H}\right)(h_1, h_2) \times \frac{\Upsilon_x(h_1, h_2)}{\Upsilon_x(h_2, h_1)}$$

$$= B(xh_1x^{-1}, xh_2x^{-1})B(h_1, h_2)^{-1}$$

$$= B((yx^{-1})^{-1}a(yx^{-1}), xh_2x^{-1})B(y^{-1}ay, h_2)^{-1} \qquad \text{(where } h_1 = y^{-1}ay)$$

$$= \frac{\beta_a(yx^{-1}, xh_2x^{-1})\beta_a(yh_2x^{-1}, xy^{-1})}{\beta_a(yx^{-1}, xy^{-1})} \times \frac{\beta_a(y, y^{-1})}{\beta_a(y, h_2)\beta(yh_2, y^{-1})}$$

$$= \frac{\beta_a(yx^{-1}, xh_2)\beta_a(yh_2, x^{-1})\beta_a(yh_2x^{-1}, xy^{-1})}{\beta_{xh_1x^{-1}}(xh_2, x^{-1})\beta_a(y, y^{-1})\beta_a(yx^{-1}, xy^{-1})\beta_a(y, h_2)\beta_a(yh_2, y^{-1})}$$

$$= \frac{\beta_a(yx^{-1}, x)\beta_a(yh_2, x^{-1})\beta_a(yh_2x^{-1}, xy^{-1})\beta_a(y, y^{-1})}{\beta_{xh_1x^{-1}}(x, h_2)\beta_{xh_1x^{-1}}(xh_2, x^{-1})\beta_a(yx^{-1}, xy^{-1})\beta_a(yh_2, y^{-1})}$$

$$= \frac{\beta_{xh_1x^{-1}}(x, x^{-1})\beta_a(yh_2, x^{-1})\beta_a(yh_2x^{-1}, xy^{-1})\beta_a(y, y^{-1})}{\beta_{xh_1x^{-1}}(x, h_2)\beta_{xh_1x^{-1}}(xh_2, x^{-1})\beta_a(y, x^{-1})\beta_a(yx^{-1}, xy^{-1})\beta_a(yh_2, y^{-1})}$$

$$= \frac{\Upsilon_x(h_1, h_2)}{\Upsilon_x(h_2, h_1)} \times \frac{\beta_a(y, y^{-1})\beta_{h_1}(x^{-1}, xy^{-1})}{\beta_a(y, x^{-1})\beta_a(yx^{-1}, xy^{-1})}$$

$$= \frac{\Upsilon_x(h_1, h_2)}{\Upsilon_x(h_2, h_1)},$$

for all $x \in G, h_1, h_2 \in H$. In the fourth through eight equalities above we used (5.12) with $(x, y, z) = (yx^{-1}, xh_2, x^{-1}), (yx^{-1}, x, h_2), (yx^{-1}, x, x^{-1}), (yh_2, x^{-1}, xy^{-1})$, and (y, x^{-1}, xy^{-1}), respectively. It follows that $(\Lambda_\omega^2 H)^G \subseteq alt'((\Omega_{H,\omega})^G)$ and the Lemma is proved. ∎

Recall that $\mathrm{E}(\mathcal{C})$ denotes the set of (equivalence classes of) braided tensor equivalences between a modular category $\mathcal{C}$ and the centers of pointed fusion categories.

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Theorem 5.2.17. *The assignment $\mathcal{M} \mapsto \iota_{\mathcal{M}}$ (defined in (5.9)) restricts to a bijection between equivalence classes of indecomposable Vec_G^ω-module categories $\mathcal{M}$ with respect to which the dual fusion category $(Vec_G^\omega)_{\mathcal{M}}^*$ is pointed and $E(Rep(D^\omega(G)))$.*

Proof. The proof is completely similar to the one given in Theorem 5.1.6. ∎

Theorem 5.2.18. *Let $\mathcal{C}_1, \mathcal{C}_2$ be group-theoretical fusion categories. Then $\mathcal{C}_1, \mathcal{C}_2$ are weakly Morita equivalent if and only if their centers $\mathcal{Z}(\mathcal{C}_1)$ and $\mathcal{Z}(\mathcal{C}_2)$ are equivalent as braided fusion categories.*

Proof. The "if" part is true for all fusion categories by [EO]. For the "only if" part, let $(G_1, \omega_1), (G_2, \omega_2)$ be two pairs of groups and 3-cocycles such that $\mathcal{C}_1$ is weakly Morita equivalent to $Vec_{G_1}^{\omega_1}$ and $\mathcal{C}_2$ is weakly Morita equivalent to $Vec_{G_2}^{\omega_2}$. If $\mathcal{Z}(\mathcal{C}_1) \cong \mathcal{Z}(\mathcal{C}_2)$ (as braided fusion categories) then $\mathcal{Z}(Vec_{G_1}^{\omega_1}) \cong \mathcal{Z}(Vec_{G_2}^{\omega_2})$ (as braided fusion categories) and therefore, $Vec_{G_1}^{\omega_1}$ and $Vec_{G_2}^{\omega_2}$ are weakly Morita equivalent by Theorem 5.2.17 and hence, $\mathcal{C}_1$ and $\mathcal{C}_2$ are weakly Morita equivalent. ∎

Corollary 5.2.19. *Let G, G' be finite groups, $\omega \in Z^3(G, k^\times)$, and $\omega' \in Z^3(G', k^\times)$. Then the representation categories of twisted doubles $D^\omega(G)$ and $D^{\omega'}(G')$ are equivalent as braided tensor categories if and only if G contains a normal Abelian subgroup H such the following conditions are satisfied:*

(1) $\omega|_{H \times H \times H}$ is cohomologically trivial,

(2) there is a G-invariant (see (5.28)) 2-cochain $\mu \in C^2(H, k^\times)$ such that that $\delta^2\mu = \omega|_{H \times H \times H}$, and

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(3) there is an isomorphism $a : G' \xrightarrow{\sim} \widehat{H} \rtimes_\nu (H \backslash G)$ such that $\varpi \circ (a \times a \times a)$ and

ω' are cohomologically equivalent.

Here ν is a certain 2-cocyle in $Z^2(H \backslash G, \widehat{H})$ coming from the G-invariance of μ and

ϖ is a certain 3-cocycle on $\widehat{H} \rtimes_\nu (H \backslash G)$ depending on ν and on the exact sequence

$1 \to H \to G \to H \backslash G \to 1$ (see Theorem 3.3.8 for precise definitions).

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