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Understanding abstract algebra concepts

Anna S. Titova
University of New Hampshire, Durham

UNDERSTANDING ABSTRACT ALGEBRA CONCEPTS

BY

ANNA S. TITOVA

BS, Russian State Pedagogical University, St. Petersburg, Russia, 1998

MS, University of New Hampshire, 2005

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This dissertation has been examined and approved.

Sonia Hristovitch

Dissertation Director, Dr. Sonia Hristovitch

Assistant Professor of Mathematics

Karen J Graham

Dr. Karen J. Graham

Professor of Mathematics

Dr. Dmitri Nikshych

Associate Professor of Mathematics

Edward Hinson

Dr. Edward Hinson

Associate Professor of Mathematics

Maria Basterra

Dr. Maria Basterra

Assistant Professor of Mathematics

05/10/2007

Date

DEDICATION

for Dasha and Andrei Smuk;
Ludmila Titova, Sergei Titov;
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who make me and my life possible,
for their continued encouragement and support

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ABSTRACT

UNDERSTANDING OF ABSTRACT ALGEBRA CONCEPTS

by

Anna Titova

University of New Hampshire, May, 2007

This study discusses various theoretical perspectives on abstract concept formation. Students' reasoning about abstract objects is described based on theoretical proposition that abstraction is a shift from abstract to concrete. Existing literature suggested a theoretical framework for the study. The framework describes process of abstraction through its elements: assembling, theoretical generalization into abstract entity, and articulation. The elements of the theoretical framework are identified from students' interpretations of and manipulations with elementary abstract algebra concepts including the concepts of binary operation, identity and inverse element, group, subgroup, cyclic group. To accomplish this, students participating in the abstract algebra class were observed during one semester. Analysis of interviews conducted with seven students and written artifacts collected from seventeen participants revealed different aspects of students' reasoning about abstract objects. Discussion of the analysis allowed formulating characteristics of processes of abstraction and generalization.

The data showed that the students often find it difficult to reason about abstract algebra concepts. They prefer to deal with "concrete" objects and often are confused if the problem is stated in more general terms. Moreover, number of students based their arguments about a certain object on their understanding of a concrete structure. For example, some students said that if integer 1 does not belong to a given structure then this structure cannot be a group. Also, since abstract algebra concepts are complex structures, participating students repeatedly missed some elements of these structures during problem solving. One of the frequently missed elements was quantification of objects. Students often were confused how to use quantifiers.

The study elaborates on these problems and offers theoretical explanations of the difficulties. The explanations, therefore, provide implications for instructions and future research.

INTRODUCTION

Abstract thought is considered to be the highest accomplishment of the human's intellect as well as its most powerful tool (Ohlsson, Lehitinen, 1997). Most people consider mathematics as abstract and it is difficult to argue with this opinion. Even though mathematical problems can be solved by guessing, trial and error, experimenting (Halmos, 1982), there is still a need for abstract thought.

One can notice that there exists a fear of mathematics, when students believe that mathematics is not for them because they failed to understand something on the first try. There is support (Ferguson, 1986) to the hypothesis that abstraction anxiety is an important factor of mathematics anxiety, especially in topics which are introduced in the middle grades. Students' statements such as "I understand 2 and 3, but I don't understand x and y." were observed by Ferguson (1986). I think if we understand the nature of abstraction, its acquisition, we can help students not to reduce the level of abstraction but, to the contrary, to bridge the gap from the abstract to concrete. This study aims to explore the process of abstraction, to give a description of its components and outcomes. My goal is to understand certain cognitive processes through empirical observations including classroom observations, interviews, and written artifacts collection. During the study I observed certain learning phenomena and made a theoretical analysis of these phenomena. "A theory should help us say that if certain phenomena are observed, then other phenomena, are likely to occur as consequences" (Dubinsky, 2000a; p.11). I select a qualitative approach to my research. I attempt to analyze students' construction of knowledge (knowledge of abstract/mathematical object, in particular) in the tradition of a grounded theory (Charmaz, 2003; Glaser & Strauss, 1967) in the content of group theory.

The importance of knowing abstract algebra and group theory in particular, is widely acknowledged. Undergraduate students use mathematical ideas, learned in these courses, in many scientific areas, such as physics, computer science and chemistry. Abstract algebra is also is an essential part of middle and secondary school teachers' preparation. However, students often find the course difficult and many researches indicate problems and gaps in students' understanding of group theory concepts (Dubinsky et al, 1994, 1997; Hazzan, 1999; Nardi, 2000). Only recently has serious research been directed towards learning and teaching of abstract algebra: "A literature search revealed 15 articles on the learning of abstract algebra. Eleven of them had been published since 1994, of which 9 grew from the work of Dubinsky, Leron, and their collaborators" (Findell, 2001; p. 6). These arguments reason my choice of investigating students' cognitive processes of abstraction and generalization in the context of group theory. The goal of this study is not to fill all possible gaps but to suggest some theoretical constructs which may help to understand and possibly avoid some of students' difficulties in the future.

The following chapter reviews theoretical perspectives on the notion of abstraction and generalization and on students' learning of group theory concepts. Chapter 2 describes theoretical guiding of the study or the theoretical framework. Chapter 3 and 4 propose research questions and methodology for searching answers. Detailed data analysis, illustrated by the data excerpts is described in Chapter 5, followed by the discussion of the findings and main conclusions in Chapter 6. Finally, I share my ideas for implications of the study for teaching and future research in Chapter 7.

CHAPTER I

LITERATURE REVIEW

This chapter includes a review of recent and classical theories on learning abstract objects. Studies that were guided by these perspectives are also discussed.

Philosophical View

There exist many definitions of abstraction. I would like to start this literature review with a brief description of the first known perspectives on this notion. Attempts by philosophers and cognitive psychologists to explore the meaning of the process of abstraction date back to Aristotle and Plato. If we think about an object as concrete as an apple it does not mean we necessarily must see the object in front of us. In other words, we would call this mental "apple" abstract. Abstract entity, in a philosophical sense, is "an object lacking spatiotemporal properties, but supposed to have being, to exist... Abstracta, sometimes collected under the category of universals, include mathematical objects, such as numbers, sets, and geometrical figures, propositions, properties, and relations... The abstract triangle has only properties common to all triangles" (Audi, 1999, p.3). From the empirical point of view, abstract ideas are universals, i.e. relations, types or properties of objects:

> We should only know what is now present to our senses: we could not know anything about the past - not even that there was a past - nor could we know any truths about our sense-data, for all knowledge of truths, as we shall show, demands acquaintance with things which are of an essentially different character from sense-data, the things which are sometimes called 'abstract ideas', but which we shall call 'universals'. We have therefore to consider acquaintance with other things besides sense-data if we are to obtain any tolerably adequate analysis of our knowledge. (Russell, 1998, p.22)

For example (Russell, 1998) in geometry, when we wish to prove something about all triangles, we draw a particular triangle yet we do not to use any characteristics which it does not share with every other triangle. According to Russell,

> The beginner, in order to avoid error, often finds it useful to draw several triangles, as unlike each other as possible, in order to make sure that his reasoning is equally applicable to all of them. But a difficulty emerges as soon as we ask ourselves how we know that a thing is a triangle. If we wish to avoid the universal triangularity, we shall choose some particular triangle, and say that anything is a triangle if it has the right sort of resemblance to our chosen particular. But then the resemblance required will have to be a universal (p.46).

Hence, for this philosopher, abstractions are created by extracting all properties common for all objects (universals), which the individual has experienced. These abstractions are needed to recognize a specific object among other objects.

Classical View by J. Piaget

Respecting the classico-philosophical point of view, Piaget (1970a, 1970b) says that mathematical abstraction (as also abstraction, marked by extraordinarily detailed and vivid recall of visual images, which gives rise to the knowledge of universals) implies certain operations, which were not considered by Aristotle.

Piaget (1970a) is searching for the answer to the very difficult and important question about the formation of human knowledge. Referring to the classical view of the problem researchers wonder if all cognitive information has its source in objects, so that the learner is "instructed" by the objects or another individual in the world outside him; or, on the contrary, the subject possesses a form produced or growing from within structures which the subject imposes on objects. The first assumption is coming from the traditional empiricism; the second is maintained by the varieties of a priories or innatism. Although these are two different statements, Piaget noticed common trends of established epistemologies: there exists a subject aware of its power in various degrees (even just a perception of objects); objects exist for a subject (even such object as 'phenomena'); and the most important is a mediation between the subject and the object and vice versa.

Further, Piaget (1970a) studies characteristics of cognition: association and assimilation. He criticizes the concept of association by claiming that this concept only refers to an external bond between the associated elements (it does not give a learner a deep knowledge about object and its properties), "whereas the idea of assimilation implies that of the integration of the given within a prior structure or even a formation of a new structure under the elementary form of a scheme" (p.22).

Piaget distinguishes three aspects of the process of assimilation: repetition, recognition and generalization, which can closely follow each other. By repetition he understands a reproduction of the same movement (or reproductive assimilation) and the formation of the beginning of a cognitive scheme; recognition or recognitive assimilation is the applying of previously created scheme to a new object or situation; and when the subject repeats the action in this new situation we deal with generalizing assimilation or generalization.

Even in the very beginning of an individual's cognitive development one can observe the construction of new combinations by a union of abstractions derived either from objects themselves, or from schemes of actions applied to them. For Piaget (1970a) the child's recognition of an object, as something having specific properties, requires an abstraction starting from objects. On the other hand learners' coordination of means and ends, taking to account the proper sequence of required movements, is a new form of behavior "compared with the global acts..." (p.23); but this new behavior is naturally acquired from such acts by a process of deriving from them the relations of order, overlapping, etc., necessary for this coordination. This coordination Piaget does no longer consider as appearance of abstraction from concrete object but rather as an abstraction, which derives higher-order structures from the previously acquired lower-order structures.

Working further, Piaget (1970b) studies the details of the notion of abstraction and discusses it in terms of mathematical knowledge formation and logic. The author agrees that logical and mathematical structures are abstract, whereas physical knowledge – the knowledge based on experience – is concrete. He is struggling to find answers to the question about human knowledge formation and, since mathematical and logical structures are defined as abstract structures, he asks: "*What are these structures abstracted from?*" One view on abstraction comes from an empirical ideology, that is – our knowledge is derived from the concrete object itself; in this case the question remains – What are the concrete objects in mathematics? A second view claims that, since the transformation of the object can be carried mentally, we can take into the account the actions itself or our actions upon the object. In that way the abstraction is derived from the action itself. This position seems for Piaget as a basis of logical and mathematical abstraction. The first type of abstraction from object he defined as a *simple abstraction*, for instance, when a child lifts objects and realizes that smaller objects are usually lighter then bigger objects, so the idea of weight, as a characteristic of an object is abstracted from the objects themselves. The second type he called the *reflective abstraction,* for example when a child counts five marbles in different ways, he notices that it does not matter how he places them, he always gets the same number. This way the child discovers the mathematical property of addition – commutativity, and this knowledge is drawn not from the physical properties of marbles but from the actions the child carried out on the marbles. Piaget uses the word "reflective" in a double sense in terms of psychology and, in addition, of physics. The following citation helps to understand how Piaget makes a distinction between the two types of abstraction,

> On the one hand, there are individual actions such as throwing, pushing, touching, rubbing. It is these Individual actions that give rise most of the time to abstraction from object. This is the simple type of abstraction...Reflective abstraction, however, is based not on individual actions but on coordinated actions. Actions can be coordinated in a number of different ways. (Piaget, 1970, p. 18)

The intermediate step between empirical and reflective abstraction that occurs after the action(s) have taken place on the object, Piaget calls a pseudo-empirical abstraction. During pseudo-empirical abstraction, the subject engages with an external object and extracts properties of the actions introduced into the object during empirical abstraction.

From this argument it is necessary to suppose that abstraction starting from actions and operations – reflective abstraction – differs from abstraction from perceived objects – simple abstraction (in the book "Mathematical Epistemology and Psychology" (Beth, Piaget, 1966) Piaget calls simple abstraction – "empirical abstraction", p. 188-189) - in the sense that reflective abstraction is constructive, while on the contrary simple or empirical abstraction consists of deriving commonalities from class of objects by combination of abstraction and simple generalization. By generalization Piaget (1966, p.243) means "the simple observation that several objects posses a common character."

Thus, Piagetian position on the process by which the subject derives new knowledge from the results of his/her own actions is as follow:

a) logico-mathematical experience consists of observing actions performed upon any object;
b) the results are determined by the schemes of the actions;
c) in order to observe these results, the subject has to perform other actions using the same schemes as those the product of which must be examined;
d) the knowledge acquired is new for the subject;

e) the abstraction, by means of which the subject acquires new knowledge as a result of his actions – involves some construction. (Beth, Piaget, 1966)

From Piagetian (1966, 1970a, 1970b) view, the axiomatization is based on the reflective abstraction. It occurs when a thinker derives conceptually elementary principles, for example identities. In the early stages the axioms were still accepted as intuitive and were borrowed from the natural thought, but later theories become less and less intuitive. More precisely,

> If we analyze the reflective abstraction into 'reflection' in the quasy-geometric sense of the projection of certain previously given relationships on to a new plane of thought, and 'reflection' in the noetic (originating in the intellect) sense of reorganization necessitated by the reconstruction of these relationships on this new plane, then this later aspect prevails over the former (p. 64).

Therefore, according to Piaget, even before the first mathematical entity was formed, the process of reflective abstraction gives rise to the initial concepts and operations in mathematics.

Theories Based on Piagetian Idea of Reflective Abstraction

In his papers about advanced mathematical thinking, Dubinsky (1991a, 1991) proposes that the concept of reflective abstraction, introduced by Piaget, can be a powerful tool in the process of investigating mathematical thinking and advanced thinking in particular.

Dubinsky (1991a) presents a brief description of his theory of mathematical knowledge and its acquisition in the area of mathematics that is more advanced then Piaget considered for his research. He takes a view that knowledge and its acquisition are not easily distinguishable. According to

Dubinsky there are three aspects which must be investigated in order to understand mathematical knowledge and its acquisition: problem situations, schemas (more or less coherent collections of cognitive objects and internal mental processes for manipulating these objects), and responses.

> One must consider the difference in the problem situation as it is intended by the observer and as it appears to the subject. One must understand the nature of schemas and the means by which they are constructed. Finally, it is necessary to explain how the subjects select the schema to be used in the response and what determines the kinds of new constructions (if any) that are made (p.5).

Elaborating further, Dubinsky (1991a) uses two observations made by Piaget to form his general theory: 1) reflective abstraction is present in the very early ages in the coordination of sensory-motor structures; 2) the entire history of the development of mathematics may be considered as an example of the process of reflective abstraction. Although Piaget concentrated on the development of mathematical knowledge at the early age, Dubinsky suggests that the same approach can be extended to more advanced undergraduate mathematical topics such as mathematical induction, predicate calculus, functions, topological spaces and vector spaces, etc., so they all can be analyzed in terms of extensions of the same notions Piaget used. Dubinsky lists various kinds of construction in reflective abstraction, which is heavily based on the work by Piaget: interiorization - the process of construction internal processes as a way of making sense out of perceived phenomena; coordination of two or more processes to construct a new one; encapsulation of a dynamic process into a static object; generalization of existing construction to a wider collection of phenomena; reversing the original processes to construct a new process. The final construction process of reflective abstraction is proposed by Dubinsky (not Piaget).

Dubinsky (1991) extends Piagetian ideas and reorganizes them into a general theory of mathematical knowledge and relates his theory in specific mathematical topics, such as vector spaces and functions, to explain some processes which may occur in the learning process. He proposes a notion of genetic decomposition. According to Dubinsky (1991, p. 96), the genetic decomposition of a concept is a description of the mathematics involved and how a subject might make the construction(s) that would lead to an understanding of the concept. Dubinsky describes examples of genetic decomposition for mathematical induction, predicate calculus, and function.

In terms of educational implications, Dubinsky's position is that learning consists of applying reflective abstraction to existing schemas in order to construct new schemas for understanding concepts, thus the schema can not be constructed in the absence of previously existed schemas. In this way Dubinsky suggests the following instructional approach to foster conceptual thinking in mathematics: 1) observe students to see their developing of concept images; 2) analyze the data and develop a genetic decomposition for each topic; 3) design instruction that moves students through the steps of the genetic decomposition, develop activities that will induce students to make a specific reflective abstractions; 4) repeat the process and continue until stabilization occurs.

Another line of research based on Piagetian theory of reflective abstraction is a study conducted and developed by Goodson–Espy (1998). She observed abstraction process during problem solving and examined the transition that students make from arithmetic to algebra. Although her study is framed in Sfard's (1991) theory of reification, Goodson–Espy's framework includes an idea of reflective abstraction proposed by Piaget and, further, the notion of levels of reflective abstraction, proposed by Cifarelli (as cited in Goodson–Espy, 1998). The first level is defined as Recognition - the ability to recognize characteristics of a previously solved problem in a new situation and to believe that one can do again what one did before. The second level of reflective abstraction is Re-presentation. At this level the student becomes able to run through a problem mentally and is able to anticipate potential sources of difficulty and promise. The third level of reflective abstraction is Structural abstraction. Structural abstraction occurs when a student evaluates solution prospects based on mental 'run-throughs' of potential methods as well as methods that have been used previously. Goodson-Espy in her paper suggests that there are strong relations between the theory of reification and the levels of reflective abstraction. Moreover, she suggests that Cifarelli's levels of reflective abstraction may be used to illustrate how the transition from one stage of concept formation (from reification theory) to the next stage could take place.

Theoretical framework of Nardy's (2000) study lies primarily in the Piagetian concept of reflective abstraction. The author explores the difficulties in students understanding of abstract algebra, group theory, in particular. She refers this understanding to advanced mathematical thinking in Dubinsky's (1991, 1991a) sense. The study reported conceptual difficulties with some concepts of group theory, which resonates with the findings in the area of investigation of learning advanced mathematics. The author and her colleagues are involved in further research in this area.

In his unpublished work, Harel (1995) elaborates more on the notion of abstraction and exemplifies Piagetian empirical (or simple) and reflective abstractions using epistemology of the concept of function. In this paper Harel illustrates some aspects of the theory using examples of interviews he conducted with college students and his observations.

Alternative views on the notion of abstraction

Recently several authors have critically analyzed classical and Piagetian approach and proposed alternative outlook on the notion of abstraction. (Hershkowitz, Schwarz, Dreyfus, 2001; Ohlsson, Lehitinen, 1997; Mitchelmore and White, 1995, 1999; Harel and Tall, 1991).

Making a fresh start, Ohlsson and Lehitinen (1997) approached the problematic of high-order cognition via distinguishing abstraction and generalization. This observation led Ohlsson and Lehitinen (1997) to the reevaluation of the role of generality in learning process. For them "to generalize" means to extract "commonalities from exemplars" (p. 38), while the main cognitive function of abstraction is to enable the assembly of previously existed ideas into more complex structure. Ohlsson and Lehitinen (1997) suggest that "people experience particulars as similar precisely to the extent that, and because, those particulars are recognized as instances of the same abstraction" (p. 41).

By reviewing the classical Aristotelian ideas Ohlsson and Lehitinen (1997) bring our attention to the fact that scientific theories (as good examples of higher order knowledge) do not fit the generalization idea:

> Consider for example, the law of mechanical motion. On the Aristotelian view, Isaac Newton should have arrived at the equation
>
> $$F = m \times a$$
>
> by measuring *F*, *m* and *a* many times in different situations and noticing that the product of *m* and *a* equals *F* in each instance. (p. 38)

The formulation of Darwin's theory, for example, preceded its application to particular cases; hence, this theory can not be generalization in the classical sense. Ohlsson and Lehitinen (1997) conclude:

> In summary, important examples of higher order knowledge in science, mathematics, and other fields are not, in fact, created by extracting commonalities across particular objects or events. In case after case, key ideas...were not as a matter of historical fact, discovered via generalization and could not, even in principle, be discovered that way (p. 40).

Ohlsson and Lehitinen (1997) claim that in order to recognize an object as an instance of an abstraction, the learner must already *possess* that abstraction. In other words, adopting Hayek's (1952, 1978) terminology, the abstract has *primacy* over the concrete. According to Hayek (1952, pp. 42-43; 1978, pp. 165-172), the general concept is a "presupposition" of experience rather than the product of abstraction from what is presented in experience.

The authors assume that the deep idea is complex, i.e. has other ideas as parts. For example, to understand the idea of a group the learner must have an idea of a set, function, and a binary operation. The observation about complex idea suggests that new ideas are created by assembling previously acquired ides into new structure. As a result of assembling we have a new structure, more complex then its components, and it follows that if these components are abstract so is their combination. Hence, according to Ohlsson and Lehitinen, we create a new abstraction, operating on existed abstractions, not on concrete experience.

In this case assembling is not mere association – a link between cue and associate, where activation of the cue evokes the association (Halford at al., 1997). Halford distinguishes two types of associations: (1) Elemental association, which comprises links between pairs of entities; and (2) Configural association, which entails two stimuli each of which modifies the link between the other stimulus and the response. The first type does "not require any representation other than input and output and, therefore, cannot achieve any abstraction" (p.22); the second type "cannot support transfer between problem isomorphs. Therefore, a configural association can achieve only the minimal level of abstraction" (p.22); on the contrary, the result of assembling is a more complex idea – an abstract idea. The application of this complex abstract idea moves from the abstract toward the concrete. The term *articulation* is used by Ohlsson and Lehitinen (1997) to refer to this process. In other words, articulation is a process through which "abstract schema (knowledge structures, which regulate thinking which goes beyond immediate experience) functions as a plan, a form to be filled with content" (Ohlsson, 1993, p. 61).

In the present view, abstraction is prerequisite for learning, whereas generalization is a product of learning process. In fact, abstract ideas are generated from other abstract ideas. But how do learners acquire these abstractions in the first place? Ohlsson and Lehitinen (1997) review several possible answers:

(1) Postulating innate abstraction.

The universal structure of a single object becomes a figure. For example, "mathematicians know only one number system, but careful analysis has produced a representation that has become an object of inquiry in its own right. The result is a new field, abstract algebra" (p.44).

(2) Seeking the origin of initial abstraction in discourse.

(3) Process of induction.

In summary, two cognitive processes were postulated by Ohlsson and Lehitinen (1997). First, to learn a complex idea is to assemble available abstract ideas into new structure. This process moves from the simple toward complex (not from concrete to general). Second, abstractions are applied to concrete objects via articulation. This process moves from abstract toward concrete.

A soviet educator Davydov (1972/1990) approaches the problem of human cognition by distinguishing empirical and theoretical thoughts. To clarify the meaning and the difference between the two kinds, the author starts with the analysis of the nature of human thought in traditional formal logic. In this view, subject is extracting similarities from the set of particular objects (which exists independently of subject), so that particular objects can be combined into a class after comparison according to the sort of similar properties. A class is a mental formation – repeating properties of many objects, which has become a particular and independent object of thoughts. Thus, formal logic defines (formal) abstraction and (formal) generalization as processes of identifying sensorially given, observable, external properties of an individual object.

Formal logic considers a thought as a transition from concrete and individual to the abstract together with the reverse transition. Concrete (as distinct from abstract) is defined as individually given, directly observable object itself. The thought that accomplishes previously mentioned transitions through formal generalization and abstraction forms empirical concepts. This theory is usually called the empirical theory of thinking (the notion is proposed by an English philosopher John Locke) and in these terms formal abstraction and generalization are called empirical abstraction and generalization. These processes solve problems of classifying objects by the external attributes and problems of identifying these attributes.

Davydov (1972/1990) is concerned with limitations of the empirical interpretation of abstraction and generalization. It follows from the definition, that in science, for example, traditional empirical abstraction and generalization is limited by directly observable phenomena. In general, the fundamental weakness of empirical theory is that every concept can be reduced to some concrete data. It means that we can find the appropriate concrete for any abstract attribute. From this position students can learn only what they can observe and experience (together with the teacher's knowledge, which is imposed on students' life experience).

Davydov (1972/1990) argues that scientific knowledge is not a simple extension and expansion of people's everyday experiences. "It requires the cultivation of particular means of abstraction, a particular analysis, and generalization, which permits the internal connections of things, their essence and particular ways of idealizing the objects of cognition to be established" (p. 86). Following this argument, the author proposes a theoretical approach to the theory of thinking – theoretical abstraction and generalization.

Theoretical abstraction is a theoretical analysis of objects (concrete or previously abstracted) and construction of a system that outlines the whole picture of the new concept being studied so it is ready to be applied for the correct recognition of particular objects. Theoretical generalization defined as a process of identification of deep, structural similarities, which identify the inner connections with previously learned ideas. According to Davydov (1972/1990), theoretical abstraction is linked to theoretical generalization in a following way: theoretical abstraction starts from initial abstracts – ready-made empirical abstractions; "the investigator can find it only in studying actual data and their relationships" (p.289). Further, from the simple, undeveloped, inconsistent first form of abstraction, the development proceeds with the analysis of these initials to obtain the necessary (theoretical) generalizations, which then will be synthesized to obtain a consistent final form – abstract idea. So, for Davydov, the process of abstraction does not proceed from concrete to abstract, but from undeveloped to developed abstraction, which allows learners to see the new features in concrete objects, when this abstraction is applied.

If the transition from general and abstract to particular has been mastered, then students bridge the gap between the concrete and the abstract. For

Davydov (1972/1990), "the more abstract the initial generalization, the more concretization its thorough mastery requires" (p.23). In other words, the stage of transition from abstract to particular should include more concrete problems for better articulation. The development of abstract, thus, depends on the accumulation of conceptions and perceptions.

In summary, Davydov (1972/1990) distinguishes two types of abstraction and generalization: theoretical and empirical. However, theoretical and empirical processes are linked to each other. From his argument it follows that for the learning of mathematics empirical theory is not enough. The main characteristics of empirical and theoretical thoughts are summarized in Table 1:

Empirical	Simple	External
Theoretical	Complex	Internal

Table 1. Characteristics of Thoughts.

Mitchelmore and White (1999) constructed the theoretical framework following Davydov's principles of generalization and abstraction, borrowing the notion of content-related or theoretical generalization. Also Hershkowitz, Schwarz, Dreyfus (2001) proposing an approach to the theoretical and empirical identification of a process of abstraction, build their functional definition of abstraction on Davydov's theory.

Mitchelmore and White (1995) noted that students often divide mathematical problems into two categories: "abstract" and "real-life" or "concrete" problems. Although "concrete" usually means an easier problem – a problem involving concrete objects, students seemed to prefer "abstract" exercises, given in a symbolic form. On the other hand, according to the authors, there is strong evidence that abstractness of mathematics is its well-known difficulty. Ideas become more difficult as they become more abstract. Mitchelmore and White see a conflict here and to find an explanation for this phenomenon they turn to the very definitions of abstraction and abstract:

> *Abstract* (adj): Apart from the concrete; general as opposed to particular; expressed without references to particular examples.
> *Abstract idea:* Mental representation or concept that isolates and generalizes an aspect of an object or group of objects from which relationships may be perceived.
> *Abstract* (ver.): To consider apart from particular instances; to form a general notion of.

In terms of these definitions, the authors distinguish two types of abstraction: "abstract-general" – when a mathematical idea is linked to concrete objects (or other mathematical ideas); and "abstract-apart" – when a mathematical idea is separated from the context.

Reviewing the theories and studies of abstraction conducted by other researchers, the authors derived a general abstraction cycle: recognition → manipulation → reification, where by reification they mean a process of converting a concept into an object of thought, extending Sfard's definition. Mitchelmore and White believe that the degree of abstraction increases as their "abstraction cycle" is repeated several times:

> For example, the physical act of counting is reified to whole numbers. Other numbers such as fraction and negative numbers are then reified and a general concept of number emerges. Study of the properties of number systems and other similar structures leads eventually to concepts such as groups, fields and spaces, each concept encapsulating a specific set of properties present in the various systems. Finally, the concept of a category is formed to abstract the common features of all such structures. Each step is an abstraction, and each new concept is experienced as more abstract than the concepts from which it is abstracted.

Dienes (as cited in Mitchelmore and White, 1995), states that the degree of abstraction of a concept is in direct proportion to the amount of variety of the experiences from which it has been abstracted. Hiebert and Lefevre (as cited in Mitchelmore and White, 1995) state that abstractness increases as knowledge becomes freed from specific contexts. In Mitchelmore's and White's opinion those statements are equating abstractness and generality.

Mitchelmore (1994), stimulated by Skemp's (as cited in Mitchelmore 1994) work, proposed a model of conceptual development, consisting of two important phases: abstraction and generalization. Generalization appears as a never-ending process as more and more situations are brought in under the same abstraction. Later, however, White and Mitchelmore (1999) critically analyzed generalization as a shift from concrete to abstract, where students are involved into their pattern-seeking activities. They pointed out that, from this perspective, teaching model states "always proceed from particular to general". This model is criticized by White and Mitchelmore with support of Davydov's definition of generalization. The critique comes from the statement that classification of objects on the basis of external characteristics does not identify inner connections.

In summary, the authors believe in "abstract – to – concrete" ("general – to – particular") learning order. Moreover, they do not equate abstraction and generalization

> We note that the terms generalization and abstraction are often used interchangeably in the literature. The essential difference, as we see it, is that abstraction creates a new mental object (a concept) whereas generalization extends the meaning of an existing concept. The act of abstracting is based on generalizing, but is seen as qualitatively different from simply identifying patterns

in a set of examples. It is a many to one function where generalizations are synthesized from many inputs to form a new abstraction (p. 5).

Harel and Tall (1991) were also investigating the meaning of the terms abstraction and generalization. They defined generalization as a process of applying a given argument in a broader context. In mathematics, in particular, this process depends on the individual's current knowledge. From their observations of the students, the authors distinguish three different kinds of generalization, depending on individual's mental constructions:

1. Expansive generalization occurs when the subject expands the applicability range of an existing schema without reconstructing it.
2. Reconstructive generalization occurs when the subject reconstructs an existing schema in order to widen its applicability range.
3. Disjunctive generalization occurs when, on moving from the familiar context to a new one, the subject constructs a new, disjoint schema to deal with the new context and adds it to the array of schemas available. (p. 38)

The last type of generalization is not considered by Harel and Tall as a cognitive generalization "in the sense that the earlier examples are not seen by the individual as special cases of the general procedure" (p. 38), however the first two seem for them more appropriate for cognitive development. Also they argue that in a short term expansive generalization is cognitively easier, but in a long run there are times when reorganization of knowledge becomes essential which means that reconstructive generalization becomes more appropriate.

Harel and Tall (1991) define abstraction as a process which occurs when subject focuses attention on specific properties of a given object and considers these properties as isolated from the original. The authors attribute the abstraction theory to reconstructive generalization, "because the abstracted properties are reconstructions of the original properties, now applied to broader domain" (p.39). In mathematics, abstraction of specific properties to form the basis of the definition of a new mathematical object is one constituent of the two distinct processes which form the process of formal definition. The second process is construction of an abstract concept through logical deduction from the definition. Harel and Tall call the first process formal abstraction – the abstraction of a new concept through the selection of properties of one or more specific situations. They, however, admit the difficulty of a formal abstraction for the learner and to help students to pass the difficulties they introduce another form of abstraction – generic abstraction. In this case concept formation starts with so-called prototypes – more specific examples, so students can see the properties required for the new concept and apply it to a wider range of examples, embodying an abstract concept.

The next section discusses how some of the theoretical perspectives can be applied to the abstract algebra content. It reviews studies of abstract knowledge acquisition focusing on abstract algebra courses.

Mathematical Knowledge Acquisition: Learning Abstract Algebra Concepts

The first undergraduate course of Abstract Algebra is always the great concern of mathematics department communities. It is explained by the importance of concepts and methods of problem solving in abstract algebra and the obvious learning difficulties which most of students usually experience. Clark et al. (1997) assumes that "perhaps even more troubling is the fact that during this course many of these students come to dislike mathematics even though, for a variety of reasons, mathematics continues to be their major. It seems this is especially the case for many pre-service secondary mathematics teachers." Indeed, the students develop a negative attitude toward mathematics in general and a fear of abstraction. With their mathematical background the students often have little experience thinking about the concepts that are dealing with structures, or proving theorems. Secondary education reforms or/and specially designed undergraduate courses are aimed to bridge this gap in students' learning. Another approach to solving the problems with the abstract algebra course was started in the late 1980s by Ed Dubinsky and his colleagues (Clark at al., 1997). They chose to confront the problems they saw in the content and pedagogy of traditional abstract algebra courses by applying a framework for curriculum development and research in mathematics education that they had been developing for several years. This section introduces the main ideas of the framework.

APOS Theory and Researches Based on It

In recent years, mathematics education community started to work on developing a theoretical framework and a curriculum for undergraduate mathematics education. Asiala et al. (1996) reported the results on their work in this area. The authors are concerned with theoretical analyses which model mathematical understanding, instruction based on the results of these analyses, and empirical data, both quantitative and qualitative, that can be used to refine the theoretical perspective and assess the effects of the instruction. Finally, the authors of this article are in the process of producing a number of studies of topics in calculus and abstract algebra (Zazkis & Dubinsky, 1996; Dubinsky at al, 1994; Brown at al, 1997; etc) using their framework.

For the author's research framework, research begins with a theoretical analysis. This initial analysis is based primarily on the researchers' understanding of the concept in question and on their experiences as learners and teachers of the concept. Then the analysis informs the design of instruction. Implementing the instruction provides an opportunity for gathering data and for reconsidering the initial theoretical analysis with respect to this data. These repetitions are continued for as long as it appears to be necessary to achieve stability in the researchers' understanding of the epistemology of the concept.

The authors noted that each time the researcher cycles through the components of the framework, every component is reconsidered and, if possible, revised. In other words, the research builds on previous implementations of the framework.

Based on the theories of cognitive construction developed by Piaget for younger children, Dubinsky proposed APOS (action – process – object – schema) theory. In terms of this framework, understanding of a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas. This assumption about mental constructions is based on a specific notions described by Piaget.

In a more detailed discussion, the authors explain that "an action is a transformation of objects which is perceived by the individual as being at least somewhat external", (p.9). In the context of Abstract Algebra, for example, if the elements of a group can be listed explicitly, then it is not difficult to find its subgroup and work with cosets. "Understanding a coset as a set of calculations that are actually performed to obtain a definite set is an action conception." (p.10). However, more is required to work with cosets in a group such as *Sn*, the group of all permutations on n objects where simple formulas are not available. In terms of the theoretical framework being discussed, students who have no more than an action conception will have difficulty in reasoning about cosets: "In the context of our theoretical perspective, these difficulties are related to a student's inability to interiorize these actions to processes, or encapsulate the processes to objects." (p.10)

Further, when an action is repeated, and the individual reflects upon it, it may be interiorized into a process. In abstract algebra, a process understanding of cosets includes thinking about the formation of a set by operating a fixed element with every element in a particular subgroup.

When an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations, can act on it, and is able to actually construct such transformations, then he or she is thinking of this process as an object. In an abstract algebra context, given an element x and a subgroup H of a group G, "if an individual thinks generally of the (left) coset of x modulo H as a process of operating with x on each element of H, then this process can be encapsulated to an object *xH*. Then, cosets are named, operations can be performed on them and various actions on cosets of *H*, such as counting their number, comparing their cardinality, and checking their intersections can make sense to the individual", (p.11).

A collection of processes and objects can be organized in a structured manner to form a schema. Schemas, at the same time, can be treated as objects and included in the organization of "higher level" schemas.

In order to illustrate that the discussed mental constructions take place during learning mathematical concepts, Asiala et al. (1996) suggest to gather data using three kinds of instruments: written questions and answers in the form of examinations in the course or specially designed question sets; in-depth interviews of students; and a combination of written instruments and interviews. Their written instruments contain fairly standard questions about the mathematical content and they are analyzed in relatively traditional ways. This information shows what the students may or may not learn. It also illustrates the possible mental constructions. To access the full range of understanding, the authors select interviewee's group by including students who gave correct, partially correct, and incorrect answers on the written instruments. They also routinely select students who appear to be in the process of learning some particular idea rather than those who have clearly mastered it or those who had obviously missed the point.

The most important part of qualitative research is to collect the needed data and analyze it properly. Asiala, et al. (1996), in their framework, divided data analysis into 5 steps: 1. Script the transcript. 2. Make the table of content. 3. List the issues. By an issue they mean some very specific mathematical point, an idea, a procedure, or a fact, for which the interviewee may or may not construct an understanding. For example, in the context of group theory one issue might be whether the student understands that a group is more than just a set, that is, it is a set together with a binary operation. 4. Relate to the theoretical perspective. At this step theoretical perspectives are revised. 5. Summarize performance. "The mathematical performance of the students as indicated in the transcripts is summarized and incorporated in the consideration of performance resulting from the other kinds of data that are gathered"(p. 27). It should be noted that in the design of instruction, the authors specially highlight the use of cooperative learning and computer programming language.

To highlight the importance of created framework, Dubinsky (2000) makes the point that working with abstraction in mathematics in general can help students to understand some complex situations with which they have to deal in everyday life, so that APOS theory can help to explain why people have difficulty in understanding some aspects of everyday life in society. The described theoretical framework is illustrating the theoretical and methodological approach which was used by the number of authors in their study of students' knowledge acquisition in abstract algebra. According to Clark et al. (1997), there are two central questions which can be explored in light of these theoretical perspectives: (1) if students' attitude toward mathematics in general and abstract algebra in particular has been improved as a result of a new treatment; and (2) what students understand about the content (basic structures of abstract algebra) as a result of the instructions. The following reports describe students' understanding of basic notions of abstract algebra course.

Dubinsky et al. (1994) explore the nature of students' understanding of group theory, focusing on the concepts of groups, subgroups, cosets, normality and quotient groups. They attempt to illustrate the APOS pattern of learning. The observations were collected during a workshop for high-school teachers who had taken a course in Abstract Algebra. In their analysis, the authors categorized students' ways of understanding concepts of group theory, considering the role of misconceptions. For example, the understanding of a group was categorized as: (1) Group as a set; (2) Group as a set with operation. The first category demonstrates the assimilation of a new idea to existing schema of sets, before it was reconstructed to achieve a higher level of abstraction. The important step toward avoiding this situation is the introducing binary operation as a function of two variables. "...the conclusion of this development is the encapsulation of two objects, a set and a function (binary operation) coordinated in a pair which may be students' first real understanding of a group" (p.292). For pedagogical implications, the authors suggest that it may be effective to go through APOS steps in instructional design. Their computer activities (designed, following APOS) showed a certain amount of success. They also feel that an essential requirement is that students reflect on the action they are performing, during the study. To reach this goal, Dubinsky et al. suggest practicing a small group class work together with computer based activities. Burn (1996) criticizes this article by arguing about the listed concepts being fundamental. He also discusses disadvantages of the computer based learning. Dubinsky et al. (1997) have tried to clarify the distinction between their and Burn's approaches to this important work and look forward to seeing a continuation of this exchange in appropriate forums.

Further, Brown et al. (1997) explore students' learning of binary operations, groups and subgroups, using APOS perspectives. Asiala et al. (1998) study the nature of abstract algebra students' understanding of permutations and symmetries. They claim that APOS is useful for understanding the mental constructions made by students learning about permutations and symmetries, and serves to increase our understanding of how learning about permutations and symmetries might take place. Zazkis and Dubinsky (1996) are interested in mathematical and psychological aspects of constructing dihedral group. The authors observe students' intension to connect the dihedral group and the group of permutations (for example D_4 and S_4).

Studies framed in different perspectives

The research, described in Hazzan's (1999) paper explores possible trends of undergraduate students' mental processes as they get involved in problem solving activities in abstract algebra course. In this research, the data have guided the theory organization, in the spirit of grounded theory. The author focused on five fundamental abstract algebra concepts: groups, subgroups, cosets, Lagrange's theorem, and quotient groups. The data for the study was collected via written research questionnaires, regular classroom tests and homework collection; incidental discussions with students.

Theoretical framework was developed during the study. The author titled his framework as “reduction of the level of abstraction”. The term “reducing abstraction”, presented in this paper, is based on three interpretations for levels of abstraction discussed in literature. First interpretation defines the abstraction level as the quality of the relationships between the object of thought and the thinking person. This interpretation is illustrated in the paper by discussing students' tendency to base their argument on more familiar mathematical objects, with which they have had previous mathematical experience. The second interpretation is - abstraction level as reflection of the process-object duality. “The more one works with an unfamiliar concept initially being conceived as a process, the more familiar one becomes with it and may proceed toward its conception as an object.” (p.79) Hazzan noticed two additional aspects of process conception: (a) students' personalization of formal expressions and logical arguments by using first-person language, and (b) students' tendency to work with canonical procedures in problem solving situations. Finally, abstraction level, defined as the degree of complexity of the concept of thought. The working assumption here is that the more compound the entity is, the more abstract it is.

In summary, the author claims that the lack of time for activities which may help students grasp abstract algebra concepts, many students fail in constructing mental objects for the new ideas and in assimilating them with their existing knowledge. The mental mechanism of reducing the level of abstraction enables students to base their understanding on their current knowledge, and to proceed towards mental construction of mathematical concepts conceived on higher level of abstraction.

Leron et al. (1995) are concerned with students' understanding the group isomorphism. Reported research is part of a series of studies in advanced mathematical thinking.

Nardi (2000) reports about students' difficulties with mathematical abstraction while studying abstract algebra concepts. She observed undergraduate students during weekly tutorials (30 – 60 minute sessions given to one or two students; tutor and students discussed problems). Nardi focused on the following group theory concepts: coset, order of an element, and isomorphism. She explains her choice by stating that "the concept of group is an example of a new mental object the construction of which causes fundamental difficulties in the transition from school to university mathematics" (p. 169). Particularly, the concept of coset emerged as paradigmatically problematic during observations. While constructing cosets the students appeared to be in difficulty with the abstract nature of the operation between elements of a group: references to the properties of numerical operations were observed to generate a concern of the tutors who discourage the students from using metaphorical expressions such as "divided by", for example in the context of quotient groups. However, the author noticed that they do not discourage them from saying "multiply with the inverse" in the context of group operations. Similarly problematic turned out to be the use of expressions such as "times" and "powers of" - also used sometimes vaguely interchangeably by the students - with regard to cyclic groups.

Nardi (2000) notices that "linguistic condensation of meaning" causes difficulties, for instance in the context of the concept of order of an element of a group. As an implication of the theorem $|<g>| = |g|$, the term "order of an element of a group" can be seen as an abbreviation for the term "the order of the group generated by an element".

Another problem, which was observed with the notion of order of an element, seems for Nardi to be in "static and operational duality": $|g|$ is the number of elements in $<g>$ and, at the same time, the number of times the power of g has to be taken in order to cover all the elements of $<g>$. So, in a sense, order of an element is a notion containing both information about a static characteristic of $<g>$ (its cardinality) and information about a way to construct $<g>$ (take the power of g, $|g|$ times). "This type of duality is commonly seen as a source of cognitive strain for students and it is likely that order of an element is not an exception." (p. 185)

The students often inquire about the "raison-d'-être" (the reason for justification or existence) of the concepts. For example, the author observed students' try to understand the notion of cosets, using geometrical images, aiming to construct a meaning of the new concept. The students also appeared to be in difficulty in conceptualizing a mapping between elements of a group and sets of elements of the group. The difficulty is explained by the shift of abstraction from one level to another, involved in the definition of a mapping between elements of a group or the cosets of a subgroup and the elements of the group.

CHAPTER II

THEORETICAL FRAMEWORK

The literature review shows that although there are few published studies on abstract algebra knowledge acquisition, a theoretical base still exists. This chapter is aimed to illustrate the theoretical and conceptual perspectives that guided this study. The literature review included different opinions, views, and theories on the problem of human cognition, and on human ability to abstract and generalize, in particular. This section explains how the theoretical constructs described in the previous chapter helped me to form my theoretical perspectives. For the purpose of this study, I intend to explore the relationships between processes of abstraction and generalization and, further, the construction of students' knowledge about abstract mathematical object.

Theoretical approach, described by Davydov (1972/1990) is highly relevant to educational research and practice. His theory seems incompatible with the classical Aristotelian theory, where abstraction is considered to be a mental shift from concrete objects to its mental representation – abstract objects. In contrary, for Davydov, as well as for Ohlsson, Lehitinen (1997), Mitchelmore and White (1994, 1999), Harel and Tall (1991, 1995), abstraction is a shift from abstract to concrete. Ohlsson and Lehitinen provide us with historical example of scientific theories development; Davydov also gives historical examples and, at the same time criticizes the empirical view on instruction by claiming that empirical character of generalization may cause difficulties in students' mathematical understanding.

Following Piaget (1970 a), I consider the process of abstraction as a derivation of higher-order structures form the previously acquired lower-order structures. Further, I distinguish two types of abstraction. One of these types is simple or empirical abstraction – from concrete instances to abstract idea. The second type then is more isolated from the concrete. Davydov (1972/1990) calls this type of abstraction – theoretical abstraction. Theoretical abstraction, based on Davydov's theory, is the theoretical analysis of objects (concrete or previously abstracted) and the construction of a system that summarizes the previous knowledge into the new concept (mathematical object) so it is ready to be applied to particular objects. This abstraction appears from abstract toward concrete and its function is the object's recognition. According to present research the second type of abstraction is commonly accepted as essential in the process of learning deep mathematical ideas. Similarly, I distinguish two types of generalization – generalization in a sense of Ohlsson and Lehitinen perspectives (which coincides with empirical perspective, described by Davydov); and theoretical generalization. Theoretical generalization is the process of identification of deep, structural similarities, which identify the inner connections with previously learned ideas. The process of theoretical abstraction leads us to the creation of a new mental object, while the process of theoretical generalization extends the meaning of this new object, searching for inner connections and connections with other structures. To understand the role these processes play in students' knowledge construction and to analyze the connections between them is the goal of a future study. However, working with literature I formed my perspectives on knowledge formation in mathematics. I will introduce the framework, grounded in discussed theoretical perspectives, which will guide the study.

First, I assume the existence of initial abstraction – empirical abstraction or generalization. Second, I assume that to understand a complex idea we must have other ideas as parts. When students understand the reason why and how particular ideas are connected to each other, when, through the process of generalization, inner connections between them are established in students' minds – then the new, more complex idea is formed. This is the description of assembling (Ohlsson and Lehitinen, 1997), however, for the convenience of this theoretical guide I divide the assembling into two processes (these processes can follow each other or simultaneously take place): assembling (or grouping) ideas and seeking for inner connections between them (generalizing), to make sense of different ideas participating in a certain whole. Finally, the individual "completes" this new structure by applying the newly created idea to concrete (examples, problems), extracting properties and making connections within the new concept.

In summary, the genesis of new abstract idea looks like following: (0) initial abstractions; (1) grouping previously acquired abstractions (initial abstractions in a very elementary level); (2) generalization to identify inner connections with previously learned ideas; (3) the shift from abstract idea to a particular example to articulate a new concept. Note that at some level of cognitive development initial abstractions become obsolete, since enough more complex and concrete-independent ideas are already acquired. The result of this genesis is a new structure which is more complex and of higher abstraction comparing to the assembled ideas. Hence, we have hierarchical construction of knowledge, where every next idea is more advanced than the previous one. Moreover, cognitive function of abstraction (from now, by abstraction and generalization I mean theoretical abstraction and generalization, defined above) is to enable the assembly of previously existed ideas into more complex structure. The main function of abstraction is recognition of the object as belonging to a certain class; while construction of a certain class is the main function of generalization, which is making connections between objects. Figure 1 shows the theoretical construct described above:

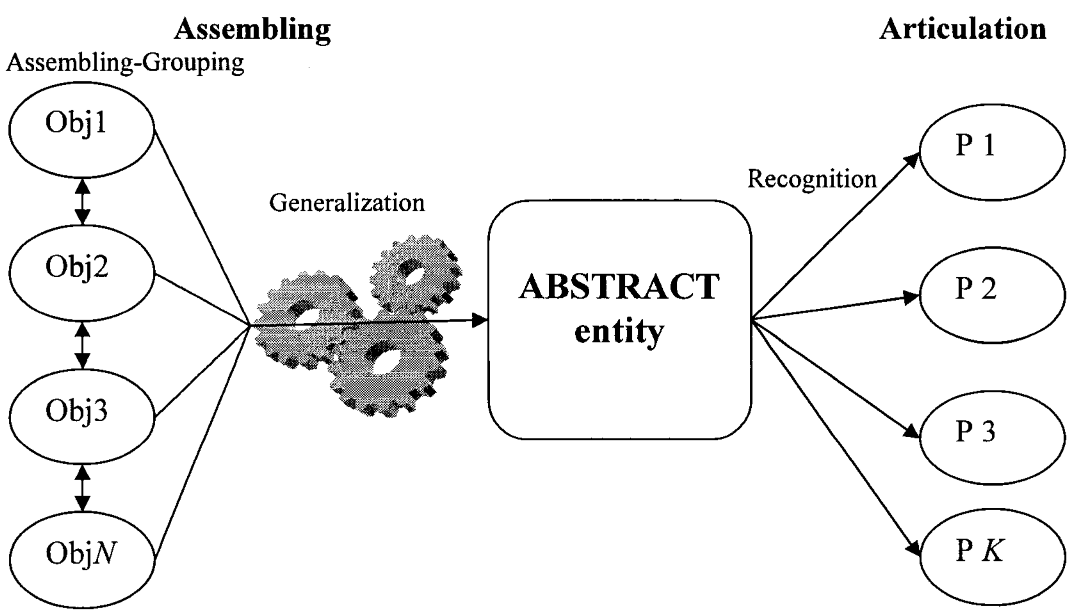


Figure 1. The Process of Abstraction.

The theoretical perspectives evolved from the literature. It helps to identify the key concepts and conceptual relationships that set the stage for framing the research questions that follow. Further, the framework suggests the design of study and helps to ground the methodology and data collection.

CHAPTER III

RESEARCH QUESTIONS

The above theoretical framework claims that if a concrete goal or problem was stated, then it can be resolved by describing new abstract constructions (formulas or mathematical objects), understanding its inner and outer connections and applying the new knowledge to the concrete situations to check the validity of the construction for the stated goal. By the statements of the theoretical framework this abstraction is created by assembling of previously learned ideas into a new structure and then by articulating this new structure, moving toward concrete examples.

In order to understand students' mathematical knowledge and its acquisition we need to investigate problem situations, an ability to recognize a new object, and an ability to work with different contexts where the object is presented or can be used as an additional construct. The main questions for my study follow from theoretical ideas and needs:

- **What are the main characteristics of the cognitive processes involved in the development of students' understanding of Group Theory concepts?**
- **What notions and ideas do students use when they recognize**

a mathematical object, and why? (what are students using: definitions, properties, visualization, previously learned constructs, or something else)

- **What are the characteristics of students' mathematical knowledge acquisition in the transition from more concrete to more theoretical problem solving activity?**

Seeking answers to my questions I attempt to clarify the place of generalization and abstraction in the process of learning. It is important to find answers to these questions because I believe they can help me to construct an understanding of what students need when studying mathematical concepts. Also it may suggest a new approach for classroom practices. To define and clarify my research questions I performed a pilot study, which helped me to identify some specific problems and phenomena which I explore more closely.

CHAPTER IV

METHODOLOGY

Introduction

To find answers to the research questions proposed in this study I chose to analyze actions of undergraduate students participating in a mathematical course. According to Strauss (1987), without grounding in data, any hypothesis or theoretical assumptions will be speculative, hence ineffective. To collect the needed data, undergraduate students participating in an abstract algebra course will be observed. My concern, as follows from the research questions, is to understand certain cognitive processes, involved in construction of knowledge and give them a plausible explanation. To understand these processes I analyzed students' actions during learning a new concept and problem solving activity. I elaborate on the theory saturation in an empirically grounded way. From Glaser's and Strauss's (1967) point of view, generating grounded theory is a way of arriving at theory suited to its supposed uses. The methodology of Grounded Theory approach from this perspective gives a possibility to generate a theory that would be functional for the intended purposes.

Settings and Instructional Context

Students participating in the abstract algebra course, which content includes basic properties of groups, rings, fields, and their homomorphism, were observed. Abstract algebra concepts were selected because of my own observations and experiences as an algebra course participant, and because of research reports (such as Dubinsky (1991), Hazzan (1999), Nardi (2000), Asiala at al. (1997, 1998), etc.) which highlighted students' difficulties in learning certain abstract algebra concepts. Another reason was that students taking this course were assumed to have a good background in Mathematics, since several other courses such as "Mathematical Proofs", "Calculus", "Discrete Mathematics" were prerequisites for this abstract algebra course. The knowledge of students' background can clarify some misinterpretations or possible learning obstacles.

The course was offered in the fall term of 2005. The class met for 50 minute sessions three times a week. A standard text (Fraleigh, 2003) was used as a source for explanation, examples, homework problems and self-study projects. The instructor was a professor of mathematics at the Department of Mathematics and Statistics of University of New Hampshire. The class did not require teaching assistant. The instructional approach was considered to be classical without the use of technology. No special treatment was suggested to instructional approach or problems' design.

Participants

The participants' pool for the study comprised UNH students taking the undergraduate abstract algebra course. Most of them were mathematics majors and students who planned to receive high school teaching certificates. The students were asked to sign a consent form (see Appendix D) to show their decision to participate in the study. Methodology excludes any specific risks of effects on participants.

Data Collection Procedure

To answer my research questions and to gain as wide and varied image as possible, I collected the data from various sources. I observed the classroom, took field notes and audio recorded the lectures. I conducted a series of 3 semi-structured interviews (see Appendix B for questions sampler) with key participants, collected written work (homework, quizzes, tests) and copied one student's lecture notes.

Students Artifacts

Non-graded copies of students' homework, quizzes, and exams (including final exam) were collected for analysis. Students' work was collected through the whole semester. The assignments reflected on requirements of the course. I expected to observe some repeated phenomena and refine my interview questionnaires based on these phenomena. I also expected to follow up on some issues raised by the written artifacts. 20 students agreed to participate in the study by giving the permission to collect their written work.

Students Interviews

Out of 20 students 7 agreed to participate in the interview sequence. I created 3 semi-structured interviews. Semi-structuring gave me the desired flexibility, since qualitative interviews must be flexible and exploratory.

Researcher adjusts later questions depending on how the interviewee answers earlier questions in order to clarify the responses or to probe for more details. Each interview consisted of scripted questions, common to all participants, and non-scripted, individual questions, raised by student's response.

I conducted three interviews (about 50 minutes each) in a form of a dialog. The interviews will focus on five fundamental abstract algebra notions and structures: binary operations, groups, cyclic groups and subgroups. The first interview focused on students' understanding of a binary operation; the second – on groups and cyclic groups, the third – on subgroups. Questionnaires were created based on pilot study findings as well as on classroom observations and my experience.

Interviews took place outside of the classroom. I audiotaped each interview for future analysis. I also collected students' scrap paper they used during the interviews. During the interviews, the students were encouraged to explain every action they perform to answer a question or solve a problem. In other words, students were asked to "think aloud". I included two different types of questions:

1) Content related questions – mainly mathematical problems on key concepts, open-ended questions about definitions and properties of the key concepts.
2) Attitudes and Believes questions – questions about students' personal experiences and relationships to the concept they just learned.

Observations

Although I narrowed my study to a specific abstract algebra part - understanding fundamental concepts of group theory, I also observed classes on other topics. I tried to get more information about students' background and make students comfortable with my presence. During the observations, I took field notes and audiotaped classes to have a chance to relate on observations during the analysis.

Instruments

To create a scripted part of my questionnaires, I followed my observations and experiences in the Abstract Algebra course. I reviewed a number of Abstract Algebra text books and carefully selected my questions.

Data Analysis

Strauss (1987) suggests several types of qualitative data coding. Initial type is termed open coding – unrestricted coding of the data. This is done by critically inspecting the field notes, interview transcripts, etc. very carefully: line by line, word by word. The aim is to produce concepts that seem to fit the data. Initial coding categorizes the data. Axial coding consists of intense analysis done around one category at a time. It results in knowledge about relationships between categories and subcategories. Selective coding is concerned with analysis of the core category.

Students' artifacts and interview transcripts were analyzed and categorized. According to Strauss (1987), the generation of theory occurs around a core category, "since a core category accounts for most of the variation in a pattern of behavior" (p.34), and most other categories are related to it.

To understand the data I started with transcribing the interviews. The transcriptions suggested some initial codes and categories for further analysis. Moreover, during this process I finally narrowed my research to the selected topics of group theory. The second step was to analyze the written work. I chose to analyze the data in chronological order. Each quiz and Exam 1 was analyzed by categories. However, Exam 2 included more complicated problems and I chose to analyze it problem by problem. After written work was considered, I returned to the analysis of the interviews. I reviewed my notes, codes and categories I generated before for more detailed analysis.

CHAPTER V

DATA ANALYSIS

Quiz 1

Quiz 1 covered the introductory topics of group theory. It concentrated on students' understanding of a binary operation and its properties. In general, all the responses showed that students struggle with understanding the connections and relations between set, its elements and a binary operation, assigned to the set.

A set together with its operation is not a simple collection of elements but a structure with specific properties. For the students it is not a new concept. They already know several operations, such as addition, multiplication, subtraction, division, defined on different sets, such as whole numbers, integers, rational numbers, set of matrices, etc. However, the data suggested that students' previous mathematical experience did not include a necessity of defining an operation within a certain set. Moreover, the operation is usually pre-assumed to be defined on a suitable set. For example, if the problem is asking to divide two integers, the result is not necessarily an integer and the problem is referred to the set of rational numbers. In the context of abstract algebra we want to generalize the concept of operation and stress the importance of its connection to the set it is defined on. The data suggested a lack of understanding of the restrictions that a certain set puts on its operation or the extensions that are required for the set to be appropriate for its operation. The textbook, used in class (Fraleigh, 2003), attempted to clarify the notion of a binary operation using “every day” language. In the following citation, the author discusses the idea how the operation and the set are connected:

> “In our attempt to analyze addition and multiplication of numbers, we are, thus, led to the idea that addition is basically just a rule that people learn, enabling them to associate, with two numbers in a given order, some number as the answer. Multiplication is also such a rule, but a different rule. Note finally that in playing this game with students, teachers have to be a little careful of what two things they give to the class. If a first grade teacher suddenly inserts *ten, sky*, the class will be very confused. The rule is only defined for pairs of things from some specified set.” (p. 1)

In the textbook, examples of sets together with binary operations are given in a rather descriptive manner: “On ***Q***, let $a * b = a/b$”. (p. 37); or “Let + and $\cdot$ be the usual binary operations of addition and multiplication on the set ***Z***” (p.33). The data shows that the notations are symbolically new and difficult for students to interpret and to translate into familiar terms. As a result, the students seem to be so concentrated on understanding of a certain operation that they miss the part about the set completely. In other words, students’ previous experience suggests that the result of an operation is always in the set. After defining a binary structure, the author uses the following notation for the description of a binary structure: $< S, * >$. This notation stresses, that students must see operation and set together, as a whole. However, I have noticed that, although some students adopted this notation, they still did not think about a set’s closure under the operation.

Set – operation relations: universal quantifications

First, I attempt to analyze the students' understanding of connections between a set and its binary operation. I have noticed that some students have difficulty understanding "universality" of binary structure properties. In other words, the students do not see that all properties that a binary structure has are distributed over all its elements. Consider the following figure:

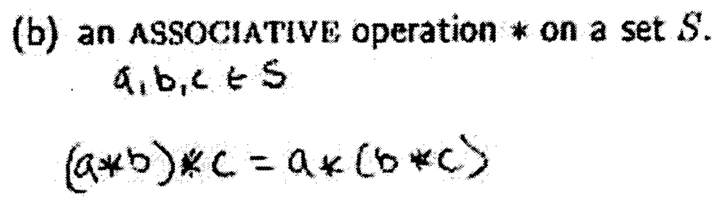


Figure 2. Student's definition of an associative operation.

In the excerpt we observe that the student is missing the part about "universality" of associativity. Indeed, it is not clear for the reader if the student meant associativity to hold for every triplet or it is enough to check associativity for a single triplet. It seems like associativity is not a universal quantification for a binary structure. In other words, the students do not clarify if associativity is the property which holds for all elements in the set. The missing quantifier actually raises a big problem for the interpreter. Do the students understand that associativity is universal quantification or not? If we turn to the students' background, we could find evidence to support their understanding of universality. All students in an abstract algebra course are familiar with operations such as addition and multiplication of real numbers and their subsets. Without loss of generality inside the students' pool, I may assume that these operations, as well as their properties, are well known. Students would not have difficulty answering the question about commutativity (or associativity) of addition of integers (Baroody, Herbert, Waxman, 1983). Previously it was enough to show one or two computations, or, possibly, create an operation table to make a general conclusion that addition (or multiplication) is associative for any three numbers of your choice. So, it is possible that the students understand the universality of associativity for a given binary structure. In this case they would have no doubts that, as soon as associativity is proven for some triplets, every triplet from the binary structure will satisfy the property. The question remains, however, how the definition of associativity is applied to problems where the students are asked to prove or disprove that a given binary structure is associative. In other words, the definition we see in Figure 2 could mislead the students toward acceptance of associativity for the whole binary structure by only checking it for one triplet. Even further, students' interpretation of the definition could possibly include the condition of existence: if there exist a, b, c from S such that $a * (b * c) = (a * b) * c$ then the operation is associative. From students' responses, it looks like the students may have difficulty in distinguishing and understanding the meaning of quantifiers "for all", and "there exists". Unfortunately, Quiz 1 did not provide a strong data support for this assumption. However, Figure 3 explicitly illustrates the problem of quantifiers the students are dealing with.

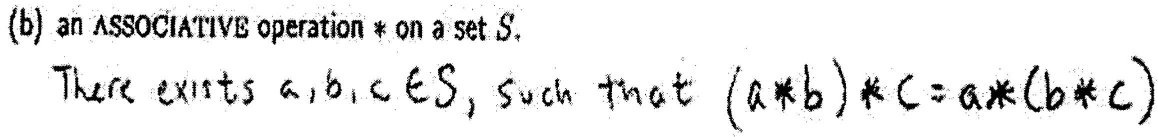


Figure 3. Student's reasoning with the quantifier.

In Figure 2 it seemed like the student did not think about universality of associativity. Probably, at this point students do not recognize the importance of universality so they do not mention it in their responses. It is also possible that they think about universality as being given for granted, meaning that no matter what the set is, if it works for one set of elements then it automatically works for other sets. So, the property is distributed on all elements of the set. In Figure 3, however, the student insists on existence of at least one triplet for which the property holds. This definition can be explained by a simple misinterpretation of logical quantifiers. Again, the phrases "there exists" and "for all" (or "for every") perhaps are not accepted by the students as valuable mathematical arguments. However, it is possible that the student recognizes an operation to be associative even if the statement $(a * b) * c = a * (b * c)$ is true for at least one triplet. I already discussed that the students' connection to familiar binary structures is very strong. They may not show it explicitly, but many instances in their responses suggest that they reason using familiar structures. Nevertheless, they do not necessarily see the conceptual differences that occur between familiar structures and other binary structures. I may assume that for the student in Figure 3 the existence means the existence "for all" elements of a given set. In other words, it is enough to check if the statement is true for only one triplet but it means for the student that, if true, the statement holds for all triplets of a given binary structure (if one triplet exists then all possible triplets satisfy).

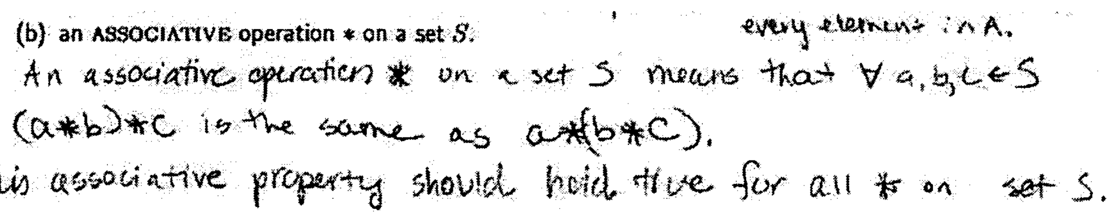

(b) an ASSOCIATIVE operation $*$ on a set S. every element in A.
An associative operation $*$ on a set S means that $\forall a, b, c \in S$
$(a*b)*c$ is the same as $a*(b*c)$.
This associative property should hold true for all $*$ on set S.

Figure 4. Student's definition and quantifiers.

The problem of quantifiers appears in different places in students' responses. In the excerpt shown in Figure 4 the use of quantifier "for all" in the last row suggests that, in the student's view, if associativity holds for all triplets in the binary structure (S, $*$), then it holds *for all* possible *operations* on S. It is clear that the sentence "for all *" is incorrect. However, I would like to discuss what it could mean in terms of students' understanding of a binary operation in general. I discussed earlier that the shift from "concrete" or familiar examples of operation to general, more abstract operations is very complicated for students. I think Figure 4 illustrates student's understanding of binary operation in the general sense. Symbolically, "$*$" represents an operation defined on the set and this operation can be defined as addition, or multiplication. It could be defined as a combination of familiar operations or as none of the above. Perhaps, by using the quantifier "for all" the student wanted to stress that she/he knows that "$*$" represents all possible operations on S.

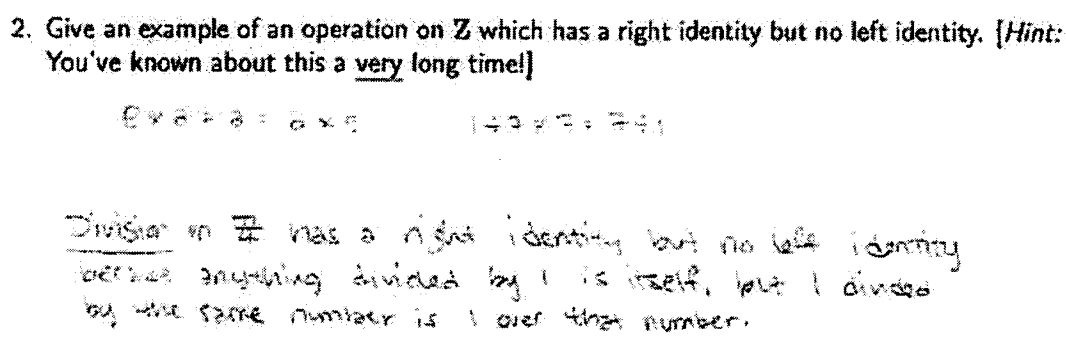
On the other hand, the first part of the student's response is correct; however, it did not seem enough for her/him and the student added the comment about universality of associative property for all "$*$", defined on *S*. The data demonstrated that almost all students understand associativity as a property of operation defined on the set. In the case illustrated in Figure 4, it looks like the student assigns the property to the set. In other words, in the set *S*, if associativity holds for the defined operation $*$, then it would hold for all other operations. It follows that if a particular binary structure is associative then all binary structures, defined on the set *S* are associative. In this case we observe the following logical contract: if $(a * b) * c = a * (b * c)$, for all *a, b, c* from *S*, then any operation $*$ on *S* is associative.

It is also possible that the role of quantifiers is unclear to the student. She/he may not fully understand the strength and value of quantifiers in mathematical statements. It may be difficult to see the difference in two statements with different quantifiers but similar equations or formulas. In this case, statements with or without quantifiers are not significantly different from one another and, for the students, using the quantifiers does not change the meaning of the mathematical sentence.

In general, for most students the relations within binary structures are confusing. Students' knowledge about sets, their elements and operations often is not systematized and thus is not represented as an object. This causes the students to struggle with understanding the set and the binary operation defined on it as a whole - as a single mathematical construct.

Set – operation relations: closure

The concept of closure is one important connection between a set and its binary operation. When working with binary structures, it is crucial to understand that in a binary structure a result of the operation performed on a pair of elements of the set must belong to the set. Nevertheless, this section Figures show that sometimes students are so concentrated on finding an operation that would satisfy the conditions of the problem that they forget about an algebraic structure the problem is restricted to:


2. Give an example of an operation on **Z** which has a right identity but no left identity. [*Hint:* You've known about this a <u>very</u> long time!]

Division on Z has a right identity but no left identity because anything divided by 1 is itself, but 1 divided by the same number is 1 over that number.

Figure 5. Student's definition of operation on Z.

In Figure 5 student finds an operation which fails to have a left identity. It looks like she/he assumed that the operation is defined on ***Z***, since a chosen pair belongs to ***Z***x***Z***. It also looks like the quantification is clear for the student: an identity (left or right) must satisfy the identity condition ($a * e = a$ or $e * a = a$) for all elements of ***Z***, which means if we need to prove otherwise it is enough to find one counter example. The student found the example: it looks like 1 works as a right identity on (***Z***, $\div$) and $1 \in$ ***Z***. However, "Division on ***Z***..." is not defined, since the result is not always in ***Z*** and division by 0 is undefined. It does not seem to be a significant part of the problem. It is more important to find a familiar operation which would satisfy the problem's conditions. The student wrote that "1 divided by the same number is 1 over that number", or 1/*a*. In Figure 5, she/he is dividing 1 by 7, so the result is 1/7. Obviously, 1/7 is not an integer and the student does not claim that it is. It suggests that she/he simply missed the part about closure. In other words, it seems that for the student a binary operation on ***Z*** is understood as a map $\mathbf{Z}\times\mathbf{Z} \to A$, where *A* could be any set, not necessarily ***Z***. A similar problem was observed in Figure 6.

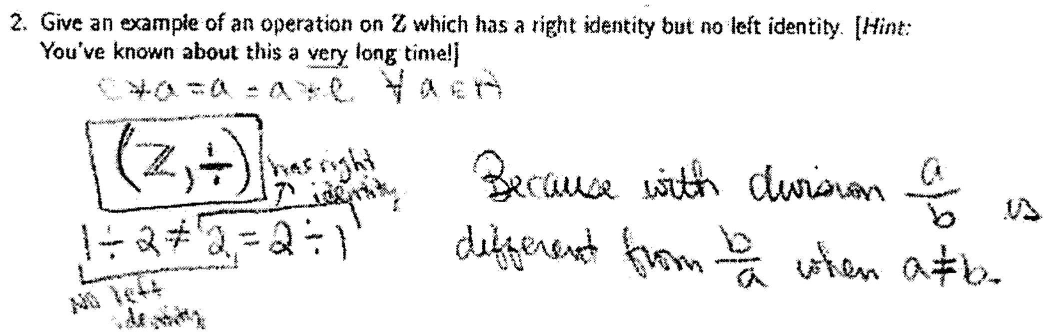


Figure 6. Student's reasoning about division as a binary operation on Z.

The student chooses division as an operation on ***Z***, which has right but no left identity. In Figure 6, the student is reasoning about identity using the fact that division is not commutative. Indeed, a/b is not the same as b/a for all but one element of ***Z***. Again, the reasoning about existence of a right but not a left identity in the structure (***Z***, ÷) is logically correct. However, the closure is missed again. Note that the symbolic notation (***Z***, ÷), which the student is using to describe the binary structure, suggests that for her/him the operation is connected to the set. They form a structure together rather than separately. It could mean that for the student a set is connected to its operation only in "half": a pair must be from the set but the result of operation is not important. In Figure 7 we observe the same inaccuracy.

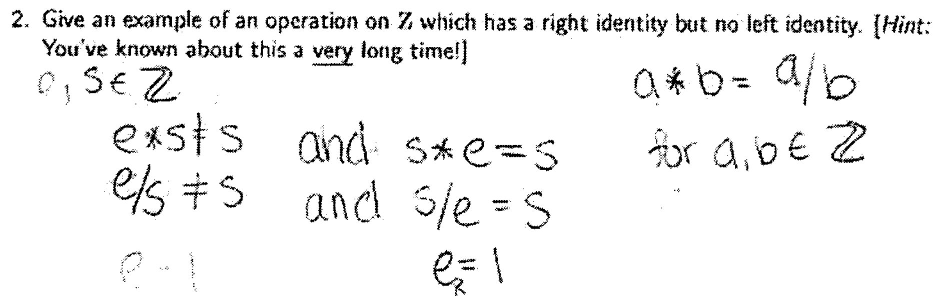


Figure 7. Student's symbolical reasoning about division being binary operation on Z.

The student used more general symbols to represent an identity element and operation. It suggests that the student understood the abstraction of the problem but still did not carefully think about the result of division of elements of ***Z***. Nonetheless, during Interview 1, which I conducted after Quiz 1, the student defines a binary operation:

> Question 1: Define what it means to say that * is a binary operation on a set *A*.
> S3: It means that if *A*×*A*...OK...if *a* and *b* belong to...if (*a, b*) belongs to *A*×*A*, then * is *a***b* in...in *S* and...so, its closed and it has to be well defined.

The student recalls that the set must be closed under its binary operation, and $a * b$ is in *S*. She/he might not realize however, that it is not always the case. Perhaps the student assumes that if an operation is defined on ***Z***, and a, b are from ***Z***, then $a * b$ is necessarily in ***Z*** for every ordered pair of elements of ***Z***. In this case there is no need to check for closure. However, her/his answer to the quiz problem suggests otherwise and I came out with another assumption. It is possible that for the student the most important part of the binary operation definition was that *(a, b)* must belong to ***Z****x****Z***. Alternatively, the student may think that the closure must hold only for SOME, but not necessarily for all ordered pairs from ***Z****x****Z***. In this case it is clear why the student answered the question in Figure 7 this way. She/he was satisfied with the answer since a/b is in ***Z*** at least for some *a, b* from ***Z***. Note that she/he uses neither a quantifier "∀", nor gives a verbal explanation for her/his choice of elements.

The hint, given in the problem, suggests that students think of familiar operations. In Figure 8 we can see the list of familiar operations most students had in mind.

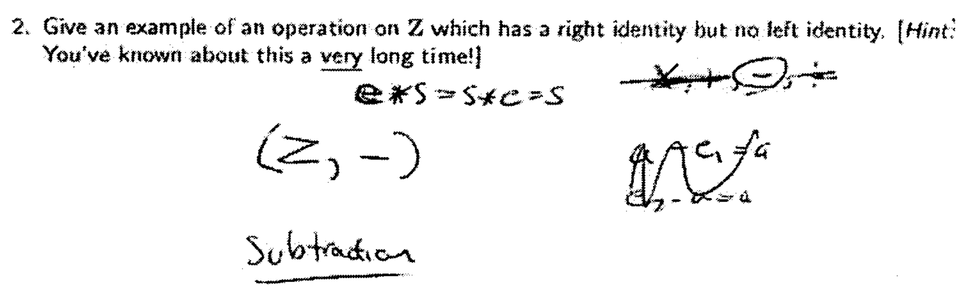


Figure 8. Choosing an operation.

The student considered each of them and crossed out ones that did not work. Most likely, the students did not consider addition and multiplication in this problem since they know these operations are commutative and then have both right and left identity. (Note that these two operations are crossed with one line). So the reasonable candidates are subtraction and division. The student didn't provide any explanation why she/he crosses out division. I may only assume that it is because ***Z*** is not closed under division. The student's argument about subtraction seems to be misleading for her/him, and the student decided to cross it out. The notation e_1, e_2 suggests that the student was looking for different elements to the right and left identity. She/he perhaps thought that the right identity exists only if the left identity also exists but in this problem they cannot coincide. Still, the student gives subtraction as an example of such operation on ***Z*** but it is possible that the participant answers this way rather by exclusion.

In Figure 9 the student tries to determine which operation would work by listing all the possible familiar operations. The only exclusion is addition. I think that it is just too obvious for the student that it is commutative and thus cannot satisfy the problem's conditions:

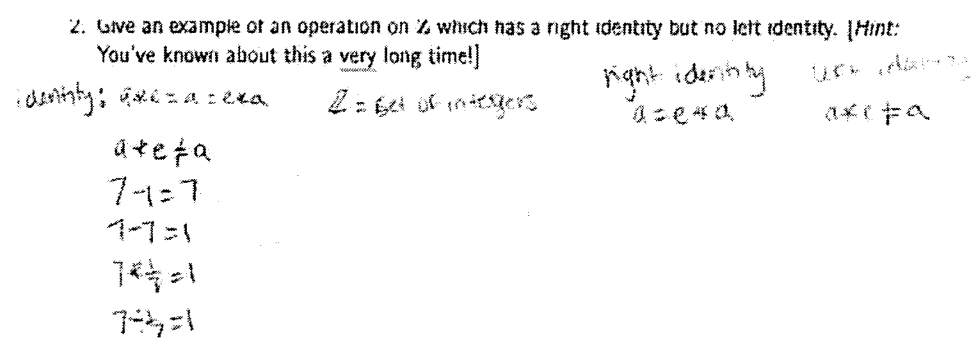


Figure 9. Student's definition of identity and reasoning about binary operations.

First, I would like to discuss the definition of the right/left identity, given by the student. It looks like, by "right identity", the student means the right side of equality in her/his identity definition ($a = a * e$); consequently, the left identity is defined by the left equality ($a * e = a$). It seems like for the student the words "left" and "right" are most significant in the definition of identity and she/he understands left identity as a left side of equation $a * e = a = e * a$ and the right side respectively defines right identity. In general, the concept of right or left operations is new to the students, while an identity element is the concept they studied before but in different settings. The students are used to work with the notions of "right side", "left side" only when working with equations. Moreover, addition and multiplication, as "standard" and well known operations, are commutative and the addition/multiplication from left or right side did not make a difference. Further in Figure 9, the student is trying to find an operation which would satisfy problem's condition. The response suggests that she/he confuses subtraction with division: when writing $7 - 1 = 7$, the student really means $7 \div 1 = 7$. Alternatively, the student may be confusing an additive identity 0 with multiplicative identity 1. Then her/his line $7 - 1 = 7$ really means $7 - 0 = 7$ and "1" in the first equation is the symbolic representation for an identity. Unfortunately, we see that the student left the problem unanswered which made a further analysis impossible or speculative.

Note, that in this problem it was not asked explicitly to check if the operation is binary. It is possible that for the students the notions of a binary operation and operation in general are not the same. I have mentioned that during the first interview, when asked to state the definition of a binary operation almost all of the interviewees recalled that the set must be closed under a binary operation, defined on it. Perhaps the sentence "binary operation" sounds like an alarm for the students to check for closure, since they are in the process of learning what a binary operation is. Nevertheless, the term "operation" is a well known and familiar term. It suggests thinking of familiar operations - overcoming set restrictions. If the problem is asking to check if the set is closed under the operation, the students would not miss the part about closure. However, for many students closure is still not a part of a binary structure as is in the case of Problem 2 (Quiz 1) where students need to define an operation on ***Z***. As I observed, the students are not concerned with the result of the operation. For them to "give an example of an operation on ***Z***" means taking two elements of ***Z*** and perform an operation of students' choice. In this quiz, 14 out of 19 students thought of either subtraction or division, while 5 out of 14 decided that division is the required operation. The excerpts showed that some students do not recognize the algebraic structure – a set together with its operation - as a whole. They separate static object – a set, from structural or operational part – an operation on the set.

<u>Interesting response</u>

The uniqueness of the following response is that the student ignored the hint and did not consider familiar operations:

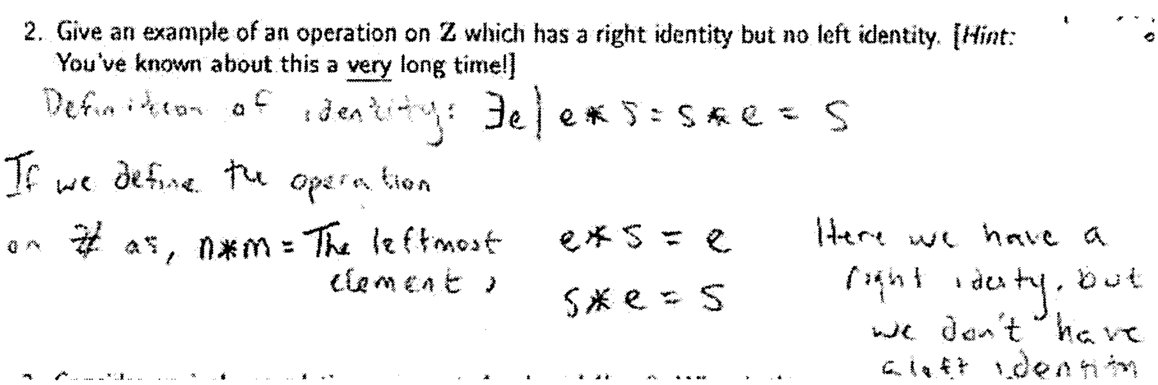


Figure 10. Student's definition of binary operation with special property on Z.

As I mentioned above, the hint gave the students a clear idea what operations to consider as candidates. However, Figure 10 shows a rather unusual way of thinking. There may be several reasons for that. Perhaps the student did think about familiar operations but decided that none of them would work: both addition and multiplication are commutative; division does not satisfy closure condition. Possibly, subtraction was a suspect in terms of closure but the student did not want to make a mistake and tried a new approach. Another possibility may be that the student really understands the definition of a binary operation and appreciates its "freedom": he could actually create an operation to satisfy the conditions of problem 2, without tying it to the familiar operations. Note, that in the student's solution the concept of closure is secured – the result is always a member of an ordered pair from **Z.** The last observation suggests that the student is, in fact, thinking in terms of a binary operation in connection to the set it is defined on. For her/him these notions are inseparable and the understanding of a binary structure is complete.

Quiz 2

Quiz 2 concentrated on students' understanding of a concept of subgroup. The analysis of the quizzes revealed several interesting aspects of students reasoning about groups and subgroups. Since groups and subgroups are not only related as sets but also connected by a binary operation defined on the sets, I found a great deal of overlap between categories I discussed previously and problems I found in this quiz.

Set – operation relations. Groups and their subgroups

A group (as well as a subgroup) is, first of all, a binary structure and the understanding of relations between the set and the operation defined on the set is crucial for understanding of the concept. For many students this is a huge obstacle. Analyzing this quiz, I distinguished several types of students' understanding of binary structures and "set – operation" relations in particular. I grouped them in three categories. First, some students do not consider an operation in relation to the set at all (or the other way around). For this group of students it is enough to have a set and for them the operation does not really affect the elements of the set; or, the student could be concerned with the operation but ignore the set completely. Second, for some students it is difficult to decide what operation is plausible for the given set. For this group it is problematic to understand that there is only one operation defined on a binary structure and as soon as the operation is defined it cannot be changed. Finally, the third group is formed of students who comprehend a binary structure as a set, closed under its operation but still do not completely understand properties that the operation assigns to the set. In other words, they have difficulty understanding closure of a binary structure. This misunderstanding implies difficulties in reasoning about properties of a binary structure that depend on both operation and the set.

Students' responses to the questions about groups and subgroups supported the partition I described above. Consider the following fragment:

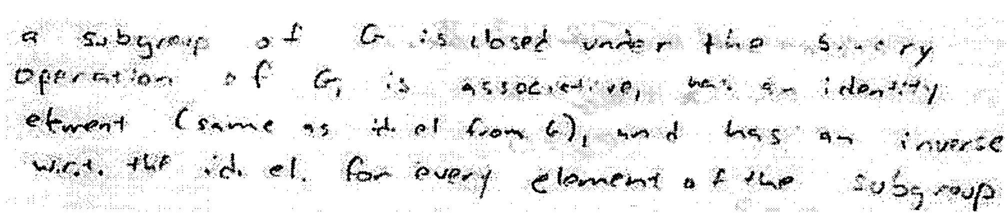


Figure 11. Student's definition of a subgroup.

The student carefully described what a subgroup is and how it is connected to the group. However, the connection is viewed only in terms of operation $*$. It looks like the student understands that a group and its subgroup have the same operation, since "a subgroup of G is closed under the binary operation of G". Nonetheless, she/he only connects a group G and its subgroup H via the operation and identity element. The student's understanding of a subgroup is rather unclear. The word "subgroup" suggests that it exists somehow "inside" the group. Perhaps, to the student, the term "subgroup" implies automatically that a subgroup is a subset of the group under the group operation. I found 2 responses (out of 5 "problematic" responses) that address the same issue.

Figure 12 illustrates the example from the same group of students (group 1). This time, however, the student misses the "operational" part of a subgroup definition.

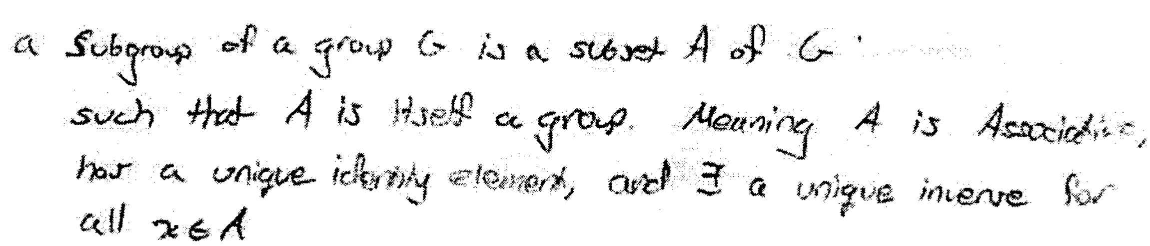

a Subgroup of a group G is a subset A of G such that A is itself a group. Meaning A is Associative, has a unique identity element, and ∃ a unique inverse for all x ∈ A

Figure 12. Missing "operational" part in subgroup definition.

In the excerpt the student connected a group with its subgroup in terms of elements but did not mention how they are connected operationally. I think that by saying that A is a group, the student somehow tried to indicate that the operation on A is defined, but still the relations between this operation and a group operation are not described in any way. This response suggests that the student possibly does not perceive a binary structure as a set together with the binary operation. It implies that she/he would not be concerned with both, set and operation, connections between a group and its subgroup. It is possible, that the student does have a good sense of what a binary structure is but does not think that a group and its subgroup must have the same binary operation. In this case it is important that a subgroup is a subset of the group but the operation may be different.

Figure 13 (definition of the General Linear Group *GL(n, **Q**)*) illustrates one of the responses with a similar problem.

Figure 13. Student's definition of the General Linear Group *GL (n, Q)*.

Out of 17 students' quizzes I have analyzed, 13 happened to have this problem – ignoring the operation. In Figure 13 the student is trying to recall what the elements of a group *GL(n, **Q**)*; are; however, she/he does not think about an operation, while the question is to define a general linear group. It appears possible that the student's group concept formation is excluding the operation part. It is only important to describe the elements of the set *GL(n, **Q**)* in this case. Perhaps for the students who gave similar responses, the operation was given for granted, since *GL(n, **Q**)* was presumed to be a structure rather then a set. In general, only 2 out of 17 students did not ignore the operation part and described *GL(n, **Q***) as a group of all invertible n by n matrices with entries from Q, under the ordinary matrix multiplication. Other students (12) struggled with the description of the set and omitted the operation completely. Two students left the problem blank. One of the responses, however, I could not relate to any of the previously described groups. I found it rather interesting and worth considering:

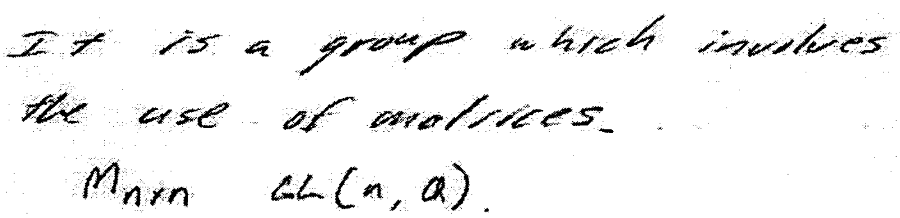
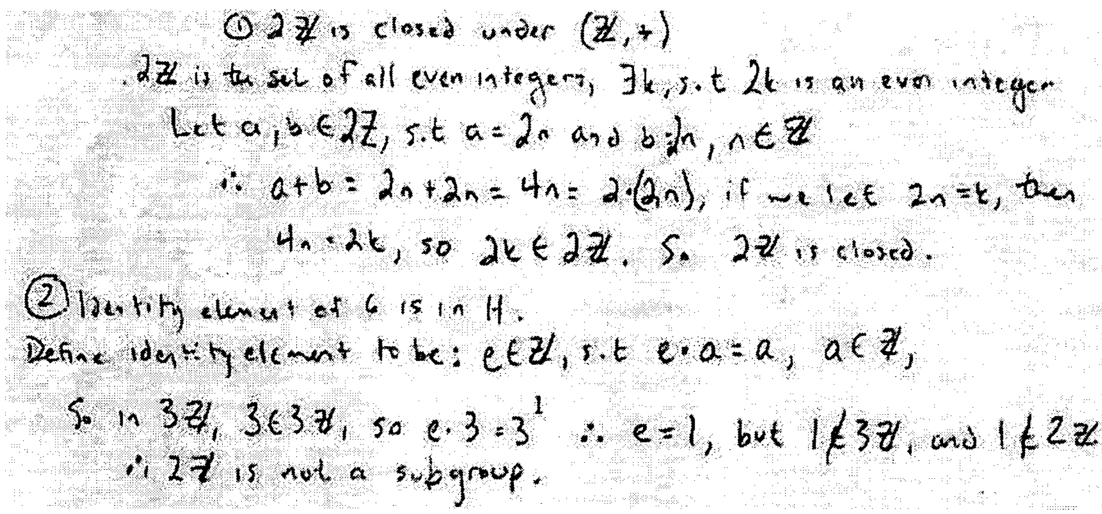


Figure 14. Student's definition of the General Linear Group GL (n, Q).

Note that the words "involves the use of matrices" suggest that the student, perhaps, understands the group as a process. She/he still did not identify the operation and did not describe the set explicitly, but defined the binary structure via its elements which are "involved" and "used" somehow.

The following responses give support for the second category I described. The example shows the confusion students often have about an operation defined in a binary structure. Out of 17 quizzes, 4 papers demonstrated an uncertainty when dealing with a question which operation to use.

Question 3. Recall that $n\mathbf{Z}$ is precisely the set of integers which are multiples of the given integer n. Use the "subgroup criterion" to determine whether or not the set $2\mathbf{Z} \cup 3\mathbf{Z}$is a subgroup of ($\mathbf{Z}$, +)



① $2\mathbb{Z}$ is closed under $(\mathbb{Z}, +)$
. $2\mathbb{Z}$ is the set of all even integers, $\exists k$, s.t $2k$ is an even integer
Let $a, b \in 2\mathbb{Z}$, s.t $a = 2n$ and $b = 2n$, $n \in \mathbb{Z}$
$\therefore a+b = 2n+2n = 4n = 2\cdot(2n)$, if we let $2n = k$, then
$4n = 2k$, so $2k \in 2\mathbb{Z}$. So $2\mathbb{Z}$ is closed.
② Identity element of G is in H.
Define identity element to be: $e \in \mathbb{Z}$, s.t $e \cdot a = a$, $a \in \mathbb{Z}$,
So in $3\mathbb{Z}$, $3 \in 3\mathbb{Z}$, so $e \cdot 3 = 3^1$ $\therefore e = 1$, but $1 \notin 3\mathbb{Z}$, and $1 \notin 2\mathbb{Z}$.
$\therefore 2\mathbb{Z}$ is not a subgroup.

Figure 15. Student's reasoning about subgroups. Confusion with operation.

In this problem, the group is defined to be (***Z***, +) – the group on integers under addition. We observe that, while working on the problem, the student suddenly switched from addition to multiplication. Note that the switch took place as soon as the student started to think about identity element. I could think of several reasons for that. Perhaps the student recalled the definition of an identity element (Equation 1) and interpreted the symbol $*$ as multiplication.

$$a * e = a = e * a \qquad (1)$$

Moreover, some books, as well as most of mathematicians do not use a symbol between elements of a set at all (write it as *ab*). So, the notation may suggest using operation of multiplication. In the beginning the student seems to be more concentrated on the defined binary structure but later in the process she/he "looses" the structure and concentrates on elements of the given sets. Another reason for the confusion could be that the symbolic representation of the structure is misleading the student. The set, the problem describes, is given by

$2\mathbf{Z} \cup 3\mathbf{Z}$ where each element is a multiple of 2 or 3. In other words each element is described via multiplication of an integer and 2 or 3: $x = 2z$ or $x = 3z$. Thus, an element of the given set is not a "static" object, but rather a process – the result of multiplication of an integer by 2 or 3, and, while the group operation is still addition, the student switched to multiplication when he needed to think about an identity element. Note that in the first part of the solution the student wrote $a + b = 2n + 2n = 4n = 2 \cdot 2n$. So, the problem of addition is now the problem of multiplication. After that the student started to use multiplication. From this point, it is clear why the student chose 1 to be an identity and her/his conclusion about $2\mathbf{Z} \cup 3\mathbf{Z}$ not being a subgroup of ($\mathbf{Z}$, +) is accurate (1 is not an element of $2\mathbf{Z} \cup 3\mathbf{Z}$).

As students learn more abstract algebra objects, the problems become more and more abstract. By getting more abstract I mean that the problems include more and more abstract objects the students must analyze while solving them. Previously they only needed to think about one or two concepts to solve a problem. In the problem above, however, the students need to understand a concept of a binary structure, group, subgroup, identity, inverse elements, etc. Moreover, it is not enough to know the definitions; the students must understand how to apply them to the concrete objects defined in the problem.

The next example illustrates similar difficulty:

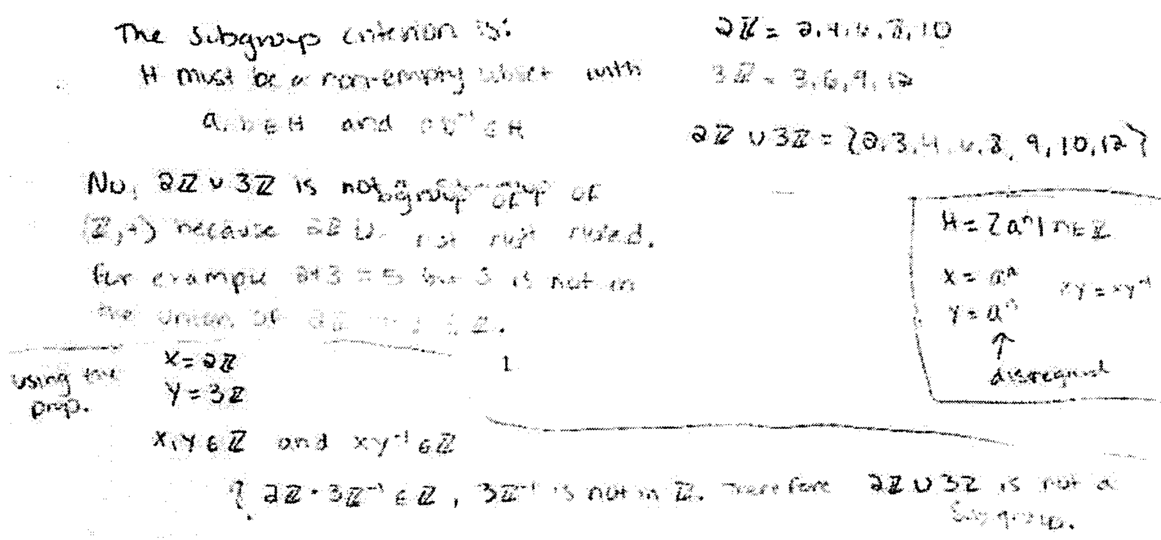


Figure 16. Example of the change of the operation.

The student finds it confusing to choose the operation when working with a "subgroup criterion". When analyzing this fragment I noticed how the student formulates the criterion: $x, y \in \boldsymbol{Z}$ and $xy^{-1} \in \boldsymbol{Z}$. It looks like the student misunderstands the criterion conditions. Perhaps, the student is determined to see if a certain product of elements from $2\boldsymbol{Z}$ and $3\boldsymbol{Z}$ will belong to $\boldsymbol{Z}$. Obviously, in the student's view, expression xy^{-1} is a product of two elements. It suggests that the criterion is isolated from the defined operation. After the student switched to multiplication, it was reasonable for her/him to argue that $3\boldsymbol{Z}^{-1}$ was not an element of $\boldsymbol{Z}$ and from the argument it followed that the given structure was not closed.

Note that the student correctly solved the problem, using the definition of a subgroup. After this, however, the student noticed that the problem asked to determine if a given set was a subgroup of ($\boldsymbol{Z}$, +) using a "subgroup criterion". From the excerpt it looks like the student got confused by the criterion and could not properly prove the statement once again. I have noticed that for some study participants it was always easier to use definition while others found the criterion more useful since sometimes it made the proof shorter.

In my opinion, Figure 17 also supports the second category rather than the first:

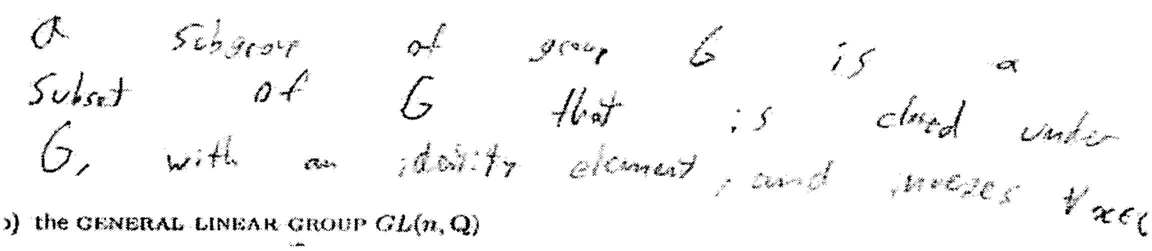


Figure 17. Student's definition of a subgroup. Missing "group" part.

At first it may look like the student is missing "operation" part and the response seems to be close to the one in Figure 12. However, I think that the student understands the role of operation in a binary structure. She/he also seems to understand that a subgroup is connected to the group operationally. However, the student still finds it difficult to determine what operation she/he should think of if no operation is given explicitly. The phrase "a subset of *G* that is closed under *G*" suggests that the student perhaps thinks about the operation of *G*. Since the operation is not given explicitly, she/he chose to say that a subgroup must be closed under the whole binary structure. However, it appears possible that the student does not perceive operational connection between a group and its subgroup and the phrase "a subset of *G* that is closed under *G*" is just an attempt to recall a correct definition without understanding the meaning of it.

Finally, sometimes students think about a subgroup as a subset together with an operation of a group. However, they ignore special conditions which an operation must assign to the binary structure.

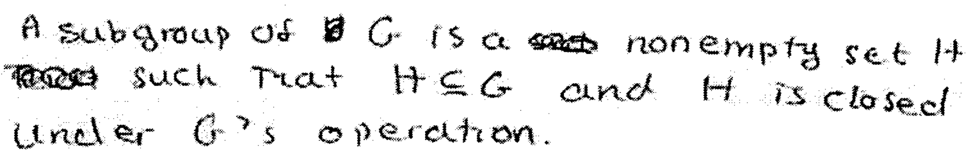

A subgroup of G is a nonempty set H such that H ⊆ G and H is closed under G's operation.

Figure 18. Missing "group" part example.

In Figure 18 the student mentioned closure under the operation of *G*. The student understands that a subgroup *H* is also a subset of *G*. However, it looks like she/he is not sure what kind of structure *H* is. Probably, for the student, the fact of being closed and being a subset gives *H* a required structure and in the student's opinion it is a group without any additional conditions.

In general, the students' responses for group/subgroup questions were rather problematic. I think that a subgroup concept formation is more complicated since it involves many other concepts. The responses showed a correspondence between students' understanding of a binary structure and a subgroup.

Exam 1

The exam covered the following concepts of group theory: binary operation, identity and inverse elements, and isomorphism. The analysis of the responses revealed similar problems in students' understanding of set–operation relations, previously discussed. In addition, I observed students' difficulties working with such important concepts of group theory as identity and inverse elements.

Generally speaking, the concept of identity element is familiar to students. Indeed, from early grades the students know that adding 0 to any number does not change the number. Similarly, multiplication by 1 does not change a number. However, in the context of group theory, the identity element is a more complicated and structured concept. In the attempt to learn the concepts, it is important to understand the relations between an identity element, the set it belongs to, and a binary operation defined on the set. Another problematic issue of the students' responses was the use of quantifiers. In many cases the students did not use quantifiers at all; some misplaced quantifiers and by all means changed the whole logical construct they described; finally, some students used one quantifier instead of another. I want to discuss the problem of quantifiers first since it seems to have a strong connection with later responses.

<u>Identity – quantifiers</u>

Understanding of quantifiers is crucial in the context of abstract algebra. Very often students find it difficult to decide why and when they should use quantifiers. Sometimes it is purely a symbolic problem. It is not a requirement to use quantifiers symbolically, and the students are simply not used to adding quantifiers into their responses. Moreover, most textbooks do not use quantifiers symbolically but rather in a descriptive manner: For example in Fraleigh (2003),

> 3.12 DEFINITION (Identity Element for $*$). Let $<S, *>$ be a binary structure. An element *e* of *S* is an identity for $*$ if $e * s = s * e = s$ for **all** $s \in S$. (p.32)

In some responses the incorrect use of quantifiers in definitions later affected problem solving. Consider different students' responses to the question: "define an identity element in a set with binary operation $*$" in Figure 19:

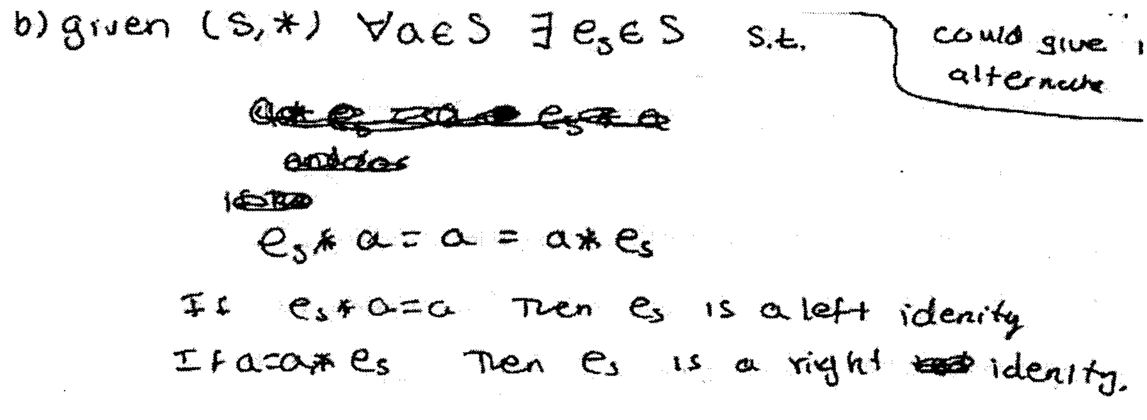


Figure 19. Illustration of ∀∃ – ∃∀ problem in identity definition.

In Figure 19 the student is saying that for all elements in the given set (*S*, ∗), there exists an element e, such that $e*a=a*e=a$. Clearly, the response states that there can be more then one identity element in the set, in fact, there could be infinitely many of them, if the set is infinite. It looks like the student is thinking that if a binary structure has an identity, then the identity must exist for every element of the set. Assuming that the student placed the quantifiers in such a way consciously, I present several explanation of possible students reasoning about the notion of identity. On one hand, the student may understand that there is (if exists) only one identity in a binary structure. Then the way she/he used the quantifiers stresses that this identity element must work for every element. In this case the quantifiers play a significant role in student's definition of identity while the order of quantifiers is not important and does not change the meaning of the definition. On other hand, if we take a look at Figure 19 and interpret it literary, it would mean that for the student an identity element depends on elements of the set, or that for every element of the set there exists a special element, called identity (*e*), with a certain property. In this case, it does not matter how many distinct identity elements a binary structure has, it is important that every element has one. Thus, generally speaking, the statement "∃∀" is being replaced by "∀∃". It could mean that the student does not understand and, thus, ignores the change of "dependent/independent" element; or, in contrary, stresses the dependence of identity element on set's elements. Students may not fully understand the meaning and strength of quantifiers. It may not occur to them that the order in which they write the quantifiers can change the meaning of a mathematical expression. In this case, the student's way of thinking in Figure 19 is not different from the other student's response in Figure 20 and the mistake we observe in Figure 19 is nothing else than misplacing symbols (quantifiers).

Figure 20. Correct order of quantifiers.

Note, in Figure 20 the student does not really give a definition of identity but rather states the axiom (group axiom) which we need to check in order to show that a binary structure has an identity element. I have noticed that students often do not distinguish definitions from theorems, properties and axioms. It suggests that the students often keep in mind only the "formula" part of a definition or property, theorem, axiom and do not try to analyze the descriptive part.

Nevertheless, coming back to Figure 19 and recalling that "∀∃" is a logical construct for the notion of inverse, it is possible that the student confuses the concept of identity and the concept of inverse. Unfortunately, there are not enough evidences in the data at this point to verify or contradict this assumption.

Analyzing the response in Figure 19, I should mention that the problem could be caused by students' lack of ability to use quantifiers. In this case, the student uses quantifiers because she/he probably has seen others (textbook, the instructor or peers) did so and for her/him the statement "$\forall\exists$" or "$\exists\forall$" does not make any difference in statements as if the student would not use quantifiers at all. The students are not required to take any mathematical logic courses except "Mathematical Proofs" prior to the abstract algebra course. However, quantifiers, such as $\forall$ (for every, or for all) and $\exists$ (there exists), have been used in many classes they have taken before. The textbook uses the "quantifying words and phrases *only, there exists, for all, for every, for each, and for some.* (Fraleigh, 1998, p.5)".

Figure 21 also illustrates disregardful use of quantifiers:

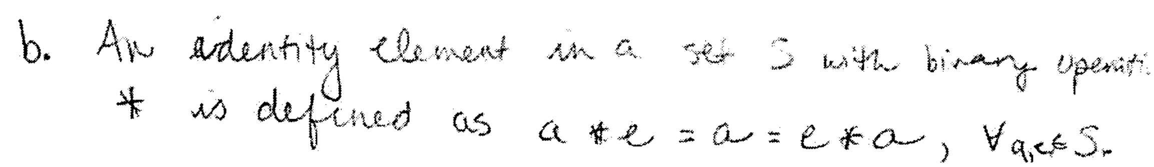


Figure 21. Response with misplaced quantifier.

Again, it is not clear why the student used quantifier this way. If we accept the assumption that *a* represents "any" element of *S* and "*e*" is reserved for identity, then it looks like the student may think that there is more then one identity element. So, if there are several identities "*e*" in S then they <u>all</u> must satisfy the equation. This definition cannot be accepted as correct and accurate;

however, it does not really contradict the concept of identity. In the textbook used for the class, the theorem about uniqueness of identity is coming after the definition. However, it is possible that the student did not use the symbol "∀" in the sense I described above. She/he perhaps only meant to say that *e* is also an element of S and the quantifier does not play any role in the definition.

While reading students' responses, I was surprised by the fact that many students used quantifiers in their answers. In spite of the fact that many of them did not use quantifiers in an appropriate way, the responses show that the students started to appreciate the role quantifiers play in the content. In the previous section (Quiz 1) universal quantification of properties or definitions of binary structures was a major problem. In Exam 1, many students used quantifiers to show that an identity is a universal quantification:

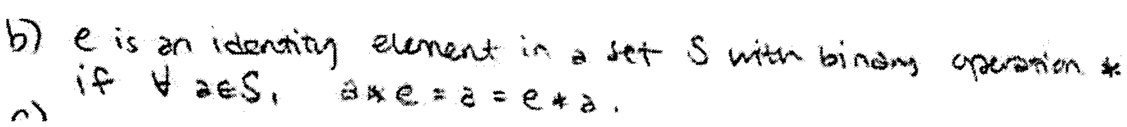


Figure 22. Illustration of student's use of universal quantifier.

However, the problem of universal quantification still persists in some responses:

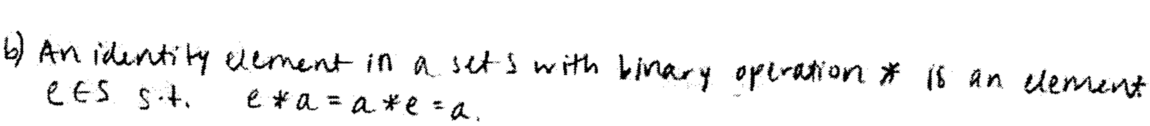


Figure 23. Missing quantifier.

As we can see the student did not specify at all where the element "a" comes from. I think that the student was very concentrated on recalling the equation, which describes identity property and she/he simply ignored the fact that there is no information about "*a*". Moreover, I think that the fact that the student did not use quantifier "for every" could in fact mean that "*a*" is "any" element of the set *S*, not a specific or the only element of *S* which satisfies the property above.

As we can see, the problem of quantifiers continues through the students' learning process. If in the first set of students' work (Quiz 1) the main problem was "to use or not to use" quantifiers, then in the second set (Exam 1) the dilemma is more complicated: "to use or not to use and if to use, then how".

Identity – uniqueness

The problem of quantifiers, observed in the responses, suggests that some students do not see an identity element as a unique element for a binary structure. When the students were asked to define an identity, all the students wrote some form of Equation 1. However, looking at the equation it is difficult to say if the students understand that an identity element is unique for every binary structure. The use of quantifiers clarified some responses, while made others even more confusing. In the next problem the students were asked to find an identity, using the definition and properties of identity.

Consider first the following definition of identity element:

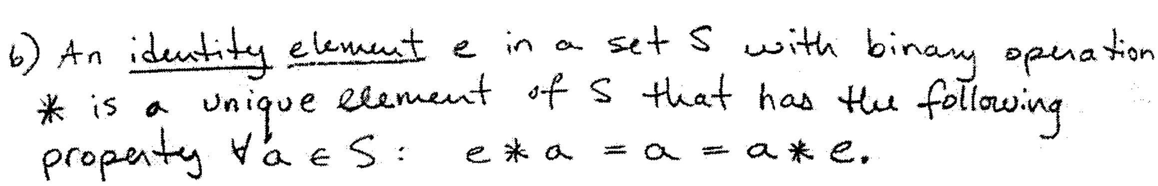

b) An identity element e in a set S with binary operation * is a unique element of S that has the following property $\forall a \in S$: $e * a = a = a * e$.

Figure 24. Student's definition of an identity element, stressing its uniqueness.

Note that the student explicitly added to the definition that an identity element is unique. The statement about uniqueness, however, requires a proof. I think that sometimes students want to represent all their knowledge about an object they define, although a uniqueness part is not a part of the definition and it needs to be proven. Nevertheless, it is possible that for the student uniqueness of an identity element is obvious. Figure 25, however, contradicts the assumption. Consider the same student's response to the following question:

Define a binary operation $*$ on $\mathbf{Q}^*$ by the rule:

$$a * b = \frac{1}{(ab)^k}$$

where k is an integer. For which value(s) of k will ($\mathbf{Q}^*$, $*$) have an identity element, and what will that identity (or those identity) element(s) would be?

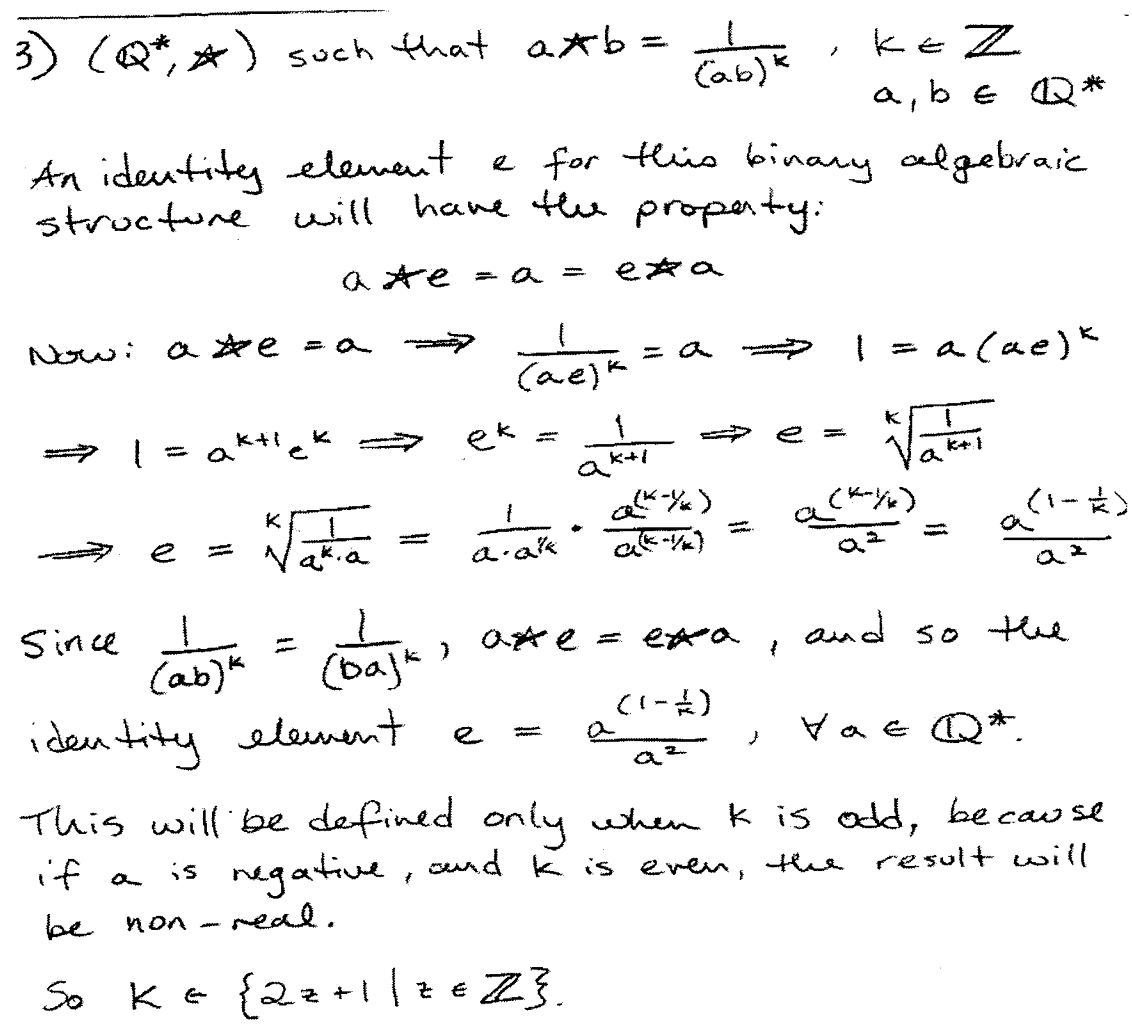


Figure 25. Student's solution for identity.

The student is looking for an element of the given set with the defined operation, which would satisfy Equation 1. She/he is solving the equation for *e*. When the result is found the student is analyzing it. She/he noted that the result

$e = \dfrac{a^{\left(1-\frac{1}{k}\right)}}{a^2}$ must be satisfied by $\forall a \in \boldsymbol{Q^*}$. Universality of identity element is explicitly stated in the definition exemplified above (Figure 24). With this condition, the student concludes that *k* must be an odd number, since otherwise it would not work for negative elements of ***Q****. However, for such *k*, the result would still depend on the element *a*. In other words, the element, which the student found to be the identity, could only satisfy Equation (1) for a single element – *a* itself. Note that in this case we observe misunderstanding of correspondence of quantifier "for every" in the identity definition: for the student, "for every *a* from ***Q****" means that every *a* from ***Q**** must satisfy the equation for *e*, while the actual meaning is the opposite – the element *e* must satisfy Equation (1) for every *a* from ***Q****. So, together with making sure that every element of ***Q**** would satisfy the formula for *e*, the student had to make sure that Equation (1) is satisfied by this *e* for every *a* from ***Q****, instead, the conclusion is based on formula for *e* <u>only</u>. Thus, in this problem, the identity element depends on the element itself and on the operation, defined on the set. It looks like the student is losing the uniqueness part in her/his response, while it is added to the student's definition of identity. Due to these controversial responses, it seems like the student does not connect a uniqueness of identity element and its independence of other elements of the set. Possibly for her/him the uniqueness is corresponded to an element of the binary structure. In other words, for every element of a binary structure there is (if exists) a unique identity element which satisfies Equation 1. If we look back at the definition, the final explanation I gave would look opposing. However, I think that the way the student used the definition to find an identity element in the problem above suggests that the assumption makes sense. In general, I have found 3 similar responses to the problem when analyzing Exam 1. However, only one more exam showed the disagreement between the solution for this problem and student's definition of an identity element. In Figure 26, the student's statement about uniqueness of an identity element is not included in the definition at first.

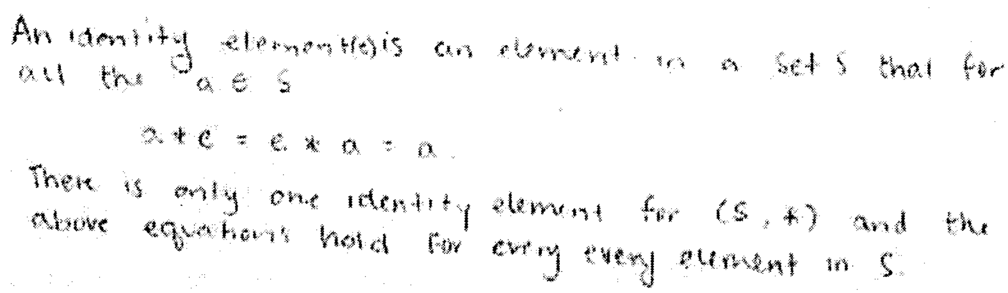


Figure 26. Student's definition of an identity, stressing uniqueness.

By the last sentence the student perhaps wanted to clarify the definition, to show more details, or to represent her/his own understanding and knowledge. So, together with the definition the student gives additional information, which is, however, contradicted by her/his response to the next problem:

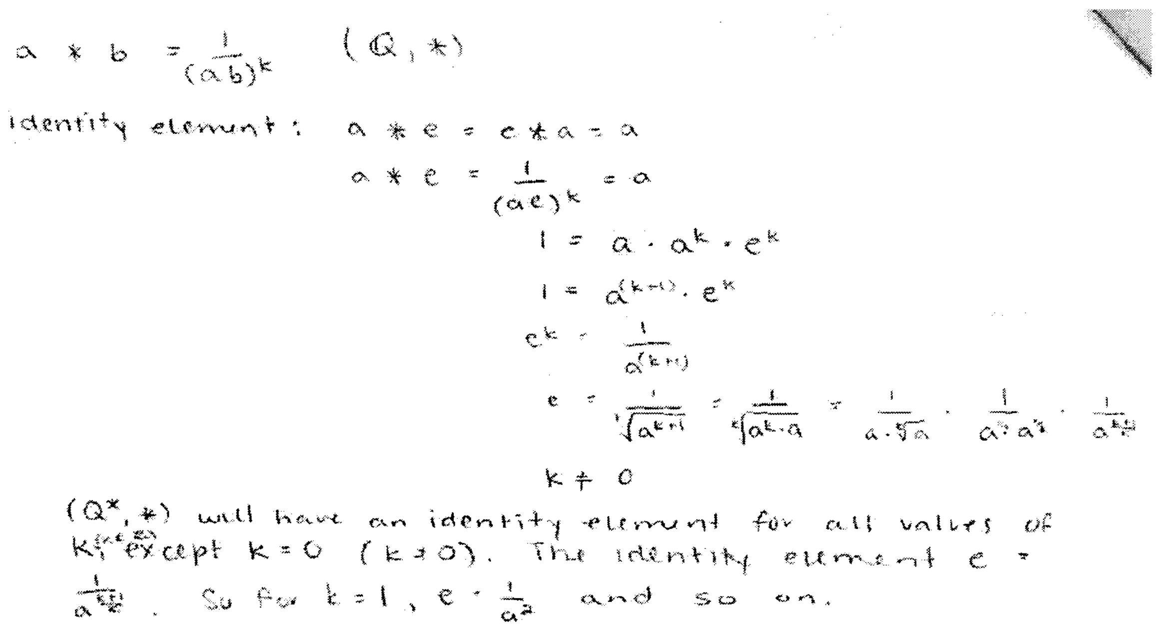


Figure 27. Multiple identity solution.

In the response the identity element again depends on *a*, which means that there could be more then one identity element. Moreover, in this case even values of *k* would not work for negative elements of ***Q****. Most likely the student did not use the whole definition of an identity element during problem solving. Perhaps the definition and uniqueness theorem did not come to the student's mind, since the problem was not asking to check if there is an identity explicitly. Equation (1), however, is used in both cases and I assume it is considered to be the most important part of the identity definition. Equation (1) gives the problem an algebraic (computational) solution and it was treated as an algebraic problem which did not require more analysis then finding a domain for k.

The "definition – problem" sequence in Figure 28 does not look controversial.

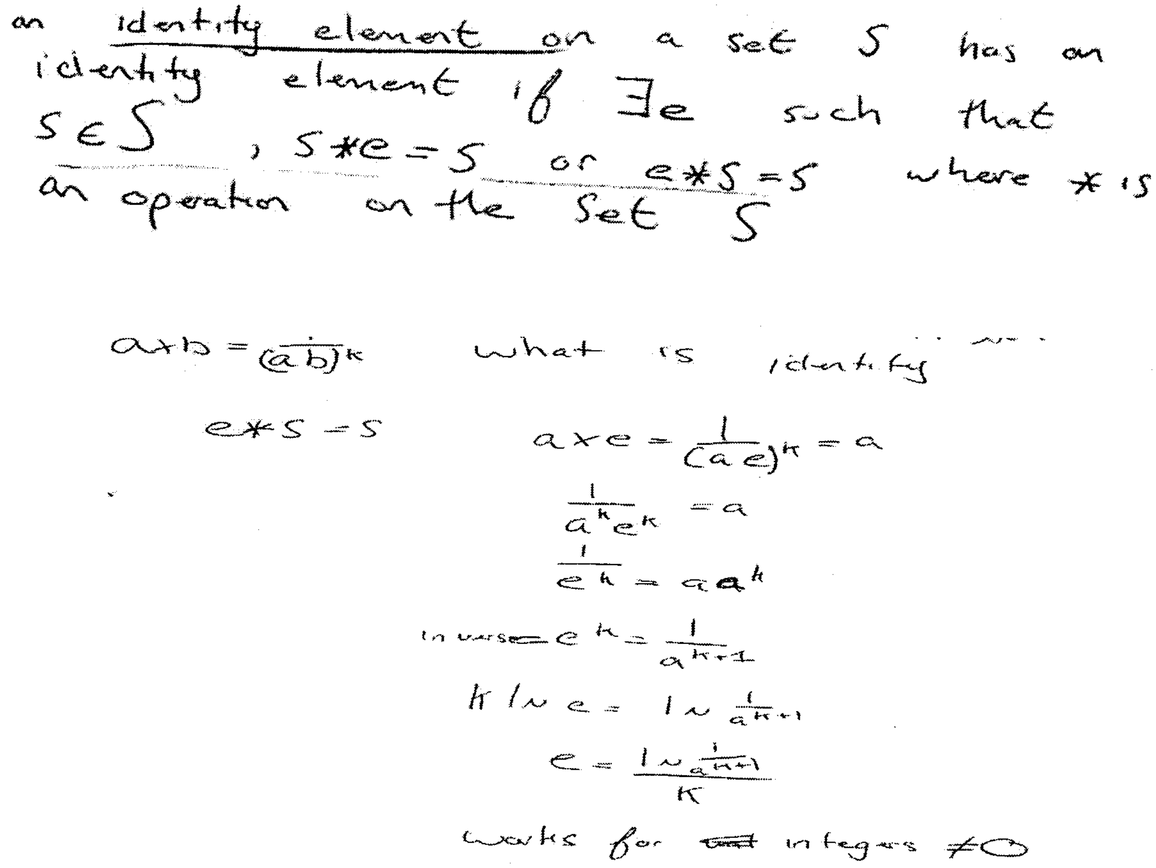


Figure 28. Student's definition identity and its application.

Definition of identity, the student stated, claims the existence of an element *e* which for *s* from *S* would satisfy the condition $s * e = s$ or $e * s = s$. The definition does not specify at all how an identity element is related to the set elements, other than *s* (missing quantifier). In this case, in spite of the symbolical problems (I assume the student is taking natural logarithm of both sides of equation since she/he unconsciously saw *e* as an exponent, which also is interesting and would require more investigation in the future), the student solved the problem without contradicting the definition above. She/he proved that there exists an identity element which satisfies Equation 1 for some element *a*.

In general, only one student states explicitly the fact that identity cannot depend on the element *a*.

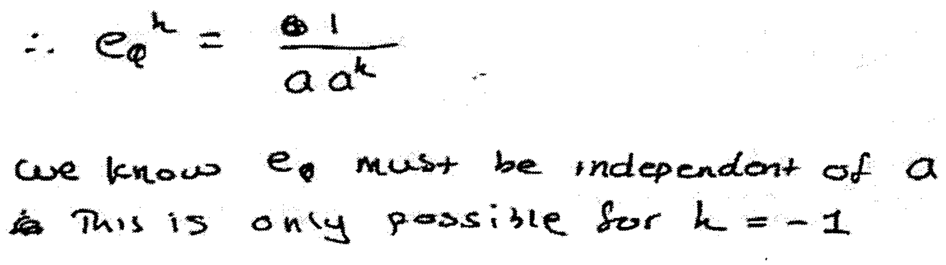


Figure 29. Independent Identity element.

However, the student's definition of identity I analyzed earlier (Figure 19) was not exactly correct. The definition suggested the opposite conclusion for the problem above: $\forall a \in S\ \exists e_S \in S$ implies that identity element depends on *a*. The disagreement suggests that the student is having difficulty understanding quantifiers, rather than the concept of identity in a binary structure. By adding a subscript "*s*" to identity symbol, I assume, the student wanted to stress that the identity *e* is a property of a binary structure and it is independent of an elements of the structure.

<u>Set – operation relations. Closure</u>

The analysis of Exam 1 revealed that the problem of closure of a binary structure under its operation still persists. Students' responses show misunderstanding of the concept of closure or simply overlooking this part of a binary structure definition. In Figure 25 and Figure 27, the students were so concentrated on finding an identity element that they missed the important and unfortunately more "global" problem of closure. While defining an identity element of a binary structure all the students wrote that *e*, identity element, is an element of a binary structure. However, in both Figure 25 and Figure 27 the students did not think about a binary structure, element of which *e* is. Obviously, the obtained result does not satisfy the definition of identity but also does not always belong to ***Q****. When a binary structure is defined, it looks like the students combine (all or some) elements of the binary structure by using the defined operation but not always checking if the result of such combination belongs to the binary structure. Moreover, in Figure 25 I discussed that the student tried to make the formula to work for all elements of ***Q****, while she/he completely ignored the result of the formula – the identity element per say. It looks like the student did not connect the identity element and the binary structure. Only one response shows explicitly the student's reasoning about the connection between the identity element and *a* binary structure (***Q****, $*$), given in the problem:

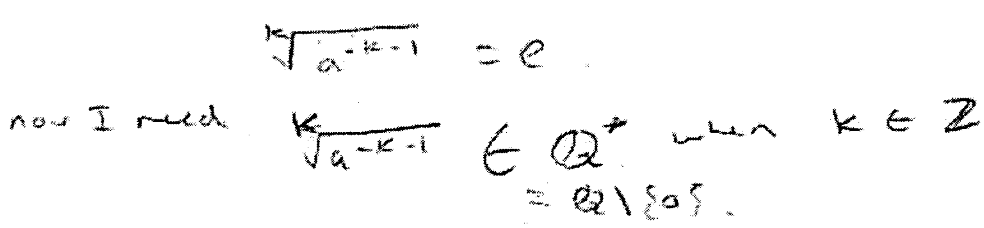
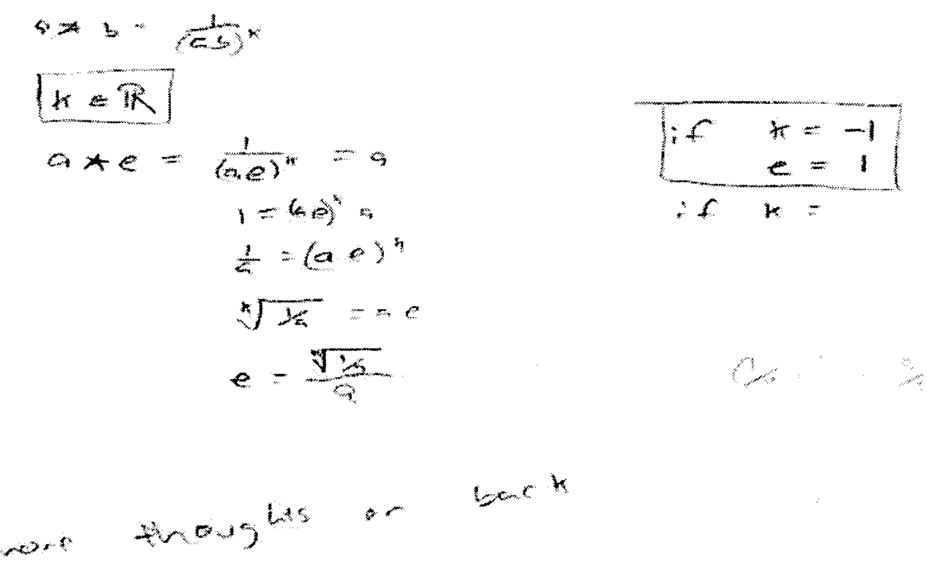


Figure 30. Consideration of the closure.

The problem that the student is trying to solve suggests that the idea of closure is more or less clear to the student. I also noticed that when the question is to define a binary operation or structure (such as group), students' responses include the part about closure: a result of the operation performed on two elements of the binary structure must belong to the set. However, the responses to other questions (Figure 25, Figure 27) show that the identity element is not a candidate for closure consideration. The last assumption suggests that students sometimes do not consider an identity element as an element of the set. This could be the reason why students often say that 1 is an identity element for a structure that does not include the element 1; or conclude that a binary structure is not a group since 1 does not belong to it. So, in a way, identity element is more connected to the operation of a binary structure rather then to the set.

Concrete examples

When solving problems in mathematics, it is natural to try to lower a degree of mathematical abstraction. It did not surprise me that some of the students found it difficult to reason about an abstract set or binary operation. Often they need to work with a couple of "concrete", more familiar examples before they actually solve a problem in a more general form. However, I was surprised to find some responses which illustrated the reverse way of thinking. In the following excerpt the student felt that there is a need for "more thoughts" after she/he already solved the problem and had the answer:



Back:

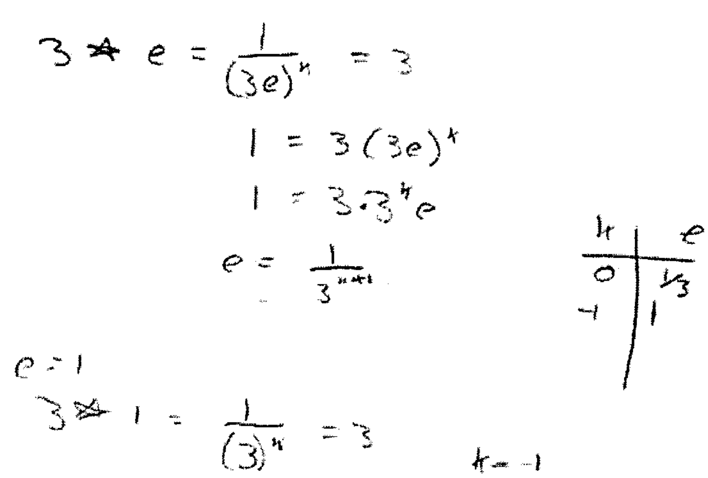


Figure 31. Student's reasoning with concrete examples.

The solution demonstrates that the student solved the problem rather intuitively, although the conclusion is correct. There is no elaboration on the result that shows the student's analysis of the binary structure closure with respect to the result or identity uniqueness. It is possible that the student could not confirm the result rigorously but the concrete examples made her/him more confident that the final conclusion is right. Note that in the example the student began with k = 0 and for a = 3 got e = 1/3. The fact that the student did not consider this answer as a correct one advocates her/his understanding of identity uniqueness for a binary structure and its independency of an element a.

In the following excerpt, however, the turn to concrete numbers was rather misleading:

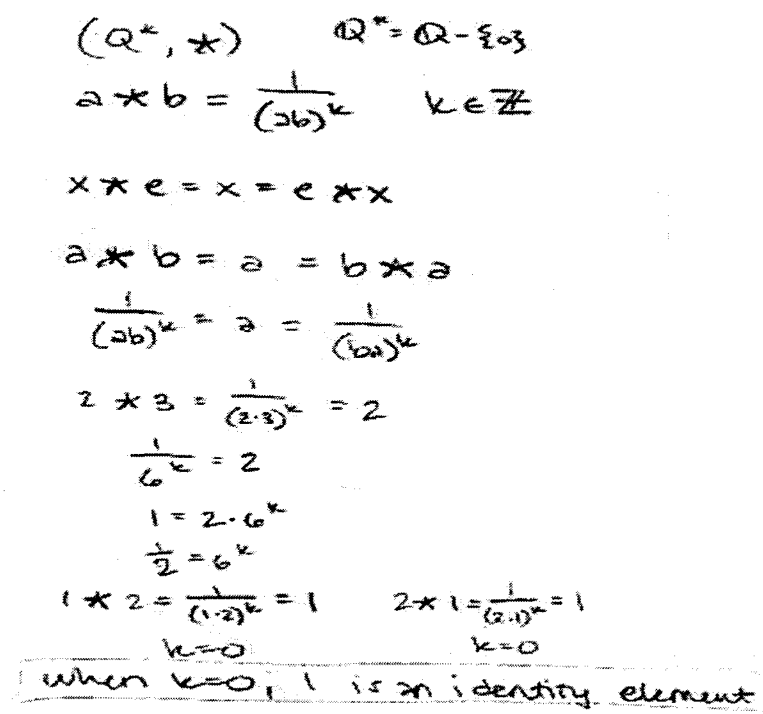


Figure 32. Misleading "concrete" argument.

It looks like by switching to concrete numbers, the student is trying to make sense of the problem, to understand the operation. I think that her/his last argument is rather interesting. First, note that from the beginning the student presumed that b = 3 is an identity. At this point it looks like the strategy is to find an identity by simply trying several pairs of numbers. I think, however, that the original understanding of an identity is rather ambiguous; even though Equation 1 is stated correctly. The student is trying to find $k \in \boldsymbol{Z}$, such that it would satisfy equation $\frac{1}{2} = 6^k$. The next row suggests that she/he realized that it is not possible and tries a different pair of rational numbers. She/he chose a pair (1, 2). There could be several reasons for that: first, the presence of the number ½ in her/his previous equation may have affected the choice; second, she/he might have chosen to work with numbers (1, 2) since they looked simple to the student; finally, the student could simply presume that 1 is an identity, since the operation "star" is defined in terms of multiplication. In the last case the choice of 2 is just random. Note that the last two equations both have 1 on the left side of the equality sign. That could mean a simple misinterpretation of the definition of identity. Also, the student might think about the inverse element instead. The last equations have a solution for k in $\boldsymbol{Z}$, so the student concluded that k must be 0, and 1 is an identity. Unfortunately, the student did not come back to the problem to verify her/his answer for the general formula and it is difficult to say if the switch to concrete numbers confused the student or the problem originally was not clear at all.

Identity element – group axioms

Identity is an important concept of abstract algebra because one of axioms in the definition of a group requires a binary structure to have an identity, in order to be a group. However, during the data analysis I have noticed that some students did not distinguish between the definition of identity and one of the group axioms. The following example illustrates the confusion:

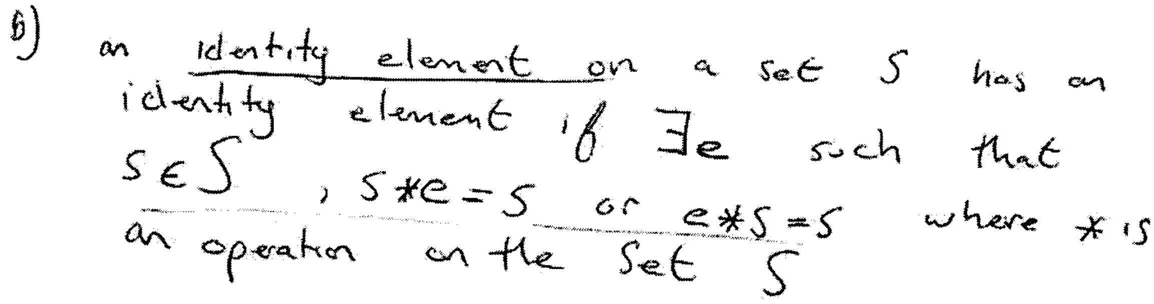


Figure 33. Definition – group axiom controversy

I think that the reason for the uncertainty is that both definition of identity and "identity" group axiom use the same equation - Equation 1. As I discussed earlier, the "algebraic" part of both mathematical statement seems to be the most important part for the students and the difference between the definition and the axiom could be really veiled for them. Students use the definition of identity when they need to figure out if a given element is the identity of (S, ∗) or not. During the problem solving activities, students are usually dealing with problems including "determine if the certain set has an identity" or "find an identity…". In other words, they are asked to check IF there is an identity and what element of the set is the identity. This could explain the confusion. I do not think that it automatically implies misunderstanding of an identity element, but rather misunderstanding of the particular problem's conditions and questions. However, it could imply even stronger misinterpretation of problems and dramatically affect the way of thinking. The next excerpt (Figure 34) illustrates how the student is trying to answer the question while using the incorrect concept in the solution process:

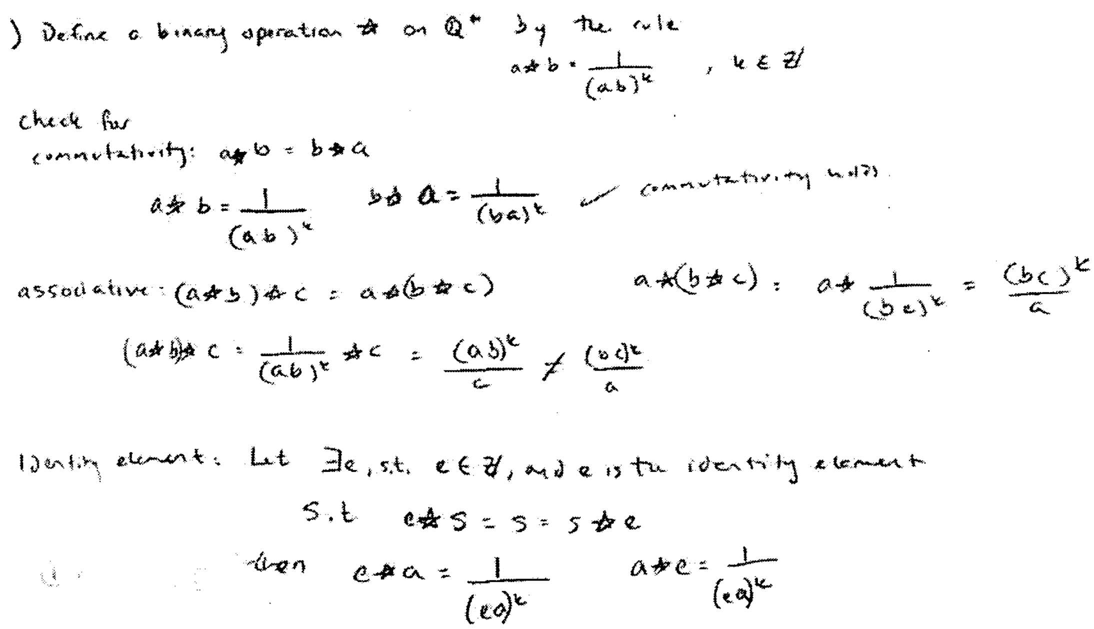


Figure 34. Student's reasoning about binary operation starting with group axioms.

The attempt to check for commutativity and associativity of the defined operation is noticed in several responses. I think the reason is that an identity element is not considered as a concept itself but rather a part of a bigger concept (group) and cannot exist without some special restriction on a binary structure. The textbook (Fraleigh, 2003, p. 37) defines a group in the following way:

> 4.1 Definition. A group <*G*, *> is a set *G*, closed under a binary operation *, such that the following axioms are satisfied:
> G1: For all $a, b, c \in G$, we have
> $(a*b)*c = a*(b*c)$. **associativity** of *
> G2: There is an element *e* in *G* such that
> $e*x = x*e = x$. **identity element** *e* for *
> G3. Corresponding to each $a \in G$, there is an element *a'* in *G* such that
> $a'*a = a*a' = e$. **inverse** *a'* of *a*

It is possible that for the student a binary structure can have an identity element if and only if the previous axioms (G1 in this case) hold. So, without checking for associativity, the identity problem would not make sense. I guess, for the student, the fact that the operation is not associative does not affect existence of identity.

The definition of an identity in students' minds has a strong connection to commutativity of an operation. Obviously, if the operation is commutative and a binary structure has an identity, then it must be both right and left identity. This observation may suggest that problems about identity require checking commutativity of operation first. However, the idea of commutativity may lead the student to the wrong conclusion about the existence of an identity element. Even if an operation is not commutative it is still possible to have a two-sided identity.

Exam 2

The second exam was centered on the concepts of a subgroup and a cyclic group. At this point the students seemed to feel more comfortable dealing with binary structures. Some students, for example, did not write a binary operation as $a*b$ in the responses but used the notation ab to represent an operation, without automatically assigning multiplication to it.

In this section I discuss students' reasoning about subgroups and students' understanding of a cyclic group (subgroup). At this point the students have to deal with a lot of new concepts including the order of a group, generator, and "degree" in the context of group theory. New theorems were introduced. The data suggested that for some students it was important to see all the elements of a subgroup to understand it and its connections to the group, while some students used the new theorems and avoided listing all possible outcomes. Sometimes it was necessary for the students to choose a concrete example and then consider the general case. The problem of quantifiers still persisted in students' work. Since problems became more abstract and structurally more complicated, the connection between quantifiers and students' reasoning about the problems became even more significant. The use of quantifiers considerably affected students' responses. Because of the complexity of the problems, I decided to analyze the exam problem by problem rather then category by category.

Analysis of Problems 1 and 2

Problem 1: Give a definition of a cyclic group that is as complete and accurate as possible.

Problem 2: Exhibit all the subgroups of $(\mathbf{Z}_{12},+_{12})$*, and indicate which of these are subgroups of others of these.*

In general, students easily accept the concept of a cyclic group. I observed that they showed good understanding of its elements. It also looks like the students understood the connection between the elements of a group and its generator. However, I think that in some cases their understanding of a cyclic group was rather intuitive. The students tried to adjust the concept of a cyclic group to the ones they had learned before. Perhaps, for some of them, reasoning about a cyclic group is more or less like observing patterns in middle school when students are studying functions. I think that is why some responses (as in Figure 35) were intuitively descriptive:

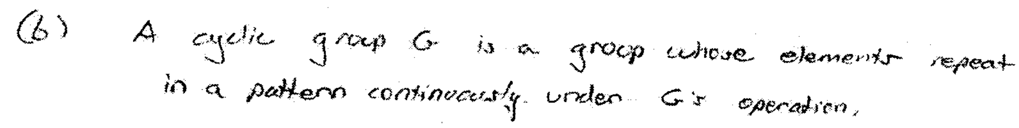


Figure 35. Student's description of a cyclic group

This response cannot be considered as a definition of a cyclic group, since the student did not describe how exactly the "elements repeat in pattern". It looks like the student tried to express the idea of a cyclic group.

In most responses, however, I observed more symbolical approach to the definition. All but 3 responses imply that the students are familiar and comfortable with the symbolic construction of a cyclic group ($G = \langle a \rangle$ and $G = \{a^n, n \in Z\}$) while there is evidence of obstacles in understanding of its elements (Figure 36).

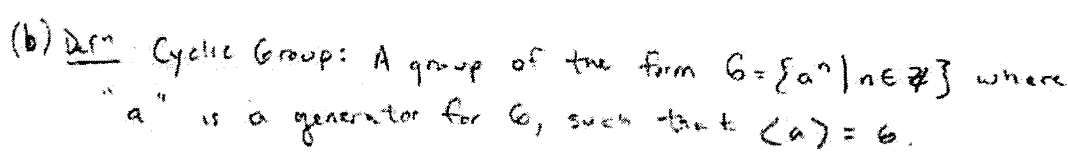


Figure 36. Student's definition of a cyclic group.

The student did not indicate if the element *a* belongs to the group. Perhaps it is obvious for the student. She/He states that *a* is a generator for *G* and $G = \langle a \rangle$. It looks like the student is confident that *a* belongs to the group. Out of 18 students, 9 did not state explicitly that *a* is from *G*. I still think that for the most students *a*, in fact, is an element of *G*. It is simply too obvious for them to write it down, since no other set is involved in the definition. I assume that the students' reasoning is based on the symbolic representation of a cyclic subgroup

$G = \langle a \rangle$. The representation stresses that group G is generated in some way by a certain element and this element cannot be considered apart from the group.

Nevertheless, understanding of the fact that a is an element of G is not enough for understanding a concept of a generator. Consider the following response:

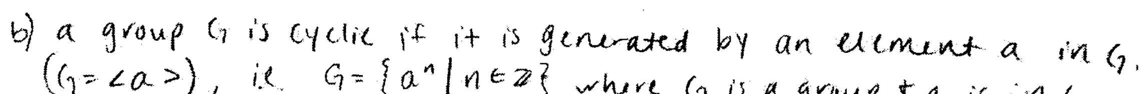


Figure 37. Student's definition lacking quantifier.

From the response it looks like any element of our choice a of G is a generator. Choose one element of G = $\mathbf{Z}_{12}$, say 4, and, by the definition above, $\mathbf{Z}_{12}$ is not a cyclic group since 4 does not generate $\mathbf{Z}_{12}$. I see two possible explanations for the case. The first one is due to lack of understanding of quantifiers. When we use a definition to determine if a group is cyclic or not, we need to show that there EXISTS an element such that $G = \langle a \rangle$. The student may have not formed the concept of a generator yet and does not understand its "existence" in the group. If every element of a group does not generate the group, then it is not cyclic. Second explanation is coming out of a definition of a cyclic subgroup of a group. Cyclic subgroup is a subgroup which is generated by an element of G. Moreover, any element of G is a generator for a certain cyclic subgroup and it looks like the definition above satisfies this concept. The above response could be considered an example of confusion between a cyclic group and a cyclic subgroup. These concepts are very close to each other and very easy to confuse. Every cyclic group has only cyclic subgroups whereas a cyclic subgroup is not necessarily a subgroup of a cyclic group. The analysis of problem 4 will provide more data support for the assumption. Actually, this idea was suggested by several students' responses:

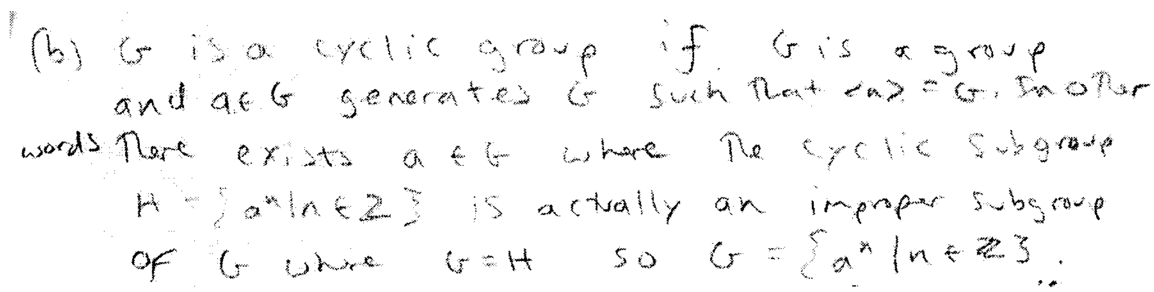

(b) G is a cyclic group if G is a group and $a \in G$ generates G such that $\langle a \rangle = G$. In other words there exists $a \in G$ where the cyclic subgroup $H = \{a^n \mid n \in \mathbb{Z}\}$ is actually an improper subgroup of G where $G = H$ so $G = \{a^n \mid n \in \mathbb{Z}\}$.

Figure 38. Defining cyclic group as improper cyclic subgroup.

While describing a cyclic group and its generator, the student probably recalled the definition of a cyclic subgroup. It might have confused the student at first but her/his last sentence shows how she/he made sense of both cyclic group and subgroup. Thus, for some students the difference between a cyclic group and cyclic subgroup of a group (not necessarily cyclic) is not clear. This uncertainty could be a result of misunderstanding of the role of generator: in the first case it must EXIST and in the second case ANY element of a group generates its cyclic subgroup. Again we deal with quantification. This time however, the situation is different from previously considered quantifier problems. Previously I observed how use of quantifiers affected the concept formation and now we see how the concept controls the quantifiers.

The analysis of the second problem cannot be distinguished from the first problem analysis in "subgroup" sense. Almost all the students described subgroups of $\boldsymbol{Z}_{12}$ explicitly (Figure 39).

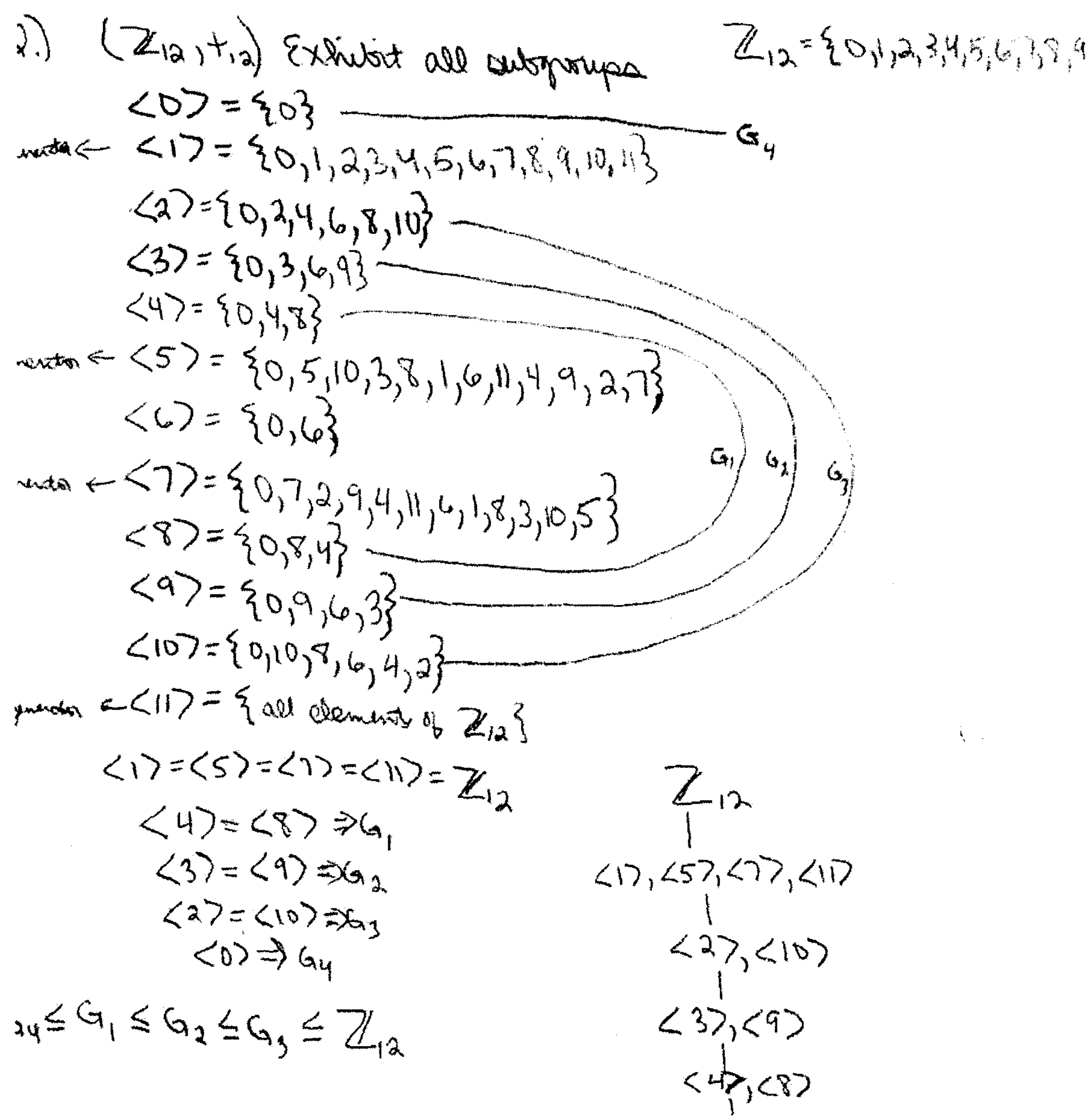


Figure 39. Student's reasoning about subgroups of Z_{12}

The student took each element of the group and generated a subgroup using operation $+_{12}$. It looks like the student' way of reasoning about cyclic groups/subgroups satisfies the definition. The fact that some of the subgroups are not the same as the group itself did not suggest that the definition may not be accurate. The student in the excerpt above realized that some of the subgroups coincide with the group. It could affect the definition of a cyclic group. Cyclic group can have more than one generator while the quantifier "there exists" could mean "exactly one" for the students and they do not think that it is appropriate to use it.

The next excerpt (Figure 40) supports "cyclic group/subgroup" assumption symbolically:

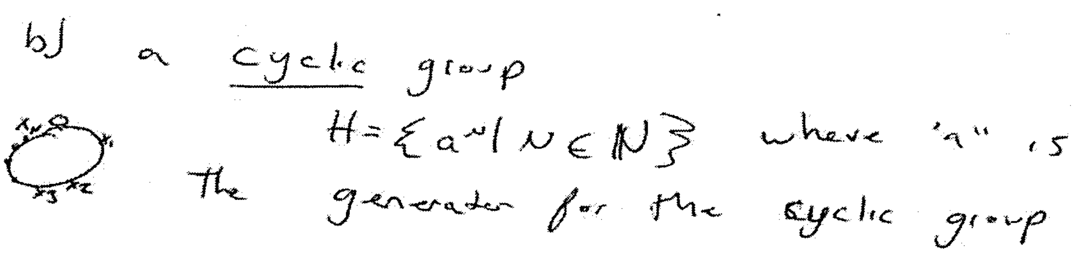


Figure 40. Student's definition of a cyclic group.

Commonly, a group is denoted by *G* and its subgroup is denoted by *H*. The way the student denoted a group this time suggests that she/he may confuse the definition of a cyclic group with the definition of a cyclic subgroup. In this case, "*a* is a generator for the cyclic group" means generator of the subgroup. Note the picture the student added to her/his response. The picture illustrates a cycle and suggests that the student understands the way a cyclic group/subgroup is generated. I think that the picture also implies that the student is thinking in terms of a finite cyclic group/subgroup. In further analysis I will discuss that sometimes the notions of a finite group and a cyclic group coincide in students' view.

During data analysis I observed a phenomenon, which occurred in many students' responses. It looks like the most important part of all definitions or theorems is the formula (if there is a formula) or algebraic expression. The conditions, for which the formula is true, are not considered to be as important as the formula. In all previous examples the students are in a hurry to write the definition algebraically, using symbols without describing them. Often the formula does not make sense if the description is not given or given in wrong terms. However, in some cases (Figure 41) the students did describe the conditions but did not give the formula:

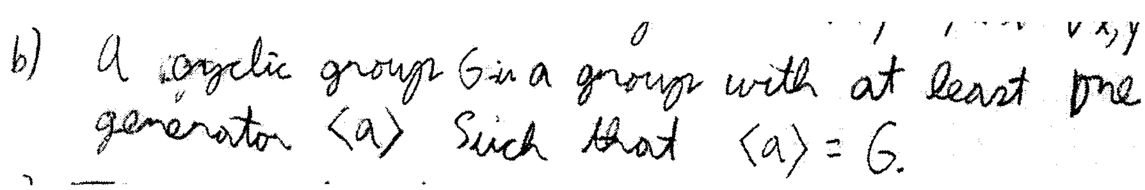


Figure 41. Definition of a cyclic subgroup without symbolic description of its elements.

The definition does not give the way of generating a cyclic group. This is rather unusual. I found 5 out of 18 responses which did not include precise description of *G* (in terms of its generator). Still, it looks like the student understands the generation of a cyclic group. Out of 5 responses of this type, 4 responses contained explicitly generated cyclic subgroups of $\boldsymbol{Z}_{12}$, using its elements as generators. Figure 39 exemplifies the same student's response for problem 2 where she/he generated the subgroups using the formula. This response suggests that the students who did not state the algebraic part of the definition of a cyclic group still understand how to generate the group. Absence of the formula could be a result of misunderstanding of symbols. For example, the notation a^n suggests that element a is multiplied by itself n times while the operation could be different (as in problem 2 the students needed to use addition to *generate* the subgroups of $\boldsymbol{Z}_{12}$). The students perhaps were confused by the notation and preferred to avoid writing what they did not completely understand. Another reason could be that the students simply forgot the algebraic notation. However, the correct solution of problem 2 suggests that the assumption is not accurate. The students may have difficulty in symbolic interpretation of the formulas. They seem to understand the concept but find it difficult to operate the symbols which they need for the concepts description. Nevertheless, the difficulties disappear as soon as the students are solving a "concrete" problem (by concrete problem I mean a problem that involves concrete numbers and operations rather then symbols), when they are dealing with a concrete cyclic group and listing all the elements of the group and performing the group operation on them.

The analysis of problems 1 and 2 suggested that the concepts of a cyclic group and its subgroups make sense for the students and they can operate with the notions while solving "concrete" problems. Further analysis however, revealed difficulties when the students are dealing with more abstract settings.

I already mentioned that almost all the students explicitly generated all the subgroups of $\boldsymbol{Z}_{12}$ and then concluded that some of them coincide and that some are the subgroups of others. At this stage, however, the students already covered the theorems which were aimed to eliminate the unneeded steps. The fact that the students prefer not to use the theorems suggests that they are not used to the theorems and prefer to use definitions; or, they do not understand and, thus, do not appreciate conclusions of the theorems. Consider the following series of examples.

Explicit List of Elements

In this group the students preferred to generate all the subgroups explicitly, using all group elements. After the exercise some students noticed some kind of "pattern" in their subgroups and recalled several theorems. In Figure 42 the subgroups are generated for all elements of $\boldsymbol{Z}_{12}$:

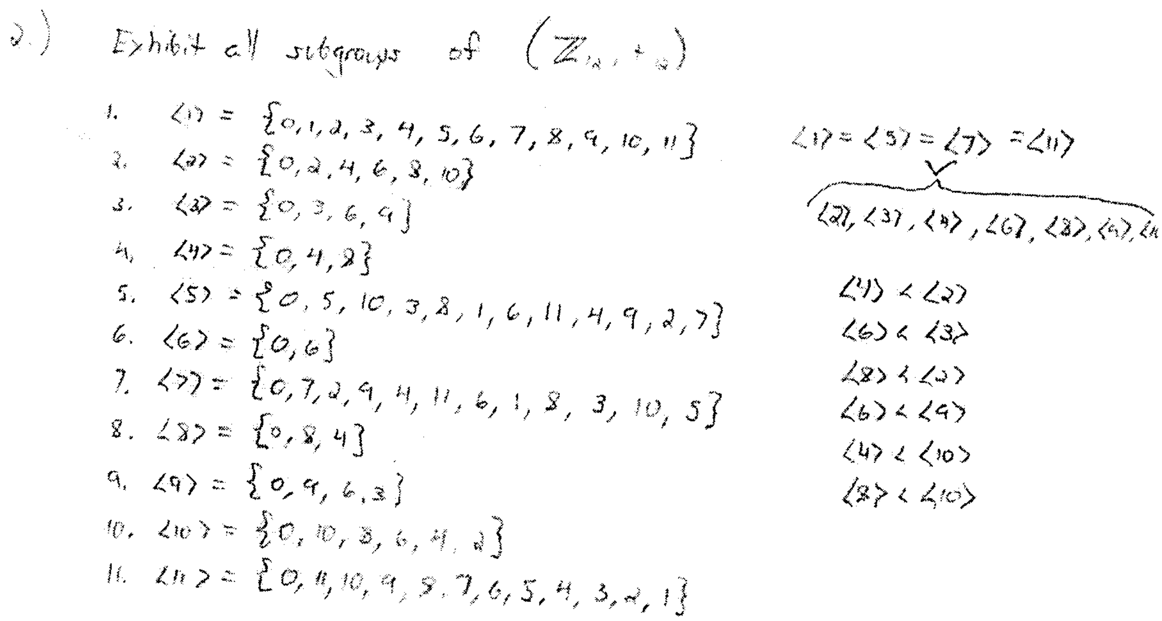


Figure 42. Computations of subgroups of Z_{12}.

The student listed all subgroups and made the final conclusion based on the list she/he developed. Out of 18 responses, 12 students chose the direct way of reasoning about subgroups of $\boldsymbol{Z}_{12}$. Note that the students did not explain why the listed subgroups represent the complete list of subgroups. They generated them using all elements of $\boldsymbol{Z}_{12}$ and definition of a cyclic group. However, if the given group would not be cyclic, the list of all subgroups would be incomplete. Only one student looked confident that these are all possible subgroups of $\boldsymbol{Z}_{12}$:

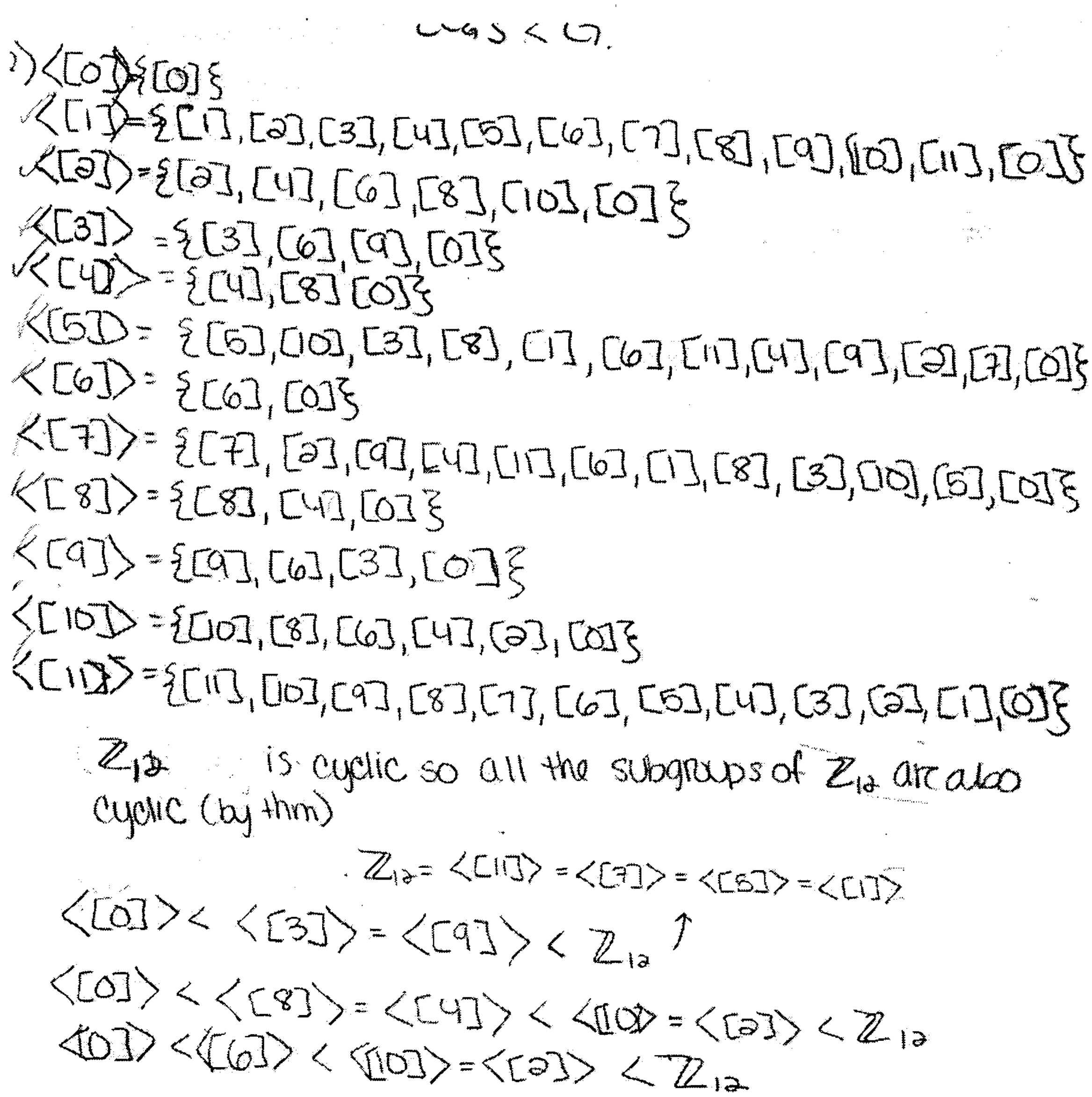


Figure 43. Computations of subgroups of Z_{12} with references to some theorems.

One student in the group of 12 recalled another theorem about cyclic group order and its generators (but first the student generated all the subgroups explicitly):

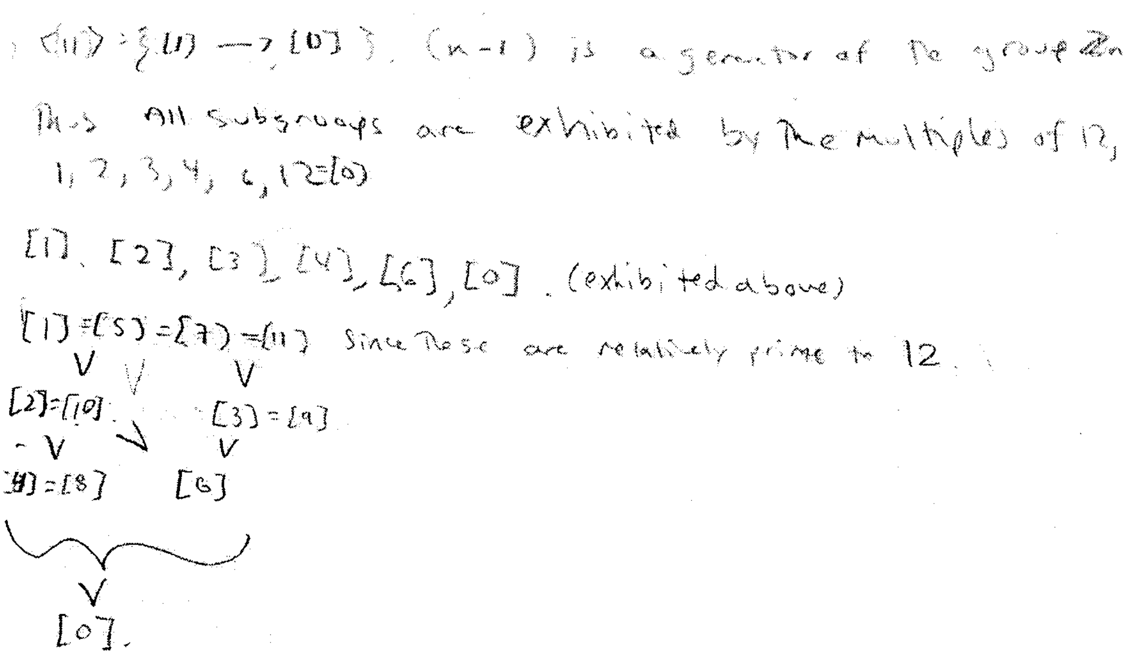


Figure 44. Reasoning about subgroups of Z_{12}, using the theorems.

Note, however, that the student recalled the theorem after she/he listed all subgroups of $\boldsymbol{Z}_{12}$. It suggests that the student might notice her/himself that 1, 5, 7, and 11 are relatively prime to 12 and this is the reason the subgroups generated by the elements coincide. By performing these steps the student recalled the theorem but she/he needed to "prove" it for $\boldsymbol{Z}_{12}$ first. After that the student stated the theorem's condition and confirmed the conclusion.

<u>Use of theorems</u>

Nevertheless, some students recalled several helpful theorems. The following response suggests that the student still did not feel very comfortable using the theorems and needed to write theorem conditions for each case she/he considers:

2) Exibit all the subgroups of $(\mathbb{Z}_{12}, +_{12})$ and indicate which of these are subgroups of others of these.

We know the generators of a cyclic group G of order n are the elements of that group G of the form a^r where r is relatively prime to n.

and so the generators of $(\mathbb{Z}_{12}, +_{12})$ are

1, 5, 7, 11

and so we have

$\langle 1 \rangle = \langle 5 \rangle = \langle 7 \rangle = \langle 11 \rangle = \mathbb{Z}_{12}$

so we look at $\langle 2 \rangle$

$\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$

$\therefore |\langle 2 \rangle| = 6$

and so we look for the other generators of 2

$\langle 2^r \rangle$ st. r is relatively prime to 6

r = 5

$\langle 2^5 \rangle = \langle 10 \rangle = \langle 2 \rangle$

$\langle 2 \rangle$ is closed under $+_{12}$ since $\forall a,b \in \langle 2 \rangle$ $a +_{12} b \in \langle 2 \rangle$

and $\langle 2 \rangle \subseteq \mathbb{Z}_{12}$

$\therefore \langle 10 \rangle \subseteq \mathbb{Z}_{12}$

so now we look at

$\langle 4 \rangle = \{0, 4, 8\}$

$|\langle 4 \rangle| = 3$

$4^2 = 8$

and 2 is relatively prime to 3

$\therefore \langle 8 \rangle = \langle 4 \rangle$

but $\langle 4 \rangle \leq 2$

$\langle 4 \rangle$ is closed under $+_{12}$ since $\forall a,b \in \langle 4 \rangle$ $a +_{12} b \in \langle 4 \rangle$

and so $(\langle 4 \rangle, +_{12}) \leq (\mathbb{Z}_{12}, +_{12})$

Figure 45. Reasoning with theorems.

Figure 45 illustrates only a partial solution since the rest is repetition of the same steps with the other number. We can observe what theorems the student used and how. It looks like the student had a deep understanding of a cyclic group and its subgroups. For the student a cyclic group is not merely a list of elements each of which is some power of the generator, but rather a structure which has various characteristics such as group order, or element order. The responses showed the understanding of relations between generators, group order, operation and element order which implies the understanding of connections between a cyclic group and its subgroups. In other words the concept of a cyclic group and its subgroups is learned in a coherent form.

Analysis of Problem 3

*Problem 3: Is it possible to find two nontrivial subgroups H and K of (**Z**, +) such that $H \cap K = \{0\}$? If so, give an example. If not, why not?*

The problem was aimed to verify students' understanding of subgroups and subgroups of ***Z*** in particular. Most of the students recalled that all subgroups of (***Z***, +) are described by (*n**Z***, +), where *n* is an integer. However, I found several interesting responses with misleading ideas. The major problem was again a subgroup concept. First, I would like to discuss the misconceptions caused by the situation when a certain group is a subgroup of another group. The fact that a structure is a group and a subset of another structure still does not imply that the first structure is a subgroup of the second. As I discussed previously, a "set – operation" problem was very common for all binary structure concepts. Consider Figure 46:

3) This is possible looking back at (2)

$(\mathbb{Z}_{12}, +) < (\mathbb{Z}, +)$ (notation issue here with $+$ and $+_{12}$ please bear with me as I am about out of time)

and we found

$(\langle 8 \rangle, +_{12}) < (\mathbb{Z}_{12}, +)$

$\therefore (\langle 8 \rangle, +_{12}) < (\mathbb{Z}, +)$

but also

$(\langle 3 \rangle, +_{12}) < (\mathbb{Z}_{12}, +)$

$\therefore (\langle 3 \rangle, +_{12}) < (\mathbb{Z}, +)$

but

$\langle 8 \rangle = \{0, 4, 8\}$

$\langle 3 \rangle = \{0, 3, 6, 9\}$

and $\langle 8 \rangle \cap \langle 3 \rangle = \{0\}$

$\therefore$ for $H = (\langle 3 \rangle, +_{12})$

and $K = (\langle 8 \rangle, +_{12})$

Then $H \cap K$ is the trivial subset of $(\mathbb{Z}, +)$

Figure 46. Using groups that are not subgroups.

It looks like the student ignored operational part of the structures ($\boldsymbol{Z}$, +) and ($\boldsymbol{Z}_{12}$, $+_{12}$). Note that she/he does not use $+_{12}$ at first but uses the symbol when she/he talks about subgroups of $\boldsymbol{Z}_{12}$. However, it seems like the symbolical difference in operations the student indicated did not provide a sufficient reason for distinguishing the two structures. The student calls it a “notation issue” (in parenthesis on the right hand side) rather then operational difference. It looks like the response was in a big way affected by the previous problem. As I discussed in analysis of Figure 45 (solution of Problem 2 by the same student), the student wrote all the details in the solution of Problem 2 and spent a lot of time on the problem. Therefore, the student's mind was still occupied with the previous problem and she/he was in a hurry to complete the work. Nonetheless, solid understanding of a binary structure as a set, together with the defined operation is not formed yet and one of the parts (set or operational) of a binary structure is often sacrificed due to certain circumstances such as time limit.

Another issue I observed analyzing this problem is illustrated in Figure 47.

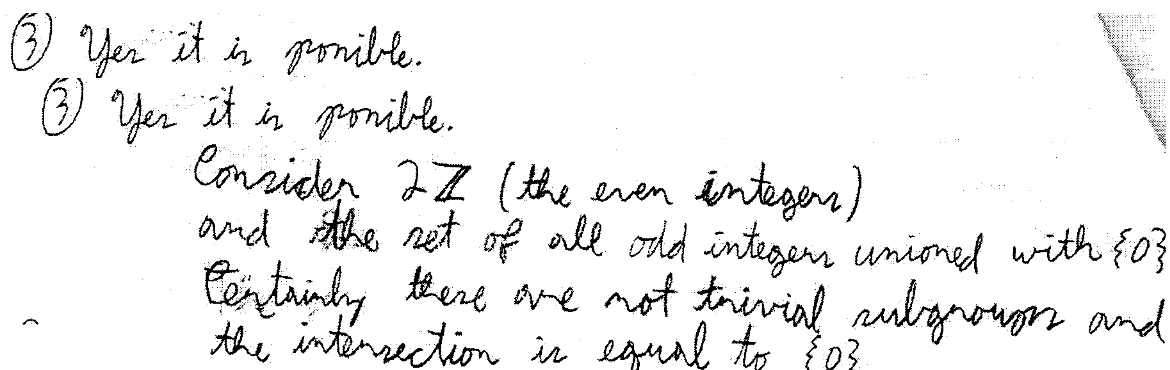

(3) Yes it is possible.
(3) Yes it is possible.
Consider $2\mathbb{Z}$ (the even integers)
and the set of all odd integers unioned with {0}
Certainly there are not trivial subgroups and
the intersection is equal to {0}.

Figure 47. Using sets which are not subgroups.

The student described two nontrivial sets whose intersection is 0. However, only one of the sets is actually a subgroup of (***Z***, +). The set of odd integers together with 0 is not a group to begin with. In this case we observe the lack of understanding of a binary structure closure under the operation. Indeed, the set, described by the student, is not closed under addition of integers. Perhaps the definition of a cyclic group (subgroup) and its generator is confusing for the student: it looks like she/he thinks about generation of the set of odd numbers by adding 2 to the next element. Probably, in the student's view, 2 is a generator of the set of odd numbers and even numbers, so, the sets are generated in a similar way. Consider the definition of a cyclic group given by the student (Figure 41). The definition does not explain how a cyclic group is generated, so it is possible that, for the student, number 2 generates the set of odd number in a following way: 1 + 2 = 3, 3 + 2 = 5, etc. Further, the student adds one more element (element 0) to the set, since the intersection of the two sets must be 0. Zero is an identity element for (**Z**, +) and does not affect other elements in terms of closure. Thus, we observe not only a misleading argument about a group (closure matter) but also one about cyclic groups and their generators. Figure 48 demonstrates corresponding problem.

nontrivial subgroups H + K, so $H \neq \{e_H\}$ + $K \neq \{e_K\}$.
Since H + K both closed under +, e = 0 in both subgroups.
So $H \neq \{0\}$ + $K \neq \{0\}$
However, since H + K are subgroups of $(\mathbb{Z}, +)$, they must contain the identity element.
Take $H \neq K$, where H contains at least $\{a, a^{-1}, e\}$ and K at least $\{b, b^{-1}, 0\}$ because $H + K \neq \{0\}$ they must have e, and an inverse for every element $\neq e$ since they are subgroups of $(\mathbb{Z}, +) \neq \{0\}$. Then if $a \neq b$, $H \cap K = \{0\}$ because it's the only element resulting from intersection.
Thus, it is possible to find two nontrivial subgroups H + K of $(\mathbb{Z}, +)$ s.t. $H \cap K = \{0\}$.

Figure 48. Using sets that are not closed under addition.

Two structures the student described are not closed under addition for any integers. Moreover, they are finite, which suggests that the student probably thought about cyclic groups. Note how the student is checking that the structures are subgroups: "they must have *e* and an inverse for every element $\neq e$ since they are subgroups." Indeed, identity and an inverse are important elements of a subgroup but there are more conditions to discuss. Also the student did not connect the structures she/he described to the given group (***Z***, +). This is rather interesting. I observed students attempts to give concrete examples for the problem (Figure 49).

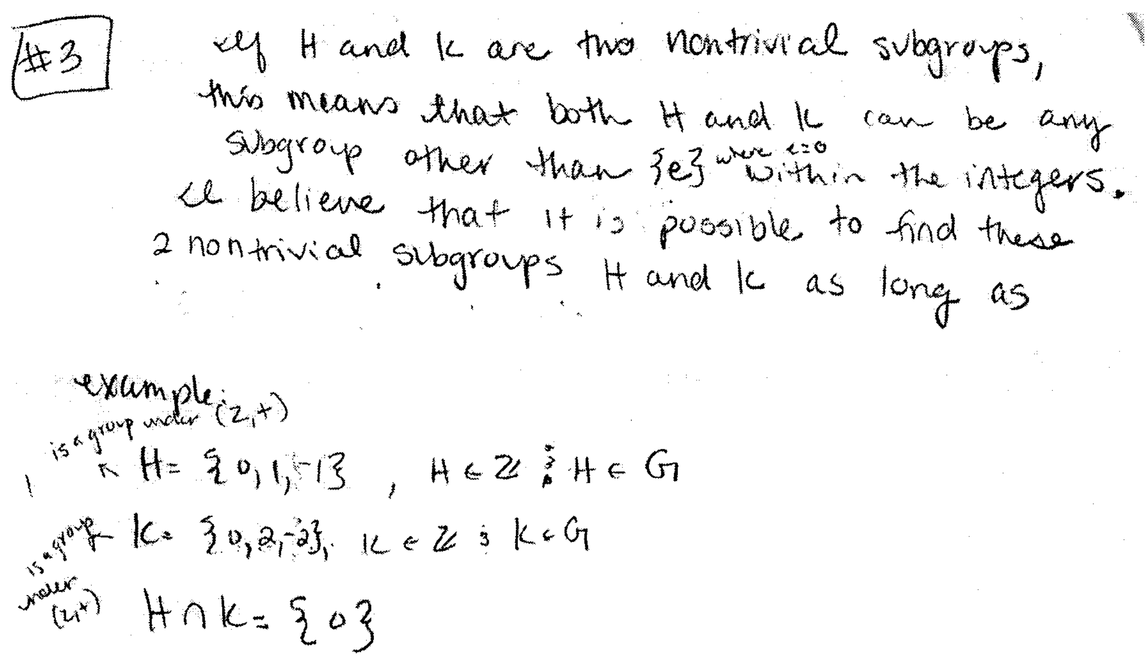
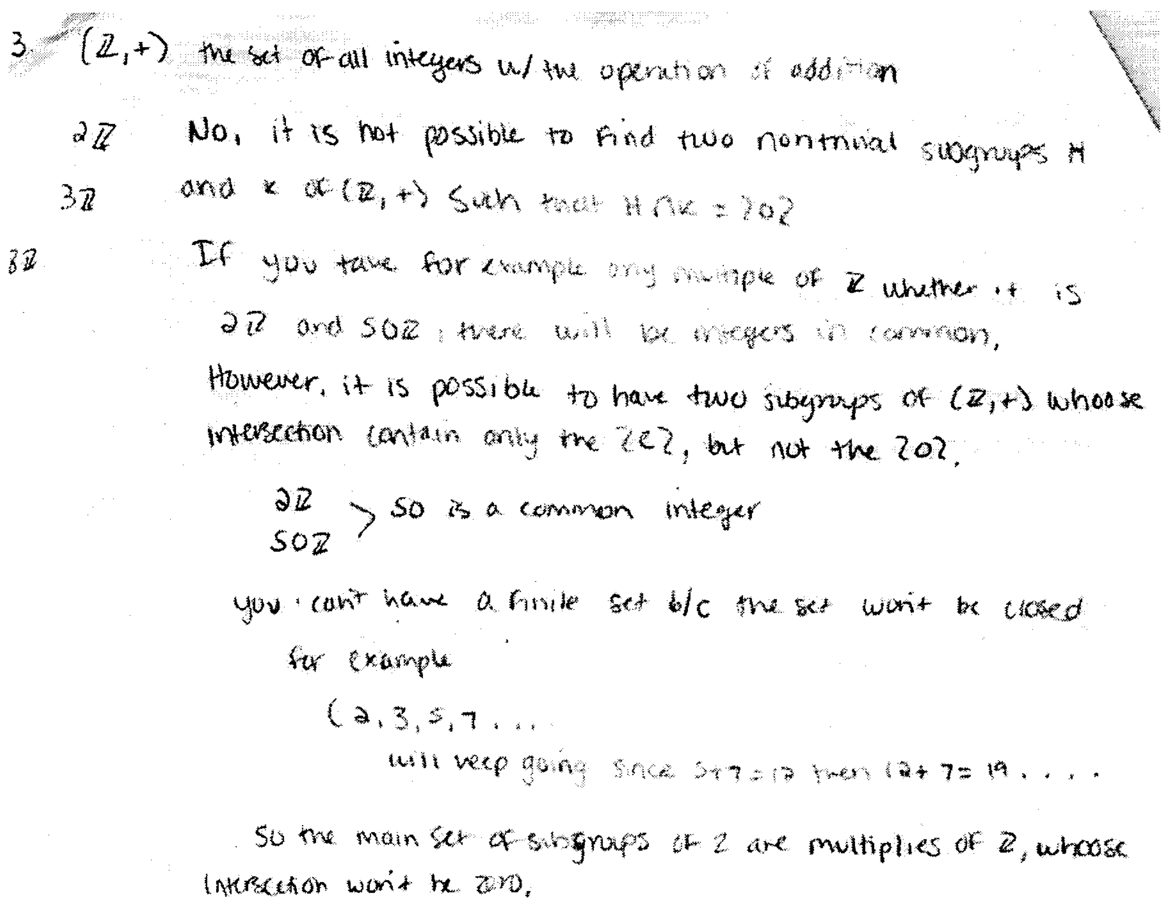


Figure 49. Student's Reasoning about subgroups.

In Figure 49 we observe the same misleading argument but in terms of concrete numbers. In Figure 48, however, the student seems to describe a more general case. If we assume that by *a, b* she/he meant some numbers (not necessarily integers) then the problem of closure is not similar to one in fragment 15. Note, that if we apply standard addition of integers to the elements *{0, a, -a}* only, without repetition, then the structure would be closed. However, a binary operation is applied to all pairs of elements of a set and to a pair (*a, a*) as well.

The student did not think about it. Thus, the problem of understanding of closure in this case is rather the problem of understanding of a binary operation.

Analyzing the problem I have noticed that many arguments (correct or incorrect) are based on concrete numerical examples. I already illustrated the misleading argument in Figure 49. I want to discuss more responses with concrete examples where the examples helped the students to understand the problem and led them to a more or less accurate conclusion (Figure 50).


3. (Z,+) the set of all integers w/ the operation of addition

2Z No, it is not possible to find two nontrivial subgroups H
3Z and K of (Z,+) such that H ∩ K = {0}
8Z If you take for example any multiple of Z whether it is
2Z and 50Z there will be integers in common,
However, it is possible to have two subgroups of (Z,+) whose intersection contain only the {e}, but not the {0}.

2Z
50Z > 50 is a common integer

you can't have a finite set b/c the set won't be closed
for example
(2, 3, 5, 7 . . .
will keep going since 5+7=12 then 12+7=19 . . . .

So the main set of subgroups of Z are multiplies of Z, whose intersection won't be zero.

Figure 50. Student's argument with concrete examples.

In Figure 50 the student comes out with two concrete examples of subgroups of ***Z*** and finds it impossible to get the {0} intersection. Further, she/he checks if there is a possibility for finite subset. Note that the argument again involves concrete numeric examples. Finally, the conclusion is more or less general. The conclusion does not involve symbolical notations of subgroups of ***Z*** (multiples of ***Z***). It suggests that the student may not feel comfortable using symbolic notation and prefers to give a concrete or descriptive solution. However, in Figure 51 the student's final argument looks accurate and reasonable and it is based on the concrete example:

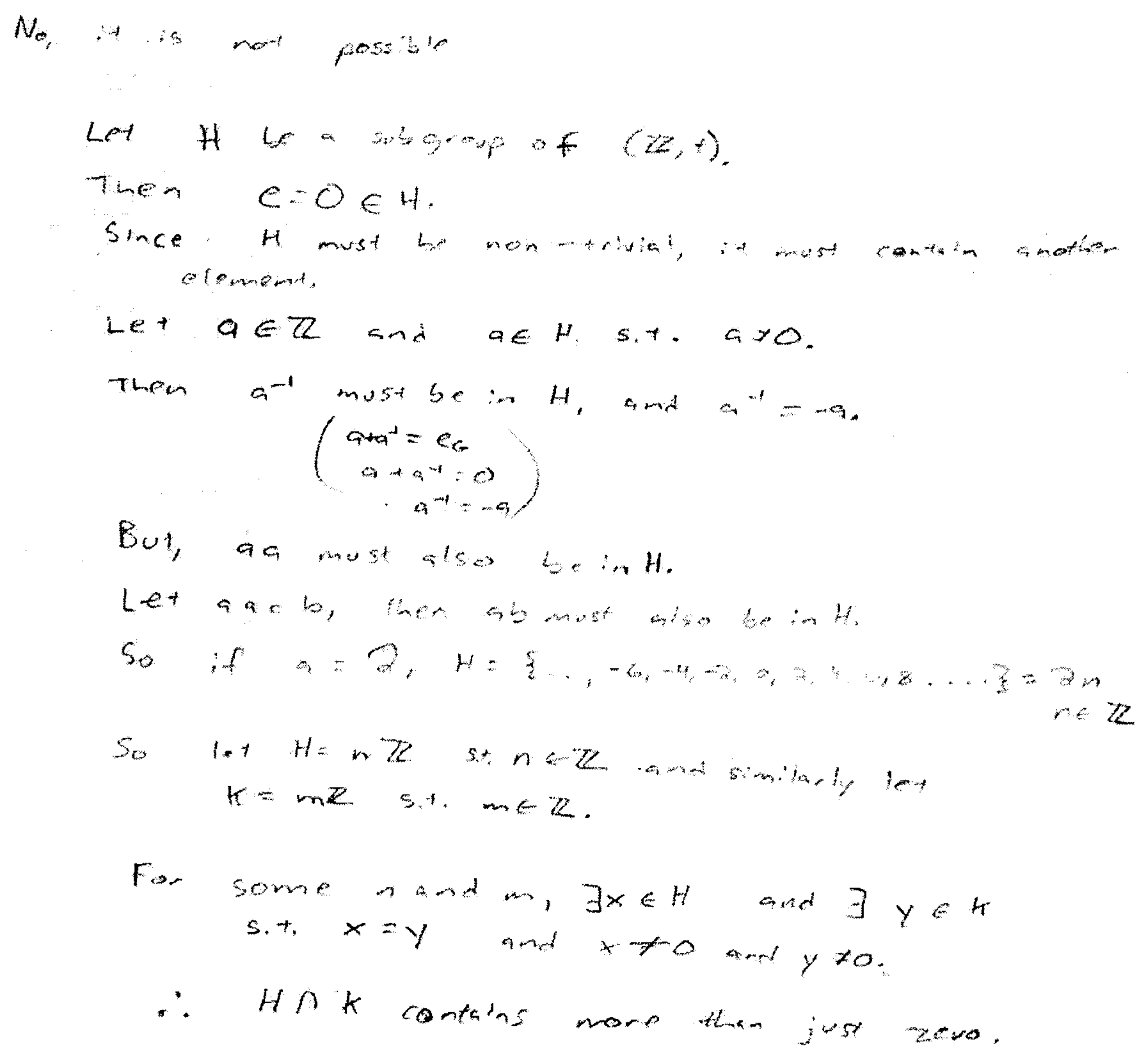


Figure 51. Student's attempt to prove her/his statement.

It looks like the student did not know how to solve the problem although the conclusion is given in the very beginning. It also looks like none of the theorems, which could be used in the problem, emerged in the beginning. The student simply reasons about a subgroup of (***Z***, +), as she/he tries to imagine how it should look. At first she/he tries to avoid concrete numbers and works with generic element of (***Z***, +). However, later it became difficult to observe all the elements in terms of *a* and *b*, so the student decided to make the argument more concrete and clear. After that a more general conclusion comes to mind and the student continues the solution in a more abstract manner.

I already discussed that for some students a general conclusion based on previously learned theorems or previously solved problems does not seem valid enough. I think it is possible that the students cannot fully perceive the general argument and try to give a concrete example to justify the answer. Maybe they just want to illustrate the accurateness of the argument, to show that it is, indeed, correct since the concrete example supports the case (Figure 52).

3 $H \cap K = \{0\}$

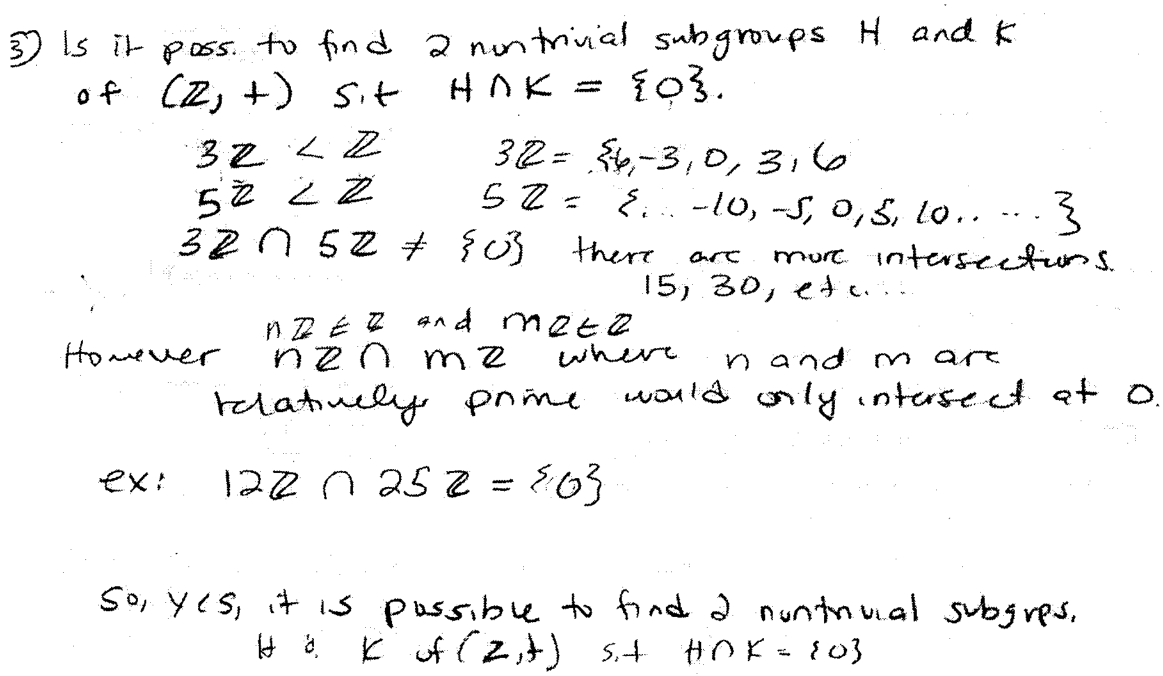
$n\mathbb{Z} \leq \mathbb{Z}$ $\quad \mathbb{Z}^+ \leq \mathbb{Z}, \; 0 \notin \mathbb{Z}^+$ $\quad \mathbb{Z}^* \leq \mathbb{Z} \quad 0 \notin \mathbb{Z}^*$

This is not possible because the nontrivial subgroups that contain 0 of $\mathbb{Z}$ are $n\mathbb{Z}$, $n \in \mathbb{Z}$. However, the intersection of two of these cannot be $\{0\}$ because they will all have other common elements. If $H = 3\mathbb{Z}$ and $K = 5\mathbb{Z}$, $H \cap K = \{\ldots, 0, \ldots, 15, \ldots\}$. There are not cyclic subgroups possible either since -1 and 1 are the only generators and they form $\mathbb{Z}$.

Figure 52. Supporting of the argument.

The student proves the statement in general terms. It is possible that the student gives the example because she/he is not sure how to describe the conclusion symbolically.

Nevertheless, sometimes the concrete examples could be misleading rather than helpful (Figure 53).



5) Is it poss. to find 2 nontrivial subgroups H and K of $(\mathbb{Z}, +)$ s.t $H \cap K = \{0\}$.

$3\mathbb{Z} < \mathbb{Z}$ $3\mathbb{Z} = \{-6, -3, 0, 3, 6$

$5\mathbb{Z} < \mathbb{Z}$ $5\mathbb{Z} = \{\ldots -10, -5, 0, 5, 10 \ldots\}$

$3\mathbb{Z} \cap 5\mathbb{Z} \neq \{0\}$ there are more intersections. 15, 30, etc...

$n\mathbb{Z} \in \mathbb{Z}$ and $m\mathbb{Z} \in \mathbb{Z}$

However $n\mathbb{Z} \cap m\mathbb{Z}$ where n and m are relatively prime would only intersect at 0.

ex: $12\mathbb{Z} \cap 25\mathbb{Z} = \{0\}$

So, yes, it is possible to find 2 nontrivial subgrps. H & K of $(\mathbb{Z}, +)$ s.t $H \cap K = \{0\}$

Figure 53. Finding counterexample.

The student gives the concrete example of two subgroups with nontrivial intersection. Further, the student attempts to discuss the solution in more general terms and that is when the concrete example confuses her/him. It looks like the reason for confusion is coming from the student's background. Perhaps she/he is uncertain about the definitions of a common multiple and a common factor of two numbers. This is a widespread confusion. I observed it many cases when teaching operations on fractions. To simplify fraction we need to find common factors of numerator and denominator, while to add fractions we need a common denominator which is a common multiple of the denominators. My personal teaching experience suggests that the confusion could still cause problems in students' understanding of more advanced mathematical concepts. Note that the first example in Figure 53 also involves relatively prime numbers (5 and 3). However, the student makes correct conclusion and finds common multiples. For some reason she/he makes the wrong statement right after the example. It may be difficult for the student to operate with more abstract terms and she/he simply got confused and referred to the next example with bigger numbers. I think that in the second example the multiples are less obvious because of the choice of numbers.

Analysis of Problem 4

Problem 4: Prove or disprove: If G is a finite group and $a \in G$, *then there is some integer n such that* $a^n = e$.

Considering the fact that the students just learned the concept of cyclic groups, the common problem in the solutions for problem 4 is more or less understandable. Symbolically, an element *a* of a finite group *G*, raised to some power *n* is associated with students' definitions of a cyclic group. Not surprisingly, some students decided that since *G* is finite then it is cyclic. Consider Figure 54:

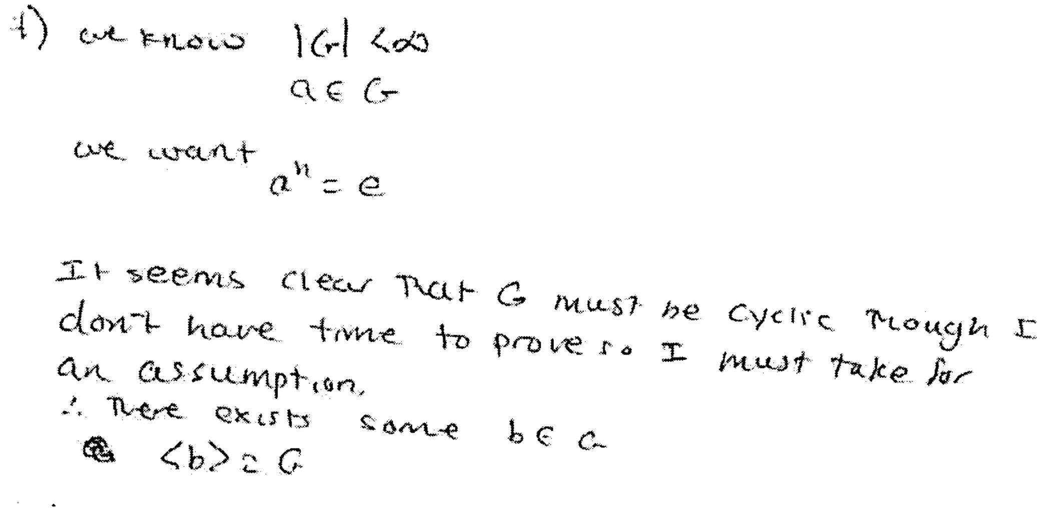


Figure 54. Argument about finite and cyclic groups.

The order of G is less than infinity and $a^n = e$, and the student concluded that G is cyclic, while her/his definition of a cyclic group states: "A group G is cyclic if and only if there exists some element $a \in G$ such that $< a >= G$, where $< a >= \{a^n | \forall n \in Z\}$."

It seems that for the student (as for many other students) the most significant part of the definition is the one that states that a^n belongs to the group for every integer *n*. She/he understands that if *a* belongs to *G* then every power of *a* must be in *G*, so the concept of closure does not appear to be a problem for the student. However, it looks like she/he misses the opposite inclusion. That is, if an element *g* belongs to the group *G* and *G* is cyclic and generated by *a*, then there exists n such that $a^n = g$. Also, for a finite cyclic group generated by *a* of order *n*, $a^n = e$. I would like to note that at this point the students are starting to feel more comfortable about finite groups. I discussed previously that students' connection to familiar objects (sets, operation) was very strong and the idea of a set being closed under an operation and being finite at the same time was difficult to accept and understand. It was too abstract for the students and did not have a "concrete" basis. However, with the appearance of cyclic groups of type $\boldsymbol{Z}_n$, the idea of closed finite structure started to make more sense. I already discussed in the analysis of Problem 2 that the students feel comfortable with generating finite cyclic subgroups using addition mod 12 and listed all elements of ($\boldsymbol{Z}_{12}$, $+_{12}$). The students are looking for the subgroups of $\boldsymbol{Z}_{12}$, by taking each element of $\boldsymbol{Z}_{12}$ and using it to generate a subgroup. We can see that they proceed till, for some *n*, a^n is equal to 0. So, now they really can make sense of a finite group (as a closed binary structure) based on the concrete examples: raising an element to any power, you still are inside the same cycle, since, for some *n*, $a^n = e$. Perhaps at this point the students associate any finite group with a cyclic group (such as $\boldsymbol{Z}_n$) since they do not know examples of finite groups that are not cyclic. Moreover, one of the group theory theorems states that every finite cyclic group of order *n* is isomorphic to Z_n for some integer *n*. The theorem could also suggest thinking about *G* as a cyclic group. So, the algebraic commonality: $a^n = e$ associated a finite group *G* with a cyclic group.

I also think that the knowledge about cyclic subgroups may suggest that some students could think of a group as the set of its cyclic subgroups where every element of a finite group generates one of the cyclic subgroups. Moreover, since the group is finite, its cyclic subgroups are finite and isomorphic to $\boldsymbol{Z}_n$, for some *n*. So, the group as a set which contains its cyclic subgroups (in other words $G = \{H_{a_1}, H_{a_2}, ..., H_{a_k}\}$, where by H_{a_i} I mean a subgroup generated by an element a_i of G, if $G = \{a_1, ... a_k\}$) is considered to be cyclic. Even further, a group G could be viewed as a union of its cyclic subgroups ($G = H_{a_1} \cup H_{a_2} \cup ... \cup H_{a_k}$) and considered to be cyclic for this reason. (I also discussed the opposite view in connection with Figure 48, where the student was only concerned with the result of operation on two distinct elements and did not consider a^n for any integer n). However, not all finite groups are cyclic. The data suggested (as I discussed previously) that many students need a concrete example to make sense of abstract algebra statements. Before they learn a cyclic group of integers mod n, it was difficult to accept the existence of a finite group in general. Now they are having difficulty accepting the existence of a finite group which is not cyclic. As soon as the students learn a group of permutations, they could find a way to overcome this problem.

As I discussed previously some students assume that a finite group must be necessarily cyclic. In some cases (Figure 55) the conclusion was the opposite – cyclic seems to imply finite.

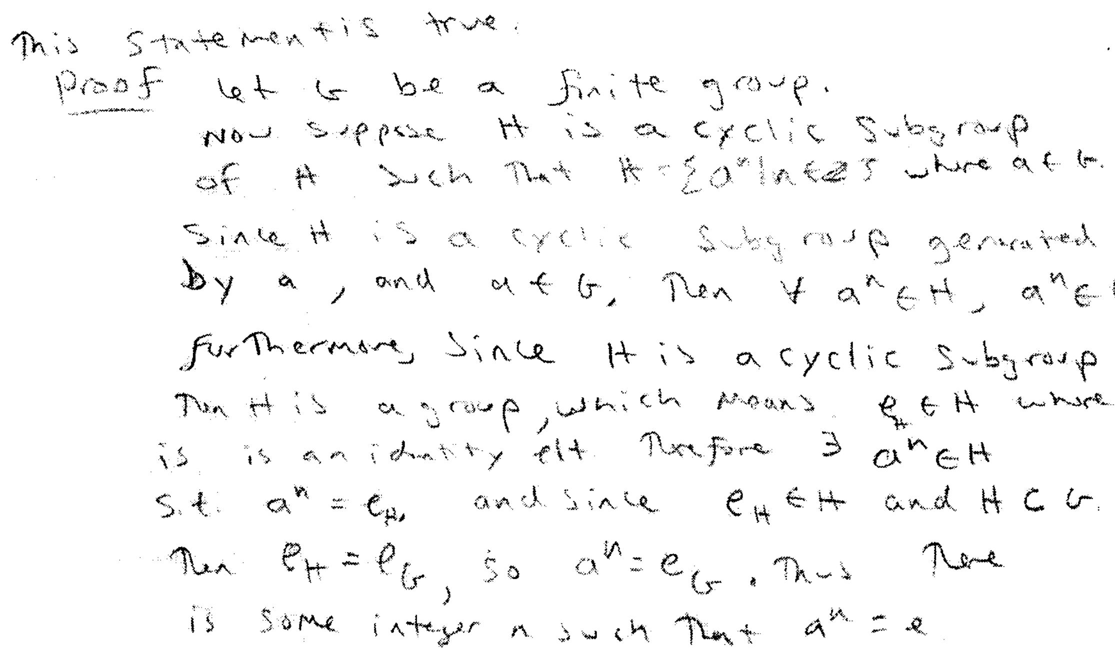

This Statement is true.
Proof let G be a finite group.
Now suppose H is a cyclic subgroup of G such that $H = \{a^n \mid n \in \mathbb{Z}\}$ where $a \in G$.
Since H is a cyclic subgroup generated by a, and $a \in G$, Then $\forall\, a^n \in H$, $a^n \in G$.
Furthermore, Since H is a cyclic subgroup Then H is a group, which means $e_H \in H$ where is is an identity elt. Therefore $\exists\, a^n \in H$ s.t. $a^n = e_H$, and since $e_H \in H$ and $H \subset G$. Then $e_H = e_G$, so $a^n = e_G$. Thus There is some integer n such that $a^n = e$

Figure 55. Reasoning about finite group and its subgroup.

The student did not explain why *H* is finite and why there exists an integer *n*, such that $a^n = e$. It is possible, however, that the student took it for granted: she/he concluded that the subgroup is finite, since the group is finite. This assumption is supported by examples that suggest that some students better understand “set” relations between a group and its subgroup then “operation” relations. It follows that for the student a subgroup of a finite group cannot be infinite since it must be a subset of the group.

I have noticed that several students tried to reason about the problem using an operation table. It looks like the students could not recall theorems which would suggest the solution strategy. In this case an operation table could give some ideas about a binary structure the problem describes. The fact that *G* is finite makes it possible to imagine the operation table. I have noticed that the students often attempted to reason through operation tables when a given structure is finite. Consider Figure 56:

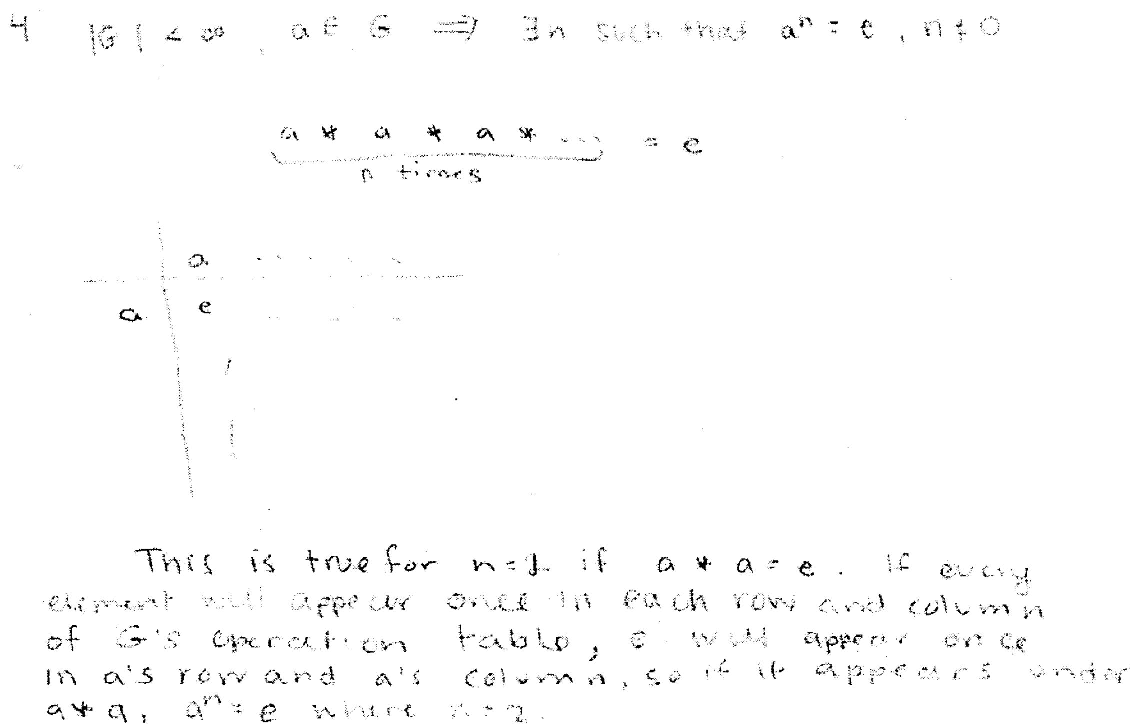
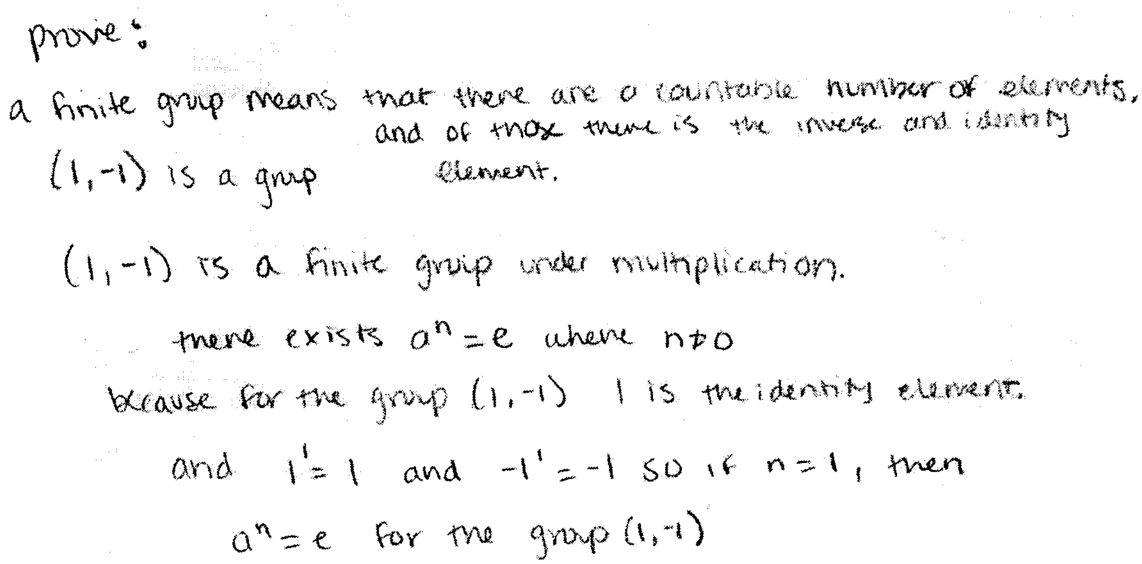


Figure 56. Table argumentation.

It looks like the student concluded that $\underbrace{aaa.....a}_{n\ times} = e$ implies $a*a=e$.

Perhaps the operation table above provoked this misleading idea. The student had difficulty expressing the operation $\underbrace{aaa.....a}_{n\ times}$ via operation table and decided that *aa* must result *e*, the identity element. Since *G* is a group the student concludes that identity element must appear once in each column and each row but then assumes that in case *a*'s row and *a*'s column the identity *e* must belong to their intersection. This discussion supported the student's assumption that $a * a = e$. I also think that the statement about that $\underbrace{aaa.....a}_{n\ times} = e$ is possible if $a * a = e$ could be explained by the student's background knowledge. If I assume that for the student the symbol "$*$" is related to the multiplication of real numbers and e corresponds to the multiplicative identity 1 (or possibly 0), then the conclusion seems to be true: $a^n = 1 \Leftrightarrow a * a = 1$.

I have noticed that for some students a concrete example often serves as a valid proof of a statement. A general conclusion is based on one or two examples. Note that I already described the cases when the students used concrete examples to understand a general proof or to justify their proofs. In the following situation (Figure 57), however, the conclusion is based on one concrete case:


prove:

a finite grup means that there are a countable number of elements, and of those there is the inverse and identity element.

(1,-1) is a grup

(1,-1) is a finite grup under multiplication.

there exists $a^n = e$ where $n \neq 0$

because for the group (1,-1) 1 is the identity element.

and $1^1 = 1$ and $-1^1 = -1$ so if $n = 1$, then

$a^n = e$ for the group (1,-1)

Figure 57. Proof of the concrete case.

First note that the student defines a finite group as a group with countable number of elements. I think that it is because the student is not sure what countable means or may be she/he does not realize that there exist countable infinite sets. This statement, however, does not affect the solution in any way. Further, the student says that the important elements of a group are the inverse and identity and comes up with the example (1, -1). I think that the example is suggested by the operation of addition of integers, since 1, and -1 are inverses of each other. Later however, the student realized that there is no additive identity and figures that the set is a group under multiplication. Note that the group is cyclic and it is easy to conclude that there is an n such that $a^n = 0$.

Analyzing Exam I have noticed that the identity element plays a crucial role in students' understanding of a binary structure. For instance, in the previous paragraph it looks like the student's reasoning about the group operation is based on the identity element the student chose. So, an identity element is not only a property of a structure but also the identity element defines the structure it belongs to.

In the following response (Figure 58) the student uses the way of contradiction to prove the statement.

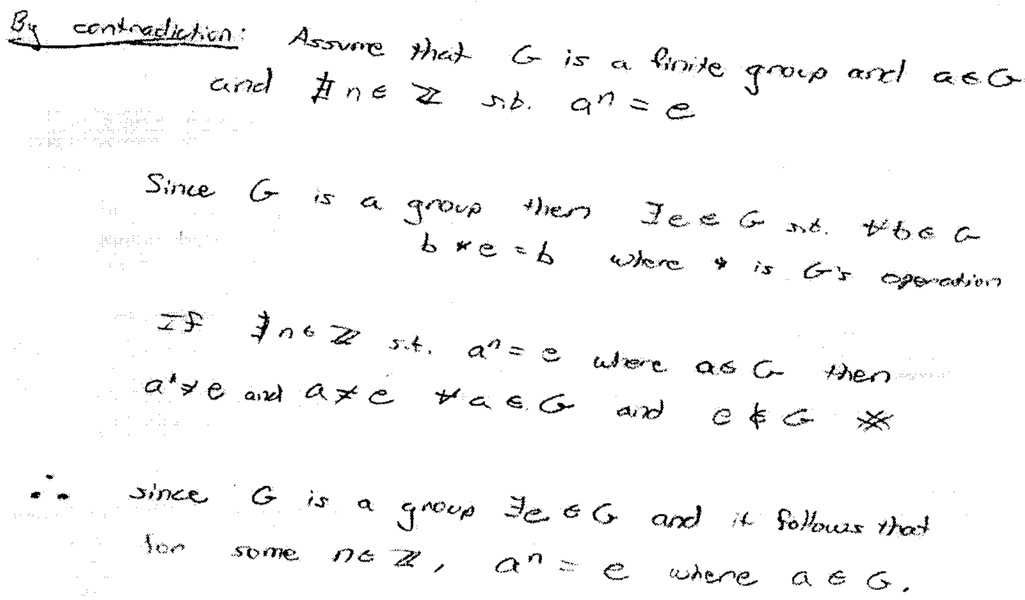


Figure 58. Proof by way of contradiction.

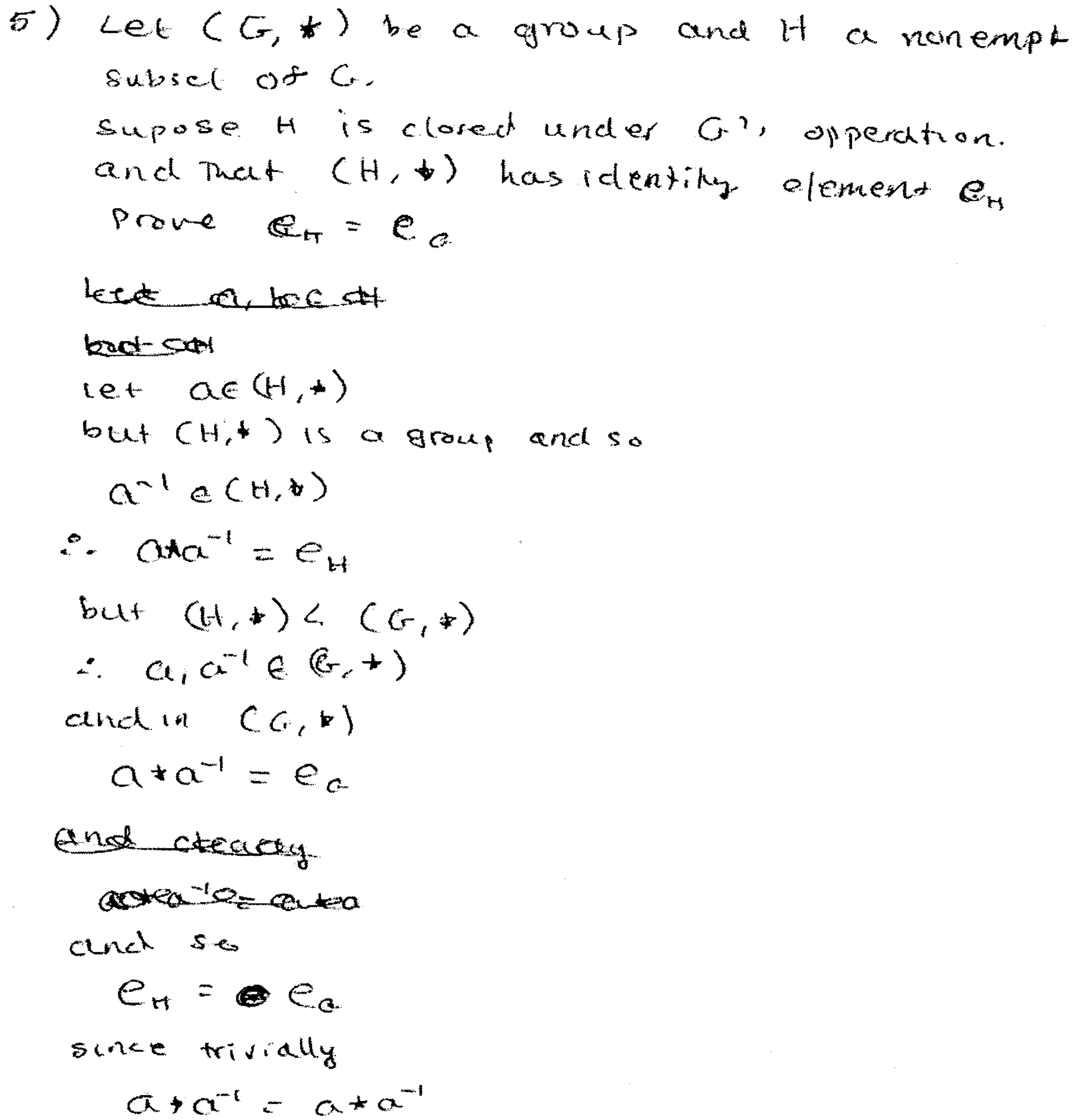
The contradictory argument is constructed correctly, however, for some reason the student considered all elements of *G* but identity *e*, while she/he states that if *G* is a group, then *e* must be in *G*. Further, the student concludes that $e \notin G$, based on the argument that $\forall n, a^n \neq e$. So, it looks like for the student, an element belongs to *G* only if it can be generated by another element of the group.

I also would like to include some additional notes about quantifiers. In Figure 58 the student's construction of opposite statement is correct in terms of quantification: she/he avoids quantifier "for every" but uses "does not exists" instead. These quantifiers are equivalent. However, "for every" is more suitable in the context. Nevertheless, it illustrates the student's understanding of quantifiers and their role in mathematical statements.

Analysis of Problem 5

*Problem 5: Let (G, *) be a group and H be a non-empty subset of G. Suppose H is closed under the operation of G and that (H, *) has an identity element* e_H. *Prove that* $e_H = e_G$.

The problem brings us back to the students' understanding of a subgroup. Almost all responses showed that the students see *H* as a subgroup (group in most cases) with some differences in argumentation. I already discussed several problematic ways of the students' reasoning about groups and their subgroups. Some students see a subgroup only as a subset, others just as a group, where set relations between *G* and *H* are unclear. In problem 5 it was given that *H* is a subset of *G*, closed under *G*'s operation. Moreover, it has an identity. It looked like all the responses that claimed that *H* is a group could be divided into two clusters. The first group of students simply stated that *H* is a group without proving it (Figure 59).



5) Let $(G, *)$ be a group and H a nonempt subset of G.
Supose H is closed under G's opperation.
and That $(H, *)$ has identity element e_H
Prove $e_H = e_G$

~~let a, b e H~~
~~bad sol~~
let $a \in (H, *)$
but $(H, *)$ is a group and so
$a^{-1} \in (H, *)$
$\therefore a * a^{-1} = e_H$
but $(H, *) < (G, *)$
$\therefore a, a^{-1} \in (G, *)$
and in $(G, *)$
$a * a^{-1} = e_G$
~~and clearly~~
and so
$e_H = e_G$
since trivally
$a * a^{-1} = a * a^{-1}$

Figure 59. Misleading subgroup argument.

It looks like structure *H*, described in the problem is associated by the student with a group (or rather a subgroup) definition. I have noticed that, in general, the students often make such conclusions. In this case, the student included inverse element in the argument, although it was not necessary for the particular problem. Since the student decided that *H* is a subgroup, then her/his proof looks like a proof for the theorem on group and subgroup identity coinciding.

The second group of students although considered *H* to be a group but still did not take it for granted and tried to prove it. All the students were concerned with the fact that it is not given that inverse elements belong to *H* for every elements of *H*. Some students made a misleading conclusion about inverse elements based on the fact that *H* has an identity (Figure 60).

5. Let $(G,*)$ be a group and H be a nonempty subset of G. Suppose H is closed under the operation of G and that $(H,*)$ has an identity element e_H. Prove that $e_H = e_G$

$(G,*)$ group, $H \leq G$
H closed under G
$(H,*)$ has id el e_H
Show H is a group, then $H < G$, then by def $(H,*)$ has id el under G.

Proof:
Let $(G,*)$ be a group and let H be a nonempty subset of G.
[Show $\exists$ $x, y \in H$ such that $xy^{-1} \in H$.]
Suppose $x, y \in H$ because H is nonempty.
Since H has identity element e_H, there exists some number z where $y * z = e_H$.
$\therefore$ z is the inverse of y, or y^{-1}.
Since H is closed, $x, y \in H$ implies $xy \in H$.
Now since $x, y^{-1} \in H$, we know $xy^{-1} \in H$.
$\therefore$ H is a group.
And since H is itself a group, and is a subset under the group G, H must be a subgroup of G.
By the subgroup criterion, we know if $H < G$, then H has an identity element under G.
$\therefore$ $e_G \in H$.
And since by definition there can only be one identity element, $e_G = e_H$.

Figure 60. Proof of the statement about *H* being subgroup of *G*.

Note that for the student the identity element *e* belongs to *H* only if *e* can be expressed as a combination of some (in this particular case of two) elements of *H*. I think that the confusion can be explained by the fact that the students often try a "concrete" approach to the closure problem. In other words they are thinking about continuous generation of elements by combining them in different ways, in different powers. Thus, if an element belongs to the set, then it can be expressed in terms of other (or only two) elements. Therefore, the identity element is $a * a^{-1}$. After this argument, it looks like the student generalized the observation and concluded that for every element of H there exists the inverse element which is also in H and that H is a subgroup by the definition or subgroup criterion. In Figure 61 the student also uses a subgroup criterion:

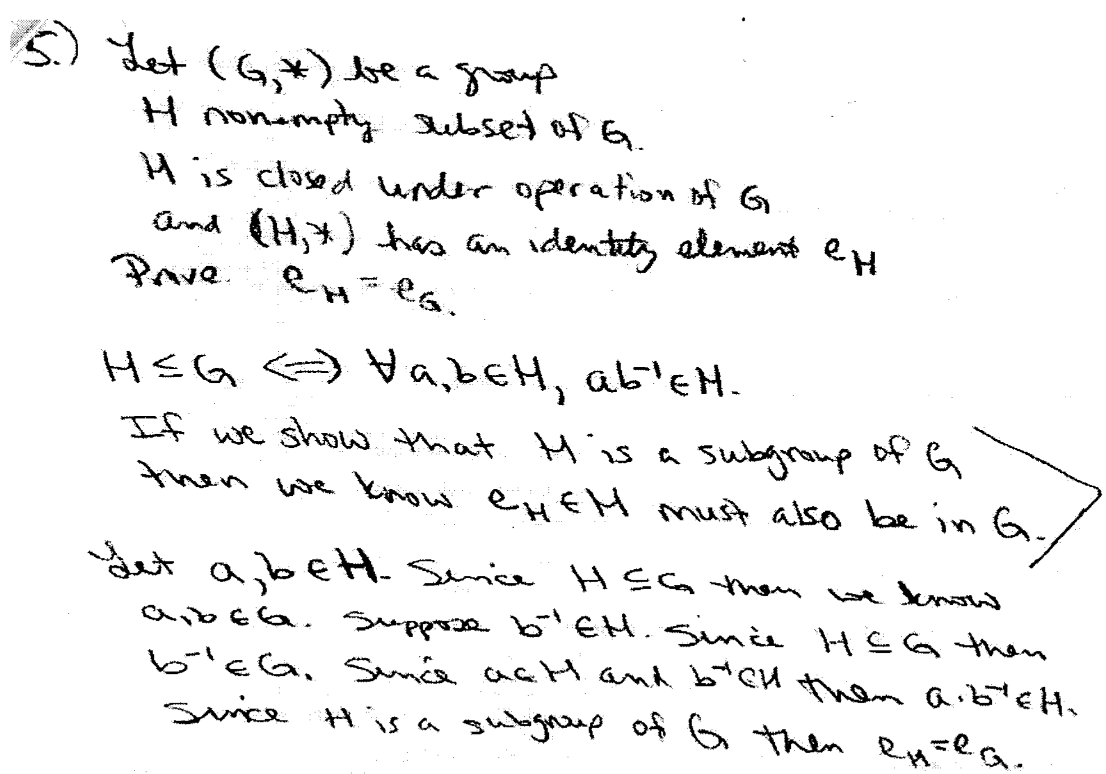


Figure 61. Proof based on H being a subgroup.

Note that the student recalls that if H is a subgroup of G they must have the same identity elements. As for inverse elements, the student "supposed" that at least for b, its inverse is inside H. It looks like the student is not sure about inverses for all elements of H and states that b^{-1} belongs to H rather carefully. Still, she/he concludes that H is a subgroup, based on the fact that b^{-1} is in H and then subgroup criterion works for this specific pair ab^{-1}. Perhaps the student has difficulty understanding the difference between a certain element of the set, namely b (or a) and "any" element b of H. These difficulties cause the conclusions which are based on non-given facts, which are impossible to prove.

Interview I

The goal of the first interview was to get familiar with the study participants, to make them comfortable when talking in front of me and to get an idea about the students' style of answering questions (for example if a student is using definitions and follows them during problem solving, gives a quick response first and then starts to recall needed definitions and statements, or does not use definitions but tries to define objects by him/herself).

In terms of content, Interview I questions helped me to get a sense of the students' knowledge of a binary operation defined on a set and connections between a set and a binary operation. In particular, I concentrated on students' understanding of a concept of a set being (or not being) closed under the operation and their understanding of "failure" of operation to be binary. Finally, based on existing study reports, I wanted to explore the students' perception of a set together with an operation defined on the set as a single object.

Asking the first question, I aimed to understand how the students define binary operation by their own words. Next questions (2 – 4) referred to the understanding of students' ability to recognize a binary operation defined on a set. Question 5 is monitoring student's understanding of "binary operation - closure" connections. In Question 6 students were asked to manipulate with concrete numbers and to define an operation on a set of concrete numbers.

I will illustrate the analysis of one typical response question by question with some additional examples from other responses to support my analysis or to add other opinions on the problems:

Question 1. Define what it means to say that $*$ is a binary operation on a set A.

S1: I am going to call it $*$ then? The asterisk? $*$ is a binary operation on a set *A*. It would mean that you take two elements in *A* then you can apply this operation it could be anything like addition multiplication. I do not know the exact definition to be honest with you. On a set *A* if you take like two elements a and b then $a * b$ is always be in the set.

I: Does this mean that it is a closed set?

S1: I do not know. What else do you want me to say? Do you want me to go to #2?

The student immediately thought about two elements from the set *A*. It looks like the student tried to imagine the situation. She/he decided not to write anything down for this question although before the interview I gave the student a piece of paper and a pencil. At this stage the students reason a lot in terms of concrete examples. It seems difficult for them to consider the objects they study without reference to a particular example. In the response above the student had to think about addition and multiplication before she/he defined a binary operation. Note that the student, as well as other interviewees, does not define a binary operation for all pairs of elements of the set. However, during the interviews, students often give less rigorous definitions compared to the ones they write on paper. I think that it simply does not appear in their minds at the time, or/and seems less important to mention compared to the structure of the definition (two elements, operation, result). In other words, universality does not seem to be a component of the binary operation definition. From the responses I could distinguish several main components of a binary operation in students' definitions: a set, two elements, a "star" ($*$) or operation, and a result. Nevertheless, universality of a binary operation for the set and closure of the set is often omitted. I observed that in this response the student needed to refer to specific binary operations to think about closure. In the following fragment

> S2: What does it mean to say that $*$is a binary operation on a set A? Just that it's closed and well defined.

the student immediately says that "it" is closed. Later, however, she/he had trouble explaining what it means for the set to be closed. Also the way the student is talking about closure suggests that she/he does not fully understand the meaning of closure. "Just that **it's** closed and well defined" does not clarify what exactly is closed: operation or set? It looks like for the students at this point a binary operation is merely a general symbolic representation of operations they have studied before. Connections between a set and a binary operation defined on it seem unclear. The idea of a binary operation seems to consist of isolated pieces of information about several objects.

The second question was very confusing for the students. It was a multiple choice question. Each part was different from another by possible number of results of a binary operation.

Question 2. Decide which of the following statements are correct, please explain your reasoning:
a) a binary operation on a set S assigns at least one element of S to each ordered pair of elements of S.
b) a binary operation on a set S assigns at most one element of S to each ordered pair of elements of S.
c) a binary operation on a set S assigns exactly one element of S to each ordered pair of elements of S.

S1: [reading] I would think that...it sounds right to my ears.

I: Just read all of them

S1: All? [reading] I'm going to get my definition confused because I know that it has to be...I am not sure it has to be one to one and onto this is kind of saying like at most one element would mean ...that it is kind of be one to one right? and onto. I think it would be at least one element of S to each ordered pair because if you look at the operation table, some times there is ...NOPE! It would be exactly one element. I think it would be it. I was thinking about it in my head to kind of see what it might look like, and I am like.. you can not have more than one element at $a * b$. This is my process even in the bathroom I am still like...Oh! I am thinking in different ways...to even get an idea but I am getting there.

The association many students had when answering the question is coming from a definition of function, one-to-one function, and bijective function. It looks like in the response above the student is starting to think about functions but then said that she/he "is going to get [her/his] definitions confused". It suggests that the student did not reflect on binary operations as on functions, while the students' background could suggest this correspondence. In the following fragment the student relied on function definition in order to get the answer:

S3: ...binary operation is a function and for any given combination of elements in S we can only have one output. And so, if we have a binary operation on the set S then we can only have "at most" one element of S to each ordered pair.

Nevertheless, the student "S1" resolved the confusion using an operation table. Reasoning with an operation table gave her/him a chance to realize that it is impossible to have more then one entry or no entries in a single cell. Reasoning with an operation table enhances the students' understanding of a binary operation. It gives the possibility to represent an abstract object in a more concrete way.

I observed two ways of thinking about the question: operation table or definition of a binary operation. Note that in question 1 only 2 students used the words "ordered pairs" to define a binary operation. Others simply said that we take two elements of the set. Question 2 is formulated in terms of "ordered pairs" and it confused some students:

> S4: I do not think about ordered pairs when I think about a binary operation.

It suggests that the students are used to think in less abstract or more descriptive terms. Perhaps the words "ordered pair of elements of *A*" makes less sense then, for instance, "two elements of *A*" or "elements *a* and *b* from *A*".

Question 3. Give an example of binary operation on **Z**.

> S1: [reading] that could be...addition, that could be multiplication. $a * b$ could be *a* times *b*; 3 times 2 is 6.

All the students answered the question correctly since it was asking about familiar objects: set of integers and operation defined on it. Note that the student felt like she/he needed to support the answer by the concrete computation. Although the question asked for one example of a binary operation, some students (3 out of 7) give both addition and multiplication examples. One of them even thought about subtraction. I think that the students simply prefer to choose familiar examples. Both addition and multiplication are such examples and the students do not know which one to choose. Also, these are typical examples of commutative and associative operations. Subtraction and division serve mostly as counter examples illustrating operations which are not binary or which are not commutative or associative.

> Question 4. Give an example of operation on ***Z*** which is not a binary operation on ***Z***?

The responses to the question could be divided into two groups. The first group of students immediately thought about division and showed a counterexample:

> S4: I'd have to say that division is not a binary operation, since if you divide 1 by 2 the result is not in ***Z*** any more.

For the second group the answer was not so obvious and they needed to think about definition of a binary operation in order to answer the question.

> S1: [reading] That could be may be something like...so it's going to be something ...that's not in the integers. Would that be like if you take a square root of something? Would that not work? Or that would work?...For me to be a binary operation that means that ...I really don't know this is confusing me for some reason.
>
> I: Why addition and multiplication were your examples for the previous question?
>
> S1: We always work with them, I kind of assumed that... Oh, because if you add two numbers together you going to get one number. If you add two integers together you get integer and if you multiply two integers together you'll get integer. So I am trying to think about something that...it is probably really easy and simple but I am a little nervous. If you take two numbers in ***Z*** you are not going to get another number in ***Z***. So, I know what does it mean but I am like...Oh! Division! I am sorry it is taking me forever. It takes me a while. The result is not integer.

It looks like the form of the question suggested that the students should be concerned with the result of the operation. In both responses the students are looking for an operation which can have an outcome which does not belong to the integers. Perhaps in the second excerpt the student's background caused her/him to think about square root since only a set of complex numbers is closed under this operation and obviously a result of taking a square root out of an integer is not necessarily in ***Z***. Then, however, the student gets confused by the operation. Something bothers her/him about the answer. Perhaps the reason is that the student understands that the operation of taking a square root is applied to a single element of a set, whereas a binary operation involves a pair of elements. However, this idea was not said aloud and I think the student simply could not formulate it. I decided to bring the student's attention to the definition of a binary operation and her/his previous responses. Thinking about well-known operations of addition and multiplication as examples of binary operations she/he created a simple logical construct for looking for an operation which is not a binary operation on ***Z***: "If you take two numbers in ***Z*** you are not going to get another number in ***Z***." Following this statement the student figures out the example.

> Question 5. Let *S* be a set. Let $*$ be an operation on *S*. What does it mean if for some elements *a, b* of *S* $a * b$ is not in *S*? In your words, what does it mean for the set to be closed under the operation?
>
> S1: [reading] It means the set isn't closed. [reading] It means that for all the elements when you apply a binary operation to it the single element that results would be in *S* also.

Although I observed that many students have difficulty understanding the concept of closure, the responses to the next question showed that they know the definition of closure. Every student answered the question similarly to "S1". None of them, though, gave the exact definition of closure. Using the required quantifiers to stress that closure is a universal quantification.

> Question 6. Define a binary operation on $S = \{0,1,2,3,4\}$.
>
> S1: [reading] So, it does not have to be closed necessarily right? or it is implying that ... may be if you took two elements in there like 0 and 1 and applied a binary operation to them...
>
> I: How did you usually define a binary operation in class?

S1: Addition or even easier then that?

I: No, not the operation but the way you defined it.

S1: This is odd, I do not remember. Oh, ok...like if you use the operation table again and you have 0, 1, 2, 3, 4 on top and 0,1, 2, 3, 4 on left hand side [filling out the table]. You could have one, a binary operation that just always returns a single one of the values. So I mean it could just be "everything comes back 0". It would **technically** be a binary operation, right?

I: Yes.

S1: It would just be ... You could have one that returns the min of two values. That would be another binary operation. [writing symbolically: $a * b = 0$ or $a * b = min\{a, b\}$].

The main problem I observed in the responses to this question was the necessity of closure. The given set is finite and the students seemed to be confused at first. Although the question is asking to define a binary operation and the definition of a binary operation was already discussed in previous questions, the student is not sure if the set is supposed to be closed or not. It looks like the student is thinking about familiar operations but feels that something like addition or multiplication would not work because of closure. I tried to suggest to the student to construct the operation table for the situation. I think that in the initial stage of learning abstract algebra concepts a table representation gives a more "visual" and concrete description of an operation. Another reason for my suggestion was that the student successfully used an operation table to reason about question 2. As soon as the student started constructing a table she/he figured out what operation could be used. However, note that although the table was created successfully and did not contradict the definition of a binary operation, the student is not sure that this operation could be considered as a correct answer. She/he asked if it "technically" is a binary operation. Perhaps the fact that an operation the student found was not one of the familiar operations, such as addition, multiplication, division, etc, was the reason for doubting the answer. I have noticed that it is very difficult for the students to become unattached from their prior experiences, especially when problems involve familiar objects such as subsets of natural numbers.

To support this assumption and to illustrate more issues, I transcribed the following response:

> S2: I do not understand what I am supposed to do. Do I need to take two elements from there? OK, so one star two equals three, I guess, if star is addition?

The student assigns the operation to only one pair of elements. Perhaps she/he feels that addition would not work for all pairs, or, it is possible that for the student the idea of a binary operation does not involve universality of the operation. In other words, the operation is not considered to be a universal quantification for a set it is defined on. Moreover, the student's response for question 1 (about binary operation) did not involve universal quantifier. However, it did not seem important because none of the interviewees used quantifiers in their responses. I assumed that it was a matter of talking about the definition, not writing the definition. As I observed in the students' written work, Quiz 1 in particular, some students used a universal quantifier in response to the question about binary operations. Nevertheless, in this response I could see a strong connection between a lost quantifier and misleading thoughts about the problem. I had to recall the definition of a binary operation and stress its universality for the set:

> S2: Oh! If I check 2 and 4, I would assign subtraction? Just any operation? Oh I have to assign a binary operation which would work for ALL of them? Ok, well not all of them would work. Like for addition... So I have to think about a binary

> operation that makes all of those work. Well, I feel like I could have a lot of them! I would have to do something that would say like...*a*, oh *a* plus the identity element? Something like that? $a * e = b$ where *e* is... I mean $a * e = a$, where *e* is the identity element or something like that.

I think that the last statement is also motivated by students understanding of a binary operation and her/his misunderstanding of universality of ordered pairs in particular. Her/his definition of a binary operation on a set A was the following: "It takes two elements from *A*, assigns star to it and has to be...Oh, confused with...Result must be in *A*". As we can see, the first and the second responses to question 6 do not contradict the definition. Addition works for 1 and 3. If it does not work for 2 and 4, then we will assign subtraction to them and so on. However, it did not seem to be right for the student and she/he attempted to find an operation which would work for all elements of the given set. Perhaps the attempt to describe the operation as "$a * e = b$" was meant to illustrate the operation on a pair of elements since two generic elements of *A* (*a* and *b*) are involved, but it did not make sense for the student and she automatically referred to the definition of an identity element. Before she/he continued, I mentioned that a binary operation is supposed to work for all possible ordered pairs not only for (*a, e*). The student reasoned further:

> S2: So, I just have to make up a binary operation, I can't just use addition, multiplication, division or subtraction, cause it's not closed. So I just have to make one up? I feel like there is no one answer. Can it be something like this...Its just taking the first element. It would just be the first element, whatever you are doing. Is it not allowed? It's not a real operation...If you always take the first element and you assign * to, you always get the element from the set.

The final part of the response suggests that again it was too difficult for the student to avoid thinking about familiar operations. In the end she/he described a binary operation but was not certain about it. The student asked me if it really worked since she/he stated that it was not a "real" operation. This link to familiar operations and consideration of these operations as the only "real" ones is a big obstacle in the students' learning. If I call the "real" operations "concrete" operations, then in most cases the student cannot bridge the gap between concrete to more abstract operations.

Question 7. Determine if the following binary operation is associative. Does it have an identity element? Decide if it is commutative. Operation is defined on Z^+ by $a * b = 2^{ab}$.

S1: [reading] The operation on the positive integers? [writing] And if it is associative it would mean that $a * b * c$ with a and b in parentheses would equal to $a * b * c$ with b and c in parentheses. [writing computations]. And I think if you do this out it would not be associative ... this might be totally wrong with my algebra...yeah this is not associative. And then, if it does have an identity element? It would be saying does there exist an "*e*" belonging to positive integers such that for all "*a*" belonging to those integers, $a * e = a = e * a$ so it would be both sides or right and left side identity. [writing computations]. E would have to equal ... is it possible?

I: This is an exponential equation.

S: Yeah, so that means that there is no...I remember doing it in homework but it was not asking about identity...

I: If you take a log from both sides?

S: Oh! Simple algebra! But I thought you cannot have an identity element that had...that was defined in terms of a? Cause would not this leave "log based 2 *a*". So $\frac{\log_2 a}{a}$ is *e*.

Identity cannot be a function of the variable, because it has to be identity element for all a belonging to those, so it would be a different element for every element. And if it's commutative? That would mean that $a * b = b * a$ and in which case it is since multiplication is commutative.

The interviewees did not have much difficulty with this question. Although the operation was not from the list of the familiar ones, it was described in terms of these operations: multiplication, in particular. I have noticed that if the problem required algebraic computations, the students felt more confident, and it looked like they were working in familiar settings. All of them easily made conclusion in terms of abstract algebra notions. In the response above, the student gives the definition of associativity, describing the solution procedure. The definition does not include quantifiers in any form and it does not affect the solution. The notion of associativity is, of course, an abstract object. However, the definition involves an algebraic expression, which is usually considered by students as more concrete. The concrete computations give the answer to the problem. Since the operation $*$ was defined in terms of multiplication, all the students recognized operation $*$ to be commutative, since multiplication of positive integers is. I honestly expected to hear some responses that $*$ is associative for the same reason it is commutative – because multiplication is associative. However, none of the interviewed students got into the trap. All but one of the interviewees noticed that computations of an identity element did not result in an answer independent of elements of the set. It means that the students were aware of uniqueness of an identity element and its independence of any element of the set. The reason I included the problem in my questionnaire was to see how the students distinguish the defined operation $*$ and the traditional operation of multiplication in which terms $*$ was defined. As I already mentioned, at this level of learning abstract algebra, the students preferred to follow the definitions rather then base their answers on first impression. However, one response began with this predicted misconception:

> S2: $a * b = b * a$ and its associative because a and b are being multiplied and associativity is a property of multiplication.…The set is of positive integers so the identity element …just going to be 1, because 1 is in the set of all positive integers.

At the beginning the student confused associativity and commutativity. However, she/he soon realized the problem and made the conclusion similar to the one I already described above. When thinking about an identity element the student preferred to base his reasoning on the operation rather than on the definition of the identity element. The first response was that the identity is 1. This response advocates a strong connection to familiar sets and operations. I asked the student to give me a definition of identity element and she/he concluded that the set of positive integers together with the defined operation does not have an identity element.

As I already mentioned, I was rather surprised by the responses, especially taking into the account students' previous answers and connection to familiar objects. In this problem they are stepping out of a rudimental way of thinking about a binary operation in terms of simple formulas and familiar operations and think on a different level - using definitions as a basis for the responses.

Interview II

The second interview was aimed to explore students' reasoning about groups. It showed continued problems with understanding of closure, identity element and inverse. I also noticed that it was difficult for the students to recognize a group as a set together with its binary operation and to understand the effect of a binary operation on the set. As was the first interview, I will present the analysis of interview II in "question by question" format, illustrating students' most significant and interesting responses.

Question 1: In your own words, what does it mean that *G* is a group?

Only 2 students out of 7 started the definition with a description of a group as a set together with its binary operation. All other students were so concerned recalling all the axioms that they simply did not consider the description of group components: a set and an operation. In the following response the student also added that a group is closed under its binary operation, which implies that she/he understands a group as a set with its binary operation, although the student does not describe them explicitly:

> S2: It has an identity, I mean...It's a ...it's closed under a binary operation, it has an identity element and an inverse and its associative?

Almost all the students used the word "it" referring to a group, or an operation, or a set. It suggested that they may be uncertain about what must be associative, what exactly has an identity element and an inverse. Later, during problem solving activities this could be the cause for misconceptions and misleading ideas. For example, the sentence "it has an identity element and an inverse" does not make it clear if the identity is a property of a binary structure, or a single element, or all elements in the set. As I already assumed in the first interview analysis, it seems that a procedural part of a definition or algebraic part of a definition is the most significant, especially for problem solving. It may seem obvious to the students that a definition of a group involves a set with an operation. The responses to question 2 support this assumption.

Question 2: Let $\frac{1}{2}\boldsymbol{Z}$ be a set $\{\frac{1}{2}z | z \in \boldsymbol{Z}\}$. (a) Is it a group? (b) Give an example of a binary operation on the set. Is it a group now? (c) Confirm that $\frac{1}{2}\boldsymbol{Z}$ is a group under addition. Is $\frac{1}{2}\boldsymbol{Z}$ a group under multiplication? Explain.

In the first part of the question no operation is defined. However, most of the students I interviewed said that it was a group since it satisfied all three group axioms. By default most of them chose addition to be the operation:

> S1: So, it's asking if the set ½ ***Z*** is associative for all elements a, b belonging to ***Z***. The impression is that it's closed and not empty. So, yes it's associative cause its addition and addition is associative. And the inverse, it would be...*z* has to belong to the integers. If *z* were 4, 4/2 is 2 and the inverse of 2 is -2 and it belongs to the integers. I am trying to answer if it is a group and I think it is a group because it satisfies all three axioms.

I have also noticed that the student had difficulty understanding the set. It looks like at first she/he reasons correctly about it but later the student states that the inverse of 2 is -2 and it belongs to the set of integers, not the one the problem defines. I think that it is a common difficulty, especially when the set is defined in terms of familiar sets. Also, as I mentioned above, the student was so concerned with finding out if the group axioms work for the set that she/he did not even notice that there was no operation defined, and started thinking about addition since it was a common example (integers and addition). Some students thought that the operation is multiplication and the operation IS defined, since the description of the set involves multiplication by $\frac{1}{2}$:

> S5: It is closed.
>
> I: Under what operation?
>
> S5: I am thinking about multiplication...so when you do ½ times ***Z***, so, multiplicative.

It looks like after my question about a binary operation, the students felt obligated to find or to define a binary operation. The next student, however, was confused by the question:

> S2: No! Wait, is it a group? No, it's not closed, cause it could be a decimal and then it would not be an integer, like 3 over 2 is 1.5 is not an integer.
>
> I: Note, the set is ½ **Z**, the set of all integers divided by 2.
>
> S2: Oh! This is the set! I was looking at the first part. OK. So, yes it's a group.
>
> I: Under what operation?
>
> S2: Multiplication? Hold on a minute. I don't see why it's...1/2 – the set of all integers divided by two. So...What do you mean "under what operation"? It could be either one.
>
> I: What operation is defined?
>
> S2: None, Oh! It's not a group!

At first the student had trouble understanding the set. Somehow, she/he recognized the set as a part of the set of integers. Perhaps at first the student decided that the operation is defined as division (or multiplication by ½) and in this case 3/2 is not in the set. This is a "background" problem - number representations. Usually, the operation is defined on a given set. However, it could not be a binary operation. In this problem I only described a set and the student said that the operation could be "either one", but I think the student tried to say that the operation is not defined. The responses suggest that for some students a group is simply a set which satisfies the axioms under an operation of their choice.

The problem with operation continues in part (b):

> S2: Under multiplication, for example, it is. Because multiplication is assoc., it has an identity element 1, which could be in the set and it has an inverse. An inverse could exist, because it is a set of all integers - positive and negative.

I: Under multiplication?

S2: Ph, I though about addition.

It looks like the student thought about multiplication first. She/he did not check if the set was closed under multiplication but started with axioms. The response again supports the assumption about the significant role of axioms in the students' reasoning about groups. The realization of the fact that multiplication would not work comes only when the student tries to describe inverse elements. Opposite numbers look like the most common example of inverse elements. The student again does not think about the operation but about plausible elements of the set: if the inverse under multiplication does not work, then additive inverse is the right one. After she/he described inverse element as opposite numbers, I asked about the operation again and the student realized that she/he was thinking about addition.

Also, I noted how the student talked about identity and inverse elements. I think it was an important detail since many students fell into the same trap. The student says that "it has an identity element". I think by "IT" she/he means the set, later, however, the student again says that "It has an inverse". It still appears that by "IT" the student means the set, not elements of the set. The phenomenon may be explained by the uncertainty of referring identity and inverse element to a certain set or to a certain element. Moreover, it is a difficult concept for students to understand that an identity element is unique to the set, meaning that it must work for all elements (in other words it is a property of the set), while the inverse is unique for each element (it is a property of the element).

In part (c) I decided to come back to the operation of multiplication in the context of the closure of ½ Z under this operation. The student "S2" did not seem to have problems answering the question since we more or less discussed the details in previous parts. In general the final part of the question did not raise any interesting issues since the students already worked out the set and understood its elements, and the operation of addition is one of the familiar and well known operations. The following is S2's response for part (c):

> S2: Same deal. It has an Identity element 0 and inverse and addition is associative.
>
> I: Is it closed?
>
> S2: Yes, it is closed. Cause if you take two elements from there and add them together you get another element from the set.
>
> I: What about multiplication?
>
> S2: No, so multiplication...well, it might be. It's not closed for all element of the set though. If you had ½ and ½ and you multiply them together you have ¼ and it's not in the set.

Although the student did not even consider closure in the previous responses, she/he understands what closure means. After I specifically asked if the set is closed under addition and later under multiplication, the interviewee answered based on concrete numbers.

I have noticed that the operation of multiplication was a very popular response. I think in this particular problem it could be explained by the form of the given set. Elements of the set involve multiplication by ½ and this could be the reason why the students think about multiplication, especially since the operation was not defined at all in part (a). I found a similar problem in the following questions.

Question 3: Let *G* be a group, let $a \in G$. How many inverses can a have? Why?

All the students recalled the theorem they learned in class about uniqueness of identity element. However, only few of them could briefly explain why it is taking place and wrote it down. Maybe the reason is that they tried to tell me the proof which was difficult, and to do so the students decided to use more general terms in order to explain the uniqueness of an inverse element. The question also did not ask to prove the uniqueness but to explain why the students think this way. Consider the following response:

> S4: Exactly one. Otherwise we would not know which one to pick... if it did (have more then one inverse element) then we would have an operation table, the role of '*a*'...that would be more then...If you would take the identity element and you operate it with '*a*' you get the identity. I think this is wrong...'*a*' and the inverse of '*a*' you get an identity and if there is another inverse you get the identity again then it would not be a group.

The student's reasoning about the uniqueness of the inverse for each element is based on an operation table. The student used the fact that G is given to be a group and concluded that if an element would have more than one inverse, then it would contradict that fact.

Question 4: Give an example of Abelian and non-Abelian group.

Almost all of the interviewed students gave a definition of Abelian and non-Abelian group. However, when giving the examples of these groups they were concentrated on exemplifying commutative and non-commutative operations, not groups:

> S1: An Abelian group, which is commutative group, would be the integers under...multiplication...there is no inverse of 0. Integers without 0, no not the integers, real numbers without 0 under multiplication is a group would be Abelian, because *a* time *b* is the same as *b* times *a*. And non-Abelian group is non–commutative group, so anything with the operation like...I mean sort of like $a - b$...if star were $a - b$, then $(\boldsymbol{R}, *)$ would be non-Abelian group.

The student gives a definition of an Abelian group and then thinks about an example of a non-Abelian group. Note that she/he thinks about multiplication of integers at first, and then the student realizes that 0 cannot have an inverse. It looks like the student feels comfortable with multiplication since she/he preferres to change the set instead of simply changing multiplication to addition. From other responses I concluded that the students usually find it easier to think about a different operation rather than to think about special restrictions or conditions for the set they chose to work with. In many cases the students chose ***Z*** to be a set and multiplication to be a group operation. Not all of them realized that it would not be a group but certainly a commutative operation on integers. Nevertheless, the student in the above excerpt did not avoid the same misconception in the second part of the problem. She/he did not check if defined structure is indeed a group. The operation was the only consideration and the student simply chose non-commutative operation on the set of real numbers. Other students had similar difficulties. In general, the most popular example for the first part was (***Z***, +) (or (***Z***,×)) and for the second part most of the students just gave me an operation they had in mind. All of the operations were non-commutative and the student claimed that they are parts of non-commutative groups (the operations were defined on the set of integers). Here are examples of these operations: division, $a * b = a + bc$, $a * b = a^2 + b$. Again, the students are concentrated on the operational or the algebraic part of the problem, leaving object's conditions outside.

In the following excerpt the student gives an example of non-commutative operation, then overcomes obstacles and gives an example of commutative group but can not give an example of a non-Abelian group and explains the difficulty:

> S2: An Abelian group is commutative and non-Abelian is not. OK. [Writing $a * b = a + bc$]. If you are talking about...
>
> I: What is the set?
>
> S2: Ok, you could do...the set of all integers under multiplication is commutative.
>
> I: is ($\mathbf{Z}$, $*$) a group?
>
> S2: Yes
>
> I: What is the inverse of 0?
>
> S2: Oh, no, it's not a group. Ok, ***Z*** under addition is. I am trying to think about not commutative...
>
> I: You defined an operation which is not commutative, what else do you need to form a group?
>
> S2: identity, inverse...form the set...Are we done number 4? I cannot think of...I just, I do not know...what I remember Abelian, I remember setting up like tables table. Finding it this way. Like giving a specific example I can tell if it is Abelian or not. I cannot think about it this way!

It looks like for the student the problem was more abstract then the one asking to check if a specific structure is an Abelian group. In case of a particular example, the problem requires the student to know the definition of commutativity. As I previously noted the students did not have trouble with definitions which involve formulas or algebraic statements. For them commutativity is merely the fact that $a * b = b * a$, and the other conditions and setting description seem unimportant. I think that the responses (giving non-commutative operation) fully support this assumption. As soon as the problem requires deeper understanding of an object (or structure) the students feel uncertain. It is also possible that the problem is that the students cannot easily stop thinking about familiar structures, such as sets of integers or real numbers and operations of addition or multiplication. It is possible that the students are so confident about their knowledge of these structures that they do not find it important to check if the structure is indeed a group with certain properties.

> Question 5: Determine whether {4, 8, 12, 16} is a group under multiplication (mod 20).

Analyzing this problem I have noticed several common approaches. At first students had difficulty understanding the set and its operation. They were used to working with groups of integers under addition mod *n*. The students already were familiar with the concept of cyclic group, and the set above suggests to think about it as a cyclic group but the operation is not suitable for this case. Another popular misconception was that it was not a group since the set did not include 1, which was necessarily a multiplicative identity, and without the identity the structure was not a group. Only one student out of 7 interviewees actually used the definition of a group, followed every step, showed that every axiom holds, and concluded that it is a group. The student did not use an operation table, just wrote the computations (underlined "16" is the found identity element) in Figure 62.

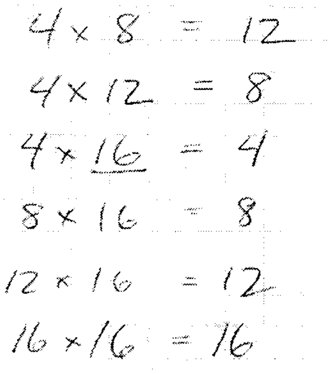


Figure 62. Student's computations.

In the following excerpt I observed how the student tried to make sense of the set and its operation:

> S1: I feel like it would be a subgroup, not a group. I do not really know what is this asking? Is it taking the integers like that and then, what is this set? Is it just a set generated by 4? If it generated by 4 wouldn't it be...oh, its multiplication. 4 squared is 16, 4 to the third is 64...which is...I do not really know how to do it multiplication; we usually do it like addition. So, it's not generated by 4...

The first sentence shows that the student starts with an idea about cyclic groups and their subgroups generated by elements of the group. It seems that at first the student confused the operation with addition mod 20. In case of addition it would not be a group (not closed) but it looks like for the student at this point a subgroup is not necessarily a group so it does not have to satisfy the axioms. I think in this response we observe the understanding of a subgroup as a subset. The given structure is not cyclic and it seems to be confusing for the student (as well as for other students), since an operation mod *n* is usually assigned to a cyclic group. Also, the students used to work with finite groups but most of them were cyclic groups of integers, mod *n*. The student in the excerpt could not overcome the difficulties and asked to move forward with the other questions. Unfortunately, we did not have time to come back to question 5.

In the following excerpt the student started with the same misleading idea – he tried to think about the structure as the one generated by 4 under addition mod 20. At first the student concluded that it was a group but then noticed that, in case of addition, the set "needed 0":

> S2: That would be yes, its 4. If you start with element 4 and mod 20, then yes it is. Well...it needs 0. If you want to say 4 mod 20: 4, 8, 12, 16, and this is...
>
> I: the operation is multiplication
>
> S2: Then its not. Does not have an identity or inverse. Wait, I am confused with this. It's not a group. Well it's under multiplication it does not have an identity or inverse. Multiplicative identity is 1.

After I reminded to him that the operation is multiplication, the student said that it was not a group since it did not have identity and inverse. It seems that the student thinks about multiplicative inverse of given integers as of inverses in the set of rational numbers where the inverses are the reciprocals. The given set only includes integers and this fact misleads the student. The final conclusion is based on absence of the multiplicative identity 1 in the set.

The phenomenon I observe in this problem is again the difficulty to think in more abstract terms, since the students' understanding of abstract structures is attached to the structures they know and they worked with. It also suggests that most students prefer to base their conclusions on their background rather then just follow the definitions of objects. I think it means that often the students do not understand the information examples are aimed to give. The examples are important for students' understanding but it appears that they usually concentrate on rather unimportant issues and cannot generalize the examples up to a certain abstract level. Somehow the examples are processed in a more static way and are generalized similarly. For example, the multiplicative identity 1 of real numbers is generalized to be an identity for other structures which involve real numbers and some type of multiplication. The next excerpt supports this assumption:

> I: "What would be identity element?"
>
> S4: "OK, Id element has to be 1, but under mod 20 it would be 19, so then it would not have an identity element so it would not be group."

To my question about identity element, I got a quick response that the identity had to be 1 without any consideration of the given operation and set.

Question 7: Give an example of cyclic and non-cyclic group.

As I expected the problem was not difficult for the students. The reason I added the problem was that I wanted to see how the students understood a cyclic group and how they defined it.

> S1: "A cyclic group is one that can be generated by an element *a*. So the only cyclic groups are integers under addition and congruence mod *n*. So, non-cyclic group - reals without 0 under multiplication.

However, in spite of the fact that all the interviewees had a more or less formed idea about a cyclic group, it seemed that one of the students confused a cyclic group with a subgroup. Consider the excerpt from the student's interview:

> S5: If *G* is a group and *H* is a subgroup, then *H* must be a group under a *G* operation. It means *H* has to be associative; *H* needs to have an identity element. There needs to be an inverse which is the same inverse as in *G*.
>
> I: Ok, this is the def of a subgroup. When a subgroup is cyclic?
>
> S5: O, wait! Cyclic...is it the one that has a generator. So the cyclic group means that...if you had $\mathbf{Z}_4$...I can't remember the exact definition...it's like the group...I mean an element of the group that generates the group?

It appears that the students may confuse a cyclic group with a subgroup since as soon as they learn a cyclic group they learn a cyclic subgroup.

Moreover, they proved a theorem that every subgroup of a cyclic group is also cyclic. It seems to make sense to the students since their written work showed no difficulty with generating subgroups of $\boldsymbol{Z}_{12}$. Another theorem about cyclic subgroups says that every element of a group generates its cyclic subgroup. This theorem is more difficult to understand and, furthermore, it bridges the concepts of a non-cyclic group, cyclic group and subgroup. Perhaps this is the reason that instead of giving a definition of a cyclic group the student started to define a subgroup. It is always possible, however, that the student gave this definition by error without really thinking about the problem.

Since the previous question discussed cyclic groups, I will jump to analysis of questions 9 and 10 as they both discuss cyclic groups.

Question 9: Find the order of the cyclic subgroup of $\boldsymbol{Z}_5$, generated by 2.

The problem could be solved by listing all the elements of a subgroup generated by 2 and then calculating how many elements there are. Only 2 students out of 7 interviewed recalled a theorem that allows bypassing unnecessary calculations and concluded that, since 2 and 5 are relatively prime, element 2 generates improper subgroup of $\mathbf{Z}_5$. This observation implies that the definition of a cyclic group and its subgroups make sense to the students and they prefer to use it instead of thinking about additional theorems.

Question 10: Is $\mathbf{Z}$ a subgroup of ($\boldsymbol{Z}_4$, +)? What are all the subgroups?

All the students answered that $\boldsymbol{Z}$ is not a subgroup of $\boldsymbol{Z}_4$, since "it has more elements". Some students, however, added that $\boldsymbol{Z}_4$ is a subgroup of $\mathbf{Z}$:

S5: No, $\boldsymbol{Z}$ is not a subgroup of $\boldsymbol{Z}_4$. $\boldsymbol{Z}_4$ is a subgroup of **Z**! Because no negatives.

It seems that the student made a quick conclusion about $\boldsymbol{Z}$ not being a subgroup of $\boldsymbol{Z}_4$, because $\boldsymbol{Z}$ is not a subset of $\boldsymbol{Z}_4$ and the word subgroup presumes it to be a subset of another group. However, the student seems to forget about the operation of $\boldsymbol{Z}_4$ and **Z**. They both are under addition but in case of $\boldsymbol{Z}_4$ it is addition mod 4. This case suggests that the operational part of a subgroup definition is not considered in the solution.

In the following response the student preferred to start with definition of a subgroup:

S4: A subgroup would mean that it is a sub...the set but we have talked about a subset of it. It's under the same operation, it has to be closed under that operation, it has to be associative, and have inverse elements, and has the identity element, the inverse element with respect to identity element. $\boldsymbol{Z}$ is not a subgroup. $\boldsymbol{Z}$ has more elements then $\boldsymbol{Z}_4$. So, the subgroups of $\boldsymbol{Z}_4$ would be... $\boldsymbol{Z}_2$ would give us 2 and 4, no 2 and 0 and it would be a subgroup. I think it would be the only subgroup, because 1 would give us like everything. So I would say the subgroup would be $\boldsymbol{Z}_4$.

In this case the student follows the definition of a subgroup. She/he did not forget about the operation, and then generated the only proper non-trivial subgroup of $\boldsymbol{Z}_4$. Note that the student calls the subgroup $\boldsymbol{Z}_2$. Perhaps the reason for this is that it is generated by 2, but then for other cyclic subgroups containing 2 this would be wrong. Another reason is the number of elements of a subgroup generated by 2. It has 2 elements under addition mod 4 and is isomorphic to $\boldsymbol{Z}_2$, but $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_4$ have different operations. Since the student did not provide any reasoning why {0, 2} is the only subgroup, I decided to talk a little more about this problem:

I: Why $\boldsymbol{Z}_3$ is not a subgroup? $\boldsymbol{Z}_3$ is a subset of $\boldsymbol{Z}_4$?

S4: I think because $\boldsymbol{Z}_3$ to be a subgroup it has to have the same operation. So, it would have...it would not be closed, because like 2 + 3 would be 5, or even 3 + 3 would be...OK, we have $\boldsymbol{Z}_4$. Like there is 4 possible cyclic subgroups. Which would be 1, which would give us all 4, and then there would be 2, which would give us whatever, and there would be 3, and there would be 4 which is 0 which would give us just the id element. So, there are 4 subgroups which are cyclic...Which are generated by 1, 2, 3, 4.

Note that at first the student doubts the operation but then tries to find something else and finally again bases the conclusion on the fact that if elements of $\boldsymbol{Z}_4$ are used to generate subgroups, then none of the subgroups would have 3 elements. Again, the student did not mention that all the subgroups of $\boldsymbol{Z}_4$ must also be cyclic. I asked the student to list all the elements of the subgroups she was just talking about. $\boldsymbol{Z}_3$ was not in the list.

S4: ...because it would generate different... $\boldsymbol{Z}_3$ has a different operation.

So, the student came back to operation reasoning to prove that $\boldsymbol{Z}_3$ is not a subgroup of $\boldsymbol{Z}_4$.

Question 8: Consider the group $(Z_5 - \{0\}, \times_5)$ = {1,2,3,4}. Please add one or two sentences explaining your answer to the following questions:
a) What is the identity element of this group?
b) What is the inverse of 3 in this group?

The question was aimed to explore students' reasoning about identity and inverse elements in a finite group. Students' background supported their responses for the first part of the question. Most of the interviewees said that 1 was an identity element of the group. Some, however, had trouble understanding the operation. The problem was that they were used to thinking about $\boldsymbol{Z}_5$ as a group under addition mod 5:

S1: Identity element for this group...I want to say 5. We say...take mod 10, 1 is a generator, go up to 10, then 10 equals to 0 - identity.

In this case I had to suggest stepping out of a cyclic group and thinking about the set and the defined operation. I also suggested creating an operation table since for this student operation table provided a good reasoning during interview 1. Following my hints the student found out that the identity is 1. In a way the student thought that the only possible finite groups were the cyclic ones under modular addition.

Things became more complicated with inverse element of 3:

S2: It should be 1/3, so I guess it does not have one. But it has to...Is it negative 2? I forgot...3 is the same thing as...not in this group. Is it 2? Because...if you keep adding 3 + 3 + 3...I do not know.

I: Is 1 inverse of 3?

S2: No, 1 is the identity element. Inverse is like if you multiply it by 3 you get one. It has to be 1/3 but its not there but it has to be...

I: What about 2?

S2: its 6 mod 5...its 1! Ok, so its 2! It makes sense! I forgot about this.

I repeatedly observed the difficulties with connecting new mathematical ideas to the old ones. The students made incorrect parallels with their previous experience and got into the same trap over and over again. In the excerpt the student understood that the identity must exist in the set since it was given to be a group under multiplication mod 5. However, the background suggests that the multiplicative inverse of 3 is its reciprocal, and it seems impossible for the student to think that the inverse element may be different. It also implies that at this stage it is difficult for the students to think about every condition of the problem: the set and the operation. In order to find the inverse element I suggested checking every element of the set. Before the student found the inverse, she/he again came back to 1/3 which showed a strong attachment to standard multiplication.

Interview III

The goal of this interview was to see how the student work with subgroups and how they understand them as mathematical objects. In the previous interview the students were asked to generate subgroups and reason about cyclic subgroups. This time they are asked to determine if a structure with specific properties is a subgroup. I also wanted to understand the students' vision of a subgroup within the group and how the students distinguish elements of a subgroup from other elements of a group. I chose the most significant and interesting questions of the interview for the analysis.

> Question 1: Let G be a group, define the center of G to be the set $Z(G) = \{x \in G \mid gx = xg \text{ for all } g \in G\}$. Determine whether $Z(G)$ is a subgroup of G.

Out of 7 students 3 used a subgroup criterion to prove that *Z(G)* is a subgroup. One of the students started to work with the definition of a subgroup but then gave a comment that a subgroup criterion would probably be easier. In general, the students did not have any problems with computational or algebraic part of the problem. However, the most difficult part was to understand the set that the problem defines. The notation confused many of the students:

> S3: So this a ***Z*** or this is "integer" ***Z***?

The student might not be familiar with the idea of the center of a group. They referred to the familiar object denoted by **Z** – set of integers.

The following excerpt represents a common way to reason about the problem using a definition of a subgroup:

> S5: In order for it to be a subgroup it has to be assoc under *G*, has to be closed, and have an identity, and inverse and an identity element has to be in *G*. So *Z(G)* is associative, cause its multiplication...

The student started with definition of a subgroup. Note that she/he said that a subgroup has to be associative "under *G*". At first I thought that this was caused by miswording but later it became clear that the student intentionally avoided the word "operation" since she/he did not see an explicit description of an operation in the problem. I asked about the operation as soon as the student said that *Z(G)* is associative under multiplication.

> I: What is the operation?
>
> S5: Multiplication. The same operation as *G*? I do not know what *G* is.
>
> I: *G* is a group. What does it mean?
>
> S5: It means that it's closed and that's associative and has identity and inverse element.
>
> I: Is operation defined on *G*?
>
> S5: No. Well there is a binary operation but it does not say it to me...
>
> I: But the group is a set together with its operation...
>
> S5: What's operation? The same operation, so that is assoc, *Z(G)* is closed because *x* is in *G* and *g* is in *G* and they both are in group so it's closed. And the identity is in *G*, so *g* must be the identity and the inverse is just...I do not know the inverse of *G*. Does inverse has to be in *G*. O, it's a group so it has an inverse.

It seems difficult for the student to think about some abstract operation that is not explicitly defined. Moreover, the notation "*xg*" probably suggested to connect the operation to multiplication as this notation was used to represent this operation before. Relying on her/his background the student concluded that multiplication must be the operation. After some discussion, the student decided to say that it is the same operation. However, she/he did not seem to understand the meaning of this. It seems that the student agreed to think about an abstract operation following the instructions but she/he still decided to think about it as multiplication, just for the sake of her/his own reasoning. Further the student was thinking about the closure. From the response it looks like the student still had trouble understanding the difference between the group G and its subset $Z(G)$. She/he proved closure based on the fact that G is a group. In other words, the student proved G's closure instead of $Z(G)$. I think the reason is that the student cannot understand how elements of $Z(G)$ are different from all other elements of G, and it follows that she/he had a misleading idea about the whole structure $Z(G)$. Furthermore, the student concluded that G has an identity element and then g must be the identity. It looks like the conclusion was coming from the definition of elements of $Z(G)$: $gx = xg$. It is possible however, that the student simply meant to say "e from G" and said "g" by accident.

I have also noticed that the student might have difficulty understanding that an inverse is a property of an element of a group, not the group itself. She/he was confused by the fact that there is no "inverse of G". Unfortunately, there is not enough evidence to understand why the student used this statement. It is possible that the student worded the statement this way since it is unusual for her/him to talk aloud about abstract mathematical objects. Perhaps the problem is more serious and conceptual. I have noticed that many students used inverse elements in reference to a group but it is still unclear what effect it may have on problem solving. It also could be a quantifier problem, since both identity and inverse elements are universal quantifications for a group, while the quantifiers are used in a different order when defining inverse and identity. For an identity it is: $\exists\, e \in G, s.t. \forall\, a \in G: ae = ea = a$; while for the inverse $\forall\, a \in G, \exists\, a^{-1} \in G, s.t.: aa^{-1} = a^{-1}a = e$. So it is a matter of using $\forall\exists$ or $\exists\forall$.

The student's concern was to understand if "the inverse is in *G*". She/he had no information on structure of *G*. It was given that *G* is a group and the student concluded that in this case "the inverse belongs to *G*". It seems that it is difficult for the student to think in abstract terms but she/he, in a way, agreed to accept that all elements of *G* have the inverse elements. I tried to encourage the student to elaborate more on identity/inverse elements of Z(G):

> I: Does an identity of *G* belong to *Z(G)*?
>
> S5: Hm...That *e* is ...*e* times *gx* equals *gx*.

This excerpt suggests that the student still had difficulty understanding elements of *Z(G)* and their properties. It looks like when the student thinks about *Z(G),* her/his attention is turned to operational description of the subset. I think that a concrete example of a group and its center would make the general construction clearer for the student:

> S5: I guess what I don't understand what a center of *G* is...
>
> I: It is just a subset of *G*, defined this way under the operation of *G*.
>
> S5: So, if *e* is *x* then *eg* equals *g* times e and this is true. So since *e* is in *G* then it is the same identity.
>
> I: Let's check if every element of *Z(G)* has an inverse.
>
> S5: So *x* is in G, then *g* times *x* inverse equals *x* inverse times *g*? So *gx* times *x* inverse, which is *g*. you have *x* times *x* inverse *g* which is just *eg* = *g*. So, inverse exists.

I: Can you repeat you computations? [The student writes down the correct computations and concludes that *Z(G)* is a subgroup of *G*.]

After making sure that the identity element belongs to *Z(G)*, it looks like the subset *Z(G)* started to make sense for the student. She/he seemed to realize what must be done to check if an element is a part of *Z(G)*. In other words, the student understood the property of elements of *Z(G)* and what exactly makes them special. The student concluded that *Z(G)* is a subgroup, however, she/he did not check if it is closed, while closure was a part of the definition the student gave in the very beginning. This situation is not unique. Often the students give the definition at first and then did not follow it. It looks like they just tried to tell me as much as they remember without really making sense of what they are talking about. In several cases (during the first interview) I asked the students to write the verbal response/definition down and not all of them could right it: "I do not really know what you want me to write". Perhaps it is a lack of ability to reason symbolically. May be the formation of the object, the students define, is not complete and it is complicated for the student to talk about the object and write its description symbolically.

In the next excerpt the student also gave the definition of a subgroup and did not forget about closure:

S2: It's closed under the operation of *G* and has an inverse and identity element in *G*.

However, she/he did not check if *Z(G)* is indeed closed under *G*'s operation. I asked the student if it is closed:

S2: I do not know what the operation of *G* is. How can I check closure?

It looks like the student initially avoided talking about closure since she/he did not understand the operation of G. The fact that the interviewee tried to define the operation in the beginning also supports this assumption:

> S2: Is it a subgroup? I do not see why not. Cause it has...We need to see if it has an identity element and an inverse in G. Doesn't it depend on...? So are they saying ...that the operation is ...you know...it depends on addition or multiplication. Yes, it has an identity element, how do you know if it has an identity element? Well, if an identity element is 1, cause we are multiplying...

As in many responses I already illustrated it looks like the students could not think about any operations other than addition or multiplication. The assumption about an identity 1 means that she/he chose multiplication. I think it happened for the reason I previously described – because of the notation *xg*.

> Question 2. Prove or find a counterexample.
>
> a) If G is an Abelian group, then the set $\{g \in G | g^2 = e\}$ is a subgroup of G.
>
> b) If G is a group, then the set $\{g \in G | g^2 = e\}$ is a subgroup of G.

Although the question was formulated differently, the method of solving this problem was similar to the question 1 solution. However, 3 out 7 students considered the problem without relying on the previous. These students were not sure at first if the situation described in the problem is possible. Consider the following responses:

> I. S1: The elements in this group...it has g and it has e, so the inverse element is g itself... So, Abelian is commutative...So G is like an Abelian group, integers under addition, I may just try and you chose little g to be 2 or something, then 2 squared equals e, which means $2 + 2 = e$ but $2 + 2$ is 4 and 4 is not identity element of **Z**. I do not think that it is going to be a subgroup of all Abelian groups.
>
> II. S2: If G is Abelian it is commutative, so we are saying g times g equals e. But that's not true because g times g inverse equals e, unless g is it' s own inverse! It is true only if g equals g inverse. It's only true if g equals 1 or 0.

In the first excerpt the student seemed confident about the objects she/he was describing. However, the whole idea is misleading. The student did not think of an example of a subgroup with the described properties within an Abelian group but on the contrary, gave an example of an Abelian group and proved that it cannot have none-zero elements g such that $g^2 = e$. It seems like the student misunderstood the idea of a subgroup as a set under the same operation and specified properties within the group. Perhaps, she/he thought about it as about an existing object but this object cannot be present in the set of integers. She/he proved that the group of integers under addition does not have non-trivial subgroups of the type: $\{g \in G | g^2 = e\}$.

In the second excerpt the student is trapped in the "familiar objects". It seems that the only operations she/he could think about are addition and multiplication, and for these concrete situations the only elements that satisfy the conditions of the problem are 0 and 1 respectively.

In the end of the analysis of the interviews I want to say several words about students' proofs. First, I have noticed the difficulties students have proving the statement following definitions or theorems. I find it interesting because it is easier to follow definitions since they suggest some kind of algorithm for the proof. Secondly, I have noticed that some students start a proof with the statement they actually need to prove (Figure 63).

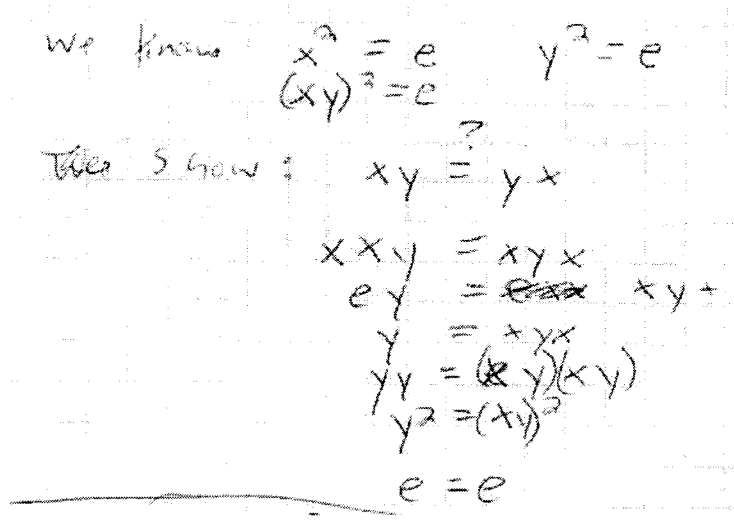


Figure 63. Student's proof.

In some cases the students noticed the problem and did the proof correctly, basing their new proof on one with the wrong order.

The discussion of the analysis and the conclusions are presented in the next Chapter.

CHAPTER VI

DISCUSSIONS AND CONCLUSIONS

The goal of this dissertation is to describe students' understanding of mathematical objects in the context of group theory at the undergraduate level. The following research questions have been formulated based on the theoretical framework of this study:

- **What are the main characteristics of the cognitive processes involved in the development of students' understanding of group theory concepts?**
- **What notions and ideas do students use when they recognize a mathematical object, and why? (what students are using: definitions, properties, visualization, previously learned constructs, or something else)**
- **What are the characteristics of students' mathematical knowledge acquisition in the transition from a more concrete to a more theoretical problem solving activity?**

To answer these questions, students' actions and reasoning about abstract algebra concepts during problem solving activities have been analyzed. The detailed analysis of the data described in the previous chapter provided answers to the research inquiry. In this section I summarize my findings and discuss them to make appropriate conclusions.

Summary

The results of my study were derived from two general types of analysis: global analysis of students' symbolical and verbal descriptions of mathematical objects and conceptual analysis of the ways in which these concepts were applied to problems. At the initial stage I analyzed students' interviews in chronological order. I formed several preliminary categories so that they could guide my further analysis. Later on I analyzed students' written work to support the categories and to form new ones. Written work is qualitatively different from verbal responses. I noticed that students tried to be more careful and more rigorous when writing down their solutions. It can be explained simply by the fact that the written work was aimed to evaluate students' progress in the course and they were more careful in their writing. I also observed that even if a student did not know the exact definition or a way to solve the problem he or she always tried to give some response. I consider these responses to be a significant source of information about students' concept formation. During data analysis I was looking for those responses that showed students' understanding of a concept rather than students' level of preparation. Moreover, sometimes the students gave a correct definition of a concept but later did not use it in the problem solving process or used it in an incorrect way. During the interviews the students were not affected by evaluation pressure and better expressed their own view of the concepts they were asked to describe or use. After a detailed analysis of students' written artifacts I came back to the interview analysis, looking for support of the new assumptions. I arranged the analysis by the students' responses: each work (quiz, exam, and interview) is analyzed problem by problem with several highlighted categories.

Problematic issues that have been found during my analysis can be summarized into several main categories: 1) a correspondence between a set and an operation defined on the set; 2) properties of sets, operations, structures, or elements; 3) use of properties of concrete objects for general conclusions; 4) understanding of abstract algebra statements involving quantifiers; 5) use of definitions – algebraic part versus structural part. In this chapter I discuss these categories in terms of the theoretical assumptions suggested by the theoretical framework that guided this study. Following the theoretical framework, an abstract object is formed via assembling previously abstracted ideas into a new, more advanced concept. Further, the main function of abstraction is recognition, and during this process the concept is articulated and it leads to the formation of the abstract idea.

Discussion

Students are familiar with binary structures from the very beginning of their learning experience. For many, if not for all, the first mathematics experience is counting. Children are getting a sense of the number system and operations on them. However, binary structure in a more general sense includes more complicated relations between elements of a set and an operation defined on this set. While previously it was necessary to apply the operation to any two elements of a given set, now a result of the operation is also an important element of the structure. The data showed that often the question of a result of a certain operation belonging to the set was not considered. The data also suggested that it was not a matter of forgetting the definition of a binary structure. In most cases, students' responses to tasks such as "Prove (or disprove) that a certain structure is a binary structure" were more or less accurate. Nevertheless, if the question did not specifically ask to check whether a certain element belongs to the structure, then this part was often omitted. In some cases the conclusion about closure was presented in the solution but was based on a different operation.

I think that the term "binary structure" and the notation (S, ∗) normally used to represent a binary structure is usually understood by students as a mathematical object with two entrees: a set and an operation. The term and notation do not imply any necessary correspondence or relations between them. Dubinsky et al. (1994) discussed this problem analyzing students' understanding of groups and their subgroups. The study proposed that there are two different visions of a group: 1) a group as a set; and 2) a group as a set with an operation. Similarly for a subgroup: 1) a subgroup as a subset; and 2) a subgroup as a subset with an operation. Analysis of the data I collected showed related trends. I discuss my results in the following section.

Understanding concept of a binary structure

Binary Operation. Closure

The first connection of a set to its operation appears when the students learn the concept of closure. That is, a set must be closed under the induced operation. The question is why the concept of closure is difficult and how students overcome the problem. First of all, the idea of a binary operation is not completely novel to the students. They have been working with operations from the very beginning of their mathematical experience. It is very well understood by every student that if you take two elements of a set and perform an operation on them the result is another element. Everything the students had to think about was the accurateness of the result. What happens in abstract algebra is that now sets are required to be closed, i.e. the result of an operation performed on any two numbers must still be in the set. A concept of closure brings a concept of a binary operation to a conceptually different level of abstraction and the previous experience is not a guarantee of success. The number of mistakes I observed in the data (see for example Figure 6 and Figure 7) suggested that the students still try to assimilate the concept of a binary operation to "operational concepts". Davydov (1972/1990) proposed that the students who experience this problem try to make sense of a binary structure using empirical thoughts (empirical generalization and abstraction). For them a binary operation defined on a set *A* is a function $f: A \times A \to B$, where *B* is some set, not necessarily *A*. The students assemble ideas of a set, its elements, an operation on any two elements, and a result of the operation on any two elements. By a simple generalization process they develop a simple abstract idea or, in other words, there is a shift from concrete operations (such that addition or multiplication, for instance) to abstract (such as operation "star" defined on set {*a*, *b*, *c*}). However, often students overlook one important and conceptually different connection among previously known sets and operations and take the definition of an operation on a set for granted. This connection is the link between the result of the operation and the original set. Realization of this connection would provide the concept of an operation with more structural meaning. I would refer this process to theoretical abstraction since there is a derivation of higher-order structures from the previously acquired lower-order structures (Piaget, 1970 a).

Thus, often the process of understanding a binary operation is empirical rather than theoretical. The data provided evidence for the failure of empirical thought about binary operation during the object recognition stage. As I illustrated in the analysis chapter, when answering the following question: "Give an example of an operation on $\boldsymbol{Z}$ which has a right identity but no left identity", the students often responded that division is this type of operation on $\boldsymbol{Z}$ (Figure 6). Indeed, division is not defined on $\boldsymbol{Z}$, since $\boldsymbol{Z}$ is not closed under division and division by 0 is undefined. However, many students recognize division as a binary operation on $\boldsymbol{Z}$ and the reason is that the idea of closure (connection between a result of the operation and a given set) was not a part of students' analysis of the structure $(\boldsymbol{Z}, \div)$. It follows that a theoretical thought is essential in the process of learning deep, structural mathematical ideas.

The interesting phenomenon occurs when students are struggling with assigning a binary operation to a finite subset of the set of integers (Interview 1, question 6, for example). It does not seem possible for many students to accept the existence of such binary structure. The reason is the closure. Students connect the given finite subset and the operation but for some reason the operation is always addition. Most of the students noticed that the subset cannot be closed under addition but concluded that it was not possible to assign the binary operation to the given set. In this case it appears that students are cognitively placing the objects of recognition to the set of objects for assembling. Operation is the object of recognition, the outcome of the theoretical thinking process. However, the students try to place the concept of operation in the initial stage - assembling process. That is why operation is not a result of the theoretical analysis of previously occurred and given ideas. The conclusion about addition for example is brought to the problem in a rather superficial way, probably due to the elemental association (Halford at al., 1997) of a subset of integers with the binary structure (***Z***, +).

Binary Structures. Group as a set of discrete elements

Understanding of a group as a structure consisting of two objects that interact with each other is complicated and novel for students. The data collected during this study suggests that some students understand a group as a set of elements. The operation in this case does not play an important role in the structure. In the previous chapter, I illustrated some responses where students switched from one operation to another (see Figure 16). It suggests that for the students operation is not an attribute of a binary structure but rather a separate object which may be used if needed.

Another example of students' responses which emphasize understanding of a group as a set occurred in Quiz 2 where the students were asked to define a General Linear group (Figure 13). During the interviews most of the students mentioned both a set and an operation when defining a group. However, only a few students defined a General Linear group *GL(n,* ***Q****)* in terms of a set and an operation. Most of them limit the response to the description of the set of invertible n×n matrices with entries from ***Q***. Further, in response to the following question from Interview 2: Determine whether {4, 8, 12, 16} is a group under multiplication (mod 20), many students noticed that it cannot be a group since 1 (or 0 in some responses) does not belong to the set. The conclusion is based on the elements of the set, not on the given binary structure.

At the early stage of understanding the binary structure concept, students construct their knowledge based on previously learned objects. In order to understand a complex idea such as binary structure, students must have other ideas as parts. So, the elements of a binary structure represent these ideas. The process of generalization initializes connections between the elements, and groups these elements in a set. Thus, the new created abstract entity simply repeats the one that already exists. In this case operation defined on a binary structure is not a part of the assembling process and exists disjointedly from the set. This is the process of generalization in Ohlsson's, Lehitinen's (1997) sense, or Davydov's (1972/1990) empirical generalization. The idea of a group as a set is formed via extraction commonalities from concrete examples, based on visual representations, symbols, discourse, etc. For example, $(\mathbf{Z}, +)$, $(\mathbf{Z}_2, +_2)$, $(\mathbf{Z}_3, +_3)$ as group examples have integer elements in common, 0 as identity element, and based on these examples the idea of a group is empirically generalized. In this case the abstract idea is not complete and further the main function of abstraction fails. According to Davydov (1972/1990), the main function of abstraction is object recognition. So, the abstraction moves from abstract (formed entity) toward concrete (recognition) (Ohlsson S, Lehitinen E., 1997). In this sense recognition is the main function of theoretical abstraction. I already illustrated how the students recognized a General Linear group as merely a set of invertible n by n matrices. Also a structure ($\{4, 8, 12, 16\}^{\times_{20}}$) was not recognized as a group since 1 (or/and 0 in some responses) is not in the set. In both cases the operation is not considered to be a part of the structure and the conclusion is based only on the elements of the set.

In these cases we deal with empirical type of generalization, or simple generalization, using Piagetian (1970) terms. Students assembled examples of groups they studied and extracted commonalities from the sets. According to Piaget (1966, 1970), simple generalization is a part of the empirical abstraction process. In the literature, abstract algebra objects, like many other objects in mathematics, require advanced thinking. Mathematical ideas are complex structures and require a theoretical thought (Davydov, 1972/90). The data illustrated a failure of empirical abstraction to recognize correct objects during problem solving (or working with concrete examples). It follows that at this level of mathematics the main function of abstraction (recognition) is not supported by empirical abstraction. Theoretical thought gives a chance to think about abstract algebra problems without extracting commonalities from previously learned ideas. Again, in the problem about the set {4, 8, 12, 16} under multiplication mod 20, assuming that students' view on this structure is bounded by the set only, we observe a process of simple generalization as merely a search for commonalities between the given set and other structures which are known to be groups. Element 1 is not in the set, so the structure is not a group. The process of assembling in situations like this is not complete and it causes the ignition of empirical generalization as a replacement for theoretical generalization.

While solving the problem about the set {4, 8, 12, 16}, many students tried to find a generator for the set to prove that this is a cyclic group. This way of thinking is unusual but correct. However, the students are thinking about the structure without considering a given operation at all. They try to play around with numbers, using the familiar operations of addition or multiplication. Some students at first stated that 4 is a generator since 4 + 4 is 8, 8 + 4 is 12, etc. This demonstrates a misconception caused by assembling wrong ideas into the generalization process. Obviously, the assembled ideas include the ideas of a set, an element, a generator, a cyclic group, addition/multiplication, closure. I think that in this case we deal with the process of theoretical generalization and further with theoretical abstraction. For this group of participants, the structure is recognized to be a group if it is isomorphic (although I do not think that the students really had a thought about isomorphic structures but they obviously had an idea about structures with similar properties: a cyclic group is a group) to a cyclic group (or itself is cyclic). Now the problem is restrained to the following: find a generator for the set; and the process of thinking goes as following: 1) find a generator; 2) if a generator is found, the structure is cyclic; 3) it is a group. It means that the process of theoretical generalization is completed, or the inner connections between the objects of assembling were analyzed. Under this assumption the recognition and conclusion are not at all controversial and we observe all stages of the process of theoretical abstraction. However, the operational part of the thinking process is misleading, since the operation is not standard addition/multiplication. Thus, the initial assembling is not quite right and the conclusion either can not be drawn, or is drawn incorrectly. Although the operation of addition/multiplication is a part of this assembling process, I still refer this understanding of groups to the "group as a set" type, since the actual operation, defined on the structure, was not considered, or at least was not considered in connection with the set.

Groups and their subgroups

Understanding of a subgroup in general

Since a subgroup is a group itself with some additional conditions, it is difficult to distinguish students' understanding of groups and subgroups. Moreover, some problems in understanding of a subgroup are caused by misunderstanding of the concept of group. Data analysis showed that even if a group or a subgroup is considered to be a binary structure (set together with the operation), the problem of "closure" often persists. Repeatedly students did not consider operation as part of the subgroup concept, or did not connect a group and its subgroup operationally. Dubinsky et al. (1994) suggests that "an individual's development of the concepts of group and subgroup may be synthesized simultaneously" (p. 273). Indeed, in order to learn the concept of a subgroup, the ideas of a group, a subset, and a group operation are assembled and theoretically generalized into an abstract entity for recognition and final concept formation. At the same time, group as a part of the assembling process has a complicated nature. Students must have an idea of a group in assembling not only because a subgroup itself is a group by definition, but also because a subgroup is a substructure of a bigger structure which is also a group. This is a very interesting issue. Assuming that the concept of a group is already learned (previously learned abstract ideas such as set, operation, closure, associativity, identity element, inverse element, etc. are assembled, generalized into an abstract entity and mastered on concrete examples), then the bigger structure which is given to be a group must be recognized as a structure with specific properties which are affecting a substructure. During data collection and analysis I noticed that if a problem stated "Let *G* be a group", students did not always realize what properties it must have. During the interviews some students said that they cannot say anything about *G* since they do not know what *G* is. So, without having a concrete structure the students had difficulty understanding group's axioms and properties. I think it means that initial assembling was not complete and/or the generalization process was empirical rather than theoretical, and concrete examples played the role of abstract ideas in the assembling process. I do not claim that concrete examples cannot be a part of the assembling process. Moreover, the analysis showed otherwise in many cases (see Figure 31): students need both abstract ideas and concrete examples to understand a more complex idea but the shift from these ideas and objects to the new abstract entity must be a result of theoretical, not empirical, generalization.

The analysis showed that students have difficulty understanding connections between a group and its subgroups, both operational and via element. Student's responses revealed three major misconceptions about the concept of a subgroup. First, for some students understanding of a subgroup is similar to the understanding of groups as sets. Unfortunately, I did not have a chance to make a deep and careful analysis of the data in "student by student" way. I did not want to limit my study to special cases only. However, I still made some connections between the responses of the same student. For instance, those students who at first understood a group as a set would not necessarily transfer this understanding onto subgroups and vice versa. For some students a group is a set with the operation while a subgroup is just a subset, a part of a bigger structure. A subgroup exists if a subset exists. For example, several students claimed that the set of odd integers is a subgroup of ($\boldsymbol{Z}$, +).

The study has also shown that students have problems seeing structural connections between groups and its subgroups. Sometimes they only comprehend elements connection. Note that this case is different from the one I described above. This time students realize that a subgroup is a group itself under an assigned operation. It is not merely a subset of a bigger set, it is a structure. Nevertheless, the assigned operation is not necessarily the group operation. For example, some of the responses defended that ($\boldsymbol{Z}_n$,$+_n$) is a subgroup of ($\boldsymbol{Z}$, +), since it is a group and $\boldsymbol{Z}_n$ is a subset of $\boldsymbol{Z}$. I also observed a change of the subgroup operation from the group operation to a different operation during problem solving activity (Figure 16).

In addition, I observed responses that not only demonstrate students' understanding of a subgroup as a subset of a given structure but, in addition, a subgroup is understood as a group and a subgroup has the group operation. However, the concept of binary operation is causing difficulty. It is well illustrated by the following response (Figure 47): the set of odd integers together with 0 is a subgroup of (***Z***, +). Note that I already referred to this example in a different category. However this time there is a conceptual difference. Element 0 is added to the set. It suggests that the students who gave this answer understand that the structure has an identity element. It follows from the fact that a subgroup is a group itself. Moreover, it looks like they understand that the operation is addition, since 0 is the additive identity. So, the only problem is the closure of the structure.

In light of theoretical perspectives it seems that in the standard learning sequence "group – subgroup" the concept of a subgroup is the merger of two concepts. One can have an abstract idea of a subgroup only if the idea of a group has already emerged. At the same time a subgroup being a group has special properties which define the subgroup, therefore it is important that students understand these characteristics. It suggests that to construct the abstract idea of a subgroup students need more ideas for assembling than for understanding the concept of groups.

When students are solving problems involving groups and subgroups they are acting upon given objects and operations using abstract ideas they already have. At this stage, the concept of a subgroup is not abstracted yet. It requires more mastery: more concrete examples. At this stage students already have the required minimum of assembled ideas, for instance: set, operation, closure, subset, identity, inverse and group. Still, assembling of various ideas is not enough. There is also the process of theoretical generalization but what if the recognition of a subgroup still fails? I think that the problem is in articulation. The concept of a subgroup is not articulated enough yet. It seems from the data that there is a lack of counterexamples in the articulation process. Theoretical generalization as a part of theoretical thought suggests that, for instance, the existence of an inverse element for every element of the set must be proved and is not given for granted. Students prove the statement using false arguments simply because they are absolutely sure that objects described in the problem cannot exist. It is difficult to recognize since the concept is not mastered to the extent when one can shift from abstractly defined structures to concrete examples of such structures. It follows that the abstract idea is not formed yet since the process of abstraction is a shift from abstract to concrete.

Cyclic groups and cyclic subgroups.

The analysis showed that students find it easy to work with concrete examples of cyclic groups. Moreover, they are very comfortable listing their subgroups and describing them. Not all the students however, appreciate theorems which help to minimize steps in the problem solving process. The fact that students often used cyclic groups as concrete examples during problem solving suggests that cyclic groups proved themselves very useful objects for the articulation process in the group concept formation. The cyclic groups of integers mod n are good examples of finite groups. Cyclic groups are also good examples for illustrating the connections between elements of a group and for many other aspects which may seem unclear at first. It also seems obvious for the students that all subgroups of a cyclic group must be cyclic. Nevertheless, sometimes this understanding is coming from empirical generalization (students observe several examples of subgroups of a cyclic group and conclude that they all must be cyclic), rather than from analysis of the inner connection within the structure. As a result, students accept the idea that if *G* is a group, then it is closed under the assigned operation. It follows that every nonidentity element generates a nontrivial cyclic subgroup. However, students' view of the inner connections is still not comprehensive and a group is perceived as a union of such cyclic subgroups (see for example Figure 54).

Data analysis and theoretical perspectives suggest that when learning concepts of cyclic groups, their subgroups and cyclic subgroups, students often rely on empirical generalization since the concepts are well illustrated by a variety of concrete examples. Instead of recognizing concepts in the examples, students are looking for commonalities via empirical thought rather than theoretical.

<u>Making conclusions based on concrete objects: more cases of empirical generalization</u>

According to the literature (including the authors Piaget, Mitchelmore, White, Davydov) abstract ideas in mathematics cannot be learned through empirical generalization. It means that it is not possible to learn by extracting commonalities from concrete examples. I should mention that "concreteness" of ideas is usually defined by students themselves. Indeed the set of integers modulo n is hardly concrete. However, in mathematics it is often considered to be concrete. Ferguson (1986), for example, reported that for students numbers are concrete and understandable. Operations with numbers as well as results of these operations are also concrete in students' sense. However, as soon as variable is involved the problem becomes more abstract and confusing. Under this assumption it is clear why during the assembling process students often use concrete objects. For example, many responses showed that no matter what group is defined, students most likely would choose 0 or 1 to be an identity element. Similar for inverse, one of the students I interviewed kept looking for 1/3 in the set of integers mod 5, since she/he "knew" that it is multiplicative inverse of 3. At the same time the student knew that $(\boldsymbol{Z}_5 - \{0\}, \times_5)$ is a group and there must be an inverse element for 3. It looks like examples of multiplicative groups such as $(\boldsymbol{Q} - \{0\}, \times)$ or $(\boldsymbol{R} - \{0\}, \times)$ are the objects in the assembling process. There is nothing wrong with this. However, it is still necessary for students to bring other ideas for assembling for understanding the inner connections of the objects and concrete examples. I have mentioned before that neither a researcher nor an instructor can predict a number and type of ideas one needs to assemble to understand a certain concept. During my teaching practice, I had a student who needed to think about a clock to understand the concept of angle. So, I assume, the assembling process is individual. It is also difficult to say if assembling of certain ideas would result in the needed abstract entity formation. Nevertheless, there are some ideas which must be learned before and must be assembled in order to understand a new concept. This result depends on students' actions upon the assembled ideas, or generalization. The example about 1/3 being the inverse element of 3 in $(\mathbf{Z}_5 - \{0\}, \times_5)$ (this common misconception was not only observed in the data I collected but also in some other reports, for example Hazzan (1999)) suggests that the student is basing the conclusion on the concrete structures. However, it looks like the concepts of inverse element and identity element are also presented in students' assembling process. Based on the definition of these concepts and several examples (such as $(\mathbf{Q} - \{0\}, \times)$ or $(\mathbf{R} - \{0\}, \times)$) students often observe that 1 is a common identity element which satisfies the definition. Similarly for inverse, a common pattern in the structures: for every element *a* from the structure its inverse element is $\frac{1}{a}$. These common features are easy to extract out of other properties and elements. The process of observing patterns and simple commonalities is described by Davydov (1972/1990) as empirical generalization. The result of this type of generalization is an empirical abstract idea. An individual, who possesses the empirical idea, has difficulty recognizing objects, since she/he is trying to find same commonalities in the object of recognition. The recognition often fails and the abstract idea is not complete or has a misleading form. This abstract idea can confuse students, as one of the interviewees, who realized that 1/3 is not an element of the structure $(\mathbf{Z}_5 - \{0\}, \times_5)$ but knew that it was a group and there must be the inverse of element 3. Sometimes the empirical thought simply misleads an individual to the wrong conclusion without realization. In this case the conclusion would be that the structure $(\mathbf{Z}_5 - \{0\}, \times_5)$ is not a group since 1/3 is the only possible inverse of 3.

Since theoretical abstraction does not go from concrete to abstract but rather backwards, is it possible to rely on concrete objects and still produce a theoretical thought. It looks like the problem is not entirely in the assembling process. Concrete examples together with previously abstracted ideas are necessary for the correct and solid concept formation. Nevertheless, generalization is the problem. What does it mean to generalize theoretically versus empirically? According to Davydov we must look for inner connections between the assembled objects. Such a connection in the illustrated example with 1/3 would be, for instance, the analysis of the question why 1 is an identity element in the exemplified structure, why 1 is also an identity element in another structure, and how the operation affects the choice.

Sometimes students use concrete examples to make sense of a problem or to help themselves to understand what to do next when solving a problem. I think that, if the examples indeed represent objects described in the problem, this way of reasoning really helps students to solve the problem. Moreover, it verifies the deep understanding of the abstract concept being used. It illustrates that the concept is already abstracted and well articulated. I would call the ability to not only recognize but also to produce a concrete object which exemplifies a certain abstract entity as a high-order articulation. Often students have difficulty not with recognition of an object but rather with producing an object with certain properties (Interview II, Question 4). This is a conceptually different problem. Students have to go from the definition of the object to some concrete representation of the object and then back to the definition. The process of abstraction in this case goes from assembling of needed ideas to concrete object production (not abstract entity as usual) and then recognition. The difference between the usual recognition problems and this type of problem is that the second one requires "two stages" of theoretical thought: first, understanding of assembled ideas and producing the concrete object; and second, recognition of the object as satisfying the given conditions. Indeed, the more complex theoretical thought is required for more advanced problems, such as producing examples and counter examples.

Definitions of objects. How students use them

The data (both written artifacts and interviews) illustrated that the students mostly used informal definitions of the concepts they study. Normally, an exam or a quiz in the course included questions about definitions, and it was always announced in class before the test so that they could study the definitions. Every quiz and exam was structured in such a way that students had to give a definition of a concept and then solve problems involving this concept. This strategy of testing helped me to identify some interesting patterns. It showed whether the students used a definition they just formulated, and if they did, in what way and what parts of the definition the students considered being the most important and significant for concept recognition and handling.

The theoretical framework suggests that a definition is the initial stage of concept formation. A definition suggests ideas for assembling. For example: a group is a set, closed under an assigned operation, the operation must be associative, an identity element must be in the set and every element of the set must have an inverse. The definition puts forward some previously abstracted ideas for assembling. Analysis of the connections between the ideas, and articulation follow the assembling. Later, when concepts are being recognized in concrete problems student also must refer to definitions to collect objects from the assembling process, which must be recognized first.

It was mentioned above that evaluation material for the class I worked with included questions to state a definition of a certain object. I also included questions of this type in the questionnaires I created for the interviews. I noticed that the students did not like these questions. One of my interviewees noted that they "just use it" but she/he was not sure how to state the requested definition. Also I noticed that even if a definition was given almost completely students infrequently used it in their problem solving; or used it partially. It suggests that there is a gap between the abstract entity students have constructed from the definition and the articulation process, the recognition per se. Recalling my assumptions that definitions suggest ideas for assembling, it follows that not all ideas are being generalized into an abstract entity. Further, when recognizing objects using definitions, students simply recognize the objects that were parts of the assembling process. However, some of these objects may not find their place in the abstract entity (and thus will not be recognized) or could be considered as unimportant. It results in failure of the recognition using the definition.

Now I would like to return to the interview questions about definitions. I think that in response to such a question students' reasoning is supposed to shift back from the object that is recognized to be, for instance, a group, to the objects that make it a group. In a way it is the reverse process of abstraction I described in the theoretical perspectives. We are looking for main ideas, which would be assembling participants in the beginning of the learning of the concept. This problem unfortunately is often difficult for students. Most often they recall the symbolic (or algebraic) part of definitions (consider excerpt in Figure 28, for example). I think it makes more sense to them, and it looks more precise to students than words and other conditions. It is always easier to understand a formula, or use a formula. It follows that if a definition includes a certain algebraic (symbolic) statement, then it is always a part of assembling process and necessarily a part of the abstract entity. This explains why students, when asked to exemplify an Abelian group, usually come up with an operation that is non-commutative but fail to notice that this operation is often not binary for the given set or the set they define.

Another important issue that came from the analysis and must be discussed is the use of quantifiers and understanding of quantification in general. The following section is addressing this issue.

Quantifiers

Understanding of quantifications is crucial for understanding of mathematical concepts, and abstract algebra concepts in particular. Indeed, when students learn the concept of a binary operation, for instance, it is important to grasp that the operation must be defined for ALL ordered pairs of elements of the set. One of my interviewees when defining a binary operation on the set {1, 2, 3, 4} assumed that it is possible to add 1 and 2, but subtract 3 and 4. The student assumed that every pair can have a different operation assigned to it. I noticed that at first almost no one used quantifiers when writing definitions or problem solution (including proofs). It does not always mean that the students do not understand quantification. It is possible that they just do not feel comfortable using standard symbols to describe quantification. Note that all students in the class I observed were required to take a course on mathematical proofs as a prerequisite for abstract algebra, so they are familiar with quantification and its role in proofs and definitions of mathematical objects. Later on many students started to use symbolical notations for quantifiers.

When I started analyzing the interviews data, I was surprised that students did not use quantifiers at all when defining objects. I already mentioned in the previous section how students usually defined objects especially when doing it verbally. However, missing quantifiers did not mean that the concept was not recognized or used properly during problem solving process. The preliminary analysis of the interviews suggested to look more carefully at the written work in terms of the presence of quantifiers. I noticed that students used quantifiers more often when writing statements but I observed some other problems. Sometimes the students changed the order of quantifiers they used. For example, instead of writing $\forall\exists$ statement they had $\exists\forall$ statement (Figure 19). I have also noticed that sometimes a quantifier corresponding to a different concept is used (Figure 21). Further analysis implied that the problem is not only symbolical but more global – misunderstanding of quantification.

Several studies that explore students' understanding of quantifiers are Dubinsky, Yirparaki (2000); Epp (2003); Durand-Guerrier, Arsac (2005). Most of them explore students' ways of proving statements and as a consequence their view on quantification since it provides the basis for formal logic. Hazzan and Leron (1996) report students' difficulty using definitions or theorems since they do not understand the quantification. Even if a person has knowledge about the logical principals used in the problem it is still difficult for students to employ the principles for the valid reasoning (Epp, 2003). Literature (Dubinsky, Yirparaki (2000); Epp (2003); Durand-Guerrier, Arsac (2005)) questions the connections and differences between formal and informal discourse concerning quantified statements.

As I previously noticed, I did not have a chance to study students' solutions in detail, since it was not a part of my research questions. However, I formed my view on students' understanding of quantification by studying their ways of defining abstract algebra objects. All objects in abstract algebra are complex structures that involve many other objects as parts and quantification helps to understand the relations between the objects and the structure that is being defined. The data revealed three major issues with understanding of quantification: missing quantifiers; $\forall\exists$ statements versus $\exists\forall$ statements; and misplaced quantifiers. The following paragraphs discuss these issues.

Since the study was aimed to understand students' formation of abstract object I did not attempt to analyze students' mathematical logic and their ways of proving statements. I believe this analysis would help me to identify students' understanding of the role of quantifiers. It looks for me that students do not comprehend the power that quantifiers have over mathematical statements that they can change the statements dramatically, from true to false. It is a thought for future studies.

Referring to students' written work I attempt to understand why the students do not use quantifiers in their responses and how it effects their problem solving. First, I have noticed that it is not possible to draw a parallel between students' use of quantifiers in the definitions of objects and their use of quantifiers in problem solving. Most often students use quantifiers in a definition and then do not employ them in the problem solving. In this case it looks like the students rather memorize the definition without making sense of every aspect of it (including language, terms, formulas, concepts). Quantifiers are a part of the logical construct. It means that, following the theoretical framework, understanding of the quantification of a certain mathematical object should probably appear during theoretical generalization process and further during articulation process when students recognize the object or understand why the given object cannot be identified as a certain structure, for instance a subgroup.

It follows that quantification is not employed during generalization process. Respectively, it is not employed in the articulation. Further, the structure is not properly recognized. So, the process of abstraction fails. Similar scenario occurs when students misplace quantifiers.

The problem with $\forall\exists$ statements versus $\exists\forall$ statements suggests that quantification is a part of generalization and articulation process. However, the data showed that often students do not see the difference between these logical statements. The most significant concepts for this matter are the concepts of identity and inverse elements. The concepts are very close to each other symbolically. Identity is a part of the definition of an inverse. Also, both of these concepts are parts of the bigger concept of group. It partially explains why students sometimes mistakenly start to think about an identity while they are asked for an inverse element. Also, sometimes they claim that if there is an identity in a certain structure then every element must have an inverse (see Figure 61). Furthermore, sometimes students confuse logical structure of the concepts of identity and inverse. The definition of identity element e in $(G, *)$ is $\exists\forall$ statement while the definition of inverse element is $\forall\exists$, but students do not pay attention to this difference and do not attend to the meaning of both statements and often do not see the difference in the meanings. As a result, students may find more then one identity element in structures defined in the problems.

Dubinsky, Yiparaki (2003) compared the use of $\forall\exists$ statements versus $\exists\forall$ statements in natural language and in mathematics. Analyzing their data they have noticed two main trends:

> Those students who did not pay attention to quantifiers in the natural language statements, continued to ignore quantifiers in the mathematical statements; and, some students who had attended to the quantifiers in the natural language statements failed to do so in the mathematical statements. There were no students who attended to quantifiers in mathematics but failed to notice them in the natural language statements. (p. 264)

I found similar trends to this phenomenon. I noticed that the students who did not pay attention to quantifiers when defining objects continued to ignore quantifiers when solving problems and applying the definition. Some students who had attended to quantifiers in the definition failed to do so during problem solving. However, Dubinsky, Yiparaki (2003) further noticed that "there were no students who attended to quantifiers in mathematics but failed to notice them in the natural language statements." (p.264), while there are cases in my data when students did not use quantifiers in definitions but they attended to quantification during problem solving.

It is difficult to make conclusions about how quantifiers (both in natural language and mathematical statements) affect concept formation and what stage of abstraction is responsible for fitting in logical construction. My data showed only the top of this iceberg and pointed out what types of problems may emerge when students work with statements involving quantification. Further data collection with modified questions is needed to come closer to understanding the role quantifiers are playing in concept formation.

Conclusions

The study was aimed to understand how students reason about abstract algebra concepts, how they operate with abstract objects, how abstract concepts are generated in general, and what connections between abstract concepts and concrete examples students see. The study is guided by the theoretical framework that is based on Piaget, Ohlsson, Lehitinen, and Davydov's view on students' reasoning and the processes of abstraction and generalization.

The study showed that one needs to have previously abstracted ideas to understand a new abstract structure. Moreover, data analysis and further discussion ascertained that an abstract concept cannot be learned without concrete examples and problems that involve the concept. In other words the articulation of an abstract concept is required for coherent structure formation. In this section I summarize the discussion of the findings and make several conclusions about students' understanding of abstract objects. I also summarize the causes of major problems in students' learning that can lead to misconceptions and inability to solve abstract algebra problems.

The data and theoretical framework suggested the model of abstract concept formation – process of abstraction (Figure 64).

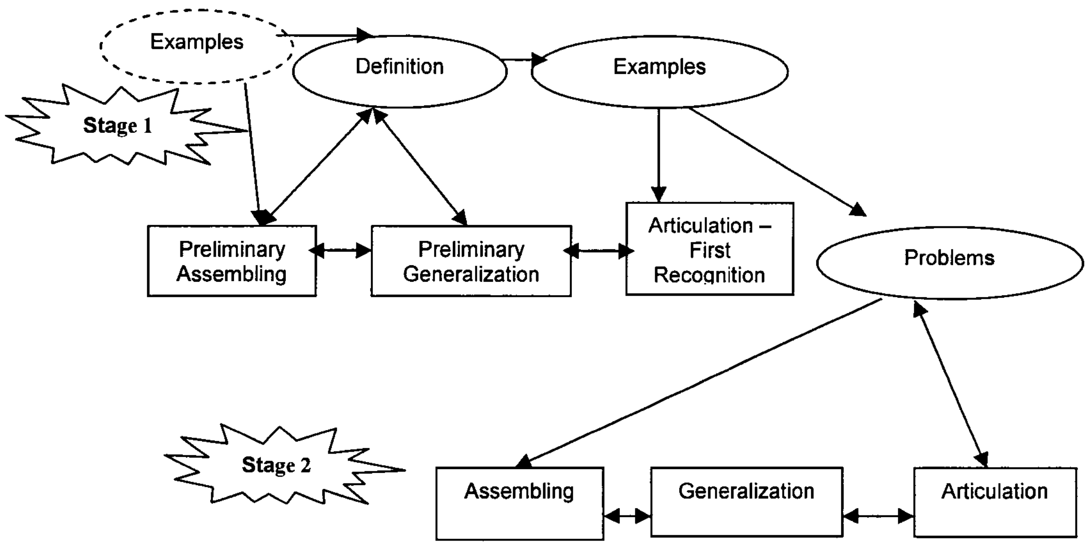


Figure 64. Process of Abstraction

At the first stage of the learning process students are often given a definition of a concept being studied. Sometimes several simple examples precede the definition. These activities give students a chance to generate a preliminary set of objects for assembling. All these objects are previously learned abstract ideas. The process of assembling is followed up by the process of theoretical generalization. Since a definition usually gives only a preliminary set of ideas for assembling, it is most likely impossible to coherently understand inner connections between the ideas and form a plausible abstract entity. For this reason, I describe this process as a preliminary generalization. The next standard instructional step is illustration of the concept via various examples. During this stage students are getting the first articulation experience and make first attempts to concept recognition. The first stage does not necessarily lead to consistent concept formation. I consider the concept to be generated if it is recognized during problem solving together with all its properties. At this stage a student should be able to exemplify and counter exemplify the concept. It means that when the concept is learned the process of abstraction of these objects gets into the following static form: 1) connected assembled ideas; 2) complete understanding of meaningful inner connections; 3) open-minded recognition of the object. At this stage, students should be able to move easily from object recognition to assembled ideas, if needed. Most of the time, if not always, the first stage does not give the result of the static form I just described. After the concept was defined and exemplified, instructions usually are followed up by a problem solving activity where the concept that is being studied interacts with other concepts and ideas. This is the second stage of the concept formation process. At this stage students are exposed to additional ideas for assembling, and make more thorough theoretical generalization for the correct recognition of the object. Also this stage provides students with understanding of properties of the concept and they again could add ideas for assembling. These stages are repeated as many times as needed.

Now I would like to summarize possible quandaries that are coming out of the theoretical configuration described above. As I have noticed all the stages of abstract concept formation are interconnected. There is a constant interaction between processes (assembling and articulation) within the process of abstraction. This observation implies that if there is a problem with one process the abstract concept cannot be appropriately formed. The discussion of my findings led me to the following summary of possible predicaments for concept formation:

1. Empirical generalization and abstraction instead of theoretical. Students are trying to learn concepts by extracting commonalities from given concrete objects and examples.
2. Assembling of unsuitable ideas. Students mistakenly assemble some ideas which are not supposed to be assembled to learn a certain concept. As a result, theoretical generalization results in a misleading abstract entity and further in false conclusions which look true under students' arguments.
3. Insufficient number of assembled ideas.
4. Making the object of recognition (during problem solving) one of the ideas for assembling (see page 172 for discussion).
5. Insufficient articulation. Students find it difficult to provide examples and especially counterexamples.
6. Isolation of concrete examples from objects of assembling. Sometimes students do not see the interaction between the concrete examples and the abstract structure. A concrete example is considered to be a static object with fixed properties. For instance, students know that (***Z***, +) is a group with identity 0 and the inverse for any element is its opposite integer, but they do not question it, they simply take it for granted.

The list above summarized my findings. The summary provides a basis for study implications. The next chapter discusses the implications and future study propositions.

CHAPTER VII

IMPLICATIONS OF THE STUDY

The focus of this section is on implications of the study for future research and for teaching abstract algebra and mathematics in general at both, college and school levels. The suggestions for future research are motivated in part by the limits of one study. Ideas for implication evolve from the data and theoretical discussion.

The analysis and further discussion elucidated the importance of the assembling process for concept formation. The study showed that it is not possible to describe or define the set of ideas for assembling when learning a new concept. In other words, the assembling process is individual. A student may have personal associations with the object of study which make sense only for her/him. However, to understand students' concept formation more deeply one must study the process of assembling in more detail. I would formulate the following questions for future exploration: Why are certain ideas are assembled? How do students connect the ideas in the assembling process to generalize them into the new abstract entity? Finally, it would be interesting to study if it is possible to create the set of objects which must be a part of the assembling process for learning a certain concept (call it concept decomposition). If yes, how it would affect instructions and students' understanding of mathematical concepts.

The data illustrated the difficulties students have when learning quantified statements. The proposed theoretical framework did not provide rational explanation for this phenomenon. The framework needs to be modified for the future exploration of quantifiers. As I mentioned before the data pointed out the problem but was not aimed to concentrate on this problem which was rather unexpected. In the future I would like to understand what the possible reasons for the difficulties are and how these difficulties effect students' concept formation. I want to find answers to the question: what is exactly missing: mathematical logic, symbolic notation, interpretation of quantifiers, or something else.

The study also suggested several implications for teaching. First, the teacher must be aware of the problems that students could possibly have when studying abstract algebra. The study is rich in data and it includes explicit examples of learning situations which are common for abstract algebra course participants. These examples can help a teacher to prepare for certain obstacles and think about problems or examples which may help the students to overcome these difficulties. Moreover, the study may suggest some problems or examples for the instructor's consideration.

The study also may suggest possible instructional approaches, following the cognitive problems I indicated in the previous chapter.

1. Writing definitions during problem solving activities and exemplifying concepts. I believe it would help to compel the

appreciation of formal logic and make definitions more explicit. It would allow students to properly use definitions when proving statements. It is especially important for teachers' preparation since NCTM Standards (2000) suggest that students should be able to develop and evaluate mathematical arguments and proofs.

2. Discussions of possible ideas for assembling. Preparation of sets of questions to stress the connections between assembled ideas.
3. Collaborative activities. Group discussions where students can put together more ideas for assembling and discuss connections between the ideas.
4. Understanding the time limits of every course it is not possible to suggest doing more problems for articulation. However, I suggest more "problem posing" activities and special attention to counterexamples.

The last suggestion also contains an additional question for possible future exploration. I have noticed that the students have difficulty with producing examples of certain objects. I would like to study how the instructional approach, based on variety of such problems would influence students' understanding of abstract algebra objects and abstract concept formation in general.

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APPENDICES

APPENDIX A

DATA COLLECTION TIME TABLE

Order	Date	Type of artifact
1	09/23/05	Quiz 1
2	09/30/05	Exam 1
3	10/17 – 10/21/05	Interview 1
4	10/19/05	Quiz 2
5	10/26/05	Exam 2
6	11/07 – 11/17/05	Interview 2
7	11/28 – 12/07/05	Interview 3

APPENDIX B

SAMPLES OF INTERVIEW QUESTIONS AND WRITTEN ASSIGNMENTS

- Quiz 1 Problems
- Quiz 2 Problems
- Exam 1 Problems
- Exam 2 Problems
- Interview I Questions
- Interview II Questions
- Interview III Questions
- Students' Questioner

MATH 761: QUIZ 1
23 September 2005

1. Define each of the following terms as completely and accurately as you can.

(a) a TRANSITIVE relation $\mathcal{R}$ on a set A.

(b) an ASSOCIATIVE operation * on a set S.

(c) a ONE-TO-ONE function *f* from A to B.

2. Give an example of an operation on Z which has a right but no left identity. [*Hint:* You've known about this a very long time!]

3. Consider equivalence relations on a set A, where $|A| = 8$. What is the greatest number of equivalence classes such an equivalence relation can have?

MATH 761: QUIZ 2
19 October 2005

1. Define each of the following terms as completely and accurately as you can.

(a) a SUBGROUP of a group *G*.

(b) the GENERAL LINEAR GROUP *GL (n, Q)*.

2. State the LEFT CANCELLATION PROPERTY for a group *G*.

3. Recall that *nZ* is precisely the set of integers which are multiples of the given integer *n*. Use the "subgroup criterion" to determine *whether or not* the set $2Z \cup 3Z$ is a subgroup of (Z, +).

Math 761: FIRST HOUR EXAM
Friday 30 September 2005

Instructions: Work all five problems on this exam as completely as you can. Show your work, since partial credit will be awarded where appropriate. Work steadily, not allowing yourself to get hung up on any one question. Please cross out any work that you do not want me to consider. Good luck!

1. (8 points each) Give a definition for each of the following terms that is as complete and accurate as possible.

(a) a PARTITION of a nonempty set S;

(b) an IDENTITY ELEMENT in a set S with binary operation *;

(c) a PERMUTATION of a nonempty set S.

2. (16 points) Let S = {*a, b, c*}. How many distinct binary operations exist on S which have "*a*" as a (two – sided) identity element? [*Hint:* Think in terms of completing

operation tables. *Warning:* This problem does not involve isomorphism in any way!]

3. (15 points) Define a binary operation $\diamond$ on Q* by the rule

$$a \diamond b = \frac{1}{(ab)^k}$$

where k is an integer. For which value(s) of k will (Q*, $\diamond$) have an identity element, and what will that identity element (or those identity elements) be?

4. (20 points) Use the cancellation property of groups to prove that every element of a group has exactly one inverse.

5. Below is a table giving a binary operation $*$ for $S = \{a, b, c\}$.

(a) (8 points) Produce an operation table for S which is not isomorphic to (S, $*$). Justify your answer.

(b) (10 points) Produce an operation table for S – different from the original one – which is isomorphic to (S, $*$). Justify your answer.

(c) (7 points) Assume the operation $*$ below is associative. Do there exist two distinct operations which make it a group? Justify your answer.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Math 761: SECOND HOUR EXAM
WEDNESDAY 26 October 2005

Instructions: Work all five problems on this exam as completely as you can. Show your work, since partial credit will be awarded where appropriate. Work steadily, not allowing yourself to get hung up on any one question. Please cross out any work that you do not want me to consider. Good luck!

1. (6 points each) Give a definition for each of the following terms that is as complete and accurate as possible.
(a) an ABELIAN group G;
(b) A CYCLIC group G;
(c) The TRIVIAL SUBGROUP of a group G.
2. (22 points) Exhibit all the subgroups of $(Z_{12}, +_{12})$, and indicate which of these are subgroups of other of these.

3. (20 points) Is it possible to find two nontrivial subgroups H and K of (Z, +) such that $H \cap K = \{0\}$? If not, why not?
4. (20 points) Prove or disprove: If G is a finite group and $a \in G$, then there is some integer n such that $a^n = e$.
5. (20 points) Let (G, $*$) be a group and H be a nonempty subset of G. Suppose H is closed under the operation of G and that (H, $*$) has an identity element e_H. Prove that $e_H = e_G$.

INTEVIEW I QUESTIONS.

TOPIC: Binary operation.

TIME: ~ 40 minutes.
Please specify all the details aloud. Say if you are using a definition, or property, or theorem to proof your statements.

I.

1. Define what it means to say that * is a binary operation on a set A.

2. Decide which of the following statements are correct, please explain your reasoning:
   a) a binary operation on a set S assigns at least one element of S to each ordered pair of elements of S.
   b) a binary operation on a set S assigns at most one element of S to each ordered pair of elements of S.
   c) a binary operation on a set S assigns exactly one element of S to each ordered pair of elements of S.

3. Give an example of binary operation on **Z**.

4. Give an example of operation on **Z** which is **not** a **binary** operation on **Z**?

5. Let S be a set. Let * be an operation on S. What does it mean if for some elements a, b of S a*b is not in S?
   In your words, what does it mean for the set to be closed under the operation?

6. Define a binary operation on S = {0,1,2,3,4}.

7. Determine if the following binary operation is associative. Does it have an identity element? Decide if it is commutative. Operation is defined on Z^+ by $a * b = 2^{ab}$.

8. What do you understand by a binary structure?

INTERVIEW II QUESTIONS

1. In your own words, what does it mean that G is a group?

2. Let $\frac{1}{2}$**Z** be the set $\{\frac{1}{2}z | z \in$ **Z**$\}$.
(a) is it a group?
(b) Give an example of a binary operation on the set. Is it a group now?
(b) Confirm that $\frac{1}{2}$**Z** is a group under addition.
(b) Is $\frac{1}{2}$**Z** a group under multiplication? Explain.

3. Let **G** be a group, let **a**$\in$**G**. How many inverses can **a** have? Why?

4. Give me an example of Abelian and non-Abelian group.

5. Determine whether {4, 8, 12, 16} is a group under multiplication (mod 20).

6. Give a table for a binary operation on the set {*e,a,b*} satisfying axioms A2, A3 for a group but not A1.(A1. the binary operation is associative;
A2. there is an element *e* in G, s.t. $e*x = x*e = x$ for all $x\in$G.
A3. for each $a\in$G, there is an element $a'\in$G, s.t. $a'*a = a*a' = e$)

7. Give an example of cyclic and non-cyclic group.

8. Consider the group $(Z_5 - \{0\}, \times_5)$ = {1,2,3,4}. Please add one or two sentences explaining your answer to the following questions:
a) What is the identity element of this group?
b) What is the inverse of 3 in this group?

9. Find the order of the cyclic subgroup of Z_5, generated by 2.

10. Is Z a subgroup of (Z_4 , +)? What are all the subgroups?

11. Is Z_3 a subgroup of Z_6?

12. If not find a subgroup of Z_6, containing 3 elements? Is this group isomorphic to Z_3? How would you answer 13 now?

13. Are Z5 and Z7 isomorphic? If yes, find an isomorphism.

14*. Let $(G,*)$ be an Abelian group, *t* is a fixed element of G. Define a binary operation ◊ by (*x, y*) → $x*y*t^{-1}; x,y \in G$ Prove or disprove that (*G*,◊) is a group.

15*. Prove that if $a^2 = e$ for any element a of a group G, then G is Abelian.

INTERVIEW III QUESTIONS

TOPICS: Subgroups, group operation, group of permutations.

1. Let G be a group, define the center of G to be the set Z(G) = {x $\in$ G | gx = xg for all g $\in$ G}. Determine whether Z(G) is a subgroups of G.

2. Prove or find a counterexample.
a) If G is an Abelian group, then the set $\{g \in G | g^2 = e\}$ is a subgroup of G.
b) If G is a group, then the set $\{g \in G | g^2 = e\}$ is a subgroup of G.

3. Find the subgroup generated by the given element in the specified group: the subgroup of S_4 generated by (134).

4. Find a subgroup of S_4 that is the same as S_3.

5. Is every subset of a group a subgroup under the induced operation?

6. Is S_3 a subgroup of S_6?

7. Determine whether the given function is a permutation of **R**, explain:

a) $f_1(x) = x + 1$
b) $f_3 = e^x$

8. Please give a brief answers to the following questions:
(a) How do you study for this math class?
(b) Do you use a textbook? How often?
(c) Do you take detailed notes? In your opinion, how helpful it is to you?
(d) What is you main source of information for the course?
(d) Can you say several words about your experience with Abstract Algebra?

9. Let H, K be two subgroups of G. Prove that H$\cup$K is a subgroup of G $\Leftrightarrow$ one subgroup contains another.

10. Give a proof or disprove the following statement: Every subgroup of a non-Abelian group is non-Abelian.

16*. If $*$ is a binary operation on a set S, an element x of S is an idempotent for $*$ if $x * x = x$. Prove that a group has exactly one idempotent element.

Questionnaire for Math 761.

Please provide the following information:

1. Name:

2. Major/concentration:

4. Past Math classes:

6. Future plans:

7. Questions, goals, hopes, concerns you have about Abstract Algebra, or mathematics in general:

APPENDIX C

MAIN DEFINITIONS AND THEOREMS USED

Definition. A **binary operation** $*$ on a set S is a function mapping $S \times S$ into S. For each $(a, b) \in G$ there exists $c \in G$ such that $a * b = c$.

Definition. A binary operation $*$ on a set S is **commutative** if (and only if) $a*b = b*a$ for all $a, b \in S$.

Definition. A binary operation $*$ on a set S is **associative** if $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$.

Definition: Let $(S, *)$ be a binary structure. An element e of S is an **identity element for** $*$ if $e*s = s*e = s$ for all $s \in S$.

Definition: A group $<G, *>$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:

G1: For all $a,b,c \in G,$ we have

$(a*b)*c = a*(b*c)$. **associativity** of $*$

G2: There is an element e in G such that

$e*x = x*e = x$. **identity element** e for $*$

G3. Corresponding to each $a \in G$, there is an element a' in G such that

$a' * a = a * a' = e$. **inverse** a' of a

Definition (as was given in class): Let G be a group. A subset H of G is a **subgroup** of G if H is itself a group under the operation of G, denoted $H < G$.

Theorem: Let G be a group, $H \neq \emptyset$, $H \subset G$. Then H is a subgroup of G if and only if $\forall a,b \in H, ab^{-1} \in H$

Definition: Let G be a group and let $a \in G$. Then the subgroup $\{a^n | n \in Z\}$ of G is called the **cyclic subgroup of G generated by *a***, and denoted by $\langle a \rangle$.

Definition: An element a of a group G **generates** G and is a **generator for** G if $\langle a \rangle = G$. A group G is **cyclic** if there is some element a in G that generates G.

Theorem: A subgroup of a cyclic group is cyclic.

Theorem: Let G be a cyclic group with generator a. If the order of G is infinite, then G is isomorphic to $(Z, +)$. If G has finite order n, then G is isomorphic to $(Z_n, +_n)$.

Theorem: Let G be a cyclic group with n elements and generated by a. Let $b \in G$ and let $b = a^s$. Then b generates a cyclic subgroup H of G containing n/d elements, where d is the greatest common divisor of n and s. Also, $\langle a^s \rangle = \langle a^t \rangle$ if and only if $gcd(s, n) = gcd(t, n)$.

Corollary: If a is a generator of a finite cyclic group G of order n, then the other generators of G are the elements of the form a^r, where r is relatively prime to n.

APPENDIX D

IRB APPROVAL AND CONSENT FORM

- IRB Approval
- Informed Consent Form

UNIVERSITY *of* NEW HAMPSHIRE

September 8, 2004

Titova, Anna S
Mathematics, Kingsbury Hall
33 College Road
Durham, NH 03824

IRB #: 3274
Study: Understanding Abstract Objects in the Context of Group Theory
Approval Date: 09/02/2004

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Expedited as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 110.

Approval is granted to conduct your study as described in your protocol for one year from the approval date above. At the end of the approval period, you will be asked to submit a report with regard to the involvement of human subjects in this study. If your study is still active, you may request an extension of IRB approval.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* (This document is also available at http://www.unh.edu/osr/compliance/IRB.html.) Please read this document carefully before commencing your work involving human subjects.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Manager

cc: File
Sonia Hristovitch

Research Conduct and Compliance Services, Office of Sponsored Research, Service Building, 51 College Road, Durham, NH 03824-3585 * Fax: 603-862-3564

UNIVERSITY *of* NEW HAMPSHIRE

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For the IRB,

Julie F. Simpson
Manager

cc: File
Sonia Hristovitch

Research Conduct and Compliance Services, Office of Sponsored Research, Service Building, 51 College Road, Durham, NH 03824-3585 * Fax: 603-862-3564

University *of* New Hampshire

Research Conduct and Compliance Services, Office of Sponsored Research
Service Building, 51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

8/21/2006

Titova, Anna S
Mathematics, Kingsbury Hall
33 College Road
Durham, NH 03824

IRB #: 3274
Study: Understanding Abstract Objects in the Context of Group Theory
Review Level: Expedited
Approval Expiration Date: 9/2/2007

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved your request for time extension for this study. Approval for this study expires on the date indicated above. At the end of the approval period you will be asked to submit a report with regard to the involvement of human subjects. If your study is still active, you may apply for extension of IRB approval through this office.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* This document is available at http://www.unh.edu/osr/compliance/irb.html or from me.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Manager

cc: File
Sonia Hristovitch

IFORMED CONSENT FORM

Dear Student:

"Understanding of Abstract Algebra concepts" is a dissertation research in the area of mathematics education. The aim of the dissertation is to explore undergraduate students' learning experiences and clarify certain cognitive processes during these experiences. I hope this dissertation will lead to better understanding of students' learning process and, consequently, to better teaching of Abstract Algebra and undergraduate mathematics in general.

I would like to ask you to participate in this study in any, all, or none of the following ways:

- by allowing copies of your written work (i.e. questioners, quizzes, homework, exams, lecture notes) to be included as data;
- by participating in 3-4 audiotaped interviews approximately one hour length with the researcher during the semester (there will be only 2 one hour interviews conducted with the pilot study participants);
- by allowing to be videotaped during the lectures.

Many students who participate in research of this type typically find the process to be helpful in their own learning. They benefit because in order to communicate with the
researcher and with other students, they reflect upon and deepen their understandings of
the mathematical concepts involved.

Audio and video data collected during this study will be transcribed and analyzed with coded names so that the identity of participants will be confidential. It will be stored in a locked cabinet for 2 years till researcher's dissertation work is complete. After the data is transcribed and analyzed audio and video tapes will be stored for one more year as evidence in support of researcher's dissertation findings after which audio/video data will be destroyed.

PLEASE READ THE FOLLOWING STATEMENTS AND RESPOND AS TO WHETHER OR NOT YOU ARE WILLING TO PARTICIPATE.

1. I understand that the use of human subjects in this project has been approved by the UNH Institutional Review Board (IRB) for the Protection of Human Subjects in Research.

2. I understand the scope, aims, and purposes of this research project and the procedures to be followed and the expected duration of my participation.

3. I have received a description of any potential benefits that may be accrued from this research and understand how they may affect me or others.

4. I understand that my consent to participate in this research is entirely voluntary, and that my refusal to participate will have no effect on my grade in Math 761.

5. I further understand that if I consent to participate, I may discontinue or modify my participation at any time with no effect on my grade in Math 761.

6. I understand that if I decline to participate in the research I will remain in the class for the lecture, but will be excluded (digitally hidden) from the videotape.

7. I confirm that no coercion of any kind was used in seeking my participation in this research project.

8. I understand that if I have any questions pertaining to the research or my rights as a research subject, I have the right to contact Anna S. Titova at titova@cisunix.unh.edu or Dr. Sonia P. Hristovitch at Sonia.Hristovitch@unh.edu (or 862-2027). I may also contact the UNH Office of Sponsored Research (862-2003) and be given the opportunity to discuss such questions.

9. I understand that I will not be paid for participation in interviews to be conducted outside of classtime. I further understand that there will be no financial compensation for other participation.

10. I understand that anonymity and confidentiality of all data records associated with my participation in this research, including my identity, will be fully maintained to the best of the researcher's ability. I understand, however, that the investigator is required by law to report certain information to government and/or law enforcement officials (e.g., child abuse, threatened violence against self or others, communicable diseases).

11. I understand that data from this study may be used in presentations for audiences of researchers and teachers.

12. I agree to respect the confidentiality and anonymity of the other participants to the best of my ability.

13. I certify that I have read and fully understand the purpose of this research project and its risks and benefits for me as stated above.

I, ________________________, CONSENT to participate in this research project in the following ways. (**Initial all that apply**.)

____ by allowing to videotape me during regular classes;

____ by allowing copies of my written work (i.e. questioners, quizzes, homework, exams, lecture notes) to be included as data;

____by participating in audiotaped interviews with the researcher periodically during the semester.

I, ______________________, DECLINE to participate in this research project.

_____________________________ ☺ ________________________

Signature of Student Date