

Fall 2006

An investigation of pre-service teachers' and professional mathematicians' perceptions of mathematical proof at the secondary school level

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**AN INVESTIGATION OF PRE-SERVICE TEACHERS' AND PROFESSIONAL
MATHEMATICIANS' PERCEPTIONS OF MATHEMATICAL PROOF AT THE
SECONDARY SCHOOL LEVEL**

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DISSERTATION

Submitted to the University of New Hampshire

in Partial Fulfillment of

the Requirements for the Degree of

Doctor of Philosophy

in

Mathematics Education

September 2006

UMI Number: 3231353

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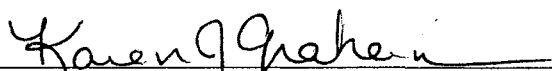
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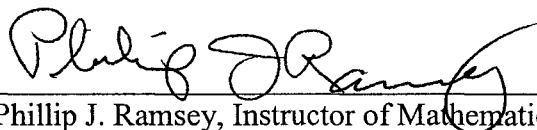
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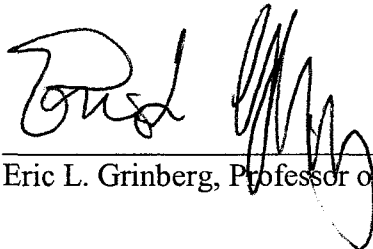
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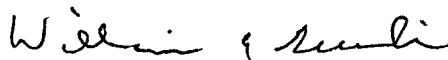
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DEDICATION

To Beth.

ACKNOWLEDGMENTS

Most of the people who contributed to this study were given the expressed assurance that their identities would remain confidential. However, this work could not have been accomplished without their participation. Therefore, I feel that I must acknowledge them at least as a group while honoring the promise of their anonymity.

First of all, I would like to thank the teachers of the three high schools who agreed to participate in the study by administering the materials to students in their classes. I am also grateful to the 58 students who completed the proof-problems. These artifacts provided me with a foundation from which to begin the research project. I wish to acknowledge the MST participants and UNH professors for their contributions to the pilot study. Your input was instrumental for the tweaking and fine-tuning of the questionnaire and for providing me with ideas that I hadn't thought of. Finally, I must acknowledge the participants of the main study. Although thirteen pre-service teachers were included in the research project, nineteen students actually completed the questionnaire and agreed to be interviewed. Additionally, I'd like to thank the mathematicians from UNH and Cal Poly for your contribution to the study. When there is no open acknowledgement and little reward for participation, I am grateful for the candor and thoroughness of your responses. Your work provided me with great insight into some of the perceptions of proof. It is trite but true that this work could not have been done without all of these participants.

I would like to thank Todd Grundmeier for helping me early in the process with comments about the proof-problems for the high school students and especially for his help in the long-distance recruiting of professional mathematicians to participate in the study. I am grateful to the members of my committee, Dr. Karen Graham, Dr. Phillip Ramsey, Dr. Sonia Hristovitch, Dr. William Geeslin, and Dr. Eric Grinberg for taking the time to read and critique this document. Your input provided me with valuable and constructive information. Thanks, too, go to Dr. Sonia Hristovich for your help during the initial phases of the study. Special thanks go to Dr. Karen Graham and to Beth Gray. As my advisor and committee chair, I am grateful to Karen for answering questions, reading innumerable drafts of my work, and in general, being a positive influence during the entire process. Lastly, many thanks to Beth, my wife, for the huge amount of support and patience you showed. Thanks for believing in me.

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ABSTRACT

AN INVESTIGATION OF PRE-SERVICE TEACHERS' AND PROFESSIONAL MATHEMATICIANS' PERCEPTIONS OF MATHEMATICAL PROOF AT THE SECONDARY SCHOOL LEVEL.

by

David M. Gray

University of New Hampshire, September, 2006

The National Council of Teachers of Mathematics [NCTM] states that by the time students graduate high school, they should learn to present “arguments consisting of logically rigorous deductions of conclusions from hypotheses” (NCTM, 2000, p. 56) in “written forms that would be acceptable to professional mathematicians” (NCTM, 2000, p. 58). Research studies indicate, however, that students and teachers have difficulty with many aspects of mathematical proof, including its nature and meaning. In addition, there appears to be a disconnect between school teachers’ and university mathematicians’ expectations for their respective students regarding mathematical proof. This study examined the perceptions of thirteen pre-service teachers and eight professional mathematicians with regard to mathematical proof in both the discipline of mathematics and proof in high school mathematics. Participants were asked to complete a questionnaire composed of open-ended questions related to their perceptions of mathematical proof on these two dimensions, for example, their perceptions about the purpose and importance of mathematical proof, their perceptions about what is acceptable

and valid as mathematical proof, and their expectations for students. The participants' responses to the questions on the questionnaire served as the primary data source.

Analysis of the data occurred in three phases: 1) a coarse reading through all the data to get a "feel" for the responses; 2) a line-by-line microanalysis and coding of the data; 3) a global analysis whereby responses were collected and "chunked" into episodes pertaining to a single concept or topic. As the data were being analyzed, hypotheses were formed. The hypotheses were then compared with the data to help bolster the plausibility of the hypotheses or to provide direction for modification of the hypotheses.

Results indicate that there are important differences in the perceptions between pre-service teachers and professional mathematicians regarding the nature and meaning of mathematical proof and its place in the high school curriculum. Some of the observed differences include: 1) professional mathematicians value the content of an argument over its form, while pre-service teachers place more importance on the details of the form of an argument, in some cases, to the exclusion of its content; 2) professional mathematicians' view of what constitutes proof is flexible and context dependent, while pre-service teachers' perceptions about what is acceptable is much less context dependent, and in some cases, rigid and unyielding; 3) pre-service teachers expect high school students to know specific derivations of formulas and particular formats for arguments, while professional mathematicians expressed the desire that high school students know about the nature of proof in mathematics.

CHAPTER 1

INTRODUCTION AND RATIONALE

Recent reform efforts have highlighted the importance of reasoning and proof in mathematics education. In particular, these reform efforts call for teachers to provide their students with mathematical experiences more closely aligned with mathematics as practiced by mathematicians (National Council of Teachers of Mathematics [NCTM], 2000; Conference Board of Mathematical Sciences [CBMS], 2001). One characteristic of the reform view of the mathematics classroom is that experimentation, analysis, conjecture, and proof should become an integral part of students' mathematical experiences. However, the way mathematics is typically presented to students is in conflict with the way in which mathematics is practiced by mathematicians, and therefore, in conflict with the intent of reform efforts in mathematics (e.g., Ernest, 1988; Alibert & Thomas, 1991; Edwards, 1997; Almeida, 2003; Weber, 2004; Weber & Alcock, 2004). To bring about reform in mathematics education requires changes in the way mathematics is presented by teachers.

Many mathematics educators have recognized that teachers' knowledge and beliefs about mathematics and mathematics teaching play a crucial role in how they present mathematics to students (e.g., Thompson, 1984; Thompson, 1992). As a result, teachers' knowledge and beliefs of mathematics become central to enacting the envisioned reforms in mathematics education. The knowledge and beliefs of teachers and

practitioners – i.e., their perceptions – are the focus of this study. More specifically, this research will examine pre-service secondary mathematics teachers and professional mathematicians’¹ perceptions of mathematical proof in the context of mathematics and in the context of high school mathematics. The purpose of this study is to identify and highlight inconsistencies between the perceptions about proof of pre-service teachers and professional mathematicians.

The objective of this chapter is to provide the rationale for and the significance of studying pre-service teachers and mathematicians’ perceptions of mathematical proof. The chapter continues below with a brief discussion of the literature related to student difficulties associated with mathematical proof, beginning with some of the difficulties connected with defining or describing mathematical proof, a discussion of expectations with respect to mathematical proof in classrooms, a brief review of the research literature about specific difficulties students have with mathematical proof, a brief review of the research literature with respect to theories on beliefs in general and student beliefs about mathematical proof, differences between proof as it is presented in the classroom and how proof is carried out by practicing mathematicians, and a section on the need for collaboration among teachers and mathematicians. This is followed by a statement of the research questions and an overview of the organization of the dissertation.

What Is Mathematical Proof?

One of the central and fundamental ideas in mathematics and mathematical thinking is mathematical proof (Ross, 1998; Schoenfeld, 1994). As Wu (1996) observes: “Proofs are the guts of mathematics. Producing a proof of a statement is the basic

¹ Hereafter, *pre-service secondary mathematics teachers* will be referred to simply as *pre-service teachers* when the context is clear.

methodology whereby we can ascertain that the statement is true” (p. 222). For many people, it is the deductive nature of mathematical proof that distinguishes mathematics from other scientific disciplines (Hoyles, 1997; Schoenfeld, 1994). But what does it mean to “prove” something? What is mathematical proof?

In *The Mathematical Experience*, Davis and Hersch (1981) construct a composite portrait of a person they refer to as the “Ideal Mathematician.” The Ideal Mathematician is meant to represent the most “mathematician-like mathematician” (Davis & Hersch, 1981, p. 34) in an effort to display some paradoxical and problematic aspects of the mathematician’s role. With regard to the Ideal Mathematician’s conception of proof, Davis and Hersch (1981) give the following description:

He rests his faith on rigorous proof; he believes that the difference between a correct proof and an incorrect one is an unmistakable and decisive difference. ... Yet he is able to give no coherent explanation of what is meant by rigor, or what is required to make a proof rigorous. In his own work, the line between complete and incomplete proof is always somewhat fuzzy, and often controversial. (p. 34)

There are many different interpretations of the word “proof.” For example, to a scientist, proof is something established by experimentation; to a jury, proof is something established by evidence “beyond a reasonable doubt”; to a statistician, proof is something calculated within a certain level of probability (Tall, 1989). What separates mathematical proof from other forms of proof is the deductive method. To a mathematician, proof consists of a finite sequence of statements each derived from previous statements according to logical rules of inference (Hanna, 1990). However, students often have different meanings associated with proof and these different meanings may hinder

students' development and understanding of the nature and meaning of mathematical proof (e.g., Recio & Godino, 2001).

What Are The Expectations Regarding Mathematical Proof In Classrooms?

The National Council of Teachers of Mathematics states that “by the end of secondary school, students should be able to understand and produce mathematical proofs – arguments consisting of logically rigorous deductions of conclusions from hypotheses – and should appreciate the value of such arguments” (NCTM, 2000, p. 56). The form of argumentation is not as important as clear communication of the mathematical ideas, however, “high school students should be able to present mathematical arguments in written forms that would be acceptable to professional mathematicians” (NCTM, 2000, p. 58).

Several questions arise from the NCTM suggestions above: first, what is meant by logically rigorous deductions of conclusions from hypotheses? And second, what written forms are acceptable to professional mathematicians?

To illustrate some of the difficulties associated with the first question, and by extension, the potential difficulties associated with teaching and learning mathematical proof, consider two proofs of the Pythagorean theorem: the first is a proof from the 12th

Figure 1. Bhaskara's Proof



century Hindu mathematician Bhaskara. Bhaskara provides the following pictorial proof (see figure 1.1) of the Pythagorean theorem with no explanation but the one-word “Behold” (Burton, 1991). It could be argued that this does not constitute proof of the theorem since there is no explanation – that is, a considerable amount of additional effort on the part of the reader is required to fill in the blanks. By way of comparison, Renz (1981) recounts that a complete proof of the Pythagorean theorem using only axioms and the rules of inference allowed in Euclid’s *Elements* comprises almost 80 pages. While the proof of Bhaskara could be considered simple and elegant, there is no explanation given as to why the theorem is true. As for the proof cited by Renz (1981), although it is rigorous and complete, it may be too long and complicated to be of much practical use. In the context of mathematics, “rigor” generally translates to mean “thorough, precise, accurate, and careful.” From these two examples alone it is evident that there is a continuum of “thoroughness, precision, accuracy, and care” that exists in mathematical proof.

The second question, “What written forms of proof are acceptable to professional mathematicians?” is closely related to the first in that the degree of rigor plays a key role as to what someone considers acceptable as mathematical proof. It differs, however, in that what one mathematician deems acceptable as a mathematical proof, another might consider unacceptable. Hoyles (1997) points out that in some situations the characteristics of mathematical proof are explicitly discussed in terms of form and presentation, while in other cases, the definitions and criteria for proof are at best implicit. In informal discussions with teachers in the United Kingdom, Hoyles (1997) reports that there are a

“multitude of opinions about how proof should and would be introduced and judged” (p. 8).

Dreyfus (1999) points out that students get mixed and confusing messages about what are acceptable forms of argumentation. Textbooks often use a combination of formal and informal or intuitive justification, generic examples, and inductive arguments. For example, calculus textbooks often introduce concepts such as limit and the slope of the tangent line with appeals to visualization and/or experimentation, after which students are asked to use ϵ - δ definitions to produce more or less formal proofs. Dreyfus (1999) notes that “students are rarely if ever given any indication whether mathematics distinguishes between these forms of argumentation or whether they are equally acceptable” (p. 97).

Dreyfus (1999) points out that the ability to express mathematical relationships or the use of language in general may also play a significant role in argumentation. Dreyfus (1999) gives the following example as an illustration of what appears to be a limitation in students’ linguistic ability to provide adequate explanations:

Determine whether the following statement is true or false, and explain:
If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then $\{v_1, v_2, v_3\}$ is also linearly independent.
Student response: True because taking down a vector does not help linear dependence. (p. 88)

The student’s use of the words “taking down” could be interpreted to mean “omitting” or “removing” and the student’s use of the word “help” might mean “doesn’t result in” or “won’t cause,” but as given, it is not clear whether the vagueness of the response is due to a lack of conceptual clarity or linguistic ability on the student’s part (Dreyfus, 1999).

In addition, the phrasing of the questions in textbooks, on exams, or questions posed by the professor can themselves be ambiguous to students. For example, what are students expected to do when asked to “explain,” “justify,” or “prove?” Does “show” mean that a formal proof is expected or that a demonstrative example is called for? Does “prove” mean “prove from first principles” or “provide a convincing argument?” For students, what mathematicians might consider acceptable as proof depends partly on how the initial questions are posed, how the questions are perceived and interpreted, what standards of argumentation are established, and when and where those standards are valid (Dreyfus, 1999; Hoyles, 1997). As Dreyfus (1999) states: “for mathematics educators, there appears to be a continuum reaching from explanation via argument and justification to proof, and the distinctions between the categories are not sharp” (p. 102).

Student Difficulties Associated With Mathematical Proof

The research literature is replete with empirical evidence of some of the difficulties students encounter with mathematical proof. The subjects involved in these studies were composed of groups of elementary school students, high school students, undergraduate students, and graduate level university students. Additional categorizations for these studies have included pre-service and in-service elementary school teachers and pre-service and in-service secondary mathematics teachers. Some of the difficulties that various groups encountered with mathematical proof will be described below.

Fischbein and Kedem (1982), for example, found that among high school students who understood, accepted, and believed a theorem and its proof, the majority of students needed additional empirical evidence to be fully convinced of the theorem’s truth.

Schoenfeld (1989) found that even when high school geometry students succeeded in

producing correct proofs of geometric propositions, the same students suggested solutions to constructions that flatly contradicted the theorems they had just proved.

Researchers found that students have difficulty with the logical aspects of proof. For example, Williams (1980) found no evidence that any of the high school students in his study understood that a statement and its contrapositive are logically equivalent or that a statement and its converse are not logically equivalent. In addition, Küchemann & Hoyles (2002) reported that 71% of the students under investigation thought that a statement and its converse were logically equivalent.

From the standpoint of strategic knowledge, Thompson (1991) found that many students exhibited an inability to recognize what needs to be shown. For example, given the problem of proving that the difference between an even and odd number is odd, students proceeded as follows:

Let $m = 2r$, $n = 2s+1$ be even and odd numbers.

Then $m - n = 2r - (2s + 1) = 2(r - s) - 1$.

Students were unable to recognize that since $r - s$ is an integer, $2(r - s) - 1$ is odd. Senk (1985) found that many high school students were unable to even begin a chain of reasoning in geometry.

Dubinsky and Yiparaki (2000) studied college students' understanding of AE (“for all, there exists”) and EA (“there exists, for all”) statements in natural language and in mathematics. Most of the students could not distinguish between AE and EA statements in mathematics. For example, two mathematical statements were shown to students: (1) For every positive number a there exists a positive number b such that $b < a$;

(2) There exists a positive number b such that for every positive number a , $b < a$. Most students viewed these as equivalent statements.

Beliefs About Mathematical Proof

The mathematics that students learn, their dispositions toward mathematics, and how they come to know mathematics is in large part a result of the mathematical experiences that teachers provide for the students (NCTM, 2000). In addition, teachers' prior experiences with, and their knowledge and beliefs about, mathematics content and mathematics pedagogy, are powerful factors in determining how teachers present mathematics to students (Thompson, 1984; Raymond, 1997). Enabling systemic reform in the way mathematics is presented and carried out in school will require fundamental changes in teachers' beliefs about mathematics and mathematics pedagogy (Ernest, 1988).

Many researchers found that their subjects believed that an inductive argument is sufficient to prove a claim (e.g., Martin & Harel, 1989a; Chazan, 1993). In many cases, the subjects believe that once a pattern is observed, the pattern will not change (Coe & Ruthven, 1994).

Other researchers found that many high school students and in-service teachers believe that a deductive proof provides evidence for the truth of a claim but it does not provide certainty. They believe that there may be counterexamples that would refute the claim (Coe & Ruthven, 1994; Chazan, 1993; Knuth, 2002b). Gholamazad, Liljedahl, & Zazkis (2004) found that the majority of participants were not satisfied with the use of a single counterexample to disprove a claim.

Mathematical Proof In School

There appears to be a disconnection between school teachers' and university mathematicians' expectations for their respective students regarding mathematical proof. Edwards (1997) notes, for example, that "the teaching of proof that takes place in many secondary level mathematics classrooms has often been inconsistent with both the purpose and practice of proving as carried out by established mathematicians" (p. 187). True understanding of mathematics, MacLane (1994) contends, is more likely to follow the sequence: intuition → trial → error → speculation → conjecture → proof. In this sequence, there is initial intuition about certain mathematical objects after which some type of experimentation may take place. This leads to speculation and conjecture about the objects under consideration which finally leads to a proof about the conjecture. "The mixture and sequence of these events differ widely in different domains, but there is general agreement that the end product is rigorous proof" (MacLane, 1994, p. 191). However, mathematics is typically taught to undergraduate students in the form: theorem → proof → examples. In this method of presentation, the theorem is stated first (preceded perhaps by pertinent lemmas), after which the proof of the theorem is given, and finally, specific examples of objects which meet the hypotheses of the theorem are shown, demonstrating the veracity of the theorem (Almeida, 1995). In this teaching approach, undergraduate students are typically presented with subject matter as a finished and polished theory where proofs are developed in a linear, deductive manner (Alibert & Thomas, 1991; Weber, 2004). At the high school level, students in traditional geometry courses typically "focus on the formal aspects of proof, and on established results, rather than on the construction and negotiation of meaning" (Edwards, 1997, p. 87).

Schoenfeld (1994) states that proof does not need to be seen as an outdated formal ritual, but can be viewed as the codification of clear thinking and a way to communicate ideas with others. Students can see and appreciate the fact that for mathematicians, proof is a way of thinking, exploring, and of coming to understand. One of the difficulties students have is that to them proof tends to have no personal meaning or explanatory power. In many cases, students are being asked to prove what is intuitively obvious to them while attending to the form of the argument rather than to its substance. Proof is not separable from mathematics as it appears to be in our curricula; it is an important part of doing mathematics. Schoenfeld (1994) states that “if students grew up in a mathematical culture where discourse, thinking things through, and *convincing* were important parts of their engagement with mathematics, then proofs would be seen as a natural part of their mathematics ... rather than as an artificial imposition” (p. 76).

Need For Collaboration

Much of the discussion in previous sections underscores the need to address pre-service teachers’ misconceptions about the nature of mathematical proof in the discipline. However, it is also clear that university mathematicians may be contributing to these misconceptions with the method by which they present mathematics to their students. Together, these observations point to the Conference Board of Mathematical Sciences (CBMS, 2001) recommendation that, “there needs to be more collaboration between [university] mathematics faculty and school mathematics teachers” (p. 10). The intent of this collaboration is to provide university mathematicians and high school teachers with opportunities to learn what each does in actual practice with the ultimate goal of improving pre-service teacher education (CBMS, 2001).

To be effective representatives of the mathematical community, prospective teachers need to develop a deep understanding about the nature and purposes of mathematical proof. They need to understand and appreciate the difficulties associated with learning about proof, how to construct a proof, when proof is needed or desired, and the purposes of proof in mathematics and in the secondary mathematics curriculum. As custodians of the discipline, professional mathematicians are in the best position to provide prospective teachers with the most realistic and genuine understanding of mathematical proof. Therefore, it is important to try to ascertain what prospective teachers believe about mathematical proof and how these beliefs relate to the beliefs of practicing mathematicians.

Research Questions

This study will investigate pre-service secondary mathematics teachers' and professional mathematicians' perceptions about mathematical proof in the context of mathematics as well as the context of secondary school mathematics. More specifically, the study will address the following questions:

- 1) How do pre-service teachers and practicing mathematicians view mathematical proof?
- 2) What do pre-service teachers and practicing mathematicians find acceptable as mathematical proof in general? Does this differ from what these two groups find acceptable as mathematical proof from high school students? If so, why?
- 3) What are the similarities and differences between what pre-service teachers consider acceptable and what mathematicians consider acceptable as mathematical proof? For

example, will the pre-service teachers and/or mathematicians prefer visual arguments over algebraic arguments?

- 4) What differences and similarities exist between the expectations pre-service teachers and mathematicians have for high school students regarding mathematical proof?
- 5) How can the observed similarities and/or differences be explained?

An initial assumption that guided this study is that there is not necessarily a “right” or “correct” set of beliefs one can hold regarding mathematical proof, but that there are different levels of expertise and experiences with mathematical proof, that for pre-service teachers and professional mathematicians become factors both in the practice and teaching of mathematics. The goal of the study is to begin to build a theory about how, when, and why pre-service teachers and professional mathematicians believe that mathematical proof fits into the secondary school mathematics curriculum.

Organization Of The Dissertation

Chapter 2 is an in-depth literature review related to difficulties associated with mathematical proof and beliefs about mathematical proof. Chapter 3 focuses on the theoretical perspective from which this study is approached. Chapter 4 provides a detailed description of the research methodology employed for this study, including the research design, methodology, and data coding and data analysis. Chapters 5 and 6 focus on a presentation of the results of the data analysis. Chapter 7 contains a discussion and interpretation of the results and the limitations of the study, and provides suggestions for pre-service teacher education and possible directions for future research.

CHAPTER 2

REVIEW OF LITERATURE

The review of literature related to mathematical proof is broken down into two categories: difficulties with mathematical proof and beliefs about mathematical proof.

Student Difficulties With Mathematical Proof

The most predominant misconception of mathematical proof held by high school and undergraduate students, as well as pre-service and in-service teachers, is that a mathematical claim can be justified by citing examples (e.g., Martin & Harel, 1989a). Citing examples as a means for justification should be expected in the primary grades (NCTM, 2000), however, deductive reasoning should begin to replace inductive reasoning and become the standard in mathematical argumentation as early as the eighth grade (Ross, 1998). In addition to the misconception that a few examples constitutes a mathematical proof, however, researchers have identified many other difficulties or problems that high school students, undergraduate students, pre-service teachers, and in-service teachers exhibit regarding mathematical proof. These difficulties and problems include affective and motivational factors related to a student's *need* to prove a conjecture (i.e., from the student's perspective the claim may be obvious, so there seems to be no need to provide proof), a lack of sufficient understanding of the concepts in the domain under consideration (e.g., knowing what a *subgroup* is in algebra), a lack of

understanding of what constitutes proof and what the proof of a statement signifies in terms of universality, deficiency in the area of notation, language, or syntactic ability or understanding, insufficient strategic knowledge for constructing proofs, and cognitive factors. Each of these will be described in more detail below, organized around the following categorization: difficulties with mathematical proof associated with content knowledge, difficulties with the nature, meaning, and conceptual understanding of mathematical proof, difficulties associated with strategic knowledge, difficulties with notation and language, affective and motivational problems associated with mathematical proof, difficulties associated with lack of exposure and practical experience with mathematical proof, and cognitive difficulties with mathematical proof.

Difficulties With Mathematical Proof Associated With Content Knowledge

Hazzan and Leron (1996) found that many undergraduate students who failed to correctly prove a claim failed because they tried to use a theorem that did not apply for that particular problem. For example, to justify the statement “ Z_3 is a subgroup of Z_6 ,” almost 30% of the wrong answers were basically of the form: “ Z_3 is a subgroup of Z_6 because 3 divides 6.”

Moore (1994) found that while undergraduate students had an intuitive understanding of set-theoretic definitions and the claim to be proved, they could not correctly prove the claims because they did not have a precise mathematical definition. For example, one student was asked to show that one set was a subset of the other. She had an intuitive understanding of “subset” but did not possess the proper formal definition: $A \subseteq B$ means $x \in A \Rightarrow x \in B \forall x \in A$.

Difficulties with the Nature, Meaning, and Conceptual Understanding of Mathematical Proof

Fischbein and Kedem (1982) provided high school students with a complete proof of the number theoretic claim: “ $n^3 - n$ is divisible by 6 for any natural number n .” Students were asked whether they understood the theorem and its proof, and whether they accepted, and believed the theorem and its proof. Of those students who understood, accepted, and believed the theorem and its proof, 75% needed additional empirical evidence to be fully convinced of the theorem’s truth.

Other researchers found that students did not understand the universality of a theorem: that once a claim is proved, it is true for all cases that meet the conditions of the theorem (Hoyles & Healy, 1999; Schoenfeld, 1989). For example, in a study by Hoyles and Healy (1999), high school students were shown a proof of the proposition that the sum of four consecutive integers is not divisible by four. The same students were then asked to find four consecutive integers whose sum was 44. Rather than referring to the proposition just proved, many of the students resorted to calculations to refute the statement.

Several researchers have reported that students have difficulty with specific methods of proof. Thompson (1991) found that most students had difficulty with the method of proof by contradiction. In some cases, she found that the students confused contradiction with counterexample. Williams (1980) found that fewer than 20% of the high school students under investigation understood the method of indirect proof and that almost 80% of the students did not understand the concept of counterexample.

Several researchers have investigated students' understanding of the method of mathematical induction. For example, among high school and college students and pre-service teachers, Fischbein and Kedem (1982), Movshovitz-Hadar (1993), and Baker (1996) found that students did not understand the concepts underlying the method of mathematical induction. In some cases the students were able to produce a correct induction proof procedurally but they didn't have the conceptual understanding of why the method works (Movshovitz-Hadar, 1993). Baker (1996) reported that high school and college students didn't have confidence in the induction method of proof. Students in this category expressed their lack of confidence in the method with comments such as: "when I do induction, I don't believe it's true" (Baker, 1996, p. 10).

Students' difficulties with conceptual understanding of mathematical proof have also been linked to the logical underpinnings of proof. Küchemann & Hoyles (2002) reported that 71% of the students under investigation thought that a statement and its converse were logically equivalent. Williams (1980) found no evidence that any of the high school students in his study understood that a statement and its contrapositive are logically equivalent or that a statement and its converse are not logically equivalent.

In an exploratory study of the way that eight mathematics and secondary education mathematics majors evaluated the validity of proofs, Selden and Selden (2003) found that the students in their study tended to focus on surface features such as algebraic representation and symbol manipulation, rather than on more global aspects such as whether the purported proof proves the converse of the statement, or whether there are fundamental and substantial gaps in the purported proof, which, if included, would prove the statement.

Riley (2003) investigated prospective secondary mathematics teachers' conceptions of proof and refutations as they were near completion of their preparation program. Results showed that only 30% of the prospective teachers correctly answered 9 or more of 12 items, for the logical underpinnings of proof. The results showed that participants have a weak understanding of the truth of a conditional statement and its related statements (e.g., the converse or negation of conditional statement). Riley (2003) concluded that prospective teachers need more experiences in determining the true values of conjectures and that there is a correlation between an individual's understanding of the logical underpinnings of proof and his or her ability to complete proofs.

Difficulties Associated with Strategic Knowledge

Senk (1985) found that only 30% of students in full-year geometry courses that teach proof reached a 75% mastery level in proof writing. More significantly, Senk (1985) found that many high school students were unable to even begin a chain of reasoning in geometry.

Selden and Selden (1995) investigated the ability of undergraduate students to translate (“unpack”) informally written mathematical statements into the language of predicate calculus. In the context of this study, a mathematical statement such as “differentiable functions are continuous” would be considered to be an informal statement, since a universal quantifier is implied but not explicitly stated. Selden and Selden (1995) hypothesized that students who cannot reliably unpack the logical structure of informally stated theorems cannot reliably validate their proofs and that being able to link a theorem with an appropriate proof framework is part of the validation process, so a student who can reliably validate proofs of theorems must also be able to recognize

which proof frameworks are appropriate for which theorems. The results indicated that for informal simplified calculus statements, only 8.5% of the students were able to successfully unpack the statements.

Weber (2001) compared doctoral students and undergraduate students constructing proofs in abstract algebra. The graduate students collectively proved 95% of the propositions while the undergraduate students proved only 30% of the same propositions. Weber (2001) hypothesized that the doctoral students appeared to know the powerful proof techniques in abstract algebra, which theorems are most important, when particular facts and theorems are likely to be useful, and when one should or should not try to prove theorems using symbol manipulation. Weber (2001) concluded that a primary cause of the undergraduates' inability to produce correct proofs may be due to a lack of strategic knowledge.

In a study of high school students in a pre-calculus and discrete mathematics class, Thompson (1991) found that many students exhibited an inability to recognize what needs to be shown. For example, given the problem of proving that the difference between an even and odd number is odd, students proceeded as follows:

let $m = 2r$, $n = 2s + 1$ be even and odd numbers.

Then $m - n = 2r - (2s + 1) = 2(r - s) - 1$.

Correct reasoning at this point would be to indicate that since $r - s$ is an integer, $2(r - s) - 1$ is an odd integer. Half of the students incorrectly asserted instead that since $2(r - s) - 1$ is an integer, $m - n$ is odd, or that since $(r - s) - 1$ is an integer, $m - n$ is odd.

Difficulties with Notation and Language

In a study of high school students in a pre-calculus and discrete mathematics class, Thompson (1991) observed that one of the common errors that occurred frequently was incorrect use of notation, such as assigning the same dummy variable to two (possibly) different integers. For example, for the problem of proving that the difference of an even and an odd integer is an odd integer, 35% of the students began with: “let $m = 2k + 1$ and let $n = 2k$.”

Dubinsky and Yiparaki (2000) studied college students' understanding of AE (“for all, there exists”) and EA (“there exists, for all”) statements in natural language and in mathematics. Most of the students could not distinguish between AE and EA statements in mathematics. For example, two mathematical statements were shown to students: (1) For every positive number a , there exists a positive number b such that $b < a$; (2) There exists a positive number b , such that for every positive number a , $b < a$. Most students viewed these as equivalent statements.

Affective and Motivational Problems Associated with Mathematical Proof

Problems with mathematical proof have also been identified as being related to affective and motivational factors. For example, Harel (1998) provided a description of intellectual need as follows:

When we encounter a situation that is incompatible with, or presents a problem that is unsolvable by our existing knowledge, we are likely to search for a resolution or a solution and construct, as a result, new knowledge. Such knowledge is meaningful to the person who constructs it, because it is a product of personal need and connects to prior experience. This...is what I call the Necessity Principle. (p. 501)

Using the theory of Harel (1998), Brown (2004) suggested that “students at the post-secondary level lack experiences that facilitate the development of an intellectual need,

on the students' behalf, for proof. In other words, it is not that students should simply write more proofs but rather that students should encounter more situations that foster, in the eyes of the student, the need for proof" (p. 629). Brown (2004) posits that developing students' understanding with respect to the Reasoning and Proof Standard (NCTM, 2000) may necessitate the development of an intellectual need for proof on the part of the students.

Reid and Dobbin (1998) found that students are unlikely to feel a need to verify that $\sqrt{2}$ is irrational since their concept image of irrationality includes $\sqrt{2}$ as an archetypal representation of the meaning of irrational. Others have found that students feel no need to prove something that is obvious to them (Dreyfus & Hadas, 1996). In addition, students believe that proof is a procedure to confirm something they already know to be true (Schoenfeld, 1994). Mathematical proof for these students is simply a mechanical exercise with no personal meaning (Schoenfeld, 1994).

By incorporating surprises in the presentation of theorems, Movshovitz-Hadar (1988) suggests that the intellectual curiosity of students is triggered, which makes them want to know more about the topic "which they could not have cared less about a moment earlier" (pp. 12-13). Presenting theorems to students in such stimulating manner, the author claims, triggers a need to prove, opening the door for presenting a proof of the theorem.

In the context of geometry, this need to prove stems from students' need for an explanation or deeper understanding of why a particular statement is true rather than or independent of their need for conviction (i.e., verification that a statement is true) (deVilliers, 1991; Mudaly & deVilliers, 2000).

Fischbein (1982) held a similar view regarding motivation to prove and affective factors in argumentation. He noted, for example, that the usual proof that the positive integers and the positive even integers are equipotent is convincing in a mathematical sense, but may not be satisfying or convincing personally. Several authors have suggested that proofs by mathematical induction can easily be placed in this category (e.g., Hanna, 1989b; Baker, 1996; Movshovitz-Hadar, 1993). Conversely, there may be situations in which the statement appears obvious even though there is no proof of it. In this case, there doesn't seem to be a need for proof since the statement is apparent. This may in part explain the apparent need of some students for additional confirmation of a proof's validity.

Difficulties Associated with Lack of Exposure and Practical Experience with Mathematical Proof

Moore (1994) noted that for undergraduate students, the transition to formal proof is abrupt. "Many students begin their upper-level mathematics courses having written proofs only in high school geometry and having seen no general perspective of proof or methods of proof" (Moore, 1994, p. 249). Students must learn how to produce a proper proof while at the same time they are learning increasingly difficult and abstract subject matter.

Hart (1994) conducted a study of students in group theory at three levels (introductory undergraduate, advanced undergraduate, and graduate level algebra) to analyze the processes, errors, and self-assessment of students at varying levels of conceptual understanding. The results of the study showed that there are indeed observable differences between experts and novices, and they also indicated that the

transition from novice to expert is developmental and unstable. Thus, it is not advisable to try to train novices abruptly in the proof-writing behavior of experts.

Weber (2001) suggested that greater success in proving comes from more extensive exposure and experience with proving theorems, including the wrong attempts made with proofs, discussions with others, and learning about other mathematical domains.

Dreyfus (1999) declared that “giving an argument or explanation is a very difficult undertaking for beginning undergraduates from at least two points of view: in most cases, they still lack the conceptual clarity to actively use relevant concepts in a mathematical argument; and more generally, they have had little opportunity to learn what are the characteristics of a mathematical explanation” (p. 91).

Weber (2003) provided an illustration of some inconsistencies that may exist among different communities with the following scenario: given the statement $f(x) = x^2 + x$, show that $f'(x) = 2x + 1$. A student in a traditional elementary calculus course would be expected to do some sort of calculation; a student in a real analysis course would be expected to use the definition of derivative to show this. In an advanced course, this fact would be considered to be so obvious that no explanation would be necessary.

Students may not be getting clear messages as to what is expected of them in terms of explanation and justification. For example, when asked to “explain,” are students expected to provide a rigorous proof? As a result of differing expectations from different mathematical situations, students have difficulty knowing what they are being asked to do. Without explicit “rules” for argumentation, students must effectively guess at what is expected of them (Dreyfus, 1999; Weber, 2003).

Herbst (2003) recounts an episode in which students were working on the task of ranking various triangles. One student noticed that if the base of a triangle is twice the base of another, while the altitude of the first is half the other, then the areas are equal. The 14-year old's reasoning was that somehow the differences between the triangles cancel each other out (in terms of the formula for area) even though the triangles didn't seem to have equal areas. In this episode, however, the teacher did not recognize the opportunity to provide a generalized argument, insisting instead that the students stick to the task of ranking by measuring the triangles. When the students measured the two triangles and calculated their areas, there was a difference due to measurement error. This provided a false counterexample to the original claim. In this case, the teacher was not prepared to deal with the novel task suggested by the student's conjecture.

Cognitive Difficulties with Mathematical Proof

Some researchers suggest that student difficulties with mathematical proof may be linked to cognitive factors. Van Dormolen (1977), Balacheff (1988), and the van Hiele (Fuys, Geddes, & Tischler, 1988) separately proposed that there are cognitive levels associated with mathematical proof, not unlike the hierarchical cognitive levels of Piaget. They separately found that students are not able to operate at a higher level unless the lower levels have first been reached. Demanding that students operate at a higher cognitive level than they have actually attained, results, at best, in students memorizing the proof procedures without understanding.

Tall (1979) notes that some students encounter cognitive difficulty due to the fact that proofs that are more general require a higher level of abstraction. For example, a

general proof that \sqrt{n} is irrational for any non-square n requires a higher level of abstraction than a proof that $\sqrt{2}$ is irrational (Tall, 1979).

Beliefs About Mathematical Proof

Some of the research results described in the previous section point out that certain difficulties with mathematical proof appear to be closely linked to one's beliefs about mathematical proof. For example, the study of Hoyles and Healy (1999) cited above suggests that because the students under investigation didn't believe that the theorem was universally proved, they needed to re-prove it when given a new example. Various aspects of high school students' and pre- and in-service teachers' beliefs about mathematical proof have been studied by researchers. Some of their findings are described below.

Martin and Harel (1989a) found that more than half of the pre-service elementary teachers in their study accepted an inductive argument as a valid mathematical proof for the statements under consideration. Chazan (1993) observed a similar result among high school geometry students. Coe and Ruthven (1994) concluded that students believe that when a pattern is observed, the pattern will not change at any unforeseen point. To these students, a few examples that verify a claim are sufficient proof of the truth of the claim (Coe & Ruthven, 1994).

In a study of preservice elementary teachers, Gholamazad, Liljedahl, and Zazkis (2004) also found that the majority of participants accepted confirming examples as a valid method of proof. In addition to accepting inductive arguments as valid proof, the number of examples required for a valid argument was observed: 27% of students crossed out some of the examples in a "proof by examples" and made statements such as

“two or three examples are enough to make sure the given statement is correct.”

Gholamazad, Liljedahl, and Zazkis (2004) found that the majority of participants accepted, as valid, proofs that consider all possible cases in a finite set and that the majority of participants were not satisfied with the use of a single counterexample to disprove a claim.

Several researchers found varying levels of confidence among students and teachers regarding their beliefs about the universal validity of mathematical proof. Fischbein (1982) found, for example, that beyond a deductive proof of a theorem, high school students needed additional examples to convince themselves of the truth of a statement. Others, however, reported just the opposite. For example, Healy and Hoyles (2000) found that students believe that a valid proof must be general, and that once a proof is given, it is not necessary to further check the validity of specific cases in the proposition's domain of validity. Other researchers found that many high school students and in-service teachers believe that a deductive proof provides evidence for the truth of a claim but it does not provide certainty. They believe that there may be counterexamples that would refute the claim (Coe & Ruthven, 1994; Chazan, 1993; Knuth, 2002b). Martin and Harel (1989b) found that for many of the undergraduate students under consideration, the given proof was particular to the given geometric figure, that is, the students believed that a new proof would be needed if a different figure were presented.

Coe and Ruthven (1994) found that few of the high school students in their study believed that proof helps one to understand why propositions are true. By contrast, Mingus and Grassl (1999) found that among pre-service secondary teachers, most of the respondents believed the role of proof was to explain why concepts work the way they do

in mathematics and that constructing proofs helps students understand the mathematics they are doing.

Healy & Hoyles (2000) found that over 60% of the high school students in their study believed that algebraic arguments (even when incorrect) would receive higher marks from teachers than narrative or non-algebraic arguments, even though they believed that non-algebraic arguments are easier to understand. The finding that in some instances form is more important than substance was also observed among in-service teachers, several of whom believed that the two-column form of proof in geometry was the epitome of proof (Knuth, 2002a).

Regarding formal proof in a dynamic geometry environment, Pandiscio (2002) found that the pre-service secondary teachers believed that high school students using dynamic geometry software would believe that proofs are not necessary. And, even though the pre-service teachers believed that a formal proof is different from “proof by examples,” they were not sure of the value of formal proof for high school students. The ability in the dynamic geometry environment to quickly and accurately examine many cases led the pre-service teachers to believe that proof is less important than it used to be (Pandiscio, 2002).

Knuth (2002a) and Almeida (1995) found that in-service teachers and undergraduate students believed that proof was appropriate only for students in advanced classes and those likely to go on to mathematics related majors in college. By contrast, all of the teachers in Knuth’s (2002a) study believed that informal proof was central to all of secondary mathematics and should be integrated into every class. Some teachers thought

more highly of arguments that provided all of the steps in an algebraic proof or more explanation of why certain steps followed from others (Knuth, 2002b).

Using a methodology similar to the work of Almeida (1995), Cyr (2004) investigated pre-service high school mathematics teachers' beliefs about the nature and role of proof and what differences there might be among the students over their four years of university training. She found that the majority of pre-service teachers held a moderate formalist view of proof, that formalism and rigor are strongly linked to writing proofs. Cyr (2004) also found that pre-service teachers thought that the form of the proof (specifically, two-column proof) was very important in high school. Cyr (2004) found very little difference between the conceptions of the participants in different years of their training. Cyr (2004) concluded that an implicit teaching of proof, in combination with the lack of the teaching of certain elements, is the cause of certain weaknesses in the conceptions of proof observed among the participants.

Proof Preference and Validation

In a recent study, Raman (2002) gave undergraduate students, graduate students in mathematics, and professors of mathematics a single mathematical statement. The participants were asked to assess each of the five given arguments of the statement for validity and to indicate for each argument a measure of the credit they would assign (an indication of what criteria the participants deemed important from a pedagogical point of view). Undergraduate students tended to favor more algebraic arguments since they “look more proof-like.” They also tended to dismiss the pictorial argument because it wasn't mathematical, and flatly rejected the empirical argument since they were taught that examples do not constitute proof. The graduate students and faculty thought that the

informal arguments (both empirical and pictorial) were good because they can lead to a formal proof. The graduate students and faculty tended not to favor the definition-based argument, even though it was correct, primarily because it did not convey the reasons why the result is true. On the other hand, the professors and graduate students both thought the pictorial argument did convey the key idea of symmetry about the y -axis. The graduate students would have preferred to have seen a combination of the pictorial and definition-based arguments. The professors and graduate students also considered the short, “clever” response as a proof, but the graduate students in particular thought that the proof did not provide clear insight into why the claim was true and therefore would be less desirable from a pedagogical point of view. Raman (2002) concluded that this group favored the arguments that conveyed the key idea of the proof, even if the purported proof was not strictly formal or “rigorous” and would in general give those types of arguments higher marks than clever proofs which do not convey understanding of the underlying concepts. In fact, many of the professors favored a pictorial argument with little explanation over an algebraic argument based on the definitions (Raman, 2002).

Weber and Alcock (2004) reported on an exploratory study that investigated how mathematicians determine whether an argument is a valid proof. Eight mathematicians were presented with six to eight arguments from number theory and were asked to think aloud while determining whether each argument constituted a valid proof. Weber and Alcock (2004) found that: a) when mathematicians questioned an assertion in an argument, they generally did not construct a full sub-proof to establish the assertion; b) the mathematicians sometimes used inductive (i.e., example-based) reasoning to validate

an assertion; and c) the mathematicians relied on their understanding of number theory to guide their reasoning. Weber and Alcock (2004) concluded that:

Many of the less formal reasoning processes described above currently receive little or no attention in proof-oriented university mathematics courses. In fact, we argue that such strategies may be implicitly discouraged. For example, in such courses, students are often told that one can never determine that a general assertion is true just by looking at examples. This may lead students to believe that such reasoning is not only inappropriate for proof presentation, but also is not applicable for proof validation (and proof construction). As students have difficulty validating proofs and do not appear to invoke these processes, giving explicit attention to these strategies in a thoughtful manner may help improve students' ability to validate proofs. (p. 626)

Summary

The NCTM (2000) and the CBMS (2001) recommend that teachers have a deep understanding of the mathematics they teach, and in particular, future secondary school teachers must have a good understanding of what it means to produce a formal proof (CBMS, 2001). Research indicates that this is not necessarily the case. The results from research point out that the problems and difficulties students and teachers experience with mathematical proof are manifold. There are at least two vantage points from which the research literature can and should be viewed with respect to pre-service teachers. First, as future classroom teachers, pre-service teachers must become aware of difficulties their students are likely to encounter and be prepared to recognize flaws in the arguments their students will produce (e.g., Selden & Selden, 1995). Second, from their present status as students, pre-service teachers need to learn and understand more about the true nature of proof in mathematics. This second aspect of proof is the primary focus of this investigation.

CHAPTER 3

THEORETICAL FRAMEWORK

This study involves an investigation of some of the aspects of teaching and learning mathematical proof. This chapter provides a description of the theoretical lens through which these aspects of mathematical proof are viewed in the context of the present study. First, there are theories about beliefs and how beliefs may contribute to one's understanding of mathematics and may affect how one teaches mathematics. These theories are described. Second, a description of a theory about learning mathematical proof in general is provided. Third, a description of some of the roles and functions of proof in mathematics is provided and how the relative importance of those roles and functions in secondary school mathematics are viewed in the context of this study. Fourth, there are several theories related to the types of arguments a person is convinced by and what types of arguments that person believes others are convinced by. These theories will be discussed. Fifth, several descriptions of mathematical proof are provided along with how mathematical proof in the context of this study is viewed. Sixth, a list of definitions of terms and concepts taken from research that will help to ground some of the observations is provided. Finally, a summary section provides an explanation about how the theories relate to the present study.

Beliefs

Hersh (1998) observes that “one’s conceptions of what mathematics is affects one’s conception of how it should be presented. One’s manner of presenting it is an indication of what one believes to be most essential in it” (p. 13).

What Are Beliefs?

Webster’s New Collegiate Dictionary (1951) defines *belief* as “a conviction or persuasion of truth, intellectual assent” and *to believe* is “to accept as true.” From this definition, one might speculate that knowledge is belief, or conversely, that belief is knowledge. However, there is a distinction between beliefs and knowledge (Thompson, 1992). The amount or strength of conviction held by individuals toward a particular belief may vary. One individual might have very strong convictions towards a certain belief while another individual might view the same thing as simply probable, but not in a particularly passionate way. By contrast, with knowledge is associated a degree of certainty and a lack of variability. I would not say, for example, that I know the Pythagorean identity *strongly* or that Newton’s laws are *extremely* factual. However, I might say that I believe strongly that there is extraterrestrial life or that I don’t believe in Santa Claus. Green (1971) asserts that “it is impossible to know something that is not the case” (p. 65). In other words, we cannot *know* something that is false. This contradicts what it means to know. However, it is possible to *believe* something that is not the case. In other words, we can (and often do) believe things that are false.

Thompson (1992) notes that:

...from a traditional epistemological perspective, a characteristic of knowledge is general agreement about procedures for evaluating and judging its validity; knowledge must meet criteria involving canons of evidence. Beliefs, on the other hand, are often held or justified for reasons that do not meet those criteria, and,

thus, are characterized by a lack of agreement over how they are to be evaluated or judged. (p. 130)

Belief Systems and the Structure of Beliefs

The accumulation of beliefs constitutes a belief system. This system is dynamic in nature as new information challenges or complements old beliefs. Of particular interest for the present study is the nature of the organization or structure of belief systems. This structure of belief systems has to do with how the beliefs are related to one another and not with the content of the beliefs themselves. Understanding the structure of teachers' beliefs, it is thought, will lead to a better understanding of how to positively influence those beliefs, and how beliefs can potentially influence teaching practice (Cooney & Wilson, 1995).

Green (1971) identified three different aspects of how belief systems are structured:

First, there is the quasi-logical relation between beliefs. They are primary or derivative. Secondly, there are relations between beliefs having to do with their spatial order or their psychological strength. They are central or peripheral. But there is a third dimension. Beliefs are held in clusters, as it were, more or less in isolation from other clusters and protected from any relationship with other sets of beliefs. Each of these characteristics of belief systems has to do not with the content of our beliefs, but with the way we hold them. (pp. 47-48)

The *quasi-logical* dimension refers to the idea that not all beliefs are completely independent of each other; they can be related to other beliefs in a way similar to the way reasons are related to conclusions (Thompson, 1992). For example, a teacher might believe that it is important to use technology to teach mathematics, and therefore, that students should be allowed to use calculators. This is a *primary* belief. The same teacher

might then believe that spending class time to teach students about calculators is important. This is a *derivative* belief.

There is implicit in the idea of belief different levels of strength. Stronger beliefs are less likely to be questioned or examined by the individual. Strongly held beliefs are called *central*, less strongly held beliefs are called *peripheral* (Thompson, 1992). In the example above, if the strength of the teacher's belief in the use of technology is weak (for example, if the teacher believes it is less important for students to learn about calculators than it is to learn the basic computational skills), the teacher may abandon the use of calculators under pressures to, say, produce higher scores on achievement tests.

Isolated *clusters* of beliefs allow for conflicting sets of beliefs to coexist (Thompson, 1992). For instance, a teacher might believe strongly that using technology for teaching mathematics is important, but the same teacher might distrust computers when it comes to calculating students' grades.

Green (1971) further categorizes beliefs as *nonevidential* and *evidential*, depending on their justification. Nonevidential beliefs are beliefs held without regard to evidence or good reasons. Nonevidential beliefs are virtually impossible to challenge with logical reasoning or by providing evidence. As a result, nonevidential beliefs are highly resistant to change. *Evidential* beliefs, conversely, can be rationally examined and criticized, and thus, are open to change in light of additional evidence or better reasons. As Green (1971) puts it: "a person may hold a belief because it is supported by the evidence, or he may accept the evidence because it happens to support a belief he already holds" (p. 49). What is clear, however, is that the degree to which beliefs are open to

change is directly related to the degree to which beliefs are held evidentially (Cooney, Shealy, & Arvola, 1998).

Teachers' Beliefs About the Nature of Mathematics

Two views that help to describe the nature of understanding mathematics were provided by Skemp (1978): relational understanding of mathematics is characterized by knowing both what to do and why, and instrumental understanding is characterized by knowing a fixed set of plans for performing mathematical tasks.

Ernest (1988) believes that teachers' beliefs about the nature of mathematics can be divided into three categories: (1) problem solving view: mathematics is dynamic and process driven, (2) Platonist: mathematics is a static but unified body of knowledge, and (3) instrumentalist: mathematics is a set of unrelated but utilitarian rules and facts.

Raymond (1997) developed a system of categories for teachers' beliefs about the nature of mathematics along with a set of descriptors for each category, drawing on the classification of Ernest (1988). The categories of beliefs are laid out in a five-point scale from what she terms "traditional" to "non-traditional." The traditional view is that mathematics is an unrelated collection of facts, rules, and skills. The primary traditional view is that mathematics is *primarily* an unrelated collection of facts, rules, and skills. The even mix of traditional and non-traditional view is that mathematics is a static but unified body of knowledge with interconnecting structures. The primarily nontraditional view is that mathematics is primarily a static but unified body of knowledge and mathematics involves problem solving. The nontraditional view is that mathematics is dynamic, problem driven, and continually expanding. Raymond (1997) developed a similar five-level scale for teachers' beliefs about teaching and learning mathematics.

These can be described along a continuum from students passively receiving mathematical knowledge with the teacher being the dispenser of knowledge, to the students being autonomous explorers learning through problem solving with the teacher guiding learning and posing challenging problems.

Beliefs and Teaching Practice, Changing Beliefs

Ernest (1988) stated that reforms such as those suggested by the NCTM:

... depend to a large extent on institutional reform. ... They depend even more essentially on individual teachers changing their approaches to the teaching of mathematics. Teaching reforms cannot take place unless teachers' deeply held beliefs about mathematics and its teaching and learning change. (p. 99)

The relationship between teachers' beliefs and their instructional decisions and behaviors is not direct nor is it a simple cause-and-effect relationship. Thompson (1984) found that teachers' views, beliefs, and preferences about mathematics and its teaching do play a significant role in how they teach. She also found that there was consistency between what the teachers professed to believe and the manner in which they presented the mathematical content.

However, Raymond (1997) found that teachers' beliefs were not always consistent with their practice. For example, while one teacher held a primarily traditional view of mathematics, her beliefs about teaching mathematics were non-traditional. Raymond (1997) found that this teacher's practice was more consistent with her beliefs about mathematics content rather than her beliefs about mathematics pedagogy. In other words, even though her beliefs about teaching were non-traditional, in practice she taught in a very traditional manner. Raymond (1997) stated that the teacher's beliefs about mathematics were highly influenced by the teacher's own experiences as a student. Raymond (1997) characterized the teacher's beliefs according to Green's (1971) model

suggesting that the teacher's more deeply held traditional beliefs were central while her non-traditional beliefs about pedagogy were peripheral or isolated. This model allows for the co-existence of contradictory and inconsistent beliefs.

Drawing upon the three views of the nature of mathematics held by teachers, Ernest (1988) suggested that teachers with an instrumentalist view of mathematics are likely to teach with the goal of skills mastery and they will tend to closely follow a set curriculum; teachers with a Platonist view of mathematics are likely to teach with explanation of concepts in mind and view students as receivers of knowledge; and teachers with a problem-solving view are likely to be facilitators of learning in the classroom and view learning as active construction through problem posing and problem solving. However, he notes that there may also exist differences in teachers' espoused beliefs and their practice. Ernest (1988) suggests two key causes for this disparity between the espoused beliefs of teachers and their actual practice: First, there is the powerful influence of social context resulting from the expectations of students, parents, colleagues, and superiors. Additionally, there are restraints placed on teachers by adopted textbooks and/or curriculum that affect teachers' practices. Secondly, there is the teachers' level of consciousness about his or her beliefs and the extent to which the teacher reflects on his or her practice.

Cooney and Wilson (1995) suggest that in addition to identifying teachers' beliefs, it is important to identify how these beliefs are structured. Non-evidential beliefs are highly resistant to change and preclude reflection on the part of the individual. Additionally, isolated beliefs not linked to central beliefs are likely not to be acted upon in the face of pressure. For example, beliefs in non-traditional approaches to pedagogy

may be held in isolation, especially as compared with more centrally-held traditional views. The result is that teachers, unsure of how to implement non-traditional approaches to teaching or when faced with pressures to cover subject matter, may revert to more centrally-held traditional methods of presentation. The implication for teacher preparation is that “accounting for the structure of beliefs enables us to create activities that encourage teachers to wonder, to doubt, to consider what might be, to reflect, and most important, to be adaptive” (Cooney, Shealy, & Arvold, 1998, p. 332).

The Nature Of Learning Proof

Balacheff (1988) distinguishes between two levels of proofs as follows:

“*Pragmatic* proofs are those having recourse to actual action or showing, and by contrast, *conceptual* proofs are those which do not involve action and rest on formulations of the properties in question and relations between them” (p. 217, italics in original). Balacheff (1988) further distinguishes four main types in the cognitive development of proof: naïve empiricism, crucial experiment, generic example, and thought experiment. Simon and Blume (1996) provide a concise description of the four levels:

At the first level [naïve empiricism] the student concludes that an assertion is valid from a small number of cases. At the second level [crucial experiment], the student deals more explicitly with the question of generalization by examining a case that is not very particular [e.g., choosing an extreme case]. If the assertion holds in that case, it is validated. At the third level, the students develop arguments based on a “generic example” [e.g., an example representative of its class]. At the fourth level [thought experiment], students begin to detach their explanations from particular examples and begin to move from practical to intellectual proofs. (p. 8)

Balacheff (1988) noted that there is a fundamental divide between the first two types and the latter two types. In the first two types, the primary purpose of the argument is to “show” that the result is true because it “works.” In the case of the generic example and

the thought experiment, there is a shift in emphasis toward establishing the necessary nature of the truth of an assertion by providing reasons. Balacheff (1988) claimed that these four forms of proof are hierarchical.

Naïve empiricism and crucial experiment share the common connection of showing empirical evidence to validate a claim. Passing from naïve empiricism to crucial experiment requires that the student recognize the need to generalize the claim. The generic example is a transitional stage between pragmatic and conceptual proofs in that it requires the student to recognize the generic character of the example.

The Role Of Proof In Mathematics

Mathematicians and mathematics education researchers have identified eight primary roles or functions that proof plays in mathematics:

- Proof as a means for verification, that is, to establish the truth of a proposition (Bell, 1976; de Villiers, 1990).
- Proof as a means of explanation, that is, why a result is true. For example, to determine the sum of first n numbers, the standard induction argument verifies that it is true but not why (Hanna, 1989b). Instead, the following argument gives insight into why it is true:

$$\text{Let } S(n) = 1 + 2 + \dots + n.$$

$$\text{Then } S(n) = n + n-1 + \dots + 2 + 1$$

$$\text{Adding gives } 2S(n) = n+1 + n+1 + \dots + n+1$$

$$\text{So, } 2S(n) = n(n+1)$$

$$\text{Or, } S(n) = \frac{1}{2} n(n+1).$$

- Proof as a means of systematization of results into a deductive system of axioms, definitions, and theorems (Bell, 1976; de Villiers, 1990).
- Proof as a means of discovery. Some results, such as non-Euclidean geometries, could not have been discovered by inductive or empirical methods (de Villiers, 1990).
- Proof as a means for communicating mathematical ideas to others.
- Proof as a means of intellectual challenge. To a mathematician, finding a proof is an intellectual challenge and a source of personal fulfillment in the same sense that solving a puzzle is for most people (de Villiers, 1996, 1999).
- Proofs that justify the use of definitions and axiomatic structures, for example, Peano's axiomatic approach to arithmetic. The purpose of proving associativity of addition for integers was not to convince mathematicians that it is true or to explain why it is true, but it was to convince mathematicians that the axiomatic system was a reasonable model for arithmetic and thus a valid topic for investigation (Weber, 2002).
- Proofs that illustrate technique. For example, the proposition that the function $f(x) = x^2$ is continuous is introduced to students in real analysis primarily to demonstrate the "delta-epsilon" technique (Weber, 2002).

These categories are not mutually exclusive (nor are they necessarily exhaustive). For example, the role of proof to convince others is also suggestive of proof as a means or method of communication. Euclid's proof of the infinitude of prime numbers is an example of a proof that convinces, one that also explains why, and a powerful illustration of the technique of proof by contradiction (Weber, 2002).

Hersh (1993) observes that students generally don't need to be convinced of a theorem's truth. In the classroom, the purpose of proof is to provide insight into why a theorem is true. The degree of formality, rigor, and completeness of a presentation of a proof should depend on the degree to which these things are likely to increase the students' understanding (Hersh, 1993). Hanna (1989b, 1990, 1995) also proposed that the main function of proof in the mathematics classroom is to promote understanding, and suggested that in order to accomplish this goal, it is essential to provide students with explanatory proofs. "There is no infidelity to the practice of mathematics if in mathematics education we focus as much as possible on good mathematical explanation (even at the expense of rigor) and highlight for the students in our proof of a theorem the important mathematical ideas that lead us to its truth" (Hanna, 1989b, p. 54).

Taxonomy Of Student Proof Schemes

The terms "proof frames" or "proof schemes" are used interchangeably and refer to the types of arguments that a person is convinced by and that that person believes others are convinced by.

Harel and Sowder (1998) define proving to mean "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). They further state that the process of proving involves two subprocesses: ascertaining and persuading. Ascertaining is the process an individual uses to remove his or her doubts about the truth of an observation, while persuading is the process an individual uses to remove others' doubts about the truth of an observation. Harel and Sowder (1998) state that "a person's proof scheme consists of what constitutes ascertaining and persuading for that person" (p. 244).

Martin and Harel (1989a) suggest that there are two distinct proof frames, inductive and deductive, constructed by students as a result of experience in everyday life and experience in the mathematics classroom. They suggest that the inductive proof frame necessarily precedes the deductive proof frame. In addition, they suggest that the deductive proof frame does not replace the inductive proof frame, but that the two, in fact, exist simultaneously. Martin and Harel (1989a) posit that activation of both inductive and deductive proof frames may be needed for some students to believe a certain conclusion.

In addition to inductive and deductive proof frames, Simon (1996) hypothesizes a third type of reasoning: transformational reasoning. Simon defines transformational reasoning as “the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or continuum of states are generated” (p. 201). This type of reasoning attempts to get a sense of how a mathematical system works. Simon (1996) cites an example from a previous study in which students were asked to determine the area of an amoeba-like planar figure. The students in the study suggested that if a string were used to establish the figure’s perimeter, that length of string could be transformed into a rectangle whose area could then be calculated. While the result of the students’ thinking was incorrect, their thinking about the situation was neither deductive (the result of a logical chain of reasoning in which each step necessarily follows from the previous step) nor inductive (generalizing a conclusion from particular instances), but transformational. Simon (1996)

postulates “that transformational reasoning is a natural inclination of the human learner who seeks to understand and validate mathematical ideas” (p. 207).

Harel and Sowder (1998) provide a detailed taxonomy of student proof schemes. The student proof schemes can be placed into three broad categories: external conviction, empirical, and analytical proof schemes.

The external proof schemes are characterized by students following formulas and memorization of prescriptions, and by students relying on the teacher or textbook or other external source for affirmation. There are three types of external proof schemes: (1) the ritual proof scheme in which students are concerned with the appearance of the form of a mathematical proof rather than the correctness of the reasoning, for example, only a proof in the two-column format is considered to be proof; (2) the symbolic proof scheme in which students treat the symbols without regard to their meaning or relation to other symbols. An example of this type of proof scheme might be to equate $\frac{a}{b} + \frac{c}{d}$ with $\frac{a+c}{b+d}$; (3) the authoritarian proof scheme in which students rely solely on outside authorities (e.g., teachers, textbooks, or other more able students) for confirmation of results (Harel & Sowder, 1998).

Empirical proof schemes are characterized by appeals to examples. There are two types of empirical proof schemes: (1) the inductive proof scheme in which students base the truth about conjectures by citing one or more specific examples; (2) the perceptual proof scheme in which students draw conclusions based on their mental images without the ability to transform or to anticipate the results of a transformation, for example, drawing conclusions about *all* triangles based on a mental image of an isosceles triangle. Students may apply the perceptual proof scheme at higher levels of mathematics as well.

For example, when given three vectors and asked to determine whether the vectors form a linearly independent set, a student might respond by saying that the vectors are linearly independent by observation (Harel & Sowder, 1998).

Analytic proof schemes are characterized by their use of logical deductions. There are two types of analytic proof schemes: (1) the transformational proof scheme in which students perform or envision operations on objects and are then able to anticipate the results of the operations, and (2) the axiomatic proof scheme in which students understand (at least in principle) that mathematical justifications start from undefined terms and axioms and then proceed by rules of logic (Harel & Sowder, 1998).

There are two types of transformational proof schemes: (1a) internalized and (1b) interiorized proof schemes. An internalized proof scheme is a transformational proof scheme that has been adopted as a proof heuristic—a method of proof. For example, when given a particular geometric figure and asked to prove that two segments of the figure are congruent, students look for two congruent triangles that include the segments. The students abstract this proof heuristic because they have used it successfully over and over again. An interiorized proof scheme is an internalized proof scheme that is recognized by the person. The awareness of the interiorized proof scheme is usually marked by the person describing it to others and/or noting when it can and cannot be used (Harel & Sowder, 1998).

Many of the students who thought in a transformational mode had certain restrictions regarding the context of the claim, the generality of the claim, or the mode of the justification (Harel & Sowder, 1998). These are described below.

In a contextual proof scheme, the claims are interpreted and proved within a certain context. For example, a student might interpret and prove the general statement “ $n + 1$ vectors in an n -dimensional vector space are linearly dependent” in the specific context of \mathbf{R}^n since he or she hasn’t abstracted the concept of linear independence beyond this “world.” In the generic proof scheme, claims are interpreted in general terms, but their proof is expressed in a particular context (this is similar to the “generic example” proof-type of Balacheff, 1988). In the constructive proof scheme, students are convinced of a claim by actual construction of an object rather than justification of the existence of the object (reminiscent of the intuitionist/constructivist philosophy of Brouwer). This proof scheme is marked by an individual’s dislike of proof by contradiction.

Axiomatic proof schemes are evident when an individual understands that a mathematical justification must have started from undefined terms and axioms. There are three types of axiomatic proof schemes: intuitive, structural, and axiomatizing. An individual who understands the distinctions between defined and undefined terms, and between statements with and without proof, may still only be able to handle axioms that are intuitive or self-evident to him or her. This is called the intuitive-axiomatic proof scheme. In the structural-axiomatic proof scheme, the focus is on the structure itself, not on the axioms. At the level of the axiomatizing proof scheme, the individual is able to investigate the implications of varying a set of axioms.

Harel and Sowder (1998) claim that these proof schemes are not mutually exclusive: a given person may exhibit various proof schemes during a short period of time. Additionally, they hypothesize that the structural proof scheme is a cognitive

prerequisite for the axiomatizing proof scheme. They view the axiomatic proof scheme as an extension of the transformational proof scheme from an epistemological point of view.

What Is Mathematical Proof?

The traditional meaning of mathematical proof relates to its connections among axioms, definitions, and previously proved propositions: “a formal proof of a given statement is a finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference, and the last statement is the one to be proved” (Hanna, 1990, p. 6; see also Hersh, 1993; Fitzgerald, 1996; Davis & Hersh, 1981).

For practicing mathematicians, a proof is an argument that convinces qualified judges (Hersh, 1993; Davis & Hersh, 1981). For the practice of proof as a means of verification or to convince, mathematicians consider other factors than absolute certainty.² For example, the reasonableness of the results carries priority over completely rigorous proof and most published proofs tend to include only those parts of proofs that are considered necessary to convince the reader. Thus, many details (including logical manipulations and many of the calculations) deemed unimportant are excluded (de Villiers, 1990; Thurston, 1995).

Mathematicians, philosophers of mathematics, and mathematics educators have come to view proof as social in nature. Ernest (1998), for example, states that

The standards of proof are never objective and ultimate, but “sufficient unto the day,” and they are perpetually open to revision. Mathematical proofs are accepted because they satisfy individuals (especially the appropriate representatives of the mathematical community) that they are adequate warrants, not because they satisfy explicit, objective logical rules

² Philosophers of mathematics have argued at length that absolute certainty in mathematics is a myth and therefore unattainable. See, for example, Lakatos (1976) and Ernest (1998).

of proof. Thus mathematical knowledge and the standards of proof that support it depend what the mathematicians of the day accept. (p. 46)

Wilder (1944) further posits that

What we call “proof” in mathematics is nothing but a testing of the products of our intuition. Obviously we don’t possess, and probably will never possess, any standard of proof that is independent of time, the thing to be proved, or the person or school of thought using it. And under these conditions, the sensible thing to do seems to be to admit that there is no such thing, generally, as absolute truth in mathematics, whatever the public may think. (p. 319)

Wilder goes on to say a mathematical theorem comes about as a result of a mathematician’s intuition. The theorem is tested by what is called a proof. If the theorem passes the test and is found to be free of contradiction, it may be passed on to colleagues for additional scrutiny.

To counter the argument that there is a certain amount of ambiguity and subjectiveness to the standards for proof just described, Ernest (1998) states that

Mathematicians accept as mathematical knowledge only that which stands up to rational public criticism and scrutiny, based on their best professional judgment and some explicitly stated rules. Furthermore, there is a continuity between the application of professional judgment (and the rules employed) from one generation to the next, and any changes are themselves publicly debated and justified. Mathematical concepts, definitions, theorems, proofs, theories, and proof-standards grow, change, and are sometimes abandoned with the passage of time, as standards of rigor and proof change. Thus their ‘objectivity’ is actually intersubjectivity, is time and community dependent, and is rooted in historical continuity and tradition. (p. 46)

Definitions

In this section, descriptions of several terms are provided. The terms described are: “concept definition,” “concept image,” “sociomathematical norms,” “proof validation,” and “proof framework.” These terms are used throughout the study (particularly in discussing the results) to help describe and explain observations.

Concept Image and Concept Definition

The term *concept image* describes “the total cognitive structure that is associated with the concept, which includes mental pictures and associated properties and processes” (Tall & Vinner, 1981. p.152). *Concept definition* can be regarded as a collection of words used to specify the concept. An individual’s concept definition may simply be the formal definition memorized by rote or it may be more meaningfully learned and/or related to other aspects of the concept. For example, some students may know the formal definition of derivative as an isolated fact that they can restate while others may understand the formal definition as the limit of the ratio of two quantities. A student’s concept definition may also be his or her personal reconstruction of the formal definition. A student’s concept definition is expected to be invoked by asking questions such as “What is the definition of derivative?” (Tall & Vinner, 1981; Vinner & Dreyfus, 1989). The part of the concept image that is activated at a particular time is called the *evoked concept image*. In a given situation, the evoked concept image may in fact conflict with a previously manifested evoked concept image. It is only when conflicting images are evoked at the same time that the student may *sense* conflict or confusion (Tall & Vinner, 1981; Edwards, 2000). Serious problems arise in cases where an individual’s evoked concept image conflicts with the formal concept definition. In these cases, students tend to trust their own interpretations over the formal theory, treating the latter as unnecessary and superfluous (Tall & Vinner, 1981).

Sociomathematical Norms

Yackel and Cobb (1996) distinguish between general classroom social norms and norms specific to mathematical aspects of students’ activity. The latter is what the authors

call a “sociomathematical norm.” For example, what counts as an acceptable mathematical explanation and justification is a sociomathematical norm. In a classroom situation, students might be expected to explain their solutions and their ways of thinking. This is a social norm. However, the collective understanding of what is an acceptable mathematical explanation is also a sociomathematical norm. In this way, each classroom is its own mathematical community. Within each classroom, norms are established for taken-as-shared knowledge. For example, suppose a student is engaged in adding 12 and 13. Out loud, the student might say “two and three make five, and one and one make two.” The students listening to this understand that when the student says “one and one make two,” the meaning is understood to be “ten and ten make twenty.” Of course, there is the possibility that the sociomathematical norms of the classroom may differ (possibly in significant ways) from accepted mathematical norms. At the very least, sociomathematical norms will almost certainly differ from one classroom to the next. The teacher in the classroom can serve as a representative of the mathematical community to steer students toward accepted mathematical explanation and an increase in mathematical sophistication.

Proof Validation

Proof validation is broadly defined as “texts that establish the truth of theorems and on readings of, and reflections on, proofs to determine their correctness” (Selden & Selden, 2003, p. 5). Proof validation includes the reading of textual material and the mental processes associated with text reading. “Validation can include asking and answering questions, assenting to claims, constructing subproofs, remembering or finding and interpreting other theorems and definitions, complying with instructions (e.g., to

consider or name something), and conscious (but probably nonverbal) feelings of rightness or wrongness” (Selden & Selden, 2003, p. 5).

Proof Framework

A proof framework (Selden & Selden, 1995) is a representation of the “top-level” logical structure of a proof that doesn’t depend on a detailed understanding of the relevant mathematical concepts. For example, the statement $\lim_{x \rightarrow 3} x^2 = 9$ might have the following as a proof framework:

Let $\epsilon > 0$. Let $\delta =$ _____. _____ so $\delta > 0$. Let x be a number.

Suppose $0 < |x - 3| < \delta$. _____. Therefore, $|x^2 - 9| < \epsilon$.

The blank spaces indicate the details of the mathematical argument needed to complete the proof.

Summary

The theories described above provide a framework for addressing the research questions and serve two major purposes: 1) as an aid to analysis; and 2) to help explain the observations. The relationship between the theories and their relevance and purpose in the context of this study is described below.

Tall and Vinner’s (1981) notion of concept image provides a convenient and appropriate construct for describing pre-service teachers’ and professional mathematicians’ perceptions of mathematical proof. Their theory also provides a construct in which to help explain certain phenomena. To begin with, an individual’s concept image comprises the totality of an individual’s cognitive structure associated with a concept, however, Tall and Vinner (1981) make clear the idea that different elements of an individual’s concept image are likely to be evoked at different times

depending on the circumstances. For example, an individual may have a traditional, hypothetico-deductive definition of mathematical proof in mind when asked to describe or define mathematical proof, but in practice the same individual might view mathematical proof as a process of experimentation and conjecture. This individual might disseminate a traditional definition in classroom settings, thereby instilling in others a particular and possibly incomplete view of mathematical proof.

Included in the theoretical framework is the idea of belief structures as posited by Green (1971) and theories about beliefs related to mathematics and beliefs related to teaching practice. In the context of this study, the theories of beliefs serve three purposes: 1) to help describe the characteristics of participants' beliefs; 2) to help explain the observations; and 3) to suggest strategies for changing beliefs.

To describe the characteristics of an individual's beliefs about mathematical proof, it is helpful to try to describe how strongly the beliefs seem to be held. Central beliefs, according to Green (1971) are more likely to be acted upon. For example, a teacher whose beliefs about the inclusion of proof in high school mathematics are central would be more likely to incorporate proof in her or his teaching.

Addressing the second purpose cited above, individuals may have conflicting and varying views about mathematical proof and these views may be explained in terms of beliefs. For example, consider an individual whose concept definition of mathematical proof can be described as traditional, hypothetico-deductive. Further investigation might reveal that the individual values experimentation and conjecture leading to proof as a method of teaching. These two seemingly incompatible beliefs about proof can coexist as isolated clusters.

Cooney and Wilson (1995) suggest that in addition to identifying what teachers believe, it is important to identify how their beliefs are structured. Identifying the structure of beliefs can be used as a tool for developing strategies for changing the community of teacher education and encouraging reform-oriented pedagogy. For example, it is possible that teachers who develop a deeper understanding of mathematical proof through genuine proof activities such as experimentation and conjecture are more likely to adopt this view more centrally, and therefore, would be more likely to teach mathematics in this manner (Cooney & Wilson, 1995; Cooney, Shealy, & Arvold, 1998).

The theories and research on beliefs suggest that what a person believes and the strength and type of those beliefs (i.e., the structure of a person's beliefs) about mathematics and the teaching of mathematics has an effect on the way that person will present the mathematics to his or her students in the classroom. Efforts to adopt the recommendations of NCTM and others may necessitate radical changes in teachers' deeply held beliefs about proof in mathematics and its place in secondary school (Ernest, 1988).

There seems to be a close relationship between the notion of concept image and beliefs. The elements of an individual's concept image are the objects associated with a concept. These elements form the facts and knowledge associated with a concept in the sense of Thompson (1992) and do not carry with them the variable conviction of beliefs. As previously indicated, however, it is possible for an individual to believe something that is false. For that individual, the object can become a fact and therefore an element of the individual's concept image for that concept. Balacheff (1988) notes that students at the naïve empiricism level conclude that an assertion is true from a small number of

cases. For these students, the assertion is a fact not because it is a fact but because they believe it is a fact. Therefore, the structure of an individual's beliefs of mathematical proof forms a different but related dimension to the concept image of mathematical proof.

Some of the roles and functions of proof will invariably receive more attention than others depending on the context. For example, many researchers have concluded that the focus of proof in secondary school should be more on "proofs that explain" rather than simply "proofs that prove," (e.g., Hanna, 1989b), however, pre-service teachers need to have a deeper understanding of proof in mathematics that should include exposure to as many of the roles of proof in mathematics as possible. In addition to providing grounding for the importance and function for proof in mathematics, the theories of the roles for proof in mathematics provide an initial framework for analysis for the present study (the framework and its connection to the roles and functions of proof will be explained in more detail in chapter four).

The hierarchical levels of Balacheff (1988) provide a framework for determining students' cognitive preparedness for deductive reasoning. In the context of the present study, however, the work of Balacheff (1988) is used as a means of categorization for analysis and as a means of explaining or defining different approaches to proof (e.g., "generic example"). In particular, this work provides a means of distinguishing between levels of sophistication evidenced among students' empirical proofs. Other researchers such as Harel and Sowder (1998) did not make this distinction. Balacheff (1988) states that passing from one level to the next requires that the student recognize the need for increasingly greater generality, a shift in emphasis from showing that a result works to providing reasons why the result follows, and finally, focusing more on structural

characteristics of the domain. The need to recognize increasing degrees of generality applies to teachers as they encounter situations in which certain assertions don't necessarily hold in more abstract settings.

The theories on proof schemes provide an additional dimension to the framework for investigating pre-service teachers' and mathematicians' perceptions of proof. For example, Harel and Sowder (1998) note that despite the fact that many adult students readily acknowledge the limitations of an inductive proof scheme, they continue to operate in this scheme. They suggest that it may be a result of passing from an authoritarian proof scheme wherein the students formerly accepted a teacher's inductive approach to theorems, to somehow accepting the inductive approach as an acceptable method for proof. A deep understanding of mathematics in the context of proof implies the ability to operate above the level of empirical proof schemes (this implication is consistent with the theory of Balacheff, [1988]). More specifically, since axiomatics play a significant role in mathematics (though not necessarily at the secondary school mathematics level), there is the added implication of ability to operate at no less than the level of the intuitive-axiomatic proof scheme.

The taxonomies of proof schemes are helpful in providing an initial framework for broadly categorizing (e.g., empirical and analytical) and describing proof-types especially for the high school student-generated arguments. Perhaps more important, however, are two other factors related to proof schemes: 1) for any individual, different proof schemes can exist almost simultaneously; and 2) proof schemes are closely tied to beliefs.

To describe the importance of the first factor related to proof schemes, consider the traditional exposition of mathematical theory in the college classroom setting. The exposition typically involves stating a theorem and then providing the proof. However, the proof of the theorem is typically rigorous only in parts that the presenter deems important (Thurston, 1995). The remaining portions of the proof are demonstrated by appeal to examples, taken on authority, or “left to the reader.” The presenter is thinking and acting within different proof schemes while at the same time assuming that the audience can do the same. For students, this may cause confusion – not so much in understanding the proof, but in understanding the process of proof. For this study, an important consequence of this phenomenon is to consider the existence of multiple proof schemes when characterizing participants’ perceptions of mathematical proof.

To explain the importance of the connection between proof schemes and beliefs, recall that Harel and Sowder (1998) define proving to mean “the process employed by an individual to remove or create doubts about the truth of an observation” (p. 241). In other words, proof is a process used by an individual to persuade someone to believe the conclusion. In this sense, proof is closely linked to belief systems. So, in addition to the type of argument an individual is convinced by (one’s proof scheme), there is an additional dimension reflective of an individual’s beliefs about proof. This dimension includes characteristics relating to proof such as: how strongly one believes an argument and whether to believe an argument based on its form. The purpose for including this dimension is for consideration of questions relating to whether participants’ perceptions of proof vary in different contexts, and in ascertaining whether participants believe an argument purported to prove an assertion.

CHAPTER 4

METHODOLOGY

Chapter Overview

This chapter provides a description of the research design and the methodology employed to investigate the perceptions of mathematical proof of pre-service teachers and professional mathematicians in the context of mathematics, and their perceptions of mathematical proof in the context of high school mathematics. The chapter begins with a description of the process employed in developing the data collection instrument used in the study. The chapter continues with a restatement of the primary research questions that guided the examination of perceptions of mathematical proof held by pre-service teachers and professional mathematicians. The next three sections describe the participants of the study and the criteria used in selecting them, the sources of data, and the method of analysis used.

Development of the Data Collection Instrument

Development of the data collection instrument (the questionnaire) took place in two distinct phases: In phase I, a set of high school student-generated arguments purported to prove various propositions was assembled; phase II involved a pilot study to assess a draft of the questionnaire and to gather data on proof-assessment tasks. These two phases culminated in the development of the final form of the questionnaire. A

description of the two phases of development and a description of the final form of the questionnaire are given below.

Student-generated arguments

Initially, the primary focus of this investigation was to determine pre-service teachers and professional mathematicians' assessment of student-generated arguments. Thus, it became important to create a database of genuine student-generated arguments to propositions. Four high schools in New Hampshire were enlisted to participate in completing proof-problems during the spring of 2004. There were four sets of four different problems distributed among the four participating high schools. The sets of problems were collated so that each student was equally likely to receive one of the four problem sets. In all, 58 completed problem sets were included in the database.

From the work of high school students, a database of responses (arguments) was put together according to type of argument. Student-generated arguments to the proof-problems were coded on two dimensions: (1) the type of argument observed, and (2) the type of errors identified. Types of arguments generally fell under two categories: empirical or analytical. Empirical arguments are those that appeal to one or more examples (Balacheff, 1988; Harel & Sowder, 1998). Balacheff (1988) identified crucial experiment as belonging to this class of argument. Here, the question of generalization is addressed by picking a case that represents an extreme in the domain. The case chosen is not very particular. For example, suppose the proposition is to prove that the sum of two even numbers is even. Consider two possible responses:

- 1) $2 + 2 = 4,$
 $2 + 4 = 6,$
 $2 + 6 = 8,$ etc.
Even number every time, so it's true.

2) Pick $1426 + 4368$. This is 5794 which is even. Hence, it's true.

In the first case, there is an appeal to examples and observing a pattern. In the second case, an arbitrary but specific case is given. It is empirical in nature, but according to Balacheff (1988), it represents a more sophisticated level of generalization than merely citing examples.

Analytic proof schemes are characterized by their use of logical deductions. Harel and Sowder (1998) suggested that there are two types of analytic proof schemes: (1) the transformational proof scheme in which students perform or envision operations on objects and are then able to anticipate the results of the operations. For example, to prove that for $x \geq 0$, $x + 1 \leq e^x$, a student might respond by saying that while both functions, $x + 1$ and e^x , are increasing, e^x is increasing *faster*; (2) the axiomatic proof scheme in which students understand (at least in principle) that mathematical justifications start from undefined terms and axioms and then proceed by rules of logic. Without explicit knowledge or evidence of the students' thinking, it may not be possible to perceive a student argument as being primarily transformational in nature. A third analytic proof scheme is the generic example of Balacheff (1988). The generic example consists of an argument based on an example that is representative of a class of objects. For example, for the proposition above, a student might respond with the following:

Consider $254 + 426$.
 $254 = 2(127)$ and $426 = 2(213)$.
So, $254 + 426 = 2(127) + 2(213)$.
Since both are divisible by 2, the answer is divisible by 2.

This argument is based on a specific example, but the example is representative of, in this case, the class of all three-digit even numbers. The generic example argument is

deductive, but is not fully generalized. Balacheff (1988) considered this type of argument to be a less sophisticated form of analytical argument.

In addition to these proof schemes, student-generated arguments were coded according to type of errors noted by the researcher. Error-types observed among responses included: notational errors such as algebraic errors or errors in assigning variables; logical errors such as proving the converse of a statement; lack of content knowledge such as not knowing what it means to square a number or what an irrational number is; incorrect use of a method of proof such as indirect proof; and strategic difficulties such as proceeding to a point and then reaching an impasse.

Pilot study

A pilot study was conducted in the summer of 2004 at the University of New Hampshire. The purpose of the pilot study was to determine what improvements and refinement could be made to the questionnaire. A draft version of a written questionnaire was administered to five in-service secondary school mathematics teachers participating in the Master of Science for Teachers program at the university as well as two professors of mathematics at the university. The purpose of the questionnaire was to gather information about participants' perceptions on two broad dimensions of mathematical proof: namely, their perceptions of proof in mathematics and their perceptions of proof in secondary mathematics. The draft version of the questionnaire was based in part on a compilation of implicit and explicit questions used in previous research on mathematical proof (e.g., Knuth, 1999) and personal observations. The questionnaire was designed to provide information about participants' perceptions concerning how they define or describe mathematical proof, their perceptions about the role and purpose of proof in

mathematics, and their views about validity of proof in mathematics. Other questions were aimed specifically at participants' perceptions about proof in high school mathematics. In particular, the questions asked participants to respond to whether they thought it was important for high school students to learn mathematical proof, their perceptions about the role and purpose of proof in high school mathematics, what expectations participants have for high school students regarding mathematical proof, and how, when, and where participants thought proof should be taught in high school mathematics.

In addition to the written questionnaire, semi-structured interviews were conducted with participants (for a list of the interview questions, see Appendix A). During the interviews, participants were shown a collection of proof-problems and a collection of student generated solutions to each of the problems (see Appendix B). The propositions and their accompanying arguments were assembled based on their coding. That is, each argument included represented a particular dimension of empirical/analytic proof scheme and may have included a particular type of error.³

In audio-taped semi-structured interviews, participants were asked to judge each argument as to whether they thought it was a valid mathematical proof, what factors were involved in their decision about the validity of each argument, what errors they detected, and what types of errors were detected. Analysis of the interview data revealed that several of the proof-problems and a great deal of the participant responses were redundant in many respects. For example, many of the student-generated responses to proof-problems were empirical in nature, were virtually the same, or didn't elicit any new

³ It should be noted that not all combinations of proof schemes and error types were represented in the database of student-generated arguments.

information from the participants. From an analysis of the responses to interview questions, an examination of the questions raised by the interviews, and the time commitment involved on the part of participants, it was decided to include fewer proof-assessment tasks and to incorporate them into the written questionnaire. The final form of the questionnaire consists of two sections with 12 and 25 questions respectively (see Appendix C).

The Final Version of the Questionnaire

The final version of the questionnaire was developed to study the main questions of the research study. The questionnaire was composed of two sections in order to assess two components of participants' perceptions of mathematical proof: mathematical proof in the context of the discipline of mathematics and mathematical proof in the context of high school mathematics.

Many of the questions were adapted from prior research. Questions 1, 2, 3, 4, 5, 8, 9, and 12 of section I were adapted from Knuth (1999), question 11, the calculus proposition and its accompanying arguments used for questions 9, 10, and 11 were adapted from Raman (2002). Questions 2, 3, 6, 8, 15, and 19 of section II were adapted from Knuth (1999). Questions 6, 7, 10 of section I and questions 1, 4, 5, 7, 9, 10, 11, 12, 13, 14, 16, 17, 18, 20, 21, 22, 23, 24, and 25 of section II were based primarily on questions raised from the analysis of the interview data obtained in the pilot study, from observations related to the literature, and personal observations and recollections. For example, observations that different levels of rigor and different degrees of acceptance of arguments exist depending on different circumstances were suggested by sources in the literature (e.g., Davis & Hersch, 1981; Weber, 2003). The idea behind question 18 (is

there a difference between “show” and “prove?”) of section II was initially suggested by prior research (e.g., Dreyfus, 1999) but was also observed in interview data.

Section I of the questionnaire included questions aimed at eliciting participants’ concept definition (Tall & Vinner, 1981) or description of mathematical proof, their perceptions about the purpose and importance of mathematical proof, how participants view validity in proof, participants’ views about who decides whether an argument is a proof, and their preferences for certain arguments. Section II of the questionnaire included questions aimed at eliciting participants’ views about the nature, importance, and purpose of proof in high school mathematics and participants’ expectations for proof in high school mathematics.

The intent of the questions in section I of the questionnaire was to elicit participants’ perceptions about the nature of proof in mathematics. Table 1 provides specific information regarding the purpose for inclusion of each item in section I of the questionnaire.

Item	Purpose for inclusion – To elicit participants’:
1	concept definition or description of mathematical proof
2	perceptions about the purpose of proof in mathematics
3	perceptions about the importance of proof in mathematics
4	perceptions about types of proof in mathematics
5	perceptions about validity of proof in mathematics
6	perceptions about formality of proof in mathematics
7	perceptions about what is acceptable as proof in mathematics
8	perceptions about authority for proof in mathematics
9	perceptions about validity of proof in mathematics
10	perceptions about validity of proof in mathematics
11	preferences for proof in mathematics
12	concept definition or description of mathematical proof

The intent of the questions in section II of the questionnaire was to elicit participants' views about the nature, importance, and purpose of proof in high school mathematics and participants' expectations for students regarding proof in high school mathematics. Table 2 provides specific information regarding the purpose for inclusion of each item in section II of the questionnaire.

Item	Purpose for inclusion – To elicit participants':
1	perceptions of the importance of proof in high school mathematics
2	perceptions about the purpose of proof in high school mathematics
3	perceptions about their expectations for students in high school mathematics
4	perceptions about their expectations for students in high school mathematics
5	perceptions about rigor of proof in high school mathematics
6	perceptions about their expectations for students in high school mathematics
7	perceptions about their expectations for students in high school mathematics
8	perceptions about their expectations for students in high school mathematics
9	perceptions about their expectations for students in high school mathematics
10	perceptions about their expectations for students in high school mathematics
11	perceptions about their expectations for students in high school mathematics
12	perceptions about their expectations for students in high school mathematics
13	perceptions about rigor of proof in high school mathematics
14	perceptions about the purpose of proof in high school mathematics
15	concept definition or description of mathematical proof
16	concept definition or description of mathematical proof
17	perceptions about the purpose of proof in high school mathematics
18	perceptions about their expectations for students in high school mathematics
19	concept definition or description of mathematical proof
20	perceptions about their expectations for students in high school mathematics
21	concept definition or description of mathematical proof
22	perceptions about the purpose of proof in high school mathematics
23	perceptions about the purpose of proof in high school mathematics
24	perceptions about their expectations for students in high school mathematics
25	perceptions about their expectations for students in high school mathematics

All of the questions contributed to some extent in providing information to help answer the primary research questions, but some questions were designed to more

directly address certain research questions. Table 3 organizes the questionnaire items with respect to each of the research questions. The table displays the items that either directly or indirectly address aspects of the research questions.

Research question	Items that address research question
1) How do pre-service teachers and practicing mathematicians view mathematical proof?	Section I: Items 1 – 12 Section II: Items 1 – 25 Especially: 11, 16, 18, 21 regarding nature 5, 12, 13, 23 regarding rigor 1, 4 regarding importance 2, 14, 17, 22, 25 regarding purpose
2) a) What do pre-service teachers and practicing mathematicians find acceptable as mathematical proof in general? b) Does this differ from what these two groups find acceptable as mathematical proof from high school students?	Section I: Items 9 – 12 Section II: Items 15, 16, 19, 20, 21 Section II: Items 5, 10, 12, 15, 16, 19, 20, 21, 24
3) What are the similarities and differences between what pre-service teachers consider acceptable and what mathematicians consider acceptable as mathematical proof?	Section I: Items 9 – 12, Section II: Items 11, 15, 16, 19, 21
4) What differences and similarities exist between the expectations pre-service teachers and mathematicians have for high school students regarding mathematical proof?	Section II: Items 3, 4, 6, 7, 8, 9, 10, 11, 12, 18, 20, 24, 25
5) How can the observed similarities and/or differences be explained?	Section I: Items 1 – 12 Section II: Items 1 – 25

Research Questions

As previously stated, there were five main questions that guided the investigation of pre-service secondary mathematics teachers' and professional mathematicians' perceptions about mathematical proof in the context of mathematics, as well as the

context of secondary school mathematics. In addition, several sub-questions served to further refine the primary research questions and provide additional direction to the research.

- 1) How do pre-service teachers and practicing mathematicians view mathematical proof?
- 2) What do pre-service teachers and practicing mathematicians find acceptable as mathematical proof in general? Does this differ from what these two groups find acceptable as mathematical proof from high school students? If so, why?
- 3) What are the similarities and differences between what pre-service teachers consider acceptable and what mathematicians consider acceptable as mathematical proof?
- 4) What differences and similarities exist between the expectations pre-service teachers and mathematicians have for high school students regarding mathematical proof?
- 5) How can the observed similarities and/or differences be explained?

Participants

Thirteen pre-service teachers and eight professional mathematicians participated in this study. In all, 19 pre-service teachers enrolled in two undergraduate mathematics education courses at the University of New Hampshire were asked to participate in the study. The thirteen pre-service teacher participants for the study were selected based on two criteria: (1) they were secondary mathematics education majors; and (2) they were near the completion of their undergraduate programs of study (junior or senior standing). By choosing participants that met these two criteria, I assumed that there would be more uniformity in their college-level experience with respect to mathematical proof, and that they would have similar backgrounds and interests in the teaching and learning of high

school level mathematics. All pre-service teachers had taken a typical four-course calculus sequence and a course in college-level geometry. Most participants had also taken a typical introduction to mathematical proof course (e.g., proof in the context of set theory and logic). The participants who did not report having taken the introductory course in mathematical proof had taken a proof-writing course set in the context of linear algebra. Most senior level pre-service teachers had taken an abstract algebra course and senior seminar (a capstone course designed to explore mathematical topics beyond a student's previous coursework). Table 4 gives the class standing of participants and a breakdown of the courses taken by participants that are proof-intensive. It is possible that the perceptions of pre-service teachers might be affected by additional exposure to proof-related coursework or maturation, the work of Cyr (2004), however, suggests that little or no change in conceptions of mathematical proof occurs over the undergraduate years. The main intent of this investigation was to highlight themes common to pre-service teachers. Differences that might exist between pre-service teachers of differing abilities, at different points in their programs, or who may have taken somewhat different coursework are generally not described.

Table 4
Proof-related coursework (beyond calculus) of pre-service teachers.

Participant	Class Standing	Proof	Linear Algebra	History of Math.	Geometry	Abstract Algebra	Senior Seminar
Pst-G ^a	Senior	X			X	X	X
Pst-D ^b	Senior	X	X		X	X	X
Pst-A	Senior	X	X	X	X		X
Pst-T	Junior		X		X		

(table continues)

Participant	Class Standing	Proof	Linear Algebra	History of Math.	Geometry	Abstract Algebra	Senior Seminar
Pst-U	Junior		X		X		
Pst-M	Senior	X	X	X	X	X	X
Pst-I	Senior	X		X	X	X	X
Pst-H	Junior	X	X	X	X	X	X
Pst-N	Senior	X		X	X	X	X
Pst-P	Junior	X	X		X		
Pst-J	Senior	X	X	X	X	X	X
Pst-K	Senior	X	X	X	X		X

^aPst-G also took courses in real analysis and topology.

^bPst-D took alternative courses to multidimensional calculus, differential equations, and linear algebra.

I initially assumed that there could be differences between the views of pure versus applied mathematicians, and so to limit any variation based on this particular dimension, only professional mathematicians whose training and degrees were in pure mathematics were asked to participate. In all, four professional mathematicians from University of New Hampshire and four professional mathematicians from California Polytechnic State University participated in this study. Table 5 gives some background information on the professional mathematicians. Again, the main intent of this investigation was to highlight themes common to professional mathematicians and how these might differ from those found of pre-service teachers, so other variables that could differ between professional mathematicians such as philosophical perspective, years of experience, or primary subject area were generally not included or considered.

Participant	Year of Ph.D.	Areas of interest
Pm-A	2000	geometric group theory
Pm-B	2000	commutative algebra
Pm-C	1987	algebra, topology, foundations, number theory
Pm-E	2001	complex analysis, functional analysis, composition operators
Pm-F	1964	analysis, operator theory
Pm-G	1995	operator theory, complex analysis, functional analysis
Pm-H	1998	algebraic topology, category theory, homological algebra
Pm-I	1964	general topology, history of general topology

Data Collection

The primary sources of data consisted of a written questionnaire (the final version) and interviews. All participants were asked to complete a written questionnaire. Participants were allowed to complete the questionnaire at home and at their own pace. In addition to the written questionnaire, pre-service teachers were also asked to complete an information form (see Appendix D) that provided information about their academic backgrounds (e.g., major, standing by year, coursework).

The questionnaire (see appendix C) included open-ended but specific questions for which participants were to provide written responses and written explanations of their responses. In addition, participants were asked to assess the validity and indicate their preference for four examples of propositions accompanied by arguments purported to prove the propositions (these correspond to questionnaire items 9 through 11 of section I, question 12 of section I, questions 15 through 17 of section II and questions 19 through 22 of section II). The written questionnaire was composed of two major components. In the first component, participants were asked about their beliefs about the nature and role of proof in mathematics. The second component was designed to gain insight into what

the participants believe about the nature and role of proof in secondary school mathematics.

There were many instances in which I was not clear what the participants intended to convey with their written responses to questionnaire items. For these instances, I followed-up by asking participants to clarify those items I didn't understand in audio-taped interviews. In all cases, I had the original copies of each of the participants' completed questionnaire for them to look at and refer to directly. An explanation of the reasons indicating the need for clarification is given below.

One participant inadvertently missed a page of the questionnaire and therefore left these questions blank. I asked this participant to respond orally to the written questions as the responses were audio-taped. In another case, I was unable to read the handwriting of the participant for a particular response.

As a part of the questionnaire, there were three propositions for which participants were asked to comment on the arguments purported to prove the proposition. In several cases, participants didn't comment on one or more of the arguments. For example, participants might respond that they liked argument one and disliked argument three but failed to comment on argument two. In audio-taped interviews, I pointed out to these participants that they didn't comment on one or more arguments and I asked these participants what they thought about that argument.

There were several cases in which the responses of participants didn't seem to answer the questions posed. For example, question ten of section II asks: Does your acceptance of a student argument as a proof depend on whether you think the student knows what s/he is talking about? One participant responded: "Yes, a logical [and]

complete proof that cannot be explained to others is essentially useless.” It appeared to me that this participant misunderstood the question.

There were many instances for which participants’ responses were in some sense incomplete. In these cases, participants were asked in audio-taped interviews to provide additional explanation to their written responses. For example, one participant responded that: “proving the obvious is most important” but failed to explain why. Participants occasionally used terminology that I was unfamiliar with. For example, when referring to different proof techniques, several participants used the terms “bootstrap” and “divide and conquer” both of which were unknown to me.

The final class of reasons for needing clarification had to do with intent of meaning. In general, attempts were made to let the responses of participants stand on their own and speak for themselves. However, in some instances, responses contained ambiguities that I thought needed clarification. In most of these cases, I could infer the intent of meaning for the responses, but I wanted to be sure that my interpretation was what the participant intended. Several examples of responses that fit into this class are presented and explained below.

As a part of the purpose for proof in mathematics, one participant responded that: “proof is important in providing a deeper understanding of mathematics. Proof goes beyond formulas and methods.” It seemed clear that the participant’s intent was that proof provides an understanding of mathematics that rote memorization of formulas and procedures does not. This was confirmed by asking the participant to explain the response. Another participant wrote that: “... proofs in lower level classes can be more informal and less structured – while proofs in higher classes should be more

understandable to everything.” Unclear to me was what the participant meant by “more understandable to everything.” As a final example, one participant responded “there are different techniques available to proof [sic] various propositions, however, proof in mathematics, its definitions and meaning to me remain stable.” The meaning of the phrase “proof in mathematics, its definitions and meaning to me remain stable” and why the participant deliberately chose to include it in her written response were not clear to me.

Interviews for clarification were conducted with eleven of the participants (interestingly, all of the participants interviewed were pre-service teachers). Seven of the participants were asked to clarify from three to five of their responses while the remaining four participants’ questionnaires required six to eight points of clarification. Interview data were transcribed verbatim and were included in the transcripts of participants’ responses to the questionnaire. The distinction between written responses and responses from interviews was noted in the transcripts for reference.

Method of Data Analysis

Recall that the goal of this research was to identify similarities and differences between pre-service teachers and professional mathematicians about their perceptions of mathematical proof. Previous research, particularly that of Knuth (1999), Raman (2002), and de Villiers (1991), provided a framework from which to begin to identify similarities and differences. For example, some of the initial high-level categories used included: nature, description, purpose, validity, importance, and expectations. Some preliminary subcategories were also adapted from previous research. For example, several researchers

identified different purposes for proof in mathematics such as “to establish truth” and “to explain” (e.g. de Villiers, 1991).

As previously indicated, the questionnaire was adapted in part from prior research (e.g., Knuth, 1999; Raman, 2002). At first, it was difficult not to categorize the data into the pre-existing framework on which the questionnaire was based. However, since this framework fit the natural structure of the questionnaire, I decided not to fight what seemed to be an obvious initial format, and so, I included the previously established categories into the preliminary coding scheme. I then looked to the data for other trends. It was also clear to me from the preliminary coarse analysis of the data that this approach could lead largely to replication and confirmation of previous results, which was not the primary goal of the study. Being mindful of potential bias in coding, I tried to allow the data to direct the actual method of analysis and to allow the coding scheme to evolve as the study progressed.

Analysis of the data took place in three phases: the first phase involved reading through all the data coarsely, the second phase involved a line-by-line microanalysis of the data, and the final phase was a more global analysis. These phases were not sequential in the sense that as one phase was completed, the next phase could begin. Instead, each phase was an integral part of a process of analysis. For example, trends and themes emerged while conducting microanalysis and codes surfaced as a result of global analysis. In this sense, the three phases of analysis overlapped as results from one phase were subsequently used in another phase. The three phases and the overall process of analysis will be described in more detail below. An additional integral component of the data analysis consisted of recording memos (Glaser & Strauss, 1967; Strauss & Corbin,

1998). Strauss and Corbin (1998) define memo as “the researcher’s record of analysis, thoughts, interpretations, questions and directions for further data collection” (p. 110). Writing memos about the ongoing analysis and referring to the record of memos became an important part of the entire analysis process.

Phase I: Reading the Data

Analysis of the data began by first reading through all the transcribed data. The data were organized and subsequently read in two different ways: 1) All data were organized by participant; 2) all data were organized by question. I made notes about my observations and initial hypotheses while reading through the transcripts. Data organized by participant provided a “feel” for the participants themselves and in some cases, provided evidence of trends that might not ordinarily have surfaced. For example, it appeared that some participants seemed to hold a somewhat narrow and unchanging view for proof, while for other participants, their responses suggested changes or evolution in their views over the course of their participation in the study. Data organized by question, on the other hand, provided more information about participants’ beliefs about specific categorical aspects of proof.

The process of reading over all the data also provided evidence of trends and allowed me to make hypotheses. For example, when describing mathematical proof, many of the pre-service teachers included action verbs such as “a way to show,” or “a process,” whereas professional mathematicians seemed to refer to proofs objectively with terms such as “a list,” or “an argument.”

Finally, while reading over participant responses, additional questions would occasionally arise. For example, many participants used the word “truth” with respect to

mathematical proof. This led me to wonder exactly what the participants meant by “truth” and whether this information could be ascertained from the data.

Microanalysis and Data Coding

The data from all questionnaires and verbatim transcriptions of interview data were entered into the spreadsheet program Excel 2000 (Microsoft Corporation, 2000). The spreadsheet program allowed the database to be easily searched and sorted according to various criteria such as keywords, categories, question number, participant, and participant type (i.e., pre-service teacher or professional mathematician). Codewords⁴ were included alongside each response as descriptors for each response. These initial codewords were taken directly from the response. Some of the codewords were qualitative in nature (such as “important” or “very familiar”) while others were more descriptive of classes of people or objects (such as “high school” or “students”). For example, one item of the questionnaire asks in what classes should mathematical proof be addressed. Pre-service teacher Pst-N responded: “In upper level classes, because lower level classes might not be able to understand the steps of proofs” (Pst-N 1030). Codewords included with this response were: “upper-level classes,” “lower-level classes,” “not able,” “understand,” and “steps.” A large number of categories emerged from carefully reading through the responses and looking for similarities and differences among the responses for the questions, the codewords, and the context of the responses. For example, for their descriptions of mathematical proof, two of the participants

⁴ Strictly speaking, there were “codephrases” as well as codewords. Codephrases were included when individual words alone did not seem to provide meaning or when the participant’s intention seemed clear. For example, the phrase “write it the way you’d say it” was used instead of the individual words. This approach had the advantages that: 1) intent of meaning was explicit which made subsequent coding easier and 2) the database was still searchable by the single codewords if necessary.

included the phrases: “to be sure” and “will always hold,” which indicated to me that these two participants explicitly ascribe a level of certainty with mathematical proof.

The questionnaire was designed with questions about mathematical proof in the mathematical discipline and in high school mathematics. This provided a natural dichotomy for top-level categorization. The items in the questionnaire also provided a natural categorization since each asked specifically for participants to respond to particular aspects of mathematical proof (i.e., description, purpose, expectations, etc.). However, the open-ended nature of the items in the questionnaire and the opportunity for participants to explain their responses resulted in a richer collection of data than might otherwise have been collected. The result was that additional or altogether new categories or subcategories were included. For example, participants used terms such as “intimidating,” “afraid of them,” and “horrible things” in their responses to various aspects of mathematical proof. This suggested a theme that could provide both instantiation and explanation of a particular characteristic that these participants associated with mathematical proof. In all, over 400 codes were identified by this process. Responses were then coded further in a hierarchical manner according to higher-level categorization. For example, responses specifically identified with the purpose of proof in high school mathematics were coded: “proof in HS – purpose – subcategory.” Not all responses fit neatly into this hierarchical scheme. For example, consider the response: “they need to understand that mathematical theorems don’t just appear, they’re there for a reason and the proofs are those reasons” (Pst-U 1392). This response contains elements related to three aspects of proof in high school mathematics: expectation of students, the nature of mathematical proof, and the purpose of proof in high school

mathematics. It seemed to me that the response was primarily concerned with expectations, and so “expectations” was assigned as the first subcategory of the code.

Global Analysis

Global analysis of the data was employed to confirm, refine, refute, or abandon working hypotheses. Responses were chunked into episodes. For the purposes of this phase of analysis, an episode was characterized as a complete, distinct response or collection of responses pertaining to or focused on a single concept or topic. In some cases, an episode might consist of a fairly large collection of sentences, at other times, an episode might consist of a single word response. Many episodes were direct responses to specific items in the questionnaire. For example, regarding the concept of validity in mathematical proof, there were items in the questionnaire that asked participants to comment on their views about the validity of a given proof. Other information about participants’ views arose as a result of participants’ responses to other items, usually as a part of their explanation of something else.

I used Excel 2000 (Microsoft Corporation, 2000) to search the transcripts of data for instances of keywords and phrases relevant to the particular concept of interest. Due to my familiarity with the data, I was also able to examine my written notes and portions of the transcripts that were likely to contain information applicable to the specific concepts being analyzed. Entire episodes of participant responses were extracted, isolated, and analyzed further to determine their relevance and fit to the working hypotheses.

The Process of Analysis

In broad terms, the coding and analysis of the data was an evolving process patterned in large part on the constant comparison method of Glaser and Strauss (1967). As the data were being analyzed, hypotheses were formed. The hypotheses were then compared with the data to help bolster the plausibility of the hypotheses or to provide direction for modification of the hypotheses. This process in turn led to the development of a theory that could both describe and explain the phenomena under investigation. A more detailed description of the process of analysis is given below.

Recall that an individual's concept image was described as "the total cognitive structure that is associated with the concept, which includes mental pictures and associated properties and processes" (Tall & Vinner, 1981, p.152) and that an individual's concept definition is the collection of words used to describe the concept. Also recall that an individual's concept definition is likely to be invoked by asking questions such as: "what is the definition of mathematical proof?" Directed questions such as item one of the questionnaire were aimed at identifying participants' concept definition of various aspects of mathematical proof with the tacit assumption that participants' concept images were likely to be more extensive and more complex.

After the initial reading of the transcripts organized by participant, a more systematic analysis of each questionnaire item was carried out. In many cases, high-level coding of the responses was relatively easy and straightforward since most responses were prompted by fairly specific questions. For example, participants were asked to define or describe mathematical proof. Responses to this item generally fell under the category "description." In many cases, participant responses to other items or participant

explanations for other items indicated descriptive or defining aspects of proof. These responses generally fell under the category “description” as well. For example, although not prompted specifically to describe his views on formal proof, Pm-I stated that: “a formal proof has clearly defined axioms and rules of inference. A formal proof is usually a list that must take a prescribed form” (Pm-I 447). In assessing the validity of an argument, Pst-J stated that: “[argument three] doesn’t explain at the beginning the goal, nor does [it] give reasons for each step” (Pst-J 717). This response provides further information about the participant’s concept image of mathematical proof. Both Pm-I and Pst-J’s responses were included under the code “description” since they both seem to indicate characteristics associated with the meaning of proof for the participants.

Responses were then further coded by their context into subcategories. As an example of this process, consider the first questionnaire item: “what does mathematical proof mean to you (i.e., how would you define or describe mathematical proof)?” Previous research (e.g., Knuth, 1999; Hanna, 1995) provided me with descriptors that seemed to accurately describe participants’ concept definition of mathematical proof. These included “traditional,” “demonstration of the truth,” and “explanation.” Other categories (e.g., “justification,” “procedure,” “argument,” and “dialog”) emerged from the data. These categories and participant responses, however, were not necessarily disjoint, since, for example, a mathematical explanation results in a demonstration of truth and a traditional, deductive proof is an explanation. Therefore, participant responses were coded according to my assessment of their principal meaning. Pm-E, for example, stated that: “a mathematical proof is a series of logical deductions that demonstrates the validity of an assertion. The deductions are based on given axioms and particular hypotheses and all

occur within an agreed upon logical framework” (Pm-E 36). While this response clearly states that a proof demonstrates the validity of an assertion, the predominant intent of meaning seems to relate to the notion of proof as a series of deductions based on a logical framework, which to me indicated a traditional definition.

In general, all questionnaire items were analyzed in the manner just described. During this process, I sometimes paused to consider whether and to what degree the results provided evidence to support a previously posed hypothesis, or in some cases, to suggest new hypotheses. The analysis of questionnaire items generally led to a collection of frequency tables that summarized the analysis. Since the goal of the research was to identify similarities and differences of the perceptions between and among groups of pre-service teachers and professional mathematicians, I included this quantitative information specifically as an indication of the relative strength of a particular characteristic discovered about the participants, and as a means of comparison between and among the two groups of participants. If a large proportion of responses fell into a particular category among a group of participants, it would indicate a level of homogeneity among those participants’ perceptions. Conversely, a small proportion would suggest a lack of homogeneity. The quantitative observations were not necessarily significant in and of themselves. However, they often led to a closer inspection of the data and, in some cases, to possible explanations for the observations. For example, in one case, three-quarters of the pre-service teachers accepted an argument that was in the form of step followed by reason, while one-half of them rejected another argument that was basically in the same format. This discrepancy in proportions led me to closer inspection of the reasons for

accepting or rejecting the two arguments. Through this process, I discovered inconsistencies among the responses of four of the pre-service teachers.

CHAPTER 5

RESULTS: PROOF IN MATHEMATICS

Chapter Overview

This chapter is organized into sections corresponding to the results of pre-service teachers' and professional mathematicians' responses to questions about their perceptions of mathematical proof in the discipline of mathematics. The primary sections include participants' perceptions on: the definition or description of mathematical proof; the purpose of proof in mathematics; the validity of proof; the importance of proof; and who or what decides that an argument is a proof. Four additional sections report the results of participants' perceptions of a set of arguments purported to prove four propositions. Relevant themes that emerged from the analysis of the data are presented. Excerpts from the questionnaires and interviews that are representative or illustrative of different categories are presented along with corresponding frequency counts.

Pre-Service Teachers' and Professional Mathematicians'

Descriptions of Mathematical Proof

This section describes the results of pre-service teachers' and professional mathematicians' descriptions of mathematical proof in mathematics based primarily on their responses to the questionnaire item: "what does mathematical proof mean to you (i.e., how would you define or describe mathematical proof)?" This section contains a table of frequencies of participants' descriptions of proof in mathematics, a description of

the categories of participants' responses, a description of several recurrent characteristics that participants included in their responses, and a summary of the results.

Table 6 provides the frequencies of participants' descriptions or definitions of mathematical proof.

	Pre-service teachers	Professional mathematicians
Demonstration of truth	5/13 (38%) ^{a,b}	1/8 (12%)
Explanation of why	3/13 (23%)	
Traditional formal	2/13 (15%)	5/8 (62%)
Other	3/13 (23%)	2/8 (25%)

^a Percentages are rounded to the nearest integer.
^b Due to rounding, percentages may not sum to 100%.

Pre-Service Teachers' Descriptions of Proof

Proof as demonstration of the truth: Pst-G's response is a typical description of proof as a demonstration or verification of the truth of an assertion: "math proof is a way to show that a theorem or a claim is true" (Pst-G 1). Almost half of the pre-service teachers described proof this way.

Traditional formal definition: By traditional formal definition is meant "a finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference, and the last statement is the one to be proved" (Hanna, 1990, p. 6). Two pre-service teachers provided responses that fit this category. For example, Pst-L stated that "mathematical proof is using previous theorems and axioms to deduce a statement" (Pst-L 10).

Explanation: Three of the 13 pre-service teachers defined proof as an explanation, albeit with different levels of sophistication. For example, Pst-H described proof as "an

explanation for why a statement holds for every element in the domain” (Pst-H 22) but then goes on to describe procedural aspects of how one writes a proof in general:

Usually, the domain is large and a variable is assigned to represent any element from the domain, and using already proven statements one can deduce that the statement does or does not hold for any arbitrary member of the domain. (Pst-H 22)

Pst-K described proof as “proving mathematical statements and being able to explain steps taken and why when completing math problems” (Pst-K 33). It appears that by “explanation,” Pst-K means explaining why one can take a certain step in the proof, rather than the larger meaning of explanation of why a proposition itself is true. Pst-U’s response, on the other hand, seems to indicate that proof is an explanation of why the proposition itself is true, “mathematical proof is being able to express mathematical reasoning behind a certain theory or mathematical principle” (Pst-U 14).

Other definitions: One pre-service teacher described proof as “a way to go beyond rote memorization of formulas and methods and provide deeper understanding of whys and hows” (Pst-T 8). One pre-service teacher described proof as a means to derive new information. One described proof as a process of generalizing properties.

Professional Mathematicians’ Descriptions of Proof

Traditional formal definition: Five of the eight professional mathematicians described mathematical proof in a traditional manner as illustrated by the response of Pm-G: “[math proof] consists of a list of statements, each known to be true or following logically from previous statements in the list, ending with what is to be proved” (Pm-G 40).

Other definitions: Three other definitions were given. One described proof as a dialog between a prover and a verifier. Another defined proof as “a logically consistent

argument” (Pm-F 38), and one defined proof as an “argument demonstrating the truth of a statement” (Pm-A 34).

Other Characteristics of Participants’ Descriptions of Proof

Many of the participants included other characteristics of proof in addition to their descriptions. These are described below.

Certainty and domain of validity: In providing their descriptions of mathematical proof, four of 13 pre-service teachers and one of eight professional mathematicians mentioned the certainty of proof. Participants who indicated this aspect of proof generally used descriptors such as “always holds” or “ensure it holds.” For example, Pst-D stated that proof “is the process of generalizing mathematical properties to ensure they hold over a given domain” (Pst-D 4).

Four of 13 pre-service teachers mention that proof deals with elements in a certain domain. Usually the response includes something to the effect that a proof “holds for every element in the domain.” In some cases, it appears that the participants were pointing to the initial hypotheses of a proposition when referring to a domain of validity as in the response: “It’s a way to show that a theorem or conjecture always holds, often under certain conditions” (Pst-I 21). In other cases, proving a result over a domain of validity refers to proving a result for all objects in the set under consideration. In this sense, there is a clear reference to the idea that a proof must generalize to the entire set of objects. None of the professional mathematicians specifically gave an indication of a domain of validity.

Logic and the objects of proof: A common thread among descriptions of mathematical proof in the literature is the deductive method (e.g., Hanna, 1990; Weber,

2003). Many participants specifically mentioned characteristics of axiomatic systems such as undefined terms, axioms, definitions, and deductive reasoning. When participants' responses specifically indicated reference to: logical inference, rules of logic, logical deductions, and/or logical reasoning, this characteristic was noted as "logic." "Objects of proof" refers to participants specifically pointing out the objects that are used in proof such as: givens, axioms, definitions, known facts, assumed facts, previously established theorems, undefined terms, hypotheses, conclusions, and results. Since the majority of professional mathematicians gave a traditional definition for mathematical proof, it is not surprising that their responses would include these characteristics. The responses of pre-service teachers, on the other hand, included these characteristics in a less precise manner.

Four pre-service teachers provided a description that included the objects of proof. Most of these descriptions (3/4) are not specific about the roles served by the objects of proof as the following examples illustrate:

I would define math proof as the process of using axioms, postulates, lemmas, corollaries, and theorems to prove something will always or never hold (Pst-A 5). I don't know, it's a means to truth... in the, I don't know, in the, like, axioms or whatever (Pst-M 19).

The response of Pm-H, on the other hand, provides more prescriptive information about the place and purpose of logic and the objects of proof:

Mathematical proof is the presentation of an argument which starts from a set of hypotheses, and reaches a conclusion by means of logical inferences based on definitions and previously established results deduced from a set of axioms according to the rules of logic (Pm-H 70).

Summary

Pre-service teachers' descriptions are much more heterogeneous than those of professional mathematicians. The majority of the professional mathematicians gave a traditional definition of proof, while the majority of pre-service teacher responses fell into three categories with almost half of the pre-service teachers' describing proof as showing the result is true, three indicating that a proof is an explanation of why a result is true, and two giving a more traditional definition.

The first and most apparent difference between pre-service teachers and professional mathematicians is the inability of pre-service teachers to clearly articulate a definition of mathematical proof. Consider the following response of pre-service teacher Pst-N: "Mathematical proof is a step process from what you're given to what the final result is" (Pst-N 27). The writer provides no clear indication for what is meant by "a step process." Whereas, professional mathematician Pm-I writes that "a mathematical proof is a justification for the validity of a mathematical statement in the context of KNOWN (or assumed) mathematical facts (truths). It consists usually of a list of mathematical statements for which each statement in the list is either a known FACT or a logical consequence of other statements in the list" (Pm-I 42).

Differences between the levels of verbal sophistication and maturity in mathematics are evident from the responses. For example, pre-service teacher Pst-L gave the following definition for proof: "mathematical proof is using previous theorems and axioms to deduce a statement when one is proving something..." (Pst-L 10). While professional mathematician Pm-E gave the following definition:

A mathematical proof is a series of logical deductions that demonstrates the validity of an assertion. The deductions are based on given axioms and particular hypotheses and all occur within an agreed upon logical framework. (Pm-E 36)

Both descriptions included the logic and the objects of proof and they are very similar in their meaning. However, Pm-E's description is much more precise than Pst-L's. It should be pointed out that while professional mathematicians were generally more sophisticated in their ability to express themselves in a verbal sense, some of the responses were vague and imprecise in their meaning. For example, Pm-A defined proof as: "an argument demonstrating the truth of a statement" (Pm-A 34). This participant may have had a specific meaning in mind with the written response, however, the meaning of the response is open to interpretation.

There are several factors that may help to explain these observed differences. The first possible explanation has to do with general learning experience. As people are exposed to higher levels of learning, they will attain more advanced and sophisticated general knowledge through exposure and experience. For example, undergraduate level students have a certain level of sophistication with regard to expressing themselves verbally or in writing. As they are exposed to higher levels of learning, they themselves become more adept at self-expression through exposure to scholarly work, feedback, and correction from others (usually teachers and mentors). As they progress, there are higher expectations and standards of scholarship for students. A second possible explanation has to do with what I call the "math sieve." Beginning early, virtually all students are exposed to mathematics. As students progress through high school and onto college, relatively smaller and smaller numbers of students advance in mathematics. At the graduate level, some students continue study, but some are lost to attrition so that by the

time a student reaches the doctoral level of study in mathematics, only the most mathematically accomplished are left. The “math sieve” is the progressive “weeding-out” of lesser and lesser mathematically accomplished students. Professional mathematicians have considerably more exposure to higher levels of experience, both in terms of general education and mathematics, than undergraduate level pre-service teachers.

Pre-Service Teachers’ and Professional Mathematicians’

Perceptions on the Purpose of Mathematical Proof

This section describes the results of pre-service teachers’ and professional mathematicians’ perceptions of the purpose of mathematical proof in mathematics based primarily on their responses to the questionnaire item: “what is(are) the purpose(s) of proof in mathematics?” This section contains a table of frequencies of participants’ descriptions of the purpose of proof in mathematics, a description of the categories of participants’ responses, and a summary of the results.

	Pre-service teachers	Professional mathematicians
Establish truth	2/13 (15%) ^{a,b}	7/8 (87%) ^{a,b}
Explanation	5/13 (38%)	
Conceptual understanding	2/13 (15%)	
Convince	1/13 (8%)	
Learn logical thinking	2/13 (15%)	2/8 (25%)
Problem solving		2/8 (25%)
Foundations	3/13 (23%)	4/8 (50%)

^a Percentages are rounded to the nearest integer.

^b Since participants’ could give multiple responses , percentages may exceed 100%.

Similar to the results reported in the previous section, the professional mathematicians, as a group, are much more homogeneous in their descriptions of the purposes of proof in mathematics than the group of pre-service teachers.

Two interesting differences between the groups can be noted: only two pre-service teachers, but almost all of the professional mathematicians (7/8), cited establishing truth as one of the main purposes for proof in mathematics. Second, while explanation and understanding have somewhat different meanings among pre-service teachers, none of the professional mathematicians mentioned understanding or explanation as a purpose for proof in mathematics. The different meanings of explanation and understanding among pre-service teachers will be explained below.

Pre-service Teachers' and Professional Mathematicians' Perceptions about the Purpose of Proof

Establish truth: Two of 13 pre-service teachers stated that the purpose of proof in mathematics is to show or verify the truth of a statement. For example, Pst-L stated that “the purpose of proofs are to show the truth (or falsehood) of a statement...” (Pst-L 99). Most (7/8) professional mathematicians stated that the purpose of proof is to establish the validity of an assertion. Five of the seven referred to “truths” and “facts” indicating certainty of the results. For example, Pm-G stated that “The main purpose [of proof] is to be sure, beyond doubt, that a statement is true” (Pm-G 148). Two of the seven indicated that proof may not be absolutely certain, but that it “allow[s] one to have as much confidence as possible about the validity of the assertions” (Pm-F 147). These two responses differ from the notion of absolute certainty that is sometimes associated with mathematical proof and reflect the idea of fallibility of results (e.g., Lakatos, 1976).

Foundations of mathematical knowledge: Three pre-service teachers gave an indication that a purpose for proof is foundational, that is, to provide a solid foundation of truths in the discipline:

Its purpose is to show the reasoning and truths that make up new pieces in the field (Pst-M 122).

The purpose of proof in math is to insure that people/mathematicians don't assume false truths (Pst-A 96).

Proof is designed to build solid truths w/in science (Pst-M 121).

Four of eight professional mathematicians indicated that the purpose of proof is to advance a verifiable body of mathematical knowledge from established facts. For example: "Once established, this fact takes its place among all those that came before it and can be used to prove still more facts and advance the total body of mathematical knowledge" (Pm-E 146).

Explain how and why: Five pre-service teachers indicated that the purpose of proof is to explain. As previously mentioned, there are different meanings ascribed to explanation. For example, two of the five pre-service teachers indicated that proof explains why a claim is true in the global sense. Three pre-service teachers gave more restricted descriptions indicating that the purpose is to explain why each step of a proof is warranted. Pst-N, for example, stated "...because from going to each step, you need to go back and find why it's OK to do that step" (Pst-N 130). The focus is on explaining how one proceeds from the premise to the conclusion rather than on an understanding of the underlying mathematical relationships (Knuth, 2002a). For example, being able to follow the algebraic steps of the derivation of the quadratic formula provides understanding as to why it is true, but the understanding may only be procedural.

To convince: One pre-service teacher indicated that the purpose of proof in mathematics is to convince (or be convinced) of the truth of a claim:

The purpose of proof in math is to understand each step from hypothesis to conclusion in order [to] believe the conclusion is true (Pst-G 80).

Conceptual understanding: Two pre-service teachers indicated that proof helps to understand mathematical concepts: “Proof is also an aid that helps individuals better understand mathematical concepts and how they function” (Pst-U 120) and Pst-K stated the purpose for proof is “to increase understanding of mathematical concepts...” (Pst-K 137). Note again that the responses appear to be somewhat vague in their meaning. It is not clear by the responses how proof helps with conceptual understanding nor is it clear what features of the mathematical concepts proof helps.

Other purposes: Two pre-service teachers stated that one purpose for proof is to learn logical thinking skills; one indicated that the purpose is to “generalize mathematical properties, allow testing of properties over generalized domains, [and] give sound arguments to allow no incorrect interpretations” (Pst-D 92). One pre-service teacher stated that “the purpose of proofs in mathematics is to derive formulas that aid in solving mathematics” (Pst-P 135). Pst-I indicated that the purpose of proof is to provide connections to other topics in a structural sense “to see where a conjecture is derived from [and] how it is interrelated to math topics previously learned” (Pst-I 125). However, this response appears to be pedagogical in nature rather than indicating mathematical foundations. This conclusion is based on several observations. First, Pst-I used the term “learned” rather than a less didactic term such as “established.” In response to a different questionnaire item, Pst-I specifically indicated that informal proof can be used “to explain the validity of a conjecture briefly to a student” (Pst-I 489). Finally, Pst-I’s used the phrase “previously learned” in response to another item that was specifically centered on an educational setting.

Two professional mathematicians indicated that proof serves as a method of investigating and solving problems. Two of the eight professional mathematicians also indicated the additional purpose of proof as a means for training people to think logically.

Summary

The homogeneity of professional mathematicians' descriptions of proof and their perceptions of the purpose of proof in mathematics suggests that there are certain common aspects of proof that over time become a prominent component of the concept image for this group of practitioners. This, coupled with the relative diversity of responses from pre-service teachers, suggests that there is a process of enculturation into the canons of proof in mathematics that takes place over time with exposure to mathematics and the community of mathematicians. Interestingly, it also appears that some beliefs held by professional mathematicians seem to be largely theoretical. By way of example, the majority of professional mathematicians gave a traditional formal definition of mathematical proof. It is evident from many of the responses of the professional mathematicians, however, that the formal definition they gave is "ideal" and that in practice, most mathematicians settle for a degree of formality. For example, Pm-I stated that "almost all of mathematics is done informally" (Pm-I 519) and Pm-G wrote that "the 'formality' varies, depending on the situation – classwork, published paper, etc. – but one almost never uses a completely formal proof" (Pm-G 518). So while professional mathematicians may outwardly profess one description of proof, it appears that they inwardly (or in practice) believe another. This particular conclusion provides substance to the (possibly) anecdotal observation that "most mathematicians hold contradictory views on the nature of their work" (Hersh, 1998, p. 40). The difference

between these publicly stated beliefs and what they inwardly believe about proof in practice may play a significant role in the formation of pre-service teachers' concept image of proof in mathematics.

Pre-Service Teachers' and Professional Mathematicians'

Perceptions on the Importance of Mathematical Proof

This section describes the results of pre-service teachers' and professional mathematicians' perceptions of the importance of mathematical proof in mathematics based primarily on their responses to the questionnaire item: "how important is mathematical proof to mathematics?" This section contains a table of frequencies of participants' descriptions for why proof in mathematics is important and a summary of the results. Virtually all participants indicated that proof is very important for mathematics. Their reasons for why they believe that proof is important varied, especially among pre-service teachers. Table 8 gives the frequencies of participants' descriptions for why proof is important in mathematics. Recall that "foundations" refers to the notion that

	Pre-service teachers	Professional mathematicians
Foundations	6/13 (46%) ^{a, b}	2/8 (25%) ^{a, b}
Essential/inseparable		5/8 (62%)
Conceptual understanding	5/13 (38%)	
Certainty of results	1/13 (8%)	1/8 (12%)
Connections	1/13 (8%)	

^a Percentages are rounded down to the nearest integer.
^b Due to rounding, percentages may not sum to 100%.

proof provides a solid foundation of truths for the discipline. Responses in the "essential/inseparable" category conveyed the idea that proof is an essential and

inseparable part of mathematics. “Conceptual understanding” refers to responses that conveyed the idea of proof as providing understanding of the concepts involved.

Pre-Service Teachers’ Perceptions on the Importance of Proof

Proof is important to the foundations of mathematics: Almost half (6/13) of the pre-service teachers indicated that proof is important to the foundations of mathematics, that is, proof provides a solid foundation of “truths” from which to begin. For example, Pst-I stated that “if mathematicians didn’t prove anything we would be doing math based on a series of assumptions [and] math wouldn’t have the solid validity that it does have [with] proofs” (Pst-I 188). And Pst-G indicated that “we assume basic math facts like $2 + 2 = 4$ [*sic*], but if we couldn’t prove it was true, it’s meaning and application in life would be irrelevant” (Pst-G 163). Pst-D also stated that proof is very important to mathematics since “proof allows us to prove that properties hold from a very sound starting point” (Pst-D 165). Pst-D’s response illustrates another phenomenon that was often observed among pre-service teachers. In their descriptions of, or their stated purposes for proof, pre-service teachers often used the word “prove” or “proof” in a circular manner for their descriptions.

Proof is important for conceptual understanding: Three of 13 pre-service teachers wrote that proof is important in mathematics because it provides a deeper understanding of mathematics, that is, proof provides explanation of “how” and “why” beyond simply following procedures. For example, Pst-T stated:

Proof is important in providing a deeper understanding of mathematics.... instead of just formulas, equations, it sort of like, it goes behind it, it sort of gives you an understanding of like how, why. (Pst-T 172)

Two other pre-service teachers indicated that proof is an important part of understanding, although their descriptions involved elements of convincing oneself or being convinced. For example, Pst-K stated that proof is “very important since true understanding of concepts is only obtained when you are able to prove it to other people” (Pst-K 206). And Pst-P stated that “proofs are used to have greater understanding of another’s work” (Pst-P 203). These responses suggest that communicating mathematical ideas through proof helps with understanding.

Certainty of results: One pre-service teacher indicated that proof is important to mathematics because it provides certainty in the truth of mathematical statements.

Connections: One pre-service teacher stated that proof is important to mathematics “because it is a way of connecting every part of mathematics to other parts” (Pst-J 205).

Professional Mathematicians’ Perceptions on the Importance of Proof

Proof is essential to mathematics: The majority of professional mathematicians (5/8) indicated that proof is an essential and inseparable part of mathematics. For example, Pm-C stated: “... one might deem the purpose of proof mathematics itself, since one has no mathematics without it” (Pm-C 158). And Pm-I stated that “[proof] is the essence of doing mathematics. Any activity in mathematics can (and should) be viewed as doing mathematics, and therefore, is the activity of either doing research to discover a proof or writing a proof” (Pm-I 221).

Proof is important to the foundations of mathematics: Two of eight professional mathematicians indicated that proof is important for the foundations of mathematics. For

example: “as a discipline, proof is foundational; without proof, there seems to be no math” (Pm-B 210).

Importance of proof depends on the context: Two professional mathematicians pointed out that the importance of proof in mathematics depends on the context. In principle, proof in mathematics is essential, but in practice, it can become less important. For example, Pm-B stated that “if one means (by “mathematics”) using math or even teaching math, certain elements of proof become less important” (Pm-B 211). And Pm-E stated that “in practice, most people don’t need or want to know why the Pythagorean [theorem] is true any more than I need or want to know why my house doesn’t fall down” (Pm-E 215).

Proof is important for the certainty of results: One professional mathematician indicated that proof is important to mathematics because it provides certainty in the truth of mathematical statements: “Many, many statements may seem true, or even be thought true. But without proof, one can never be sure” (Pm-G 219).

Summary

Professional mathematicians seemed to view doing mathematics and proving as essentially the same activities. Most pre-service teachers indicated that proof is very important to the discipline but for different reasons. Some pre-service teachers appeared to view proof as a separate part of mathematics used to establish a firm foundation of truths after which one is “allowed” to subsequently apply these truths. Pst-I, for example, stated that “if mathematicians didn’t prove anything we would be doing math based on a series of assumptions [and] math wouldn’t have the solid validity that it does have [with] proofs” (Pst-I 188). Pst-L stated that “without proof we would be doing blind math

without any idea of if it were true or not” (Pst-L 179). Finally, Pst-G stated that “we assume basic math facts like $2 + 2 = 4$ [*sic*], but if we couldn’t prove it was true, it’s meaning and application in life would be irrelevant” (Pst-G 163). These responses all suggest the idea that mathematics can still be done without proof (although perhaps not in any meaningful sense).

Other pre-service teachers indicated that proof is important in providing understanding behind mathematical formulas and concepts. On the other hand, none of the professional mathematicians specifically mentioned understanding in connection with mathematical proof.⁵ Unlike the pre-service teachers, professional mathematicians acknowledged that the importance of proof may also depend on the context. Interestingly, only one professional mathematician indicated that mathematical proof provides certainty of the result. This is in contrast to the majority of professional mathematicians who cited establishing the truth of conjectures or claims as the primary purpose of proof in mathematics.

Pre-Service Teachers’ and Professional Mathematicians’

Perceptions on Validity in Mathematical Proof

This section describes the results of pre-service teachers’ and professional mathematicians’ perceptions of the validity of mathematical proof in mathematics based primarily on their responses to the questionnaire item: “can a mathematical proof ever become invalid?” No specific meaning for the term “invalid” was given to participants. The results below include observations about differences in the participants’ meaning or interpretation for the term “invalid.”

⁵ Although, in one case it can be implied: Pm-E stated that “in practice, most people don’t need or want to know why the Pythagorean [theorem] is true” (Pm-E 215). This suggests that a proof of the Pythagorean theorem can provide deeper understanding of why it is true.

The section contains a table of frequencies of participants' responses as to whether they think a mathematical proof can become invalid. This is followed by a description of participants' reasons for why a proof can or cannot become invalid. Four pre-service teachers' responses indicated uncertainty as to whether they thought a proof can become invalid. However, while these pre-service teachers expressed doubt about the validity of proof, most of their responses suggested that they believe that a proof cannot become invalid. The numbers in brackets in table 9 reflect this different categorization. The reorganization of the frequencies in table 9 was done to point out that roughly half of the pre-service teachers believe that a proof can become invalid while only one-quarter of the professional mathematicians indicated this.

Table 9

Can a mathematical proof become invalid?

	Pre-service teachers		Professional mathematicians
Yes	6/13 (46%) ^{a, b}	[7/13 (54%) ^a]	2/8 (25%)
No	3/13 (23%)	[6/13 (46%)]	6/8 (75%)
Not sure	4/13 (31%)	[0/13 (0%)]	

^a Percentages are rounded to the nearest integer.

^b Due to rounding, percentages may not sum to 100%.

Pre-Service Teachers' Perceptions on Validity in Mathematical Proof

A mathematical proof cannot become invalid: Three pre-service teachers indicated that a proof cannot become invalid because a proof is based on previously proven theorems or assumed truths.

If a proof is done correctly then it is based on other math ideas that have been proven, therefore a proof cannot become invalid. (Pst-G 300)

No. Once proven, they are forever. They can't be disproven because they are based on truths. (Pst-M 308)

No, because a proof should work for all answers or it would not be a proof. (Pst-N 314)

A mathematical proof can become invalid: Six of 13 pre-service teachers and one professional mathematician stated that a proof can become invalid. Three pre-service teachers stated that a proof can become invalid if it is based on false claims. The following responses illustrate this:

A math proof can only become invalid if the facts used to prove the statement are proved to be invalid. (Pst-A 302)

Yes, if it were based on a proposition that was proved false. (Pst-I 310)

It can [become invalid] if one uses assumptions that are not proved to be true. (Pst-J 317)

Several pre-service teachers indicated that a proof can become invalid if an error in the proof or a counterexample to the statement is found. For example, Pst-L stated that "...a proof can become invalid if someone analyzes it and finds there is a flaw in the proof. Someone can also find a counterexample that defies the statement of the proof" (Pst-L 305). Other pre-service teachers indicated that a proof can become invalid if a necessary step of the proof is skipped or the proof is not set up correctly. For example, Pst-P wrote that:

A proof can become invalid if the person proving skips a necessary step or forgets to declare a portion of the proof. Although the writer might feel he/she has done it correctly it may not be accurate. (Pst-P 315)

A mathematical proof can maybe become invalid: Four pre-service teachers expressed doubt as to whether they believe a proof can become invalid. Their responses are given below:

If the axioms are true and the previously proved theorems are not disproven, then I do not believe proof[s] could ever become invalid. (Pst-D 301)

A proof can be wrong. I'm not sure if a proof can ever become invalid. (Pst-T 303)

I think the only way a proof can ever become invalid is if it never was a proof. Maybe there was some illegal step that was missed and people thought it was a proof and then saw the error. For a proof to become invalid, if the proof was based on other proven theorems, I would think that something else would have to change, like an undefined term was given a new meaning or an axiom was changed. (Pst-H 311)

Yes if a counterexample is obtained, but that would make the proof invalid to begin with. (Pst-K 319)

Professional Mathematicians' Perceptions on Validity in Mathematical Proof

A mathematical proof cannot become invalid: Most (6/8) of professional mathematicians indicated that a proof cannot become invalid because a proof is based on previously proven theorems or assumed truths. It is possible that a purported proof might contain an error, in which case it is not a proof to begin with. For example:

No. It can be found to have had errors, of course. (Pm-G 335)

A proof is a proof or it is not a proof, in which case it has an error. (Pm-I 338)

No. If an error is found centuries later it only means that we have been calling something a "proof" that never was. (Pm-E 328)

A mathematical proof can become invalid: Two professional mathematicians stated that a proof can become invalid. The reason had to do with the consistency of the system and recognizing the possibility of fallibility in mathematics. For example, Pm-H indicated that "if we would ever find the axiom of choice inconsistent with the set of Zermelo – Frankel axioms many mathematical proofs will become invalid" (Pm-H 377).

A mathematical proof can maybe become invalid: One of the professional mathematician's responses was originally put into the category "cannot become invalid" but upon further reflection, it became evident that the response could also be put into the category "can become invalid." I include this participant's complete response to the

question to illustrate some of the deeper philosophical difficulties associated with this seemingly simple question:

No, [a proof cannot become invalid]. But we can try to prove things and make mistakes. We might go about calling our mistake a “proof” for some time. It is not that the proof became invalid – rather it never was a proof at all. Of course, this is something of a semantic question. And, if the axioms turn out to be contradictory then all bets are off. Although in that case “proof” is suddenly not well defined (is it well defined now?) so that asking if a proof became invalid doesn’t seem to make sense. (Pm-B 321)

Several other professional mathematicians indicated similar difficulties in attempting to provide an adequate answer to the question.

Summary

To professional mathematicians, validity in proof has specific meaning: an argument is either valid and therefore is a proof, or it is not valid and hence is not a proof. To the pre-service teachers, the meaning of validity in proof is much less clear. For some pre-service teachers, the validity of a claim hinges on the truth of the assumptions on which the claim is based. Additionally, pre-service teachers seem to misunderstand the nature and role of axioms, definitions, and hypotheses in proof. For example, Pst-J stated that a proof can become invalid “if one uses assumptions that are not proved to be true” (Pst-J 317). This response suggests that initial assumptions (including axioms) can be proved (or disproved). This instance indicates a lack of understanding about what it means to make assumptions and therefore a lack of understanding of axiomatic-deductive systems.

Several of the professional mathematicians expressed uncertainty about the validity of a proof with respect to consistency of the system. They seem to recognize, at least to some degree, the possibility of change in mathematics, for example, if an axiom

is found to be inconsistent (c.f., Lakatos, 1976; Ernest, 1998). Whereas the uncertainty of the pre-service teachers is more closely associated with a lack of knowledge and understanding with respect to what constitutes mathematical proof. Several professional mathematicians also indicated that the notion of proof is not necessarily as simple as we might make it out to be. For example, after a lengthy discourse on difficulties with the term “validity” with respect to axiomatic systems, changing standards of rigor, and questions of relative and absolute truth, Pm-C concludes that “the notion of validity seems internal to the dynamic between prover and verifier. If I could find a timeless sense of valid, a given text would remain valid or not for all time. But I don't see any way to ground such an over arching notion of validity” (Pm-C 374).

There is also evidence that the understanding of validity of proof is fragile among the pre-service teachers. Many pre-service teacher responses suggested that they connect the validity of a proof with correctness of a proof. That is, they believe that a proof can become invalid if an error is found. Many of the participants stated that an argument can be “better” than others (for various reasons), but several of the pre-service teachers (4) indicated that an argument can be more valid than another. For example, Pst-H stated “I would say argument 1 is the most valid.... Argument 2 is also valid but the steps are not clearly written out and it is harder to follow” (Pst-H 649). This and other responses suggests that pre-service teachers are simply confusing “validity” with other characteristics of the argument such as, in this case, whether it is clear and easy to follow.

Pre-Service Teachers' and Professional Mathematicians'

Perceptions About Who Decides Whether an Argument Is a Mathematical Proof

This section describes the results of pre-service teachers' and professional mathematicians' perceptions about who decides whether an argument is a mathematical proof based primarily on their responses to the questionnaire item: "who decides whether an argument is a mathematical proof?" This section contains a table of frequencies of participants' descriptions about who decides whether an argument is a mathematical proof, a discussion of major points pertaining to participants' responses, and a summary of the results.

	Pre-service teachers	Professional mathematicians
External authority	9/13 (69%) ^{a, b}	
Mathematical community	3/13 (23%)	5/8 (62%) ^a
The reader	2/13 (15%)	3/8 (37%)
I don't know	2/13 (15%)	

^a Percentages are rounded to the nearest integer.

^b Since participants' could give multiple responses , percentages may exceed 100%.

Pre-Service Teachers' Perceptions About Who Decides Whether an Argument is a Mathematical Proof

I don't know: Two of 13 pre-service teachers indicated that the decision about whether an argument is a proof depends on the situation. In a learning situation, it would be the teacher or professor. In other situations, such as in professional mathematics, these two expressed ambiguity and uncertainty. Pst-H, for example, stated that "if you are proving a statement that has not yet been proved, I guess there is probably a group of professors somewhere that get to decide, but I don't really know" (Pst-H 572). One other

expressed similar ambiguity and uncertainty: “I don't know if anyone can decide if something's a mathematical proof” (Pst-U 559).

The teacher, professor, or some other external authority: Nine of 13 pre-service teachers indicated that the decision as to whether an argument is a proof is up to an external authority such as a teacher, or conformity to the rules. Five of the nine specifically indicated that a teacher or professor makes the decision. For example, Pst-T stated: “deciding whether or not an argument is a proof would ultimately be up to the teacher” (Pst-T 549). Two of the nine indicated that a mathematician makes the decision. For example, Pst-P stated: “I feel a mathematician or someone with a great deal of mathematical background should decide whether an argument is a mathematical proof” (Pst-P 587). One of nine pre-service teachers indicated that the system of rules and axioms itself decides, and another indicated that “an argument is a proof if it fits the description of what a math proof is” (Pst-J 588). There is an implied “right/wrong” judgmental impression in many of these responses. That is, these participants have to some extent come to rely on external authority for the determination of whether an argument is valid. For example, Pst-H stated:

If it is on an exam, then the teacher or professor says if it is acceptable or not. If you are proving a statement that has not yet been proved, I guess there is probably a group of professors somewhere that get to decide, but I don't really know. (Pst-H 572)

Not only is there the belief that some authority deliberately decides whether an argument is a proof or not, but in addition, Pst-H doesn't seem to know what happens outside of the school setting. This was a common theme among pre-service teachers. Hanna (1995) believes that proof is a:

... transparent argument, in which all the information used and all the rules of reasoning are clearly displayed and open to criticism. It is the very nature of proof that the validity of the conclusion flows from the proof itself, not from any external authority. Proof conveys to the students the message that they can reason for themselves, that they do not need to bow down to authority. (p. 46)

It appears from the responses of pre-service teachers that they have not achieved this level of authority with regard to proof.

The mathematical community: Three pre-service teachers indicated that it is the mathematical community who decides whether an argument is a proof. However, some responses indicated that the pre-service teachers are not completely certain about who the mathematical community is or what happens in mathematics outside of the classroom setting. Pst-H's response above illustrates this. Pst-A expressed similar uncertainty as well:

In general I don't think a math proof become[s] valid until it is somehow published and mathematicians all over the world have a chance to agree or disagree with the proof. (Pst-A 547)

The reader: Two pre-service teachers indicated that it is the reader who decides whether an argument is a proof. Proof in this context refers to an argument that convinces the reader.

Professional Mathematicians' Perceptions About Who Decides Whether an Argument Is a Mathematical Proof

The mathematical community: Six of eight professional mathematicians indicated that it is the mathematical community who ultimately decides whether to accept an argument as a proof. Pm-F, for example wrote that "it is the mathematical community that decides if a purported proof meets the requisite standards" (Pm-F 598). Pm-B responded by stating that "the math community decides whether an argument (1)

Contains the proper details to describe a string of reason leading to the conclusion; (2) The conclusion actually follows from the reasoning” (Pm-B 591).

The reader: One professional mathematician indicated that the reader decides. Two indicated that both the reader and writer of the proof decide whether to accept an argument as proof. Pm-G, for example stated that “the writer may claim to have a proof, but if the reader doesn’t accept a part, because, for example, the writer has asserted that something “is clear” or “follows easily from…” then he may not accept the proof. This sort of disagreement can usually be settled by having the proof-writer clarify the proof” (Pm-G 600).

Summary

More than half of the pre-service teachers indicated that an argument becomes a proof when an external authority of some kind decides that it is a proof. Professional mathematicians tended to view the acceptance of an argument as proof in a more flexible manner with a shared and social component attached. For example, Pm-G described the decision about an argument’s status as a process rather than a static choice. In this instance, proof is a social process consisting of negotiating meaning, rather than the application of formal criteria (Lakatos, 1976).

Among pre-service teachers, there is the impression of acceptance in an absolute sense: that a deliberate “right or wrong” decision is made by an external authority, whereas among professional mathematicians, the impression is less absolute and more akin to a debate among experts. It also appears that professional mathematicians count themselves as belonging to the mathematical community, whereas pre-service teachers seemed less inclined to consider themselves as part of the mathematical community. In

practice, professional mathematicians present their arguments to the community of their peers for validation (Thurston, 1995; Rav, 1999). For most pre-service teachers, it appears that their experience has been confined mostly to submitting arguments to an authority, not so much for validation as for judgment. The uncertainty of what constitutes proof and who decides what is a proof for pre-service teachers can be illustrated by the response of Pst-U:

... proofs are so ambiguous, because, they're up for interpretation all the time... I would say... that... I don't want to say a textbook or anything like that, because... they've got all of their own right answers. But, I mean, I still kind of think I lean towards this in that every individual is going to decide whether an argument is a proof, because it's their understanding. ... I don't know if anyone can decide if something's a mathematical proof. I mean, you've got... you know a teacher could give out a test, and determine whether that's a proof or not. But that's in just one subset of the entire mathematics, I mean, in another light you could say, well, only super genius mathematicians can verify that, yes, this is a valid math proof. (Pst-U 557)

Pre-Service Teachers' and Professional Mathematicians' Perceptions of a Calculus Proof

This section provides the results of the participants' evaluation of a proposition from calculus ("the derivative of an even function is odd") and three arguments (see Appendix C, section I, nos. 9 – 11) purported to prove the proposition. The first part of the section provides two frequency tables. The second part of the section provides results pertaining to the participants' reasons for why they considered an argument to be valid or not valid, and why they liked or disliked the arguments.

Participant Responses: Is the Argument a Valid Proof?

The following table provides the frequencies of participants' responses about whether they considered the arguments provided to be valid proofs or not.

Table 11

Participant assessment of the validity of the calculus arguments

	Pre-service teachers	Professional mathematicians
Argument 1 valid	5/12 ^a (41%) ^{b, c}	3/8 (37%) ^{b, c}
Argument 2 valid	12/13 (92%)	7/8 (87%)
Argument 3 valid	2/13 (15%)	8/8 (100%)
Argument 1 not valid	7/12 ^a (58%) ^{b, c}	2/8 (25%) ^{b, c}
Argument 2 not valid	1/13 (8%)	0/8 (0%)
Argument 3 not valid	11/13 (85%)	0/8 (0%)

^a One respondent did not comment on argument 1.^b Percentages are rounded to the nearest integer.^c Since participants' could have considered more than one argument as valid, percentages may exceed 100%.*Participant Responses: Which Argument Do You Prefer?*

The following table provides the frequency of participants' responses about which of the arguments they preferred.

Table 12

Participant preference for the calculus arguments

	Pre-service teachers	Professional mathematicians
Prefer argument 1	3/12 ^a (25%) ^{b, c}	1/8 (12%) ^{b, c}
Prefer argument 2	8/12 (67%)	1/8 (12%)
Prefer argument 3	1/12 (8%)	6/8 (62%)

^a One respondent did not have a clear preference for the three arguments.^b Percentages are rounded to the nearest integer.^c Due to rounding, percentages may not sum to 100%.

One pre-service teacher and five professional mathematicians indicated that their preference for an argument would depend to some extent on the situation, for example, what theorems might have been previously established or the mathematical level of the audience involved. Pst-D stated: "I prefer [two] for its technicality and ability to be understood by a calculus student. I prefer [three] for its ingenious[ness] and ease of understanding by those who have a calculus background" (Pst-D 741) (Pst-D's responses were not included in the tally because there was no single preference indicated). Pm-I

wrote: “there are situations when [number two] or [number three] would be convincing. [Number three] depends on knowing the chain rule. And [number two] assumes a ‘conventional definition’” (Pm-I 676). Pm-G stated that: “they involve different levels of detail. [Number one] uses some implied (but not stated) theorems about slopes of lines which are mirror images. [Number three] uses the chain-rule, which, presumably, must be proved separately. [Number two] is the least dependent on other material” (Pm-G 665).

Pre-Service Teachers’ Explanations About the Validity of and Their Preference for Three Calculus Arguments

Items nine and ten of the questionnaire asked participants to indicate which of the three calculus arguments they considered to be valid and invalid mathematical proofs. Each item also asked the participants to provide explanation. Some responses did not clearly indicate the participants’ perceptions of validity regarding the arguments but did indicate their preference for one or more of the arguments. For example, Pst-G stated: “I believe that argument [number two] is the best argument. Using the definition of $f(x)$ provides a better explanation or basis for the proof” (Pst-G 620). In some instances, participants’ reasons for indicating that an argument was valid and their reasons for preferring an argument were the same as illustrated by two responses given by Pst-T. Regarding which of the arguments he considered to be valid, Pst-T stated: “Argument [two] seems the most valid. Argument [two] uses definitions and step by step reasoning to support its claims” (Pst-T 637). Regarding which of the arguments he preferred, Pst-T wrote: “I prefer argument [two]. Argument [two] provides step by step arithmetic and algebraic work as well as relevant reasoning” (Pst-T 745). Because of the overlap, results concerning participants’ reasons for preferring one argument over another and their

reasons for judging an argument as valid or invalid will not be reported separately. Where possible and when germane to the discussion, any differences between participants' responses that indicate their judgment of validity versus their preference will be highlighted.

State the goal: Three pre-service teachers indicated that a proof must initially explicitly explain what needs to be shown. Pst-A's response illustrates this idea:

So I think what I'm saying there is, I wanted to say, for maybe at first to define what an even function or an odd function [is]. So if they say... but that's a little strict because they said ... which means it's odd, so... maybe it is better than I thought. But to me, I mean, that would be pretty strict, but I'd want them to say "we must show that a function is odd if ..." etc. etc.... (Pst-A 627) ... a proof must explain what it is we're trying to show. (Pst-A 703)

State theorems: Three pre-service teachers indicated that argument three doesn't explicitly state or explain the chain rule. For example, Pst-L stated: "The third argument references the chain rule but does not say what the chain rule is" (Pst-L 706). For this reason, these pre-service teachers indicated that argument three is not valid.

Steps and reasons: Ten of 13 pre-service teachers indicated that steps and the reasons for those steps are important in determining the validity of an argument. For example, steps perceived as missing or steps given without detailed explanation were given as reasons that an argument is not a valid proof:

Honestly I do not remember much from calculus but I shouldn't have to do the math out myself. The proof should be clear and show each step, leaving the reader confident that this is true. (Pst-H 713)

Pst-J stated that:

Both [arguments] 1 and 3 do not explain at the beginning the goal, nor does either give reasons for each step. They give a general reason but omit the individual steps. (Pst-J 717)

Pst-J's response of giving a "general reason" suggests the notion of the "key idea" of the proof (Raman, 2002), that is, the primary understanding for why the proposition is true but because there are steps and reasons perceived as missing, the proof is not valid. Pst-I stated that she thought argument two was valid because it "has a proposed theorem [with] steps [and] reasons to prove the theorem" (Pst-I 646).

Two Pre-service teachers specifically indicated their preference for argument two was because it was the most "proof-like," suggesting that for them form is an important factor in assessing validity. The notion of what form is most "proof-like" will be discussed more in subsequent sections.

Easy to follow: Five pre-service teachers pointed out that they preferred a particular argument because it was easy for them to follow. Four of the five pre-service teachers noted their preference for argument two because of the statement and reason format. Pst-P, for example, stated a preference for argument two because "I like the work that is done out. I feel it is a more concrete proof..." (Pst-P 784). Pst-A indicated that argument three was easier to follow in the sense that it conveys the same meaning as the other arguments but was shorter.

Don't understand: Six pre-service teachers pointed out that they did not understand all or part of some of the arguments, did not know the definitions of odd and even functions, and/or did not understand how or why the conclusion follows from the argument. For example referring to argument one, Pst-G stated that "maybe I just didn't understand it as well as I did the idea behind arguments two and three" (Pst-G 738). Pst-N stated that she did not know "why the steps showed that the function was odd" (Pst-K 719). Pst-A did not consider any of the arguments to be valid because the definitions of

odd and even functions were not stated: “I feel like I’m missing information in all of the statements about what makes a function even or odd” (Pst-A 626). Some of the participants chose not to judge the validity of an argument if they did not understand all or part of it (e.g., Pst-G) while others (e.g., Pst-N, Pst-K) judged the argument as not valid because there were parts of it they did not understand or were unable to follow.

Writer understands: Three pre-service teachers pointed out that their judgment of the validity for a particular argument depends on whether the argument shows that the writer of the proof has an understanding of the concepts involved. For example, Pst-T indicated that argument three is not valid because it shows only a procedural understanding of the concepts: “[Argument three] just provides the method and result. It does not provide any reasoning to support its claim” (Pst-T 704). The acceptance of an argument as a valid proof also depends on the writer’s ability to express conceptual understanding. For example, Pst-K wrote: “[Argument one] showed an understanding and process that went beyond simply applying the formula” (Pst-K 654). Regarding his preference for the arguments, Pst-U stated:

I prefer [argument one] because it represents a better understanding of the subject. It shows that mathematical principles of linearity and slope have built upon each other to formulate the student’s answer. It shows he/she has a good math foundation. (Pst-U 748)

Pst-U’s response also suggests that the “key idea” of a proof is an important factor in terms of the value of the proof itself in addition to a potential tool for assessment of student understanding.

Picture: Three pre-service teachers indicated that the diagram in argument one helped them to follow and understand the argument. For example, Pst-K stated that “the picture is a very good visual representation that explains the ideas very well and has the

supporting written backing to explain the picture” (Pst-K 787). Many pre-service teachers pointed out that a picture is not a proof. For example, Pst-N stated that “I also learned in geometry that a picture isn't supposed to be considered a proof” (Pst-N 775).

Short, elegant, and other descriptive words: Two pre-service teachers indicated they liked the third argument because it is short. For example, Pst-M observed: “I guess I like it. It’s right to the point” (Pst-M 753). One other pre-service teacher indicated that that the third argument is “ingenious” (Pst-G 742).

Professional Mathematicians’ Explanations About the Validity of and Their Preference for Three Calculus Arguments

Most of the professional mathematicians (6/8) but none of the pre-service teachers indicated that to prove anything about the derivative of a function, the function under consideration must first be assumed to be differentiable. Without this assumption, these six wrote, none of the arguments are valid. This was one of the primary reasons professional mathematicians cited for judging the arguments valid or not valid. Most other comments about the three questions (items 9 through 11 of the questionnaire) posed to the professional mathematicians about the arguments for the calculus proposition seemed to relate to the professional mathematicians’ preference for the arguments and were based on qualitative factors such as brevity or elegance, and contextual factors such as knowing what theorems had been previously established and the purpose for using the proof.

Short, elegant, and other descriptive words: Five of eight professional mathematicians indicated a preference for argument three because it is the “shortest” or “most elegant.” Other adjectives included: cleanest, concise, quickest, simplest, and

slickest. On the other end of the spectrum, one professional mathematician did not like argument two because it is “ugly.”

Picture doesn't tell whole story: Two professional mathematicians indicated that a picture can be deceiving: “[Argument one] relies on a picture, which conceivably does not tell the whole story, so it is not a valid proof” (Pm-F 723). This response differs from pre-service teacher responses (e.g., Pst-N) that indicated simply that “a picture is not a proof” without explanation as to why.

Not convincing: Two professional mathematicians indicated that they were not convinced by a particular argument, but did not specifically indicate whether they considered the arguments valid or not valid. They were not convinced because the argument in some way didn't provide the deductive reasoning necessary to prove the claim. Pm-B, for example stated that:

Numbers 2 and 3 are proofs (given the hypothesis that f is differentiable - which perhaps can be understood else f' is nonsensical). Using established, well understood theorems, they demonstrate the veracity of the given statement (in such a way that no alternative interpretation is possible). Number 1 makes me a little nervous. The reasoning is correct (as is demonstrated by arguments 2 and 3) but I am not able to as easily disregard other possibilities given the explicit reasoning given (Pm-B 658).

Content and context: Many professional mathematicians pointed to issues related to the conceptual content of the arguments and the circumstances for which the arguments are to be applied. Recall that when asked for their judgment about the validity of the three arguments, six professional mathematicians pointed out that to prove anything about a function and its derivative, the function must first be assumed to be differentiable. Two others judged arguments as not valid: both Pm-A and Pm-F indicated that argument one was not valid. Only Pm-F provided explanation as to why he judged

the argument as not valid. Interestingly, Pm-F stated that “even though I don’t consider the first [argument to be] a proof it gives me a good intuitive idea of what is going on” (Pm-F 796).

It is quite possible that pre-service teachers did not mention differentiability of the function because they have most likely dealt almost exclusively with differentiable functions (Moore, 1994; Dreyfus, 1999). Three professional mathematicians specifically noted that the third argument relies on the truth of the chain rule, which may or may not have been previously proven. Three professional mathematicians indicated that the context for the proof needs consideration. For example, the domain of the function must be considered and what prior information has been established. Two professional mathematicians expressed that argument two was good because it relies only on definitions and symbol manipulation and requires no other theorems, unlike arguments one and three.

Summary

The majority of pre-service teachers indicated that a proof must explicitly state initial goals, including clearly stating important definitions, relevant theorems and/or prior results should be clearly stated, all steps of the argument should be included, and for each step reasons should be given. There is a clear preference by pre-service teachers for the second argument. This argument most closely fits the description that most pre-service teachers seem to favor (statement followed by reason). The ability of the reader to follow and understand the steps of an argument was also a significant factor in determining validity for pre-service teachers.

Professional mathematician had a clear preference for the third argument primarily because it was “elegant” or in some sense clever. The ability of the reader to follow and/or understand the arguments did not appear to be an issue for the professional mathematicians. Of concern, however, were content issues such as: whether the function is differentiable and where the function is defined. In addition, professional mathematicians noted that issues of context, such as, what information has been previously established for the arguments, plays a role in determining preference. Only one pre-service teacher indicated that context is a factor.

Pre-Service Teachers’ and Professional Mathematicians’ Perceptions of a Trigonometric

Identity Verification

This section provides the results of the participants’ evaluation of a verification of the trigonometric identity: $\sec^2 \theta - \tan^2 \theta = 1$ (see Appendix C, section I, item 12). The first part of the section provides a frequency table of participants’ responses. The second part of the section provides results pertaining to the participants’ reasons for why they considered the verification of the identity to be a valid proof or not.

Participant Responses: Is the Verification a Valid Proof?

The following table provides the frequency of participants’ responses about whether they considered the verification to be a valid proof.

Table 13		
Is the given trigonometric identity verification a proof?		
	Pre-service teachers	Professional mathematicians
Yes	5/13 (38%) ^{a, b}	3/8 (37%) ^{a, b}
No	6/13 (46%)	4/8 (50%)
Depends	2/13 (15%)	1/8 (12%)

^a Percentages are rounded to the nearest integer.

^b Due to rounding, percentages may not sum to 100%.

Participants' Reasons for Accepting the Verification as a Valid Proof

Of the participants who considered the verification to be a proof, all five of the pre-service teachers and one professional mathematician noted that the argument is a proof because each step follows from the previous step in a clear manner, beginning with the initial step which is a well-known identity ($1 = \cos^2\theta + \sin^2\theta$). One professional mathematician stated that the argument is a proof because “every step uses correct mathematics” (Pm-A 855). Another stated that “It’s a bit spare, but is a proof” (Pm-E 859) suggesting possibly that some words of explanation would enhance the argument.

Participants' Reasons for Not Accepting the Verification as a Valid Proof

Six pre-service teachers indicated that the argument is not a proof because there was no explanation given for one or more of the steps taken. For example, Pst-L stated that:

I don't believe this is a proof because it is just work. There aren't any reasons supporting each step along the way. When someone reads a proof, they need to be able to understand why such steps were taken, and although the manipulation is apparent, the reasons are not. (Pst-L 833)

Two professional mathematicians also indicated that there was no explanation for the steps taken in the argument and therefore they did not consider the argument a proof. For example, Pm-F stated that: “I don't consider a list of equations without any accompanying explanations a proof” (Pm-F 861). Two professional mathematicians specifically stated that the argument is not a proof because $\cos \theta$ could equal zero, so dividing by $\cos \theta$ would create a problem. In all, six of eight professional mathematicians pointed out that one must at least consider what happens when $\cos \theta = 0$, but none of the pre-service teachers pointed this out.

Participants' Reasons for Ambiguity About the Verification as a Valid Proof

Two pre-service teachers expressed uncertainty as to whether the argument is a proof. One pre-service teacher indicated that there were no reasons for the steps given, and that certain steps were missing from the argument, and called it “informal.” The other pre-service teacher called the argument “incomplete” because it did not prove the Pythagorean identity. Both of these responses seemed to indicate that there were aspects of the argument missing that they would expect to see to consider it a formal proof. There was no clear indication as to whether either pre-service teacher considered the verification to be a proof. One professional mathematician wrote that: “In some context... and for some audiences, YES [it is a proof]. There are some hidden assumptions that the author expects the reader to recognize. The ‘proof’ could be strengthened with a few additional comments” (Pm-I 865).

Summary

Roughly half of pre-service teachers sampled and half of the professional mathematicians sampled indicated that the verification of the trigonometric identity that was presented was a proof. This group was able to follow the sequence of equations and supply reasons for the steps taken. Half of the pre-service teachers and half of the professional mathematicians did not consider the argument to be a proof because there was no explanation given for some or all of the steps. For this group of pre-service teachers and professional mathematician, explanation in a proof serves at least two purposes; for some of the pre-service teachers, providing reasons for steps in a proof is orthodoxy in the sense that an argument is not a proof unless all the steps have reasons.

For other pre-service teachers and professional mathematicians, supplying words of explanation helps the reader understand and follow the argument.

A contradiction was observed among the responses of pre-service teachers to this item (whether they thought the verification was a proof) and their responses to the calculus-related proposition. Four pre-service teachers indicated that the second argument for the calculus proposition was a proof and specifically stated that it provided steps and reasons for those steps. However, these four pre-service teachers indicated that the verification of the trigonometric identity was a valid proof, despite the obvious lack of reasons for the steps taken. A contradiction was also observed in Pst-N's assessment of the validity of the two arguments. Pst-N did not consider argument three of the calculus arguments to be valid because the chain rule was not explained. However, in assessing the validity of the trigonometric identity, Pst-N indicated that it was a proof because it started with a known fact (the Pythagorean identity) and proceeded to establish the conclusion. This contradiction was not observed among the sample of professional mathematicians.

Pre-Service Teachers' and Professional Mathematicians' Perceptions of the Role of Examples in Proof

This section provides results pertaining to participants' conceptions of using specific examples in mathematical proof. A number theoretic proposition ("the sum of three consecutive integers is divisible by three") along with two arguments purported to prove the claim was given to participants for their reaction. One of the arguments used two specific examples (one negative, one positive) while the other used nine positive examples. Participants were asked whether either argument was a proof and whether they

thought either argument was better than the other (see Appendix C, section II, items 15 through 18).

None of the participants considered either argument to be a valid mathematical proof. Almost all of the participants generally agreed that a finite number of examples cannot prove a statement that applies to an infinite number of objects. Interestingly, only one professional mathematician specifically stated the possibility of checking all possible examples of a proposition over a finite set. Most responses were similar to the following:

No, they do not generalize the problem sufficiently. These only show specific examples where some three integers are consecutive rather than any three integers are consecutive (Pst-D 1798).

NO – a finite # of examples does not provide a proof for a statement occurring in an infinite set (Pm-A 1841).

Purpose and Appropriate Use for Examples in Proof

The following table provides the frequencies of participants’ perceptions about when or for what purpose using examples is appropriate.

Table 14
When is it appropriate to use specific examples?

	Pre-service teachers	Professional mathematicians
To convince	5/13 (38%) ^{a, b}	3/8 (37%) ^{a, b}
To discover a proof, pattern	5/13 (38%)	6/8 (75%)
To understand or test a proof	6/13 (46%)	4/8 (50%)
To disprove by counterexample	5/13 (38%)	2/8 (25%)
Never	1/13 (8%)	3/8 (37%)
At lower levels	4/13 (31%)	
To make proof more concrete	2/13 (15%)	

^a Percentages are rounded to the nearest integer.
^b Since participants’ could give multiple responses , percentages may exceed 100%.

Both groups of participants indicated that one appropriate use for specific examples is to convince oneself or to convince someone else that the proposition might be true in general. For example, “to convince oneself of the veracity of the statement; to

convince a class of the fact so that you can use it” (Pm-B 1963). Two pre-service teachers gave the indication that an additional purpose for examples is to make the proof of the proposition more concrete and therefore more accessible to students:

It might be appropriate when one wants to demonstrate the proposition with real numbers and not just variables, this way students can see how it is working (Pst-L 1948).

Specific examples illustrate the expected goal of a proof. They are helpful in high school because students haven’t developed the ability to work solely [sic] with concepts (Pst-U 1560).

This suggests that these participants believe that high school students are not prepared to understand the generality and universality of proof.

Many participants indicated that the use of examples is helpful for experimenting with or motivating the proposition by providing illustrative examples; using examples may help in discovering a pattern that could lead to a conjecture; and the use of examples may help in discovering a proof (i.e., similar to generic example). A notable difference between the groups is that fewer than half (5/13) of the pre-service teachers indicated the use of examples to motivate or illustrate propositions, to look for patterns, or to help them discover a proof while a majority (6/8) of professional mathematicians indicated these uses for examples. It appears that the message received by many students is that examples are not to be used at all, which reflects the idea that “examples are not proof” belongs to the orthodoxy that is somehow communicated to students (Weber & Alcock, 2004). For example, when asked about the purpose of using examples with respect to proof, Pst-D responded:

Using specific examples helps to build a case for the general properties. It helps high school students, especially, who are new to proof, to see that if it works for number of them, then it will probably work for a given set. High school math is

different than college or general math because we are expected to use only general mathematics, never specific examples. (Pst-D 1550)

Pst-D is repeating a perceived canon that even though he recognizes examples as a valuable tool in mathematical experimentation, in college, the expectations change. Most professional mathematicians, on the other hand, indicated that using examples is an integral part of investigating mathematical objects. For example, Pm-C stated that “in the world of professional mathematics, theorems do not (generally?) drop out of the clear blue sky. Familiarity with a range of examples usually suggests a generality that one then tries to prove or disprove” (Pm-C 1596) (see also MacLane, 1994; Weber, 2004). It appears that students are trained with the notion of not using examples while professional mathematicians use them regularly (e.g., Weber & Alcock, 2004).

Professional mathematicians indicated that the use of examples can be an aid to understanding the meaning of why specific hypotheses are needed for the claim and the proof itself. For example, Pm-H stated that “examples should precede the statement of the theorem and non-examples i.e. examples where the hypotheses are not satisfied and the conclusion may or may not follow should be provided after the proof of the theorem” (Pm-H 1609). Several pre-service teachers indicated that actual examples of proofs themselves were important for learning how to prove or perhaps gaining insight into what is expected of them as students. For example, Pst-T stated that “in my own experience the best way for me to learn and understand proof was through examples. I don’t think that this is any different in high school and college. Generally an example is the best way to go through a proof step by step” (Pst-T 1552). And Pst-A indicated: “in both high school and college the teacher/professor should give an example of some proofs” (Pst-A 1551). Other pre-service teachers indicated that an appropriate use for examples is to double-

check that the proof itself is correct. In other words, even though a proof of a proposition is provided, the subject needs additional evidence that the proposition is true. Fischbein (1982) observed this phenomenon. Pst-N, for example stated: “specific examples show that the proof is accurate” (Pst-N 1569). Here the subject needs further evidence to be convinced that the proof itself works, not evidence to be convinced of the plausibility of the proposition.

Four pre-service teachers indicated that it would be appropriate to use examples in arguments involving students in lower grade levels. These pre-service teachers were careful not to associate the word “proof” with arguments using only examples. However, there was only one instance of a participant specifically indicating that the distinction between using examples and proving the general case should be explicitly stated.

Concerning the place for examples, Pst-K thought that elementary school was appropriate because:

I think the idea of trying to prove something without examples is a hard thing to introduce before [3rd or 4th grade] because they need to... I think that's the beginning of a proof is to use the example. And then after that, they say “OK. That isn't actually proving it. You need to go beyond that. But I think it's a good starting point (Pst-K 2440).

Pst-K did not provide any indication about how or whether she would explain to students why examples are not sufficient proof although she recognizes that: “...it would be a bad thing to say ‘you can't prove it that way,’ and not be able to show them the way they're supposed to do it” (Pst-K 2468).

Is Either Inductive Argument Better?

Recall that the participants were asked to look at a proposition (“the sum of three consecutive integers is divisible by three”) and two inductive arguments purported to

prove the proposition. The first argument consisted of two examples (one negative, one positive) and the second argument consisted of nine examples (see Appendix C). Table 15 provides the frequencies of participants' responses about whether they considered either of the two inductive arguments provided for the proposition to be better than the other and if they considered one better than the other, their primary reasons why.

	Pre-service teachers	Professional mathematicians
Neither is better	7/13 (54%) ^{a, b}	6/8 (75%)
More examples better	3/13 (23%)	2/8 (25%)
Negative & positive better	2/13 (15%)	
Not sure	1/13 (8%)	

^a Percentages are rounded to the nearest integer.
^b Due to rounding, percentages may not sum to 100%.

Are More Examples Better?: A few participants gave responses that were more or less absolute in nature. These absolute responses suggest that providing one or more examples in support of a claim is not any closer to a proof than providing a single instance of the claim. In all, three pre-service teachers and two professional mathematicians responded in this manner. For example, Pst-A stated: “[the writers of the arguments] both don’t get the concept of math proof” (Pst-A 1872). Pm-A stated that “a finite [number] of examples does not provide a proof for a statement occurring in an infinite set” (Pm-A 1841). Pm-H simply responded “No” without any further explanation.

Professional mathematicians generally didn’t consider either of the inductive arguments to be proofs because neither argument was sufficiently general to cover all the integers. However, over half of the professional mathematicians pointed out that having more examples might be better since it might result in greater confidence that the claim is

true and that more examples might help to discover a general pattern. Only one pre-service teacher recognized these characteristics.

Almost half of the pre-service teachers indicated a clear preference for one of the arguments, although the reason for their preference was not clear. For example, Pst-D stated: “I consider the first to be better. The second shows many examples, but it does not show the different case, where the integers are negative” (Pst-D 1869). Pst-D seems to value recognition on the proof writer’s part that the domain includes negative numbers, but this fact doesn’t seem to contribute in any way to discovering a proof. Other pre-service teachers specified that having more examples would be better, but did not indicate why having more examples would be better.

Pre-Service Teachers’ and Professional Mathematicians’ Perceptions of a Number

Theoretic Proposition

In the previous section, the arguments presented to the participants were purely empirical. The responses given by the participants, in large part, were related primarily to their views on the use of examples in proof. This section provides results pertaining to participants’ perceptions of the form and content of mathematical proof. A number theoretic proposition (“for any integers a and b , if ab is odd, then a is odd and b is odd”) along with two arguments purported to prove the claim was given to participants for their reaction. The first argument provided was more algebraic in nature while the second argument provided non-algebraic statements followed by examples illustrating each statement. Participants were asked whether either argument was a valid proof and if they thought that either argument was better than the other (see Appendix C, section II, items 19 through 22). The majority of pre-service teachers rejected the non-algebraic argument

simply because it contained specific illustrative examples, even though the majority of professional mathematicians considered the argument to be at least partially acceptable.

The acceptance of these two arguments as valid among pre-service teachers seems to rest more on surface features rather than the actual content of the argument. Most pre-service teachers (9/13) indicated that the second argument was not valid because it contained examples. It seems that among pre-service teachers the mere appearance of examples in an argument disqualifies it from attaining proof status. For example, Pst-I stated that: “the 1st is better, it gives clearer explanations than [argument two] and examples do not signify a proof, they merely show one instance of where it works” (Pst-I 2358) and Pst-T stated that argument two “only proves a small set of cases” (Pst-T 2344). Conversely, most professional mathematicians (6/8) considered the second argument to be more or less acceptable under certain conditions. For example, Pm-C noted that the examples in the second argument were unnecessary but that the argument itself was sound:

Student argument #2 includes extraneous examples. Ignoring those, the student makes some claims without proof ... I don't miss the contrapositive apparatus here because I accept the existence of just four possibilities: even times even, odd times even, even times odd and odd times odd (Pm-C 2202).

As previously indicated, the majority of the pre-service teachers in this study seemed to point to the two-column or statement/reason form of argument as the epitomized model of formal proof. As Pst-N indicated when discussing the second argument for the calculus proposition (see Appendix C, section I, items 9 through 11):

...argument two is probably the one that I like best because it's what I think a formal proof is. You give that, then you give the definition, and then he went here, and this is why, and then you get the conclusion. So this is what I was talking about. There's one column, then there's a column for the reasons. (Pst-N 778)

For the arguments related to this particular proposition, several pre-service teachers (4/13) also pointed to the form of the argument as being important in their responses to the questions of validity and preference. For example, Pst-U stated that “the first is more proof-like...” (Pst-U 2137), Pst-A noted that “the 1st student had the idea of how a proof should be done while the 2nd student was using a method that isn’t done often in proof” (Pst-A 2341), and Pst-D indicated that “they’re using generality in its proper form, so, even integers are $2n$, expressed as $2n$, where n is in the integers...” (Pst-D 2080). Several pre-service teachers (4/13) gave an indication that the form of the argument matters in the deeper sense of making the argument clear and understandable. For example, Pst-P noted that “argument 2 has valid points but the organization is lacking...” (Pst-P 2366) and Pst-D stated that: “...the first uses more appropriate mathematical language than the second” (Pst-D 2339).

Half of the professional mathematicians also indicated that the form of an argument is important but seemed to suggest reasons other than surface features such as format (e.g., two column) or the mere appearance of examples. These included providing explicit statements about issues of language, or how a particular logical device is employed, in this case, the contrapositive. For example, Pm-F noted that:

Before a piece of writing can be accepted as a valid proof it must be written in correct English with proper sentence structure, spelling and punctuation. The first student comes a little closer to what is needed than the second. Assuming these matters are attended to, I think neither [argument] is too bad. The first student should correct the sentence with the $a \times 2b$ and give more detail on how the contrapositive method is being used (Pm-F 2181).

Pm-C also indicated the necessity of attending to the form of an argument by providing explicit statements of certain parts of the argument:

Student argument #1 promises to use the contrapositive method, but only does so implicitly. The student never says “Assume that we don't have both a and b odd.” The student omits details about the definition of odd and even, about the associative law of multiplication and writes $a \times 2b$ instead of the intended $a \times 2n$ (Pm-C 2196).

One pre-service teacher indicated that the validity of the argument rests on the writer explicitly stating the contrapositive: “the first method should have stated the contrapositive and then proved that. However, the proof is good beside that, but without that contrapositive stated it is not valid” (Pst-L 2129).

Interestingly, almost all (7/8) of the professional mathematicians, but none of the pre-service teachers, pointed out that the first argument contains an error. It seems that professional mathematicians were able to recognize the error, interpret what the writer probably meant to write, and then validate the argument based on that interpretation. It is possible that pre-service teachers noticed the error and proceeded the same way as the professional mathematicians. However, since none of the pre-service teachers acknowledged the mistake, it appears more likely that they were unaware of the error. It seems that the surface features pre-service teachers tend to focus on appear to be those of form and protocol rather than correctness.

Chapter Summary

Two major themes emerged from the analysis of differences between and among pre-service teachers' and professional mathematicians' perceptions of proof in mathematics: the diversity and uniformity between different aspects of pre-service teachers' and professional mathematicians' perceptions of mathematical proof and; differences between and among these two groups with respect to their perceptions about the nature of mathematical proof. These will be summarized below.

Pre-service teachers' descriptions of mathematical proof, their perceptions about the purpose and importance of proof in mathematics, and their views on validity in mathematical proof were generally much more diverse than those of professional mathematicians. The uniformity of professional mathematicians' perceptions of proof in mathematics indicates that there are certain common characteristics of proof that have become important parts of the concept images for this group of professional mathematicians. Along with the relative diversity of responses from pre-service teachers, the uniformity among the perceptions of professional mathematicians suggests that there is a process of enculturation into the standards of proof in mathematics that takes place over time with exposure to mathematics and the community of mathematicians. Evidence from this study suggests that the process of enculturation for this group of pre-service teachers may be only partially realized.

There were several differences observed between professional mathematicians and pre-service teachers regarding their perceptions about the nature of proof in mathematics. To begin, professional mathematicians and pre-service teachers appeared to have differing perceptions about the notion of validity regarding proof. This group of professional mathematicians generally agreed that an argument is either a valid proof or it is not a proof at all. The notion of validity was much less clear for pre-service teachers. For some pre-service teachers, the validity of a claim hinges on the truth of the assumptions on which the claim is based. Many pre-service teachers appeared to connect the validity of a proof with the correctness of a proof. That is, they believed that a proof can become invalid if an error is found. Several of the pre-service teachers also indicated that one argument can be more valid than another. There is also evidence that the pre-

service teachers in this study seemed to misunderstand the nature and role of axioms, definitions, and hypotheses in proof.

Pre-service teachers appeared to focus on surface features such as the form of the argument rather than the actual content of the argument or issues of correctness. For example, they indicated that the simple presence of specific examples in an argument disqualifies it as a proof. Professional mathematicians, on the other hand, seemed to focus more on the content of the argument itself and whether the argument actually establishes the truth of the claim. Professional mathematicians also viewed the completeness or degree of formality of an argument as context dependent whereas pre-service teachers seemed to have a more formal and strict archetypal form of proof (e.g., the “two-column proof”) that should be employed in all situations.

Many of the participants indicated that examples can be used to motivate or illustrate propositions, to look for patterns, or to help them discover a proof. However, while a majority of professional mathematicians suggested these uses for examples, fewer than half of the pre-service teachers indicated these as uses for examples. It appears that the message received by many pre-service teachers is that examples are not to be used at all, which reflects the idea that “examples are not proof” belongs to the orthodoxy that is somehow communicated to students (Weber & Alcock, 2004). This disparity between pre-service teachers and professional mathematicians about their perceptions of the use of examples in proof also points to a difference in the way in which doing mathematics is perceived (e.g., Maclane, 1994).

More than half of the pre-service teachers indicated that an argument becomes a proof when an external authority of some kind decides that it is a proof. Professional

mathematicians tended to view the acceptance of an argument as proof in a more flexible manner with a shared and social component attached. The decision about an argument's status is viewed by professional mathematicians more as a process rather than a static choice. Proof, for many of the professional mathematicians, is a social process consisting of negotiating meaning rather than the application of formal criteria⁶ (Lakatos, 1976). For many pre-service teachers, however, an argument becomes a proof only when an external authority decides it is a proof. This decision, for pre-service teachers, seems to be absolute; that is, an argument is either right or it is wrong. For most pre-service teachers, it appears that their experience has been confined mostly to submitting arguments to an authority, not so much for validation as for judgment. It also appears that professional mathematicians count themselves as belonging to the mathematical community, whereas pre-service teachers seem less inclined to consider themselves as part of the mathematical community. Some pre-service teachers appear to view proof as a separate part of mathematics. Professional mathematicians seem to view doing mathematics and proving as essentially the same activities.

The analysis of data from this investigation points to substantial differences in the perceptions between and among pre-service teachers and professional mathematicians regarding proof in mathematics. The issues and central themes raised in this chapter will be further developed in the following chapters, beginning with the results and discussion of participants' perceptions of mathematical proof as it relates to secondary mathematics.

⁶ Recall that while most of the professional mathematicians in this study provided a traditional description of mathematical proof, most also stated that the formality associated with the traditional description is not likely to be used in practice. This apparent inconsistency will be developed further in the chapters that follow.

CHAPTER 6

RESULTS: PROOF IN HIGH SCHOOL MATHEMATICS

Chapter Overview

The focus of the preceding chapter was on the participants' perceptions of proof, specifically as they relate to the discipline of mathematics. The evidence pointed to some marked differences between the perceptions of pre-service teachers and professional mathematicians. The primary focus of this chapter is on participants' perceptions of mathematical proof as it relates to high school mathematics. The objective for focusing on proof in high school mathematics was to find out whether and to what degree differences exist between and among the perceptions of pre-service teachers and professional mathematicians with respect to their perceptions of proof in high school mathematics. The investigation into perceptions from a different viewpoint (that of proof in high school mathematics) provides an opportunity to further characterize participants' perceptions of proof and helps to explain some of the findings.

The chapter is organized into sections corresponding to the results of participants' responses to questions about their perceptions of mathematical proof in high school mathematics. The primary sections include participants' perceptions on: the purpose of proof in high school mathematics; the importance of mathematical proof in high school mathematics; expectations for high school students regarding mathematical proof;

perceptions of proof in high school algebra and high school geometry; and what is explicitly told to students regarding mathematical proof.

Pre-Service Teachers' and Professional Mathematicians'

Perceptions on the Purpose of Mathematical Proof in High School Mathematics

Pre-Service Teachers' Perceptions on the Purpose of Proof

This section describes the results of pre-service teachers' and professional mathematicians' perceptions on the purpose of mathematical proof in high school mathematics. Table 16 provides the frequencies of participants' responses to the questionnaire item: What is the purpose of proof in high school mathematics?

	Pre-service teachers	Professional mathematicians
Prepare for college	4/13 (31%) ^{a, b}	
Allay fear of proof	3/13 (23%)	
Understanding	11/13 (85%)	2/8 (25%) ^{a, b}
Learn life skills		2/8 (25%)
Learn logical thinking	4/13 (31%)	6/8 (75%)
To expose students to proof	2/13 (15%)	1/8 (12%)
Teach the nature of mathematics	3/13 (23%)	7/8 (87%)
Appreciation of mathematics	2/13 (15%)	2/8 (25%)
Problem solving	2/13 (15%)	
Communication of ideas	1/13 (8%)	

^a Percentages are rounded to the nearest integer.

^b Since participants' could give multiple responses , percentages may exceed 100%.

Pre-service teachers primarily view the purposes of proof in high school mathematics to be to promote understanding and/or to prepare students for proof in college, while the majority of professional mathematicians see the purposes of proof to be to teach students to think logically and/or to teach students about the nature of mathematics. One notable feature among pre-service teachers' responses that is

completely absent from professional mathematicians' responses is a high-level of anxiety associated with proof. These and other relevant observations are presented below.

Understanding: The majority of pre-service teachers (11/13) indicated that one purpose for proof in high school mathematics is to promote understanding. Over half of the pre-service teacher responses (6/11) pointed to understanding in the conceptual sense. But five of the eleven pre-service teachers attached a somewhat different meaning to understanding. Two aspects of these particular pre-service teacher responses are worth noting.

First, these responses imply understanding in the sense of explaining how a particular formula or equation came about in a procedural sense. For example, a derivation of quadratic formula for the roots of a second degree polynomial usually consists of a series of algebraic manipulations beginning with the general form of the polynomial and ending with expressions for the two roots of the equation. Each step of the derivation is usually justified by one or more of the "rules" of algebra such as "one can divide both sides by the leading coefficient." This example represents understanding in the procedural sense: it means understanding why one is allowed to proceed from one step to the next. This is in contrast to understanding in a conceptual or global sense, which means a deeper understanding of why the claim itself should be true or a deeper understanding of the mathematics involved in the claim. For example, consider two of the arguments given for the calculus proposition: "the derivative of an even function is odd" (see Appendix C, section I, items 9 through 11). Argument 2 uses the definition of derivative (i.e., the limit of the ratio of differences of dependent versus independent variables) and solely by manipulating the equations, arrives at the desired result.

Argument 1, on the other hand, gives a descriptive and pictorial argument that provides insight into what it means for a function to be even or odd.

Regarding the second characteristic of different meanings attached to “understanding,” there is the impression that the equations and formulas pre-exist, are known to be true, and are given to students to verify as an exercise or to simply accept. “Proof” in this case seems to mean being able to verify that the equation can be derived from some starting point. The verification of the trigonometry identity (see Appendix C, section I, item 12) is an example of this type of problem. The result is provided. The goal of the verification is to manipulate other known or given expressions or equations (in this case, $\cos^2\theta + \sin^2\theta = 1$) to match the given result ($\sec^2\theta - \tan^2\theta = 1$).

By way of example, Pst-N talked about being “given the quadratic formula, and then you're given what x equals, you don't really go through the process of how you got it to be x equals” (Pst-N 1302). Pst-N’s response highlights the notion that high school students are often given formulas and then simply asked to use them to solve related problems. Professional mathematicians’ responses indicated understanding as a purpose for proof in high school mathematics but in the more global sense of conceptual understanding. For example, Pm-E stated that:

The elegance and simplicity and intuitiveness of geometry make it the prime candidate for [learning proof]. Elsewhere, proofs should be included as flavor and insofar as they enhance understanding. (Pm-E 1043)

And Pm-C stated:

As a science, they should come to know mathematics as a body of ideas with implications for our understanding of diverse phenomena. When these ideas give rise to a technology that the students will use (a formula or an algorithm), students should strive to understand the underlying ideas. (Pm-C 995)

To contrast several responses, Pst-N indicated, for example, that the quadratic formula is, in many cases, given to students without explanation. It may also be that the formula is derived for students in detail usually by the teacher or textbook, or in some (possibly rare) cases, students may actually derive the formula themselves. The results of the quadratic formula provide information about two key ideas: 1) the zeros of a second-degree polynomial function and equivalently; 2) the linear factors of a second-degree equation. Pm-C's response suggests that it is important that high school students have an understanding of what the formula means rather than an understanding only of how it is derived. Pst-N, on the other hand, seemed to indicate understanding in the procedural sense of being able to "follow along" with the derivation without necessarily knowing or being able to connect what the formula means or represents:

To go from $ax^2 + bx + c = 0$, and then to get what x equals, in order for x to equal plus or minus $b \dots$ that would be a proof. And, I'm thinking that maybe higher level students would be able to follow along, whereas, like if you're in a track system, the lower level students would just probably be given it. And where higher level students would know why and where it came from and how to actually derive x . (Pst-N 1303)

Teach logical thinking: The majority of professional mathematicians (6/8) and four pre-service teachers indicated that one purpose of proof in high school mathematics is to teach logical thinking skills. Two professional mathematicians also stated that a purpose for proof in high school mathematics is to develop thinking skills that are applicable outside of the mathematics classroom, for example:

All high school students should learn to "justify" according to a given set of rules. I think it is also appropriate for all students to see "proof" in its stricter form, as they will be dealing with people using such ideas and need to decide their course of action appropriately. (Pm-B 1099)

And Pm-I stated that “proof” develops:(1) thinking skills; (2) writing skills; (3) ordering skills; (4) reading skills and can develop; (5) research skills (in any subject) and (6) mathematical knowledge” (Pm-I 1108).

Professional mathematicians more than pre-service teachers indicated the important contribution of logical thinking skills to the general education of students. For example, of logical thinking, Pm-E stated that “[high school students] must learn how to reason and how to present a logical argument. This skill is distilled into its purest form in mathematics” (Pm-E 1102). Pm-F stated “every student who has the capacity to understand should learn proof because it is a fundamental tool of thinking” (Pm-F 1104). Only one of the four pre-service teachers indicated this aspect of proof but in reference to other classes, not as a life skill: “in other classes, you can use other knowledge, because you know how to prove” (Pst-P 968).

Teach about the nature of mathematics: Most professional mathematicians (7/8) indicated that a purpose for proof in high school mathematics is to teach students about the nature of mathematics. The responses of professional mathematicians seem to follow three distinct themes:

- 1) The foundations of mathematics, that is, the body of mathematical knowledge exists because of and through mathematical proof. For example, Pm-E wrote that the purpose of proof in high school mathematics is: “mainly to illustrate that mathematics is not a collection of disjointed facts and algorithms, but rather a cumulative body of knowledge on a firm foundation” (Pm-E 981).
- 2) Mathematical proof provides certainty of results which distinguishes mathematics from other scholarly fields; for example, Pm-G wrote that “most high school work

does not involve proofs directly, but students must know that the certainty of the math results they learn is somehow different from that in other fields” (Pm-G 922).

- 3) Exposure to the axiomatic system, that is, establishing facts in the context of axioms and previously established theorems. “It may well be the first time a young person encounters the idea of establishing a proposition on the basis of assumptions or previously established propositions” (Pm-F 919).

Three pre-service teachers gave responses indicating that a purpose of proof in high school mathematics is to teach students about the nature of mathematics. For example, Pst-L stated that “without proofs, we would blindly be doing computations and such without an idea if what we are doing is really correct” (Pst-L 947). And PST-H stated:

All students should have some knowledge of the development of mathematics and this involves seeing how our theorems were proved from simple axioms, and they should know that without a proof, statements mean nothing. (Pst-H 1090)

Proof as a source of anxiety: Several (5/13) pre-service teachers indicated that students should be exposed to proof in high school in order to lessen the shock of encountering proof in college and/or to make proof easier in college. Some responses indicated a strong negative connotation or association with proof. Regarding apprehension for proof, for example, Pst-T stated “...when I was first exposed to it in college in math proof it was just, like a totally different language almost. I mean, luckily I sort of caught on” (Pst-T 1012). Pst-L indicated that “...you don't want to make it so that [students] think proofs are these horrible things” (Pst-L 2272). Pst-H pointed out that “if students begin to prove things and are expected to provide justification in high school, then it won't be as much of a shock when they get to college” (Pst-H 954). Similarly, Pst-

L stated that “if students are exposed to it all along it wouldn’t be intimidating anymore” (Pst-L 1017). One professional mathematician acknowledged that using the word “show” in place of “prove” can reduce fear of proof in students: “...in college math, at least, ‘show that’ is typically a code phrase to indicate that slightly less rigor is required... and perhaps more importantly, to keep from terrifying students” (Pm-B 2036).

From the response of Pm-B it seems clear that professional mathematicians (i.e. university professors) are at least aware of a level of anxiety surrounding proof that exists among undergraduate students. From the responses of pre-service teachers, it seems clear that proof is still viewed as a source of anxiety by students near or at the completion of their undergraduate study.

Summary

Pre-service teachers view the purpose of proof in high school mathematics as an aid to understanding, but many view understanding in a procedural sense. Pre-service teachers themselves may not understand or be able to connect to the deep ideas behind some of the concepts of high school mathematics. This may help to explain the apparent procedural and somewhat superficial view of proof held by some pre-service teachers. A large number of pre-service teachers view the purpose of proof in high school as preparation for further study of mathematics at the college level, and for that reason, believe that the process of proof may not be necessary for all high school students to learn.

Professional mathematicians view proof in high school as a means to teach logical thinking (which many believe to be an important component of students’ general education), and to teach students about the nature of mathematics. With respect to the

purpose of proof and the nature of mathematics, the responses of professional mathematicians indicated that it is important that high school students learn: 1) that the body of mathematical knowledge exists because and through mathematical proof; 2) that mathematical proof provides certainty of results which distinguishes mathematics from other scholarly fields; and 3) to understand the role of proof in establishing facts in the context of axioms and previously established theorems.

Pre-Service Teachers' and Professional Mathematicians'

Perceptions on the Importance of Mathematical Proof in High School Mathematics

This section describes the results of pre-service teachers' and professional mathematicians' perceptions on the importance of mathematical proof in high school mathematics. Table 17 provides frequencies of participants' responses to the question: How important is proof in high school mathematics?

	Pre-service teachers	Professional mathematicians
Very important	8/13 (62%) ^{a, b}	4/8 (50%) ^{a, b}
Somewhat important	1/13 (8%)	2/8 (25%)
More important for some	4/13 (31%)	1/8 (12%)
Not important		1/8 (12%)

^a Percentages are rounded to the nearest integer.
^b Due to rounding, percentages may not sum to 100%.

Table 18 provides frequencies of the descriptions or explanations given by participants about why they responded the way they did.

Table 18

Descriptions of the importance of proof in high school mathematics

	Pre-service teachers	Professional mathematicians
Understanding	6/13 (46%) ^{a, b}	
Mechanics, basic skills	2/13 (15%)	2/8 (25%) ^{a, b}
Time factor	1/13 (8%)	
Proof for some	4/13 (31%)	1/8 (12%)
Formal not important	1/13 (8%)	
Connected	1/13 (8%)	
Communication	1/13 (8%)	
Exposure	1/13 (8%)	2/8 (25%)
Essential part of mathematics	1/13 (8%)	2/8 (25%)
Nature/role of proof in math		2/8 (25%)

^a Percentages are rounded to the nearest integer.

^b Since participants' could give multiple responses , percentages may exceed 100%.

The majority of pre-service teachers and professional mathematicians stated that proof is moderately to very important for high school mathematics. Pre-service teachers primarily link the importance of proof with helping student understanding. Professional mathematicians cited different explanations regarding the importance of proof in high school, but most were related to students understanding the role and nature of proof in mathematics and at least providing students with exposure to proof.

Pre-Service Teachers' Perceptions on the Importance of Proof in High School

Mathematics

Six pre-service teachers indicated that the importance of proof in high school mathematics is linked to understanding. All six rated the importance of proof in high school mathematics as very high. This suggests that since they view proof as helping to explain and/or understand mathematics, they believe that it is important to include proof in high school mathematics. Four pre-service teachers indicated that proof is more important to some students than to others. For example, Pst-T stated:

I think, as far as whether or not, students are going to pursue math, mathematics later on in college, then I believe it's important for them to at least get some experience with it prior to college. But as far as just... like, students that are looking into considering English or arts or something like that, it's not... they're not going to really need mathematics as much in college.... (Pst-T 884)

Other responses reflect the idea that proof is more important to those who are going to teach mathematics, and/or to those who are going on to study mathematics in college. This notion of “proof for some” will be further examined in the section on expectations. One pre-service teacher indicated that “formal math proof is not important. However, being able to show ones [*sic*] reasoning is important” (Pst-U 896). The distinction between formal proof and reasoning for Pst-U is related to his perception of formal proof, which appears to include strict adherence to a set method or form. For example, Pst-U wrote that:

A formal proof, you know, it has to be, you know, you have to write out your claim, and assume the hypothesis, you know, contradictory statements and what not, using... utilizing the different tools you learn in a math proof class, and kind of develop a full, full proof that way. (Pst-U 405)

Pst-U seems to equate “reasoning” with what he refers to as “informal proof”:

I think I meant that in terms of... as far as formal proofs found in the high school setting, I don't think they are as critical as just basic reasoning, as like an informal proof. I'd focus more on informal proofs in high school and junior high school and leave the formal proofs that like a collegiate level, more so. Informal... I consider it just to be... it's pretty much the thoughts, the thought process, I mean, it has to be correct, but it's not as strict. (Pst-U 183)

Professional Mathematicians' Perceptions on the Importance of Proof in High School Mathematics

Four professional mathematicians indicated that the importance of proof in high school is connected to exposing students to the process of proving, understanding, and/or gaining an appreciation for the nature and role of proof in mathematics; and two others

indicated that proof is an essential and inseparable part of mathematics, and that all mathematics, whether professional or in school, involves proof. Pm-I noted, however, that:

Since I view “doing mathematics” as writing or researching a proof, all of high school mathematics is focused on writing mathematical proof. Unfortunately, “doing mathematics” in high school is not (I believe) viewed in this way. Teachers probably don’t present their activities in this context. (Pm-I 925)

It seems clear that Pm-I would like to have high school mathematics teachers present their subject in a manner more closely associated with Pm-I’s perception of the actual practice of professional mathematicians.

Two professional mathematicians indicated that it may be more important that students learn basic mathematical skills. One professional mathematician in particular stated that proof in high school mathematics is “not very important – students should learn the mechanics first” (Pm-A 909). Pm-B indicated that the importance of proof in high school mathematics depends on how one defines the purpose of high school mathematics in the first place:

This is really a question about what the purpose of high school math is. I think the purpose is... (1) Teach the tools necessary to function in our society; (2) Teach them how to think logically. The second aspect relies more heavily on proof. One can balance a checkbook with little proof involved. (Pm-B 911)

Pm-B’s responses to this and other questions suggest that he generally values proof in mathematics very highly, however, he noted that the importance of proof depends in large part on the purposes of mathematics in education and what is valued in high school mathematics.

Summary

Roughly half of the pre-service teachers and half of the professional mathematicians indicated that proof should be an important part of high school mathematics. For many pre-service teachers, proof is an aid to understanding and therefore it is important to include proof in high school mathematics. For many professional mathematicians, it is important for high school students to be exposed to proof, and to learn about the role and nature of proof in the discipline of mathematics. For a few pre-service teachers and professional mathematicians, proof in high school mathematics is not important at all. For example, Pm-A stated that proof is “not very important – students should learn the mechanics first” (Pm-A 909). These participants viewed students’ acquisition of proficiency in basic skills to be of greater importance than proof. Several pre-service teachers also indicated that the importance of proof for students depends on other factors. The notion of “proof for some” will be more completely examined in the next section (expectations).

Pre-Service Teachers’ and Professional Mathematicians’

Expectations of High School Students Regarding Mathematical Proof

This section describes the results of pre-service teachers’ and professional mathematicians’ expectations for high school students regarding mathematical proof in high school mathematics. This section is divided into three subsections. The first subsection relates information about what participants expect high school students to know or be able to do with respect to mathematical proof. Data were gathered primarily through participants’ responses to two questionnaire items: “What specifically do you think is important for high school students to learn about mathematical proof?” and

“What expectations do you have for high school students regarding mathematical proof?” (see Appendix C, section II, items eight and nine respectively). The second subsection relates to participants’ perceptions about which students and which classes should address mathematical proof. Data for this category were gathered primarily through participants’ responses to three questionnaire items: “In what high school classes should proof be addressed?”, “Should all high school students learn mathematical proof?” and “Would proof look different depending on the level of students?” corresponding to items three, four, and seven of section II of the questionnaire, respectively (see Appendix C). The third subsection relates to participants’ perceptions of when and how much rigor is necessary in high school mathematics. Data for this category were gathered primarily through participants’ responses to four items in section II of the questionnaire (see Appendix C). These questionnaire items are: “How detailed/rigorous/complete do high school students’ proofs need to be?” (item five), “Would your acceptance of either of these arguments as valid proofs depend on the class (i.e., algebra, geometry, pre-calculus) or grade level of the student?” (item 20), “When do you think this level of detail/rigor/completeness is necessary?” (item 23), and “Would you expect to see the same level of detail/rigor/completeness in high school algebra?” (item 24).

What Participants Expect High School Students to Know or be Able to do with Respect to Mathematical Proof

Table 19 provides the frequencies of participants’ responses regarding what they expect high school students to know or be able to do with respect to mathematical proof.

Table 19

Expectations of high school students about mathematical proof

	Pre-service teachers	Professional mathematicians
Basic proof skills	4/13 (31%) ^{a, b}	3/8 (37%) ^{a, b}
Construct simple proofs	3/13 (23%)	2/8 (25%)
Follow a proof	2/13 (15%)	1/8 (12%)
Exposure to proof	4/13 (31%)	5/8 (62%)
Affective aspects of proof	4/13 (31%)	2/8 (25%)
Details on demand		1/8 (12%)
Everything has a reason	2/13 (15%)	
Provide justification	4/13 (31%)	
Present a coherent argument	2/13 (15%)	2/8 (25%)
Logical thinking	1/13 (8%)	1/8 (12%)
Foundation of mathematics		2/8 (25%)
Written expression	2/13 (15%)	1/8 (12%)
Language of mathematics		2/8 (25%)
Proof & communication	3/13 (23%)	1/8 (12%)
Logic aspects of proof		3/8 (37%)
Purpose of proof	3/13 (23%)	4/8 (50%)

^a Percentages are rounded to the nearest integer.

^b Since participants' could give multiple responses, percentages may exceed 100%.

Over half of the pre-service teachers (7/13) and over half of the professional mathematicians (5/8) indicated that they expect high school students to know basic proof concepts and/or that they be able to construct simple proofs. Two professional mathematicians explicitly stated that they don't expect proficiency in proof from high school students. The remainder of responses suggest that most of these participants generally have fairly minimal expectations for high school students regarding proof. For example, Pst-I expects high school students "just to be able to show steps clearly proving a theorem. Nothing too formal or detailed" (Pst-I 1473) and Pst-A expects them "just to be able to complete basic geometrical proofs" (Pst-A 1464). From professional mathematicians, Pm-F stated that "I would hope that high school students learn the first steps toward writing mathematical proofs" (Pm-F 1486) and Pm-G indicated that "the main purpose of any sort of proof that a student may be asked to do [in high school] is

just to show him the basic process involved” (Pm-G 984). In fact, two participants specifically indicated that college is a more appropriate place to learn proof.

In some instances, what pre-service teachers and professional mathematicians indicated that they expect of high school students is related to what they perceived high school students’ attitudes about proof to be. For example, Pst-G stated “my expectations are slim because most high school students don’t want to learn proofs, they want to ‘plug & chug’ and just get an answer” (Pst-G 1458) and Pst-J stated “I expect them to dislike proof since that is a general trend” (Pst-J 1478). From the viewpoint of experience as a university professor, Pm-G indicated that as far as what he expected of high school students and proof: “right now, very little. I see many good high school students in my beginning calculus classes.... Few have any appreciation of proofs” (Pm-G 1488).

Exposure: Four of the thirteen pre-service teachers and five of the eight professional mathematicians indicated that high school students should at least be exposed to mathematical proof. Pst-A, for example stated “... my basic theory about proof in high school is that it should be there to introduce them to it” (Pst-A 1135). Pm-A indicated the expectation “that they have seen and done some proofs before they reach my classes. I do not expect them to be proficient” (Pm-A 1480).

Communication and presentation of arguments: Roughly one-quarter (3/13) of the pre-service teachers and one professional mathematician indicated that students in high school should know that proof is an important means for communicating mathematical ideas to others. The pre-service teacher responses suggest that this communication occurs in the context of a classroom (i.e., explaining to peers and/or teachers), whereas the response of the professional mathematician suggests a professional and/or academic

context. For example, Pst-J stated that “they should learn that proof is a way of explaining to others what you’ve done using logical reasoning” (Pst-J 1424). Pm-H indicated that “students should be aware that mathematical proof is the way to communicate and advance mathematics. A proof is intended for the reader” (Pm-H 1789).

Two pre-service teachers and two professional mathematicians indicated that high school students should be able to present a coherent argument. These four participants’ responses seem to suggest a primarily social aspect to argumentation rather than adherence to a particular format. For example, Pst-K stated that she expected that students “will be able to justify their arguments in a way that is logical, complete and convincing to a peer and the teacher” (Pst-K 1479). Pm-B stated that “students should learn what it means to justify something completely and learn how to reason through arguments” (Pm-B 1427).

What Should Students Know About Proof?

Professional mathematicians were generally more precise in articulating what they thought high school students should know about proof. With a few exceptions, pre-service teachers were generally much less specific about their expectations. For example, Pst-A stated that he expected students “just to be able to complete basic geometrical proofs” (Pst-A 1464). Pst-T indicated: “I believe that basic concepts such as ‘how to’ and counterexamples should definitely be covered” (Pst-T 1386), whereas, professional mathematicians in some cases provided specific examples of what they think high school students should know. For example, Pm-C stated that high school students “should learn

not to confuse a proposition with its converse” (Pm-C 1445); and Pm-E indicated “for example, that [a finite set of examples] does not prove the proposition” (Pm-E 1485).

When responses of pre-service teachers were specific, they tended to focus on the form of the argument and derivation of specific mathematical formulas. For example, Pst-T stated: “I currently have very little expectations of students regarding mathematical proof. I expect them to know the basic ‘T’ [two-column] format proof in geometry, but beyond that I don’t expect them to know anything” (Pst-T 1465). And responding to the question “what specifically do you think is important for high school students to learn about mathematical proof?” Pst-N stated: “the steps to get from 1 equation to another. The trig [*sic*] proof should be learned in high school, your argument #1 [*sic*]. Also proving the quadratic equation should be taught in high school not just memorized” (Pst-N 1414).

More generally, pre-service teacher responses indicated that they expect students to learn about the purposes and importance of proof in mathematics (i.e., to establish truth of claims, to establish the foundations of mathematics, and to provide understanding and/or explanation), trigonometric identity verifications, derivation of the quadratic formula, and the two-column format commonly employed in geometry.

Professional mathematician responses, on the other hand, tended to focus on the context in which proof takes place and the nature of mathematical proof. For example, Pm-C alluded to the idea that students should understand the “axiomatic” nature of proof when he wrote that “students must know the definitions of the terms they use, they must recognize undefined terms in their discourse, they must know the precise statements of the theorems they've already learned, and must recognize the application of known

theorems in specific contexts” (Pm-C 1198). Pm-H’s response seemed to point more to students understanding the nature and purpose of proof as well as the notation: “I think it is very important that high school students understand the role of mathematical proof and begin to acquire some of the language and syntax involve[d] in writing mathematical proofs” (Pm-H 936).

In general, professional mathematician responses indicated that they expected high school students to learn about the definitions of the terms they use, the undefined terms, precise statements of theorems, the role and purpose of proof both in terms of the foundations of mathematics and certainty of results, the structure of the language of mathematics, basic mathematical notation and syntax, when something is not a proof, the difference between a proposition and its converse, and the fundamentals of pure logic.

Several pre-service teachers (4/13) and professional mathematicians (2/8) provided responses that reflect a recognition of the affective side of mathematics. As previously noted, some pre-service teachers indicated that their expectations of high school students regarding proof are low, due to their perception of high school students’ negative attitudes toward proof. Two pre-service teachers and two professional mathematician responses suggest that students’ negative attitudes toward mathematical proof should somehow be deliberately addressed. For example, Pm-C stated that “mathematics has an emotional side too. Discovering a proof requires patience and commitment. Students often follow lines of reasoning that lead nowhere and they must learn to deal with that frustration” (Pm-C 1449) and Pm-H stated she would like students to “... view [proof] as an ‘everyday thing’ not as something esoteric” (Pm-H 1455). Pst-L and Pst-I stated respectively: “... you don’t want to make it so that they think proofs are

these horrible things...” (Pst-L 2281). “Just telling students to do a proof b/c it’s ‘In the curriculum’ will dishearten them + devalue the learning of proof” (Pst-I 1410).

Summary

Pre-service teachers seem to focus more on certain surface features and particular objects related to proof, such as the two-column format, or specific derivations of formulas, as being important for students in high school to learn. On the other hand, professional mathematicians seem to focus more on general and global characteristics about the nature of proof such as logic and recognizing fallacy as carrying greater importance for high school students.

Participants’ Perceptions About Which Students and Which Classes Should Address

Mathematical Proof

Table 20 provides the frequencies of participants’ responses to the two questions, In what high school classes should proof be addressed? and, Should all high school students learn mathematical proof?

	Pre-service teachers	Professional mathematicians
Proof in all classes	8/13 (62%) ^{a, b}	5/8 (62%) ^{a, b}
Proof in some classes	5/13 (46%)	3/8 (37%)
Proof for all students	10/13 (77%)	6/8 (75%)
Proof for some students	3/13 (23%)	1/8 (12%)

^a Percentages are rounded to the nearest integer.
^b Since participants’ could give multiple responses , percentages may exceed 100%.

The majority of pre-service teachers and professional mathematicians indicated that proof should be addressed in all classes and that all students should learn mathematical proof. A surprisingly large number of pre-service teachers indicated that proof should only be addressed in some classes and/or that proof is not for all students.

While some professional mathematicians indicated that proof may not belong in all classes, none specifically indicated that they thought proof was only for some students or that it belongs only in a certain class, such as geometry. Some responses specifically stated that geometry is the appropriate class in which to learn proof. Other responses indicated that proof should be addressed only in upper-level classes and/or only for college-bound students. Reasons for not including proof for “lower-level” classes generally related to the participants’ view that learning basic mathematical skills is perceived as more important than learning proof. When asked where proof should be taught, for example, Pm-G stated: “primarily, I think, it would be in upper level classes. Most lower level classes are difficult enough in just teaching basic [skills]” (Pm-G 1049). Pst-L stated:

It is important for students to be exposed to proofs and to be active in the process [of proving] however, not everyone is going to be a mathematician and so students need to come out of high school with the mathematical skills to succeed in the world. (Pst-L 892)

The reasons given by participants for only teaching proof to college-bound students suggest that learning proof is not necessary for students not going to college. For example, Pst-U stated that “if a student knows he or she doesn’t want to pursue a career in mathematics that it should not be required” (Pst-U 1084). Pst-T wrote: “the students that are in these lower level math classes aren’t really going to use much mathematics in college or outside of college” (Pst-T 1075).

Several responses indicated that not all students have the mathematical background to engage in mathematical proof. For example, Pst-N stated that proof should be addressed only “in upper level classes, because lower level classes might not be able to understand the steps of proofs” (Pst-N 1030). Pst-I indicated that lower-level students

generally struggle to acquire basic mathematical skills and hence, proof should be addressed only in upper-level classes, adding that “proof is more confusing in H.S. than it is helpful” (Pst-I 1024).

Participants’ Perceptions of When and How Much Rigor is Necessary in High School Mathematics.

Not all participants provided descriptions for the question: When is rigor necessary? In all, seven pre-service teachers and four professional mathematicians gave indications about where or when they thought a high level of rigor is necessary in high school proof. Responses were almost evenly divided among the categories, as well as between pre-service teachers and professional mathematicians. The main difference between the responses appears to be the reasons cited. For example, both Pst-L and Pm-H stated that high school students should be exposed to rigorous proof. Pst-L indicated that students “need to have experience with it” (Pst-L 2529) without providing any explanation as to why, while Pm-H stated:

I think students should see and be asked to provide this level of detail at least once. They should also understand that such formal proofs underlie every mathematical argument but it is not always necessary to be included in a proof. (Pm-H 2577)

Three participants specifically indicated that proofs in geometry should be rigorous (as opposed to other classes such as algebra). But, while the two pre-service teachers gave no reasons why they thought this, the professional mathematician specifically indicated that proofs should be rigorous in geometry because rigorous proof is the point of the class. Similarly, several participants indicated that it is important for students to be rigorous when first learning a subject and/or when first learning proof. One pre-service teacher explained that:

I think [proofs] should be very detailed because it may be the first experience the students have w/ proof. In any case, they are “new” at proof and should therefore pay close attention to the details w/ in their writing. (Pst-M 1165)

This response suggests a structural or syntactical aspect associated with rigor. By contrast, Pm-F stated that “when beginning the subject it is useful to see that every statement can be supported” (Pm-F 2562), suggesting that rigor can provide the student with a deeper understanding and appreciation about the nature of proof. One professional mathematician pointed out that a high level of rigor may be detrimental to students’ mathematical development, noting that:

Some [geometric] foundations take both “point” and ‘line” as primitives; other foundations regard a line as a set of points leaving only “point” undefined. These sort of distinctions should not distract a geometry beginner. They should first learn how lines and points behave. (Pm-C 1203)

When asked to comment on a rigorous geometry proof, Pm-C further stated that:

The need for this sort of [highly rigorous] proof just indicates bad foundations. Work like this makes mathematics seem disengaged from the world. It sends the wrong message to students. (Pm-C 2573)

Several pre-service teacher responses suggest that a high degree of formality or rigor is unnecessary and would hamper students’ understanding. For example, Pst-I stated that “students need to back up their conjectures + statements + proof is the way to do that, but formal proof is unnecessarily tedious” (Pst-I 1087).

Summary

No definitive definition or description of what constitutes “rigorous proof” was specifically indicated by any of the participants. However, from the responses, it appears that pre-service teachers generally equate rigor specifically with the two-column format typically found in geometry. For pre-service teachers, rigorous proof (as exemplified by pre-service teachers’ notion of formal proof) also seems to imply a high level of included

detail: most pre-service teachers seem to indicate that rigorous proof should include ALL the steps of a proof which means including steps such as “subtract 1 from both sides,” etc. For example, when commenting on the trigonometric verification (see Appendix C, section I, item 12), Pst-D stated that: “I believe that this argument is an informal proof. There are some steps that are skipped and implied (canceling and subtraction), rather than being written out completely...” (Pst-D 823) suggesting that a more formal and rigorous proof would include all of these steps.

For pre-service teachers, formal proof appears to be strictly defined by its form. Informal proof is any other kind of argument (including inductive arguments). Informal proof has an associated continuum; that is, for pre-service teachers the degree of informality of a proof can vary.

For professional mathematicians, rigorous proof seems to be equated with the level of detail included in a proof and has very little to do with form. Formal proof is also strictly defined but in terms of axioms, definitions, and rules of logic and seems to be equated with proof in the Platonic sense. Formal proof, most professional mathematicians accede, is not attainable and therefore mathematicians deal in degrees of formality.

Pre-Service Teachers’ and Professional Mathematicians’

Perceptions of Proof in High School Algebra and High School Geometry

Not all participants gave responses for questions regarding the purpose and nature of proof in high school algebra and geometry. What follows is based on those participants who did provide information.

Over half (7/13) of the pre-service teachers indicated that formal, detailed proof, like that traditionally found in geometry, doesn’t belong in high school algebra. Most of

these pre-service teachers (6/7) cited that in their own experience, there was no proof in algebra. As the pre-service teachers defined it, proof was generally only experienced in geometry. For example, Pst-A stated: “in my experience proof in algebra didn’t exist and geometry was the only proof I experienced” (Pst-A 1224). The reasons given by pre-service teachers for not including proof in algebra suggest that 1) there is too much material to cover and too little time in which to cover it, so proof in algebra carries a low priority; and 2) algebra doesn’t have as many theorems, definitions, or axioms, and thus doesn’t lend itself to the study of proof. One pre-service teacher in particular indicated that rigorous proof in algebra should only be expected of high school students “if the students were given field axioms and theorems for algebra” (Pst-N 2632), but most pre-service teachers did not indicate an explicit connection between the axiomatic structure of the real numbers as an algebraic field and high school algebra.

To pre-service teachers, there seem to be at least two meanings for proof in high school algebra: derivation and explanation. As previously indicated, the two-column (statement/reason) format is viewed as typically belonging exclusively to geometry.

Proof in algebra is seen by some pre-service teachers in terms of manipulating equations:

... algebra proofs typically have an equation involved where a student must take one side and by utilizing mathematical properties, find that in a given set, either it is or is not equal to the other side. (Pst-D 1223)

And Pst-P stated that:

... for algebra, if you’re proving an equation, or why cosine squared plus sine squared equals one, there's like a more routine, you know, you don't do a chart [i.e., two-column proof] (Pst-P 1243)

Interestingly, most “derivation” proofs (that is, those like trigonometric verification) often are developed and presented in a two-column format. Pst-P’s response seems to suggest that the two are different. In a sense, “derivation” proofs appear to hold a somewhat lower status than proofs in geometry.

Several pre-service teachers view proof in algebra as explanation. For these pre-service teachers, the axiomatic structure of the real numbers as a field seems to be reduced to the “rules” of high school algebra; namely, one can do anything to one side of an equation as long as it is done to the other (except divide by zero) with only implicit understanding that certain properties, such as commutativity, apply. For example, Pst-H stated:

Algebra proofs don’t need to be as formal and sometime can simply involve explaining your answer or the steps you took to solve a problem. (Pst-H 1239)

Pst-A brought up a commonly employed shortcut method known as “FOIL” for multiplying two binomial expressions⁷: “in my experience proof in algebra didn’t exist and geometry was the only proof I experienced (even though most students are exposed to FOIL and quad. [*sic*] Formula)” (Pst-A 1224). From Pst-A’s response, it appears that this method is used in high school algebra as an acceptable explanation for the product of two binomial expressions.

As for the professional mathematicians, two specifically indicated that formal, detailed proof doesn’t belong in high school algebra because the primary goal of high school algebra is to teach proficiency with algebraic manipulation. A third professional mathematician also indicated that the goal in algebra is proficiency with algebraic

⁷ FOIL is an acronym for First, Outside, Inside, and Last referring to the multiplication of the terms of two binomial expressions.

manipulation, adding that “the underlying ‘axioms’ for the real line as a (complete ordered) field might complicate skill development” (Pm-I 2656).

Most professional mathematicians in this sample view proof in high school algebra in terms of the axiomatic system usually associated with the real numbers as an algebraic field and for that reason don’t view proof as belonging in high school algebra. Many see manipulation of algebraic expressions and equations, skills acquisition, and systematic thinking as the primary goals of high school algebra. For example, Pm-B stated: “in algebra, we need to teach our students to manipulate equations and be systematic” (Pm-B 1254). Interestingly, none of the professional mathematicians indicated understanding or explanation in conjunction with algebraic manipulation. It may be that the “rules” are given or explained to students with the tacit assumption that they understand why they are able to manipulate expressions and equations in certain ways.

Most professional mathematicians indicated that proof is more appropriately taught in high school geometry. Reasons cited for studying proof in geometry rather than algebra include: that the objects of geometry are simple and intuitive; and the structure of Euclidean geometry lends itself to the study of axiomatic mathematics. Professional mathematicians seem to believe that the study of proof from an axiomatic viewpoint is an important aspect of mathematics learning for high school students.

For the most part, pre-service teachers seem to view proof in high school geometry as a separate but related topic. Pst-A, for example, stated that:

...you learn about the “if this implies that... then this implies that...” that's where you're introduced to it, but I know why it would have to be geometry. But that's where we learned about all... the contrapositive.... I suppose you could do it in another course, but that part of the course [the proof activity] almost seems

separate from geometry. But the thing is, after we learn that at the beginning of the geometry course, then we went into the shapes and what not, and then you would learn more theorems about the shapes, so... it kind of fit in nice there where you're learning... you're [*sic*] learn the general, like, "if it's cold outside then it's snowing" and you use the general... and then when you get to the shapes, you can use it there because there's a lot of examples there. (Pst-A 1147)

It seems that pre-service teachers' perceptions of proof are largely formed around their personal experiences with high school geometry. Specifically, they seem to view proof in a particular format (i.e., two-column, statement/reason format).

Summary

For most pre-service teachers in this sample, geometry is viewed as a better place to learn proof than algebra. Several specifically indicated that they view geometry as more visual and therefore the proofs tend to be more visual. This suggests that proofs in geometry are more immediately understandable by students because they can more easily visualize the conjectures. In addition, they perceive the objects of geometry to be more concrete to high school students. From the large number of responses indicating that for pre-service teachers there is no proof in high school algebra, it seems that another explanation that they view proof as belonging only in geometry is because that is how they were taught and/or that is the way it has always been.

For the participants in this study, proof in high school algebra is perceived quite differently than proof in high school geometry. The pre-service teachers seem to understand that there are underlying axioms, definitions, and theorems, but for the most part they view algebra as the manipulation of equations according to certain rules. There is no indication that they view the underlying rules as having been established by proof or if they were simply stated. Several responses suggest that the rules are, in general, simply

given to students. Professional mathematicians seem to have a clearer understanding of the reasons for introducing proof in geometry rather than other classes. They view geometry as a simple and intuitive setting thereby better lending itself to studying an axiomatic system. Some suggest that proof can be taught in several different subject areas with possibly better results than how it is currently being taught. For example, Pm-C noted that “geometry offers an excellent opportunity to study the axiomatic method” (Pm-C 1059). But there are other areas of mathematics in addition to geometry that offer good settings in which to develop reasoning and proof:

Combinatorics, though seldom taught in high school, offers excellent opportunities for reasoning about sets and functions. Elementary number theory, also seldom taught in high school, presents opportunities for observing phenomena that seem mysterious until one meets novel ideas. Learning some Boolean algebra might provide a context for doing proofs and also help student think about the structure of proofs. (Pm-C 1060)

From a learning standpoint, Pm-I stated:

Proof is more difficult to address when students need to learn new fundamental “ideas” – like limit. Set theory or geometry rely on ideas that are understood intuitively (even if they go undefined). The more complex the BASIC notions the more you confound proof writing as the main focus. (Pm-I 1052)

Pre-Service Teachers’ and Professional Mathematicians’

Perceptions on What is Explicitly Told to Students About Mathematical Proof

This section describes the results of pre-service teachers’ and professional mathematicians’ perceptions on what is explicitly told to students about mathematical proof. This section is divided into two subsections. The first subsection relates information about participants’ perceptions of the difference between requesting someone to “show” or asking someone to “prove” in the context of a proposition. The second

subsection relates information regarding participants' perceptions about what is explicitly told to students about mathematical proof.

Differences Between “Show” and “Prove”

Table 21 provides the frequencies of responses of participants regarding their perceptions of the difference between requesting someone to “show” or asking someone to “prove” in the context of a proposition.

	Pre-service teachers	Professional mathematicians
Show and prove are the same	6/13 ^a (46%) ^{b, c}	4/7 ^d (57%) ^b
Show and prove are different	8/13 ^a (62%) ^{b, c}	3/7 ^d (43%) ^b

^a One pre-service teacher provided a contradictory response. This will be discussed below.
^b Percentages are rounded to the nearest integer.
^c Due to a participant’s contradictory response, percentages exceed 100%.
^d One professional mathematician’s answer for this question could not be determined.

Table 21 shows that roughly half of the pre-service teachers and roughly half of the professional mathematicians indicated that the terms “show that” and “prove that” mean the same thing. The remainder of the participants indicated that the two terms mean different things. Many of the pre-service teachers equate the request “show that” with a request to provide examples that make the claim true. For example, Pst-G stated that “if a student was asked to show that this proposition was true a couple of examples would be suitable...” (Pst-G 1981). Pst-K stated that there is a difference between the terms “since show that implies an example was enough” (Pst-K 2032). Pst-A indicated that there is a difference, but didn’t provide any explanation: “when students see ‘show’ they don’t think of proof” (Pst-A 1988). In all, of the eight pre-service teachers who indicated that

there is a difference between the terms, five indicated that a request to “show” implies that to provide a few examples is sufficient. The two remaining pre-service teachers’ responses indicated that the phrase “prove that” signifies that a more formal proof is required. For example, Pst-U responded:

... prove emphasizes a more formal type of response; a formal reasoning. Whereas show emphasizes... it doesn't really emphasize anything, just that I want to know that, you know, they understand this concept. ... they can show me in some mathematical way how they got an answer. (Pst-U 2008)

And Pst-M stated:

If it were “give an example of...” but I tend to regard “show that” in much the same respect as “prove.” The only difference however, I feel is that “prove” would need a more formal writing technique. (Pst-M 2018)

Likewise, two of the three professional mathematicians who indicated that there is a difference between the two terms responded in a similar manner. For example, Pm-B stated:

In college math, at least, “show that” is typically a code phrase to indicate that slightly less rigor is required (at least stylistically)... (Pm-B 2038)

And Pm-G stated:

We often use “show that” to mean the same thing as “prove that...,” except perhaps that we want to indicate a lesser degree of formality. (Pm-G 2045)

One pre-service teacher seemed to contradict herself in her response to the difference between the terms “show that” and “prove that.” Pst-L responded:

... “show that” is just another way of saying “prove that.” If you wanted just a few examples for the students to show it would be acceptable, but not if you are asking them in general. (Pst-L 1997)

However, it appears that Pst-L does hold different meanings for these terms and that their meaning may be dictated by the context as she perceives it. The following extract was

taken from an interview with Pst-L as she was responding to the question, “why prove something obvious?”:

... because ... a lot of the time before I took senior seminar, it's like, we were doing ... we were showing things from the axioms in Euclidean geometry. And like showing that there is a unique identity element... it's like, you think it's like, really, yeah... it's obvious.... (Pst-L 2670)

Pst-L uses the word “show” in connection with the axiomatic systems of Euclidean geometry and algebra. During the same interview, I asked Pst-L how she might present a theorem for which the proof is too advanced for the students to grasp and she responded:

...if there's like maybe a way that you can show that, like... maybe like show them, like say, you can see it if you did this example, and like do a few examples of it. But not say “well, this is the proof,” because you need to have it generalized to be a proof. (Pst-L 2684)

It is likely that this is more a linguistic issue than a lack of understanding. And, it is fairly clear that Pst-L means to highlight the difference between inductive and deductive reasoning. However, because of a lack of precision in language, there may be further ambiguity about intended meaning and expectations.

What Do Teachers Tell Their Students?

Of the ten participants who considered the terms “show that” and “prove that” as synonymous, two participants specifically indicated that teachers should clearly inform their students that they regard the two terms as synonymous. None of the remaining participants gave any indication that students should be directly informed of any difference in terminology. In fact, professional mathematician Pm-B’s response of “code phrase” suggests that students come to know what might be expected of them indirectly.

Intent of meaning may also be uncertain, ambiguous, or misleading for students.

For example, when asked if a proof can become invalid, professional mathematician Pm-E stated:

No. If an error is found centuries later it only means that we have been calling something a “proof” that never was. This is largely semantic, of course, since we routinely tell people “your proof is wrong.” Speaking strictly, we mean “your attempt at a proof has been unsuccessful.” (Pm-E 328)

Students may, in fact, not understand the difference between “wrong” and “unsuccessful.” The difference between a statement and its intended meaning may be largely semantic for professional mathematicians who have the wisdom and experience to discern the difference, but for pre-service teachers (whose conceptions have been established as being less than optimal) saying “your proof is wrong” may well have a certain connotation associated with it, regardless of the intended meaning.

Several pre-service teachers indicated that in their own experience as students, there has been a great deal of variation among different professors about what they expected from students. For example, Pst-M stated:

I have [Professor A⁸]... we had to prove a geometry [conjecture]... and his meaning of proof means everything has to be there; there has to be a reason for every single step that you make. Whereas with [Professor B⁹] ... like I just had a class with him, and he's kind of like “ etc., etc. ...” it's not as formal. So informal... I mean, we're still told that it's true by him and we can see how it is, but it's not laid out... every single step isn't laid out individually... for every step, you just kind of trust it. (Pst-M 413)

There were only a few other instances in which participants specifically indicated that students be informed of expectations. Most of these responses seem to be directed toward pointing out to students the inadequacy of using examples as proof. For example, Pst-K stated:

⁸ Pseudonym

⁹ Pseudonym

I would say to present a proof that has examples that work and examples that don't work. Show them, OK, does this work or doesn't this work? And present it to the class and if some people find it works and some people find that it doesn't work, use of examples, say "OK. So, why wasn't an example enough to prove it? And why was an example enough to disprove it?" (Pst-K 2461)

Responding to the question of why all steps must be included in student proofs, one participant stated that students should be informed:

To know that each step is a valid argument in a proof and I need to see that step. I would tell them that pretend I don't know ANYTHING and to show me every step even if it is remedial, I want every step. Don't take anything for granted. (Pst-N 1688)

Summary

Regarding proof in high school mathematics, Pst-D suggested that "we should probably have a tiered system that standardizes what proofs look like in each grade, by teacher" (Pst-D 1131). This statement expresses a somewhat unrealistic and possibly unattainable goal, given the wide range of meanings associated with mathematical proof. At the same time, Pst-D's statement expresses a desire to make proof less mysterious and less difficult for students by allowing them to know in advance what is expected of them. Roughly half of the participants indicated that "show" and "prove" represent the same thing and that roughly half of the participants indicated that the two terms mean different things. This itself provides evidence that there is a range of possible expectations for students. Participants' responses further indicate that the differences in meanings of terminology and what is expected of students for mathematical proof is not generally stated explicitly. In fact, it appears that a great deal of what is learned about proof by students is learned indirectly. This might also help to explain pre-service teachers' varied and vague understandings of mathematical proof.

CHAPTER 7

DISCUSSION, CONCLUSIONS, AND IMPLICATIONS

Chapter Overview

Recall that the objective of this study was to identify similarities and differences between and among pre-service teachers and professional mathematicians with respect to their perceptions of proof in mathematics and proof in high school mathematics. The purpose for identifying these similarities and differences was to highlight inconsistencies between and among the two groups of participants with the expectation of providing information that would help to improve pre-service teacher education. The previous two chapters reported the results of participants' perceptions regarding proof in mathematics and proof in high school mathematics respectively. In this chapter, the major findings will be highlighted and discussed. This chapter is organized into sections corresponding to the major findings with respect to participants' perceptions about mathematical proof in mathematics and participants' perceptions about mathematical proof in high school mathematics. These two sections are followed by a section containing some concluding remarks and some personal observations. Finally, there are sections on the limitations of this study, implications of this study for teacher education, and a section on implications for future research.

Discussion of Participants' Perceptions of Proof in Mathematics

“I hate proofs” (Pst-F).

The focus of this section is a comparison of the differences between the two primary groups of participants regarding their perceptions of proof in mathematics. The primary differences observed fall into the following categories: descriptions of proof, form versus content, validity and the nature of proof, orthodoxy in proof, proof as a separate part of mathematics, anxiety about proof, proof as explanation, and validation of arguments.

Descriptions of Proof

Most pre-service teachers' descriptions of proof were vague, non-specific, and diverse. In addition, a large number of pre-service teacher responses provided additional characteristics of proof such as the certainty of a result, the domain of validity, and incidental mention of proof-terms such as “axioms” in their descriptions which seems to reflect a repetition of the textbook properties of proof rather than a deep understanding of proof. From their descriptions of proof in mathematics, it appears that the pre-service teachers in this study found it very difficult to describe or define mathematical proof.

Most professional mathematicians, on the other hand, were easily able to provide fairly specific and verbally sophisticated descriptions or definitions for proof. Most of the professional mathematicians described proof in the traditional formal manner; as a sequence of statements beginning with axioms, undefined and defined terms, and proceeding by rules of logic to the desired conclusion. However, while professional mathematicians outwardly defined proof formally, their responses to other items

indicated that their initial definition is largely theoretical; in practice, professional mathematicians engage primarily in less than formal proof. Hersh (1997) noted the difference between what mathematicians outwardly profess and what they actually do in practice when he observed that:

The working mathematician is a Platonist on weekdays, a formalist on weekends. On weekdays, when doing mathematics, he's a Platonist, convinced he's dealing with an objective reality whose properties he's trying to determine. On weekends, if challenged to give an account of this reality, it's easiest to pretend he doesn't believe in it. He plays formalist, and pretends mathematics is a meaningless game. (pp. 39-40)

In this study, the difference between professional mathematicians' outward descriptions of proof and how they employ proof in practice is not surprising given that mathematics is traditionally presented as a finished product without the fits and starts associated with the actual process of arriving at the finished result (e.g., Maclane, 1994). This method of presentation, however, may give pre-service teachers an incomplete understanding of proof in mathematics. In the preface to his work *Mathematics and Plausible Reasoning*, Polya (1954) states that:

Finished mathematics presented in a finished form appears as purely demonstrative, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making. You have to guess a mathematical theorem before you prove it; you have to guess the idea of the proof before you carry through the details. You have to combine observations and follow analogies; you have to try and try again. The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing. If the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing, for plausible inference. (p. vi)

In order for pre-service teachers to gain a more accurate understanding of proof in mathematics, it may be necessary to explicitly state the differences between the formal and practical aspects of proof in mathematics.

Form Versus Content

Many pre-service teachers in this study focused on the form of an argument rather than its content. For most pre-service teachers, the statement-reason format is crucial for an argument to be considered as a formal proof. The two-column format of proof typically employed in high school geometry usually involves a high level of detail and rigor, whereby no statements are given without reasons and all intermediate steps must be included however obvious or simple they might be. Recall that Harel and Sowder (1998) identified the ritual proof scheme as one for which students are concerned with the appearance of the form of a mathematical proof rather than the correctness of the reasoning. Pre-service teachers in this study seemed to judge arguments on the degree to which they conform to the two-column format of proof.

Professional mathematicians, on the other hand, attended to the content of the argument and whether it actually establishes the given proposition. Many pre-service teachers referred to the explanatory aspect of proof but in a primarily procedural sense. For these pre-service teachers, understanding means being able to explain how one is able to go from one step to the next rather than understanding in the more global sense of knowing why a claim should be true. Knuth (2002b) also observed that among in-service teachers, explanation as it related to proof pertained to understanding how one can proceed from one step to the next. Interestingly, none of the professional mathematicians specifically indicated understanding as a component or purpose for proof. It may be that professional mathematicians view proof as an explanation for why a claim is true and as a means for understanding, but none of the professional mathematicians in this study specifically indicated understanding as a component or purpose for proof.

Consistent with the findings of Knuth (2002b) and Cyr (2004), evidence from this study suggests that for pre-service teachers the two-column proof may be the primary component of their concept definition for formal proof. For this reason, virtually all other “types” of proof (i.e., paragraph, visual, etc.) for the pre-service teachers belong to the class of informal proof. Professional mathematicians seemed to indicate that formal proof is generally an ideal and that for the most part, the practice of mathematics relies on degrees of formality.

Validity and the Nature of Proof

Almost all of the professional mathematicians in this study indicated that “validity” has specific meaning in mathematical proof: an argument is either valid, in which case it is a proof, or the argument is not valid, in which case it is not a proof. While professional mathematicians’ perceptions about validity were fairly uniform, there was some division about pre-service teachers’ views regarding the validity of arguments. Half of the pre-service teachers stated that proofs can’t become invalid since they are based on assumed truths and previously established theorems. However, half stated that proofs can become invalid. There were two primary reasons cited by the pre-service teachers that a proof can become invalid: 1) a proof can become invalid if an error is found; and 2) consistent with the findings of Coe and Ruthven (1994), Chazan (1993), and Knuth (2002b), many of the pre-service teachers in this study believe that it is possible for a counterexample to a claim to be found even after a deductive proof is given. Many pre-service teachers also indicated that it is possible for axioms to be true or false and that it is possible for axioms or theorems to be proven wrong even after they have been stated or proved. This suggests that pre-service teachers have a fragile

understanding of axiomatic systems in the sense that many stated that a proof is based on “truths” and that the axioms must be true or a proof becomes invalid. There seems to be a gap in pre-service teachers’ understanding of assuming the premises of an assertion and proceeding with logic to arrive at the conclusion.

Several pre-service teachers indicated in some cases that one argument can be more or less valid than another. It is likely that they are confusing validity with other characteristics of the argument. One possibility is that an argument may be “more valid” to someone if it is personally more convincing or easier for the person to follow.

Several pre-service teachers linked the validity of proof to the understanding of the proof writer, that is, if the writer can’t demonstrate that he/she understands what he/she is doing, the proof is not valid. Several pre-service teachers indicated that certain details (such as what needs to be shown) must be explicitly stated in a proof for it to be considered valid. For example, one of the arguments participants were asked to judge used the contrapositive but only stated that fact as an afterthought (see Appendix C, section II, student argument number 1, items 19 through 21). One pre-service teacher indicated that “The first [argument] should have stated the contrapositive and then proved that. ...but without that contrapositive stated it is not valid” (Pst-L 2125). In other words, the validity of the proof rests on the author providing the proper detail for the reader. For professional mathematicians, a proof should contain an appropriate amount of detail for the audience for which it is intended. This stance suggests that if there are perceived gaps on the part of the reader, it is incumbent upon the reader, and not the writer, to supply those details. For professional mathematicians, however, the level of detail included or the background of the proof reader does not affect the validity of a proof.

Similar to pre-service teachers, many professional mathematicians indicated that an argument may contain errors, but stated that in this case the argument was not valid to begin with. Professional mathematicians generally expressed the thought that an argument is either valid in which case it is a proof, or it is not valid and thus is not a proof. Several professional mathematicians acknowledged that potential inconsistencies in one or more axioms could arise which would render proofs based on those axioms invalid, but this depends on fundamental shifts in the system itself. The primary difference between pre-service teacher and professional mathematicians' perceptions of validity is that professional mathematicians view validity with the consistency of the axiomatic system in mind, whereas pre-service teachers view validity as somehow finding fault in the argument itself either by identifying errors, finding a counterexample, or disproving previously proven (or accepted) claims. This suggests that pre-service teachers have a fragile understanding of the certainty and universality of proof. In other words, once a valid proof establishes the truth of a proposition, it is not necessary to look further for possible counterexamples to disprove the proposition, nor is it necessary to examine additional examples. Hoyles and Healy (1999) and Fischbein (1982) found that among high school students, a majority needed additional empirical evidence to convince themselves that a proposition was true even after seeing a deductive proof of the proposition. In their work with high school students, Coe and Ruthven (1994) and Chazan (1993) found that high school students thought it was possible to find counterexamples to a deductive proof. The significance of the finding in this study is that the same observation is being connected with advanced undergraduate students.

Orthodoxy in Proof

“Orthodoxy in proof” in this study refers to the accepted canons and practices of proof. Many of the orthodoxies may be considered to be sociomathematical norms relative to certain mathematical communities. For example, when presenting a proof of a theorem to a collection of topologists, it may be unnecessary to explicitly calculate a topological space’s fundamental group. In other cases, sociomathematical norms may vary from one class to the next. For example, at the undergraduate level, several pre-service teachers noted that there were different expectations regarding what is acceptable as proof among different professors and different classes. Some of the orthodoxies, however, may have more to do with misconceptions or misinterpretation of accepted canons and practices. For many pre-service teachers, the idea that “examples are not proof” is somehow translated to mean that the use of examples is never allowed or, as indicated by Weber and Alcock (2004), implicitly discouraged. For example, pre-service teachers rejected some arguments simply because of the presence of examples without regarding the content of the argument. In one case, the proposition: “For any integers a and b , if ab is odd then a is odd and b is odd” was given. One of the arguments supplied was:

an odd times an even = an even	ex $2 \times 1 = 2$	$101 \times 10 = 1010$
an even times an even = an even	ex $2 \times 2 = 4$	$98 \times 8 = 784$
an odd times and odd = an odd	ex $3 \times 3 = 9$	$27 \times 9 = 243$

so, if ab is odd, both a and b must be odd.

(see Appendix C, section II, student argument number 2, items 19 through 21). Most professional mathematicians indicated that the argument given for this proposition was valid if the first three statements were accepted as true. One professional mathematician

indicated specifically that the examples were “extraneous” but did no harm. Most pre-service teachers, however, rejected the argument simply because it contained examples. In a study of undergraduate students, Raman (2002) found that they tended to dismiss pictorial arguments because they are not mathematical; in other words, the picture does not look like a proof, even though the picture conveyed the idea of why the given proposition was true. In the present study, however, most of the pre-service teachers dismissed pictorial arguments simply because they were taught that pictures do not constitute proof. For many pre-service teachers, the two-column format seems to be the standard for formal proof. In this sense, the two-column proof appears to be a part of the orthodoxy for proof maintained by pre-service teachers. The findings concerning pre-service teachers’ absolute beliefs about the use of examples and pictures in proof and their beliefs about the two-column proof as defining formal proof, together suggest an inflexibility about the beliefs about proof held by pre-service teachers. These beliefs may be transmitted inadvertently to students by their teachers. For example, when asked when the use of examples is appropriate, one professional mathematician stated that:

Doing such calculations [i.e., demonstrating the claim by showing specific examples that work] could help a student discover a more refined statement which actually has an easier proof. But I would prefer for students at this level to suppress “private” work and only share the rewards. (Pm-C 1974)

It is possible that only the idea of “suppressing private work” (i.e., don’t include the examples) is transmitted to students and not the reasons for suppressing private work.

Consistent with Weber and Alcock (2004) who found that professional mathematicians often use inductive reasoning to validate proof, evidence from the present study indicates that professional mathematicians often use inductive arguments privately to convince themselves of the truth of a claim. Evidence from the present study also

indicates that professional mathematicians use inductive reasoning publicly when the details of the proof are deemed unnecessary (oftentimes in a setting with colleagues such as in colloquia or in classroom settings). The pre-service teachers in this study, on the other hand, were generally more inflexible in their outward beliefs regarding examples in proof and were less likely to indicate the use of examples to motivate or illustrate propositions, to look for patterns, or to help them discover a proof.

Proof is a Separate Part of Mathematics

Consistent with the findings of Knuth (2002a) regarding in-service teachers, many of the pre-service teachers in the present study view proof as separate part of mathematics. It is likely that this belief is established and/or entrenched by their previous learning experiences. For example, many pre-service teachers indicated that proof is learned in geometry, that there is no proof in algebra, and that it seems common for most college students to be exposed to proof for the first time in a separate proof-class.

In some cases, pre-service teachers cite that the purpose of proof is to establish the truth of an assertion so that it can then be applied. For example, one significant part of the experience of high school algebra is the application of the quadratic formula to find roots of quadratic equations. The quadratic formula is sometimes derived but other times it is simply given to students as illustrated by Pst-N's comment that: "...in high school, I didn't learn the derivation of [the quadratic formula] until I got to college for some reason..." (Pst-N 1309). Whether the formula is proven or not, once the formula is available, the focus shifts to procedural manipulation of equations using the formula. As Pst-N stated: "...in high school, it's just like, well, here's the equation. Just do it. And you're done" (Pst-N 1330). While this may be an important and worthwhile aspect of the

high school curriculum, it may be that what is remembered from this experience by students is the manipulation and not the reasoning behind it. The tradition of providing a formula (with or without proof) with the purpose of having students manipulate equations may further serve to emphasize the belief that proof is a separate part of mathematics. Contrast this with the view held by most professional mathematicians in this study who perceive mathematical proof as an essential and inseparable part of mathematics.

Several pre-service teachers suggested that it was some unknown mathematician or group of mathematicians who provide justification for theorems and mathematical formulas after which they become a part of the mathematical knowledge base. Non-mathematicians (or mathematicians of “lesser” ability) are then free to tap into the resources and use the previously established knowledge.

Anxiety About Proof

Many pre-service teachers expressed fear and anxiety about proof in mathematics. For example, Pst-F stated “...I'm terrified of proofs. I hate proofs. I never felt that I had a good grasp on them at all. And we didn't do them in all of high school, and I just got to, I don't know... I hate them” (Pst-F). I theorize that this anxiety stems from the fact that, for students, proof is less procedural than solving an algebra or calculus problem and it requires a deeper level of thinking (Tall, 1991) that students may not be capable of handling initially.

Proof as Explanation

Most professional mathematicians indicated that the purpose of proof in mathematics is to establish the truth of a claim. A major purpose for proof cited by pre-service teachers was that of explanation or understanding. However, for many pre-service

teachers, understanding means being able to explain how one is able to go from one step to the next rather than understanding in the more global sense of knowing why a claim should be true. On the other hand, professional mathematicians rarely mentioned the purpose of explanation, although it is implied in a few responses. When indicated, however, the responses invariably pointed to understanding and/or explanation in the global sense. A few pre-service teachers indicated that proof helps understand in the global sense, that is, why a claim is true.

Validation of Arguments

Professional mathematicians may not need convincing of the truth of an assertion because they have seen it before, they can easily see why it's true without too much work, or perhaps a quick calculation is enough to convince them (e.g., Weber & Alcock, 2004; Raman, 2002). But for pre-service teachers, most haven't seen the proposition before and have limited experience with the subject area in general, so they don't have the same experiential intuition as professional mathematicians do (e.g., Moore, 1994; Weber, 2001).

One initial aspect of proof validation is to be convinced that a claim is true. Professional mathematicians may be convinced of the truth of a claim by the initial statement of the claim whereas pre-service teachers may need to be further convinced of the truth of the claim by the argument. As a second aspect of validation, one must determine to one's satisfaction that the argument does indeed establish the veracity of the claim. This requires an understanding of the characteristics of the mathematical objects involved, the logical devices employed in the argument, and an ability to follow the lines of reasoning. In the case of the third calculus proof (see Appendix C, section I, argument

number three, items 9 through 11) for example, many pre-service teachers indicated a lack of understanding of how the application of the chain rule established the claim. Many pre-service teachers indicated that a proof must explicitly state what needs to be shown. In the case of the this calculus argument, this means to explicitly state: “what must be shown is if $f(x)$ is an even function, which means $f(x) = f(-x)$, then $f'(-x) = -f'(x)$ ” (see Appendix C, section I, items 9 through 11). Pre-service teachers also displayed a lack of understanding of how the claim was established because they weren’t explicitly told what needed to be shown.

Thus, pre-service teachers were not convinced that this argument established the claim. To be convinced by an argument, many pre-service teachers appear to need an argument that provides all the steps along with justification for the steps. “Understanding” involves the ability to understand how one step follows from the previous step (whether the justification is explicitly stated or not). The discrepancy between the third calculus proof needing steps and reasons and trigonometric proof not needing reasons provides evidence for this observation. For pre-service teachers, the steps of the trigonometric argument and the reasons for taking those steps (even if not stated) are easily understood (divide, simplify, etc.). The second argument of the calculus proposition is similar in that it doesn’t require an understanding of derivative, or even and odd functions, but rather requires only symbol manipulation. The trigonometric validation and the second calculus argument represent what Rav (1999) called “derivation” proofs. They generally involve manipulation of equations and/or symbols but do not require a deep understanding of the concepts involved. So the actual reason why the derivative of an even function is odd is missed. Interestingly, the third argument

of the calculus proposition might also be considered to be a derivation proof. In fact, only the first argument gives a hint as to why the derivative of an even function is odd, albeit in an incomplete and possibly misleading sense.

Consistent with the findings of Selden and Selden (2003), pre-service teachers in this study sometimes focus on surface features. For example, pre-service teachers in this study disqualified arguments based solely on the fact that it contained examples. On the other hand and in contrast to the findings of Selden and Selden (2003), pre-service teachers in this study did not notice errors in several of the arguments. If pre-service teachers were focusing closely on surface features of the arguments, one might expect them to detect such errors. It is possible that because pre-service teachers may have very little experience in proof validation (Selden & Selden, 1995), they are not as capable of detecting errors, while professional mathematicians, due to their experience (particularly with student work), are keyed into looking out for certain types of common errors.

Summary

In this study, the professional mathematicians seem to be considerably more flexible in their views of mathematical proof and what they will or will not accept as valid proof. However, what is acceptable and what is not acceptable is context dependent and the parameters for acceptance do not appear to be well-defined or well-articulated. There appears to be a process of enculturation into the canons of argumentation. The canons vary greatly from one area of mathematics to another, but there seem to be standard, albeit sometimes implicit, rules for argumentation. It appears that these standards of argumentation are not well understood by the majority of pre-service teachers in this study. There may be several reasons for this:

- 1) The pre-service teachers in this study have had only limited experience with mathematical proof. Much of the work in the typical college-level calculus sequence, for example, is centered around new concepts such as limit and continuity, as well as the algebraic manipulation associated with derivatives and anti-derivatives. As Dreyfus (1999) observed, “proof” as presented by professors in calculus is most likely to be an intuitive explanation rather than deductive proof. After calculus, there is a sudden shift toward proof-oriented classes (Moore, 1994) for undergraduate students. Many of the pre-service teachers’ statements in this study indicated some level of anxiety associated with proof. It may be that the abruptness of the shift toward proof-oriented classes can have an appreciable negative effect on students.
- 2) Pre-service teachers do not appear to get a genuine sense for the true nature of mathematics from their previous mathematical experiences and coursework. Pre-service teachers are typically presented with mathematics as a finished theory (Almeida, 2003; Alibert & Thomas, 1991; Weber, 2004). For pre-service teachers, there doesn’t appear to be any of the typical characteristics of scientific inquiry such as intuition, speculation, experimentation, conjecture, and finally, proof (Maclane, 1994). In addition, a great deal of the mathematics that pre-service teachers encounter is presented as being true to begin with (i.e., “prove the following...”), so for students, proof becomes an exercise involving manipulation with little personal meaning (Schoenfeld, 1994).
- 3) Pre-service teachers internalize certain beliefs as absolute doctrine. This may leave pre-service teachers with unnecessarily inflexible beliefs about the nature of how mathematics can be done, including the purpose and role of proof in mathematics.

- 4) The standards and expectations of argumentation in mathematics are not always explicitly presented or explained to pre-service teachers. In fact, many times they are implicit. Pre-service teachers must effectively guess at what is expected of them.
- 5) The standards for proof often change among different contexts (such as different classes, subject areas, or between different professors) for various reasons, which are not always understood by students. For example, one professor may have a strict set of standards for argumentation for the purpose of illustrating axiomatic systems, while another approaches argumentation in a less stringent and rigorous manner, perhaps to focus on the larger theory.

Alibert and Thomas (1991) note that the way mathematics is presented to undergraduate students may cause difficulties for students:

The apparent conflict between the practice of mathematicians on the one hand, and their teaching methods on the other creates problems for students. They exhibit a lack of concern for meaning, a lack of appreciation of proof as a functional tool and an inadequate epistemology. (p. 215)

In a study of first year university students, Alibert (1988) implemented what he referred to as “scientific debate” in the classroom. The scientific debate is initiated by the teacher first presenting mathematical statements to the students. Students are then expected to determine the validity of the statements by debate, proof, refutation, or counterexample. The majority of the students in the study provided positive responses when asked about this approach compared to more traditional approaches such as lecture.

Almeida (2003) observed positive responses from undergraduate students after they had participated in a project based on a sequence, which is, according to Maclane (1994), more faithful to the manner in which mathematics is done in practice by mathematicians.

Discussion of Participants' Views of Proof in High School Mathematics

The focus of this section is on a comparison of the differences between the two primary groups of participants regarding their perceptions of proof in high school mathematics. The primary differences observed fall into the following categories: purpose of proof in high school mathematics, expectations for high school students, and is proof for all.

Purpose of Proof in High School Mathematics

Pre-service teachers believe that the purpose of proof in high school is to promote student understanding, but in the procedural sense, that is, understanding why one can proceed from one step to the next. In this context, pre-service teachers believe it is important that high school students be able to follow the steps of a proof so that students understand, for example, where a formula comes from. Only a few professional mathematicians mentioned the role of understanding for proof in high school mathematics. Those that did used the term in a global sense, that is, why the theorem is true and what it means. There is some indication that pre-service teachers view proof as an aid to conceptual understanding (i.e., where a formula or claim comes from) and as a means of explanation (i.e., why a formula or claim is true), however, this was in only a limited number of cases. Most professional mathematicians view the purpose of proof in high school as being to teach logical thinking (consistent with Fawcett, 1938 and Fitzgerald, 1996) and to teach about the nature of proof in mathematics. Given that reform efforts like those proposed by NCTM (2000) call for students to learn mathematics with understanding, it may be significant that only a small number of pre-service teachers and professional mathematicians indicated explanation or understanding

as being the fundamental role for proof in high school. The relatively small proportion of participants who cited understanding and explanation as an important purpose for proof in high school mathematics is not encouraging, given that many mathematics educators believe that the primary purpose for proof in high school should be to provide understanding (see e.g., Hanna, 1989b; Dreyfus & Hadas, 1996; Hersch, 1993).

Expectations for High School Students

Most participants thought that proof was very important to mathematics. Yet, a smaller proportion indicated that proof is important for high school mathematics. This is particularly significant for the group of pre-service teachers, given that many pre-service teachers viewed proof in mathematics as important for understanding.

Most participants have relatively minimal expectations of high school students regarding mathematical proof. This outlook is inconsistent with the recommendation that “high expectations for mathematics learning [must] be communicated in words and deeds to all students” (NCTM, 2000, p. 13).

What Should Students Know About Mathematical Proof?

Pre-service teacher responses indicated that they expect students to learn about the purposes and importance of proof in mathematics (i.e., to establish truth of claims, to establish the foundations of mathematics, and to provide understanding and/or explanation), trigonometric identity verifications, derivation of the quadratic formula, and the two-column format commonly employed in geometry. By contrast, professional mathematician responses indicated that they expected high school students to learn about the definitions of the terms they use, and recognize the undefined terms, precise statements of theorems, the role and purpose of proof both in terms of the foundations of

mathematics and certainty of results, the structure of the language of mathematics, basic mathematical notation and syntax, when something is not a proof, the difference between a proposition and its converse, and the fundamentals of pure logic. Professional mathematicians were able to more clearly articulate specific goals that they view as important for high school students regarding mathematical proof. Pre-service teachers, on the other hand, were less specific about their expectations for high school students, particularly with respect to the nature of mathematical proof. From their own perceptions about the nature of proof in mathematics, it is possible that the pre-service teachers have unclear expectations for high school students because they are, themselves, uncertain about various aspects of mathematical proof.

Is Proof for All?

A surprisingly large proportion of pre-service teachers stated that it is not important for all students to learn proof. This is consistent with Knuth (2002a) and Almeida (1995) who found that in-service teachers and undergraduate mathematics majors believed that proof was appropriate only for students in advanced classes and those likely to go on to mathematics-related majors in college. It is likely that this idea of “proof for some” is linked to the idea of formal proof. That is, participants may view formal proof *as they define it* to be important only for some students, especially those going on to college. Other responses suggest that informal proof is important and should be included in all classes for all students. Knuth (2002a) found a similar result, however he defined informal proof as “arguments in which one provides reasons to justify one’s mathematical actions or presents examples to support one’s claims (in either case, not arguments one would consider to be valid proofs)” (p. 72). To the participants in this

study, informal proof sometimes means examples, but more often, it is considered to be a valid proof that is not as formal or rigorous.¹⁰ For example, arguments that leave out certain steps would be considered as informal, as would arguments that use natural language (i.e., using “there is a number” rather than using the symbolic form: “ $\exists x$ ”), paragraph format, etc., and in some cases, so-called “visual proofs.”

Proof in All Classes?

Most pre-service teachers and professional mathematicians in this study indicated that proof belongs in geometry, although the reasons they gave were different. For pre-service teachers, proof belongs in geometry because it has always been in geometry and because they believe that there is no proof in algebra. For professional mathematicians, proof belongs in geometry because geometry is intuitive and therefore it is easier to learn proof if the subject matter is more concrete, easier to visualize, and relatively familiar to students. Both groups view the purpose of algebra in high school as learning to manipulate expressions and equations. Proof in high school algebra as viewed by pre-service teachers means explaining what one is doing in a procedural sense and according to certain rules. Professional mathematicians perceive proof in algebra from the perspective of the axiomatic foundations of algebra, and therefore view proof as being perhaps unnecessarily difficult for typical high school students. Wu (1996) suggests that the viewpoint that proof only belongs in geometry may create or exacerbate misconceptions about mathematics:

¹⁰ Many of the participant responses were vague and in some cases circular as to their descriptions of informal proof. Consider two of the pre-service teachers responses: Pst-I wrote that an informal proof is: “A description of why a conjecture holds, not written in a formal manner” (Pst-I 416) and Pst-P stated that “An informal mathematical proof is similar to a [formal] proof yet less detailed” (Pst-P 428). Ambiguity was not restricted to pre-service teacher responses. Consider two of the professional mathematicians responses: Pm-I stated that an informal proof is “a proof that’s isn’t a formal proof” (Pm-I 461) and Pm-F wrote that “an ‘informal proof’ is one that may not meet the same standards of rigor that a ‘formal proof’ does” (Pm-F 442).

The perception that proofs in geometry are too far removed from everyday mathematics stems to a large extent from a glaring defect in the present-day mathematics education in high school, namely, the fact that outside of geometry there are essentially no proofs. Even as anomalies in education go, this is certainly more anomalous than others inasmuch as it presents a totally falsified picture of mathematics itself. (p. 228)

Summary

Collectively, professional mathematician expectations for high school students appear to be more closely aligned with expectations of reform recommendations such as NCTM (2000) and CBMS (2001) in that the reform documents call for students to learn specific aspects of proof, such as certain methods of proof, and mathematical language and notation, even though many professional mathematicians' responses suggest doubt about whether these expectations are possible or likely to be achieved.

It is unclear whether pre-service teachers' beliefs about proof in high school are due to their beliefs about proof in mathematics or whether their beliefs are largely established from their own experiences in high school. What is clear is that by teachers espousing minimal expectations, exposure to proof only in geometry, and suggesting that only college-bound students need to know proof, the visions for reform as expressed by NCTM (2000) for incorporating reasoning and proof as an integral part of all mathematics learning will be difficult to meet.

Concluding Remarks and Personal Reflections

Evidence from this study suggests that pre-service teachers' perceptions about mathematical proof are often inconsistent with the perceptions of proof held by professional mathematicians. In particular, five major results emerged from this study which highlight some of the inconsistencies observed between pre-service teachers and professional mathematicians with respect to their perceptions of mathematical proof:

- 1) Professional mathematicians value the content of an argument over its form while pre-service teachers place more importance on the details of the form of an argument, in some cases, to the exclusion of its content.
- 2) Professional mathematicians' view of what constitutes proof is flexible and context dependent, while pre-service teachers' perceptions about what is acceptable is much less context dependent, and in some cases, rigid and unyielding.
- 3) Pre-service teachers expressed that they have mixed and confusing ideas about what constitutes proof. For some pre-service teachers, there are contradictory ideas about what is acceptable as proof.
- 4) The results from this study indicate that pre-service teachers have an incomplete and naïve understanding about the nature of proof in mathematics. Pre-service teachers indicated that they expect high school students to know specific derivations and particular formats for arguments, while professional mathematicians expressed the desire that high school students know about the nature of proof in mathematics.
- 5) Results indicate that pre-service teachers don't always know what is expected of them. There doesn't seem to be explicit instruction between professional mathematicians and pre-service teachers at the undergraduate level regarding the canons and terminology of mathematical proof.

Evidence from this study and observations from previous research (e.g., Alibert & Thomas, 1991; Moore, 1994; Weber, 2001) suggest that these inconsistencies may be largely due to a lack of sufficient exposure to and experience with authentic proof-activities and a lack of explicit training and information about the nature of mathematical proof.

From a more personal perspective, this study grew out of my own experiences with mathematical proof. It seemed to me that the standards for proof at the undergraduate level changed from one classroom to the next or from one professor to the next. While learning about mathematical proof, the differences in standards seemed capricious and didn't seem to carry any explanation as to why the differences existed. As a result of exposure to more mathematics and the culture of mathematicians, as well as the work done on this study, my perspective on mathematical proof has changed. Mathematical proof is a complex domain. Like the domains of literature or art, there are no specific formulas that can be applied in an algorithmic manner. At any given time, there are certain strategies that apply in many situations and certain accepted ways of doing things. Like literature and art, what is "good" and what is "acceptable" depends upon context. Qualitative judgment may even depend on an individual's personal opinion. Also, like literature and art, one generally learns about the domain of mathematics and improves one's ability within the domain with practice and with exposure to the community of mathematicians.

Limitations of the Study

There were two limitations to this study. First, because no in-depth interviews were conducted with participants, I was unable to get answers to many of the questions raised throughout the data analysis. For example, most of the participants gave ambiguous descriptions of the differences between formal and informal proof. I couldn't completely determine the reason for participants' vagueness from the data. For many of the participants, there also seemed to be a difference between "formal proof" and

“reasoning.” It would have been beneficial to determine what meaning participants attached to these (and other) terms.

A second limitation of this study is the relatively small sample size of both pre-service teachers and professional mathematicians. For example, even though the coursework taken by the pre-service teachers in this study was fairly uniform, neither the class standing (i.e., junior or senior) of participants nor their experience prior to college were considered. However, placing pre-service teachers into smaller groups based on other characteristics could have weakened the strength of conclusions drawn from a comparison of the smaller groups.

There were factors that were not considered regarding professional mathematicians that also could have added to the data. For example, since each mathematical community has its own set of norms for argumentation, categorizing professional mathematicians’ by their area of specialty (i.e., algebra, analysis, topology, etc.) or their years of experience might have yielded important information, however, given the sample size available, any observable differences might have been individually idiosyncratic.

Implications For Teacher Education.

[The] context in which students meet proofs in mathematics may greatly influence their perception of the value of proof. By establishing an environment in which students may see and experience first-hand what is necessary for them to convince others, of the truth or falsehood of propositions, proof becomes an instrument of personal value which they will be happier to use [or teach] in the future. (Alibert & Thomas, 1991, p. 230)

Results from this study and from previous research point out that the problems and difficulties students and teachers experience with mathematical proof are manifold. By simply looking at the number of difficulties and misconceptions connected with

mathematical proof, a great deal of evidence emerges that points out the complexity associated with the teaching and learning of mathematical proof. Some implications for pre-service teacher education as a result of the present study are given below.

Pre-service teachers should be explicitly taught about proof and argumentation. First, pre-service teachers need to know what constitutes acceptable forms of argumentation. They should know under what conditions the standards of argumentation change and why. For example, pre-service teachers need to learn when it is necessary to include a high level of detail and/or rigor, and quite as importantly, *why* a particular level of detail or rigor in a proof is necessary or desired. They should also learn under what circumstances a particular level of rigor is not necessary and why. These goals might be accomplished by having students analyze, critique, and discuss authentic mathematical arguments.

Second, pre-service teachers should be exposed to genuine activities in reasoning and proof in their undergraduate work. These activities should include the type of investigative work typically done by professional mathematicians: intuition, experimentation, conjecture, and proof (MacLane, 1994). As noted by several researchers, college mathematics is typically presented in a traditional or finished manner (e.g., Almeida, 2003; Alibert & Thomas, 1991). Thus, incorporating changes in how the content is presented to students by professional mathematicians may be necessary. For example, rather than always presenting material as a finished and polished theory, topics could be presented in an incomplete form thereby allowing pre-service teachers to experiment, make conjectures, and attempt to validate or refute their conjectures.

Third, pre-service teachers should be engaged in discussions about and exposure to the real nature of proof in mathematics, including how proof is practiced in different subdisciplines of mathematics and exposure to the philosophical and historical aspects associated with mathematical proof. These issues could be addressed in a course on the history of mathematics.

Finally, pre-service teachers need to develop their understanding of the mathematics they will teach beyond a procedural understanding. Pre-service teachers need to develop a deeper understanding of high school mathematics and be able to engage their students in mathematical reasoning and discourse (Ferrini-Mundy & Findell, 2001; Schoenfeld, 1994; NCTM, 2000; CBMS, 2001; Hanna, 2000). This might ultimately necessitate changes in the focus and presentation of pre-service content courses to specifically address issues related to high school mathematics. In particular, how and where proof fits into the high school curriculum and how proof promotes understanding (Hanna, 1995).

In its reasoning and proof standard, NCTM (2000) states that:

Instructional programs from prekindergarten through grade 12 should enable all students to -

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof. (p. 56)

In light of this study, attaining these goals may require changes in how the content of mathematics is presented to pre-service teachers. Some of the changes to pre-service teacher courses might include: engaging in genuine mathematical activities such as the “scientific debate” (Alibert, 1988) or other projects in which students are guided to look for patterns, make conjectures, and ultimately search out an explanation (i.e., proof) for

their observations and getting explicit education in the nature of mathematical proof is needed that addresses the difficulties identified by this study and previous research. Rather than offering proof as an abstract subject in a separate class, proof could be introduced in a mathematical context such as number theory. Introduction to the fundamentals of axiomatic systems could be accomplished through investigation of other subject areas such as finite geometries where the focus is on the axiomatic system and not the objects within the system. To address issues related to the nature and appreciation of mathematical proof, Almeida (2003) argues that offering a course design that is faithful, to some extent, to the historical genesis of modern mathematics would positively affect students' proof attitudes and enable them to discover for themselves a "sense of proof" (p. 479).

Implications For Future Research

This study revealed discrepancies between and among the two groups of participants about mathematical proof and its place in the secondary mathematics curriculum. In particular, it was concluded that pre-service teachers' perceptions of mathematical proof are largely formed by their experiences with proof. This being the case, it follows that exposing pre-service teachers to more genuine mathematical experiences would change their perceptions to be more aligned with the actual practice of professional mathematicians. The question naturally arises as to whether and to what degree exposure to more genuine mathematical experiences would change pre-service teachers' perceptions of mathematical proof.

In addition, the pre-service teachers were purposefully chosen to be a relatively homogeneous group as far as their mathematical backgrounds were concerned and this

study represents only a snapshot in time. There remains the question of if and how these participants' perceptions will change over time as they gain experience in actual teaching. Will the teachers become more or less likely to incorporate mathematical proof in their teaching? Does the level of their incorporation of proof coincide with their previous beliefs? How do the perceptions of pre-service teachers compare with perceptions of in-service teachers?

The pre-service teachers in this study appear to place a high level of importance on understanding, explanation, and justification regarding mathematical proof in high school mathematics. How will these teachers present mathematical proof to their future students? Research aimed at answering these questions could provide insight into how to best prepare teachers for teaching.

To foster communication and understanding among high school teachers and professional mathematicians, it would be beneficial to determine more about professional mathematicians' perceptions of proof in high school mathematics.

In many university settings, professional mathematicians are largely responsible for teaching mathematical content courses taken by undergraduates including pre-service teachers. What would professional mathematicians' perceptions be to changes in how mathematics is presented to undergraduate students?

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APPENDICES

APPENDIX A

QUESTIONS FOR SEMI-STRUCTURED INTERVIEWS

Questions for semi-structured interviews regarding student-generated proofs.

- 1) Would you accept this argument as a valid mathematical proof? Why or why not?
- 2) What (if anything) do you think is wrong with this argument? Why?
- 3) Would your acceptance of this argument as a valid proof depend on the grade-level of the student? How? *Probe: which of the errors in (2) would you excuse? Why?*
- 4) Would your acceptance of the argument depend on the class (i.e., algebra, geometry, pre-calculus) of the student? How? *Probe: which of the errors in (2) would you excuse? Why?*
- 5) Would you use this argument in your own instruction if you were going to teach a proof of the original proposition? Explain why or why not.
- 6) Of the 3, 4 or 5 arguments for this proposition, is there any one argument you prefer over all the others? Why?
- 7) Were any of the arguments more convincing than others? Explain.
- 8) What factors do you consider in preferring one argument over another?
- 9) Is there anything you would like to add about this argument?

APPENDIX B

STUDENT-GENERATED ARGUMENTS

Proposition 1: Prove that the sum of 3 consecutive integers is divisible by 3.

Student argument # 1

$\begin{array}{r} -1 + -2 + -3 \\ -6 \\ \hline \frac{-6}{3} = -2 \checkmark \end{array}$	$\begin{array}{r} 7 + 8 + 9 \\ 24 \\ \hline \frac{24}{3} = 8 \checkmark \end{array}$
--	--

Student argument # 2

let n be any integer
 $n-1, n, n+1 = 3n$ so there's a factor of 3, therefore it's divisible by 3.

Student argument # 3

Examples: $1+2+3=6$ $6/3=2$ and $11+12+13=36$ $36/3=12$
 By adding 3 consecutive integers, it is the same as multiplying the second integer by three.
 Ex $1+2+3$ (subtract 1 from 3) now you have $2+2+2$ (which is the same as dividing by 3).

Student argument # 4

$1+2+3=6$ $6/3=2$	$16+17+18=51$ $51/3=17$
$4+5+6=15$ $15/3=5$	$19+20+21=60$ $60/3=20$
$6+7+8=21$ $21/3=7$	$22+23+24=69$ $69/3=23$
$10+11+12=33$ $33/3=11$	$30+31+32=93$ $93/3=31$
	$34+35+36=105$ $105/3=35$

Student argument # 5

$\frac{x+(x+1)+(x+2)}{3} = 0$ $2x+1+x+2=0$ $3x+3=0$	$3x = -3$ $3(x+1)=0$
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Proposition 2: Prove that for any integers a and b , if ab is odd then a is odd and b is odd.

Student argument # 1

let $a=2n$, $n \in \mathbb{I}$ and b any integer
 So $ab = 2n \cdot b$ which is an integer with a factor of 2 and therefore is even. Similarly, if b were even $a \cdot 2b$ would be even. If ab is odd then a and b must be odd.
 I used the contrapositive method here.

Student argument # 2

an odd times an even = an even ex $2 \times 1 = 2$ $101 \times 10 = 1010$
 an even times an even = an even ex $2 \times 2 = 4$ $98 \times 8 = 784$
 an odd times an odd = an odd ex $3 \times 3 = 9$ $27 \times 9 = 243$
 so if ab is odd both a and b must odd.

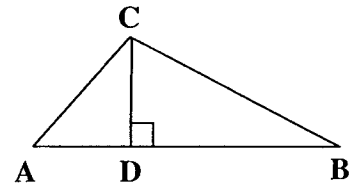
Student argument # 3

$7(3) = 21$	$21(7) = 147$	$15(7) = 105$	$7(9) = 63$
$1(3) = 3$	$15(3) = 45$	$7(11) = 77$	$9(11) = 99$
$5(1) = 5$	$17(5) = 85$	$13(11) = 143$	$9(13) = 117$
$11(3) = 33$	$17(3) = 51$	$15(11) = 165$	$9(9) = 81$
$7(5) = 35$	$19(1) = 19$	$21(11) = 231$	$9(5) = 45$
$13(3) = 39$	$19(3) = 57$		

Student argument # 4

An odd number times another odd number always equals an odd number.
 Examples: $5 \cdot 7 = 35$, $3 \cdot 9 = 27$, $11 \cdot 3 = 33$

Proposition 3: Given triangle ABC . Suppose CD is perpendicular to AB , and the length of CA is not equal to the length of CB . Prove that the length of AD is not equal to the length of DB .



Student argument # 1

$AC \neq CB$
 $AC = c_1$ (Hypotenuse) of $\triangle ACD$ $CB = c_2$ (hypotenuse) of $\triangle DCB$
 $A^2 + B_1^2 = c_1^2$ $A^2 + B_2^2 = c_2^2$
 $DB = B_2$
 $AD = B_1$ Therefore, since $c_1 \neq c_2$, The sides that make up the hypotenuses' do not equal as well $B_1 \neq B_2$

Student argument # 2

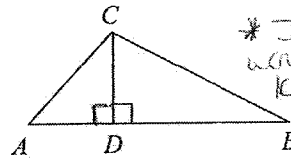
If I were to use a ruler
 and measure it, I could be showing the
 the lengths were not equal.

Student argument # 3

what I was think was
 that since $AC \neq CB$, this means that the
 overall triangle can't be equilateral because not all
 the side are the same. Since all side are not the
 same I would say that the CD could not split
 AB in half because in an equilateral triangle all
 of the vertices are directly halway along the opposite
 side. Since this is not an equilateral triangle the
 line CD could not split AB in half making
 $AD \neq DB$

Student argument # 4

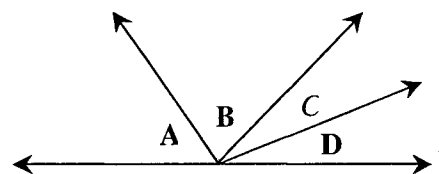
$\overline{CD} \perp \overline{AB}$. $\angle COA \cong \angle CDB$
~~both~~ both 90° each.
 If $CA \neq CB$, $AD \neq DB$



* I think I
 would need to
 know what
 kind of \triangle
 ABC was;
 and if CD
 was bisecting
 \overline{AB}

CD would have to be a bisector
 in order for \overline{AB} to be split
 in half w/ congruent sides \overline{AD} \overline{DB} .

Proposition 4: Given that l is a line, $\angle A \cong \angle B$ and $\angle C \cong \angle D$, prove that $m\angle A + m\angle C = 90^\circ$.



Student argument # 1

Statements	Reason
1) l is a line.	1) given
2) $\angle A \cong \angle B$	2) given
3) $\angle C \cong \angle D$	3) given
4) $m\angle A + m\angle B + m\angle C + m\angle D = 180^\circ$	4) angle addition property
5) $m\angle A + m\angle B = \text{obtuse } \angle$	5) def. of obtuse \angle
6) $m\angle C + m\angle D = \text{acute } \angle$	6) def. of acute \angle
7) $\angle B = \text{midpoint of line } l$	7) def. of midpoint
8) $m\angle B + m\angle D = 90^\circ$	8) def. of right \angle
9) $m\angle A + m\angle C = 90^\circ$	9) " "

Student argument # 2

$$m\angle A + m\angle C + m\angle B + m\angle D = 180^\circ$$

$$m\angle A + m\angle A + m\angle C + m\angle C = 180^\circ$$

$$m\angle A + m\angle C = 90^\circ$$

Student argument # 3

If any two \angle 's are \cong w/ each other and the other two are as well. The line = 180°
 then any two \angle 's added together = 90° because of a Theorem that was stated in Geometry.

APPENDIX C

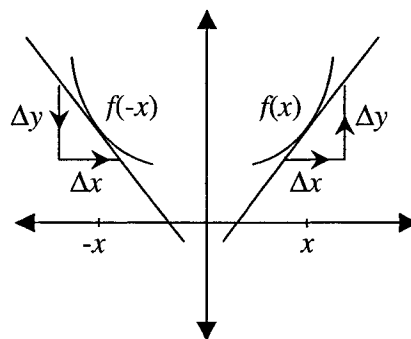
QUESTIONNAIRE

Please look over the following proposition and the three accompanying arguments for the proposition, and answer the questions that follow:

Proposition: The derivative of an even function is odd.

Argument # 1

If $f(x)$ is an even function, it is symmetric over the y axis. So the slope at any point x is the opposite of the slope at $-x$. In other words, $f'(-x) = -f'(x)$, which means the derivative of the function is odd.



Argument # 2

Want to show that if $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} \quad \text{by the definition of derivative.}$$

$$= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} \quad \text{since } f \text{ is even. Now, let } t = -h. \text{ Then,}$$

$$= \lim_{t \rightarrow 0} \frac{f(x+t) - f(x)}{-t}$$

$$= -\lim_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t}$$

$$= -f'(x) \text{ as desired.}$$

Argument # 3

Given $f(x)$ is even, so $f(x) = f(-x)$. Take the derivative of both sides. $f'(x) = -f'(-x)$ by the chain rule. So $f'(x)$ is odd.

The following questions refer to the proposition “The derivative of an even function is odd” given on page 3.

9) Which of these three arguments do you consider to be valid mathematical proofs? Please explain.

10) Which of these three arguments do you consider not to be valid mathematical proofs? Please explain.

11) Which of the three arguments do you prefer the most? Why?

Please look over the following problem and the accompanying argument, and answer the questions below:

Verify the identity: $\sec^2 \theta - \tan^2 \theta = 1$.

Argument # 1

$$1 = \cos^2 \theta + \sin^2 \theta$$

$$\frac{1}{\cos^2 \theta} = \frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$\frac{1}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} = 1$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

12) Is this argument a proof? Why or why not?

11) When working in the realm of proof, what is the purpose of using specific examples? Is this different in high school than in college or mathematics in general? Please explain.

12) When is it acceptable for students to leave out certain steps in a proof? Please explain.

13) Suppose a student leaves out a step in a proof that you consider to be important. What is your reasoning for wanting the student to include the step?

14) Why should students prove something that to them is obvious?

Please look over the following proposition and the two accompanying student arguments for the proposition, and answer the questions that follow.

Proposition: The sum of three consecutive integers is divisible by 3.

Student argument # 1

$$-1 + -2 + -3$$

$$-6$$

$$\frac{-6}{3} = -2 \checkmark$$

$$7 + 8 + 9$$

$$24$$

$$\frac{24}{3} = 8 \checkmark$$

Student argument # 2

$$1 + 2 + 3 = 6 \quad 6/3 = 2$$

$$4 + 5 + 6 = 15 \quad 15/3 = 5$$

$$6 + 7 + 8 = 21 \quad 21/3 = 7$$

$$10 + 11 + 12 = 33 \quad 33/3 = 11$$

$$16 + 17 + 18 = 51 \quad 51/3 = 17$$

$$19 + 20 + 21 = 60 \quad 60/3 = 20$$

$$22 + 23 + 24 = 69 \quad 69/3 = 23$$

$$30 + 31 + 32 = 93 \quad 93/3 = 31$$

$$34 + 35 + 36 = 105 \quad 105/3 = 35$$

The following questions refer to the proposition: “The sum of three consecutive integers is divisible by 3” given on page 9.

15) Would you consider either of these two arguments to be valid mathematical proofs? Please explain.

16) Would you consider either of these two arguments to be better than the other? Please explain.

17) When (if ever) would it be appropriate to use either of these two arguments?

18) Would your view of these arguments change if the original question was phrased as “Show that...” instead of “Prove that...?” Please explain.

Please look over the following proposition and the two accompanying student arguments for the proposition, and answer the questions that follow.

Proposition: For any integers a and b , if ab is odd then a is odd and b is odd.

Student argument # 1

let $a=2n$, $n \in \mathbb{I}$ and b any integer
So $ab = 2n \cdot b$ which is an integer with a factor of 2 and therefore is even. Similarly, if b were even $a \cdot 2b$ would be even. If ab is odd then a and b must be odd.
I used the contrapositive method here.

Student argument # 2

an odd times an even = an even ex $2 \times 1 = 2$ $101 \times 10 = 1010$
an even times an even = an even ex $2 \times 2 = 4$ $98 \times 8 = 784$
an odd times an odd = an odd ex $3 \times 3 = 9$ $27 \times 9 = 243$
so if ab is odd both a and b must odd

The following questions refer to the proposition: “For any integers a and b , if ab is odd then a is odd and b is odd” given on page 11.

19) Would you consider either of these two arguments to be valid mathematical proofs? Please explain.

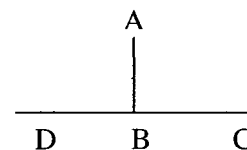
20) Would your acceptance of either of these arguments as valid proofs depend on the class (i.e., algebra, geometry, pre-calculus) or grade level of the student? Please explain.

21) Would you consider either of these two arguments to be better than the other? Please explain.

22) When (if ever) would it be appropriate to use either of these two arguments?

Please look over the following proposition and the accompanying argument, and answer the questions that follow.

Proposition:
If $\angle ABC$ is a right angle, then $AB \perp BC$.



Proof: <i>Statement</i>	<i>Reason</i>
1. $\angle ABC$ is a right angle	1. Given.
2. BC can be extended to DC	2. Postulate: 2 points determine a line.
3. $\angle DBC$ is a straight angle	3. Definition of a straight angle.
4. $m(\angle DBC) = 180^\circ$	4. Postulate: the measure of a straight angle is 180° along with statement 3.
5. $m(\angle ABC) = 90^\circ$	5. Definition of a right angle along with statement 1.
6. $m(\angle DBC) = m(\angle ABC) + m(\angle ABD)$	6. Postulate: the whole is equal to the sum of its parts.
7. $180^\circ = 90^\circ + m(\angle ABD)$	7. Postulate: substituting equals for equals gives equals along with statements 4 and 5.
8. $90^\circ = m(\angle ABD)$	8. Postulate: subtracting equals from equals gives equals (subtract 90° from both sides of equation in statement 7).
9. $m(\angle ABD) = m(\angle ABC)$	9. Postulate: substituting equals for equals gives equals along with statements 5 and 8.
10. $AB \perp BC$	10. Definition: two lines that form equal adjacent angles with each other are perpendicular.

23) When do you think this level of detail/rigor/completeness is necessary? Why?

24) Would you expect to see the same level of detail/rigor/completeness in high school algebra? Why or why not?

25) This proposition may seem obvious to some students. Why prove it?

APPENDIX D

GENERAL INFORMATION FORM

General Information

Please complete the following information as accurately as possible. All information will be confidential. Thank you for participating in this research study!

Name: _____

Email: _____ Phone: _____

Please check (✓) your appropriate class level:

- Freshman Sophomore Junior Senior Other _____

Please check (✓) which mathematics degree program you are currently seeking:

Mathematics Education:

- Secondary
 Junior High
 Elementary

Mathematics:

- B.S.
 B.A.
 other _____

Please check (✓) the courses (or equivalent transfer courses) you are **currently taking** or have **completed**:

- | | |
|---|---|
| <input type="checkbox"/> Math 425. Calculus I | <input type="checkbox"/> Math 622. Geometry for Teachers |
| <input type="checkbox"/> Math 426. Calculus II | <input type="checkbox"/> Math 639. Intro. to Statistical Analysis |
| <input type="checkbox"/> Math 525 Linearity I | <input type="checkbox"/> Math 656. Introduction to Number Theory |
| <input type="checkbox"/> Math 526. Linearity II | <input type="checkbox"/> Math 657. Geometry |
| <input type="checkbox"/> Math 527. Differential Equations | <input type="checkbox"/> Math 658. Topics in Geometry |
| <input type="checkbox"/> Math 528. Multidimensional Calculus | <input type="checkbox"/> Math 698. Senior Seminar |
| <input type="checkbox"/> Math 531. Mathematical Proof | <input type="checkbox"/> Math 761. Abstract Algebra |
| <input type="checkbox"/> Math 532. Discrete Mathematics | <input type="checkbox"/> Math 762. Linear Algebra |
| <input type="checkbox"/> Math 545. Linear Algebra and Math Proof | <input type="checkbox"/> Math 767. One-Dim. Real Analysis |
| <input type="checkbox"/> Math 619. Historical Foundations of Math | <input type="checkbox"/> Math 784. Topology |
| <input type="checkbox"/> Math 621. Number Systems for Teachers | <input type="checkbox"/> Math 791. Teaching of Mathematics, 7-12 |

Other mathematics courses you have taken: _____

APPENDIX E

IRB APPROVAL AND CONSENT FORM

- Institutional Review Board approval
- Consent Form



UNIVERSITY of NEW HAMPSHIRE

May 21, 2004

Gray, David M
Mathematics, Kingsbury Hall
P.O. Box 95
Henniker, NH 03242

IRB #: 3161
Study: An investigation of preservice teachers' and mathematicians' perceptions of mathematical proof in secondary school
Approval Date: 03/29/2004

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study with the following comment(s):

- *The investigator may involve students at Pembroke Academy in this study.*

Approval is granted to conduct your study as described in your protocol for one year from the approval date above. At the end of the approval date you will be asked to submit a report with regard to the involvement of human subjects in this study. If your study is still active, you may request an extension of IRB approval.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, *Responsibilities of Directors of Research Studies Involving Human Subjects*. (This document is also available at <http://www.unh.edu/osr/compliance/IRB.html>.) Please read this document carefully before commencing your work involving human subjects.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Manager

cc: File
Karen Graham

**Research Conduct and Compliance Services, Office of Sponsored Research, Service Building,
51 College Road, Durham, NH 03824-3585 * Fax: 603-862-3564**

Consent Form

Dear student:

I am conducting a research project to find out how pre-service secondary mathematics teachers think about mathematical proof at the secondary school level and how the perceptions of these teachers compare to the perceptions held by professional mathematicians. I am writing to invite you to participate in this project.

If you agree to participate in this study, you will be asked to complete a general information form and a questionnaire about your beliefs regarding proof in mathematics and proof in secondary school mathematics. The questionnaire should take no more than a few hours to complete. Your responses to the questionnaire will be used to investigate similarities and differences between pre-service teachers and mathematicians. You may be asked to participate in interviews with me for clarification or additional information as a part of this research project. Interviews will be audiotaped for later transcription. Audiotapes will be destroyed at the end of the study. You may also choose whether you wish to participate in the interviews.

You will not receive any compensation to participate in this project, however, the anticipated benefit is an increase our understanding of teachers' and mathematicians' perceptions of mathematical proof in the context of high school mathematics.

Participation is strictly voluntary; refusal to participate will involve no prejudice, penalty, or loss of benefits to which you would otherwise be entitled. If you agree to participate and then change your mind, you may withdraw at any time during the study without penalty.

The investigator seeks to maintain the confidentiality of all data and records associated with your participation in this research. You should understand, however, there are rare instances when the investigator is required to share personally-identifiable information (e.g., according to policy, contract, regulation). For example, in response to a complaint about the research, officials at the University of New Hampshire, designees of the sponsor(s), and/or regulatory and oversight government agencies may access research data. Data (including audiotapes) will be kept in a locked file cabinet at the University of New Hampshire; only Dr. Karen Graham and I will have access to the data.

If you have any questions about this research project or would like more information before, during, or after the study, you may contact David Gray at dmgray@cisunix.unh.edu or 603-428-6490, or Dr. Karen Graham at karen.graham@unh.edu or 603-862-1943. If you have questions about your rights as a research subject, you may contact Julie Simpson in the UNH Office of Sponsored Research at 603-862-2003 to discuss them.

I have enclosed two copies of this letter. Please sign one indicating your choice and return it along with the remainder of the packet. The other copy is for your records. Thank you for your consideration.

Sincerely,

David M. Gray
Department of Mathematics and Statistics
University of New Hampshire

Yes, I, _____ AGREE to participate in this research project in the following ways (initial all that apply):

_____ By allowing copies of my written work to be included as data; and

_____ By participating in audiotaped interviews with the researcher (if needed).

No, I, _____ DO NOT AGREE to participate in this research project.