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Nonunital multiplier pairs and remarks on generalized group C*-algebras

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Nonunital Multiplier Pairs and Remarks on Generalized Group C*-algebras

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Dedication

I would like to dedicate this thesis to my wife Jen and my advisor Don. Throughout the entire process your love and encouragement have carried me through the many frustrations and set backs I have suffered. Thank you for all your support. Oh and thanks Don for all your jokes, I will always remember why Mickey Mouse wanted a divorce from Minnie Mouse.$\ddot\smile$

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ABSTRACT

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In the first part of this paper we will consider a generalization of D. Hadwin and E. Nordgren's work on multiplier pairs. Here we will not assume the existence of an identity, but rather just ask for the existence of a bounded approximate identity. Without the assumption of the identity, we find a new result concerning the relationship between the norm closure of the left multiplication operators and the approximate double commutant of the left multiplication operators.

In the second part we will suppose $f, g : \mathbb{T} \to \mathbb{T}$ are continuous functions on the unit circle $\mathbb{T}$ and let $\mathbb{B}(f, g)$ denote the universal C*-algebra generated by U and V subject to the conditions that U and V are a unitary, and $Uf(V)U^{-1} = g(V)$. We then will prove that this C*-algebra may be represented as a crossed product. Next we will show that under certain conditions on f or g, $\mathbb{B}(f, g)$ will be nuclear, weakly quasidiagonal and we will be able to compute its *Ext* group. In the last two sections we will give a partial description of the K_1-group of $\mathbb{B}(f, g)$ and then using the results from [DH] calculate the free entropy dimension of $\mathbb{B}(f, g)$.

In the third and last part of this paper we show that the standard family of independent unitary $n \times n$ random matrices remains an asymptotically free Haar unitary with respect to any state $\varphi : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$. The result was originally stated by Voiculescu for the normalized trace. Our work here will follow the modified version of Voiculescu's theorem given by D. Hadwin and M. Dostál in [DH].

Chapter 1

Nonunital Multiplier Pairs

D. Hadwin and E. Nordgren in [HN] constructed a general setting in which to study composition operators and multiplication operators. Their work covered examples in many fields including measure-theoretic, function-theoretic and noncommutative measure theory for finite von Neumann algebras. In this work we will generalize their work to include many examples their work did not address. Specifically we will not assume the existence of an identity but rather just ask for the existence of a bounded net.

We will call a pair (X, Y) a *nonunital multiplier pair* provided X is a Banach space, Y is a Hausdorff topological vector space, $X \subset Y$, and the inclusion map is continuous. Moreover, we suppose we have a bilinear map (multiplication) $m : X \times X \to Y$, with the notation $m(u, v) = u \cdot v$ such that

1. m is separately continuous.
2. The sets $\mathcal{L}_0 = \{x \in X \mid x \cdot X \subset X\}$ and $\mathcal{R}_0 = \{x \in X \mid X \cdot x \subset X\}$ are dense in X.
3. There is a net $\{e_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{L}_0 \cap \mathcal{R}_0$ that such that ,
   (a) for every $x \in X$ we have $\lim\limits_{\lambda} \|x - x \cdot e_\lambda\| = 0$ and $\lim\limits_{\lambda} \|x - e_\lambda \cdot x\| = 0$,
   (b) $\sup\{\|e_\lambda x\| + \|x e_\lambda\| \mid \lambda \in \Lambda, x \in X, \|x\| \leq 1\} < \infty$.
4. There are dense subsets $E \subset \mathcal{L}_0$, $F \subset X$, $G \subset \mathcal{R}_0$ such that $(u \cdot v) \cdot w = u \cdot (v \cdot w)$ whenever $u \in E$, $v \in F$, $w \in G$.

If $x \in X$ we define L_x and R_x on X by

$$L_x w = x \cdot w \text{ and } R_x w = w \cdot x,$$

where the domain of L_x is $\mathrm{Dom}(L_x) = \{w \in X \mid x \cdot w \in X\}$ and the domain of R_x is $\mathrm{Dom}(R_x) = \{w \in X \mid w \cdot x \in X\}$. We define $\mathcal{L} = \{L_x \mid x \in \mathcal{L}_0\}$ and $\mathcal{R} = \{R_x \mid x \in \mathcal{R}_0\}$.

Theorem 1 *The following are true:*

1. *The multiplication $\cdot$ is jointly continuous from $X \times X$ to Y.*

2. *For every $x \in X$, L_x and R_x are densely defined closed operators.*

3. *L_x is bounded on $\mathcal{R}_0$ if and only if $x \in \mathcal{L}_0$, and R_x is bounded on $\mathcal{L}_0$ if and only if $x \in \mathcal{R}_0$.*

4. *$\mathcal{L}$, $\mathcal{R} \subset B(X)$.*

5. *If u , $v \in \mathcal{L}_0$ or v , $w \in \mathcal{R}_0$ or $u \in \mathcal{L}_0$, $w \in \mathcal{R}_0$, then*

$$(u \cdot v) \cdot w = u \cdot (v \cdot w).$$

6. *$\mathcal{L}' = \overline{(\mathcal{R})}^{sot}$ and $\mathcal{R}' = \overline{(\mathcal{L})}^{sot}$.*

7. *$L_v L_w = L_{v \cdot w}$ if $v, w \in \mathcal{L}_0$ and $R_v R_w = R_{v \cdot w}$ if $v, w \in \mathcal{R}_0$.*

Proof. The proofs for 1, 3, 4, 5 and 7 are identical as those given by D. Hadwin and E. Nordgren in [HN]. Thus we need only show the proofs of 2 and 6.

2. For any $w \in \mathcal{R}_0$ we see that $X \cdot w \subseteq X$, thus $\mathcal{R}_0 \subseteq \mathrm{Dom}(L_x)$ and therefore L_x is densely defined. Suppose $\{w_n\}$ is a sequence in $\mathrm{Dom}(L_x)$, $w, v \in X$, $\|w_n - w\| \to 0$ and $\|x \cdot w_n - v\| \to 0$. From our assumption that $\cdot$ is separately continuous we see that $x \cdot w_n \to x \cdot w$ in Y. and since the inclusion map is continuous $x \cdot w_n \to v$ in Y. But Y is Hausdorff, so $x \cdot w = v$. Thus L_x is closed. The proof for R_x is similar.

6. Suppose $T \in B(X)$ and $T \in \mathcal{L}'$. For any $x \in \mathcal{L}_0$,

$$\begin{aligned}
\lim_\lambda \|R_{Te_\lambda} x - Tx\| &= \lim_\lambda \|x \cdot Te_\lambda - Tx\| \\
&= \lim_\lambda \|L_x Te_\lambda - Tx\| \\
&= \lim_\lambda \|TL_x e_\lambda - Tx\| \\
&= \lim_\lambda \|T(x \cdot e_\lambda) - Tx\| \\
&\leq \lim_\lambda \|T\| \|x \cdot e_\lambda - x\| = 0.
\end{aligned}$$

Since $\{e_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{L}_0 \cap \mathcal{R}_0$, then $\{R_{e_\lambda}\}_{\lambda\in\Lambda} \subset \mathcal{R}$. Thus it follows that $T \in \overline{(\mathcal{R})}^{sot}$ and therefore $\mathcal{L} \subset \overline{(\mathcal{R})}^{sot}$. The reverse inclusion follows from (5). The proof for $\mathcal{R}' = \overline{(\mathcal{L})}^{sot}$ is similar.

■

To prove the main result of this section we need to consider a special case of a result contained in [CM]. It concerns a class of normed associative complex algebras $\mathcal{A}$ which satisfies, for some positive constant γ,

$$\gamma \operatorname{dist}(A, Z(\mathcal{A})) \leq \sup_{\substack{\|X\|\leq 1 \\ X\in\mathcal{A}}} \|AX - XA\|$$

for all $A \in \mathcal{A}$, where $Z(\mathcal{A})$ is the center of $\mathcal{A}$. The class is that of ultraprime normed algebras which may be defined as follows:

Definition 2 *Suppose $\mathcal{A}$ is a normed associative complex algebra, then $\mathcal{A}$ is said to be* ultraprime *if there is a positive number K such that*

$$K\|A\|\|B\| \leq \|M_{A,B}\|$$

for all A, B in the algebra, where $M_{A,B}$ denotes the linear operator defined by $M_{A,B}(X) = AXB$. The largest possible K for which the above is true is called the constant of ultra-primeness.

The theorem M.Cabrera and J. Martínez prove says that this collection will satisfy the above property for a γ dependent only on the constant of ultraprimeness. For a more precise statement along with the history of the problem the reader should refer to [CM]. For our purposes we need only the following proposition.

Proposition 3 *For any ultraprime algebra $\mathcal{A}$ whose associated constant K satisfies $0 < K \leq 1$, then*

$$dist\left(A, Z\left(\mathcal{A}\right)\right) \leq \sup_{\|X\| \leq 1} \|AX - XA\|$$

for any $A \in \mathcal{A}$,

Corollary 4 *For $T \in B\left(X\right)$, $dist(T, \mathbb{C} \cdot 1) \leq \sup_{\|S\| \leq 1} \|TS - ST\|$.*

Proof. Suppose $x, y \in X$ such that $\|x\| = 1$ and $\|y\| = 1$ also let $A, B \in B\left(X\right)$. By the Hahn-Banach theorem choose $\alpha \in X^{\sharp}$ such that $\alpha\left(By\right) = \|By\|$, and $\|\alpha\| = 1$. Then it follows

$$\begin{aligned}
\|A\left(x \otimes \alpha\right) B\| &\geq \|A\left(x \otimes \alpha\right) By\| \\
&= \|A\left(\alpha\left(By\right) x\right)\| \\
&= \|\alpha\left(By\right) Ax\| \\
&= \|By\| \|Ax\|.
\end{aligned}$$

Since $\|x \otimes \alpha\| = \|x\| \|\alpha\| = 1$

$$\begin{aligned}
\|M_{A,B}\| &\geq \|M_{A,B}\left(x \otimes \alpha\right)\| \\
&= \|Ax\| \|By\|
\end{aligned}$$

and this holds for all $x, y \in X$. Therefore $B\left(X\right)$ is ultraprime, $K \leq 1$ and our result follows from the paper of M. Cabrera and J. Martínez.[CM]. ■

The main result in this chapter shows a relationship with the approximate double commutant which is a notion developed by D. Hadwin in [Had3]. From this paper we have the following definition and proposition.

Definition 5 *The* approximate double commutant *of $\mathcal{S} \subset B\left(X\right)$ is the set of operators T for which*

$$\lim_{\lambda} \|A_{\lambda}T - TA_{\lambda}\| = 0$$

whenever $\{A_\lambda\}$ *is a bounded net such that*

$$\lim_\lambda \|A_\lambda S - SA_\lambda\| = 0$$

for every $S \in \mathcal{S}$.

We will denote the approximate double commutant of $\mathcal{S}$ by *appr* $(\mathcal{S})''$. The next proposition shows the relation between the approximate double commutant and the double commutant. The result can be found in [Had3].

Proposition 6 *If* $\mathcal{S} \subseteq B(X)$, *then appr* $(\mathcal{S})''$ *is a (norm) closed subalgebra of* $\mathcal{S}''$.

We now prove the "approximate" version of part 6 of Theorem 1.

Theorem 7 *If* e_λ *is an idempotent for every* $\lambda \in \Lambda$, $\lim_\lambda \|L_{e_\lambda x - x}\| = 0$, $\lim_\lambda \|L_{xe_\lambda - x}\| = 0$ *and* $\mathcal{R}_0(e_\lambda X e_\lambda) \subseteq e_\lambda \mathcal{R}_0 e_\lambda$, *then appr* $(\mathcal{L})'' = \overline{(\mathcal{L})}^{\|\|\|} + \mathbb{C} \cdot 1$.

Proof. Suppose $T \in$ *appr* $\{\mathcal{L}\}''$. From our assumption it follows that $TR_{e_\lambda} = R_{e_\lambda}T$ for all $\lambda \in \Lambda$ and thus

$$\begin{aligned} Te_\lambda x e_\lambda &= TR_{e_\lambda} e_\lambda x e_\lambda \\ &= R_{e_\lambda} T(e_\lambda x e_\lambda) \\ &= T(e_\lambda x e_\lambda) e_\lambda. \end{aligned}$$

This calculation along with the fact that

$$L_{e_\lambda} T(e_\lambda x e_\lambda) = e_\lambda T(e_\lambda x e_\lambda) e_\lambda,$$

and

$$\begin{aligned} \lim_\lambda \|L_{e_\lambda} T(e_\lambda x e_\lambda)\| &= \lim_\lambda \|TL_{e_\lambda}(e_\lambda x e_\lambda)\| \\ &= \lim_\lambda \|T(e_\lambda x e_\lambda)\|, \end{aligned}$$

implies that

$$L_{e_\lambda} T(e_\lambda X e_\lambda) \subseteq e_\lambda X e_\lambda.$$

Since $e_\lambda x e_\lambda \in \mathcal{R}_0 (e_\lambda X e_\lambda)$ and we are assuming $\mathcal{R}_0 (e_\lambda X e_\lambda) \subseteq e_\lambda \mathcal{R}_0 e_\lambda$, we can find a $w \in \mathcal{R}_0$ such that $e_\lambda x e_\lambda = e_\lambda w e_\lambda$. Because of the commuting relationship,

$$L_{e_\lambda} T R_{e_\lambda w e_\lambda} \mid_{e_\lambda X e_\lambda} = R_{e_\lambda w e_\lambda} L_{e_\lambda} T \mid_{e_\lambda X e_\lambda}$$

and thus we can apply the results from D. Hadwin and E. Nordgren [HN] to show

$$L_{e_\lambda} T \mid_{e_\lambda X e_\lambda} = L_v \mid_{e_\lambda X e_\lambda}$$

for some $v \in \mathcal{L}_0 (e_\lambda X e_\lambda)$. Thus we can write $v = e_\lambda u_\lambda e_\lambda$ where $u_\lambda \in \mathcal{L}_0$ and therefore

$$L_{e_\lambda} T \mid_{e_\lambda X e_\lambda} = L_{e_\lambda u_\lambda e_\lambda} \mid_{e_\lambda X e_\lambda} .$$

For any $W \in B(X)$ and any $x \in \mathcal{L}_0$

$$\lim_\lambda R_x (1 - L_{e_\lambda} R_{e_\lambda}) W (1 - L_{e_\lambda} R_{e_\lambda}) = 0$$

and

$$\lim_\lambda (1 - L_{e_\lambda} R_{e_\lambda}) W (1 - L_{e_\lambda} R_{e_\lambda}) R_x = 0.$$

Thus if we let $T_\lambda = (1 - L_{e_\lambda} R_{e_\lambda}) T (1 - L_{e_\lambda} R_{e_\lambda})$, we find for all $W \in B(X)$ that T_λ commutes asymptotically with $(1 - L_{e_\lambda} R_{e_\lambda}) W (1 - L_{e_\lambda} R_{e_\lambda})$. Hence if we choose a bounded net $\{W'_\lambda\} \subset B(X)$ and let $W_\lambda = (1 - L_{e_\lambda} R_{e_\lambda}) W'_\lambda (1 - L_{e_\lambda} R_{e_\lambda})$ it follows

$$\lim_\lambda \|T_\lambda W_\lambda - W_\lambda T_\lambda\| = 0.$$

This statement along with our Lemma gives us a net $\{n_\lambda\} \subset \mathbb{C}$ such that

$$\lim_\lambda (1 - L_{e_\lambda} R_{e_\lambda}) T (1 - L_{e_\lambda} R_{e_\lambda}) \mid_{(I - L_{e_\lambda} R_{e_\lambda})(X)} - n_\lambda = 0$$

thus proving our theorem. ■

The first example comes from our original motivating example from measure theory.

Example 8 *Suppose (Ω, Σ, μ) is a measure space such that, for every $E \in \Sigma$ with $\mu(E) > 0$ there is an $F \subset E$ such that $0 < \mu(F) < \infty$. Let $1 \le p < \infty$ and $X = L^p(\mu)$. Let Y be the set of all (equivalence classes of) measurable functions on Ω topologized by convergence* *in measure. Let* Λ *be the collection of all sets of finite measure directed by* $\subset$. *It is clear that* $\{\chi_E \mid E \in \Lambda\}$ *is an approximate identity in* $L^P(\mu)$. *It is also clear that* $\mathcal{L}_0 = \mathcal{R}_0 = L^\infty(\mu) \cap L^p(\mu)$. *Thus* (X, Y) *is a multiplier pair and* $\{\chi_E \mid E \in \Lambda\}$ *satisfies the conditions of Theorem 7. Moreover, it is clear that* $\|L_f\| = \|R_f\| = \|f\|_\infty$ *for every* $f \in \mathcal{L}_0$. *It follows from Theorem 1 that* $\mathcal{L}^{-sot} = \{L_f \mid f \in L^\infty(\mu)\}$ *is a maximal abelian subalgebra of* $B(L^p(\mu))$. *Also appr* $(\mathcal{L})''$ *is the set of* L_f *with* f *in the* $\| \ \|_\infty$*-closure of* $L^\infty(\mu) \cap L^p(\mu)$, *which is the set of all* $f \in L^\infty(\mu)$ *such that, for every* $\varepsilon > 0$ *we have* $\mu(\{\omega \in \Omega \mid |f(\omega)| \geq \varepsilon\}) < \infty$.

The next example comes from an operator analogue of the discrete measure-theoretic example above.

Example 9 *Suppose* H *is a separable infinite-dimensional Hilbert space. The minimal nonzero two-sided ideal in* $B(H)$ *is* $\mathcal{F}(H)$, *the set of all finite-rank operators. The largest proper ideal in* $B(H)$ *is* $\mathcal{K}(H)$, *the set of all compact operators. There are many two-sided ideals between* $\mathcal{F}(H)$ *and* $\mathcal{K}(H)$, *many of which are Banach spaces with respect to some natural norms. Among these are the ideals defined in terms of unitarily invariant norms. We say that* $||| \ |||$ *is a* unitarily invariant *norm on* $\mathcal{F}(H)$ *provided*

1. $|||UTV||| = |||T|||$ *for all* $T \in \mathcal{F}(H)$ *and all unitary operators* U, V.

2. $|||P||| = 1$ *whenever* P *is a rank-one projection.*

The completion of $\mathcal{F}(H)$ *with respect to such a norm is a two-sided ideal* $\mathcal{I}_{||| \ |||}$ *in* $B(H)$ *such that* $\mathcal{I}_{||| \ |||} \subset \mathcal{K}(H)$. *Since* $|||T||| = \left|\left|\left|(T^*T)^{\frac{1}{2}}\right|\right|\right|$ *for every* $T \in \mathcal{F}(H)$ *(polar decomposition), and since* $(T^*T)^{\frac{1}{2}}$ *is unitarily equivalent to a diagonal operator,* $||| \ |||$ *is completely determined by its values on the diagonal operators. This way* $||| \ |||$ *induces a permutationally invariant norm on a sequence Banach space* $Y_{||| \ |||}$ *contained in* c_0 *(the null sequences) and containing* c_{00} *(the finitely nonzero sequences) such that the norm is unchanged when the entries of the sequence are replaced by their absolute values. Conversely, every such Banach space yields a unitarily invariant norm on* $\mathcal{F}(H)$ *that, in turn, gives a two-sided ideal.*

It follows from condition 1 above that

$$|||ATB||| \leq \|A\| \, |||T||| \, \|B\|$$

for every $A, B \in B(H)$ *and every* $T \in \mathcal{I}_{||| \; |||}$. *It follows that if* $A, B \in B(H)$ *and* $L_A, R_B : \mathcal{I}_{||| \; |||} \to \mathcal{I}_{||| \; |||}$ *are defined by* $L_A(T) = AT$ *and* $R_B(T) = TB$, *then* $\|L_A\| \leq \|A\|$ *and* $\|R_B\| \leq \|B\|$. *It follows from condition 2 above that* $\|L_A\| = \|A\|$ *and* $\|R_B\| = \|B\|$.

We let $X = \mathcal{I}_{||| \; |||} = Y$ *with* $\cdot$ *defined to be the usual operator product. If* M *is a closed linear subspace* H, *let* P_M *be the orthogonal projection onto* M. *Let* Λ *denote the directed (by inclusion) set of all nonzero finite-dimensional linear subspaces of* H. *If* $T \in \mathcal{F}(H)$ *and* M *contains* $ran(T) \cup ran(T^*)$, *then* $P_M T = T P_M = T$. *Since* $\|L_{P_M}\| = 1$ *and* $\mathcal{F}(H)$ *is* $||| \; |||$*-dense in* $\mathcal{I}_{||| \; |||}$, *it follows that* $\{P_M \mid M \in \Lambda\}$ *is an approximate identity in* $\mathcal{I}_{||| \; |||}$ *that satisfies the hypothesis of Theorem 7.*

Suppose $T \in B(H)$. *Then* $L_T \in B(X)$ *and* L_T *is the strong limit in* $B(X)$ *of the net* $\{L_{P_M T}\}_{M \in \Lambda}$, *which is in* $\mathcal{L}$. *Conversely, suppose* $S \in \mathcal{L}^{-sot}$. *Then there is a net* $\{F_\iota\}_{\iota \in I} \in \mathcal{F}(H)$ *such that* $L_{F_i} \to S$ *in the strong operator topology on* $B(X)$. *Suppose* $M \in \Lambda$; *then* $L_{P_m F_i P_M} \to L_{P_M} S L_{P_M}$ *in the strong operator topology. Since* $P_M \mathcal{F}(H) P_M$ *is finite-dimensional, it follows that there is an* $F_M \in P_M \mathcal{F}(H) P_M$ *such that*

$$L_{P_M} S L_{P_M} = L_{F_M}.$$

Since $\|F_M\| = \|L_{P_M} S L_{P_M}\| \leq \|S\|$, *it follows that there is a subnet of* $\{F_M\}_{M \in \Lambda}$ *that converges in the weak operator topology to an operator* $T \in B(H)$. *It clearly follows that* $S = L_T$. *Hence the strong operator closure of* $\mathcal{L}$ *is* $\{L_T \mid T \in B(H)\}$. *It follows from the fact that* $\|L_T\| = \|T\|$ *whenever* $T \in B(H)$ *and the norm closure of* $\mathcal{F}(H)$ *is* $\mathcal{K}(H)$ *that* $appr(\mathcal{L})''$ *is* $\{L_T \mid T \in \mathcal{K}(H) + \mathbb{C} \cdot 1\}$. *The analogues for* $\mathcal{R}$ *hold as well.*

The next example contains aspects of both of the preceding two examples.

Example 10 *For an elementary introduction to noncommutative* L^p*-theory, we refer the reader to the paper of Nelson [Nel]. Suppose* $\mathcal{M}$ *is a* II_∞ *factor von Neumann algebra on* *a separable Hilbert space, and* $1 \leq p < \infty$. *Then there is type* II_1 *factor* $\mathcal{N}$ *on a separable Hilbert space* K *such that* $\mathcal{M}$ *is isomorphic to* $B\left(\ell^2\right) \otimes \mathcal{N} \subset B\left(\ell^2 \otimes K\right)$, *which is the same as the algebra of all of the bounded operators* A *on* $H = K \oplus K \oplus \cdots$ *with an operator matrix* (A_{ij}) *with each* $A_{ij} \in \mathcal{N}$. *Let* $\tau : \mathcal{N} \to \mathbb{C}$ *be the unique faithful trace on* $\mathcal{N}$, *and define* $tr : \mathcal{M}^+ \to [0, \infty]$ *by*

$$tr\left((A_{ij})\right) = \sum_{n=1}^{\infty} \tau\left(A_{nn}\right).$$

It is well-known that tr *is invariant under unitary conjugation and that the set* $\mathcal{I}_p$ *of all elements* T *of* $\mathcal{M}$ *such that*

$$\|T\|_p = \left[tr\left((T^*T)^{\frac{p}{2}}\right)\right]^{\frac{1}{p}} < \infty$$

is a two-sided ideal in $\mathcal{M}$ *and that* $\| \ \|_p$ *is a norm on* $\mathcal{I}_p$. *Moreover, there is a notion of* convergence in measure on $\mathcal{M}$ *and the completion* Y *of* $\mathcal{M}$ *in this topology has many useful properties. In particular, the completion* $L^p(\mathcal{M}, tr)$ *of* $\left(\mathcal{I}_p, \| \ \|_p\right)$ *is naturally contained in* Y. *It is also well known that* $\mathcal{F}\left(\ell^2\right) \otimes \mathcal{N}$ *is dense in* $L^p(\mathcal{M}, tr)$. *We define* Λ *(parallel to the preceding examples) as the net of all projections* P *in* $\mathcal{M}$ *with* $tr(P) < \infty$. *Then* Λ *is an approximate identity in* $L^p(\mathcal{M}, tr)$. *The natural multiplication (from* $\mathcal{M}$*) on* $\mathcal{I}_p$ *extends to a multiplication on* $L^p(\mathcal{M}, tr)$, *where the product is in* Y. *Thus* $(X, Y, \cdot)$ *is a nonunital multiplier pair. It is well-known that*

$$\|ATB\|_p \leq \|A\| \, \|T\|_p \, \|B\|$$

for all $A, B \in \mathcal{M}$ *and* $T \in \mathcal{I}_p$. *Following the unpublished paper of Hadwin and Nordgren [HN2], it is easy to show that* $\|L_A\| = \|A\| = \|R_A\|$ *whenever* $A \in \mathcal{M}$, *where* $L_A, R_A \in B(L^p(\mathcal{M}, tr))$. *Following the arguments in the preceding example, we can show that* $\mathcal{L}^{-sot} = \{L_T \mid T \in \mathcal{M}\}$ *and* $appr(\mathcal{L})'' = \left\{L_T \mid T \in \left(\mathcal{F}\left(\ell^2\right) \otimes \mathcal{N}\right)^- + \mathbb{C} \cdot 1 = C^*\left(\mathcal{F}\left(\ell^2\right) \otimes \mathcal{N}\right)\right\}$, *where* $\left(\mathcal{F}\left(\ell^2\right) \otimes \mathcal{N}\right)^-$ *is the spatial* C^**-algebraic tensor product* $\mathcal{K}\left(\ell^2\right) \otimes \mathcal{N}$. *Similarly,* $\mathcal{R}^{-sot} = \{R_T : T \in \mathcal{M}\}$. *Thus we have* $\{L_T \mid T \in \mathcal{M}\}$ *and* $\{R_T \mid T \in \mathcal{M}\}$, *which extends von Neumann's double commutant theorem (when* $p = 2$*). Similarly, when* $p = 2$, *our approximate results reduce to Hadwin's approximate double commutant theorem for* C^**-algebras [Had3].*

Chapter 2

Preliminaries on C*-algebras

2.1 Generators and Relations

In this chapter we wish to study $\mathbb{B}(f,g)$, the universal group C*-algebra generated by two unitaries U and V subject to the one relation $U^*f(V)U = g(V)$, where f and g are continuous functions from the circle to the to the circle. This class of algebras contains many important examples, such as the Baumslag-Solitar algebra and the irrational rotation algebra.

To begin, we will review generators and relations in both a group and then in our primary object of study, a C*-algebra. In the next section we will show that our C*-algebra may be represented as a crossed product. With this representation and certain restrictions on the functions f and g we will be able to discuss the amenability, quasidiagonality and *Ext* groups of $\mathbb{B}(f,g)$. Following this we will give a partial result concerning the K_1-group of $\mathbb{B}(f,g)$ and then in the last section discuss the free entropy dimension of $\mathbb{B}(f,g)$.

Suppose X is a subset of a group G. Then the subgroup of G generated by X is denoted by $\langle X\rangle$, and is by definition the least subgroup of G containing X. It follows that

$$\langle X\rangle = \{x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n} \mid x_i \in X, \varepsilon_i = \pm 1\}.$$

We call the expression of the form

$$x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}$$

an X-word, or simply a word. A word is termed reduced if

$$x_i = x_{i+1} \Rightarrow \varepsilon_i + \varepsilon_{i+1} \neq 0, \ for \ i = 1,\dots,n-1.$$

If $G = \langle X \rangle$ and every non-empty reduced word is not equal to the identity then we term X a *free set of generators* of G and G itself is termed *free*. A group is termed *finitely generated* when the cardinality of X is finite, and denoted $\mathbb{F}_n$ when it is both free and finitely generated, where n is the cardinality of X.

Let G again be a group, F a free group on a set X and θ a map from X into G such that

$$G = \langle \theta(X) \rangle .$$

There will then exist a group epimorphism $\varphi : F \to G$, such that $\varphi_{|X} = \theta$. Suppose there is a subset R in F such that

$$\ker\varphi = < R > .$$

We then write

$$G = \langle X \mid R \rangle \tag{1}$$

and term $\langle X \mid R \rangle$ a *presentation* of G with *relations* R. Notice that such a presentation (1) comes with an explicit map θ such that the extension of θ to the free group F on X yields an onto homomorphism φ with kernel $\langle R \rangle$. Also if we identify X with its image in G then (1) simply means that X generates G and everything about G can be deduced from the fact that $r = 1$ in G for every $r \in R$.

Definition 11 *A group is finitely presented if it has a finite presentation, i.e. if*

$$G = \langle X \mid R \rangle$$

where X and R are both finite.

Example 12 *Suppose $m, n \in \mathbb{Z}^+$ and let*

$$G = \left\langle u, v \mid u^{-1} v^m u v^{-n} \right\rangle .$$

Note that in G

$$u^{-1} v^m u = v^n .$$

We will notate this group $B(m,n)$ and refer to the collection $B(m,n)$, for all choices of $m, n \in \mathbb{Z}^+$, as the Baumslag-Solitar groups.

2.2 Generators and Relations in a C*-algebra

In the following $\mathcal{A}$ will be an algebra over $\mathbb{C}$ and H will be a complex Hilbert space, $B(H)$ the algebra of operators on H and $\mathcal{U}(H)$ group of unitaries on H.

Definition 13 *We say that $\mathcal{A}$ is a* C*-algebra *if $\mathcal{A}$ is a Banach algebra and $\mathcal{A}$ has an* involution, *denoted by $*$ satisfying*

1. $(ab)^* = b^*a^*$
2. $(a+b)^* = a^* + b^*$
3. $(a^*)^* = a$
4. $(\lambda a)^* = \bar{\lambda}a^*$, *for all* $\lambda \in \mathbb{C}$
5. *For all* $a \in \mathcal{A}$, $\|a^*a\| = \|a\|^2$.

Example 14 *Suppose H is a Hilbert space and $B(H)$ is the algebra of (bounded linear) operators on H. For each $T \in B(H)$ there is a unique operator $T^* \in B(H)$ defined by*

$$(Tx, y) = (x, T^*y)$$

for every $x, y \in H$. With this involution and the operator norm

$$\|T\| = \sup \{\|Tx\| \mid x \in H, \|x\| \leq 1\},$$

$B(H)$ becomes a C-algebra. A famous theorem of Gelfand and Naimark and Segal says that every C*-algebra is isomorphic to a subalgebra of $B(H)$ for some Hilbert space H. We will refer this as the GNS construction.*

When defining a C*-algebra by generators and relations, we must be very careful. Precisely what does "the universal C*-algebra generated by X with relations R" actually mean? We will discuss the precise notion of "relation" last, so, for the moment, assume that you know what a "relation" on elements of a C*-algebra means. Suppose X is the generating set for our "universal C*-algebra" and $\mathcal{R}$ is a family of relations on the elements of X. A *representation* of the relations $\mathcal{R}$ is a function $f : X \to B(H)$ for some Hilbert space H so that the relations $\mathcal{R}$ hold with each $x \in X$ replaced with $f(x)$. Note that a representation is a function on X, but if X is a singleton each representation corresponds to a single operator, and if X has two points, each representation corresponds to a pair of operators, etc.... The universal C*-algebra $C^*(X \mid \mathcal{R})$ generated by X subject to the relations $\mathcal{R}$ should satisfy the property that, whenever f is a representation of the relations, then there is a unital $*$-homomorphism $\pi : C^*(X \mid \mathcal{R}) \to B(H)$ such that $\pi|_X = f$.

Suppose $\{f_\iota \mid \iota \in I\}$ is a family of functions $f_\iota : X \to B(H_\iota)$. If $\sup\{\|f_\iota(x)\| \mid \iota \in I\} < \infty$ for each $x \in X$, we can define the function $f : X \to \oplus_{\iota\in I} H_\iota$ by $f(x) = \oplus_{\iota\in I} f_\iota(x)$. We call f the *direct sum* of the f_ι's. Suppose $\{f_\lambda : \lambda \in \Lambda\}$ is a net of functions from X to $B(H)$. We say that $f_\lambda \to f$ in the *point-norm topology* if and only if

$$\|f_\lambda(x) - f(x)\| \to 0$$

for every $x \in X$.

Now let us turn to the notion of a "relation". Are relations just equations in the variables? In some sense the answer is affirmative. However, the condition "$x = x^*$ and $\sigma(x)$ is contained in the Cantor set" is a relation, and it's expression as an equation would be complicated and not too useful. However, "$\sigma(x)$ is contained in the Cantor set" is not a relation. So how do we tell what a relation is? This question was answered in [HKN] where it was shown that to be a relation it must be preserved under unitary equivalence, direct sums and direct summands, and it must be preserved under norm limits. In [HKN] a necessary and sufficient condition was given on a set $\mathcal{R}$ of relations in order for the universal C*-algebra generated by X defined by the relations $\mathcal{R}$ to make sense.

Proposition 15 *Suppose X is a nonempty set and $\mathcal{R}$ is a set of relations on X. Then $C^*(X \mid \mathcal{R})$ exists if and only if all of the following are true:*

1. *A direct sum of functions is a representation of $\mathcal{R}$ if and only if each summand is.*

2. *A point-norm limit of representations of $\mathcal{R}$ is a representation of $\mathcal{R}$.*

3. For each $x \in X$, we have $\sup \{\|f(x)\| \mid f$ is a representation of $\mathcal{R}\} < \infty$.

4. There is at least one representation of $\mathcal{R}$.

5. A function that is unitarily equivalent to a representation of $\mathcal{R}$ is a representation of $\mathcal{R}$.

Also in [HKN] the notion of "noncommutative continuous function" was introduced, and it was shown that relations could be describe completely in terms of equations of the form $\phi = 0$ where ϕ is a noncommutative continuous function of the variable in X.

The necessity of these conditions comes from the fact that a direct sum of representations of a C*-algebra is a representation, that every unital $*$-homomorphism of a C*-algebra is a contraction, and that a point-norm limit of $*$-homomorphisms is a $*$-homomorphism.

To illustrate the conditions in the preceding proposition, consider the following examples.

Example 16 *Let $\mathcal{A}$ be the C*-algebra generated by a, such that a is nilpotent, that is*

$$\mathcal{A} = C^*(\{a\} \mid a^n = 0 \text{ for some } n \in \mathbb{N}).$$

Let

$$a_n = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}.$$

Then a_n is a representation of the nilpotent relation, so there should be a unital $*$-homomorphism $\pi_n : \mathcal{A} \to \mathcal{M}_2(\mathbb{C})$ with $\pi_n(a) = a_n$. However, this forces $n = \|a_n\| \, \|\pi\| \leq \|a\|$ for every $n \in \mathbb{N}$. We might try to remedy this by adding the restriction $\|a\| \leq 1$. However, on a separable infinite-dimensional Hilbert space, the norm closure of the set of nilpotent operators with norm at most 1 contains every normal operator T with $\|T\| \leq 1$ whose spectrum is connected and contains 0 [Her]. Moreover, if J_n is the $n \times n$ nilpotent Jordan cell, then each J_n is nilpotent, $\|J_n\| = 1$, but $\oplus_{n \in \mathbb{N}} J_n$ is not nilpotent. This example also shows that $C^*(\{a\} \mid \sigma(a) = \{0\})$ is not defined.

Example 17 *Let $\mathcal{A}$ be the C*-algebra generated by x and y subject to the following relations $\mathcal{R}$:*

1. $\|x\| \leq 1, \|y\| \leq 1$,

2. $xy = yx$,

3. $\|xy^* - y^*x\| \leq \frac{3}{4}$.

It is easily seen that the conditions in Proposition 15 are satisfied and thus the universal C^*-algebra, $C^*(\{x,y\} \mid \mathcal{R})$ will exist.

2.3 Group C*-algebras and Crossed Products

If we insist that the generators are unitary elements, then there is automatically a bound on the norms of each of the generators, so we need only consider the remaining conditions.

Definition 18 *Suppose G is a discrete group and $G = \langle X \mid R_0 \rangle$, then we define the* group C*-algebra *as*

$$C^*(G) = C^* \langle X \mid R \rangle,$$

where R is R_0 along with $x^* = x^{-1}$.

It is clear that a representation of the relations R amounts to a unitary representation of G. It follows that every unitary representation of G extends uniquely to a unital $*$-homomorphism on $C^*(G)$. This last property is the usual defining property for $C^*(G)$. If G is an abelian group, then $C^*(G)$ is an abelian C*-algebra and $C^*(G) \approx C(X)$, where X the dual group of G, i.e., the group of homomorphisms from G to the circle group $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

Suppose G is a discrete group. We will define

$$\ell^2(G) = \left\{ f : G \to \mathbb{C} \mid \|f\|_2^2 = \sum_{g \in G} \left|f(g)^2\right| < \infty \right\}.$$

For any $f, g \in \ell^2(G)$ if we define their inner product as

$$(f, g) = \sum_{h \in G} f(h)\overline{g(h)},$$

we then see that $\ell^2(G)$ is a Hilbert space. Identifying each $g \in G$ with the characteristic function $\mathcal{X}_{\{g\}}$in $\ell^2(G)$, it follows that $\{\mathcal{X}_{\{g\}}\}_{g\in G}$ is an orthonormal basis for $\ell^2(G)$. We now define for each $g \in G$, L_g on the orthonormal basis $\{\mathcal{X}_{\{g\}}\}_{g\in G}$ as

$$L_g\left(\mathcal{X}_{\{h\}}\right) = \mathcal{X}_{gh}.$$

We can now extend L_g to a unitary operator on $\ell^2(G)$. Equivalently, we may define L_g on $\ell^2(G)$ as the unitary operator

$$\left(L_g(f)\right)(h) = f\left(g^{-1}h\right).$$

Thus we have a $*$-representation of G into $B(H)$. This representation is known as the *left regular representation.* This shows that there is at least one representation of the relations defining $C^*(G)$.

Definition 19 *The* reduced group *C^*-algebra of G is defined as*

$$C^*\langle L_g \mid g \in G\rangle = \overline{Sp(\{L_g \mid g \in G\})}^{\|\ \|}$$

and is denoted $C_r^*(G)$.

Crossed products in C*-algebra's were introduced to study the action of groups on compact Hausdorff spaces, and later group-actions on C*-algebras. They provide a larger algebra which encodes the original C*-algebra and the group action. To ease our explanation, we will assume G is a discrete group with identity e, $\mathcal{A}$ is a unital C*-algebra, and $\alpha : G \to Aut(\mathcal{A})$ is a group homomorphism. We now will give a definition of the crossed product C*-algebra for this special case.

Definition 20 *We define the* crossed product *C^*-algebra, denoted by $\mathcal{A}\rtimes_\alpha G$ as $C^*(\mathcal{A} \sqcup G \mid \mathcal{R})$ where $\sqcup$ is the disjoint union and $\mathcal{R}$ consists of :*

1. *all relations of $\mathcal{A}$ that are true for $\mathcal{A}$,*

2. *$g^* = g^{-1}$ for all $g \in G$,*

3. $e = 1$, *and*

4. for all $a \in \mathcal{A}$ *and for* $g \in G$, $gag^{-1} = \alpha(g)(a)$.

Note that the last condition in our definition is known as the covariance relation.

As in the case of the group C*-algebra, the crossed product C*-algebra also has a left regular representation. This will give rise to the reduced crossed product.

A standard construction in the theory of C*-algebra's is that of the Gelfand-Naimark-Segal, for short the GNS construction. It says that given an abstract C*-algebra $\mathcal{A}$, there exists a Hilbert space H, and an isometric $*$-isomorphism $\pi : \mathcal{A} \to B(H)$. Thus we may use the GNS construction to view an abstract C*-algebra, as a concrete algebra of operators.

Suppose $\mathcal{A}$ is a unital C*-algebra, G is a discrete group and $\alpha : G \to Aut(\mathcal{A})$ is a group homomorphism. From the above remark we may assume that $\mathcal{A} \subseteq B(M)$, where M is a Hilbert space. Let

$$\begin{aligned} H &= \ell^2(G, M) \\ &= \left\{ f : G \to M \mid \|f\|_2^2 = \sum_{g \in G} \|f(g)\|^2 < \infty \right\}. \end{aligned}$$

H is easily seen to be a Hilbert space. Similar to the construction for the reduced group C*-algebra we define for each $g \in G$ an operator L_g on $B(H)$ as

$$L_g(f)(x) = f(g^{-1}x).$$

To see that each L_g is a unitary operator, consider the orthonormal basis $\{\mathcal{X}_{\{x\}}\}_{x \in G}$ of H and the following computation

$$\begin{aligned} L_g\left(\mathcal{X}_{\{x\}}(h) \cdot M\right) &= \mathcal{X}_{\{x\}}\left(g^{-1}h\right) \cdot M \\ &= \mathcal{X}_{\{gx\}}(h) \cdot M. \end{aligned}$$

Since L_g maps an orthonormal basis to itself, it follows that

$$\{L_g \mid g \in G\} \subseteq \mathcal{U}(H).$$

We now need to see that the covariance relation holds, thus for each $A \in \mathcal{A}$, define $\hat{A} \in B(H)$ by

$$\hat{A}(f)(x) = \left(\alpha\left(x^{-1}\right) A\right) f(x).$$

Now from the calculation,

$$\begin{aligned}
\left(L_g \hat{A} L_{g^{-1}}\right)(f)(x) &= L_g\left(\hat{A}\left(L_{g^{-1}}(f)\right)\right)(x) \\
&= \hat{A}\left(L_{g^{-1}}(f)\right)\left(g^{-1}x\right) \\
&= \left(\alpha\left(\left(g^{-1}x\right)^{-1}\right) A\right)\left(L_{g^{-1}}(f)\right)\left(g^{-1}x\right) \\
&= \left(\alpha\left(x^{-1}g\right) A\right)(f(x)) \\
&= \left(\left(\widehat{\alpha(g) A}\right)(f)\right)(x),
\end{aligned}$$

we see that the covariance relation 4 holds. Therefore, if $\pi : \mathcal{A} \to B(K)$ is a unital $*$-homomorphism and K a Hilbert space, $\rho : G \to \mathcal{U}(K)$ a group homomorphism and for all $a \in \mathcal{A}$ and for all $g \in G$,

$$\rho(g)\,\pi(a)\,\rho\left(g^{-1}\right) = \pi(\alpha(g)(a)),$$

then there exists $\tau : \mathcal{A} \rtimes_\alpha G \to B(K)$ such that $\tau|_{\mathcal{A}} = \pi$ and $\tau|_G = \rho$. We define the reduced group C*-algebra of $\mathcal{A} \rtimes_\alpha G$ as

$$\overline{\{L_g \mid g \in G\}}^{\|\ \|},$$

and denote it as $\mathcal{A} \rtimes_{\alpha,r} G$.

It is clear that there is a natural $*$-homomorphism from $\mathcal{A} \rtimes_\alpha G$ onto $\mathcal{A} \rtimes_{\alpha,r} G$. If G is amenable, this $*$ -homomorphism is actually an isomorphism. We can say a little more. Suppose $\gamma : \mathcal{A} \to B(M)$ is a unital $*$-embedding. Then $\beta = \gamma\alpha(\cdot)\gamma^{-1} : G \to Aut(\gamma(\mathcal{A}))$ is a homomorphism. The following proposition says that if G is amenable, then the left regular representation $\mathcal{A} \rtimes_{\alpha,r} G$ is independent of the faithful representation of $\mathcal{A}$.

Proposition 21 *If G is an amenable discrete group and $\gamma : \mathcal{A} \to B(M)$ is a unital $*$-embedding, then $\mathcal{A} \rtimes_{\alpha,r} G$ and $\gamma(\mathcal{A}) \rtimes_{\beta,r} G$ are naturally isomorphic.*

Chapter 3

The algebras $\mathbb{B}(f,g)$ and $\mathbb{B}(m,n)$.

3.1 The Basic Properties

We now will introduce the main ideas in this paper. If $m,n \in \mathbb{Z}$, the *Baumslag-Solitar* group $B(m,n)$ is the group generated by u, v with the relation $uv^m u^{-1} = v^n$. We let $\mathbb{B}(m,n) = C^*(B(m,n))$.

Suppose $f, g : \mathbb{T} \to \mathbb{T}$ are continuous functions on the unit circle $\mathbb{T}$ in the complex plane with $f(\mathbb{T}) \cap g(\mathbb{T}) \neq \varnothing$. Let $\mathbb{B}(f,g)$ denote the universal C*-algebra generated by U and V subject to the conditions: U and V are unitary, and $Uf(V)U^{-1} = g(V)$. Note that if $f(z) = z^n$ and $g(z) = z^m$, then $\mathbb{B}(f,g) = \mathbb{B}(m,n)$. If $f(z) = z$ and $g(z) = e^{2\pi i\theta}z$ where θ is an irrational real number, then $\mathbb{B}(f,g)$ is the *irrational rotation algebra* $\mathbb{A}_\theta$.

Let $\mathbb{A}(f,g)$ denote the universal C*-algebra generated by unitaries W_n $(n \in \mathbb{Z})$ subject to the conditions $f(W_{k-1}) = g(W_k)$ $(k \in \mathbb{Z})$. It is clear that there is an automorphism $\alpha : \mathbb{A}(f,g) \to \mathbb{A}(f,g)$ such that $\alpha(W_k) = W_{k+1}$ $(k \in \mathbb{Z})$. In the C*-crossed product $\mathbb{A}(f,g) \rtimes_\alpha \mathbb{Z}$ there is a unitary U_α such that

$$\alpha(A) = U_\alpha A U_\alpha^{-1}$$

for every $A \in \mathbb{A}(f,g)$.

Theorem 22 *There is a $*$-isomorphism from $\mathbb{B}(f,g)$ to $\mathbb{A}(f,g) \rtimes_\alpha \mathbb{Z}$ that sends U to U_α and V to W_0. The inverse of this isomorphism sends W_k to $U^k V U^{-k}$ $(k \in \mathbb{Z})$.*

Proof. We will define a group homomorphism π from $\mathbb{B}(f,g)$ to $\mathbb{A}(f,g) \rtimes_\alpha \mathbb{Z}$, by this we mean on the group generated by generators, as follows,

$$U^{-k}VU^k \to W_k$$
$$U \to U_\alpha.$$

Using the Spectral Mapping Theorem we can then verify the condition on $\mathbb{A}(f,g) \rtimes_\alpha \mathbb{Z}$ with a simple calculation,

$$\begin{aligned} f(W_{k-1}) &= f(U^{-(k-1)}VU^{k-1}) \\ &= U^{-(k-1)}f(V)U^{k-1} \\ &= U^{-k}(Uf(V)U)U^k \\ &= U^{-k}g(V)U^k \\ &= g(U^{-k}VU^k) \\ &= g(W_k). \end{aligned}$$

We may now extend the above map to a $*$-homomorphism of $\mathbb{B}(f,g)$ to $\mathbb{A}(f,g) \rtimes_\alpha \mathbb{Z}$ such that when restricted to the group of generators it is the same and we will call this map π.

To see that π is a $*$-isomorphism, we first note that by defining the map from the group generated by the unitaries W_n into $\mathbb{B}(f,g)$ by

$$W_k \to U^{-k}VU^k,$$

we will get a group homomorphism. This then will induce a $*$-homomorphism from

$$C^*(\langle W_n \mid f(W_{k-1}) = g(W_k)\rangle$$

into $\mathbb{B}(f,g)$, which we will call τ. Since α acts as conjugation in the C*-crossed product, it follows that

$$\begin{aligned} \tau(\alpha(W_k)) &= \tau(W_k) \\ &= U^{-(k+1)}VU^{k=1} \\ &= U^{-1}(U^{-k}VU^k)U \\ &= U^{-1}\tau(W_k)U. \end{aligned}$$

Hence there will exist a $*$-homomorphism $\rho : \mathbb{B}(f,g) \to \mathbb{A}(f,g) \rtimes_{\alpha} \mathbb{Z}$ such that

$$\rho(W_k) = U^{-k} V U^k$$

and

$$\rho(U_\alpha) = U.$$

Thus $\rho = \pi^{-1}$ and $\mathbb{B}(f,g)$ is isomorphic to $\mathbb{A}(f,g) \rtimes_{\alpha} \mathbb{Z}$. ■

Suppose $f, g : \mathbb{T} \to \mathbb{T}$. Let $\mathbb{T}^\infty = \prod_{n\in\mathbb{Z}} \mathbb{T}$ with the product topology, and let $X_{f,g} = \{\{\lambda_n\} \in \mathbb{T}^\infty \mid f(\lambda_k) = g(\lambda_{k+1})$ for $k \in \mathbb{Z}\}$. Define a homeomorphism $\beta : X_{f,g} \to X_{f,g}$ by

$$\beta(\{\lambda_k\}) = \{\lambda_{k+1}\}.$$

Proposition 23 *Let $\mathcal{I}$ be the commutator ideal of $\mathbb{A}(f,g)$. Then $\mathbb{A}(f,g)/\mathcal{I}$ is isomorphic to $C(X_{f,g})$ and the automorphism φ on $\mathbb{A}(f,g)/\mathcal{I}$ induced by α is given on $C(X_{f,g})$ by*

$$\varphi(F) = F \circ \beta.$$

Lemma 24 *Thus $C(X_{f,g}) \rtimes_{\varphi} \mathbb{Z}$ is a $*$-homomorphic image of $\mathbb{B}(f,g)$. If $\mathbb{A}(f,g)$ is commutative, then $\mathbb{B}(f,g)$ is isomorphic to $C(X_{f,g}) \rtimes_{\varphi} \mathbb{Z}$.*

Proof. Let Σ be the maximal ideal space of $\mathcal{A}(f,g)/\mathcal{I}$ and ϕ a nonzero functional in Σ. It then follows by the Spectral Mapping Theorem that

$$\begin{aligned} f(\phi(W_k)) &= \phi(f(W_k)) \\ &= \phi(g(W_{k+1})) \\ &= g(\phi(W_{k+1})) \end{aligned}$$

and thus $\{\phi(W_n)\}$ is an element of $X_{f,g}$. Given a $\{\lambda_n\} \in X_{f,g}$ we can easily see that it easily gives rise to ϕ in Σ, by defining

$$\phi(W_n) = \lambda_n.$$

Thus the isomorphism follows. Now we note that the automorphism α gives us that,

$$\varphi(\{\phi(W_k)\}) = \{\phi(W_{k+1})\},$$

for any ϕ in Σ and hence for any F in $C(X_{f,g})$,

$$\begin{aligned}\varphi(F(\{\phi(W_n)\})) &= F(\varphi(\{\phi(W_n)\})) \\ &= F(\beta(\{\phi(W_n)\})).\end{aligned}$$

■

Corollary 25 *If either f or g is injective, then $\mathbb{B}(f,g)$ is isomorphic to $C(X_{f,g}) \rtimes_{\varphi} \mathbb{Z}$.*

Proof. In the proof of Theorem 22 we have $f(W_{k-1}) = g(W_k)$, and if either f or g is injective, it follows that W_k commutes with W_{k-1} (i.e., either $W_{k-1} = f^{-1}(g(W_k))$ or $W_k = g^{-1}(f(W_{k-1}))$). ■

Lemma 26 *If h is an injective continuous complex function on $f(\mathbb{T}) \cap g(\mathbb{T})$, then $\mathbb{B}(f,g)$ is $*$-isomorphic to $\mathbb{B}(h \circ f, h \circ g)$.*

Proof. To see this we will define a map ϕ from $\mathbb{B}(f,g)$ to $\mathbb{B}(h \circ f, h \circ g)$ by

$$\begin{aligned} U &\to h(U) \\ V &\to h(V). \end{aligned}$$

Since,

$$g(V) = Uf(V)U^{-1} \to h(Uf(V)U^{-1}) = U(h(f(V))U^{-1} = h(g(V)),$$

and

$$g(V) \to h(g(V))$$

it is clear that ϕ is a $*$-homomorphism. The inverse will follow easily from the fact that h is injective, and thus is invertible on a restricted range. ■

Lemma 27 .*$\mathbb{B}(f,g)$ is isomorphic to $\mathbb{B}(g,f)$.*

Proof. This lemma will follow by defining a map ϕ from $\mathbb{B}(f,g)$ to $\mathbb{B}(g,f)$ by

$$\begin{aligned} U &\to U^{-1} \\ V &\to V. \end{aligned}$$

■

3.2 Nuclearity and Amenability

The next result will enable us to conclude that in certain instances, the *Baumslag-Solitar* groups are amenable, but to conclude this we will need the concept of nuclearity. While this concept is of much importance in the study of operator algebras, it will only play a small role in our work. Thus, we will only mention the definition and results needed to conclude our work. We would however suggest to the reader to seek out one of the many books on operator algebras to further study this topic.

Definition 28 *A C*-algebra $\mathcal{A}$ is nuclear when, for each C*-algebra $\mathcal{B}$, there is only one norm on $\mathcal{A} \otimes \mathcal{B}$. such that $\mathcal{A} \otimes \mathcal{B}$ is a C*-algebra.*

The following proposition is a result of J Rosenberg and can be found in [Ros].

Proposition 29 *If $\mathcal{A}$ is an abelian C*-algebra with an automorphism α, then $\mathcal{A} \times_\alpha \mathbb{Z}$ is nuclear.*

The next result establishes the equivalence of discrete amenable groups and the nuclearity of their related group C*-algebra. This result is due to U. Haagerup and can be found in [Ha]

Theorem 30 *Let G be a discrete group. Then the following are equivalent:*

1. *G is amenable*

2. *$C_r^*(G) \approx C^*(G)$*

3. *$C^*(G)$ is nuclear.*

It should be noted that the above theorem can be extended to a more general situation, but for our purposes the above presentation will suffice.

Theorem 31 *If f or g is injective, then $\mathbb{A}(f,g)$ is abelian, $\mathbb{B}(f,g)$ is isomorphic to $\mathbb{B}(z,h)$ for some h, and $\mathbb{B}(f,g)$ is nuclear.*

Proof. Without loss of generality we may assume f is injective and thus $f(z) = z$ on a restricted domain. Clearly $\mathbb{B}(f, g)$ is isomorphic to $\mathbb{B}(z, h)$, where $h = f^{-1} \circ g$. It then follows that the relation on $\mathbb{A}(f, g)$ becomes

$$f(W_{k-1}) = W_{k-1} = g(W_k),$$

and

$$\begin{aligned} W_{n-1}W_n &= g(W_n)g(W_{n+1}) \\ &= g(W_{n+1)}g(W_n) \\ &= W_nW_{n-1}. \end{aligned}$$

Therefore $\mathbb{A}(f, g)$ is abelian and it follows then that $\mathbb{A}(f, g) \rtimes_\alpha \mathbb{Z}$ is nuclear and thus from Proposition 29 $B(f, g)$ is nuclear. ■

Corollary 32 *If $m = 1$ or $n = 1$, then $B(m, n)$ is amenable.*

While the following result is known, we include here a new and harder proof. Each element of $B(m, n)$ can be uniquely written as:

$$V^l U^{e_1} V^{l_1} U^{e_2} V^{l_2} \cdots U^{e_k} V^{l_k}, \text{where } l \in \mathbb{Z}, e_i = \pm 1,$$

$$|l_i| \le m - 1, \text{ if } e_i = 1$$

and

$$|l_i| \le n - 1, \text{ if } e_i = -1.$$

Thus it easily follows that $B(m, n)$ is an i.c.c. group.

Lemma 33 *The group homomorphism $\alpha : B(m, n) \to B(m, n)$ defined by mapping $U \to U$ and $V \to V^{-1}$ will extend to an outer autormorphism on $\mathcal{L}(B(m, n))$, such that $\alpha^2 = id$.*

Proof. Clearly α is a group automorphism and $\alpha^2 = id$. We may then extend α to an automorphism of the group von Neumann algebra, $\mathcal{L}(B(m, n))$ such that $\alpha^2 = id$. To see that α is an outer automorphism, suppose α is inner. Thus there is a $W \in \mathcal{L}(B(m,n))$ such that W is unitary and $\alpha(A) = W^*AW$ for all $A \in \mathcal{L}(B(m,n))$. We may then write $W = \sum_{g \in B(m,n)} \lambda_g g$, and note that $\alpha(U) = U = W^*UW$ and $\alpha(V) = V^* = W^*VW$. Hence,

$$\begin{aligned} UWU^* &= \sum_{g \in B(m,n)} \lambda_g U g U^{-1} \\ &= \sum_{g \in B(m,n)} \lambda_{UgU^{-1}} g \\ &= \sum_{g \in B(m,n)} \lambda_g g \\ &= W \end{aligned}$$

and,

$$\begin{aligned} VWV &= \sum_{g \in B(m,n)} \lambda_g V g V \\ &= \sum_{g \in B(m,n)} \lambda_{V^{-1}gV} g \\ &= \sum_{g \in B(m,n)} \lambda_g g \\ &= W. \end{aligned}$$

Since $\|W\| = 1$, then there exists a $g \in B(m,n)$ such that $\lambda_g \neq 0$ and thus $\lambda_g = \lambda_{U^{-1}gU} = \lambda_{V^{-1}gV}$. Thus we see that for all n,

$$g = U^{-1} g U = U^{-n} g U^n$$

and,

$$g = V^{-1} g V^{-1} = V^{-n} g V^{-n}.$$

With $g = U^{-n}gU^{n}$, and from the canonical form for an element in $B(m,n)$ we see that $g = U^{k}$, for some k. But if $k > 0$, then

$$\begin{aligned} U^{k} &= V^{-2}U^{k}V^{-2} \\ &= V^{-2}\underbrace{U\cdots U}_{k}V^{-2} \\ &= V^{-5}(UV^{-1})^{k-1}U, \end{aligned}$$

which is a contradiction. If $k < 0$, then a similar argument will also produce a contradiction. Thus we see that $k = 0$, and $g = 1$. Therefore α is an outer automorphism. ∎

To apply the work of A. Connes we will need to show that α is not in the *sot*-closure of the inner automorphisms of the group von Neumann algebra $\mathcal{L}\left(B\left(m,n\right)\right)$, notated as $\alpha \notin \overline{Inn}^{SOT}(\mathcal{L}(B(m,n)))$. In other words we will need to prove that for every $\varepsilon > 0$ there is a unitary $W_{\varepsilon} \in \mathcal{L}\left(B\left(m,n\right)\right)$ such that $\|W_{\varepsilon}UW_{\varepsilon}^{*} - U\| < \varepsilon$ and $\|W_{\varepsilon}VW_{\varepsilon}^{*} - V^{*}\| < \varepsilon$ or $\|U^{*}W_{\varepsilon}U - W_{\varepsilon}\| < \varepsilon$ and $\|VW_{\varepsilon}V - W_{\alpha}\| < \varepsilon$.

Theorem 34 *$\mathbb{B}(m,n)$ is a nuclear C^{*}-algebra if and only if* $\min(|m|,|n|) = 1$.

Proof. As stated prior to this theorem we will show that $\alpha \notin \overline{Inn}^{SOT}(\mathcal{L}(B(m,n)))$ and then this theorem will follow from A. Connes work in [Con].

Suppose $\varepsilon > 0$ and that there exists a unitary $W_{\varepsilon} \in \mathcal{L}(B(m,n))$ such that $\|U^{*}W_{\varepsilon}U - W_{\varepsilon}\| < \varepsilon$ and $\|VW_{\varepsilon}V - W_{\alpha}\| < \varepsilon$. Define the following sets:

$$\begin{aligned} S_1 &= \{g \in B(m,n) \mid g = U^{k}V\cdots, \text{ where } k > 0\} \\ S_2 &= \{g \in B(m,n) \mid g = U^{k}V\cdots, \text{ where } k < 0\} \\ S_3 &= B(m,n)\backslash\{S_1 \cup S_2\}. \end{aligned}$$

It is easily seen that the sets partition G thus we can write

$$\begin{aligned} W_{\varepsilon} &= \sum_{g\in B(m,n)} \lambda_g g \\ &= \sum_{g\in S_1} \lambda_g g + \sum_{g\in S_2} \lambda_g g + \sum_{g\in S_3} \lambda_g g, \end{aligned}$$

and

$$\begin{aligned} U^*W_\varepsilon U &= \sum_{g\in B(m,n)} \lambda_g U^{-1}gU \\ &= \sum_{g\in S_1} \lambda_g U^{-1}gU + \sum_{g\in S_2} \lambda_g U^{-1}gU + \sum_{g\in S_3} \lambda_g U^{-1}gU. \end{aligned}$$

From our assumption,

$$\left\| \left(\sum_{g\in S_2} \lambda_g g + \sum_{g\in S_3} \lambda_g g \right) - \sum_{g\in S_2} \lambda_g U^{-1}gU \right\| < \varepsilon,$$

and using a version of the triangle inequality it follows that

$$\left\| \sum_{g\in S_2} \lambda_g g + \sum_{g\in S_3} \lambda_g g \right\| - \left\| \sum_{g\in S_2} \lambda_g U^{-1}gU \right\| < \varepsilon.$$

Since $\left(\sum_{g\in S_2} \lambda_g g \right) \perp \left(\sum_{g\in S_3} \lambda_g g \right)$, and $\left(\sum_{g\in S_2} \lambda_g g \right)$ is unitarily equivalent to $\left(\sum_{g\in S_2} \lambda_g U^{-1}gU \right)$, it will follow via some simple calculations that

$$\left\| \sum_{g\in S_3} \lambda_g g \right\| < 3\varepsilon.$$

If we now define the sets

$$\begin{aligned} S_1' &= \{g \in B(m,n) \mid g = V^k U \cdots, \text{ where } k > 0\} \\ S_2' &= \{g \in B(m,n) \mid g = V^k U \cdots, \text{ where } k < 0\} \\ S_3' &= B(m,n) \backslash \{S_1 \cup S_2\}, \end{aligned}$$

and express W_ε and $VW_\varepsilon V$ as above, we can conclude using a similar argument that

$$\left\| \sum_{g\in S_1'} \lambda_g VgV \right\| < 3\varepsilon.$$

From what we have shown, we see that the part of W_ε beginning with either $V^{\pm}$ or $U^{\pm}$has norm less than one. Thus the norm of W_ε is less than one, which contradicts the fact that W_ε is a unitary. Therefore the result follows from A. Connes in [Con]. ■

3.3 Quasidiagonality

The concept of a quasidiagonal operator was first introduced by P.R. Halmos. Unlike quasitriangularity, quasidiagonality will not be preserved under similarity. Also, if T is a quasidiagonal operator, then so is every operator in the unital C*-algebra generated by T.

The notion of quasidiagonality was later extended to a quasidiagonal C*-algebra of operators and of a quasidiagonal representation of an arbitrary C*-algebra. Since there are $*$-isomorphic C*-algebras such that one is quasidiagonal and the other is not, quasidiagonality is not a C*-algebraic property. However, a weaker version was later introduce and then further expanded on by D. Hadwin in [Had2]. It will be from [Had2] which most of our work here will follow.

Definition 35 *Let H be a separable Hilbert space, then a C*-subalgebra $\mathcal{A}$ of $B(H)$ is* quasidiagonal *if there is an orthogonal sequence $\{P_n\}$ of finite-rank projections whose sum is* 1 *such that $A - \sum_n P_n A P_n$ is a compact operator for every A in $\mathcal{A}$.*

To see an example of quasidiagonal C*-algebra, suppose H is ℓ^2 and let S be the unilateral shift operator of multiplicity 1. Let $A = S \oplus S^*$. Since A is the sum of of a diagonal operator and a compact operator A is quasidiagonal and hence $C^*(A)$ is a quasidiagonal C*-algebra.

Next we introduce the notion of a quasidiagonal representation of a C*-algebra and a weakly quasidiagonal C*-algebra.

Definition 36 *A representation of an arbitrary C*-algebra is* quasidiagonal *if its range is quasidiagonal.*

Definition 37 *A separable C*-algebra is* weakly quasidiagonal *if it is $*$-isomorphic to a quasidiagonal C*-algebra of operators.*

With these notions, we will now consider the results in this section.

Suppose $f, g : \mathbb{T} \to \mathbb{T}$. Let $\mathbb{T}^\infty = \prod_{n\in\mathbb{Z}} \mathbb{T}$ with the product topology, and let $X_{f,g} = \{\{\lambda_n\} \in \mathbb{T}^\infty \mid f(\lambda_k) = g(\lambda_{k+1}) \text{ for } k \in \mathbb{Z}\}$. Define a homeomorphism $\beta : X_{f,g} \to X_{f,g}$ by

$$\beta(\{\lambda_k\}) = \{\lambda_{k+1}\}.$$

Theorem 38 *If $g : \mathbb{T} \to \mathbb{T}$ satisfies either g or g^{-1} has no nontrivial closed invariant subsets of $\mathbb{T}$, then $\mathbb{B}(z, g)$ is weakly quasidiagonal, and has a faithful quasidiagonal representation π so that $\pi(V)$ is a diagonal operator on $\ell^2(\mathbb{Z})$ and $\pi(U)$ is the bilateral shift operator.*

Proof. We know from theorem 31 that $\mathbb{B}(z, g)$ is $*$-isomorphic to $\mathcal{A} = C(X_{z,g}) \ltimes_\varphi \mathbb{Z}$. where the action φ is given by a homeomorphism on $X_{z,g}$. We define a $*$-representation $\pi : \mathcal{A} \to B(\ell^2(\mathbb{Z}))$ by

$$\pi_x(h) = diag\{h(\varphi^n(x))\}$$

for all $h \in C(X)$, $n \in \mathbb{Z}$ and

$$\pi_x(U) = W,$$

where W is the bilateral shift. It follows from Proposition 21 that the direct sum of all the π_x's is faithful, i.e., $\bigcap_{x\in X_{z,g}} Ker(\pi_x) = 0$. There is then an $x_0 \in X_{z,g}$, such that for all $n_0 > m$, $\{\varphi^n(\varphi^{n_0}(x_0)) \mid n \geq 0\}$ is dense in $X_{z,g}$. Thus for any $x \in X_{z,g}$, there is a subsequence such that $\varphi^{n_k}(x_0) \to x$. Hence it follows that $W^{n_k}\pi_{x_0}(\cdot)W^{-n_k}$ converges $*$-strongly to $\pi_x(\cdot)$. This says that $Ker(\pi_{x_0}) \subseteq Ker(\pi_x)$ and thus $Ker(\pi_{x_0}) \subseteq \bigcap_{x\in X_{z,g}} Ker(\pi_x) = 0$. Therefore π_{x_0} is faithful on $\mathcal{A}$, and our result follows from [Smu]. ∎

Since $B(z, g)$ is weakly quasidiagonal we note for $x_0 \in X_{f,g}$ and for all $m \geq 0$, $\{\varphi^n(x_0) \mid n > m\}$ is dense in $X_{f,g}$. We will then define π on U and V as,

$$\pi(V) = diag\{g(\varphi^n(x_0))\}$$

and $\pi(U)$ is the bilateral shift.

Corollary 39 *If $m = 1$ or $n = 1$, then $\mathbb{B}(m, n)$ is weakly quasidiagonal*

3.4 *Ext* Group of $C^*(B(m,n))$

Brown, Douglas and Fillmore initiated the study of the semigroup *Ext* in [BDF]. They were interested in the question of unitary equivalence in the Calkin algebra, $\mathcal{C} = B(H)/\mathcal{K}$, where H is a separable complex Hilbert space and $\mathcal{K}$ is the compact operators. The question arises naturally from the work of Weyl, von Neumann and Berg who showed that normal operators S and T in $B(H)$ will have the same limit points in their spectrum if and only if there is a compact operator K such that $S+K$ and T are unitarily equivalent.

If $\pi : B(H) \to \mathcal{C}$ is the Calkin map, then it is clear that the essential spectrum of T, $\sigma_e(T) = \sigma(\pi(T))$, is a unitary invariant. Also if N is normal in $B(H)$ then $\sigma(\pi(N))$ is the set of limit points of the spectrum of N. Thus we see that the spectrum is a unitary invariant for elements of $\mathcal{C}$ that are determined by normal operators from $B(H)$.

Brown, Douglas and Fillmore considered all operators T in $B(H)$ that give rise to normal operators in $\mathcal{C}$. These operators we will call *essentially normal* operators. While it may be shown that some of these operators arise as compact perturbations of normals, not all of them do. An example of this would be of the unilateral shift operator S. If $S = N+K$, with N normal and K compact, then the (Fredholm) index of S, that is $\dim\ker S - \dim\ker S^*$, would be -1, which would say that index of N is -1. But this is a contradiction since $\|N(h)\| = \|N^*(h)\|$ for every $h \in H$ which says that $\ker N = \ker N^*$ and hence must have index 0. Brown, Douglas and Fillmore classified unitary equivalence, in $\mathcal{C}$, of essentially normal operators solely in terms of the essential spectrum and the Fredholm index.

With this motivation, we will now define the object of study in this section.

Note that we will say that $\mathcal{A}$ is a separable C*-algebra if there is a faithful $*$-representation of $\mathcal{A}$ into $B(H)$, where H is a separable Hilbert space.

Definition 40 *Suppose that $\mathcal{A}$ is a separable C^*algebra. We then define $Ext(\mathcal{A})$ to be the equivalence classes of faithful $*$-representations of $\mathcal{A}$ into $\mathcal{C}$, where the equivalence of faithful $*$-representations τ_1 and τ_2 are equivalent if there is a unitary U on H such that $\tau_2 = \pi^*(U)\tau_2\pi(U)$. We define the sum of two equivalence classes $[\tau_1]+[\tau_2] = [\tau_1 \oplus \tau_2]$.*

From Voiculescu's generalization of the Weyl-von Neumann-Berg Theorem in [Voi6] Ext will always have an identity. Thus $Ext(\mathcal{A})$ will always be a unital semigroup. Arveson proved that if $\mathcal{A}$ is nuclear, then $Ext(\mathcal{A})$ is a group.

Theorem 41 *Suppose $\mathcal{A}$ is a unital separable C^*-algebra, $\alpha \in Aut(\mathcal{A})$ such that*

1. $Ext\left(C\left(S^1\right) \otimes \mathcal{A}\right) = Ext\left(C\left(S^1\right)\right) \times Ext(\mathcal{A})$
2. $Ext(\mathcal{A} \ltimes_\alpha \mathbb{Z})$ *is homotopy invariant.*

Then the map $\alpha : Ext(\mathcal{A} \ltimes_\alpha \mathbb{Z}) \to (Ext(\mathcal{A})) \times \mathbb{Z}$ is injective.

Proof. Let $\pi : B(H) \to \mathcal{C}(H)$ be the Calkin map. Suppose τ is invertible in $Ext(\mathcal{A} \ltimes_\alpha \mathbb{Z})$, i.e. $\tau : \mathcal{A} \ltimes_\alpha \mathbb{Z} \to \mathcal{C}(H)$ is a $*$-embedding, $\left[\tau_{|\mathcal{A}}\right] = 0$ in $Ext(\mathcal{A})$ and $ind(\tau(U)) = 0$, where U is the unitary conjugate as α in $\mathcal{A} \ltimes_\alpha \mathbb{Z}$.

We wish to show $[\tau] = 0$ in $Ext(\mathcal{A} \ltimes_\alpha \mathbb{Z})$. To see this we will begin by choosing a representation $(\rho, \nu) : \mathcal{A} \ltimes_\alpha \mathbb{Z} \to B(H)$ which is faithful. It then follows $\rho \otimes I = \rho \oplus \rho \oplus \rho \cdots$, $\nu \otimes I = \nu \oplus \nu \oplus \nu \cdots$ and $\pi \circ (\rho \otimes I)$ are all faithful representations. Since $\left[\tau_{|\mathcal{A}}\right] = 0$ and the representations $\tau_{|\mathcal{A}}$ and $\pi \circ (\rho \otimes I)$ are unitarily equivalent, we find that $[\pi \circ (\rho \otimes I)] = 0$. So without of loss of generality we may assume $\tau_{|\mathcal{A}} = \pi \circ (\rho \otimes I)$. Let N be a diagonal unitary in $B(H)$ such that, $\sigma(N) = S^1$, 1 is in the point spectrum of N and $N = diag(1 = \lambda_1, \lambda_2, \lambda_3, \cdots)$ where each λ_i has infinite multiplicity. Consider $\nu \otimes N = \lambda_1 V \oplus \lambda_2 V \oplus \lambda_3 V \oplus \cdots$ and let

$$\begin{aligned}\delta &= (\rho \otimes I, \nu \otimes N) \\ &= (\rho, \nu) \oplus (\rho, \lambda_2 v) \oplus (\rho, \lambda_3 \nu) \oplus \cdots .\end{aligned}$$

Since each $(\rho, \lambda_i \nu)$ is faithful, it follows that δ is faithful and thus $[\pi \circ \delta] = 0$ in $Ext(\mathcal{A} \ltimes_\alpha \mathbb{Z})$. It will therefore be enough to show that $[\tau \oplus (\pi \circ \delta)] = 0$, since this equivalent to showing $[\tau] = 0$ in $Ext(\mathcal{A} \ltimes_\alpha \mathbb{Z})$.

With $ind(\tau(U)) = 0$, we can find a $\overline{U}$ in $B(H)$ such that $\pi\left(\overline{U}\right) = \tau(U)$. Let $U_1 = \overline{U} \oplus (v \otimes N)$ and $W = (\nu \otimes I) \oplus (\nu \otimes I)$, and consider the unitary $U_1 W^* = U(\nu \otimes I)^* \oplus$

$(I \otimes N)$. Then $\left(\pi(U_1W^*), \pi \circ (\rho \otimes I)_{|\mathcal{A}}\right)$ is a faithful representation of $C(S^1) \otimes \mathcal{A}$ and $ind(\pi(U_1W^*)) = 0$. Since we know $[\pi \circ (\rho \otimes I)] = 0$ in $Ext\mathcal{A}$, then

$$[\pi(U_1W^*), \pi \circ (\rho \otimes I)] = 0$$

in $Ext(C(S^1)) \otimes \mathcal{A}$, and therefore we can find a representation $\beta : C(S^1) \otimes \mathcal{A} \to B(H)$ such that $\pi \circ \beta = (\pi(U_1W^*), \pi \circ (\rho \otimes I))$. We will write β as $\left(X, \beta_{|\mathcal{A}}\right)$, where $X \in \beta(\mathcal{A})'$ and X is a unitary on $C(S^1)$. Since $\beta(\mathcal{A})'$ is a von Neumann algebra, there is a self adjoint T in $\beta(\mathcal{A})'$ such that $X = e^{iT}$. Thus

$$\begin{aligned} \pi(U_1W^*) &= \pi\left(e^{iT}\right) \\ &= e^{i\pi(T)}. \end{aligned}$$

together with

$$\pi(\beta(\mathcal{A})) = (\pi \circ (\rho \otimes I))(\mathcal{A})$$

and that the exponent is a norm continuous function gives us $\pi(T) \in (\pi \circ (\rho \otimes I))(\mathcal{A})'$. Hence $\pi(U_1W^*) = e^{i\pi(T)}$ and this implies $\pi(U_1) = e^{i\pi(T)}\pi(W)$.

We now define for $0 \leq s \leq 1$,

$$\sigma_s = \left(\left(\tau \oplus (\pi \circ \delta)_{|\mathcal{A}}\right), e^{is\pi(T)}\pi(W)\right)$$

which clearly defines a homotopic path from

$$\begin{aligned} \sigma_0 &= \left(\left(\tau \oplus (\pi \circ \delta)_{|\mathcal{A}}\right), \pi(W)\right) \\ &= \left(\left(\tau \oplus (\pi \circ \delta)_{|\mathcal{A}}\right), V\right) \end{aligned}$$

and

$$\begin{aligned} \sigma_1 &= \left(\left(\tau \oplus (\pi \circ \delta)_{|\mathcal{A}}\right), e^{i\pi(T)}\pi(W)\right) \\ &= \left(\left(\tau \oplus (\pi \circ \delta)_{|\mathcal{A}}\right), \pi(U_1)\right) \\ &= \left(\left(\tau \oplus (\pi \circ \delta)_{|\mathcal{A}}\right), \tau(U) \oplus V\right) \\ &= \tau \oplus (\pi \circ \delta). \end{aligned}$$

Therefore we $[\tau \oplus (\pi \circ \delta)] = 0$ and thus $[\tau] = 0$. ■

The following results follows from 41.

Corollary 42 *$Ext(C^*(B(m,n)) = \mathbb{Z} \times \mathbb{Z}$, if $m = 1$ or $n = 1$..*

3.5 $\mathbf{K}_1\left(\mathbb{B}\left(f,g\right)\right)$.

The first notions of K-Theory arose in the setting of Grothendieck's work on the Riemann-Roch theorem. Atiyah and Hirzebruch then developed K-Theory in algebraic topology as a way to study vector bundles. Operator algebraists further generalized the K-Theory, setting it in a "noncommutative" topological space.

Noncommutative spaces arise in a natural way. If X is a compact Hausdorff space, then associated with it is a commutative C*-algebra of continuous functions on X. Since this relation is bijective, we can then view these spaces as the "commutative" topological spaces. Replacing the commutative C*-algebra with one that is noncommutative gives us a way to view noncommutative spaces.

In the context of algebraic topology, one considers first the K-group K_0, and then by using the suspension is able to define the other K-groups. Bott periodicity limits the need to define K-groups of higher order than six. For operator algebra's, Bott periodicity limits us to just two groups K_0 and K_1. Furthermore, due to the properties inherent in C*-algebras, we can define K_1 completely in terms of direct limits, without defining suspensions, K_0 or the Grothendieck construction.

Suppose $\mathcal{A}$ is a unital C*-algebra and n is a positive integer. Let $M_n(\mathcal{A}) \subset B(H^n)$ be the C*-algebra of $n \times n$ matrices with entries from $\mathcal{A}$ and standard matrix operations and norm. Then let $\mathcal{U}_n(\mathcal{A})$ be the unitary elements of $M_n(\mathcal{A})$, and $\mathcal{U}_n(\mathcal{A})_0$ the connected component of the identity. We can define an embedding of the group $\mathcal{U}_n(\mathcal{A}) \hookrightarrow \mathcal{U}_{n+1}(\mathcal{A})$ by

$$u \longrightarrow \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

We then define $\mathcal{U}_\infty(\mathcal{A})$ to be the C*-algebraic direct limit group(being ascending unions). Thus $\mathcal{U}_\infty(\mathcal{A}) = \cup_{n=1}^{\infty}\mathcal{U}_n(\mathcal{A})$. The distance given by the norm defines a natural topology on $\mathcal{U}_\infty(\mathcal{A})$. We now will give a definition of $K_1(\mathcal{A})$.

Definition 43 *Suppose $\mathcal{A}$ is a unital C*-algebra. Then*

$$K_1(\mathcal{A}) = \mathcal{U}_\infty(\mathcal{A})/\mathcal{U}_\infty(\mathcal{A})_0$$

where $\mathcal{U}_\infty(\mathcal{A})_0$ is the connected component of the identity. Note that when u is a unitary $n \times n$ matrix, $[u] \in K_1(\mathcal{A})$ denotes image of u in $K_1(\mathcal{A})$.

The following is a list of basic facts concerning K_1:

1. K_1 is a functor from the category of unital C*-algebras with unital $*$-homomorphisms as morphisms to the category of abelian groups.

2. For each unital C*-algebra $\mathcal{A}$, there is a canonical homomorphism $\kappa_{\mathcal{A}} : \mathcal{U}(\mathcal{A}) \to K_1(\mathcal{A})$ such that whenever $\pi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism and $\pi_* : K_1(\mathcal{A}) \to K_1(\mathcal{B})$ is the induced map, we have

$$\pi_* \circ \kappa_{\mathcal{A}} = \kappa_{\mathcal{B}} \circ \left(\pi|_{\mathcal{U}(\mathcal{A})}\right).$$

3. If $\mathcal{C}$ is the Calkin algebra, then $K_1(\mathcal{C}) = (\mathbb{Z}, +)$ and $\kappa_{\mathcal{C}}$ is the Fredholm index map.

4. $K_1(\mathcal{A} \oplus \mathcal{B}) = K_1(\mathcal{A}) \oplus K_1(\mathcal{B})$.

While we are unable to compute the K_1 group exactly, we can say something about its structure.

Theorem 44 *If $\mathcal{A} = \mathbb{B}(f, g)$, then $\kappa_{\mathcal{A}}(U)$ has infinite order and generates a direct summand of $K_1(\mathcal{A})$*

Proof. Let D be a diagonal unitary operator with eigenvalues $\lambda_1, \lambda_2, \ldots$ satisfying

$$f(\lambda_k) = g(\lambda_{k+1}),$$

and let S be the unilateral shift operator. We then have

$$S^* f(D) S = g(D).$$

The operator S is not unitary, but its image s in the Calkin algebra is unitary. Thus the elements s^* and d (the image of D in the calkin algebra) satisfy the defining relations of $\mathbb{B}(f,g)$. Hence there is a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{C}$ with $\pi(U) = s^*$ and $\pi(V) = d$. However, $\pi_*(\kappa_{\mathcal{A}}(U)) = \kappa_{\mathcal{C}}(s^*) = ind(s^*) = 1$. Thus $\pi_* : K_1(\mathcal{A}) \to \mathbb{Z}$ is a surjective homomorphism, and the theorem now follows from elementary group theory, i.e.,

$$K_1(\mathcal{A}) \simeq \langle \kappa_{\mathcal{A}}(U) \rangle \oplus \ker \pi_* \simeq \mathbb{Z} \oplus \ker \pi_*.$$

■

3.6 Free Entropy Dimension and $\mathbb{B}(f,g)$.

D. Voiculescu [Voi1]-[Voi4] introduced the notion of free entropy and free entropy dimension in a finite von Neumann algebra with a faithful trace. Voiculescu and later Ge [Ge1],[Ge2] and Ge and Shen [GS] used free entropy to solve many old open problems. We do not need to define the free entropy dimension $\delta_0(\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$; we need only quote a result of Hadwin [Had1], which is a modification of a result of Ge and Shen [GS]. Here $\sigma_p(A)$ denotes the *point spectrum* (i.e., set of eigenvalues) of the operator A.

Proposition 45 *Suppose $\mathcal{M}$ is a finite von Neumann algebra, with faithful trace τ, and $\mathcal{M}$ is generated by a (finite or infinite) sequence $\{T_1, T_2, \ldots\}$ of nonzero operators such that*

1. *$W^*(T_1)$ is hyperfinite*
2. *For each $j \in \mathbb{N}$, there are normal elements $A_j, B_j \in W^*(\{T_1, \ldots, T_j\})$ such that, for $j = 1, 2, \ldots,$*

    (a) $A_j T_{j+1} = T_{j+1} B_j$

(b) A_j *and* B_j *have no common eigenvalues, i.e.* $\sigma_p(A_j) \cap \sigma_p(B_j) = \varnothing$.

Then $\delta_0(\mathcal{M}) \leq 1$.

Theorem 46 *If* $\pi : \mathbb{B}(f,g) \to \mathcal{M}$ *is a unital* $*$*-homomorphism where* $\mathcal{M}$ *is a* II_1 *factor von Neumann algebra, and*

$$\sigma_p(\pi(V)) = \sigma_p(\pi(f(V))) \cap \sigma_p(\pi(g(V))) = \varnothing$$

then the free entropy dimension of $\pi(\mathbb{B}(f,g))''$ *is at most* 1.

Proof. Let $T_1 = \pi_\tau(V)$, $T_2 = f(\pi_\tau(V))$, $T_3 = g(\pi_\tau(V))$, and $T_4 = U$. Since $T_2T_1 = T_1T_2$, $T_3T_1 = T_1T_3$, and $T_4T_2 = T_3T_4$, the conditions of the preceding theorem are satisfied. ∎

The following corollary was proved in [GS] (Ge-Shen)

Corollary 47 *If* $\mathcal{M}$ *is the von Neumann algebra generated by the left regular representation of* $B(m,n)$, *then* $\delta_0(\mathcal{M}) \leq 1$.

Chapter 4

Observation about Random Matrices

In [Voi5] D. Voiculescu proved a theorem about the asymptotic freeness in the limit as $n \to \infty$ of a randomly chosen (with respect to Haar measure) k-tuple of $n \times n$ unitary matrices, with respect to the normalized trace $\tau_n$on $\mathcal{M}_n(\mathbb{C})$ averaged with respect to Haar measure. A very elementary proof of this result was given by Dostal and Hadwin [DH]. In this section we show that Voiculescu's result holds when τ_n is replaced with any state on $\mathcal{M}_n(\mathbb{C})$.

Let $\mathcal{U}_n$ denote the group of unitary $n \times n$ matrices with Haar measure μ_n. If $k \in \mathbb{N}$, let $\mathcal{U}_n^k$ denote the Cartesian product of k copies of $\mathcal{U}_n$, and let μ_n^k denote the corresponding product measure. Let $\vec{u} = (u_1, \ldots, u_k) \in \mathcal{U}_n^k$; then we can view $u_1, \ldots, u_k$ as matrix-valued variables on $\mathcal{U}_n^k$. Voiculescu's result concerns limits of expressions of the form

$$\int_{\mathcal{U}_n^k} \tau_n \left(u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m}\right) d\mu_n^k(\vec{u}),$$

where $1 \leq s_1, \ldots, s_m \leq k$ and $t_1, \ldots, t_m \in \mathbb{Z}$.

Theorem 48 *Suppose $\varphi : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ is a state and $1 \leq s_1, \ldots, s_m \leq k$ and $t_1, \ldots, t_m \in \mathbb{Z}$. Then*

$$\int_{\mathcal{U}_n^k} \tau_n \left(u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m}\right) d\mu_n^k(\vec{u}) = \int_{\mathcal{U}_n^k} \varphi \left(u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m}\right) d\mu_n^k(\vec{u}).$$

Proof. First we note that any linear functional φ on $\mathcal{M}_n(\mathbb{C})$ can be represented as

$$\varphi(T) = trace\,(TK)$$

for some unique $n \times n$ matrix K. Saying that φ is a state is the same as saying that $K \geq 0$ and $trace(K) = 1$. We can write $K = WDW^{-1}$ with $W \in \mathcal{U}_n$ and D a diagonal matrix with nonnegative diagonal entries $p_1, \ldots, p_n$. Note that τ_n is the special case where $p_1 = \cdots p_n = 1/n$. Let D_j be the diagonal $n \times n$ matrix with a 1 in the j^{th} diagonal entry and 0 elsewhere. Then there is a unitary $n \times n$ matrix V_j such that $D_j = V_j D_1 V_j^{-1}$. Since μ_n is Haar measure, an integral $\int_{\mathcal{U}_n^k} h(\vec{u})\, d\mu_n^k(\vec{u})$ is unchanged when each u_j is replaced by $a_j u_j b_j$, where $a_1, b_1, \ldots, a_k, b_k \in \mathcal{U}_n$. It follows that

$$
\begin{aligned}
& \int_{\mathcal{U}_n^k} \varphi\left(u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m}\right) d\mu_n^k(\vec{u}) \\
= & \int_{\mathcal{U}_n^k} trace\left(u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m} WDW^{-1}\right) d\mu_n^k(\vec{u}) \\
= & \int_{\mathcal{U}_n^k} trace\left(W^{-1} u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m} WD\right) d\mu_n^k(\vec{u}) \\
= & \int_{\mathcal{U}_n^k} trace\left(\left(W^{-1} u_{s_1} W\right)^{t_1} \left(W^{-1} u_{s_2} W\right)^{t_2} \cdots \left(W^{-1} u_{s_m} W\right)^{t_m} D\right) d\mu_n^k(\vec{u}) \\
= & \int_{\mathcal{U}_n^k} trace\left(u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m} D\right) d\mu_n^k(\vec{u}) \\
= & \sum_{j=1}^{n} p_j \int_{\mathcal{U}_n^k} trace\left(u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m} V_j D_1 V_j^{-1}\right) d\mu_n^k(\vec{u})
\end{aligned}
$$

and, using the same reasoning, each of the integrals in the last sum, and thus the whole sum, equals

$$\int_{\mathcal{U}_n^k} trace\left(u_{s_1}^{t_1} u_{s_2}^{t_2} \cdots u_{s_m}^{t_m} D_1\right) d\mu_n^k(\vec{u}).$$

Hence the integral does not depend on K. ∎

Chapter 5

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