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Burnside factors, amenability defects and transitive families of projections in factors of type II(1)

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**Burnside Factors, Amenability Defects and Transitive
Families of Projections in Factors of Type II_1**

BY

Jon P. Bannon

B.S., University of New Hampshire (1999)

DISSERTATION

Submitted to the University of New Hampshire
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

in

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Dedication

To Tenaya

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I would like to express the deepest gratitude to my advisor, Professor Liming Ge, for helping me grow to be a better mathematician and a better person. Without his encouragement, discipline and care, my writing this dissertation would not have been possible.

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Table of Contents

Dedication	iii
Acknowledgments	iv
Abstract	vii
1 Introduction and Background	1
1.1 Introduction	1
1.2 Background	3
2 Transitive Families of Projections in Factors of Type II_1	7
2.1 Main results on transitive families	9
3 Burnside Group Factors	16
3.1 An operator algebra view of a theorem of Burnside	17
3.2 The Burnside groups are I.C.C.	22
4 Correspondences	24
4.0.1 W^* -algebras and von Neumann algebras	24
4.0.2 First definition of correspondence	25
4.0.3 Second definition of correspondence	27
4.0.4 Third definition of correspondence	28
4.0.5 Normal left W^* -modules	29
4.0.6 Fourth definition of correspondence	36
4.0.7 Fifth definition of correspondence	38
4.0.8 Equivalence and containment of correspondences	46

4.0.9	Examples of correspondences	47
4.1	Amenability and correspondences	50
4.1.1	Topology on correspondences	50
4.2	Correspondences and the amenability of von Neumann algebras	57
5	Følner Invariants	59
5.1	Connes' Følner-type condition	61
5.2	Group von Neumann algebras	65
	Bibliography	82

ABSTRACT

Burnside Factors, Amenability Defects and Transitive Families of Projections in Factors of Type II_1

by

Jon P. Bannon

University of New Hampshire, May, 2005

We introduce a notion of transitive family of projections in a type II_1 factor and prove that there exists (i) a 5 element transitive family in the hyperfinite II_1 and (ii) a 12 element free transitive family. We then prove that the group von Neumann algebras of the known infinite free Burnside groups are all type II_1 factors. Our investigation of weak-amenability properties of Burnside groups leads us to consider the Connes theory of correspondences. From this investigation we are able to define a new Følner invariant for type II_1 factors. We prove a monotonicity result and find a positive lower bound for the free group factor $L(\mathbb{F}_2)$.

Chapter 1

Introduction and Background

1.1 Introduction

In this dissertation, we investigate various questions about the structure of type II_1 factors. The way we present the material in the paper respects the order in which it was investigated, to give the reader a sense of the genesis of the ideas. We begin in Chapter 2 by introducing the notion of transitive family of projections in a type II_1 factor. We prove, by generalizing a method of Halmos, that in certain classes of type II_1 factors there always exists a transitive family of five nontrivial projections. We then prove that one can find a transitive family of five nontrivial projections which are free with respect to the trace of the factor they generate. In Chapter 3 we present a short proof of a theorem proved originally by W. Burnside in 1902 which states that the only homomorphism of an infinite group of finite exponent into the general linear group of a finite dimensional vector space is the zero homomorphism. The main original contribution of our proof is to highlight which parts of the proof are intrinsically operator-algebraic and which are not, helping to more clearly expose the obstruction to finding out whether or not the infinite Burnside groups provide a counterexample to the Connes embedding conjecture. After this, we go on to prove that the group von Neumann algebras of infinite free Burnside groups of large enough odd exponent are type II_1 factors. Our proof relies heavily on results from group theory. In Chapter 4 we follow closely the unpublished notes of Sorin Popa on the

Connes theory of correspondences, filling in details in order to provide a readable, introductory account of this theory. The development of Chapter 4 leads us to use the Connes-Følner condition in chapter 5 to define a new Følner invariant $F\phi(M)$ for type II_1 factors, about which we obtain various results, culminating in a proof that $F\phi(L(\mathbb{F}_2)) \geq \frac{1}{7}$.

Our main result in the first chapter came about by trying to apply the technique of Murray and von Neumann's proof that the free group factors do not possess Property Γ to the generator question for type II_1 factors. The generator question of von Neumann asks if every von Neumann algebra acting on a separable Hilbert space can be generated, as a von Neumann algebra, by a single element. This is a central open question in the subject. The work in the second chapter was inspired by two questions of Liming Ge. The first question, paraphrased, asks whether or not the group von Neumann algebras of infinite free Burnside groups provide a counterexample to the Connes embedding conjecture. The proof we give of the theorem Burnside in the first part of Chapter 3 may be a good starting point for attacking Ge's question. The second part of Chapter 3 deals with Ge's question of whether or not the group von Neumann algebra of an infinite free Burnside group is a factor, and whether or not it must contain a noncommutative free group subfactor. This is motivated by the famous question of von Neumann in group theory asking if every non-amenable group must contain a nonabelian free subgroup, which was answered in the negative with the infinite free Burnside groups as a class of counterexamples. We solve the first part of this question in the affirmative. It is a well-known result of Adian that all of the free Burnside groups we consider are non-amenable groups. Recently the work of the group theorists Osin, Arzhantseva, Burillo, Lustig, Reeves, Short and Ventura has shown that there exist various notions of weak amenability for groups, and that the free Burnside groups are not, with respect to these various notions, weakly amenable (c.f.

[19], [4]). These authors construct several invariants that measure the amenability defect of a finitely generated group. This inspired us to consider whether or not there may be analogous notions of weak amenability for finitely generated type II_1 factors, and whether we could define new notions of amenability defect for these factors. The work of Osin uses Huliniski's representation-theoretic characterization of amenability, which leads us to consider the non-commutative representation theory provided by Connes's theory of correspondences. In Chapter 4 we closely follow Popa's unpublished notes on correspondences and fill in many details. This work provided us with a powerful point of view for thinking about amenability questions for type II_1 factors and, in conjunction with the technique of Murray and von Neumann used in the first chapter, motivated the original work in chapter 5.

This research was partially supported by a University of New Hampshire dissertation fellowship.

1.2 Background

The basics of the theory of operator algebras can be found in [16]. In this dissertation, we will provide a brief overview below, of related topics, for the sake of completeness. Other ideas will be introduced later in the text as needed. For Burnside groups, we refer to Adian [1].

Let H be a Hilbert space, and $B(H)$ the algebra of all bounded linear operators from H into itself. By the Riesz representation theorem, there is a natural involution $*$ on $B(H)$, where if $T \in B(H)$ then T^* is defined to be the unique operator in $B(H)$ satisfying $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$. A subalgebra \mathcal{A} of $B(H)$ is said to be self-adjoint if $T \in \mathcal{A}$ implies that $T^* \in \mathcal{A}$. Given $T \in B(H)$ and $x, y \in H$, define $\omega_{x,y}(T) = \langle Tx, y \rangle$. The weak-operator topology on $B(H)$ is the coarsest topology with respect to which each of the linear functionals $\omega_{x,y}$ is continuous. A

von Neumann algebra is a self-adjoint subalgebra of $B(H)$ that is closed in the weak-operator topology. John von Neumann discovered that just as one can decompose any semisimple algebra as a direct sum of simple algebras “indexed” by central elements, one can decompose any von Neumann algebra as a direct integral over its center, with the analogues of the simple summands played by von Neumann algebras, each having trivial center $\mathbb{C}I$. For this reason, he defined a factor to be a von Neumann algebra that has a trivial center. From 1935 to 1942 in the series of papers entitled “On Rings of Operators”, von Neumann and F.J. Murray developed the theory of von Neumann algebras beginning from the point of view of the Wedderburn structure theory of semisimple algebras.

All von Neumann algebras are generated by the self-adjoint projections they contain, a fact that motivated Murray and von Neumann to use properties of the projection lattice to classify the factors. Murray and von Neumann compared the ranges of two given projections in the factor by a partial isometry also in the factor. From this idea they obtained an equivalence relation on the set of projections, and a total ordering \preceq on the set of equivalence classes. More precisely, two projections P, Q in a factor are equivalent if there exists an operator V in the factor so that $V^*V = P$ and $VV^* = Q$; we have $P \preceq Q$ if there is a subprojection Q_0 of Q in the factor so that P is equivalent to Q_0 . Imitating set theory, Murray and von Neumann defined a finite projection in a factor to be one that is not equivalent to any proper subprojection in the factor. They also defined the notion of minimal projection in the natural way. They classified the factors into three broad types, type *I* factors are those with a minimal projection, type *II* factors are those containing no minimal projection but containing a finite projection, and type *III* factors are those in which every nonzero projection is infinite.

It should be noted that a general von Neumann algebra may also be called type *I*

(respectively *II* or *III*) if it has a direct integral decomposition into factors that are all of type *I* (respectively *II* or *III*).

All von Neumann algebras are algebras with identity, so a finer classification of factors is available. If the identity of a factor is a finite projection, then we say that the factor is finite. It is not hard to show that every type *I* factor contains a finite collection of mutually orthogonal equivalent minimal projections that sum to the identity element, and from this we may construct a $*$ -isomorphism of any finite type *I* factor with some $M_n(\mathbb{C})$. The only other finite factors are the infinite-dimensional ones, which are called type II_1 factors. An important equivalent criterion for a factor to be finite is that there exist a unique faithful tracial state on the factor. The range of this tracial state is $\{0, 1, 2, \dots, n\}$ for a type *I* factor $*$ -isomorphic to $M_n(\mathbb{C})$ and is $[0, 1]$ for any type II_1 factor. Two projections in the projection lattice of a finite factor are equivalent if and only if they have the same trace. Any factor that is not finite is called properly infinite, and must be of one of the distinct types $I_\infty (\cong \mathcal{B}(H))$, II_∞ or *III*. Although there is no trace on an infinite factor, we may define a $[0, \infty]$ -valued dimension function on the projection lattice of a factor that behaves like the trace does on a finite factor. In particular two projections are equivalent if and only if they have the same dimension. The range of the dimension function is $\{0, 1, 2, \dots, \infty\}$ for a type I_∞ factor, $[0, \infty]$ for a type II_∞ factor and $\{0, \infty\}$ for a type *III* factor. In the first paper on rings of operators, Murray and von Neumann were able to construct examples of factors of every type using various actions of discrete groups on measure spaces.

Perhaps the most important motivation for studying type II_1 factors lies in the further classification of all factors up to $*$ -isomorphism, thanks to the remarkable work of A. Connes and M. Takesaki. In his 1973 Ph.D. thesis, Connes classified the type *III* factors into type III_λ , with $\lambda \in [0, 1]$. Takesaki proved his duality theorem

by showing that every type III factor is an abelian extension (by \mathbb{R}) of a type II_∞ factor. Since it can be easily shown that every type II_∞ factor is the tensor product of a type II_1 factor by $\mathcal{B}(H)$, the problem of completely classifying von Neumann algebras essentially reduces to classifying the type II_1 factors.

Chapter 2

Transitive Families of Projections in Factors of Type II_1

Let \mathcal{H} be a complex, separable Hilbert space, and $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a factor of type II_1 . If S is a non-empty set, we say that a family of norm-closed subspaces $\{\mathcal{H}_i\}_{i \in S}$ of \mathcal{H} is transitive relative to \mathcal{M} if for each $i \in S$, the projection P_i of \mathcal{H} onto \mathcal{H}_i lies in \mathcal{M} and only the scalar operators leave all of the \mathcal{H}_i invariant. In this case, we also say that the family $\{P_i\}_{i \in S}$ is a transitive family of projections relative to \mathcal{M} . When $\dim(\mathcal{H}) \geq 3$, a transitive family cannot contain only two nontrivial projections P and Q , since in this case for any $0 < \lambda < 1$,

$$\lambda P(I - Q) + Q(I - P)$$

leaves the ranges of both P and Q invariant, but cannot commute with P unless $PQP = QP$. In this paper we first prove that if \mathcal{M} is a type II_1 factor and is generated by two self-adjoint elements, then there is a transitive family of five projections relative to $\mathcal{M} \otimes M_2(\mathbb{C})$. This leads us to the question of whether or not there is a transitive family of three or four projections relative to some factor of type II_1 ? To shed light on this question we consider free families of projections. A family $\{P_i\}_{i=1}^n$ of projections in a factor of type II_1 is free if each P_i has trace $\frac{1}{2}$ and the P_i are free with respect to the trace (in the sense of Voiculescu, see [21] and [5]). We shall exhibit a free transitive family of twelve projections.

In $\mathcal{B}(\mathcal{H})$, a family of norm-closed subspaces is transitive if the only bounded operators on \mathcal{H} that leave every subspace in the family invariant are scalars. Transitive families of subspaces were first considered by Paul Halmos in his 1970 paper “Ten problems in Hilbert space” [13]. In this paper Halmos studied medial subspace lattices, which are families of subspaces that contain $\{0\}$, \mathcal{H} , and at least two nontrivial subspaces of \mathcal{H} , with the additional property that any pair of nontrivial subspaces K_1, K_2 in the lattice are topologically complementary (that is, $K_1 \cap K_2 = \{0\}$ and $\overline{\text{span}}\{K_1, K_2\} = \mathcal{H}$). Halmos constructed a finite-dimensional example of a transitive medial subspace lattice having five nontrivial elements, and raised the question of how small a transitive medial subspace lattice could be. In 1971, Harrison, Radjavi and Rosenthal found that, in a separable, infinite-dimensional Hilbert space, there is a transitive medial subspace lattice having four nontrivial elements[14]. It has become apparent since, that the construction of a medial subspace lattice having three elements is a difficult problem. In fact, even finding a transitive family of three nontrivial norm-closed subspaces is hard. Lambrou and Longstaff have shown that in finite (≥ 3) dimensional \mathcal{H} , the smallest possible cardinality of a transitive family of subspaces is four[17]. Hadwin, Longstaff and Rosenthal have (when $\dim \mathcal{H}$ is infinite) found a transitive family of two norm-closed subspaces and a linear manifold, and have shown that the existence of a three element transitive family of norm-closed subspaces would follow from the existence of two dense operator ranges in \mathcal{H} such that the only bounded operators leaving both of the ranges invariant are scalars[11].

We note that the questions considered in this paper are closely related to the generator question of von Neumann algebras, which asks if every von Neumann algebra acting on a separable Hilbert space is generated by two self-adjoint elements. The last example in this note shows that free families of projections that generate factors can be transitive.

2.1 Main results on transitive families

For basic information about von Neumann algebras, we refer the reader to [16].

Definition 1 *Let \mathcal{H} be a complex, separable Hilbert space, let I denote the identity in $\mathcal{B}(\mathcal{H})$, and let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a factor of type II_1 . Let S be a nonempty set. A family $\{P_i\}_{i \in S}$ of projections in $\mathcal{B}(\mathcal{H})$ is transitive relative to \mathcal{M} if each P_i is in \mathcal{M} and the only operators $T \in \mathcal{M}$ that satisfy $(I - P_i)TP_i = 0$ for all $i \in S$ are scalars.*

Remark 2 *When there is no danger of confusing which factor we are considering, we say that a family of projections $\{P_i\}_{i \in S} \subseteq \mathcal{M}$ is transitive, when $\{P_i\}_{i \in S}$ is transitive relative to \mathcal{M} .*

Proposition 3 *Let \mathcal{H} be a complex, separable Hilbert space, and let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a factor of type II_1 such that \mathcal{M} is generated, as a von Neumann algebra, by three projections P_1, P_2 and P_3 . Then the family $\{P_1, I - P_1, P_2, I - P_2, P_3, I - P_3\}$ is transitive relative to \mathcal{M} .*

Proof. If $T \in \mathcal{M}$ leaves the ranges of each of these projections invariant, then $TP_i - P_iT = P_iTP_i + (I - P_i)TP_i - P_iTP_i - P_iT(I - P_i) = 0$ for $i = 1, 2, 3$. It follows that $T \in \mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$. ■

We now extend an idea of Halmos[13].

Proposition 4 *Let \mathcal{H} be a complex, separable Hilbert space, and let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a factor of type II_1 such that \mathcal{M} is generated, as a von Neumann algebra, by two self-adjoint elements A, B . There is a transitive family of 5 projections relative to $\mathcal{M} \otimes M_2(\mathbb{C})$.*

Proof. We realize $\mathcal{M} \otimes M_2(\mathbb{C})$ as $M_2(\mathcal{M})$ acting on $\mathcal{H} \oplus \mathcal{H}$. Let I denote the identity in \mathcal{M} , and I_2 the identity in $M_2(\mathcal{M})$. Now each of the projections

$$P_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & \frac{1}{2}I \end{bmatrix}$$

lies in $M_2(\mathcal{M})$. If an operator $T \in M_2(\mathcal{M})$ leaves the ranges of each of these three projections invariant, then T must have the form

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix}$$

with $T_1 \in \mathcal{M}$. We consider the matrix

$$A_1 = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}A & \frac{1}{2}A \end{bmatrix} \in M_2(\mathcal{M}).$$

Letting $\lambda = \|A_1 A_1^*\|^{-1}$, we note that there is the following equality of range projections $R(\lambda A_1 A_1^*) = R(A_1 A_1^*) = R(A_1)$. Now notice that $\lambda A_1 A_1^*$ is a positive element of norm 1, and therefore $0 \leq \lambda A_1 A_1^* \leq I_2$, and by Lemma 5.15 in the first volume of [16], the sequence $\{(\lambda A_1 A_1^*)^{\frac{1}{n}}\}$ converges in the strong-operator topology to $R(\lambda A_1 A_1^*)$ and therefore $R(A_1) \in M_2(\mathcal{M})$, since $M_2(\mathcal{M})$ is a von Neumann algebra. Note that the range of the operator A_1 is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$, since it is the graph of the bounded operator A . Now if T leaves the ranges of P_1, P_2, P_3 , and A_1 invariant, then for any $x \in \mathcal{H}$ it must be that

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix} \begin{bmatrix} x \\ Ax \end{bmatrix} = \begin{bmatrix} T_1 x \\ T_1 Ax \end{bmatrix} = \begin{bmatrix} T_1 x \\ AT_1 x \end{bmatrix}.$$

It follows that $T_1 A = AT_1$.

Similarly, we consider

$$B_1 = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}B & \frac{1}{2}B \end{bmatrix} \in M_2(\mathcal{M})$$

and see that $R(B_1) \in M_2(\mathcal{M})$, and if T leaves the ranges of P_1, P_2, P_3 and B_1 invariant, then $T_1 B = B T_1$. Hence T_1 commutes with both generators of the factor \mathcal{M} , and hence $T_1 \in \mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$. Thus the family of projections

$$\{P_1, P_2, P_3, R(A_1), R(B_1)\} \subseteq M_2(\mathcal{M})$$

is transitive. ■

Corollary 5 *There is a transitive family of 5 projections relative to the hyperfinite II_1 factor \mathcal{R} .*

Proof. It is well known that $\mathcal{R} \simeq \mathcal{R} \otimes M_2(\mathbb{C})$, and that \mathcal{R} is generated by two self-adjoint elements. From the above proposition, $\mathcal{R} \otimes M_2(\mathbb{C})$ contains a transitive family of five projections. It follows that \mathcal{R} contains a transitive family of five projections.

■

We now exhibit a free transitive family of projections.

Let $\{G_i\}_{i=1}^n$ be groups, and e_i is the identity of G_i for $i = 1, 2, \dots, n$. Let $\underset{i=1}{*}^n G_i$ denote the group free product of the G_i , and let e denote its identity element. Recall that elements in $\underset{i=1}{*}^n G_i$ are given, in reduced form, by elements in the set $\{e\} \cup \bigcup_{k \in \mathbb{N}} \{g_{i_1} g_{i_2} g_{i_3} \dots g_{i_k} : i_j \in \{1, 2, \dots, n\}; i_j \neq i_{j+1} \text{ for } j \in \{1, 2, \dots, (k-1)\}; g_{i_j} \in G_{i_j} \setminus \{e_{i_j}\}\}$.

Let G denote the group free product $\underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 \dots * \mathbb{Z}_2}_{12 \text{ times}}$, and let $\mathbb{A} = \{a_1, a_2, a_3, \dots, a_{12}\}$ denote the canonical set of generators of G . This group is I.C.C., therefore the group von Neumann algebra \mathcal{L}_G acting on $l_2(G)$ is a factor of type II_1 . Recall that each element in \mathcal{L}_G has the form L_x , where $x \in l_2(G)$ and the action on a function $y \in l_2(G)$ is defined by $(L_x y)(g) = \sum_{h \in G} x(gh^{-1})y(h)$. With $g \in G$, let x_g be the function in $l_2(G)$ that takes the value 1 on g and 0 on every other group element. To avoid excessive use of subscripts, we everywhere write L_g in place of L_{x_g} . Since $L_{a_i}^2 = I$ for each $i \in \{1, 2, 3, \dots, 12\}$, it is evident that $P_i = \frac{I + L_{a_i}}{2}$ is a projection in \mathcal{L}_G , and the family $\{P_1, P_2, \dots, P_{12}\}$ is free with respect to the trace on \mathcal{L}_G .

Theorem 6 In \mathcal{L}_G , the family $\{P_1, P_2, P_3, \dots, P_{12}\}$ is transitive.

Proof. Suppose that $L_f \in \mathcal{L}_G$ is a solution to the system $(I - P_i)TP_i = 0$, ($i = 1, 2, 3, \dots, 12$), and therefore

$$(I - L_{a_i})L_f(I + L_{a_i}) = 0 \quad (i = 1, 2, 3, \dots, 12).$$

Let both sides of this equation act on x_e , to obtain

$$L_f x_e = L_{a_i} L_f x_e - L_f L_{a_i} x_e + L_{a_i} L_f L_{a_i} x_e \quad (i = 1, 2, 3, \dots, 12).$$

We see that

$$\begin{aligned} (L_f x_e)(g) &= \sum_{h \in G} f(gh^{-1})x_e(h) = f(g), \\ (L_{a_i}(L_f x_e))(g) &= (L_{a_i} f)(g) = \sum_{h \in G} x_{a_i}(gh^{-1})f(h) = f(a_i g), \\ (L_f(L_{a_i} x_e))(g) &= (L_f x_{a_i})(g) = \sum_{h \in G} f(gh^{-1})x_{a_i}(h) = f(ga_i), \\ (L_{a_i}(L_f L_{a_i} x_e))(g) &= (L_{a_i} L_f x_{a_i})(g) \\ &= \sum_{h \in G} x_{a_i}(gh^{-1}) \left(\sum_{k \in G} f(hk^{-1})x_{a_i}(k) \right) = f(a_i g a_i). \end{aligned}$$

From these it follows that for all $g \in G$

$$f(g) = f(a_i g) - f(ga_i) + f(a_i g a_i) \quad (i = 1, 2, 3, \dots, 12).$$

By the triangle inequality, we see that

$$|f(g)| \leq |f(a_i g)| + |f(ga_i)| + |f(a_i g a_i)| \quad (i = 1, 2, 3, \dots, 12),$$

and by the well known inequality $(x_1 + \dots + x_k)^2 \leq k(x_1^2 + \dots + x_k^2)$ for non-negative real k , we see that

$$|f(g)|^2 \leq 3(|f(a_i g)|^2 + |f(ga_i)|^2 + |f(a_i g a_i)|^2) \quad (i = 1, 2, 3, \dots, 12).$$

With $i, j \in \{1, 2, \dots, 12\}$ given, let $S_{ij} = \{g \in G : g \text{ begins with } a_i \text{ and ends with } a_j \text{ in its reduced form in the free product}\}$.

With $g \in S_{ij}$, let $\{b_{ij}^1, \dots, b_{ij}^{10}\} = \mathbb{A} \setminus \{a_i, a_j\}$. We then have

$$\begin{aligned} |f(g)|^2 &\leq 3(|f(b_{ij}^1 g)|^2 + |f(g b_{ij}^1)|^2 + |f(b_{ij}^1 g b_{ij}^1)|^2), \\ |f(g)|^2 &\leq 3(|f(b_{ij}^2 g)|^2 + |f(g b_{ij}^2)|^2 + |f(b_{ij}^2 g b_{ij}^2)|^2), \\ &\vdots \\ |f(g)|^2 &\leq 3(|f(b_{ij}^{10} g)|^2 + |f(g b_{ij}^{10})|^2 + |f(b_{ij}^{10} g b_{ij}^{10})|^2). \end{aligned}$$

Adding these inequalities, we obtain that

$$10|f(g)|^2 \leq 3 \sum_{k=1}^{10} (|f(b_{ij}^k g)|^2 + |f(g b_{ij}^k)|^2 + |f(b_{ij}^k g b_{ij}^k)|^2).$$

Summing over $g \in S_{ij}$, we have

$$\sum_{g \in S_{ij}} |f(g)|^2 \leq \frac{3}{10} \sum_{g \in S_{ij}} \sum_{k=1}^{10} (|f(b_{ij}^k g)|^2 + |f(g b_{ij}^k)|^2 + |f(b_{ij}^k g b_{ij}^k)|^2).$$

Suppose that $g_1 \neq g_2$ are elements in S_{ij} . Note that by construction, all elements of the form $b_{ij}^k g_1$, $b_{ij}^k g_1 b_{ij}^l$ or $g_2 b_{ij}^k$ are in reduced form in the free product, for $k, l \in \{1, 2, \dots, 10\}$. Consider any $k_1, k_2, k_3, k_4 \in \{1, 2, \dots, 10\}$. Since $g_1 \neq g_2$, it follows that $b_{ij}^{k_1} g_1 \neq b_{ij}^{k_2} g_2$, $b_{ij}^{k_1} g_1 b_{ij}^{k_2} \neq b_{ij}^{k_3} g_2 b_{ij}^{k_4}$, $b_{ij}^{k_1} g_1 \neq g_2 b_{ij}^{k_2}$, $b_{ij}^{k_1} g_1 \neq b_{ij}^{k_2} g_2 b_{ij}^{k_3}$, and $g_1 b_{ij}^{k_1} \neq b_{ij}^{k_2} g_2 b_{ij}^{k_3}$. Therefore the right hand sum above can have no repeated terms, meaning for any $g_0 \in G$, the term $|f(g_0)|^2$ shows up at most once on the right hand side of the inequality. We sum over all i, j to obtain

$$\sum_{i,j=1}^{12} \sum_{g \in S_{ij}} |f(g)|^2 \leq \frac{3}{10} \sum_{i,j=1}^{12} \sum_{g \in S_{ij}} \sum_{k=1}^{10} (|f(b_{ij}^k g)|^2 + |f(g b_{ij}^k)|^2 + |f(b_{ij}^k g b_{ij}^k)|^2).$$

Let $S = \sum_{i,j=1}^{12} \sum_{g \in S_{ij}} \sum_{k=1}^{10} (|f(b_{ij}^k g)|^2 + |f(g b_{ij}^k)|^2 + |f(b_{ij}^k g b_{ij}^k)|^2)$. We now note that there can be repeated terms in S . We shall list the ways that a given term $|f(g_0)|^2$ may be repeated in the sum S . Suppose that $a, b \in \mathbb{A}$, and that g_0 begins with a and ends with b in its reduced form. Each occurrence of the term $|f(g_0)|^2$ in S corresponds to an appearance of $|f(g_0)|^2$ on the right side of an inequality of the form $|f(g')|^2 \leq 3(|f(a_i g')|^2 + |f(g' a_i)|^2 + |f(a_i g' a_i)|^2)$, where $a_i \in \mathbb{A}$ and g' is one of the group elements

ag_0 , g_0b or ag_0b . If $a \neq b$, then there can only be two occurrences of $|f(g_0)|^2$ in the sum S , one coming from the inequality

$$|f(ag_0)|^2 \leq 3(|f(g_0)|^2 + |f(ag_0a)|^2 + |f(g_0a)|^2),$$

and one from the inequality

$$|f(g_0b)|^2 \leq 3(|f(bg_0b)|^2 + |f(g_0)|^2 + |f(bg_0)|^2).$$

If $a = b$, then $|f(g_0)|^2$ may occur three times. Once in

$$|f(ag_0)|^2 \leq 3(|f(g_0)|^2 + |f(ag_0a)|^2 + |f(g_0a)|^2),$$

again in the inequality

$$|f(g_0a)|^2 \leq 3(|f(ag_0a)|^2 + |f(g_0)|^2 + |f(ag_0)|^2),$$

and finally, in the inequality

$$|f(ag_0a)|^2 \leq 3(|f(g_0a)|^2 + |f(ag_0)|^2 + |f(g_0)|^2).$$

We therefore note that any term $|f(g_0)|^2$ in S may occur at most three times. We call the number of times the term $|f(g_0)|^2$ appears in the sum S the multiplicity of $|f(g_0)|^2$ in S .

Let script \mathcal{T} denote $\{t : t \text{ is a term in } S\}$. Then, since all terms in S are non-negative, $S = \sum_{t \in \mathcal{T}} n_t t$ where n_t is the multiplicity of the term t in S .

We now have that

$$\begin{aligned} \sum_{g \in G \setminus \{e\}} |f(g)|^2 &= \sum_{i,j=1}^{12} \sum_{g \in S_{ij}} |f(g)|^2 \\ &\leq \frac{3}{10} S = \frac{3}{10} \sum_{t \in \mathcal{T}} n_t t \leq \frac{9}{10} \sum_{t \in \mathcal{T}} t \\ &\leq \frac{9}{10} \sum_{g \in G \setminus \{e\}} |f(g)|^2. \end{aligned}$$

Therefore $\sum_{g \in G \setminus \{e\}} |f(g)|^2$ is necessarily zero and $f(g) = 0$ when $g \neq e$. We have now that only $f(e)$ may be nonzero, and hence L_f must be a scalar. It follows that the family $\{P_1, \dots, P_{12}\}$ is transitive. ■

Remark 7 *The number of projections in the above theorem may be reduced. We believe that 4 such free projections should form a transitive family, but new techniques may be needed to prove this.*

Chapter 3

Burnside Group Factors

In 1902, William Burnside raised the question of whether a finitely generated group must be finite if each of its elements has order dividing a given natural number n , called the exponent of the group [3]. Along with this question Burnside provided cases in which it had an affirmative answer, namely for any group of exponent 2 or 3 and for all groups of exponent 4 that can be generated by 2 elements.

No further progress was made on this problem until 1940, when I.N. Sanov showed that all finitely generated groups of exponent 4 must be finite [20]. Seventeen years later, Marshall Hall [12] demonstrated that this was also true for groups of exponent 6.

In 1964 Golod discovered the first example of a finitely generated infinite group with the property that every element in the group has finite order. This finding suggested the existence of infinite groups of large exponent.

In 1968, Novikov and Adian published a ground breaking series of papers [2] in which they proved that there are infinite periodic groups with odd exponent $n \geq 4381$. Their proof followed from a complicated inductive method to present the free Burnside groups $B(m, n) = \mathbb{F}_m / \mathbb{F}_m^n$ by relations of the form $A^n = 1$ with specially chosen elements A in \mathbb{F}_m . In 1975, Adian [1] improved the method and showed that there are infinite periodic groups of odd exponent $n \geq 665$. Beyond proving that the groups $B(m, n)$ are infinite, Adian and Novikov were able to prove much more. For example, they determined that the word and conjugacy problems are solvable in $B(m, n)$, that

any finite or abelian subgroup of $B(m, n)$ is cyclic, and that the centralizer of any non-identity element in $B(m, n)$ is a cyclic group of order n . This last result will be the main ingredient in the proof that the group $B(m, n)$ is an *I.C.C.* group.

3.1 An operator algebra view of a theorem of Burnside

In 1905, Burnside proved that if G is a subgroup of $Gl(k, \mathbb{C})$ having finite exponent n , then the order of G must be finite. The proof of this result is over 90 pages long but contains some deep ideas that we should consider. We include here a short proof of this theorem with an operator-algebraic flavor. The main reason to consider an old question like this is that it may provide insight into a question of Ge which asks whether or not infinite Burnside group von Neumann algebras can be embedded into an ultrapower of the hyperfinite type II_1 factor.

Below, regard $M_k(\mathbb{C})$ as acting on $V = \mathbb{C}^k = \text{span}\{e_1, \dots, e_k\}$, where

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow (i\text{'th position}).$$

All groups $G \leq Gl(k, \mathbb{C}) \subseteq M_k(\mathbb{C})$, i.e. we consider the groups represented already for simplicity.

Definition 8 *A group G is reducible if V has a nontrivial G -invariant subspace, otherwise G is said to be irreducible.*

Definition 9 *Let $\text{End}(V)$ ($\cong M_k(\mathbb{C})$) denote the ring of all linear maps of V to itself. We denote by $\text{End}_G(V)$ the subring of $\text{End}(V)$ consisting of linear maps that are also G -linear.*

Remark 10 If $\psi : \text{End}(V) \rightarrow M_k(\mathbb{C})$ is an isomorphism, then $\psi(\text{End}_G(V)) = (\text{span}(G))'$.

Lemma 11 (Schur) If G is irreducible, then $\text{End}_G(V)$ is a division ring.

Proof. Suppose that G is irreducible and $0 \neq \varphi \in \text{End}_G(V)$. Since $\ker \varphi$ and $\varphi(V)$ are G -invariant subspaces of V , $\ker \varphi = \{0\}$ and $\varphi(V) = V$ and φ is a G -linear automorphism of V . ■

Theorem 12 (Wedderburn Reciprocity) If G is irreducible, then $(\text{span}(G))'' = \text{span}(G)$.

Proof. It is clear that $\text{span}(G) \subseteq (\text{span}(G))''$. The bijection $\psi : M_k(\mathbb{C}) \rightarrow \mathbb{C}^{k^2}$ sending

$$[\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_k] \mapsto \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_k \end{pmatrix}$$

gives us that $\psi T \psi^{-1}$ agrees with

$$\begin{bmatrix} T & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & T \end{bmatrix} = T \otimes I \in M_{k^2}(\mathbb{C})$$

in its action on \mathbb{C}^{k^2} . Let

$$G^\perp = \{S \in M_k(\mathbb{C}) | \text{Tr}(S^*g) = 0 \text{ for all } g \in G\}.$$

Decompose $M_k(\mathbb{C}) = \text{span}(G) \oplus G^\perp$ as an orthogonal direct sum with respect to the inner product $\langle A, B \rangle = \text{Tr}(B^*A)$. Let P be the orthogonal projection in $M_{k^2}(\mathbb{C})$ of \mathbb{C}^{k^2} onto $\psi(\text{span}(G))$. Suppose $v \in M_k(\mathbb{C})$, $v = v_0 + v_1$, where $v_0 \in \text{span}(G)$ and $v_1 \in G^\perp$. Note that $\text{span}(G)$ and G^\perp are both G -invariant subspaces. If $T \in \text{span}(G)$,

then

$$\begin{aligned} P\psi T\psi^{-1}\psi(v) &= P\psi T(v_0 + v_1) = P\psi(Tv_0 + Tv_1) \\ &= P\psi Tv_0 = \psi Tv_0 \end{aligned}$$

and

$$\begin{aligned} \psi T\psi^{-1}P\psi(v) &= \psi T\psi^{-1}(P\psi(v_0) + P\psi(v_1)) \\ &= \psi T\psi^{-1}(P\psi(v_0)) \\ &= \psi T\psi^{-1}\psi(v_0) = \psi Tv_0. \end{aligned}$$

It follows that P commutes with $\psi T\psi^{-1} = T \otimes I$ for all $T \in \text{span}(G)$, hence $P \in (\text{span}(G))' \otimes M_k(\mathbb{C}) \subseteq M_{k^2}(\mathbb{C})$.

Given $S \in (\text{span}(G))''$, $S \otimes I = \psi S\psi^{-1}$ commutes with P , hence leaves the range $\psi(\text{span}(G))$ of P invariant. However, $I \in \text{span}(G)$, so

$$\psi S\psi^{-1}\psi(I) = \psi(S) \in \psi(\text{span}(G))$$

and therefore

$$S \in \text{span}(G)$$

and $(\text{span}(G))'' \subseteq \text{span}(G)$. ■

Lemma 13 (*Burnside*) *If G is irreducible then $\text{span}(G) = M_k(\mathbb{C})$.*

Proof. It is clear that $\text{span}(G) \subseteq M_k(\mathbb{C})$. By Schur's lemma, $\text{End}_G(V)$ is a division ring, so $(\text{span}(G))'$ is as well. If $0 \neq T \in (\text{span}(G))'$ and $\lambda \in \sigma(T) (\neq \emptyset)$, then $T - \lambda I = 0$ and $T = \lambda I$. Hence $(\text{span}(G))' = \mathbb{C}I$, and $(\text{span}(G))'' = M_k(\mathbb{C})$. By the Wedderburn Reciprocity Theorem, $\text{span}(G) = (\text{span}(G))''$, and the proof is complete.

Remark 14 *If we considered only subgroups of unitary elements in $M_k(\mathbb{C})$, the above proof would follow from the von Neumann Double Commutant Theorem, which is a strengthening of Wedderburn Reciprocity to the infinite-dimensional setting.*

■

Proposition 15 (Burnside) *If G is irreducible, and there is a positive integer n such that for all $g \in G$, $g^n = I$, then $|G| \leq n^{k^3}$.*

Proof. Given $h \in G$, if $\lambda \in \sigma(h)$ then $x = h^n x = \lambda^n x$ for any eigenvector x and therefore $(\lambda^n - 1) = 0$, so λ is an n 'th root of unity. It follows that $Tr(h)$ can take on at most n^k different values. (let $Tr(h)$ be the usual non-normalized trace on $M_k(\mathbb{C})$). Choose a basis $(g_1, \dots, g_{k^2}) \subseteq G$ for $M_k(\mathbb{C})$. We prove that

$$x = y \text{ in } G \Leftrightarrow (Tr(x^* g_1), \dots, Tr(x^* g_{k^2})) = (Tr(y^* g_1), \dots, Tr(y^* g_{k^2})) \text{ in } \mathbb{C}^{k^2}.$$

The " \Rightarrow " direction is trivial. To prove the " \Leftarrow " direction, note that if

$$Tr(x^* g_i) = Tr(y^* g_i) \text{ for all } i$$

then $Tr((x - y)^* g_i) = 0$ for all i and hence $Tr((x - y)^* z) = 0$ for all $z \in \text{span}(G) = M_k(\mathbb{C})$, and therefore $x - y = 0$ and $x = y$. It follows that

$$|G| = \#\{(Tr(x^* g_1), \dots, Tr(x^* g_{k^2})) | x \in G\} \leq (n^k)^{k^2} = n^{k^3}.$$

■

Theorem 16 (Burnside) *If $G \leq Gl(k, \mathbb{C})$ such that for some positive integer n for every $g \in G$, $g^n = I$, then $|G| \leq n^{k^3}$.*

The case where G is irreducible was the subject of the previous proposition. Suppose that V has a nontrivial G -invariant subspace. The proof of the result will be by induction on k . The case where $k = 1$ is trivial. Suppose that all cases less than k have been settled. If $W \leq V$ is a nontrivial G -invariant subspace, that $\dim(W) = r$,

and there is an $S \in M_k(\mathbb{C})$ such that $S^{-1}GS = G_0 \leq Gl(k, \mathbb{C})$ such that each element of G_0 has the form

$$\begin{bmatrix} x_1 & x_3 \\ 0 & x_2 \end{bmatrix}$$

where x_1 is an $r \times r$ matrix, x_2 is a $(k-r) \times (k-r)$ matrix. The matrices $x_1 \in G_1$ (a representation of G on W) and the $x_2 \in G_2$ (a representation of G on a complement of W). Since $0 < r < k$, G_1 and G_2 satisfy the induction hypothesis, so $|G_1| \leq n^{r^3}$ and $|G_2| \leq n^{(k-r)^3}$. Given $A, B \in G_0$ with

$$A = \begin{bmatrix} x_1 & x_3 \\ 0 & x_2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} x_1 & x'_3 \\ 0 & x_2 \end{bmatrix}$$

we have that

$$B^{-1}A = \begin{bmatrix} I_r & T \\ 0 & I_{k-r} \end{bmatrix}$$

for some matrix T . We have that

$$I = (B^{-1}A)^n = \begin{bmatrix} I_r & nT \\ 0 & I_{k-r} \end{bmatrix}$$

implies that $T = 0$ and hence that $B^{-1}A = I$. We have proven that given $x_1 \in G_1$ and $x_2 \in G_2$ that there exists a unique x_3 so that

$$\begin{bmatrix} x_1 & x_3 \\ 0 & x_2 \end{bmatrix} \in G_0.$$

It follows that

$$|G| = |G_0| \leq |G_1||G_2| \leq n^{r^3+(k-r)^3} \leq n^{r^3},$$

since

$$k^3 = (r + k - r)^3 = r^3 + (k - r)^3 + 3r^2(k - r) + 3(k - r)^2r \geq r^3 + (k - r)^3.$$

3.2 The Burnside groups are I.C.C.

In this section we prove a result that we're sure is known to group theorists, but we cannot find in the literature. This result is the first that ties together two deep areas of mathematics, the study of infinite Burnside groups and that of type II_1 factors.

A group G is said to be an *I.C.C.* group if the conjugacy class of each non-identity element of G is infinite. In what follows $G \setminus S$ will denote the set theoretic complement of S in G . If $S \subseteq G$, and $g \in G$, we define $gS = \{gs : s \in S\}$. The centralizer of g in G is $C(g) = \{h \in G | hg = gh\}$.

If m, n are positive integers, the group $B(m, n)$ will denote the free Burnside group of exponent n on m generators, and let 1 be its identity element. This group is given by a set of generators a_1, a_2, \dots, a_m with defining relations of the form $g^n = 1$ for every word g in the group alphabet $\{a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_m^{\pm 1}\}$.

We now quote a result of Adian [1].

Theorem 17 (*Adian*) *The centralizer of an arbitrary non-identity element in $B(m, n)$ is a cyclic group of order n , for odd $n \geq 665$.*

Lemma 18 *Let G be an infinite group with identity element e . If $C(g)$ is finite for every $g \in G \setminus \{e\}$, then G is I.C.C.*

Proof. Let $g \neq e$, and suppose that $C(g)$ is finite. Define g_0 to be e . There is an element $g_1 \in G \setminus C(g)$, since G is infinite. Suppose $k > 1$ and that we have found g_0, \dots, g_k so that whenever $i, j \in \{0, \dots, k\}$ and $i \neq j$ we have that $g_i g g_i^{-1} \neq g_j g g_j^{-1}$, or equivalently, $g_i \notin g_j C(g)$. For each $i \in \{0, \dots, k\}$ the set $g_i S_1$ is finite, and hence the set $S = \bigcup_{i=0}^k g_i S_1$ is a finite union of finite sets, and is finite. It follows that there exists $g_{n+1} \in G \setminus S$ so that g_0, \dots, g_n, g_{n+1} have the property that $g_i g g_i^{-1} \neq g_j g g_j^{-1}$ whenever $i, j \in \{0, \dots, n+1\}$ and $i \neq j$. In this way, we construct a sequence $\{g_n\}_{n=0}^{\infty}$

of elements in G such that $g_i g g_i^{-1} \neq g_j g g_j^{-1}$ whenever $i \neq j$. Consequently, there are infinitely many distinct conjugates of g in G . ■

Theorem 19 *If $n \geq 665$ is odd, the group $B(m, n)$ is an I.C.C. group.*

Proof. By Adian's result the centralizer of any non-identity element in $B(m, n)$ is a cyclic group of order n . By the lemma, it follows that $B(m, n)$ is an I.C.C. group. ■

Chapter 4

Correspondences

4.0.1 W^* -algebras and von Neumann algebras

A von Neumann algebra is a unital C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology. A W^* -algebra is an (abstract) C^* -algebra that is a dual Banach space when viewed as a Banach space with respect to its C^* -norm. In other words, a C^* -algebra \mathcal{A} is a W^* -algebra if there exists a Banach space \mathcal{B} , so that $\mathcal{B}^\# = \mathcal{A}$. Note that, if $N \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, the set $N_\#$ of all linear functionals ρ on N that are continuous on $(N)_1$ with respect to the weak operator topology is a norm-closed subspace of the Banach space $N^\#$, and is therefore a Banach space. We call $N_\#$ the predual of N . Since a $*$ isomorphism of one von Neumann algebra onto another induces a linear isometric isomorphism between their preduals, the Banach space $N_\#$ is, up to isometric isomorphism, independent of the faithful representation of the von Neumann algebra N . The importance of the predual is that any von Neumann algebra N is isomorphic, as a Banach space, to $(N_\#)^\#$ (every norm-continuous linear functional on $N_\#$ has the form \widehat{T} , for some $T \in N$, where $\widehat{T}(\rho) = \rho(T)$ for all $\rho \in N_\#$). It follows that every von Neumann algebra is a W^* -algebra. The well-known stronger result of Sakai states that a C^* -algebra \mathcal{A} is $*$ -isomorphic to a von Neumann algebra if and only if \mathcal{A} is a W^* -algebra. This theorem enables us to speak interchangeably of W^* algebras and von Neumann algebras in what follows.

The ultraweak topology on a W^* -algebra N is the weak* topology on N obtained from $N_\#$. In terms of the convergence of nets, $T_\lambda \rightarrow T$ ultraweakly if and only

if $T_\lambda(\rho) \rightarrow T(\rho)$ for each $\rho \in N_\#$. If we regard N as a von Neumann algebra, then $T_\lambda \rightarrow T$ ultraweakly if and only if $\rho(T_\lambda) \rightarrow \rho(T)$ for each linear functional ρ on N that is weak operator continuous on the unit ball $(N)_1$ of N . A $*$ -homomorphism between two W^* -algebras N, M shall be called normal if it is continuous with respect to the ultraweak topology on N and the ultraweak topology on M . For a more extensive treatment on the ultraweak topology and normal functionals, we refer the reader to [16].

4.0.2 First definition of correspondence

We now introduce the first notion of correspondence.

Definition 20 *Let N, M be W^* -algebras. A correspondence between N and M (an N - M Hilbert W^* -bimodule) is a Hilbert space H equipped with bilinear product maps*

$$\begin{aligned} N \times H &\rightarrow H : (T, \xi) \mapsto T\xi && (\text{ultraweak} \times \|\cdot\|) - \|\cdot\| \text{ continuous} \\ H \times M &\rightarrow H : (\xi, S) \mapsto \xi S && (\|\cdot\| \times \text{ultraweak}) - \|\cdot\| \text{ continuous} \end{aligned}$$

such that

- (i) $I_N \xi = \xi I_M = \xi$
- (ii) $T_1(T_2 \xi) = T_1 T_2 \xi$
- (iii) $(\xi S_1) S_2 = \xi(S_1 S_2)$
- (iv) $(T \xi) S = T(\xi S)$

for all $\xi \in H$, all $T, T_1, T_2 \in N$, and all $S, S_1, S_2 \in M$.

We now make a few comments about this definition. If $\xi_\lambda \xrightarrow{\|\cdot\|} \xi$ in H , then for any $T \in N$, we have that $(T, \xi_\lambda) \xrightarrow{u.w. \times \|\cdot\|} (T, \xi)$, so by continuity, $T\xi_\lambda \xrightarrow{\|\cdot\|} T\xi$ in H . Therefore the operator L_T on H defined by $L_T \xi = T\xi$ is bounded, and clearly preserves the unit.

If $T \in N$ is unitary, then for all $\xi \in H$,

$$\begin{aligned} \langle T\xi, \xi \rangle &= \langle T\xi, TT^*\xi \rangle = \langle L_T\xi, L_T T^*\xi \rangle \\ &= \langle L_T^*(L_T\xi), T^*\xi \rangle = \langle T^*T\xi, T^*\xi \rangle = \langle \xi, T^*\xi \rangle. \end{aligned}$$

therefore $L_T^* = L_{T^*}$ in this case. since we may write any element in N as a finite linear combination of unitary elements, this also holds in general as well, by the bilinearity of the inner product. Now if $T_\lambda \xrightarrow{u.w.} T$, then for a given $\xi \in H$ we have that $T_\lambda\xi \xrightarrow{\|\cdot\|} T\xi$ and therefore $L_{T_\lambda} \xrightarrow{S.O.T.} L_T$, so if T_λ is a monotone increasing net of positive operators with least upper bound T , then L_{T_λ} is a monotone increasing net of positive operators with least upper bound L_T . We note that the ultraweakly continuous states ω on $range(L)''$ are exactly the normal states, i.e. those for which $\omega(H_\lambda) \rightarrow \omega(H)$ whenever $\{H_\lambda\}$ is a monotone increasing net of self-adjoint operators with least upper bound H . We may use this fact to prove that the map $L : T \rightarrow L_T$ is u.w.-u.w. continuous from N into $range(L)''$ since for every normal state φ on $range(L)''$, we have that $L \circ \varphi$ is a normal state on N . Every ultraweakly continuous linear functional on $range(L)''$ can be written as a linear combination of at most four normal states, a fact which follows from the polarization identity and the fact that each normal state on $B(H)$ can be written $\omega = \sum_{i=1}^{\infty} \omega_{x_i}$ with $\sum_{i=1}^{\infty} \|x_i\|^2 = 1$. We obtain that for every u.w. continuous linear functional φ on $range(L)''$, that $L \circ \varphi$ is an u.w. continuous linear functional on N , and this gives us that the map $L : N \rightarrow range(L)''$ is u.w.-u.w. continuous.

Note that $(N)_1 = ((N_{\#})^{\#})_1$ is weak*-compact, and therefore compact in the ultraweak topology. If $\varphi : N \rightarrow \mathcal{B}(H)$ is a normal *-representation, then φ is continuous from the ultraweak topology on N to the weak operator topology on $\mathcal{B}(H)$. It follows that $\varphi((N)_1)$ is W.O.T. compact, and hence W.O.T. closed. Since φ takes N onto $\varphi(N)$, which is a C^* -algebra and therefore a Banach space, it follows that φ is an open mapping, so there exists $r > 0$ such that $(\varphi(N))_r \subseteq \varphi((N)_1)$. By the Kaplansky

density theorem,

$$(\varphi(N)^-)_r \subseteq (\varphi(N))_r^- \subseteq (\varphi(N))_1^- = (\varphi(N))_1 \subseteq \varphi(N),$$

where $()^-$ denotes closure in the weak-operator topology. Hence $\varphi(N)^- \subseteq \varphi(N)$, and $\varphi(N)$ is W.O.T. closed, and therefore a von Neumann algebra. In particular, $range(L)'' = range(L)$ in the above discussion.

In summary, we have that the map $L : T \mapsto L_T$ defines a normal, unital $*$ -representation of N on $\mathcal{B}(H)$.

We define M^{op} to be equal to M as a Banach space, but with the product

$$S_1 \circ S_2 = S_2 S_1,$$

where $S_2 S_1$ is the product in the W^* -algebra M . Natural Banach and C^* -algebra structures on M^{op} are inherited from the Banach and C^* -algebra structures on M . Since $M = M^{op}$ as a Banach space, $M_{\#} = (M^{op})_{\#}$, and therefore M^{op} is a W^* -algebra. We call M^{op} the opposite W^* -algebra of M .

We see, via an argument nearly identical to the one above, that the map $S \mapsto R_S$, where $R_S \xi = \xi S$ defines a normal $*$ -representation of M^{op} on $\mathcal{B}(H)$.

Finally, property (iv) gives that $(T\xi)S = T(\xi S)$, and hence $R_S L_T \xi = L_T R_S \xi$ for all $\xi \in H$, $T \in N$, and $S \in M$. Therefore the representations $S \mapsto R_S$ and $T \mapsto L_T$ commute.

4.0.3 Second definition of correspondence

This brings us to our second definition of correspondence.

Definition 21 *Let N, M be W^* -algebras. A correspondence between N and M is a pair $(\pi_N, \pi_{M^{op}})$, where $\pi_N : N \rightarrow \mathcal{B}(H)$ and $\pi_{M^{op}} : M^{op} \rightarrow \mathcal{B}(H)$ are normal, unital $*$ -representations of N and M^{op} on the same Hilbert space H such that $\pi_N(N) \subseteq \pi_{M^{op}}(M^{op})'$.*

Note that the Hilbert space H in the above definition is naturally an N - M Hilbert W^* -bimodule with multiplications defined by

$$\begin{aligned}(T, \xi) &\mapsto \pi_N(T)\xi, \\ (\xi, S) &\mapsto \pi_{M^{op}}(S)\xi.\end{aligned}$$

Given a correspondence $(\pi_N, \pi_{M^{op}})$, we may construct a unital $*$ -representation $\pi_N \otimes \pi_{M^{op}}$ of the algebraic tensor product $*$ -algebra $N \otimes M^{op}$ on $\mathcal{B}(H) \otimes \mathcal{B}(H)$:

$$[(\pi_N \otimes \pi_{M^{op}})(T \otimes S)](\xi \otimes \eta) = (\pi_N(T) \otimes \pi_{M^{op}}(S))(\xi \otimes \eta) = \pi_N(T)\xi \otimes \pi_{M^{op}}(S)\eta,$$

which is normal when restricted to $N \otimes I_{M^{op}}$ and also normal when restricted to $I_N \otimes M^{op}$. Normality here means that if $T_\lambda \xrightarrow{u.w.} T$ in N , then $\pi_N(T_\lambda \otimes I_{M^{op}}) \xrightarrow{u.w.} \pi_N(T \otimes I_{M^{op}})$, and similarly in the other coordinate.

Conversely, given a unital $*$ -representation $\pi : N \otimes M^{op} \rightarrow \mathcal{B}(H)$ such that $\pi|_{N \otimes I_{M^{op}}}$ and $\pi|_{I_N \otimes M^{op}}$ are each normal, the pair $(\pi|_{N \otimes I_{M^{op}}}, \pi|_{I_N \otimes M^{op}})$ is a correspondence. In this manner we obtain a third definition of correspondence.

4.0.4 Third definition of correspondence

Definition 22 *Let N, M be W^* -algebras. A correspondence between N and M is a unital $*$ -representation of the algebraic tensor product $N \otimes M^{op}$ that is normal when restricted to $N \otimes I_{M^{op}}$ and is normal when restricted to $I_N \otimes M^{op}$.*

We shall now motivate a fourth definition of correspondence, in the case where N is a W^* -algebra, and M is a countably-decomposable factor von Neumann algebra. This fourth point of view will help us gain some intuition for Connes' view of correspondences as morphisms in the category of von Neumann algebras in his book [7].

4.0.5 Normal left W^* -modules

In order to motivate the fourth definition of correspondence, we revisit the theory of normal left M Hilbert W^* -modules, which we shall simply call left M -modules. Please note that, although it is in most cases natural to first introduce modules and then bimodules, we chose to introduce bimodules first because they shall be the primary objects of our study.

Definition 23 *Let M be a W^* -algebra. A (left) M -module is a Hilbert space H equipped with an (ultraweak \times $\|\cdot\|$) - $\|\cdot\|$ continuous, bilinear product map*

$$\begin{aligned} M \times H &\rightarrow H \\ (T, \xi) &\mapsto T\xi \end{aligned}$$

such that

$$\begin{aligned} i) \quad I_M \xi &= \xi \\ ii) \quad T_1(T_2 \xi) &= (T_1 T_2) \xi \end{aligned}$$

for all $\xi \in H$, and all $T_1, T_2 \in M$.

Proceeding as we did above in the case of bimodules, we have the more convenient equivalent definition of left module. This next definition is the one we will use.

Definition 24 *A (left) M -module is a pair (H, π) , where H is a Hilbert space, and $\pi : M \rightarrow \mathcal{B}(H)$ is a unital, normal $*$ -representation of M on H .*

We shall now define the concept of M -submodule, and the concept of isomorphism of M -modules.

Definition 25 *An M -module (K, π_K) is an M -submodule of the M -module (H, π) , if $K = PH$ for some projection $P \in (\pi(M))' \subseteq \mathcal{B}(H)$, and $\pi_K = \pi_P$, where $\pi_P(T) = \pi(T)P$ is the natural representation of M on PH .*

Definition 26 Let (H_1, π_{H_1}) and (H_2, π_{H_2}) be M -modules. A linear map $\varphi : H_1 \rightarrow H_2$ is an M -module homomorphism if it is M -linear, that is,

$$\varphi(\pi_{H_1}(T)\xi) = \pi_{H_2}(T)\varphi(\xi)$$

for all $\xi \in H_1$ and all $T \in M$. An M -module isomorphism is a unitary M -module homomorphism, meaning that as a linear map it is norm-preserving and invertible.

We now demonstrate a basic link between M -modules and Murray-von Neumann equivalence of projections.

Proposition 27 Let (H, π) be an M -module. Two M -submodules (PH, π_P) and (QH, π_Q) of (H, π) are isomorphic if and only if $P \sim Q$ in $\pi(M)'$.

Proof. Suppose that $P \sim Q$ in $\pi(M)'$. Let $V \in \pi(M)'$ satisfy $V^*V = P$ and $VV^* = Q$, then V is a partial isometry of PH onto QH , and the map φ defined by $\varphi(P\xi) = VP\xi$ is an M -module isomorphism. Conversely, suppose that $\varphi : PH \rightarrow QH$ is an M -module isomorphism, then by definition, φ is onto QH and $\|\varphi(P\xi)\| = \|P\xi\|$ for all $\xi \in H$, so $V\xi = \varphi(P\xi)$ defines a partial isometry V in $\mathcal{B}(H)$ with initial space PH and final space QH , hence $P \sim Q$ in $\pi(M)'$. ■

We now give a basic example of an M -module, and then show that any separable M module can be described using this example. A von Neumann algebra M is countably decomposable if any family of mutually orthogonal projections in M has cardinality at most \aleph_0 . Recall that a von Neumann algebra M is countably decomposable if and only if there is a faithful, normal state on M . The proof of this result can be found in [15], but we include it for completeness.

Example 28 Let M be a countably decomposable von Neumann algebra. If ρ is a faithful normal state on M , then $(L^2(M, \rho), \pi_\rho)$ is a left M -module, where π_ρ is the GNS representation of M on $L^2(M, \rho)$ associated to the state ρ .

Let $l^2(\mathbb{N})$ be the Hilbert space of all square summable sequences of complex numbers. Note that the Hilbert space $L^2(M, \varphi) \otimes l^2(\mathbb{N})$ is naturally an M -module when the action of M is given, for $T \in M$ and $\xi \otimes \eta \in L^2(M, \varphi) \otimes l^2(\mathbb{N})$, by $T(\xi \otimes \eta) = (T \otimes I)(\xi \otimes \eta) = T\xi \otimes \eta$. If $\{(H_i, \pi_i)\}_{i=1}^{\infty}$ are M -modules, then $(\bigoplus_{i=1}^{\infty} H_i, \bigoplus_{i=1}^{\infty} \pi_i)$ is an M -module, called the direct sum of the $\{(H_i, \pi_i)\}_{i=1}^{\infty}$. Here, the action of M on $\bigoplus_{i=1}^{\infty} H_i$ is given by $(\bigoplus_{i=1}^{\infty} \pi_i(T))(\xi_i)_{i=1}^{\infty} = (\pi_i(T)\xi_i)_{i=1}^{\infty}$. An M -module is called separable if it is separable as a Hilbert space. For notational purposes, in the following proposition we shall not write $(L^2(M, \rho), \pi_{\rho})$ for the GNS M -module associated to the state ρ , but instead we shall refer to "the M -module $L^2(M, \rho)$ ". Also, we shall write the GNS action of M on $L^2(M, \rho)$, for each $T \in M$ and $\xi \in L^2(M, \rho)$, by $T\xi$, rather than $\pi_{\rho}(T)\xi$. The same conventions will be held for modules closely related to $L^2(M, \rho)$, for example direct sums of copies of $L^2(M, \rho)$.

Proposition 29 *Let M be a von Neumann algebra, and ρ be a faithful, normal state on M . If H is a separable M -module, then there exists an M -submodule of $L^2(M, \rho) \otimes l^2(\mathbb{N})$ that is isomorphic to H as an M -module.*

Proof. Decompose H as $\bigoplus_{i=1}^{\infty} [\pi(M)\xi_i]$ for a sequence of vectors in H , and we see that $\pi = \bigoplus_{i=1}^{\infty} \pi|_{[\pi(M)\xi_i]}$. If each $([\pi(M)\xi_i], \pi|_{[\pi(M)\xi_i]})$ were isomorphic to an M -submodule of $L^2(M, \rho) \otimes l^2(\mathbb{N})$, then H would be isomorphic to an M -submodule of $\bigoplus_{i=1}^{\infty} (L^2(M, \rho) \otimes l^2(\mathbb{N}))$, where the action of M on this Hilbert space would be the direct sum of the standard (GNS) action. But with this action of M , any Hilbert space isomorphism of $\bigoplus_{i=1}^{\infty} (L^2(M, \rho) \otimes l^2(\mathbb{N}))$ onto $L^2(M, \rho) \otimes l^2(\mathbb{N})$ naturally gives an M -module isomorphism from $\bigoplus_{i=1}^{\infty} (L^2(M, \rho) \otimes l^2(\mathbb{N}))$ onto $L^2(M, \rho) \otimes l^2(\mathbb{N})$. So we need to prove the theorem only in the case where H is cyclic.

We introduce the following notation: if ϕ is a state on M , then $\langle \cdot, \cdot \rangle_{\phi}$ (resp. $\| \cdot \|_{\phi}$) shall denote the GNS inner product (resp. norm), associated to the state on $L^2(M, \omega)$.

Suppose now that H is cyclic. In this case, there is a unit vector $\eta \in H$ such that $[\pi(M)\eta] = H$, and $(H, \pi) \cong L^2(M, \psi)$, where $\psi(T) = \langle \pi(T)\eta, \eta \rangle_\psi$ is the normal state on M associated to η . We also may regard M as represented on $\mathcal{B}(L^2(M, \rho))$. From this viewpoint there exists a sequence $\{\xi_i\}_{i=1}^\infty$ of unit vectors in $L^2(M, \rho)$, such that $\sum_{i=1}^\infty \|\xi_i\|_\rho^2 = 1$, and $\psi(T) = \sum_{i=1}^\infty \langle T\xi_i, \xi_i \rangle_\rho$ for all $T \in M$. We shall also by ψ denote the obvious extension of ψ to all of $\mathcal{B}(L^2(M, \rho))$.

Note that for all $i \in \mathbb{N}$,

$$\|T\xi_i\|_\rho^2 = \langle T^*T\xi_i, \xi_i \rangle_\rho \leq \psi(T^*T) = \langle \pi(T^*T)\eta, \eta \rangle_\psi = \|\pi(T)\eta\|_\psi^2.$$

It follows that $\pi(T)\eta \mapsto T\xi_i$ defines a bounded M -linear map from $\pi(M)\eta$ into $L^2(M, \rho)$ that extends to a map $R_i : L^2(M, \psi) \rightarrow L^2(M, \rho)$.

Let $\{e_i\}_{i=1}^\infty$ be the standard orthonormal basis for $l^2(\mathbb{N})$, and define

$$\vartheta : L^2(M, \psi) \rightarrow L^2(M, \rho) \otimes l^2(\mathbb{N})$$

by $\vartheta(\xi) = \sum_{i=1}^\infty (R_i\xi \otimes e_i)$. This map is M -linear since each R_i is M -linear. Let the norm and inner product on $L^2(M, \rho) \otimes l^2(\mathbb{N})$ be written as $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ respectively.

Now for all $T \in M$,

$$\begin{aligned} \|\vartheta(\pi(T)\eta)\| &= \left\| \sum_{i=1}^\infty (R_i(T\eta) \otimes e_i) \right\| = \left\| \sum_{i=1}^\infty (T\xi_i \otimes e_i) \right\| \\ &= \left\langle \sum_{i=1}^\infty (T\xi_i \otimes e_i), \sum_{j=1}^\infty (T\xi_j \otimes e_j) \right\rangle^{1/2} \\ &= \left(\sum_{i=1}^\infty \langle T\xi_i, T\xi_i \rangle_\rho \right)^{1/2} = \left(\sum_{i=1}^\infty \langle T^*T\xi_i, \xi_i \rangle_\rho \right)^{1/2} \\ &= (\psi(T^*T))^{1/2} = (\langle \pi(T^*T)\eta, \eta \rangle_\psi)^{1/2} \\ &= (\langle \pi(T)^* \pi(T)\eta, \eta \rangle_\psi)^{1/2} = (\langle \pi(T)\eta, \pi(T)\eta \rangle_\psi)^{1/2} \\ &= \|\pi(T)\eta\|_\psi. \end{aligned}$$

It follows that ϑ is isometric, and $H \cong L^2(M, \psi) \cong \vartheta(L^2(M, \psi))$ which is an M -submodule of $L^2(M, \rho) \otimes l^2(\mathbb{N})$. ■

By the previous two propositions, classifying the separable M -modules up to isomorphism amounts to classifying all M -submodules of the single M -module $L^2(M, \rho) \otimes l^2(\mathbb{N})$ up to isomorphism, which amounts to classifying the projections in $(M \overline{\otimes} CI)' = M' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})) \subseteq \mathcal{B}(L^2(M, \rho) \otimes l^2(\mathbb{N}))$ up to Murray-von Neumann equivalence (Here we denote by M' the commutant of M in $\mathcal{B}(L^2(M, \rho))$). If M is a factor von Neumann algebra, then either M is of type *III*, in which case there is only one separable M -module up to isomorphism, or there is a faithful normal semi-finite tracial weight τ_∞ on $(M' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})))_+$ such that $\tau_\infty(I_M \otimes E_{11}) = 1$ where $E_{11} = \langle \cdot, e_1 \rangle e_1 \in \mathcal{B}(l^2(\mathbb{N}))$. In the latter case, the isomorphism classes of separable M modules are classified by the values of τ_∞ taken on the projection lattice in $M' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N}))$. Thus, the classification of separable modules over factors comes directly from the type classification of factors.

In particular, if M is a finite factor, we define the M -dimension, or coupling constant, of the M -module (H, π) by

$$\dim_M((H, \pi)) = \tau_\infty(P),$$

where P is any projection in $M' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ such that $(H, \pi) \cong P(L^2(M, \rho) \otimes l^2(\mathbb{N}))$ as an M -module. In the case where $N \subseteq M$ is an inclusion of a subfactor N of a finite factor M with trace τ , one may naturally view $L^2(M, \tau)$ as an N module, and in this situation we define the Jones index $[M, N]$ of the inclusion to be $\dim_N(L^2(M, \tau))$.

In general, it is not true that if two von Neumann algebras are $*$ -isomorphic, their commutants are as well. The following proposition will allow us to see that in the above situation, $\pi(M)$ is $*$ -isomorphic to $(M \overline{\otimes} CI)P$ (since M is a factor, it is algebraically simple and therefore $\pi(M)$ is $*$ -isomorphic to M) and also $\pi(M)' \subseteq \mathcal{B}(H)$ is $*$ -isomorphic to $((M \overline{\otimes} CI)P)' = P((M \overline{\otimes} CI))P = P(M' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})))P \subseteq \mathcal{B}(P(L^2(M, \rho) \otimes l^2(\mathbb{N})))$.

Proposition 30 *Let M be a von Neumann algebra. If two M -modules (H_1, π_1) and*

(H_2, π_2) are isomorphic as M -modules, then $\pi_1(M)'$ is $*$ -isomorphic to $\pi_2(M)'$.

Proof. Suppose that $\varphi : H_1 \rightarrow H_2$ is an M -module isomorphism. It follows that $\varphi(\pi_1(T)\xi) = \pi_2(T)\varphi(\xi)$ for all $T \in M$, hence that $\pi_1(T) = \varphi^{-1}\pi_2(T)\varphi = (Ad(\varphi) \circ \pi_2)(T)$ for every $T \in M$. Since φ is a unitary linear map from H_1 onto H_2 , the mapping $Ad(\varphi) : \mathcal{B}(H_2) \rightarrow \mathcal{B}(H_1)$ sending $A \in \mathcal{B}(H_2)$ to $\varphi^{-1}A\varphi \in \mathcal{B}(H_1)$ is a surjective $*$ -isomorphism. We shall show that $Ad(\varphi)|_{\pi_2(M)'}$ gives a $*$ -isomorphism of $\pi_2(M)'$ onto $\pi_1(M)'$. If $T' \in \pi_2(M)'$, then $T'\pi_2(T) = \pi_2(T)T'$ for all $T \in M$. We now show that $Ad(\varphi)|_{\pi_2(M)'}(T') \in \pi_1(M)'$. Let $\pi_1(T) \in \pi_1(M)$, then

$$\begin{aligned} Ad(\varphi)|_{\pi_2(M)'}(T')\pi_1(T) &= \varphi^{-1}T'\varphi\pi_1(T) \\ &= \varphi^{-1}T'\varphi\varphi^{-1}\pi_2(T)\varphi \\ &= \varphi^{-1}T'\pi_2(T)\varphi = \varphi^{-1}\pi_2(T)T'\varphi \\ &= \varphi^{-1}\pi_2(T)\varphi\varphi^{-1}T'\varphi = \pi_1(T)\varphi^{-1}T'\varphi \\ &= \pi_1(T)Ad(\varphi)|_{\pi_2(M)'}(T'). \end{aligned}$$

It is easy to show that the map $Ad(\varphi)|_{\pi_2(M)'}$ preserves adjoints, and is bijective. ■

Amplifications of Factors

Suppose that M is a countably decomposable factor, and let H be a separable Hilbert space. A factor $*$ -isomorphic to $P(M\overline{\otimes}\mathcal{B}(H))P$ with P some projection in $M\overline{\otimes}\mathcal{B}(H)$ is called an amplification of M . If P is an infinite projection in $M\overline{\otimes}\mathcal{B}(H)$, then $P(M\overline{\otimes}\mathcal{B}(H))P$ and $M\overline{\otimes}\mathcal{B}(H)$ are $*$ -isomorphic. To see this, note that if $VV^* = I$ and $V^*V = P$ in $M\overline{\otimes}\mathcal{B}(H)$, then the linear map

$$\theta : T \mapsto PV^*TVP$$

satisfies $\theta(T_1)\theta(T_2) = PV^*T_1VPV^*T_2VP = PV^*T_1VV^*VV^*T_2VP = PV^*T_1T_2VP = \theta(T_1T_2)$ and preserves adjoints. Since $M\overline{\otimes}\mathcal{B}(H)$ is a factor, θ is an injective map.

Given $PTP \in P(M\overline{\otimes}\mathcal{B}(H))P$, we have that $VTV^* \in M\overline{\otimes}\mathcal{B}(H)$ and $\theta(VTV^*) = PTP$ and hence θ is onto. From the reasoning above, we see that in the case where M is properly infinite, every amplification of M is isomorphic to $M\overline{\otimes}\mathcal{B}(H)$.

In the case where M is a finite factor, we have that $M\overline{\otimes}\mathcal{B}(H)$ possesses a faithful normal tracial weight $\tau_\infty = \tau \otimes Tr$. If P is a finite projection in $M\overline{\otimes}\mathcal{B}(H)$, suppose that $\tau_\infty(P) = \alpha < \infty$. Choose $k \geq \alpha$ so that we may find a projection $Q_1 \in M$ with $\tau(Q_1) = \frac{\alpha}{k}$. Find a finite projection $Q_2 \in \mathcal{B}(H)$ so that $Tr(Q_2) = rank(Q_2) = k$. We now show that $Q_2\mathcal{B}(H)Q_2$ must be a finite type I factor. Finiteness follows since Q_2 is the identity of $Q_2\mathcal{B}(H)Q_2$, and is a finite projection. We know that a projection E in a factor M is minimal if and only if $EME = \mathbb{C}E$. Suppose that E is a minimal projection in $\mathcal{B}(H)$. Since $\mathcal{B}(H)$ is a factor, the Murray-von Neumann order \preceq on its projection lattice is a total ordering. Since E is a minimal projection, we have that $E \preceq Q_2$, that is, there is a subprojection E_0 of Q_2 such that $E \sim E_0$ in $\mathcal{B}(H)$. It follows that E_0 is also a minimal projection in $\mathcal{B}(H)$, since for any non-zero projection $Q \in \mathcal{B}(H)$, we have that $E_0 \sim E \preceq Q$. We claim the $Q_2E_0Q_2$ ($= E_0$) is a minimal projection in $Q_2\mathcal{B}(H)Q_2$. It is clear that E_0 is a projection, we must show that it is minimal in $Q_2\mathcal{B}(H)Q_2$. To show that $Q_2E_0Q_2$ is minimal, consider $Q_2TQ_2 \in Q_2\mathcal{B}(H)Q_2$, then

$$(Q_2E_0Q_2)(Q_2TQ_2)(Q_2E_0Q_2) = Q_2 \underbrace{E_0Q_2TQ_2E_0}_{\lambda E_0} Q_2 = \lambda Q_2E_0Q_2.$$

It follows that $(Q_2E_0Q_2)(Q_2\mathcal{B}(H)Q_2)(Q_2E_0Q_2) = \mathbb{C}(Q_2E_0Q_2)$, and $Q_2E_0Q_2$ is a minimal projection. We see that $Q_2\mathcal{B}(H)Q_2$ is a finite type I factor with identity Q_2 . We have chosen Tr so that if E_0 is a minimal projection in $\mathcal{B}(H)$, then $Tr(E_0) = 1$. We can therefore write Q_2 as the sum of k equivalent, pairwise orthogonal projections E_1, \dots, E_k in $\mathcal{B}(H)$, and therefore $\sum_{i=1}^k Q_2E_iQ_2 = Q_2(\sum_{i=1}^k E_i)Q_2 = Q_2$ in $Q_2\mathcal{B}(H)Q_2$, and hence Q_2 is the sum of k such projections in $Q_2\mathcal{B}(H)Q_2$, and hence $Q_2\mathcal{B}(H)Q_2$ is a factor of type I_k , and $*$ -isomorphic to $M_k(\mathbb{C})$. It follows

that if P is a finite projection such that $\tau_\infty(P) = \alpha$, then $P(M\overline{\otimes}\mathcal{B}(H))P \cong (Q_1 \otimes Q_2)(M\overline{\otimes}\mathcal{B}(H))(Q_1 \otimes Q_2) \cong (Q_1 \otimes I)(M\overline{\otimes}M_k(\mathbb{C}))(Q_1 \otimes I)$. For this reason, we call a factor $*$ -isomorphic to

$$Q(M\overline{\otimes}M_k(\mathbb{C}))Q$$

an amplification of M when $0 < \alpha \leq k$, and $\tau_k(Q) = \frac{\alpha}{k}$, where $\tau_k = \tau \otimes \frac{1}{k}Tr$ is the normalized trace on the factor $M\overline{\otimes}M_k(\mathbb{C})$.

Tomita-Takesaki Theory

We now recall some basic facts about the Tomita-Takesaki modular theory that will be needed in the sequel. Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, with commutant M' . Suppose, in addition, that there is a joint cyclic and separating unit vector $\xi \in \mathcal{H}$ for M and M' . In this case, the mapping

$$S : A\xi \mapsto A^*\xi \quad (A \in M)$$

extends to a closable operator on \mathcal{H} , having polar decomposition $S = J\Delta^{1/2}$, where Δ is a positive invertible operator on \mathcal{H} , and J is an order two antilinear isometry of \mathcal{H} onto itself. The mapping

$$\phi : A \mapsto JA^*J$$

defines a $*$ -anti-isomorphism from M onto M' . It follows that M^{op} and M' are $*$ -isomorphic under these circumstances, an isomorphism given by $\phi \circ id^{-1}$.

4.0.6 Fourth definition of correspondence

We assume in this section that N is a W^* -algebra and that M is a countably decomposable factor. Let ρ be a faithful normal state on M^{op} . Suppose that $(\pi_N, \pi_{M^{op}})$ is a correspondence between N and M , such that the underlying Hilbert space H of the correspondence to be separable. We may consider only the action of

M^{op} for a moment, and view the underlying Hilbert space H as a left M^{op} -module $(H, \pi_{M^{op}})$. From our work on separable left modules, there is a projection $P \in (M^{op})' \otimes \mathcal{B}(l^2(\mathbb{N})) \subseteq \mathcal{B}(L^2(M^{op}, \rho) \otimes l^2(\mathbb{N}))$ so that $(H, \pi_{M^{op}})$ is isomorphic as an M^{op} module to $P(L^2(M^{op}, \rho) \otimes l^2(\mathbb{N}))$, and $(\pi_{op}(M^{op}))' \subseteq \mathcal{B}(H)$ is $*$ -isomorphic to $P((M^{op})' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})))P \subseteq \mathcal{B}(L^2(M^{op}, \rho) \otimes l^2(\mathbb{N}))$, say ϕ gives a $*$ -isomorphism of $(\pi_{op}(M^{op}))'$ onto $P((M^{op})' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})))P$. Note that this is a $*$ -isomorphism between von Neumann algebras, and is hence automatically normal. Since $\pi_N(N) \subseteq (\pi_{op}(M^{op}))'$, the map ϕ restricted to $\pi_N(N)$ gives a normal $*$ -homomorphism from $\pi_N(N)$ into $P((M^{op})' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})))P$.

Now by the Tomita-Takesaki theory, whenever a von Neumann algebra has a joint cyclic and separating vector, then it is $*$ -anti-isomorphic to $(M^{op})'$. We know that M^{op} is also $*$ -anti-isomorphic to M . If $\varphi_1 : M^{op} \rightarrow (M^{op})'$ and $\varphi_2 : M^{op} \rightarrow M$ are $*$ -anti-isomorphisms, then $\varphi_2^{-1} \circ \varphi_1$ is a $*$ -isomorphism of $(M^{op})'$ onto M . From these we obtain a $*$ -isomorphism Φ from $P((M^{op})' \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})))P$ onto $P_1(M \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})))P_1$ for some P_1 (Note that $L^2(M^{op}, \rho)$ may be naturally identified with $L^2(M, \rho)$). As a result, we have that $\Phi \circ \phi|_{\pi_N(N)}$ is a $*$ -homomorphism from $\pi_N(N)$ into $P_1(M \overline{\otimes} \mathcal{B}(l^2(\mathbb{N})))P_1$. This motivates the following definition.

Definition 31 *Let N be a W^* -algebra, and M be a countably decomposable factor. A correspondence between N and M is a normal, unital $*$ -homomorphism from N into an amplification of M .*

In motivating this definition, we have already shown how to get such a $*$ -homomorphism from a (separable) correspondence. We should show how to get a (separable) correspondence from such a $*$ -homomorphism. Suppose that $M^\alpha = P(M \overline{\otimes} \mathcal{B}(H))P$ is an amplification of the factor M (with H separable) and that $\varpi : N \rightarrow M^\alpha$ is a normal, unital $*$ -homomorphism. Reversing the process above, we get that $P(M \overline{\otimes} \mathcal{B}(H))P \cong P((M^{op})' \overline{\otimes} \mathcal{B}(H))P$ via a $*$ -isomorphism ϕ_1 . We have that $\phi_1 \circ \varpi = \pi_N$ gives a nor-

mal representation of N on $P(L^2(M^{op}, \rho) \otimes H)$, and $\pi_{M^{op}} = \phi_2$ sending $T \in M^{op}$ to $(T \otimes I)P \in (M^{op} \otimes CI)P$ gives the normal representation of M^{op} on $P(L^2(M^{op}, \rho) \otimes H)$. These two representations commute by construction, since $P((M^{op})' \overline{\otimes} \mathcal{B}(H))P$ is the commutant of $(M^{op} \otimes CI)P$.

4.0.7 Fifth definition of correspondence

We now set out to show that if N is a W^* -algebra and M is a countably decomposable finite factor then one can, via Stinespring dilation, define a correspondence between N and M as a unital, normal, completely positive map between N and M . For this we include a short discussion of completely positive maps which can be found, for the most part, in [8].

Completely positive maps

Let N, M be C^* -algebras. A linear map

$$T : N \rightarrow M$$

naturally induces a linear map $T^{(n)}$ ($= T \otimes id_n$) from $N \otimes M_n(\mathbb{C})$ into $M \otimes M_n(\mathbb{C})$ defined by $T^{(n)}(a \otimes A) = Ta \otimes A$ for $a \in N$ and $A \in M_n(\mathbb{C})$. Alternatively, viewing $N \otimes M_n(\mathbb{C})$ and $M \otimes M_n(\mathbb{C})$ as $M_n(N)$ and $M_n(M)$ respectively, we see that

$$T^{(n)}(a_{ij})_{i,j=1}^n = (Ta_{ij})_{i,j=1}^n$$

for $(a_{ij})_{i,j=1}^n \in M_n(N)$.

A linear map $T : N \rightarrow M$ is said to be completely positive if $T^{(n)}$ is positive for all $n \in \mathbb{N}$.

The transpose map on $M_2(\mathbb{C})$ is the standard example of a positive linear mapping that is not completely positive, since if $A \in M_2(\mathbb{C})$ has the property that $\langle Ax, x \rangle \geq 0$

for all $x \in \mathbb{C}^2$, then $\langle A^T x, x \rangle \geq 0$ for all $x \in \mathbb{C}^2$, but

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has non-negative eigenvalues 0, 0, 0, 2, but the map $(\cdot)^T$ applied to B yields the matrix

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}^T & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

that has eigenvalues -1, 1, 1, 1, one of which is negative.

Since the positive elements in $N \otimes M_n(\mathbb{C})$ (and $M \otimes M_n(\mathbb{C})$) are those that can be written as A^*A , and since $*$ -homomorphisms are multiplicative and $*$ -preserving, it is routine to verify that any $*$ -homomorphism from N into M is a completely positive map.

Another example of a completely positive map, this time from a C^* -algebra N to itself, is given by $*$ -conjugation $Ad(T)(\cdot) = T^* \cdot T$ by any element T in N . The complete positivity of this map can be seen as follows: regard N as faithfully represented on a Hilbert space H , and regard $N \otimes M_n(\mathbb{C})$ as acting on the Hilbert space $H \otimes \mathbb{C}^n$. If $\sum(A_i \otimes B_i) \in N \otimes M_n(\mathbb{C})$ is such that

$$\left\langle \sum(A_i \otimes B_i)(\xi \otimes \eta), (\xi \otimes \eta) \right\rangle = \sum \langle A_i \xi, \xi \rangle \langle B_i \eta, \eta \rangle \geq 0$$

for all $\xi \otimes \eta$ in $H \otimes \mathbb{C}^n$, then

$$\begin{aligned}
& \langle (Ad(T) \otimes id_n) \left(\sum (A_i \otimes B_i) \right) (\xi \otimes \eta), (\xi \otimes \eta) \rangle \\
&= \langle \sum (T^* A_i T \otimes B_i) (\xi \otimes \eta), (\xi \otimes \eta) \rangle \\
&= \sum \langle A_i T \xi, T \xi \rangle \langle B_i \eta, \eta \rangle \geq 0.
\end{aligned}$$

It follows that $Ad(T)$ is completely positive. Note that if N is a C^* -algebra with unit I_N , that $Ad(T)(I_N) = T^*T$ does not necessarily equal I_N .

Given a representation π of a C^* -algebra on a Hilbert space H , and P a projection in $\mathcal{B}(H)$, the mapping $\pi_P = Ad(p) \circ \pi|_{PH}$ is called the compression of the representation π to the closed subspace PH . Since the map $T \mapsto T|_{PH}$, by an argument similar to the one above, is completely positive, the map π_P is the composition of three completely positive maps, hence is completely positive.

The compression of a representation is not generally a representation. In fact, any state ρ on a unital C^* -algebra N may be regarded as the compression of the universal *GNS* representation obtained from ρ to the one-dimensional subspace spanned by the vector v for which $\rho(A) = \langle \pi(A)v, v \rangle$ for all $A \in N$. The following computation shows why. Suppose that $w \in span\{v\}$, and that P is the projection onto $span\{v\}$. We have that

$$\begin{aligned}
P\pi(A)Pw &= \langle w, v \rangle P\pi(A)v \\
&= \langle \pi(A)v, v \rangle \langle w, v \rangle v \\
&= \langle \pi(A)v, v \rangle Pw \\
&= \rho(A)Pw \\
&= \rho(A)w.
\end{aligned}$$

It follows from the Stinespring theorem (a special case of which we prove in the next section) that every unital completely positive map from a unital C^* -algebra N

into some $\mathcal{B}(H)$ must be the compression of some representation of π on a Hilbert space containing H as a subspace.

Correspondences and completely positive maps

In this section we show how to associate a correspondence between a W^* -algebra N and a countably decomposable finite factor M with a trace τ to a normal, unital, completely positive map from N into M and how to obtain a correspondence between N and M from such a completely positive map. The construction below should work for any factor, but having a faithful trace τ instead of just a state ρ considerably shortens the construction. We shall mainly concern ourselves with the case where N and M are type II_1 factors later on however, so the cost of this assumption is not too great. First, we shall show how to make $N \otimes M$ into a correspondence between N and M using an inner product obtained from a normal, unital completely positive map $\phi : N \rightarrow M$.

Lemma 32 *Let N be a W^* -algebra and M a countably decomposable factor with faithful normal state ρ . Regard $M \subseteq \mathcal{B}(L^2(M, \rho))$ and let $\phi : N \rightarrow M$ be a unital completely positive map. The form $\langle \cdot, \cdot \rangle_\phi$ defined on simple tensors by*

$$\langle S_1 \otimes T_1, S_2 \otimes T_2 \rangle_\phi = \langle \phi(S_2^* S_1) T_1, T_2 \rangle$$

extends to a positive conjugate-bilinear form $\langle \cdot, \cdot \rangle_\phi : N \otimes M \rightarrow \mathbb{C}$ on the algebraic tensor product. Here $\langle \cdot, \cdot \rangle$ denotes the inner product on the GNS Hilbert space $L^2(M, \rho)$.

Proof. The fact that the form $\langle \cdot, \cdot \rangle_\phi$ extends to a conjugate-bilinear form on the algebraic tensor product $N \otimes M$ is straightforward. To prove that this map is positive we must show that, for any element $\sum_{i=1}^n (S_i \otimes T_i) \in N \otimes M$, that

$$\left\langle \sum_{i=1}^n (S_i \otimes T_i), \sum_{i=1}^n (S_i \otimes T_i) \right\rangle_\phi = \sum_{i,j} \langle \phi(S_j^* S_i) T_i, T_j \rangle \geq 0.$$

To do this, consider the algebraic tensor product $M \otimes M_n(\mathbb{C})$ and the algebraic tensor product $N \otimes M_n(\mathbb{C})$. The algebraic tensor products admit C^* -norms from those of N and $M_n(\mathbb{C})$, and are complete with respect to these norms, and may hence be viewed as C^* -algebras. Therefore, to show that an element A in $M \otimes M_n(\mathbb{C})$ is positive, it suffices to show that we may write $A = B^*B$ for some $B \in M \otimes M_n(\mathbb{C})$. Let $\{E_{ij}\}_{i,j=1}^n$ denote the standard system of $n \times n$ matrix units in $M_n(\mathbb{C})$. The element $\sum_{l,k=1}^n (S_l^* S_k) \otimes E_{lk} \in N \otimes M_n(\mathbb{C})$ is positive, since

$$\sum_{l,k=1}^n (S_l^* S_k) \otimes E_{lk} = \left(\sum_{l=1}^n S_l \otimes E_{1l} \right)^* \left(\sum_{k=1}^n S_k \otimes E_{1k} \right).$$

Since the map $\phi : N \rightarrow M$ is completely positive, it follows that $\phi \otimes id_n : N \otimes M_n(\mathbb{C}) \rightarrow M \otimes M_n(\mathbb{C})$ is positive, and hence the element $\sum_{l,k=1}^n \phi(S_l^* S_k) \otimes E_{lk}$ is positive in $M \otimes M_n(\mathbb{C})$. Regard $M_n(\mathbb{C})$ as acting on \mathbb{C}^n with standard orthonormal basis $\{e_1, \dots, e_n\}$. It follows that for any $\sum_{i=1}^n (T_i \otimes e_i)$ in $M \otimes \mathbb{C}^n \subseteq L^2(M, \rho) \otimes \mathbb{C}^n = H$ we have

$$\begin{aligned} \left\langle \left(\sum_{l,k=1}^n \phi(S_l^* S_k) \otimes E_{lk} \right) \left(\sum_{i=1}^n (T_i \otimes e_i) \right), \left(\sum_{i=1}^n (T_i \otimes e_i) \right) \right\rangle_H &= \sum_{i,j} \sum_{k,l} \langle \phi(S_j^* S_i) T_i, T_j \rangle \langle E_{lk} e_i, e_j \rangle_{\mathbb{C}^n} \\ &= \sum_{k,l} \langle \phi(S_l^* S_k) T_k, T_l \rangle \\ &= \left\langle \sum_{k=1}^n (S_k \otimes T_k), \sum_{l=1}^n (S_l \otimes T_l) \right\rangle_{\phi}. \end{aligned}$$

This shows that the conjugate bilinear form $\langle \cdot, \cdot \rangle_{\phi}$ is positive. ■

Assume now that ϕ is normal, and let H_{ϕ} denote the completion of the normed space $(N \otimes M)/W$, where W is the subspace $\{v | v \in N \otimes M \text{ and } \langle v, v \rangle_{\phi} = 0\}$. This is a subspace because the form $\langle \cdot, \cdot \rangle_{\phi}$ is positive, and hence admits a Cauchy-Schwartz inequality. The subspace is closed due to the continuity of the norm $\| \cdot \|_{\phi} = \langle \cdot, \cdot \rangle_{\phi}^{1/2}$.

We consider the natural action of $N \otimes M^{op}$ on the space $N \otimes M$ defined by $(S \otimes T)(\sum_{i=1}^n (S_i \otimes T_i)) = \sum_{i=1}^n (SS_i \otimes T_i T)$. Here the products SS_i are taken in N and the products $T_i T$ in M .

We now show that the subspace W of $N \otimes M$ is invariant under this action of $N \otimes M^{op}$ and that this action induces an action of $N \otimes M^{op}$ on H_ϕ . Note, if $A \in W$ and $B \in N \otimes M^{op}$, regard B as in $N \otimes M$, and then $|\langle A, B \rangle_\phi| \leq \langle A, A \rangle_\phi^{1/2} \langle B, B \rangle_\phi^{1/2} = 0$. Now for $S \otimes T \in N \otimes M^{op}$ and $v = \sum_{k=1}^n (S_k \otimes T_k) \in N \otimes M$,

$$\begin{aligned}
\langle (S \otimes T)v, (S \otimes T)v \rangle_\phi &= \left\langle \sum_k (SS_k \otimes T_k T), \sum_l (SS_l \otimes T_l T) \right\rangle_\phi \\
&= \sum_{l,k} \langle \phi(S_l^* S^* SS_k) T_k T, T_l T \rangle = \sum_{l,k} \tau(T^* T_l^* \phi(S_l^* S^* SS_k) T_k T) \\
&= \sum_{l,k} \tau(T_l^* \phi(S_l^* S^* SS_k) T_k T T^*) = \sum_{l,k} \langle \phi(S_l^* S^* SS_k) T_k T T^*, T_l \rangle \\
&= \left\langle \sum_k (S^* SS_k \otimes T_k T T^*), \sum_l (S_l \otimes T_l) \right\rangle_\phi = \langle (S^* S \otimes T T^*)v, v \rangle_\phi \\
&= \langle (S \otimes T)^*(S \otimes T)v, v \rangle_\phi.
\end{aligned}$$

(Note: This computation was shortened by the traciality assumption.)

Now we see that if $v \in W$, then

$$\begin{aligned}
0 &\leq \langle (S \otimes T)v, (S \otimes T)v \rangle_\phi = \langle (S \otimes T)^*(S \otimes T)v, v \rangle_\phi \\
&\leq \langle (S \otimes T)^*(S \otimes T)v, (S \otimes T)^*(S \otimes T)v \rangle_\phi^{1/2} \langle v, v \rangle_\phi^{1/2} = 0,
\end{aligned}$$

therefore $|\langle (S \otimes T)v, (S \otimes T)v \rangle_\phi| = 0$ and $(S \otimes T)v \in W$. We have proven that W is invariant under the action of $N \otimes M^{op}$. It follows that the action of $N \otimes M^{op}$ on $(N \otimes M)/W$ given by

$$(S \otimes T)(v + W) = (S \otimes T)v + W$$

is well-defined.

We are able to obtain more from the above computations, however. Given $v = \sum_{k=1}^n (S_k \otimes T_k)$, the positivity of the form $\langle \cdot, \cdot \rangle_\phi$ tells us that the linear functional on N defined by

$$S \otimes I_{M^{op}} \mapsto \sum_{l,k} \langle \phi(S_l^* SS_k) T_k, T_l \rangle = \langle (S \otimes I_{M^{op}})v, v \rangle_\phi$$

is positive, and hence bounded with norm $\langle v, v \rangle_\phi$. Since τ and ϕ are both normal, this linear functional is again normal. Similarly the linear functional on M^{op} given by

$$I_N \otimes T \mapsto \sum_{l,k} \langle \phi(S_l^* S_k) T_k T, T_l \rangle = \langle v(I_N \otimes T), v \rangle_\phi$$

is positive, normal and has norm $\langle v, v \rangle_\phi$.

From these, we have that for any $v = \sum_{k=1}^n (S_k \otimes T_k) \in N \otimes M$, we have that

$$\begin{aligned} \|(S \otimes I_{M^{op}})v\|_\phi^2 &= \langle (S \otimes I_{M^{op}})v, (S \otimes I_{M^{op}})v \rangle_\phi \\ &= \langle (S \otimes I_{M^{op}})^*(S \otimes I_{M^{op}})v, v \rangle_\phi = \langle (S^* S \otimes I_{M^{op}})v, v \rangle_\phi \\ &\leq \langle v, v \rangle_\phi \|S^* S\|_N, \end{aligned}$$

so that each $S \otimes I_{M^{op}}$ is a bounded operator with norm at most $\|S^* S\|_N^{1/2} = \|S\|_N$, with $\|\cdot\|_N$ the operator norm on N . Similarly, we have that each $I_N \otimes T$ is bounded with norm at most $\|T\|_{M^{op}}$. It follows that each $S \otimes T = (S \otimes I_{M^{op}})(I_N \otimes T)$ acts as a bounded operator on $N \otimes M$, and therefore all finite linear combinations do as well, so the action extends by continuity to all of H_ϕ .

By the normality of the functionals above, if $S_\lambda \otimes I_{M^{op}} \xrightarrow{\text{ultraweak}} S \otimes I_{M^{op}}$ in N , then for every vector $v \in H_\phi$, we have that

$$\langle (S_\lambda \otimes I_{M^{op}})v, v \rangle_\phi \rightarrow \langle (S \otimes I_{M^{op}})v, v \rangle_\phi$$

and hence $S_\lambda \otimes I_{M^{op}} \xrightarrow{W.O.T.} S \otimes I_{M^{op}}$ as acting on H_ϕ , so the action is normal when restricted to $N \otimes I_{M^{op}}$, and similarly normal when restricted to $I_N \otimes M^{op}$.

It follows that H_ϕ is a correspondence between N and M .

We now show how to recover ϕ from H_ϕ . Given $T \in M \subseteq L^2(M, \tau_M)$, define $\Xi : M \rightarrow H_\phi$ such that $\Xi(T) = I_N \otimes T$. By the above remarks this extends to a contractive linear operator on all of $L^2(M, \tau_M)$, which we shall also call Ξ . We have,

for each $T_1, T_2 \in M$, and $S \in N$, that

$$\begin{aligned} \langle \Xi^*(S \otimes I_{M^{op}})\Xi T_1, T_2 \rangle_{\tau_M} &= \langle (S \otimes I_{M^{op}})(I_N \otimes T_1), (I_N \otimes T_2) \rangle_{\phi} \\ &= \langle (S \otimes T_1), (I_N \otimes T_2) \rangle_{\phi} = \tau_M(T_2^* \phi(S) T_1) = \langle \phi(S) T_1, T_2 \rangle_{\tau_M}. \end{aligned}$$

Hence the ‘‘coefficients’’ of $\Xi^*(S \otimes I_{M^{op}})\Xi$ match those of $\phi(S)$, and $\phi(S) = \Xi^*(S \otimes I_{M^{op}})\Xi$.

The previous paragraph suggests how we may canonically associate a completely positive map to a given correspondence H . Now choose a bounded vector ξ in H , that is, let ξ be a unit vector with the property that there is a $c > 0$ so that for all $T \in M$, $\langle \xi T T^*, \xi \rangle \leq c \tau_M(T T^*)$. To see that such bounded vectors exist, consider first the case where $H = L^2(M, \tau_M)$, and choose $\xi \in M \subseteq H$ unitary, then appeal to the classification of right modules parallel to the earlier treated classification of left modules to obtain the general case. (Rough-hewn hint: We know that H is isomorphic to $(L^2(M) \otimes L^2(\mathbb{N}))P$, and that P cannot kill all simple tensors of the form $U \otimes S$ with U unitary in M^{op} and $S \in L^2(\mathbb{N})$. Choose such a simple tensor, and note that given $T \in M$,

$$\begin{aligned} \|(U \otimes S)P(T \otimes I)P\|_H^2 &\leq \|(U \otimes S)(T \otimes I)\|_H^2 \\ &= \|TU\|_{L^2(M)}^2 \|S\|_{L^2(\mathbb{N})}^2 = c \|T\|_{L^2(M)}^2 \end{aligned}$$

where $c = \|S\|_{L^2(\mathbb{N})}^2$.

Define $\Xi : L^2(M, \tau_M) \rightarrow H$ by extending the map on M defined by $\Xi(T) = \xi T$. Since H is a correspondence, Ξ is a bounded operator such that

$$\begin{aligned}
\langle \Xi^* \pi_N(S) \Xi (JT^* J) T_1, T_2 \rangle_{\tau_M} &= \langle \pi_N(S) (\xi T_1 T), (\xi T_2) \rangle_{\tau_M} \\
&= \langle \pi_{M^{op}}(T) \pi_N(S) (\xi T_1), (\xi T_2) \rangle_{\tau_M} \\
&= \langle \pi_N(S) (\xi T_1), \pi_{M^{op}}(T)^* (\xi T_2) \rangle_{\tau_M} \\
&= \langle \pi_N(S) (\xi T_1), \pi_{M^{op}}(T^*) (\xi T_2) \rangle_{\tau_M} \\
&= \langle \pi_N(S) (\xi T_1), (\xi T_2 T^*) \rangle_{\tau_M} = \langle \Xi^* \pi_N(S) \Xi T_1, T_2 T^* \rangle_{\tau_M} \\
&= \langle (JT^* J) \Xi^* \pi_N(S) \Xi T_1, T_2 \rangle_{\tau_M}.
\end{aligned}$$

This shows that $\Xi^* \pi_N(S) \Xi$ commutes with $JMJ = M'$, and hence $\Xi^* \pi_N(S) \Xi \in M'' = M$. It follows that the range of the map $Ad(\Xi) \circ \pi_N : N \rightarrow \mathcal{B}(L^2(M, \tau_M))$ is contained in M . Note also that the map defined by $\phi(S) = \Xi^* \pi_N(S) \Xi$ is a composition $Ad(\Xi) \circ \pi_N$ of two normal completely positive maps, $*$ -conjugation by Ξ and the normal representation π_N , hence ϕ is also completely positive and normal. We have thus obtained from the correspondence H between N and M a normal, unital, completely positive map ϕ from N into M . This motivates the following fifth interpretation of correspondence.

Definition 33 *Let N be a W^* -algebra, and M be a countably decomposable factor. A correspondence between N and M is a normal, unital completely positive map from N into M .*

4.0.8 Equivalence and containment of correspondences

In this section we define what it means for two correspondences between W^* -algebras N and M to be equivalent. Suppose that H, H' are correspondences between N and M , and that π_H and $\pi_{H'}$ are the representations of $N \otimes M^{op}$ on H and H'

respectively. Recall that π_H is unitarily equivalent to $\pi_{H'}$ if there exists a unitary linear map $U : H \rightarrow H'$ so that $\pi_H = \text{Ad}(U) \circ \pi_{H'}$.

Definition 34 *Let N, M be W^* -algebras and let H, H' be correspondences between N and M . We say that H is equivalent to H' , written $H \sim H'$, if the associated representations of $N \otimes M^{op}$ are unitarily equivalent.*

If H is a correspondence, we denote by \widehat{H} its equivalence class under \sim . We denote the set of all equivalence classes of correspondences by $\text{Corr}(N, M)$. When no confusion is likely, we will sometimes omit the hat $\widehat{}$ and use the same notation for a correspondence and its class.

We now introduce the notion of subcorrespondence as an $N \otimes M^{op}$ -invariant subspace of H , i.e. a subrepresentation of $N \otimes M^{op}$ on H .

Definition 35 *A subcorrespondence of a correspondence H is a subrepresentation of $N \otimes M^{op}$ on H . A correspondence K is said to be subequivalent to H if K is equivalent to a subcorrespondence of H .*

4.0.9 Examples of correspondences

We now give a few examples of correspondences. The first two shall play a key role in what follows. The last example demonstrates that the “category” of correspondences is, in some sense, much richer than that of left-modules. Although these examples may be considered in full generality (using a faithful normal weight in place of a faithful normal state), we shall only include what we need below.

Example 36 *(the identity correspondence, or trivial correspondence) Let M be a W^* -algebra with faithful normal state ρ , and let $L^2(M, \rho)$ denote the standard GNS Hilbert space. By the Tomita-Takesaki theory, M acts on this Hilbert space by left*

and right multiplication,

$$(T, \xi) \mapsto T\xi$$

$$(\xi, T) \mapsto JT^*J\xi = \xi T$$

The left multiplication gives a normal representation of M , and the right multiplication gives a normal representation of M^{op} . These representations commute, and hence $H_{id} = L^2(M, \rho)$ becomes a correspondence between M and itself, called the trivial or identity correspondence. By basic GNS considerations, using any other state at the outset shall yield an equivalent correspondence, hence the identity correspondence is unique up to equivalence.

It should be noted that if M is finite and countably decomposable, then a correspondence H between M and itself contains the identity correspondence H_{id} as a direct summand if and only if there exists a separating central vector for M in H , that is, a vector $\xi \in H$ such that for any $T \in M$, if $T\xi = 0$ then $T = 0$. To see this, note that if $H = H_{id} \oplus K$, then the vector $(\xi_0, 0)$ is the desired separating vector with ξ_0 the separating vector for the action of M on H_{id} . Conversely, if ξ is a separating vector for the action of M on H , then we may write $H = [M\xi] \oplus [M\xi]^\perp$, and $[M\xi]$ is isomorphic to H_{id} .

Example 37 (The coarse correspondence) Let M and N be W^* -algebras with faithful, normal states ρ_N, ρ_M respectively. The coarse correspondence between N and M is the Hilbert space $L^2(N, \rho_N) \otimes L^2(M, \rho_M)$, such that N acts via left multiplication in the first coordinate, and M acts via right multiplication in the second coordinate, that is, for $S \in N$ and $T \in M$,

$$(S \otimes T)\xi = S(JT^*J)\xi = S\xi T.$$

We may also identify the coarse correspondence with the Hilbert space of all Hilbert-Schmidt operators $L^2(L^2(N, \rho_N), L^2(M, \rho_M))$ by identifying simple tensors with "rank-

one operators". Let $\xi \in L^2(N, \rho_N)$, $S \in N$, $T \in M$, then

$$(S \otimes T)(\xi) \equiv \langle \xi, S \rangle_{\rho_N} T \in L^2(M, \rho_M).$$

The left action of N is given by $S_0(S \otimes T)(\xi) \equiv \langle \xi, S_0 S \rangle_{\rho_N} T$ and the right action of M by $(S \otimes T)T_0(\xi) \equiv \langle \xi, S \rangle_{\rho_N} T T_0$, which are the natural choices.

If M is a factor, let $\text{Aut}(M)$ denote the group of all $*$ -automorphisms of M . Two such automorphisms θ_1 and θ_2 are said to be *outer conjugate* if there exists a unitary element $W \in M$ and $\phi \in \text{Aut}(M)$ so that $\phi^{-1} \circ \theta_1 \circ \phi = \text{Ad}(W) \circ \theta_2$. We say that θ_1 and θ_2 are *trivially outer conjugate* if $\phi = \text{id}$.

Example 38 Let M be a factor of type II_1 with faithful normal trace τ and let $\theta \in \text{Aut}(M)$. Consider the Hilbert space $H_\theta = L^2(M, \tau)$. We define an M - M bimodule structure on H_θ as follows. Let $S, T \in N$ and $\xi \in H_\theta$. Define the left and right actions by

$$S \cdot \xi \cdot T = S(J\theta(T^*)J)\xi.$$

We claim that if θ_1 and θ_2 are distinct automorphisms of M , then H_{θ_1} is isomorphic to H_{θ_2} if and only if θ_1 is trivially outer conjugate to θ_2 . If $\varphi : H_{\theta_1} \rightarrow H_{\theta_2}$ is an isomorphism of correspondences, then for $T \in M$ and $\xi \in H_{\theta_1}$ we have that

$$\varphi(J\theta_1(T^*)J\xi) = J\theta_2(T^*)J\varphi(\xi)$$

and hence that $J\theta_1(T^*)J = \varphi^{-1}J\theta_2(T^*)J\varphi$. Multiplying on the left and right by J and using the facts that $J^2 = I$ and M is a self-adjoint algebra, we have that for all $T \in M$,

$$\theta_1(T) = (J\varphi J)^{-1}\theta_2(T)(J\varphi J).$$

Note that $J\varphi J \in M''$, since φ^* (as well as φ) commutes with the left action of M , and hence lies in $M' \subseteq \mathcal{B}(L^2(M, \tau))$. Conversely, suppose that there is a unitary element $U \in M$ so that $\theta_1(T) = \theta_2(U^*TU)$ for every $T \in M$. We then have that the unitary

operator $J\theta_2(U^*)J : H_{\theta_2} \rightarrow H_{\theta_1}$ is an M -module isomorphism, since $\theta_2(U)\theta_1(T) = \theta_2(TU)$

$$\begin{aligned}\xi\theta_2(T)\theta_2(U) &= \xi\theta_2(TU) \\ &= \xi\theta_2(U)\theta_1(T)\end{aligned}$$

and this action commutes with left multiplications.

From this example, we see that there are at least as many isomorphism classes of correspondences between a factor of type II_1 and itself as there are trivially outer conjugacy classes of automorphisms of that factor.

4.1 Amenability and correspondences

4.1.1 Topology on correspondences

Unless otherwise explicitly noted, from now on assume that all W^* -algebras considered have separable predual. We now define a topology on the set of equivalence classes $\text{Corr}(N, M)$ of separable correspondences between W^* -algebras N and M . This topology is defined in a way analogous to the standard topology on the space of unitary representations of a group.

Definition 39 Let $\widehat{H}_0 \in \text{Corr}(N, M)$ and $\varepsilon > 0$. Let $F \subseteq N$, $E \subseteq M$ and $X = \{\xi_1, \dots, \xi_p\} \subseteq H_0$ be finite sets. We define

$$U(\widehat{H}_0; \varepsilon, F, E, X) \subseteq \text{Corr}(N, M)$$

to be the set of classes of correspondences $\widehat{H} \in \text{Corr}(N, M)$ for which there exists $\{\eta_1, \dots, \eta_m\} \subseteq H$ such that

$$|\langle S\xi_i T, \xi_j \rangle - \langle S\eta_i T, \eta_j \rangle| < \varepsilon$$

for all $S \in F$, $T \in E$ and $1 \leq i, j \leq p$. The U -topology on $\text{Corr}(N, M)$ is the one for which the sets U form a basis of open neighborhoods.

We may define another topology on this class that turns out to be equivalent to the U -topology on $\text{Corr}(N, M)$.

Definition 40 Let $\widehat{H}_0 \in \text{Corr}(N, M)$ and $\varepsilon > 0$. Let $F \subseteq N$, $E \subseteq M$ and $X = \{\xi_1, \dots, \xi_p\} \subseteq H_0$ be finite sets. We define

$$V(\widehat{H}_0; \varepsilon, F, E, X) \subseteq \text{Corr}(N, M)$$

to be the set of classes of correspondences $\widehat{H} \in \text{Corr}(N, M)$ for which there exists a correspondence H_1 in the class \widehat{H} such that $H_1 = H_0$ as a Hilbert space and such that if $S \cdot \xi \cdot T$ denotes the bimodule structure on H_1 (with $\xi \in H_1 = H_0$) and $S\xi T$ the bimodule structure on H_0 , then

$$\|S \cdot \xi \cdot T - S\xi T\| < \varepsilon$$

for all $S \in F$, $T \in E$ and $\xi \in X$. The V -topology on $\text{Corr}(N, M)$ is the one for which the sets V form a basis of open neighborhoods.

Proposition 41 The U and V topologies on $\text{Corr}(N, M)$ are equivalent.

As a result of the above proposition, we shall not need to specify which topology on $\text{Corr}(N, M)$ we are speaking of from now on. We now introduce an important notion for our consideration of amenability in terms of correspondences.

Definition 42 Let H_0, H be equivalence classes of separable correspondences between W^* -algebras N and M . We say that H_0 is weakly contained in H if H is in the closure of H_0 , i.e. there exist bimodule structures \cdot_λ on H_0 such that in the V -topology picture

$$\|T \cdot_\lambda \xi \cdot_\lambda S - T\xi S\| \rightarrow 0.$$

We write this as $H \in H_0$.

Neighborhoods of H_{id}

In this section we describe the neighborhoods of the identity correspondence in more detail. Let M be a finite factor throughout this section. Given a positive real number ε and a finite subset E of M , we define $W(\varepsilon, E)$ to be the set of equivalence classes of correspondences \widehat{H} between M and itself such that there exists a unit vector $\xi \in H$ with the property that $\|T\xi - \xi T\| < \varepsilon$ for all $T \in E$. This is to say that $W(\varepsilon, E)$ is the set of classes of correspondences that have a vector which almost commutes with every element in the finite set E .

Proposition 43 *The sets W form a basis of neighborhoods of \widehat{H}_{id} in $\text{Corr}(M, M)$.*

Proof. Let F be a finite subset of M so that $I_M \in F$, and ξ_0 the trace vector in $L^2(M, \tau) = H_{id}$. If $\widehat{H} \in V(\widehat{H}_{id}; \frac{\varepsilon}{2}, F, F, \{\xi_0\})$, then there exists a vector $\eta \in H$ such that $\|T_1\xi_0T_2 - T_1 \cdot \eta \cdot T_2\| < \frac{\varepsilon}{2}$ for all $T_1, T_2 \in F$. Recall that $H = H_{id}$ as a Hilbert space, but with different left and right actions. We use a “ \cdot ” to denote the actions in H . Since $I_M \in F$, we have that $\|T\xi_0 - T \cdot \eta\| < \frac{\varepsilon}{2}$ and $\|\xi_0T - \eta \cdot T\| < \frac{\varepsilon}{2}$ for all $T \in F$, therefore by the fact that ξ_0 is a trace vector,

$$\begin{aligned} \|T \cdot \eta - \eta \cdot T\| &= \|T \cdot \eta - T\xi_0 + \xi_0T - \eta \cdot T\| \\ &\leq \|T \cdot \eta - T\xi_0\| + \|\xi_0T - \eta \cdot T\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

therefore $\widehat{H} \in W(\varepsilon, F)$. We have shown that $V(\widehat{H}_{id}; \frac{\varepsilon}{2}, F, F, \{\xi_0\}) \subseteq W(\varepsilon, F)$.

We shall now show that for any $\varepsilon > 0$ and finite subset F of M , there exist $\varepsilon' > 0$ and a finite subset F' of M such that $W(\varepsilon', F') \subseteq U(\widehat{H}_{id}; \varepsilon, F, F, \{\xi_0\})$.

To this end, suppose that ε and F are given. The Dixmier approximation theorem tells us that for any $\varepsilon' > 0$, there exist unitary operators U_1, \dots, U_m in M such that

$$\left\| \frac{1}{m} \sum_{i=1}^m U_i^* S_1 S_2 U_i - \tau(S_1 S_2) I_M \right\| < \varepsilon'$$

for every S_1 and S_2 in F . Define the set $F_0 = F \cup \{U_i^* S_1 S_2 : S_1, S_2 \in F\}_{i=1}^m \cup \{U_i\}_{i=1}^m$, and then $F' = F_0 \cup F_0^*$. Now suppose that $\widehat{H} \in W(\varepsilon', F')$. In this case we have that there exists a unit vector $\xi \in H$ so that $\|T \cdot \xi - \xi \cdot T\| < \varepsilon'$ for all $T \in F'$. If $T_1, T_2 \in F'$, then

(i)

$$\begin{aligned}
|\langle T_1 \cdot \xi \cdot T_2, \xi \rangle - \langle T_1 \xi_0 T_2, \xi_0 \rangle| &= |\langle T_1 \cdot \xi \cdot T_2, \xi \rangle - \tau(T_1 T_2)| \\
&= |\langle T_1 \cdot \xi \cdot T_2, \xi \rangle - \langle T_1 T_2 \cdot \xi, \xi \rangle + \langle T_1 T_2 \cdot \xi, \xi \rangle - \tau(T_1 T_2)| \\
&\leq |\langle T_1 \cdot \xi \cdot T_2, \xi \rangle - \langle T_1 \cdot (T_2 \cdot \xi), \xi \rangle| + |\langle T_1 T_2 \cdot \xi, \xi \rangle - \tau(T_1 T_2)| \\
&= |\langle T_1 \cdot (\xi \cdot T_2 - T_2 \cdot \xi), \xi \rangle| + |\langle T_1 T_2 \cdot \xi, \xi \rangle - \tau(T_1 T_2)| \\
&\leq \|T_1\| \varepsilon' + |\langle T_1 T_2 \cdot \xi, \xi \rangle - \tau(T_1 T_2)|
\end{aligned}$$

(ii)

$$\begin{aligned}
&|\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 T_1 \cdot \xi, \xi \rangle| \\
&= |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi \cdot T_1, \xi \rangle + \langle T_2 \cdot \xi \cdot T_1, \xi \rangle - \langle T_2 T_1 \cdot \xi, \xi \rangle| \\
&\leq |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi \cdot T_1, \xi \rangle| + |\langle T_2 \cdot \xi \cdot T_1, \xi \rangle - \langle T_2 \cdot (T_1 \cdot \xi), \xi \rangle| \\
&= |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi \cdot T_1, \xi \rangle| + |\langle T_2 \cdot (\xi \cdot T_1 - T_1 \cdot \xi), \xi \rangle| \\
&\leq |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi \cdot T_1, \xi \rangle| + \|T_2\| \varepsilon' \\
&= |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi, \xi \cdot T_1^* \rangle| + \|T_2\| \varepsilon' \\
&= |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi \cdot T_1^*, \xi \rangle + \langle T_2 \cdot \xi \cdot T_1^*, \xi \rangle - \langle T_2 \cdot \xi, \xi \cdot T_1^* \rangle| + \|T_2\| \varepsilon' \\
&\leq |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi \cdot T_1^*, \xi \rangle| + |\langle T_2 \cdot \xi \cdot T_1^*, \xi \rangle - \langle T_2 \cdot \xi, \xi \cdot T_1^* \rangle| + \|T_2\| \varepsilon' \\
&= |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi \cdot T_1^*, \xi \rangle| + |\langle T_2 \cdot \xi \cdot (T_1^*, \xi - \xi \cdot T_1^*) \rangle| + \|T_2\| \varepsilon' \\
&\leq |\langle T_1 T_2 \cdot \xi, \xi \rangle - \langle T_2 \cdot \xi \cdot T_1^*, \xi \rangle| + 2\|T_2\| \varepsilon' \\
&= 2\|T_2\| \varepsilon'
\end{aligned}$$

Now we let $T_1 = U_i^* S_1 S_2$ and $T_2 = U_i$ in (ii) to yield, for each $i = 1, 2, \dots, m$

$$\begin{aligned} & |\langle U_i^* S_1 S_2 U_i \cdot \xi, \xi \rangle - \langle U_i U_i^* S_1 S_2 \cdot \xi, \xi \rangle| \\ &= |\langle U_i^* S_1 S_2 U_i \cdot \xi, \xi \rangle - \langle S_1 S_2 \cdot \xi, \xi \rangle| \\ &\leq 2\varepsilon' \|U_i\| = 2\varepsilon' \leq 2\varepsilon' \max_{T \in F'} \|T\| = 2\varepsilon' c. \end{aligned}$$

Note that

$$\left| \sum_{i=1}^m \langle U_i^* S_1 S_2 U_i \cdot \xi, \xi \rangle - m \langle S_1 S_2 \cdot \xi, \xi \rangle \right| \leq \sum_{i=1}^m |\langle U_i^* S_1 S_2 U_i \cdot \xi, \xi \rangle - \langle S_1 S_2 \cdot \xi, \xi \rangle| \leq 2m\varepsilon' c$$

and therefore

$$\left| \frac{1}{m} \sum_{i=1}^m \langle U_i^* S_1 S_2 U_i \cdot \xi, \xi \rangle - \langle S_1 S_2 \cdot \xi, \xi \rangle \right| \leq 2\varepsilon' c.$$

From this follows

(iii)

$$\begin{aligned} & |\tau(S_1 S_2) - \langle S_1 S_2 \cdot \xi, \xi \rangle| \\ &\leq |\tau(S_1 S_2) \langle I_M \cdot \xi, \xi \rangle - \frac{1}{m} \sum_{i=1}^m \langle U_i^* S_1 S_2 U_i \cdot \xi, \xi \rangle| + \left| \frac{1}{m} \sum_{i=1}^m \langle U_i^* S_1 S_2 U_i \cdot \xi, \xi \rangle - \langle S_1 S_2 \cdot \xi, \xi \rangle \right| \\ &\leq \left\| \frac{1}{m} \sum_{i=1}^m U_i^* S_1 S_2 U_i - \tau(S_1 S_2) I_M \right\| + 2\varepsilon' c \\ &\leq \varepsilon' + 2\varepsilon' c \\ &\leq \varepsilon' c + 2\varepsilon' c = 3\varepsilon' c, \end{aligned}$$

where the last inequality follows because there are unitary operators in F' .

Combining (i) and (iii) with $\varepsilon' = \frac{\varepsilon}{4c}$, we obtain

$$\begin{aligned} |\langle T_1 \cdot \xi \cdot T_2, \xi \rangle - \langle T_1 \xi_0 T_2, \xi_0 \rangle| &\leq \|T\| \varepsilon' + |\langle T_1 T_2 \cdot \xi, \xi \rangle - \tau(T_1 T_2)| \\ &\leq \|T\| \varepsilon' + 3\varepsilon' c \\ &\leq 4\varepsilon' c = \varepsilon \end{aligned}$$

for every T_1 and T_2 in F' . It follows that $\widehat{H} \in U(\widehat{H}_{id}; \varepsilon, F, F, \{\xi_0\})$ and the proposition is proven. ■

Amenable groups

A locally compact group G is said to be amenable if there exists a finitely additive measure μ on the Borel subsets of G which is invariant under the left translation action of G on itself and satisfies $\mu(G) = 1$. It is a well-known fact that the amenability of a group can be characterized by its representation theory, in the sense that a group is amenable if and only if the left regular representation of the group weakly contains the trivial representation. We explain what is meant by this below. In what follows, $L^2(G)$ shall denote the Hilbert space of square-integrable functions with respect to normalized Haar measure on G . Recall that the left regular representation of G on $L^2(G)$ is defined by $hf(g) = f(h^{-1}g)$ for all $g, h \in G$ and $f \in L^2(G)$.

Definition 44 *The left regular representation of a locally compact group G on the Hilbert space $L^2(G)$ is said to weakly contain the trivial representation if for any $\varepsilon > 0$ and any compact subset $S \subseteq G$, there exists $v \in L^2(G)$ such that $\|v\| = 1$ and*

$$|\langle sv, v \rangle - 1| < \varepsilon$$

for any $s \in S$.

Theorem 45 (Hulaniski) *A locally compact group G is amenable if and only if the left regular representation of G weakly contains the trivial representation.*

Proof. (see A. Hulaniski, Means and Følner conditions on locally compact groups, *Studia Math.*, 27 (1966), 87-104.) ■

Correspondences and group representations

Let N be a von Neumann algebra. If $M = N \rtimes G$ is the von Neumann algebra crossed product of N by an action of a locally compact group G , then there is a canonical way to associate a correspondence between M and itself to a given strong-operator continuous unitary representation π of G on a Hilbert space H_π .

Let $H = H_\pi \otimes L^2(M)$. We take the actions on $L^2(M)$ from the left and right regular representations of M along with the action of G on H_π and define a bimodule action on H . The right action is given, for $T \in M$, $\xi \in L^2(M)$, and $\eta \in H_\pi$ by

$$(\eta \otimes \xi)T = \eta \otimes (\xi T).$$

The left action is given, for $S \in N$, by

$$S(\eta \otimes \xi) = \eta \otimes (S\xi)$$

and for $g \in G$ by

$$g(\eta \otimes \xi) = (\pi(g)\eta) \otimes \xi.$$

Now that we have a canonical way to assign correspondences to group representations, let us look at some examples.

Example 46 Recall that the trivial representation of G is the representation $\pi_{id} : G \rightarrow \mathbb{C}$ such that $\pi_{id}(g) = 1$ for all $g \in G$. We see that the correspondence canonically assigned to the trivial representation is the identity correspondence $\mathbb{C} \otimes L^2(M) = L^2(M) = H_{id}$.

Example 47 Now consider the left regular representation $\lambda : G \rightarrow L^2(G)$ such that $\lambda(g)f(h) = f(g^{-1}h)$ for all $g, h \in G$. In this case we get the correspondence $L^2(G) \otimes L^2(N \rtimes G)$. If $N = \mathbb{C}$ and G is a discrete group, then since the elements of G form an orthonormal basis for the dense subspace M of $L^2(M)$, we have that $L^2(G) = L^2(\mathbb{C}G) = L^2(M)$. Under the assumption $N = \mathbb{C}$ we also obtain that the left action of N on $L^2(G) \otimes L^2(N \rtimes G)$ becomes trivial. From the previous two sentences it follows that the correspondence canonically associated to the left regular representation is the coarse correspondence $L^2(M) \otimes L^2(M) = H_c$.

4.2 Correspondences and the amenability of von Neumann algebras

In this section we restrict our attention to type II_1 factors since our goal is to develop an invariant for type II_1 factors which will measure the “amenability defect” of such a factor. The examples of the previous section suggest that the natural definition for amenability for a type II_1 factor should shadow the representation-theoretic definition for groups we have given. This is to say that a von Neumann algebra M should be called amenable if the identity correspondence is weakly contained in the coarse correspondence. Although this is only one of myriad equivalent characterizations for the amenable von Neumann algebras, it seems to provide the right point of view for bootstrapping notions of amenability defect for groups to get notions in the type II_1 factor setting.

In this section we prove the equivalence of this notion of amenability with injectivity in the case of type II_1 factors. This essentially recasts the celebrated theorem of Connes in the language of correspondences.

Definition 48 *A von Neumann algebra M is said to be amenable if the identity correspondence H_{id} of M is weakly contained in the coarse correspondence H_{co} of M .*

Recall that a von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ is called injective if there is a norm-one Banach space projection (conditional expectation) of $\mathcal{B}(\mathcal{H})$ onto M . A special case of the famous theorem of Alain Connes states that a type II_1 factor is injective if and only if the factor is hyperfinite. We now give a proof, due to Sorin Popa, that a type II_1 factor is amenable if and only if it is hyperfinite. To prove the theorem, we cite part of Theorem 5.1 of [6], but do not prove it here.

Proposition 49 *Let M be a factor of type II_1 with trace τ acting standardly on \mathcal{H} ($= L^2(M, \tau)$). The following are equivalent:*

(i) M is injective.

(ii) There is a hypertrace on M , that is a state Φ on $B(\mathcal{H})$ that restricts to a normal trace-state on M .

(iii) Given $\{x_1, x_2, \dots, x_n\} \subset M$ and $\varepsilon > 0$, there exists a nonzero finite-rank projection $e \in B(\mathcal{H})$ such that $\forall j \in \{1, 2, \dots, n\}$

$$\| [x_j, e] \|_{H.S.} \leq \varepsilon \|e\|_{H.S.} \text{ and } \left| \tau(x_j) - \frac{\langle x_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}} \right| \leq \varepsilon.$$

Proposition 50 A type II_1 factor M is amenable if and only if M is injective.

Proof. Suppose that M is injective. Given any finite subset $F \subseteq M$, by the result of Connes, there exist finite-rank projections η_n on $L^2(M)$ so that

$$\| T\eta_n - \eta_n T \|_{H.S.} \rightarrow 0$$

for any $T \in F$. These finite rank projections are Hilbert-Schmidt operators on $L^2(M)$, and hence may be regarded as elements of the coarse correspondence H_c . As a result, we see that H_{id} is weakly contained in H_c .

Conversely, suppose that M is amenable. Let Λ denote the directed set of finite subsets of M . Using the hypothesis, find a net $\{\eta_i\}_{i \in \Lambda} \subseteq H_{co} = L^2(M) \otimes L^2(M)$ of unit vectors so that $\| T\eta_i - \eta_i T \| \rightarrow 0$ for all $T \in M$. Let $\Phi(T) = \lim_{\Lambda} \langle T\eta_i, \eta_i \rangle$ for $T \in \mathcal{B}(L^2(M))$ denote a Banach limit over Λ . It follows that Φ is a state on $\mathcal{B}(L^2(M))$, and that given any unitary element $U \in M$, we have that

$$\begin{aligned} \Phi(UTU^*) &= \lim_{\Lambda} \langle UTU^* \eta_i, \eta_i \rangle \\ &= \lim_{\Lambda} \langle TU^* \eta_i, U^* \eta_i \rangle = \lim_{\Lambda} \langle T\eta_i U^*, \eta_i U^* \rangle \\ &= \lim_{\Lambda} \langle T\eta_i, \eta_i \rangle = \Phi(T). \end{aligned}$$

Hence Φ is a hypertrace on $M \subseteq \mathcal{B}(L^2(M))$, and therefore M is injective. ■

Chapter 5

Følner Invariants

In the previous chapter we saw how the condition in theorem 5.1 of [6] could be used to show that injectivity of a type II_1 factor is equivalent to the amenability of that factor. In this chapter we will more closely examine this condition and then use it to define a nontrivial invariant for type II_1 factors. The main task after having defined this number is to show that it is computable and can distinguish two non-hyperfinite factors from one another. The latter task will provide us with a lot of work in the future.

In [9], Følner used combinatorial methods to obtain a condition on a discrete group that is equivalent to amenability of the group. Later I. Namioka was able to obtain Følner's condition using functional analytic methods due to Day [18]. In his classification of injective factors, Alain Connes exploits an analogy between an invariant mean on a group and a hypertrace on a type II_1 factor M acting standardly on a Hilbert space \mathcal{H} . More specifically, Connes follows Namioka and applies Day's idea to the hypertrace of the II_1 factor to get a Følner-type condition for type II_1 factors. This condition is satisfied for the factor M if and only if M is injective, that is, there is a norm-one Banach space projection from $B(\mathcal{H})$ onto M , vindicating the analogy between injectivity of factors of type II_1 and amenability of discrete groups. Recently, group theorists have devised ways to measure the degree of non-amenability of a group. In particular Arzhantseva, Burillo, Lustig, Reeves, Short and Ventura [4] have discovered the notion of universal Følner invariant for a finitely generated group

G , denoted by $F\phi l(G)$, which for an n -generated group satisfies $0 \leq F\phi l(G) \leq \frac{2n-2}{2n-1}$. In particular, $F\phi l(G) = 0$ whenever G is amenable and $F\phi l(G) = \frac{2n-2}{2n-1}$ if and only if $G = \mathbb{F}_n$. This group invariant is based on the Følner condition, so one hopes that via the connection with functional analysis the universal Følner invariant has an analogue for type II_1 factors. In this chapter, we develop such an analogue.

Heuristically, a space satisfies a Følner-type condition if it can be exhausted by a family $\{A_n\}$ of sets of finite volume, with boundaries $\{\partial A_n\}$ of finite volume, such that

$$\lim_{n \rightarrow \infty} \frac{\text{vol}(\partial A_n)}{\text{vol}(A_n)} = 0. \quad (5.1)$$

In the case of discrete groups, the volume of a subset is just its cardinality. Various notions of boundary have been considered for subsets of finitely generated discrete groups, among which are the Cheeger boundary, the interior boundary and the exterior boundary. Remarkably, the usual notion of amenability of a finitely generated discrete group G , i.e. the existence of an invariant mean on G , is equivalent to G satisfying condition 5.1 for any of the competing definitions of boundary.

The notion of boundary of a subset of a finitely generated group G generally depends on a given finite generating subset X . Arzhantseva, Burillo, Lustig, Reeves, Short and Ventura define

$$F\phi l(G, X) = \inf_{\substack{A \subseteq G \\ \text{finite}}} \frac{\#\partial_X A}{\#A} \quad (5.2)$$

where $\partial_X A = \{a \in A \mid ax \notin A \text{ for some } x \in X^{\pm 1}\}$ is the interior boundary of A with respect to X in G . They go on to define the universal Følner invariant

$$F\phi l(G) = \inf_X F\phi l(G, X) \quad (5.3)$$

where the infimum is taken over all finite generating subsets X of G . If $F\phi l(G) = 0$, the group G is said to be weakly-amenable and if not, G is said to be uniformly non-amenable.

Our motivation for considering these numbers for type II_1 factors comes from the fact that in the above group theory paper, the authors are able to prove that for $n \geq 665$ odd and $m > 1$, $B(m, n)$ is uniformly non-amenable. Our hope is to use the corresponding invariant for type II_1 factors to glean some information about Burnside factors.

5.1 Connes' Følner-type condition

From Theorem 2.5 in [6], the first condition in the following proposition is equivalent, without the stipulation that M is finitely generated, to injectivity of a II_1 factor M with trace τ acting standardly on \mathcal{H} ($= L^2(M, \tau)$). Let $\mathcal{U}(M)$ denote the group of unitary elements in M . Throughout this paper we only consider von Neumann algebras that can be generated by finitely many elements; we do this in order to consider the lower Følner number.

Proposition 51 *Let M be a factor of type II_1 with trace τ acting standardly on \mathcal{H} ($= L^2(M, \tau)$). The following are equivalent:*

- i** *Given $\{x_1, x_2, \dots, x_n\} \subset M$ and $\varepsilon > 0$, there exists a nonzero finite-rank projection $e \in B(\mathcal{H})$ such that $\forall j \in \{1, 2, \dots, n\}$*

$$\|[x_j, e]\|_{H.S.} \leq \varepsilon \|e\|_{H.S.} \text{ and } \left| \tau(x_j) - \frac{\langle x_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}} \right| \leq \varepsilon.$$

- ii** *Given $\{U_1, U_2, \dots, U_n\} \subset \mathcal{U}(M)$ and $\varepsilon > 0$, there exists a nonzero finite-rank projection $e \in B(\mathcal{H})$ such that $\forall j \in \{1, 2, \dots, n\}$*

$$\|[U_j, e]\|_{H.S.} \leq \varepsilon \|e\|_{H.S.} \text{ and } \left| \tau(U_j) - \frac{\langle U_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}} \right| \leq \varepsilon.$$

Proof. ($i \Rightarrow ii$) Suppose i holds and ii follows directly.

(ii \Rightarrow i) Suppose ii holds. Let $\{x_1, x_2, \dots, x_n\} \subset M$ and $\varepsilon > 0$. Each x_i can be written as a linear combination of at most four unitary elements, as $x_i = \sum_{j=1}^4 \lambda_j^{(i)} U_j^{(i)}$, with $\lambda_j^{(i)} \in \mathbb{C}$. Let $S = \bigcup_{i,j} \{U_j^{(i)}\} \subset \mathcal{U}(M)$. Let $C = \max\{|\lambda_j^{(i)}| \mid i \in \{1, 2, \dots, n\}; j \in \{1, 2, 3, 4\}\}$. By ii, given $\varepsilon' = \frac{\varepsilon}{4C}$ we can find a finite rank projection e such that $\forall U \in S$, $\| [U, e] \|_{H.S.} \leq \varepsilon' \|e\|_{H.S.}$ and $|\tau(U) - \frac{\langle Ue, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}}| \leq \varepsilon'$. Note that if we apply the triangle inequality

$$\begin{aligned} \| [x_i, e] \|_{H.S.} &= \| (\sum_{j=1}^4 \lambda_j^{(i)} U_j^{(i)})e - e(\sum_{j=1}^4 \lambda_j^{(i)} U_j^{(i)}) \|_{H.S.} \\ &= \| \sum_{j=1}^4 \lambda_j^{(i)} (U_j^{(i)}e - eU_j^{(i)}) \|_{H.S.} \leq 4C \| [U_j^{(i)}, e] \|_{H.S.} \leq 4C\varepsilon' \|e\|_{H.S.} \\ &= \varepsilon \|e\|_{H.S.} \end{aligned}$$

and

$$\begin{aligned} |\tau(x_j) - \frac{\langle x_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}}| &= |\tau(\sum_{j=1}^4 \lambda_j^{(i)} U_j^{(i)}) - \frac{\langle \sum_{j=1}^4 \lambda_j^{(i)} U_j^{(i)} e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}}| \\ &\leq 4C |\tau(U_j^{(i)}) - \frac{\langle U_j^{(i)} e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}}| \leq 4C\varepsilon' \\ &= \varepsilon. \end{aligned}$$

therefore i holds. ■

Remark 52 For the remainder of this paper, we shall refer to property (ii) in the above theorem as the *Connes-Følner condition*.

Definition 53 Let M be a von Neumann algebra, and $X \subset \mathcal{U}(M)$ be a finite subset of M . We define the property $Q(X, \varepsilon)$ to be “there exists a nonzero finite-rank projection $e \in B(\mathcal{H})$ such that $\forall j \in \{1, 2, \dots, n\}$, $\| [U_j, e] \|_{H.S.} \leq \varepsilon \|e\|_{H.S.}$ and $|\tau(U_j) - \frac{\langle U_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}}| \leq \varepsilon$.”

Definition 54 Let M be a von Neumann algebra, and $X \subset \mathcal{U}(M)$ be a finite subset of M . Define

$$F\phi(M, X) = \inf\{\varepsilon > 0 : Q(X, \varepsilon)\}.$$

Remark 55 Note that for the element $I \in M$, we have that $[I, e] = 0$ and $\tau(I) - \frac{\langle Ie, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}} = 0$, so we may disregard the element I in the sets X for which we consider $Q(X, \varepsilon)$. More specifically, $Q(X, \varepsilon)$ holds if and only if $Q(X \setminus \{I\}, \varepsilon)$ holds.

Definition 56 Let M be a von Neumann algebra. We define the universal Følner invariant $F\phi(M) = \sup_X F\phi(M, X)$, where the supremum is taken over all finite sets $X \subset \mathcal{U}(M)$.

Proposition 57 M is injective if and only if $F\phi(M) = 0$.

Proof. By Theorem 2.5 of [6] and Proposition 51, $F\phi(M, X) = 0$ for all finite sets X of unitary elements of M if and only if M is injective. ■

We now prove a monotonicity result.

Proposition 58 Let M be a von Neumann algebra. If X_1 and X_2 are finite subsets of $\mathcal{U}(M)$ that generate M , and $X_1 \subseteq X_2$, then $F\phi(M, X_1) \leq F\phi(M, X_2)$.

Proof. We have that for any $\varepsilon > 0$ that $Q(X_2, \varepsilon) \Rightarrow Q(X_1, \varepsilon)$, hence $\inf\{\varepsilon > 0 : Q(X_1, \varepsilon)\} \leq \inf\{\varepsilon > 0 : Q(X_2, \varepsilon)\}$. ■

Remark 59 Note that if the type II_1 factor M is not hyperfinite, then there is a finite subset X for which $\{\varepsilon > 0 : \neg Q(X, \varepsilon)\}$ is not empty, and hence

$$\inf\{\varepsilon > 0 : Q(X, \varepsilon)\} = \sup\{\varepsilon > 0 : \neg Q(X, \varepsilon)\}$$

under these circumstances.

Suppose $L^2(M)$ is separable. For a positive operator $T \in B(L^2(M))$, let $Tr(T) = \sum_{i=1}^{\infty} \langle Te_i, e_i \rangle$, where $\{e_i\}_{i=1}^{\infty}$ is any orthonormal basis for $L^2(M)$. The Hilbert-Schmidt norm of an operator $T \in B(L^2(M))$ is given by $\|T\|_{H.S.} = Tr(T^*T)^{1/2}$. We say that $T \in B(L^2(M))$ is in the Hilbert-Schmidt class when $\|T\|_{H.S.} < \infty$. The class of all such operators in $B(L^2(M))$ may be regarded as a Hilbert space when equipped with the inner product $\langle A, B \rangle = Tr(B^*A)$.

Proposition 60 *Let X be a finite subset of $U(M)$ and e a rank $l (< \infty)$ projection in $B(L^2(M))$. For all $U \in X$,*

$$\frac{\|[U, e]\|_{H.S.}}{\|e\|_{H.S.}} = \sqrt{2} \sqrt{1 - \|eUe\|_{\tau_l}^2}.$$

where $\|\cdot\|_{\tau_l}$ is the Hilbert-Schmidt norm on $M_l(\mathbb{C})$ with respect to the normalized trace $\tau_l \equiv \frac{1}{l}Tr_l$ on $M_l(\mathbb{C})$.

Proof. First note that $\|e\|_{H.S.}^2 = \langle e, e \rangle_{H.S.} = (Tr(e^*e)^{1/2})^2 = Tr(e) = l$. The following computation proves the proposition:

$$\begin{aligned} \|[U, e]\|_{H.S.} &= \|Ue - eU\|_{H.S.} \\ &= (Tr((Ue - eU)^*(Ue - eU)))^{1/2} \\ &= (Tr((eU^* - U^*e)(Ue - eU)))^{1/2} \\ &= (Tr(e) - Tr(U^*eUe) - Tr(eU^*eU) + Tr(U^*eU))^{1/2} \\ &= \sqrt{2}(Tr(e) - Tr(eU^*eUe))^{1/2} \\ &= \sqrt{2} \sqrt{l - \|eUe\|_{H.S.}^2} = (\sqrt{2} \sqrt{1 - \|eUe\|_{\tau_l}^2}) \|e\|_{H.S.}, \end{aligned}$$

where in the fourth equality we have used the fact that Tr is a trace (the reason this is justified may be found in Kadison and Ringrose vol. II). ■

Proposition 61 *For any type II_1 factor M , $F\phi(M) \leq 2$.*

Proof. First suppose that X is a finite set of unitary elements in M , such that $\varepsilon > 2$ and $\neg Q_0(X, \varepsilon)$ holds. In particular, $\varepsilon > \sqrt{2}$ under these circumstances. If $\sqrt{2}\sqrt{1 - \|eUe\|_{\tau}^2} > \varepsilon$ then $\|eUe\|_{\tau}^2 < 0$, which cannot happen. It therefore must be that for every finite-rank projection e in $B(L^2(M))$, there exists $U \in X$ such that

$$|\tau(U) - \tau(eUe)| > \varepsilon.$$

However, using the triangle and Cauchy-Schwartz inequalities,

$$2 < \varepsilon < |\tau(U) - \tau(eUe)| \leq |\tau(U)| + |\tau(eUe)| \leq 2,$$

a contradiction. ■

5.2 Group von Neumann algebras

Let $\{g_i\}_{i=1}^{\infty} = G$ be a countable *I.C.C.* discrete group, and $1 \in G$ be its identity element. In this section we shall explore $F\bar{\phi}l(\mathcal{L}_G, X)$ for sets X of unitary generators of the factor von Neumann algebra \mathcal{L}_G associated to G .

Given $g \in G$, let $L_g \in B(L^2(G))$ be the operator defined by $L_g h = gh$ for $h \in G$. Assume $X = \{U_1, U_2, \dots, U_k\}$ and that $U_j = \sum_{t \in G} \mu_t^{(j)} L_t$ for each $j \in \{1, 2, \dots, k\}$, where the sum is a strong-operator limit of finitely indexed sub-sums. Let e be a finite-rank projection in $B(L^2(G))$, and suppose that $\{\xi_1, \dots, \xi_l\}$ is an orthonormal basis for the closure of the range of e in $L^2(G)$. Define the operator $\xi_i \otimes \xi_j \in B(L^2(G))$ so that for $v \in L^2(G)$, $(\xi_i \otimes \xi_j)v = \langle v, \xi_j \rangle \xi_i$. It is straightforward to check that $e = \sum_{i=1}^l \xi_i \otimes \xi_i$. Furthermore, suppose that $\xi_i = \sum_{g \in G} \lambda_g^{(i)} g$, so that $\sum_{g \in G} \overline{\lambda_g^{(j)}} \lambda_g^{(i)} = \delta_{ij}$ for $i, j \in \{1, 2, \dots, l\}$.

Lemma 62 For all $j \in \{1, 2, \dots, k\}$,

$$\|eU_j e\|_{H.S.}^2 = \|[\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l\|_{Tr_l}^2,$$

where $\|\cdot\|_{\tau}$ is the Hilbert-Schmidt norm on $M_l(\mathbb{C})$ with respect to the (non-normalized) trace Tr_l on $M_l(\mathbb{C})$.

Proof. To begin with,

$$\begin{aligned} \text{Tr}(U_j^* e U_j e) &= \sum_{i=1}^{\infty} \langle U_j^* e U_j e g_i, g_i \rangle = \sum_{i=1}^{\infty} \langle U_j e g_i, e U_j g_i \rangle \\ &= \sum_{i=1}^{\infty} \sum_{t \in G} \sum_{s \in G} \mu_t^{(j)} \overline{\mu_s^{(j)}} \langle L_t e g_i, e s g_i \rangle. \end{aligned}$$

However for any $h \in G$, we have

$$\begin{aligned} e h &= \sum_{j=1}^l (\xi_j \otimes \xi_j)(h) = \sum_{j=1}^l \langle h, \xi_j \rangle \xi_j = \sum_{j=1}^l \langle h, \sum_{g \in G} \lambda_g^{(j)} g \rangle \xi_j \\ &= \sum_{j=1}^l \sum_{g \in G} \overline{\lambda_g^{(j)}} \langle h, g \rangle \xi_j = \sum_{j=1}^l \overline{\lambda_h^{(j)}} \xi_j. \end{aligned}$$

Hence $e g_i = \sum_{p=1}^l \overline{\lambda_{g_i}^{(p)}} \xi_p$ and $e s g_i = \sum_{q=1}^l \overline{\lambda_{s g_i}^{(q)}} \xi_q$. It follows that

$$\sum_{i=1}^{\infty} \langle U_j e g_i, e U_j g_i \rangle = \sum_{i=1}^{\infty} \sum_{t \in G} \sum_{s \in G} \sum_{p=1}^l \sum_{q=1}^l \mu_t^{(j)} \overline{\mu_s^{(j)}} \overline{\lambda_{g_i}^{(p)}} \lambda_{s g_i}^{(q)} \langle L_t \xi_p, \xi_q \rangle,$$

But

$$\langle L_t \xi_p, \xi_q \rangle = \langle L_t \sum_{h \in G} \lambda_h^{(p)} h, \sum_{v \in G} \lambda_v^{(q)} v \rangle = \sum_{h \in G} \sum_{v \in G} \lambda_h^{(p)} \overline{\lambda_v^{(q)}} \langle t h, v \rangle = \sum_{h \in G} \lambda_h^{(p)} \overline{\lambda_{t h}^{(q)}},$$

and therefore

$$\begin{aligned} &\sum_{i=1}^{\infty} \sum_{t \in G} \sum_{s \in G} \sum_{p=1}^l \sum_{q=1}^l \mu_t^{(j)} \overline{\mu_s^{(j)}} \overline{\lambda_{g_i}^{(p)}} \lambda_{s g_i}^{(q)} \langle L_t \xi_p, \xi_q \rangle \\ &= \sum_{i=1}^{\infty} \sum_{t \in G} \sum_{s \in G} \sum_{p=1}^l \sum_{q=1}^l \sum_{h \in G} \mu_t^{(j)} \overline{\mu_s^{(j)}} \overline{\lambda_{g_i}^{(p)}} \lambda_{s g_i}^{(q)} \lambda_h^{(p)} \overline{\lambda_{t h}^{(q)}} \\ &= \sum_{g \in G} \sum_{t \in G} \sum_{s \in G} \sum_{p=1}^l \sum_{q=1}^l \sum_{h \in G} \mu_t^{(j)} \overline{\mu_s^{(j)}} \overline{\lambda_g^{(p)}} \lambda_{s g}^{(q)} \lambda_h^{(p)} \overline{\lambda_{t h}^{(q)}} \end{aligned}$$

Changing the order of summation,

$$\begin{aligned}
& \sum_{g \in G} \sum_{t \in G} \sum_{s \in G} \sum_{p=1}^l \sum_{q=1}^l \sum_{h \in G} \overline{\mu_t^{(j)} \mu_s^{(j)} \lambda_g^{(p)} \lambda_{sg}^{(q)} \lambda_h^{(p)} \lambda_{th}^{(q)}} \\
&= \sum_{p=1}^l \sum_{q=1}^l \left(\sum_{h \in G} \sum_{t \in G} \overline{\mu_t^{(j)} \lambda_h^{(p)} \lambda_{th}^{(q)}} \right) \left(\sum_{g \in G} \sum_{s \in G} \overline{\mu_s^{(j)} \lambda_g^{(p)} \lambda_{sg}^{(q)}} \right) \\
&= \sum_{p=1}^l \sum_{q=1}^l \left| \sum_{h \in G} \sum_{t \in G} \overline{\mu_t^{(j)} \lambda_h^{(p)} \lambda_{th}^{(q)}} \right|^2 \\
&= \sum_{p=1}^l \sum_{q=1}^l |\langle \xi_q, U_j \xi_p \rangle|^2
\end{aligned}$$

since

$$\begin{aligned}
\sum_{h \in G} \sum_{t \in G} \overline{\mu_t^{(j)} \lambda_h^{(p)} \lambda_{th}^{(q)}} &= \left\langle \left(\sum_{t \in G} \overline{\mu_t^{(j)}} L_{t-1} \right) \sum_{h \in G} \lambda_h^{(q)} h, \sum_{g \in G} \lambda_g^{(p)} g \right\rangle \\
&= \langle U_j^* \xi_q, \xi_p \rangle = \langle \xi_q, U_j \xi_p \rangle.
\end{aligned}$$

■

Proposition 63 For all $j \in \{1, 2, \dots, k\}$,

$$\| [U_j, e] \|_{H.S.} = \sqrt{2} \sqrt{l - \| [\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l \|_{Tr_l}^2}.$$

Proof. We combine the previous lemma with an earlier result, and this is immediate.

■

Remark 64 Suppose for a moment that G is generated by $S = \{g_1, g_2, \dots, g_m\}$ In the above proof, if we consider the finite rank projection $e = g_1 \otimes g_1 + \dots + g_k \otimes g_k$ and the unitary element L_{g_j} , we compute $Tr(L_{g_j^{-1}} e L_{g_j} e) = \#S_1^{(j)}$, where

$$S_1^{(j)} = \{g_i \in \{g_1, g_2, \dots, g_m\} : g_j g_i \in \{g_1, g_2, \dots, g_m\}\}.$$

We obtain that $\sqrt{2}(Tr(e) - Tr(U_j^* e U_j e))^{1/2} = \sqrt{2}(m - \#S_1^{(j)})^{1/2}$ in this case. Defining $S_2^{(j)} \subseteq \{g_1, \dots, g_m\}$ as

$$S_2^{(j)} = \{g_i \in \{g_1, g_2, \dots, g_m\} : g_j g_i \in g_j S \Delta S\},$$

since $S = S_1^{(j)} \dot{\cup} S_2^{(j)}$ we get that $\sqrt{2}(m - \#S_1^{(j)})^{1/2} = \sqrt{2}\sqrt{\#S_2^{(j)}} = \sqrt{2}\sqrt{\frac{\#S_2^{(j)}}{\#S}}\|e\|_{H.S.}$.

Note that

$$\bigcup_{j=1}^k S_2^{(j)} = \{g_i \in \{g_1, g_2, \dots, g_m\} : \exists g_j \in \{g_1, g_2, \dots, g_m\} \text{ s.t. } g_j \in g_j S \Delta S\} = \partial_{\{g_1, \dots, g_k\}} S,$$

the (left) interior boundary of S . This is a relation of our concept with the classical Følner sets.

Proposition 65 For all $j \in \{1, 2, \dots, k\}$, $|\tau(U_j) - \frac{\langle U_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}}| = |\tau(U_j) - \tau_l([\langle \xi_q, U_j^* \xi_p \rangle]_{q,p=1}^l)|$, where $\tau_l = \frac{1}{l} \text{Tr}_l$, the normalized trace on $M_l(\mathbb{C})$.

Proof. We show that $\frac{\langle U_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}} = \tau_l([\langle \xi_q, U_j^* \xi_p \rangle]_{q,p=1}^l)$. Note that

$$\langle e, e \rangle_{H.S.} = (\text{Tr}(e^* e)^{1/2})^2 = \text{Tr}(e) = l$$

and that

$$\begin{aligned} \langle U_j e, e \rangle_{H.S.} &= \text{Tr}(e^* U_j e) \\ &= \text{Tr}(U_j e) = \sum_{i=1}^{\infty} \langle U_j e g_i, g_i \rangle \\ &= \sum_{i=1}^{\infty} \langle U_j \sum_{k=1}^l \overline{\lambda_{g_i}^{(k)}} \xi_k, g_i \rangle \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^l \overline{\lambda_{g_i}^{(k)}} \langle U_j \xi_k, g_i \rangle \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^l \overline{\lambda_{g_i}^{(k)}} \langle U_j \xi_k, g_i \rangle \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^l \overline{\lambda_{g_i}^{(k)}} \langle U_j \sum_{g \in G} \lambda_g^{(k)} g, g_i \rangle \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^l \sum_{g \in G} \lambda_g^{(k)} \overline{\lambda_{g_i}^{(k)}} \langle U_j g, g_i \rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} \sum_{k=1}^l \sum_{g \in G} \sum_{t \in G} \mu_t^{(j)} \lambda_g^{(k)} \overline{\lambda_{g_i}^{(k)}} \langle t g, g_i \rangle \\
&= \sum_{g \in G} \sum_{k=1}^l \sum_{t \in G} \mu_t^{(j)} \lambda_{t^{-1}g}^{(k)} \overline{\lambda_g^{(k)}} \\
&= \sum_{k=1}^l \sum_{g \in G} \sum_{t \in G} \mu_t^{(j)} \lambda_{t^{-1}g}^{(k)} \overline{\lambda_g^{(k)}} = \sum_{k=1}^l \langle \xi_k, U_j^* \xi_k \rangle,
\end{aligned}$$

since

$$\begin{aligned}
\sum_{g \in G} \sum_{t \in G} \mu_t^{(j)} \lambda_{t^{-1}g}^{(k)} \overline{\lambda_g^{(k)}} &= \langle (\sum_{t \in G} \mu_t^{(j)} L_t) \sum_{h \in G} \lambda_h^{(k)} h, \sum_{t \in G} \lambda_t^{(k)} t \rangle \\
&= \langle U_j \xi_k, \xi_k \rangle = \langle \xi_k, U_j^* \xi_k \rangle.
\end{aligned}$$

■

Corollary 66 For all $j \in \{1, 2, \dots, k\}$,

$$|\tau(U_j) - \frac{\langle U_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}}| = |\tau(U_j) - \overline{\tau_l([\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l)}|,$$

where $\tau_l = \frac{1}{l} Tr_l$, the normalized trace on $M_l(\mathbb{C})$.

Proof. Since $a_{ij} = \langle \xi_j, U^* \xi_i \rangle = \overline{\langle \xi_i, U \xi_j \rangle} = \overline{b_{ji}}$, we have that $Tr_l([a_{ij}]_{i,j}) = \sum_{i=1}^l a_{ii} =$

$$\sum_{i=1}^l \overline{b_{ii}} = \overline{\sum_{i=1}^l b_{ii}} = \overline{Tr_l([b_{ij}]_{i,j})}. \quad \blacksquare$$

Remark 67 Let $\varepsilon > 0$. Since $\|e\|_{H.S.} = \sqrt{l} \neq 0$, we have that

$$\begin{aligned}
&\frac{\sqrt{2} \sqrt{l - \|[\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l\|_{Tr_l}^2}}{\sqrt{l}} \\
&= \sqrt{2} \sqrt{1 - \frac{1}{l} \|[\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l\|_{Tr_l}^2} \\
&= \sqrt{2} \sqrt{1 - \frac{1}{l} Tr([\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l * ([\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l))} \\
&= \sqrt{2} \sqrt{1 - \frac{1}{l} \|[\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l\|_{\tau_l}^2},
\end{aligned}$$

therefore $\| [U_j, e] \|_{H.S.} \leq \varepsilon \| e \|_{H.S.}$ if and only if

$$\sqrt{2} \sqrt{1 - \| [\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l \|_{\tau_l}^2} \leq \varepsilon.$$

We see also that $|\tau(U_j) - \frac{\langle U_j e, e \rangle_{H.S.}}{\langle e, e \rangle_{H.S.}}| \leq \varepsilon$ if and only if

$$|\tau(U_j) - \overline{\tau_l([\langle \xi_q, U_j \xi_p \rangle]_{q,p=1}^l)}|.$$

This gives us the following definitions and proposition.

Definition 68 Given a finite set X of unitary elements in a II_1 factor M acting standardly on \mathcal{H} , and $\varepsilon > 0$, we define the property $Q_0(X, \varepsilon)$ to be "there exists a finite orthonormal set $\{\xi_1, \dots, \xi_l\} \in \mathcal{H}$ such that $\forall U \in X$,

$$\sqrt{2} \sqrt{1 - \| [\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l \|_{\tau_l}^2} \leq \varepsilon \text{ and } |\tau(U) - \overline{\tau_l([\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l)}| \leq \varepsilon."$$

Remark 69 Since we may in general assume that $I \notin X$, we have in the case of a group von Neumann algebras where the generating set X are group elements that $\tau(L_{g_i}) = 0$ for all $g_i \in X$, and that property $Q_0(X, \varepsilon)$ reduces to "there exists a finite orthonormal set $\{\xi_1, \dots, \xi_l\} \in \mathcal{H}$ such that $\forall U \in X$,

$$\sqrt{2} \sqrt{1 - \| [\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l \|_{\tau_l}^2} \leq \varepsilon \text{ and } |\tau_l([\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l)| \leq \varepsilon."$$

Remark 70 Since $0 \leq \frac{\| [U, e] \|_{H.S.}}{\| e \|_{H.S.}} = \sqrt{2} \sqrt{1 - \| [\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l \|_{\tau_l}^2}$, we have that $0 \leq 1 - \| [\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l \|_{\tau_l}^2$, hence if $\varepsilon > 0$, then $\sqrt{2} \sqrt{1 - \| [\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l \|_{\tau_l}^2} \leq \varepsilon$ is equivalent to $\| [\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l \|_{\tau_l} \geq \sqrt{1 - \frac{\varepsilon^2}{2}}$. We therefore may rewrite property $Q_0(X, \varepsilon)$ as "there exists a finite orthonormal set $\{\xi_1, \dots, \xi_l\} \in \mathcal{H}$ such that $\forall U \in X$,

$$\| [\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l \|_{\tau_l} \geq \sqrt{1 - \frac{\varepsilon^2}{2}} \text{ and } |\tau(U) - \overline{\tau_l([\langle \xi_q, U \xi_p \rangle]_{q,p=1}^l)}| \leq \varepsilon."$$

Furthermore, imitating Voiculescu, we may define $\Omega_R^0(\{U_1, \dots, U_k\} : l, n, \varepsilon) = \{(W_1, W_2, \dots, W_k) \in \mathcal{U}(M_n(\mathbb{C}))^k \mid R \leq \| W_i \|_{\tau_n} ; |\tau(U_{i_1} U_{i_2}, \dots, U_{i_j}) - \overline{\tau_n(W_{i_1} W_{i_2}, \dots, W_{i_j})}| \leq \varepsilon$

for $i_j \in \{1, 2, \dots, k\}$ and $1 \leq j \leq l$. We may then rewrite property $Q_0(X, \varepsilon)$ as "there exists a finite orthonormal set $\{\xi_1, \dots, \xi_n\} \in \mathcal{H}$ such that $\forall U \in X$,

$$[\langle \xi_q, U \xi_p \rangle]_{q,p=1}^n \in \Omega^0_{\sqrt{1-\frac{\varepsilon^2}{2}}}(\{U_1, \dots, U_k\} : 1, n, \varepsilon)".$$

Notice, that the quantity we have introduced here is **not** directly related to the Γ_R sets of Voiculescu, we simply imitate his notation. We easily have the following proposition, and omit the proof.

Proposition 71 *If M is a type II_1 factor associated to an I.C.C. discrete group, then for all finite sets X of unitary elements that generate M and all $\varepsilon > 0$, property $Q(X, \varepsilon)$ is equivalent to property $Q_0(X, \varepsilon)$.*

We now attempt to compute explicit lower bounds for $F\phi l(\mathcal{L}_{\mathbb{F}_2}, X)$, for various finite subsets X .

Remark 72 *Let $X = \{L_a, L_b\}$ is the set of standard unitary generators of $\mathcal{L}_{\mathbb{F}_2}$, If ξ_1, \dots, ξ_k is an orthonormal set in $L^2(M)$, then we know that if the quantity*

$$\frac{1}{k} \sum_{i,j=1}^k |\langle L_a \xi_j, \xi_i \rangle|^2$$

is sufficiently close to 1, then the corresponding quantity

$$\frac{1}{k} \sum_{i,j=1}^k |\langle L_b \xi_j, \xi_i \rangle|^2$$

is close to 0 and vice-versa. To see why, note that if we could write each $\xi_i = \xi_i^a + \xi_i^b$, where ξ_i^a denotes the part of ξ with support a... and ξ_i^b the part of ξ with support b... Notice that $\langle L_a \xi_j, \xi_i \rangle = \langle L_a \xi_j, \xi_i^a \rangle$, so that $|\langle L_a \xi_j, \xi_i \rangle|^2 \leq \|\xi_i^a\|^2$. Since $\|\xi_i^a\|^2 + \|\xi_i^b\|^2 = 1$, we have that if the ξ_i are concentrated on the ξ_i^a parts then the $\|\xi_i^b\|^2$ parts are small. It follows that if $\frac{1}{k} \sum_{i,j=1}^k |\langle L_a \xi_j, \xi_i^a \rangle|^2$ is large (near 1), then $\frac{1}{k} \sum_{i,j=1}^k |\langle L_b \xi_j, \xi_i^b \rangle|^2 \leq C \max(\|\xi_i^b\|^2)$ will be small (near 0). This suggests a "balancing type" result should hold for the standard generators of $\mathcal{L}_{\mathbb{F}_2}$. The least

amount of information is obtained by the above reasoning when the “weights” $\|\xi_i^a\|^2$ and $\|\xi_i^b\|^2$ are all $\frac{1}{2}$.

Lemma 73 Let ξ_1 and ξ_2 be vectors in $L^2(\mathbb{F}_2)$, then $\langle L_a \xi_1, \xi_2 \rangle = \langle L_a \xi_1, \xi_2^a \rangle + \langle (L_a \xi_1^{a^{-1}})^b, \xi_2^b \rangle + \langle (L_a \xi_1^{a^{-1}})^{b^{-1}}, \xi_2^{b^{-1}} \rangle + \langle (L_a \xi_1^{a^{-1}})^e, \xi_2^e \rangle$.

Proof. Let $S_a \subseteq \mathbb{F}_2$ be the set of reduced words in \mathbb{F}_2 that begin with the letter a . Similarly define $S_{a^{-1}}, S_b, S_{b^{-1}}$. Writing ξ_1 as $\xi_1^a + \xi_1^b + \xi_1^{a^{-1}} + \xi_1^{b^{-1}} + \xi_1^e$, we get that L_a sends $\xi_1^a + \xi_1^b + \xi_1^{b^{-1}} + \xi_1^e$ to vectors with supports in $\overline{\text{span}}(S_a)$, and $\xi_1^{a^{-1}}$ to vectors with supports in the orthocomplement of this set. This gives the desired decomposition.

■

Lemma 74 Let ξ_1 and ξ_2 be vectors in $L^2(\mathbb{F}_2)$, then $\langle L_{a^{-1}} \xi_2, \xi_1 \rangle = \langle L_{a^{-1}} \xi_2, \xi_1^{a^{-1}} \rangle + \langle (L_{a^{-1}} \xi_2^b), \xi_1^b \rangle + \langle (L_{a^{-1}} \xi_2^a)^{b^{-1}}, \xi_1^{b^{-1}} \rangle + \langle (L_{a^{-1}} \xi_2^e), \xi_1^e \rangle$.

Lemma 75 Let ξ_1 and ξ_2 be vectors in $L^2(\mathbb{F}_2)$, then

$$\begin{aligned} & \langle L_a \xi_1, \xi_2^a \rangle + \langle (L_a \xi_1^{a^{-1}})^b, \xi_2^b \rangle + \langle (L_a \xi_1^{a^{-1}})^{b^{-1}}, \xi_2^{b^{-1}} \rangle + \langle (L_a \xi_1^{a^{-1}})^e, \xi_2^e \rangle \\ &= \langle L_a \xi_1^{a^{-1}}, \xi_2 \rangle + \langle \xi_1^b, (L_{a^{-1}} \xi_2^a)^b \rangle + \langle \xi_1^{b^{-1}}, (L_{a^{-1}} \xi_2^a)^{b^{-1}} \rangle + \langle \xi_1^e, (L_{a^{-1}} \xi_2^a)^e \rangle. \end{aligned}$$

Proof. Follows from the properties of the inner product, adjoint, and the previous two lemmas. ■

Lemma 76 Let ξ_1 and ξ_2 be vectors in $L^2(\mathbb{F}_2)$, then $\langle L_a \xi_1, \xi_2^a \rangle$ and $\langle L_a \xi_1^{a^{-1}}, \xi_2 \rangle$ share only the term

$$\langle L_a \xi_1^{a^{-1}}, \xi_2^a \rangle.$$

Proof. We have

$$\langle L_a \xi_1, \xi_2^a \rangle = \langle L_a \xi_1^a, \xi_2^a \rangle + \langle L_a \xi_1^b, \xi_2^a \rangle + \langle L_a \xi_1^{a^{-1}}, \xi_2^a \rangle + \langle L_a \xi_1^{b^{-1}}, \xi_2^a \rangle + \langle L_a \xi_1^e, \xi_2^a \rangle$$

and

$$\langle L_a \xi_1^{a^{-1}}, \xi_2 \rangle = \langle L_a \xi_1^{a^{-1}}, \xi_2^a \rangle + \langle L_a \xi_1^{a^{-1}}, \xi_2^b \rangle + \langle L_a \xi_1^{a^{-1}}, \xi_2^{a^{-1}} \rangle + \langle L_a \xi_1^{a^{-1}}, \xi_2^{b^{-1}} \rangle + \langle L_a \xi_1^{a^{-1}}, \xi_2^e \rangle.$$

■

Lemma 77 *Let ξ_1 and ξ_2 be unit vectors in $L^2(\mathbb{F}_2)$, then*

$$|\langle L_a \xi_1, \xi_2 \rangle| \leq \|\xi_2^a\| + \min(\|(L_a \xi_1^{a^{-1}})^b\|, \|\xi_2^b\|) + \min(\|(L_a \xi_1^{a^{-1}})^{b^{-1}}\|, \|\xi_2^{b^{-1}}\|) + \min(\|(L_a \xi_1^{a^{-1}})^e\|, \|\xi_2^e\|).$$

Proof. This is an application of the triangle and Cauchy-Schwartz inequalities to the above decomposition, along with the fact that L_a is unitary. ■

Lemma 78 *Let ξ_1 and ξ_2 be unit vectors in $L^2(\mathbb{F}_2)$, then*

$$|\langle L_a \xi_1, \xi_2 \rangle| \leq \|\xi_2^a\| + \|\xi_2^b\| + \|\xi_2^{b^{-1}}\| + \|\xi_2^e\|.$$

Proof. This is an easy consequence of the fact that L_a is unitary, and the above lemma. ■

Lemma 79 *Let ξ_1 and ξ_2 be unit vectors in $L^2(\mathbb{F}_2)$, then*

$$\frac{1}{4} |\langle L_a \xi_1, \xi_2 \rangle|^2 \leq \|\xi_2^a + \xi_2^b + \xi_2^{b^{-1}} + \xi_2^e\|^2.$$

Proof. We have that $\|\xi_2^a\|^2 + \|\xi_2^b\|^2 + \|\xi_2^{b^{-1}}\|^2 + \|\xi_2^e\|^2 = \|\xi_2^a + \xi_2^b + \xi_2^{b^{-1}} + \xi_2^e\|^2$, since these are orthonormal pieces. ■

Remark 80 *If we project all vectors onto the orthocomplement of the identity, these bounds all become smaller. We cannot do this without changing the property we consider, though.*

Lemma 81 *Let ξ_1 and ξ_2 be unit vectors in $L^2(\mathbb{F}_2)$, then*

$$\langle L_a \xi_1, \xi_2 \rangle = \sum_{\epsilon_1, \epsilon_2 \in \{a, a^{-1}, b, b^{-1}, e\}} \langle L_a \xi_1^{\epsilon_1}, \xi_2^{\epsilon_2} \rangle$$

Proof. This is straightforward. ■

Lemma 82 *Let conditions be as above, then*

$$|\langle L_a \xi_1^b, \xi_2^a \rangle| \leq \|\xi_1^b\| \|\xi_2^a\| \leq \min(\|\xi_1^b\|, \|\xi_2^a\|).$$

Similarly for all other terms in the sum of the above lemma.

Proof. Note, since L_a is unitary, it is an isometry, so that $\|L_a \xi_1^b\| = \|\xi_1^b\|$. The rest follows from the Cauchy-Schwartz inequality and the fact that both $\|\xi_1^b\|$ and $\|\xi_2^a\|$ are less than or equal to 1. ■

Lemma 83 *Let ξ_1 and ξ_2 be unit vectors in $L^2(\mathbb{F}_2)$, then*

$$|\langle L_a \xi_1, \xi_2 \rangle|^2 = 25 \sum_{\varepsilon_1, \varepsilon_2 \in \{a, a^{-1}, b, b^{-1}, e\}} |\langle L_a \xi_1^{\varepsilon_1}, \xi_2^{\varepsilon_2} \rangle|^2, \text{ and}$$

$$|\langle L_a \xi_1, \xi_2 \rangle|^2 \leq 25 \sum_{\varepsilon_1, \varepsilon_2 \in \{a, a^{-1}, b, b^{-1}, e\}} \|\xi_1^{\varepsilon_1}\|^2 \|\xi_2^{\varepsilon_2}\|^2$$

Tighter bounds hold if we replace each eligible term like $\|\xi_1^{a^{-1}}\|^2 \|\xi_2^b\|^2$ by $\|(L_a \xi_1^{a^{-1}})^b\|^2 \|\xi_2^b\|^2$, and we get looser bounds if we replace all terms on the RHS according to $\|\xi_1^a\|^2 \|\xi_2^a\|^2 \leq \min(\|\xi_1^a\|, \|\xi_2^a\|)$.

Proof. Uses an elementary inequality, and the previous lemmas. ■

Definition 84 *Let M_n be finite factors with traces τ_n , and let $\prod M_n$ denote their C^* -product, i.e. the C^* -algebra of uniformly norm-bounded sequences equipped with pointwise operations and the supremum norm. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Then*

$$\mathcal{I}_\omega = \{(A_i)_i \in \prod M_n : \lim_{i \rightarrow \omega} \tau_i(A_i^* A_i) = 0\}$$

is a closed two-sided ideal in $\prod M_n$, and by a result of Sakai, the quotient $(\prod M_n)/\mathcal{I}_\omega$ is a factor von Neumann algebra $\prod^\omega M_n$ with a faithful, normal trace τ_ω defined by $\tau_\omega((A_i)_i + \mathcal{I}_\omega) = \lim_{i \rightarrow \omega} \tau_i(A_i)$. The factor $\prod^\omega M_n$ will be called an ultraproduct of the M_n with respect to the free ultrafilter ω , or simply an ultraproduct of the M_n .

Lemma 85 *If $F\phi(M, X) = 0$, then whenever $Q(X, \varepsilon)$ via a rank- n projection e , the following is true: for each $U \in X$ and $\delta > 0$ there exists a unitary $n \times n$ matrix $W \in eB(L^2(M))e$ such that*

$$\|eUe - W\|_{\tau_n} \leq \delta.$$

Proof. Suppose that $F\phi(M, X) = 0$. Thus for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ so that $0 < \frac{1}{n} \leq \varepsilon$ and $Q(X, \frac{1}{n})$ and therefore there is an $e_n = \sum_{i=1}^{l(n)} \xi_i^{(n)} \otimes \xi_i^{(n)}$, where the $\{\xi_i^{(n)}\}_{i=1}^{l(n)}$ are an orthonormal system in $L^2(M)$, with

$$0 \leq \frac{\| [U, e_n] \|_{H.S.}}{\|e_n\|_{H.S.}} = \sqrt{2} \sqrt{1 - \|e_n U e_n\|_{\tau_l(n)}^2} \leq \frac{1}{n}$$

for all $U \in X$. With $e_n U e_n = A_n = [\langle \xi_q^{(n)}, U \xi_p^{(n)} \rangle]_{q,p=1}^{l(n)}$, we have

$$1 - \frac{1}{2n^2} \leq \tau_{l(n)}(A_n^* A_n) = \tau_{l(n)}(A_n A_n^*) = \|e_n U e_n\|_{\tau_l(n)}^2.$$

Furthermore, since e_n is a projection, $\|e_n\| \leq 1$ and hence

$$\|A_n\| = \|e_n U e_n\| \leq \|U\| \|e_n\|^2 \leq 1,$$

and hence $\|A_n A_n^*\| = \|A_n\|^2 \leq 1$. Let ω be a free ultrafilter, and $\prod_{l(n)}^{\omega} M_{l(n)}(\mathbb{C})$ denote the ultraproduct factor as defined above. We have a sequence

$$(A_n) \equiv \{A_n \mid n_1 < n_2 \Rightarrow l(n_1) < l(n_2)\}_{n=1}^{\infty}$$

of matrices satisfying

$$\begin{aligned} \tau_{\omega}((A_n^* A_n) + \mathcal{I}_{\omega}) &= \tau_{\omega}((A_n A_n^*) + \mathcal{I}_{\omega}) = 1 \\ &= \tau_{\omega}((I_n) + \mathcal{I}_{\omega}) \end{aligned}$$

so by faithfulness of τ_{ω} and the fact that $(I_n - A_n A_n^*)_n \geq 0$ for all n ,

$$\tau_{\omega}((I_n - A_n A_n^*)_n + \mathcal{I}_{\omega}) = 0,$$

so indeed (A_n) represents a unitary element in the ultraproduct $\prod^\omega M_{l(n)}(\mathbb{C})$. Recall that if

$$(A_n) + \mathcal{I}_\omega \text{ and } (B_n) + \mathcal{I}_\omega$$

represent distinct elements of $\prod^\omega M_{l(n)}(\mathbb{C})$, then the 2-norm distance between them is given by

$$\begin{aligned} & \| (A_n - B_n) + \mathcal{I}_\omega \|_2 \\ &= \tau_\omega(((A_n^* - B_n^*) + \mathcal{I}_\omega) ((A_n - B_n) + \mathcal{I}_\omega))^{1/2} \\ &= \left[\lim_{l(n) \rightarrow \omega} \tau_{l(n)}((A_n^* - B_n^*)(A_n - B_n)) \right]^{1/2} \\ &= \left[\lim_{l(n) \rightarrow \omega} \|A_n - B_n\|_{\tau_{l(n)}}^2 \right]^{1/2} \\ &= \lim_{l(n) \rightarrow \omega} \|A_n - B_n\|_{\tau_{l(n)}}. \end{aligned}$$

Suppose that $\delta > 0$ and that for every unitary $l(n) \times l(n)$ matrix W_n , $\|A_n - W_n\|_{\tau_{l(n)}} > \delta$, it then follows that $\| (A_n - W_n) + \mathcal{I}_\omega \|_2 > \delta$ in $L^2(\prod^\omega M_{l(n)}(\mathbb{C}), \tau_\omega)$. Since every sequence (W_n) represents a unitary element in $\prod^\omega M_{l(n)}(\mathbb{C})$, and every unitary element is represented by such a sequence, a contradiction follows, since (A_n) represents a unitary element in $\prod^\omega M_{l(n)}(\mathbb{C})$. Therefore, for all $\delta > 0$ there exists a unitary $l(n) \times l(n)$ matrix W_n so that $\|A_n - W_n\|_{\tau_{l(n)}} \leq \delta$, hence we may view W_n as a unitary element of $e_n B(L^2(M)) e_n$ (that is, a unitary operator on $\text{span}\{\xi_i^{(n)}\}_{i=1}^{l(n)} \cong \mathbb{C}^{l(n)}$). ■

We can, by a method of Ravichandran, prove the following theorem.

Theorem 86 *If $X = \{L_a, L_b\}$ is the set of standard unitary generators of $\mathcal{L}_{\mathbb{F}_2}$, then $F\phi l(\mathcal{L}_{\mathbb{F}_2}, X) > 0$.*

Proof. Suppose that $F\phi l(\mathcal{L}_{\mathbb{F}_2}, X) = 0$. It follows that $Q(X, \frac{1}{7})$, so there exists a positive integer $l(= l(n))$ and a rank l projection $e = \sum_{i=1}^l \xi_i \otimes \xi_i$, where the $\{\xi_i\}_{i=1}^l$ is an orthonormal system in $L^2(\mathbb{F}_2)$, such that for both $U \in X$

$$0 \leq \frac{\| [U, e] \|_{H.S.}}{\| e \|_{H.S.}} = \sqrt{2} \sqrt{1 - \| e U e \|_n^2} \leq \frac{1}{7}.$$

Henceforth let U be the generator L_a and U' the generator L_b . Let $A = eUe$ and $Ue = B$. We have that $B^*A = A^*A = A^*B$ and we may formally write

$$\begin{aligned}
& \|eUe - Ue\|_{\tau_l}^2 \\
&= \|A - B\|_{\tau_l}^2 \\
&= \tau_l((A^* - B^*)(A - B)) \\
&= \tau_l(A^*A - B^*A - A^*B + B^*B) \\
&= \tau_l(A^*A) - \tau_l(B^*A) \\
&\quad - \tau_l(A^*B) + \tau(B^*B) \\
&= 1 - \tau_l(A^*A)
\end{aligned}$$

and since $\frac{97}{98} \leq \|eUe\|_{\tau_l}^2 = \tau_l(A^*A)$, we have that

$$\frac{1}{98} \geq 1 - \tau_l(A^*A) = \|eUe - Ue\|_{\tau_l}^2.$$

By the above lemma, there is an $l \times l$ unitary matrix $W \in eB(L^2(\mathbb{F}_2))e$ such that

$$\|A - W\|_{\tau_n} \leq \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{7}.$$

Now we have that $\|A - Ue\|_{Tr} \leq \frac{\sqrt{l}}{7\sqrt{2}}$ and $\|A - W\|_{Tr} \leq \left(1 - \frac{1}{\sqrt{2}}\right) \frac{\sqrt{l}}{7}$, so that

$$\|Ue - W\|_{Tr} \leq \frac{\sqrt{l}}{7}.$$

so we may write formally,

$$\|Ue - W\|_{\tau_l}^2 \leq \frac{1}{49}.$$

Note that since e is the projection onto ξ_1, \dots, ξ_l and $W \in eB(L^2(\mathbb{F}_2))e$ we have

$$\|Ue - W\|_{\tau_l}^2 = \frac{1}{l} \sum_{i=1}^l \|(Ue - W)\xi_i\|^2.$$

Suppose that $i \in \{1, 2, \dots, l\}$, and that $\xi_i = \sum_{g \in \mathbb{F}_2} \lambda_g^{(i)} g$, where we write g in place of

$\chi_{\{g\}}$ to keep notation simpler, we obtain

$$\begin{aligned} \|(Ue - W)\xi_i\|^2 &= \|(U - W) \sum_{g \in \mathbb{F}_2} \lambda_g^{(i)} g\|^2 \\ &= \sum_{g \in \mathbb{F}_2} |\lambda_{a^{-1}g}^{(i)} - \sum_{k=1}^{l(n)} w_{ik} \lambda_g^{(k)}|^2. \end{aligned}$$

For S a non-empty subset of \mathbb{F}_2 and $\eta = \sum_{g \in \mathbb{F}_2} \mu_g g \in L^2(\mathbb{F}_2)$, define $\|\eta\|_S^2 = \sum_{g \in S} |\mu_g|^2$. It follows that

$$\|(U - W)\xi_i\|_S^2 = \sum_{g \in S} |\lambda_{a^{-1}g}^{(i)} - \sum_{k=1}^{l(n)} w_{ik} \lambda_g^{(k)}|^2.$$

We have that

$$\begin{aligned} |(\|U\xi_i\|_S - \|W\xi_i\|_S)| &\leq \|(U - W)\xi_i\|_S \\ &\leq \|(U - W)\xi_i\|. \end{aligned}$$

and

$$\begin{aligned} &|(\|U\xi_i\|_S^2 - \|W\xi_i\|_S^2)| \\ &\leq |(\|U\xi_i\|_S - \|W\xi_i\|_S)|(\|U\xi_i\|_S + \|W\xi_i\|_S) \\ &\leq 2|(\|U\xi_i\|_S - \|W\xi_i\|_S)| \\ &\leq 2\|(U - W)\xi_i\|_S \\ &\leq 2\|(U - W)\xi_i\|. \end{aligned}$$

so that by the triangle inequality,

$$\begin{aligned} &|\frac{1}{l} \sum_{i=1}^l \|U\xi_i\|_S^2 - \frac{1}{l} \sum_{i=1}^l \|W\xi_i\|_S^2| \\ &\leq \frac{1}{l} \sum_{i=1}^l |(\|U\xi_i\|_S^2 - \|W\xi_i\|_S^2)| \\ &\leq \frac{2}{l} \sum_{i=1}^l \|(U - W)\xi_i\| \end{aligned}$$

so

$$\begin{aligned}
& \left| \frac{1}{l} \sum_{i=1}^l \|U\xi_i\|_S^2 - \frac{1}{l} \sum_{i=1}^l \|W\xi_i\|_S^2 \right|^2 \\
& \leq \frac{4}{l^2} \left(\sum_{i=1}^l \|(U-W)\xi_i\| \right)^2 \\
& \leq \frac{4l}{l^2} \sum_{i=1}^l \|(U-W)\xi_i\|^2 \\
& = \frac{4}{l} \sum_{i=1}^l \|(U-W)\xi_i\|^2 \\
& = 4\|U-W\|_{\tau_l}^2 \leq \frac{4}{49}
\end{aligned}$$

If $\eta = \sum_{g \in \mathbb{F}_2} \mu_g g \in L^2(\mathbb{F}_2)$, define $\eta|_S = \sum_{g \in S} \mu_g g \Omega \in L^2(S)$. Note that $\|\eta|_S\|_{L^2(S)} = \|\eta\|_S$. We have that

$$\begin{aligned}
W\xi_i|_S &= \sum_{k=1}^l w_{ik} \xi_k|_S = \sum_{g \in S} \left(\sum_{k=1}^l w_{ik} \lambda_g^{(k)} \right) g \\
&= (W\xi_i)|_S.
\end{aligned}$$

so that

$$\begin{bmatrix} W & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W \end{bmatrix} \begin{bmatrix} \xi_1|_S \\ \vdots \\ \xi_l|_S \end{bmatrix} = \sum_{g \in S} (W \begin{bmatrix} \lambda_g^{(1)} \\ \vdots \\ \lambda_g^{(l)} \end{bmatrix}) g.$$

Since W is a unitary operator on \mathbb{C}^l , we have that

$$\begin{aligned} \left\| \begin{bmatrix} W & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W \end{bmatrix} \begin{bmatrix} \xi_{1|S} \\ \vdots \\ \xi_{l|S} \end{bmatrix} \right\|_{L^2(S)^l}^2 &= \sum_{g \in S} \left\| W \begin{bmatrix} \lambda_g^{(1)} \\ \vdots \\ \lambda_g^{(l)} \end{bmatrix} \right\|_{\mathbb{C}^l}^2 \\ &= \sum_{g \in S} \left\| \begin{bmatrix} \lambda_g^{(1)} \\ \vdots \\ \lambda_g^{(l)} \end{bmatrix} \right\|_{\mathbb{C}^l}^2 \\ &= \left\| \begin{bmatrix} \xi_{1|S} \\ \vdots \\ \xi_{l|S} \end{bmatrix} \right\|_{L^2(S)^l}^2. \end{aligned}$$

We may conclude that

$$\begin{aligned} \sum_{i=1}^l \|W\xi_i\|_S^2 &= \sum_{i=1}^l \|(W\xi_i)|_S\|_{L^2(S)}^2 \\ &= \sum_{i=1}^l \|W\xi_i|_S\|_{L^2(S)}^2 \\ &= \sum_{i=1}^l \|\xi_i|_S\|_{L^2(S)}^2 = \sum_{i=1}^l \|\xi_i\|_S^2. \end{aligned}$$

We also have that for each i ,

$$\begin{aligned} (U\xi_i)|_S &= \left(\sum_{g \in \mathbb{F}_2} \lambda_{\alpha^{-1}g}^{(i)} g \right) |_S \\ &= \sum_{g \in S} \lambda_{\alpha^{-1}g}^{(i)} g \\ &= U\xi_i|_S. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{i=1}^l \|U\xi_i|_S\|_{L^2(S)}^2 &= \sum_{i=1}^l \|(U\xi_i)|_S\|_{L^2(S)}^2 \\ &= \sum_{i=1}^l \|U\xi_i\|_S^2. \end{aligned}$$

Notice that

$$\begin{aligned}
\|(U\xi_i)|_S\|_{L^2(S)}^2 &= \sum_{g \in S} |\lambda_{a^{-1}g}^{(i)}|^2 \\
&= \sum_{g \in a^{-1}S} |\lambda_g^{(i)}|^2 \\
&= \|\xi_i|_{a^{-1}S}\|_{L^2(a^{-1}S)}^2 = \|\xi_i\|_{a^{-1}S}^2.
\end{aligned}$$

We have that

$$\begin{aligned}
\left| \frac{1}{l} \sum_{i=1}^l (\|\xi_i\|_{a^{-1}S}^2 - \|\xi_i\|_S^2) \right| &= \\
&= \left| \frac{1}{l} \sum_{i=1}^l \|U\xi_i\|_S^2 - \frac{1}{l} \sum_{i=1}^l \|\xi_i\|_S^2 \right| \\
&\leq \frac{4}{49}.
\end{aligned}$$

Now we shall choose a subset S for which the above inequality will give us a contradiction. For simplicity of notation, let us define

$$c_S \equiv \frac{1}{l} \sum_{i=1}^l \|\xi_i\|_S^2.$$

The above inequality becomes

$$|c_{a^{-1}S} - c_S| \leq \frac{4}{49}.$$

If we carry out the above analysis using U' in place of U , we obtain

$$|c_{b^{-1}S} - c_S| \leq \frac{4}{49}.$$

Since S was arbitrary, we could replace S by aS (resp. bS) to get

$$\begin{aligned}
|c_S - c_{aS}| &\leq \frac{4}{49} \\
(\text{resp. } |c_S - c_{bS}| &\leq \frac{4}{49}).
\end{aligned}$$

Choose the set S to be all reduced words in \mathbb{F}_2 that begin with a^{-1} . Then $S \cup aS = \mathbb{F}_2$ and also S , bS and $b^{-1}S$ are pairwise disjoint. Since $S \cup aS = \mathbb{F}_2$, we have that c_S

or c_{aS} exceeds $\frac{1}{2}$. Since S , bS and $b^{-1}S$ are pairwise disjoint, at least one of c_S, c_{bS} or $c_{b^{-1}S}$ must be $\leq \frac{1}{3}$. With no loss of generality, we may assume that $\frac{1}{2} \leq c_{aS}$. It follows that

$$\frac{1}{2} \leq c_{aS} = |c_S| \leq |c_S - c_{aS}| + |c_S| \leq \frac{4}{49} + c_S$$

so that

$$\frac{1}{2} - \frac{4}{49} \leq c_S.$$

Let us assume, again with no loss of generality, that $c_{bS} \leq \frac{1}{3}$, then

$$c_S \leq |c_S - c_{bS}| + c_{bS} \leq \frac{1}{3} + \frac{4}{49}.$$

It follows that

$$\frac{5}{12} < \frac{1}{2} - \frac{4}{49} \leq c_S \leq \frac{1}{3} + \frac{4}{49} < \frac{5}{12}$$

which is a contradiction. ■

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