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FREE PRODUCTS OF OPERATOR SPACES AND FREE MARKOV PROCESSES

BY

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DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirments for the Degree of

Doctor of Philosophy
in
Mathematics

May, 2004

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ABSTRACT

FREE PRODUCTS OF OPERATOR SPACES AND FREE MARKOV PROCESS

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University of New Hampshire, May 2004

Certain (reduced) free product is introduced in the framework of operator spaces. Under the construction, the free product of preduals of von Neumann algebras agrees with the predual of the free product of von Neumann algebras. This answers a question asked by Effros affirmatively. An example is presented to show that the C^*-algebra reduced free product of two C^*-algebras may be contractively isomorphic to a proper subspace of the operator space reduced free product of the two C^*-algebras.

Free Markov processes are also investigated in Voiculescu's free probability theory. This highly non-commutative notion generalizes that of free Brownian motion and free Lévy processes. Some free Markov processes are realized as solutions to free stochastic differential equations driven by free Lévy processes. A special and rather interesting kind of free Markov processes, free Ornstein-Uhlenbeck processes, is studied in some details. It is shown that a probability measure on $\mathbb{R}$ is free self-decomposable if and only if it is the stationary distribution of a stationary free Ornstein-Uhlenbeck process (driven by a free Levy process). Finally, the notion of free fractional Brownian motion is introduced. Examples of fractional free Brownian motion are given, which are based on creation and annihilation operators on full Fock spaces. It is proved that the Langevin equation driven by fractional free Brownian motion has a unique solution. We call the solution a *fractional free Ornstein-Uhlenbeck process.*

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Chapter 1

INTRODUCTION

A von Neumann algebra is a *-algbra of operators on a Hilbert space, containing the identity operator, and closed in the *strong operator topology*. Some Von Neumann algebras appear as group algebras of (infinite) groups and provide an indispensable tool for the study of the representations of these groups as well as harmonic analysis on them. On the other hand, they provide a mathematical model for the study of infinitely extended quantum systems (a study in the style indicated, first, by P. A. M. Dirac [16]). As a result, the subject of von Neumann algebras has undergone an intensive and massive development since its introduction in 1929.

F. J. Murry and von Neumann ([41, 42, 43, 46]) showed that each von Neumann algebra is a "direct integral" over its center of certain von Neumann algebras whose centers consist of scalar operators (these von Neumann algebras were called *factors*). The factors were recognized as the key components of the subject. Murry and von Neumann separated factors into three basic types: those with a minimal projection, the type I factor; those without minimal projections but admitting a functional resembling the trace of a matrix, the type II factors; and all the rest, type III factors. The different types of factors are now known to be closely related to one another by means of general operations known as "tensor product" and "crossed product" by groups. The focus of much research has returned to the factors of type II_1 from the type III factors (most visible in the von Neumann

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algebra approach to quantum field theory and quantum statistical mechanics), and to applications of operator algebras to other areas of mathematics.

A factor of type II_1 is *hyperfinite* if it is the weak operator closure of the union of an increasing sequence of finite dimensional subfactors of it. This kind of type II_1 factors can be realized as the group von Neuman algebra of the group of those permutations of the integers that move at most a finite number of integers. Another kind of examples of type II_1 factors is the free group factor, the group von Neumann algebra of the free non-abelian group on n (≥ 2) generators. Murry and von Neumann proved in [43] that all hyperfinite type II_1 factors on a separable Hilbert space are isomorphic to one another. They also proved in the same paper that the hyperfinite type II_1 factor is not isomorphic to free group factors. One of the longstanding (still open) questions in the theory of von Neumann algebras is

> whether free group factors on different numbers of generators are isomorphic to one another.

To study free group factors (and answer the question above), Voiculescu introduced and developed the theory of *free probability* in early 1980s in the context of von Neumann algebras. Free probability is one kind of non-commutative probability theory, where the classical independence is replaced by free independence. Independence of random variables, in classical probability theory, corresponds to certain tensor product relation of the polynomial rings generated by the variables; while free independence is based on the free product relation of the (noncommutative) polynomial rings generated by random variables. The notion of free product of algebras (or groups) existed long time ago. The free product used in free probability theory reflects certain topological structures of the algebras of random variables. A special case of such examples can be traced back to [14], where W. M. Ching introduced the notion of (reduced) free products of von Neumann algebras with traces. Later, it was generalized to C*-algebras with given states by D. Voiculescu in [58]. Some fundamental results were obtained by Voiculescu, e.g., the free central limit theorem. In early 1990s, Voiculescu, Ge, Redulescu were able

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to solve old problems in von Neumann algebras using free probability techniques (see [53, 61, 25, 26, 27]).

The study on stochastic processes is a vast research area in free probability. The analogues of classical Brownian motion and Levy processes in free probability were introduced in 1990s. Voiculescu introduced the concept of free Markov processes in 1999 (see [61]). So far, Most of the research work on stochastic processes in free probability is on free Brownian motion and free Levy processes (see [1, 3, 4, 8, 9, 10, 11]). There is not much work on the general free Markov process, which is a focus of our study. In this dissertation, it is shown that free Brownian motion and free Levy processes are examples of free Markov processes. One of our results says that, for a free Markov process, the "future" subalgebra and the "past" subalgebra are "conditionally perpendicular" with respect to the "present" algebra. We call this "conditional perpendicularity" of the past algebra and the "future" algebra a *weak Markov property.* We proved that, in classical case, the weak Markov property is the same as classical Markov property. It is shown that a stochastic process with weak Markov property has transition functions. The transition functions have very similar properties to those of a classical Markov process.

Certain free stochastic differential equations driven by free Brownian motion were studied by Biane and Spicher in 2001. They showed that the free stochastic differential equations have solutions, and the solutions have free Markov property (see [11]). We consider the similar free stochastic differential equations driven by *free Levy processes.* We proved that the equations have solutions. The solutions are free Markov processes consisting of random variables with non-compactly supported distributions. The proof of our result relies on a free Burkholder-Gundy type inequality in L^2-norm (for the Lévy case) proved by M. Anshelevich [1]. A similar inequality in operator norm for stochastic integrals with respect to free Brownian motion was obtained in [10]. Our results provide a method to find examples of free Markov processes with non-compactly supported distributions.

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Biane and Speicher [11] studied the solution to the following stochastic differential equation (a special case of the free stochastic diffrential equations mentioned previously)

$$X_t = X_0 - \lambda \int_0^t X_s ds + S_t, t \geq 0, \tag{1.1}$$

where $\lambda > 0$, $\{S_t : t \geq 0\}$ is a free Brownian motion, and the initial variable X_0 and $\{S_t : t \geq 0\}$ are free. They proved that the unique solution to (1.1) has the following form

$$X_t = e^{-\lambda t} X_0 + e^{-\lambda t} \int_0^t e^{\lambda s} dS_s, t \geq 0. \tag{1.2}$$

The process given in (1.2) is called a *free Ornstein-Uhlenbeck process* (Briefly free OU process). They also showed that its limit distribution is a semicircular law.

Barndorff-Nielsen and Thorbjornsen [4] mentioned free OU processes driven by free Levy processes (but there were no details given). In this dissertation, we study similar equations to (1.1), driven by free Levy processes. It is proved that the solution of the equation has the same form as (1.2), a free OU process driven by a free Levy process. One of our results says that a probability measure on $\mathbb{R}$ is freely self-decomposable if and only if it is the limit distribution of a free OU process driven by a free Levy process. Periodic OU processes were introduced in classical probability by Pedersen [49] in 2002, and the class of the stationary distributions of periodic OU processes was studied in [50], 2003. In this dissertation, we consider the same questions in free probability and obtain similar results to the classical case. Fractional OU processes driven by fractional Brownian motion were studied recently in classical probability theory (see [13]). We introduce the notion of fractional free Brownian motion with examples based on creation and annihilation operators on a full Fock space. We show that the equation, similar to (1.1), driven by fractional free Brownian motion has a unique solution. We call the solution a *fractional free OU process.*

Another research topic in this dissertation is free products in operator spaces. The existence of free products in operator spaces was conjectured by Effros (see [18]). In this dissertation, we prove the conjecture affirmatively.

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Our ideas of the free product construction of operator spaces can be traced back to that of operator algebras. W. M. Ching and D. Voiculescu's ideals for reduced free products of operator algebras with given states are based on the free product of Hilbert spaces and the GNS construction (see [14, 58, 62, 64]). The free product is more or less an algebraic construction. The difficulty arises when operator spaces are considered, where any algebraic structure is absent. By [21], an operator space may be viewed as the quotient space of the space of all trace class operators on a Hilbert space. It is natural to associate the space of all trace class operators with certain Hilbert space structure. Based on free products of Hilbert spaces, we give, in this dissertation, certain free products of operator spaces, which proves Effors's conjecture mentioned above. In general, free products of C^*-algebras (or von Neumann algebras) are not nuclear (or injective). It is proved, in this dissertation, that the reduced free product of operator spaces does not preserve the local lifting property, a notion introduced by S.-H. Kye and Z.J. Ruan to characterize the pre-dual of an injective operator space (see [19] and [35]). On the other hand, operator spaces with the local lifting property have certain property of completely isometric embedding into their free product. An example is presented to compare the C^*-algebra reduced free product with the operator space reduced free product for two C^*-algebras.

The dissertation is divides into four chapters besides this introduction. Chapter two is a chapter of preliminaries. We review in this chapter some basic concepts and results in free probability and operator spaces used in the sequel. In Chapter three, we deal with certain free products of operator spaces. Chapter four is devoted to the study of free Markov processes. Finally, in Chapter five, we discuss a special class of stochastic processes—free Ornstein-Uhlenbeck processes.

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Chapter 2

PRELIMINARIES

We review some basic concepts and results, which are used in the sequel, in free and classical probability theory and operator spaces. Certain unbounded operators affiliated with a von Neumann algebra and operator-valued functions on (unbounded) operator algebras will be discussed.

2.1 Free Probability

We refer to [64], [34] and [4] for basics on free probability, operator algebras, and unbounded operators affiliated with a von Neumann algebra and the convergence of unbounded operators in distribution, respectively.

A non-commutative probability space is a pair $(\mathcal{A}, \tau)$ consisting of a unital algebra $\mathcal{A}$ over the complex field $\mathbb{C}$, and a linear functional τ on $\mathcal{A}$ with value 1 at the unit I of algebra $\mathcal{A}$. Elements in $\mathcal{A}$ are called random variables. The *distribution* of a random variable A in a non-commutative probability space $(\mathcal{A}, \tau)$ is a linear functional $\mu(A)$ on $\mathbb{C}[x]$, the polynomial algebra in variable x. The linear functional is defined by $\mu(A)(p) = \tau(p(A))$, for p in $\mathbb{C}[x]$. Positivity is one important property for random variables in classical probability. To study the positivity for random variables in non-commutative probability spaces, we may assume that $\mathcal{A}$ is a unital *-algebra and τ is a state (i. e. $\tau(A^*A) \geq 0$, for $A \in \mathcal{A}$). The element A^*A is said to be positive, for $A \in \mathcal{A}$. State τ is tracial

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if $\tau(AB) = \tau(BA)$, τ is faithful if $\tau(A^*A) = 0$ implies that A is zero. Examples of non-commutative probability spaces come from operator algebras on a Hilbert space and the states used are usually vector states. In this dissertation, we always assume that non-commutative probability spaces are W^*-probability spaces.

A *W^*-probability space* is a pair $(\mathcal{A}, \tau)$ consisting of a von Neumann algebra $\mathcal{A}$ and a normal state τ. Throughout the dissertation, we assume that $\mathcal{A}$ has a separable predual and τ is a faithful normal tracial state. Define $\|X\|_2 = \tau(X^*X)^{1/2}$, for all X in $\mathcal{A}$. Let $L^2(\mathcal{A}, \tau)$ be the completion of $\mathcal{A}$ with respect to $\|\cdot\|_2$. Then $L^2(\mathcal{A}, \tau)$ is a Hilbert space with respect to inner product $\langle X, Y\rangle = \tau(XY^*)$. Suppose $\mathcal{S}$ is a subset of $\mathcal{A}$. We use $W^*(\mathcal{S})$ to denote the von Neumann subalgebra of $\mathcal{A}$ generated by $\mathcal{S}$. The classical independence is replaced by *free independence* in free probability. A family $\{\mathcal{A}_i : i \in \mathbf{I}\}$ of von Neumann subalgebras of $\mathcal{A}$ is *free* with respect to τ if $\tau(A_1A_2\cdots A_n) = 0$ whenever $A_j \in \mathcal{A}_{i_j}$, $i_j \in \mathbf{I}$, $i_1 \neq \cdots \neq i_n$ and $\tau(A_j) = 0$ for $1 \leq j \leq n$ and every n in $\mathbb{N}$. A family of subsets (or elements) of $\mathcal{A}$ is *free* if the family of the von Neumann subalgebras they generate is free.

Let $\mathcal{B}$ be a subalgebra of algebra $\mathcal{A}$. A *conditional expectation* $\mathbf{E}$ of $\mathcal{A}$ onto $\mathcal{B}$ is a $\mathcal{B}$-bi-module map (that is, $\mathbf{E}(B_1AB_2) = B_1\mathbf{E}(A)B_2$, for $A \in \mathcal{A}$, $B_1, B_2 \in \mathcal{B}$). If $\mathcal{B}$ is a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{A}$, there is a unique trace-preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$.

In free probability, *freeness with amalgamation* seems to be the right replacement of the classical conditional independence. Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C}$ and $\mathcal{D}$ be von Neumann subalgebras of von Neumann algebra $\mathcal{A}$, $\mathbf{E}_{\mathcal{B}}$ be the trace-preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$. We say $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{B}$-free (or freely independent with amalgamation) if $\mathcal{C}$ and $\mathcal{D}$ are free with respect to $\mathbf{E}_{\mathcal{B}}$, that is, $\mathcal{C}$ and $\mathcal{D}$ satisfy the same condition as that in the definition of freeness (in this case, τ is repalced by conditional expectation $\mathbf{E}_{\mathcal{B}}$) (see [62] and [64]).

Note that, when $\mathcal{B} = \mathbb{C}I$, $\mathcal{B}$-free independence is the same as free independence. Freeness, in general, does not imply $\mathcal{B}$-freeness, when $\mathcal{B} \neq \mathbb{C}I$.

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The freeness of subalgebras can be obtained from free products. Let $(\mathcal{A}_1, \tau_1)$ and $(\mathcal{A}_2, \tau_2)$ be W^*-probability spaces. Suppose $\mathcal{A}_0$ is the algebraic free product of $\mathcal{A}_1$ and $\mathcal{A}_2$. There is a unique tracial state τ on $\mathcal{A}_0$ such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are free with respect to τ and the restriction of τ on $\mathcal{A}_j$ is τ_j, $j = 1, 2$. Let $\mathcal{A}$ be the weak operator closure of $\mathcal{A}_0$ acting on $L^2(\mathcal{A}_0, \tau)$. Then $\mathcal{A}$ is called the *(reduced von Neumann algebra) free product* of $\mathcal{A}_1$ and $\mathcal{A}_2$, denoted by $\mathcal{A}_1 * \mathcal{A}_2$. For example, the free group factor $\mathcal{L}_{F_2}$ is *-isomorphic to $L^\infty([0,1]) * L^\infty([0,1])$. Similarly, one may define *(reduced C^*-algebra) free product* of two C^*-algebras with given states (Roughly speaking, the free product is the uniform closure of $\mathcal{A}_0$ acting on $L^2(\mathcal{A}_0, \tau)$).

In free probability, Gaussian law is replaced by semicircular law. In classical probability, Gaussian law is the limit distribution of the normalized partial sums of an i.i.d. sequence of random variables. In free probability, semicircular law is the limit distribution of the normalized partial sums of a freely i.i.d. sequence of random variables. A *semicircle law* (or distribution) is a probability measure on $\mathbb{R}$ whose density function is

$$\gamma_{c,r}(t) = \frac{2}{\pi r^2}\sqrt{r^2 - (t-c)^2}\chi_{[c-r,c+r]}(t),$$

where c and $r > 0$ are real numbers. The mean of the semicircular law $\gamma_{c,r}$ is c and the variance is $\frac{r^2}{4}$.

Free Brownian Motion. (cf. [11], [12]) Let $(\mathcal{A}, \tau)$ be a W^*-probability space with filtration $\{\mathcal{A}_t : t \geq 0\}$ (that is, $\{\mathcal{A}_t : t \geq 0\}$ is a family of von Neumann subalgebras of $\mathcal{A}$ such that $\mathcal{A}_t \subseteq \mathcal{A}_s$, when $0 \leq t \leq s$). A family $\{S_t : t \geq 0\}$ of self-adjoint operators in $(\mathcal{A}, \tau)$ is called an *$(\mathcal{A}_t)$-free Brownian motion*, if, $X_0 = 0$, and, for $0 \leq s < t$, $S_t - S_s$ and $\mathcal{A}_s$ are free, and $S_t - S_s$ has semicircular distribution of mean zero and variance $t - s$.

Unbounded Operators and Convergence in Distribution. Let $(\mathcal{A}, \tau)$ be a W^*-probability space with $\mathcal{A}$ acting on the Hilbert space $\mathcal{H}$ $(= L^2(\mathcal{A}, \tau))$ by left multiplications. A self-adjoint (unbounded) operator A, defined on a dense subspace of $\mathcal{H}$, is said to be affiliated with $\mathcal{A}$, if all spectral projections of A lie

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in $\mathcal{A}$. Generally, a closed densely defined operator T on $\mathcal{H}$ is said to be affiliated with $\mathcal{A}$, if $T = UA$, for some U in $\mathcal{A}$, and self-adjoint operator A affiliated with $\mathcal{A}$, where $T = UA$ is the polar decomposition of T. Denoted by $\widetilde{\mathcal{A}}$ the algebra of all densely defined and closed (unbounded) operators affiliated with $\mathcal{A}$ (see [6], [34] and [44] for details). Elements in $\widetilde{\mathcal{A}}$ are again called random variables (in general, with non-compactly supported distributions).

Let $\widetilde{\mathcal{A}}_{sa}$ be the set of all self-adjoint elements in $\widetilde{\mathcal{A}}$. Given X in $\widetilde{\mathcal{A}}_{sa}$, let $C^*(X)$ be the unital C^*-algebra generated by $\{f(X) : f \in BC(\mathbb{R})\}$, where $BC(\mathbb{R})$ is the space of all bounded continuous functions on $\mathbb{R}$, $W^*(X)$ be the von Neumann subalgebra of $\mathcal{A}$ generated by $C^*(X)$. Let $U|A|$ be the polar decomposition of A, $W^*(A)$ denote the von Neumann subalgebra of $\mathcal{A}$ generated by U and $W^*(|A|)$. For $X_i \in \widetilde{\mathcal{A}}$, $i \in \Lambda$, let $X_i = U_i|X_i|$ be the polar decomposition of X_i. The family $\{X_i : i \in \Lambda\}$ is said to be *free* if $\{W^*(X_t) : i \in \Lambda\}$ forms a free family. Similarly, we can define freeness with amalgamation for elements in $\widetilde{\mathcal{A}}$ (see [64]). We may view $\mathcal{H}$ as a subset of $\widetilde{\mathcal{A}}$ as given by left multiplication (defined on the dense subspace $\mathcal{A}$ of $\mathcal{H}$). Thus an unbounded random variable $X \in \widetilde{\mathcal{A}}$ is given by an element in $\mathcal{H}$ (as a left multiplication operator) if and only if the domain of X can be enlarge to contain $\mathcal{A}$. In this case, we identify X with the left multiplication operator L_x given by $x = X \cdot I$ in $\mathcal{H}$. We also use $\|X\|_2$ to denote $\|x\|$.

The distribution of element $X \in \widetilde{\mathcal{A}}_{sa}$, denoted by $\mu(X)$, is a linear functional on $BC(\mathbb{R})$, which maps function f in $BC(\mathbb{R})$ to $\tau(f(X))$. Let $A, B \in \widetilde{\mathcal{A}}_{sa}$ be freely independent elements with distributions $\mu(A)$ and $\mu(B)$, respectively. We call the distribution μ of $A + B$ the *free additive convolution* of $\mu(A)$ and $\mu(B)$, denoted by $\mu(A) \boxplus \mu(B)$. A probability measure on $\mathbb{R}$ is $\boxplus$ *(or free)-infinitely divisible*, if for any natural number n, there exists a probability measure μ_n on $\mathbb{R}$ such that

$$\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}}.$$

Let f and g be independent random variables on a probability space (Ω, Σ, μ) with distributions $\mu(f)$ and $\mu(g)$, respectively. The distribution $\mu(f + g)$ of $f + g$

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is called the *convolution* of $\mu(f)$ and $\mu(g)$, denoted by $\mu(f) * \mu(g)$. A probability measure μ on $\mathbb{R}$ is *infinitely divisible* if, for any natural number n, there is a probability measure μ_n such that

$$\mu = \underbrace{\mu_n * \cdots * \mu_n}_{n \text{ times}}.$$

We use $\mathcal{ID}(\boxplus)$ and $\mathcal{ID}(*)$ to denote the set of all $\boxplus$-infinitely divisible distributions on $\mathbb{R}$ and that of all infinitely divisible measures on $\mathbb{R}$, respectively.

***R*-transform.** Let μ be a probability measure on $\mathbb{R}$ with all moments finite, one may define the Cauchy transform of μ as follows.

$$z = G_\mu(\zeta) = \zeta^{-1} + \sum_{k=0}^{\infty} \mu_k \zeta^{-k-1},$$

where μ_k is the kth moment of μ. Let $K_\mu(z) := G_\mu^{-1}(z) = \zeta$. We say that $R_\mu(z) := K_\mu(z) - \frac{1}{z}$ is the *R-transform* of μ. It was proved in [58], [39] and [6] that a probability measure $\mu \in \mathcal{ID}(\boxplus)$ if and only if there exist a finite measure σ on $\mathbb{R}$ and a real number γ, such that

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t} \sigma(dt), z \in \mathbb{C}^+,$$

where $\phi_\mu(z) = R_\mu(1/z)$, R_μ is the R-transform of μ, and $\mathbb{C}^+$ is the the set of all complex numbers with positive real parts. We call (γ, σ) the free generating pair of μ. In classical probability, $\mu \in \mathcal{ID}(*)$ if and only if there exist a finite measure σ on $\mathbb{R}$ and a real number γ such that

$$\log f_\mu(z) = i\gamma z + \int_{\mathbb{R}} (e^{izt} - 1 - \frac{izt}{1+t^2}) \frac{1+t^2}{t^2} \sigma(dt), z \in \mathbb{R}, \tag{2.1.1}$$

where f_μ is the characteristic function of μ (see [55]). Similarly, (γ, σ) is called the generating pair of μ. The following is another representation of a measure μ in $\mathcal{ID}(*)$ (see Theorem 8.1 in [55]):

$$\log f_\mu(z) = i\gamma' z - \frac{1}{2} a z^2 + \int_{\mathbb{R}} (e^{izt} - 1 - iz\chi_{[-1,1]}(t)) v(dt), \tag{2.1.2}$$

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where γ' is a real number, $a \geq 0$ and v is a measure on $\mathbb{R}$ satisfying

$$v(\{0\}) = 0, \int_{\mathbb{R}} \min\{1, t^2\} v(dt) < \infty. \tag{2.1.3}$$

This measure v is called the Levy measure of μ. (A measure μ on $\mathbb{R}$ is a *Levy measure* if it satisfies (2.1.3)). We call (γ', A, v) the generating triple of measure μ. The relationship between measures σ and v is the following:

$$v(dt) = \frac{1+t^2}{t^2} \chi_{\mathbb{R}-\{0\}}(t)\sigma(dt). \tag{2.1.4}$$

We can define the Levy measure of $\mu \in ID(\boxplus)$ by (2.1.4). Bercovici and Pata [5] defined a bijection Λ from $\mathcal{ID}(*)$ onto $\mathcal{ID}(\boxplus)$ as follows. For $\mu \in \mathcal{ID}(*)$ with generating pair (γ, σ), Λ maps μ to be the measure in $\mathcal{ID}(\boxplus)$ with free generating pair (γ, σ).

A probability measure μ on $\mathbb{R}$ is said to be *free (or $\boxplus$) self-decomposable* if, for any $c \in (0,1)$, there exists a probability measure μ_c on $\mathbb{R}$ such that $D_c\mu \boxplus \mu_c$, where measure $D_c\mu$ is defined by $D_c\mu(B) := \mu(c^{-1}B)$, for Borel set $B \subseteq \mathbb{R}$. A sequence (σ_n) of finite measures on $\mathbb{R}$ is said to *converge weakly* to a finite measure σ on $\mathbb{R}$, denoted by $\sigma_n \xrightarrow{\text{w}} \sigma$ if, for all f in $BC(\mathbb{R})$,

$$\int_{\mathbb{R}} f(t)\sigma_n(dt) \longrightarrow \int_{\mathbb{R}} f(t)\sigma(dt),$$

as $n \to \infty$. For X_n, X in $\widetilde{\mathcal{A}}_{sa}$, $\{X_n\}_{n=1}^{\infty}$ is said to converge to X in distribution, denoted by $X_n \xrightarrow{\text{d}} X$, if $\mu(X_n) \xrightarrow{\text{w}} \mu(X)$. Given X_n, X in $\widetilde{\mathcal{A}}$, $\{X_n\}_{n=1}^{\infty}$ is said to converge to X in probability, denoted by $X_n \xrightarrow{\text{p}} X$, if $|X_n - X| \xrightarrow{\text{d}} 0$. By [4], for $X_n, X \in \widetilde{\mathcal{A}}_{sa}$, $X_n \xrightarrow{\text{p}} X$ if and only if $X_n - X \xrightarrow{\text{d}} 0$, and $X_n \xrightarrow{\text{p}} X$ implies that $X_n \xrightarrow{\text{d}} X$. For $X, Y \in \widetilde{\mathcal{A}}_{sa}$, $X \stackrel{\text{d}}{=} Y$ means X and Y have the same spectral distribution.

A family $\{S_t : t \geq 0\}$ of elements in $\widetilde{\mathcal{A}}_{sa}$ is a *free Levy process*, if $S_0 = 0$, it has free increments (that is, $S_{t_0}, S_{t_1} - S_{t_0}, \cdots, S_{t_n} - S_{t_{n-1}}$ are free, for $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$), it is stationary (that is, $\mu(S_{t+s} - S_s) = \mu(S_t)$, for $s, t \in (0, \infty)$) and $S_t \xrightarrow{\text{d}} 0$, as $t \to 0$ (see [1], [3], [4] and [9]).

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A free Levy process $\{S_t : t \geq 0\}$ is adapted to the filtration $\{\mathcal{A}_t : t \geq 0\}$ of $\mathcal{A}$, denoted by $\mathcal{A}_t$-free Levy process, if $W^*(S_t) \in \mathcal{A}_t$, for $t \geq 0$, and $S_t - S_s$ and $\mathcal{A}_s$ are free, for $0 \leq s < t$ (see [1]).

2.2 Operator Spaces

For the basics of operator spaces, we refer to [21] and [51].

Let $\mathcal{B}(\mathcal{H})$ be the space of all bounded linear operators on Hilbert space $\mathcal{H}$. For each $n \in \mathbb{N}$, there is a canonical norm $\|\cdot\|_n$ on $M_n(\mathcal{B}(\mathcal{H}))$ given by identifying $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$. We call this family of norms an *operator space matrix norm* on $\mathcal{B}(\mathcal{H})$. An *operator space* is a norm closed subspace of $\mathcal{B}(\mathcal{H})$ equipped with the operator space matrix norm inherited from $\mathcal{B}(\mathcal{H})$. The morphisms in the category of operator spaces are completely bounded linear maps. Given operator spaces V and W, a linear map $\varphi : V \to W$ is *completely bounded* if the corresponding linear maps $\varphi_n : M_n(V) \to M_n(W)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ are uniformly bounded, i. e.

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\} < \infty.$$

A map is *completely contractive* (respectively, *completely isometric*, a *complete quotient*) if $\|\varphi\|_{cb} \leq 1$ (respectively, for each n in $\mathbb{N}$, φ_n is an isometry , a quotient map).

Let M_r be the algebra of all $r \times r$ matrices over $\mathbb{C}$, E be a Banach space,

$$b_r(E) = \{T : E \to M_r : T \text{ is a linear operator and } \|T\| \leq 1\},$$

$$b(E) = \bigcup_{r \in \mathbb{N}} b_r(E).$$

With S an element in $M_n(E)$, let

$$\|S\|_{\min} = \sup\{\|f_n(S)\| : f \in b_1(E)\}, \|S\|_{\max} = \sup\{\|f_n(S)\| : f \in b(E)\}.$$

Moreover, we have

$$\|S\|_{\min} \leq \|S\| \leq \|S\|_{\max},$$

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where $\|\cdot\|$ is an operator space norm on $M_n(E)$ (see (3.3.6) in [21]). Operator space $(E, \|\cdot\|_{\min})$ (or $E, \|\cdot\|_{\max}$) is called *minimal (or maximal) quantization* of E. Proposition 3.3.1 in [21] states that an operator space $\mathcal{S}$ is minimal (i. e. $\|S\| = \|S\|_{\min}$, for any $S \in M_n(\mathcal{S}), n \in \mathbb{N}$) if and only if it is completely isometric to a subspace of a commutative C^*-algebra. Moreover, $(\max E)^* = \min E^*$ (see (3.3.13) in [21]).

Effros and Ruan [21] showed that an operator space $\mathcal{S}$ is the dual space of a Banach space (in this case, $\mathcal{S}$ is called a *dual operator space*) if and only if there are a Hilbert space $\mathcal{H}$ and a W^*-homeomorphic and completely isometric map φ from $\mathcal{S}$ into $\mathcal{B}(\mathcal{H})$. Take a unit vector $\xi \in \mathcal{H}$ as a distinguished vector, we call $(\varphi, \mathcal{H}, \xi)$ a *realization* of $\mathcal{S}$.

Recall that an operator space $\mathcal{S}$ is *injective*, if for operator spaces $\mathcal{W}_0 \subseteq \mathcal{W}$, each completely bounded linear map $\varphi_0 : \mathcal{W}_0 \to \mathcal{S}$ has a completely bounded linear extension $\varphi : \mathcal{W} \to \mathcal{S}$ satisfying $\|\varphi\|_{cb} = \|\varphi_0\|_{cb}$ (see [21]). A von Neumann algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is *injective* if there is a conditional expectation Π of $B(\mathcal{H})$ onto $\mathcal{A}$. Given an operator space $\mathcal{S}$, an injective operator space $\mathcal{W}$ and a completely isometry κ of $\mathcal{S}$ into $\mathcal{W}$, we say that $(\mathcal{W}, \kappa)$ is an *injective envelope* of $\mathcal{S}$ if there is not an injective proper subspace of $\mathcal{W}$, which contains $\kappa(\mathcal{S})$ (see Chapter 15 in [48]).

2.3 Classical Markov Processes

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\{f_t : t \geq 0\}$ a family of measurable functions from $(\Omega, \mathcal{F}, \mu)$ into a locally compact Hausdorff space X with a Borel σ-algebra $\mathcal{B}$. Define $\mathcal{F}_{\leq t}$ to be the σ-subalgebra of $\mathcal{F}$ generated by $f_s^{-1}(B)$ for all Borel subsets B of X and $s \leq t$. Similarly, one may define $\mathcal{F}_{=t}$ and $\mathcal{F}_{\geq t}$. The family $\{f_t : t \geq 0\}$ is a *Markov process* if

$$P(AB|\mathcal{F}_{=t}) = P(A|\mathcal{F}_{=t})P(B|\mathcal{F}_{=t}),$$

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for all A in $F_{\leq t}$, B in $\mathcal{F}_{\geq t}$, where $P(\cdot|\mathcal{F}_{=t})$ is the conditional probability with respect to $\mathcal{F}_{=t}$. Given $s \leq t$, $x \in X$ and Borel subset $\Gamma \subseteq X$, we can define a transition function $P(s,x,t,\Gamma) = P(f_t \in \Gamma | f_s = x)$. Then $\{f_t : t \geq 0\}$ is a Markov process if and only if $P(s,x,t,\Gamma)$ has the following properties(see 8.1.3 and 8.2.3 in [65]).

1. When s,t,x are given, $P(s,x,t,\cdot)$ is a probability measure on $\mathcal{B}$;

2. when s,t,Γ are given, $P(s,\cdot,t,\Gamma)$ is a measurable function on $(\mathbb{R},\mathcal{B})$;

3. $P(s,x,s,\Gamma) = \chi_\Gamma(x)$.

2.4 Operator-Valued Lipschitz functions

A map $Q : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ is called *Lipschitz* (or *operator-valued Lipschitz*) with respect to $\|\cdot\|_2$, if there exists a constant $C > 0$ such that

$$\|Q(X_1,\cdots,X_k) - Q(Y_1,\cdots,Y_k)\|_2 \leq C\sum_{i=1}^{k}\|X_i - Y_i\|_2 \tag{2.4.1}$$

for all operators $X_1, Y_1, \cdots, X_k, Y_k$ in $\mathcal{A}_{sa}$. A map $Q : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ is *locally Lipschitz* (or *locally operator-valued Lipschitz*) with respect to $\|\cdot\|_2$, if for all $M > 0$ there exists constants $C_M > 0$ such that (2.4.1) holds for all X_i, Y_i in $\mathcal{A}_{sa}$ with $\|X_i\|_2$ and $\|Y_i\|_2$ less than M, $1 \leq i \leq k$. Similar definitions of (locally) Lipschitz maps with respect to operator norm can be found in Section 2.3 in [11].

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Chapter 3

CERTAIN FREE PRODUCT OF OPERATOR SPACES

We study certain free product of operator spaces in this chapter.

This chapter is organized as follows. We give the definition of certain (reduced) free product for operator spaces in Section 3.1 (Definition 3.1.2). We show that the (reduced) free product satisfies Effros's requirement on the free product of preduals of von Neumann algebras acting on separable Hilbert spaces (Theorem 3.1.4). Section 3.2 is devoted to the study of the properties of the (reduced) free product of operator spaces. It is proved that the free product of two operator spaces does not have an operator space local lifting property, even if the two operator spaces have the operator space local lifting property (Theorem 3.2.4). On the other hand, operator spaces with the operator space local lifting property have certain property of completely isometrically embedding into their free product (Theorem 3.2.5 and Corollary 3.2.6). Finally, in Section 3.3, an example is presented to show that the C^*-algebra reduced free product of two C^*-algebras may be contractively isomorphic to a proper subspace of the operator space (reduced) free product of the two C^*-algebras (Theorem 3.3.1).

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3.1 The Definition of the Free Product

In this section, we give the definition of certain (reduced) free product for operator spaces and show that the free product satisfies Effros's requirement on the free product of preduals of von Neumann algebras.

Let $\mathcal{T}(\mathcal{H})$ be the space of all trace class operators on Hilbert space $\mathcal{H}$. $\mathcal{H}_1 \tilde{\otimes} \mathcal{H}_2$ denotes the algebraic tensor product of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\mathcal{H}_1 \otimes \mathcal{H}_2$ the completion of $\mathcal{H}_1 \tilde{\otimes} \mathcal{H}_2$. For $\sum_{i=1}^n x_i \otimes y_i \in \mathcal{H} \tilde{\otimes} \mathcal{H}$ and $z \in \mathcal{H}$, define $\sum_{i=1}^n x_i \otimes y_i(z) = \sum_{i=1}^n \langle z, y_i \rangle x_i$. Thus, $\sum_{i=1}^n x_i \otimes y_i$ gives rise to a finite rank operator and thus is of trace class. Hence, $\mathcal{H} \tilde{\otimes} \mathcal{H}$ may be viewed as a dense subspace of $\mathcal{T}(\mathcal{H})$. On the other hand, $\mathcal{T}(\mathcal{H})$ is the predual of $\mathcal{B}(\mathcal{H})$. Hence, we may identify $x = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{H} \tilde{\otimes} \mathcal{H}$ with the linear functional $\sum_{i=1}^n w_{x_i, y_i}$, where w_{x_i, y_i} is the vector state on $\mathcal{B}(\mathcal{H})$ corresponding to vectors x_i, y_i in $\mathcal{H}$.

Given an operator space $\mathcal{S}$, Proposition 4.2.3 in [21] states that there are a Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_i : i \in \Lambda\}$ and a completely quotient map $\varphi : \mathcal{T}(\mathcal{H}) \to \mathcal{S}$. Thus, $\varphi : \mathcal{H} \tilde{\otimes} \mathcal{H} \mapsto \mathcal{S}$ has dense image and

$$|\varphi(x \otimes y)| \leq \|x\| \cdot \|y\|,$$

for x, y in $\mathcal{H}$. Moreover, suppose $\mathcal{S}$ is separable. Let $[e_i \otimes e_j] = \varphi(e_i \otimes e_j)$ be the image of $e_i \otimes e_j$ in $\mathcal{S}$, for all i, j in Λ. We may choose a countable subset Λ_0 of Λ such that $\{[e_i \otimes e_j] : i, j \in \Lambda_0\}$ spans a dense subspace of $\mathcal{S}$. Let $\mathcal{H}_0$ be the separable Hilbert space spanned by $\{e_i : i \in \Lambda_0\}$. Then $\varphi(\mathcal{H}_0 \tilde{\otimes} \mathcal{H}_0)$ is dense in $\mathcal{S}$.

From the above discussions, we have

Proposition 3.1.1. *Let $\mathcal{S}$ be an operator space. Then there are a Hilbert space $\mathcal{H}$ and a linear map $\Psi : \mathcal{H} \tilde{\otimes} \mathcal{H} \to \mathcal{S}$ such that the image of Ψ is dense in $\mathcal{S}$ and $\|\Psi(x \otimes y)\| \leq \|x\| \cdot \|y\|$, for all x, y in $\mathcal{H}$. Moreover, we may choose $\mathcal{H}$ separable, when $\mathcal{S}$ is separable.*

By Proposition 4.2.3 in [21], φ is a completely quotient map and $\varphi^* : \mathcal{S}^* \to B(\mathcal{H})$ is a dual representation of $\mathcal{S}^*$ (that is, φ is a weak *-continuous and complete

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isometry from $\mathcal{S}^*$ into $\mathcal{B}(\mathcal{H})$). We use $(\mathcal{S}, \varphi, \mathcal{H}, \xi)$ to denote a tuple of an operator space $\mathcal{S}$ and a dual representation φ of the dual space $\mathcal{S}^*$ on a Hilbert space $\mathcal{H}$ with a distinguished vector ξ, or $(\mathcal{S}, \mathcal{H}, \xi)$ to denote the tuple, when $\mathcal{S}^*$ is viewed as a subspace of $\mathcal{B}(\mathcal{H})$.

Let $(\mathcal{S}_1, \mathcal{H}_1, \xi_1)$ and $(\mathcal{S}_2, \mathcal{H}_2, \xi_2)$ be such two tuples. Define

$$\mathcal{W} = ((\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2))\widetilde{\otimes}((\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)).$$

Let $\mathcal{M}$ be the subspace of $\mathcal{W}$ of all elements $\sum_{i=1}^n x_i \otimes y_i \in \mathcal{W}$ such that

$$\sum_{i=1}^n \langle \lambda_{i_1}(T_1) \cdots \lambda_{i_m}(T_m) x_i, y_i \rangle = 0, \forall T_j \in \mathcal{S}_{i_j}^*, j = 1, 2, \cdots m, i_1 \neq \cdots \neq i_m,$$

where $i_j \in \{1, 2\}, m \in \mathbb{N}$, if $m > 0$; $\lambda_{i_1}(T_1) \cdots \lambda_{i_m}(T_m) = I$, if $m = 0$, and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}$, where $(\mathcal{H}, \xi) = (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$. We define a functional on $\mathcal{W}/\mathcal{M}$ as follows. For $[\sum_{i=1}^n x_i \otimes y_i] \in \mathcal{W}/\mathcal{M}$, we define

$$\|[\sum_{i=1}^n x_i \otimes y_i]\| := \sup\{|\sum_{i=1}^n \langle T x_i, y_i \rangle| : T \in \mathcal{S}_1^* \odot \mathcal{S}_2^*, \|T\| \leq 1\},$$

where $\mathcal{S}_1^* \odot \mathcal{S}_2^*$ is the ultra-weak operator topology closure of the linear span $\mathcal{L}$ of

$$\{\lambda_{i_1}(T_1) \cdots \lambda_{i_m}(T_m) : T_j \in \mathcal{S}_{i_j}^*, j = 1, 2, \cdots m, i_1 \neq \cdots \neq i_m, i_1, \cdots, i_m \in \{1, 2\}\},$$

where $m = 0, 1, 2, \cdots$. First, we should verify that the functional $\|\cdot\|$ is well defined. Suppose $[\sum_{i=1}^n x_i \otimes y_i] \in \mathcal{W}/\mathcal{M}$ is zero, then, by definition,

$$T([\sum_{i=1}^n x_i \otimes y_i]) = \sum_{i=1}^n \langle T x_i, y_i \rangle = 0,$$

for all $T \in \mathcal{L}$. Therefore, $T([\sum_{i=1}^n x_i \otimes y_i]) = 0$, for all $T \in \mathcal{S}_1^* \odot \mathcal{S}_2^*$, since $\sum_{i=1}^n x_i \otimes y_i$ is continuous with respect to ultra-weak operator topology and $\mathcal{L}$ is dense in $\mathcal{S}_1^* \odot \mathcal{S}_2^*$ with respect to this topology. Thus, $\|[\sum_{i=1}^n x_i \otimes y_i]\| = 0$. Moreover, it is easy to verify that $\|\cdot\|$ is a norm on $\mathcal{W}/\mathcal{M}$.

Since $\mathcal{S}_1^* \odot \mathcal{S}_2^*$ is a norm closed subspace of $B(\mathcal{H})$, where $\mathcal{H}$ is the free product of $\mathcal{H}_1$ and $\mathcal{H}_2$, and the completion of $\mathcal{W}/\mathcal{M}$ with respect to the norm $\|\cdot\|$ defined

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above is a norm closed subspace of the dual space of $\mathcal{S}_1^* \odot \mathcal{S}_2^*$, the completion has the operator space structure induced from $(\mathcal{S}_1^* \odot \mathcal{S}_2^*)^*$.

Now we are in a position to give the definition of the (reduced) free product of operator spaces.

Definition 3.1.2. *Let $(\mathcal{S}_1, \mathcal{H}_1, \xi_1)$ and $(\mathcal{S}_2, \mathcal{H}_2, \xi_2)$ be two tuples. The completion of $\mathcal{W}/\mathcal{M}$ with respect to the norm $\|\cdot\|$ defined above with the operator space structure described before the definition is called the (reduced) free product of operator spaces $\mathcal{S}_1$ and $\mathcal{S}_2$, denoted by $(\mathcal{S}_1, \mathcal{H}_1, \xi_1) * (\mathcal{S}_2, \mathcal{H}_2, \xi_2)$, briefly, by $\mathcal{S}_1 * \mathcal{S}_2$.*

The following result shows that we can answer the question affirmatively asked by E. Effros by the (reduced) free product introduced above.

Theorem 3.1.3. *Let $(\mathcal{R}_1, \omega_1, \mathcal{H}_1, \xi_1)$ and $(\mathcal{R}_2, \omega_2, \mathcal{H}_2, \xi_2)$ be two W^*-tuple, and $(\mathcal{R}_1)_*$ and $(\mathcal{R}_2)_*$ be the preduals of von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Then, as operator spaces,*

$$((\mathcal{R}_1)_*, \mathcal{H}_1, \xi_1) * ((\mathcal{R}_2)_*, \mathcal{H}_2, \xi_2) = ((\mathcal{R}_1, \omega_1, \mathcal{H}_1, \xi_1) * (\mathcal{R}_2, \omega_2, \mathcal{H}_2, \xi_2))_*.$$

Proof. Let $(\mathcal{H}, \xi)$ be the free product of $(\mathcal{H}_1, \xi_1)$ and $(\mathcal{H}_2, \xi_2)$,

$$\mathcal{T} = \{\sum_{i=1}^{\infty} w_{x_i, y_i} : x_i, y_i \in \mathcal{H}, \sum_{n=1}^{\infty}(\|x_i\|^2 + \|y_i\|^2) < \infty\}.$$

Let

$$\mathcal{R}^{\perp} = \{\sum_{i=1}^{\infty} w_{x_i, y_i} \in \mathcal{T} : \sum_{i=1}^{\infty} w_{x_i, y_i}(A) = 0, \forall A \in \mathcal{R}\},$$

where $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ is the reduced free product of von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$. Then $(\mathcal{R}_1 * \mathcal{R}_2)_* = \mathcal{T}/\mathcal{R}^{\perp}$ (see Section 7.4 in [34]). Now we prove that the reduced free product of $(\mathcal{R}_1)_*$ and $(\mathcal{R}_2)_*$ is $\mathcal{T}/\mathcal{R}^{\perp}$. Recall that $\mathcal{L}$ in the definition of free product of operator spaces is the linear span of

$$\{\lambda_{i_1}(A_1) \cdots \lambda_{i_m}(A_m) : A_j \in \mathcal{R}_{i_j}, j = 1, 2, \cdots, m, i_1 \neq \cdots \neq i_m, i_1, \cdots, i_m \in \{1, 2\}\},$$

where $m \in \mathbb{N}$. Note that $\mathcal{R}_1$ and $\mathcal{R}_2$ are unital *-algebras and λ_i is *-isomorphism, for $i = 1, 2$. Thus, $\mathcal{L}$ is a unital *-sub-algebra of $B(\mathcal{H})$. Hence, $\mathcal{R}_1 \odot \mathcal{R}_2$ in

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the construction of free products of operator spaces is the von Neumann algebra generated by $\lambda_1(\mathcal{R}_1)$ and $\lambda_2(\mathcal{R}_2)$, that is, $\mathcal{R}_1 \odot \mathcal{R}_2 = \mathcal{R}_1 * \mathcal{R}_2$. It follows that

$$(\mathcal{R}_1)_* * (\mathcal{R}_2)_* \subseteq \mathcal{T}/\mathcal{R}^{\perp}.$$

On the other hand, for $x = \sum_{i=1}^{\infty} w_{x_i,y_i} \in \mathcal{T}$, let $[\sum_{i=1}^{\infty} w_{x_i,y_i}]$ be the image of x in $\in \mathcal{T}/\mathcal{R}^{\perp} = (\mathcal{R}_1 * \mathcal{R}_2)_*$ and $n \in \mathbb{N}$, we have $[\sum_{i=1}^{n} w_{x_i,y_i}] \in (\mathcal{R}_1)_* * (\mathcal{R}_2)_*$ and

$$\begin{aligned} &|\sum_{i=1}^{\infty} w_{x_i,y_i}(A) - \sum_{i=1}^{n} w_{x_i,y_i}(A)| = |\sum_{i=n+1}^{\infty} w_{x_i,y_i}(A)| \\ \leq &\sum_{i=n+1}^{\infty} \|x_i\| \cdot \|y_i\| \leq \sum_{i=n+1}^{\infty} (\|x_i\|^2 + \|y_i\|^2) \\ \rightarrow &0, \end{aligned}$$

as $n \rightarrow \infty, \forall A \in \mathcal{R}_1 * \mathcal{R}_2, ||A|| \leq 1$. It implies that $[\sum_{i=1}^{\infty} w_{x_i,y_i}] \in (\mathcal{R}_1)_* * (\mathcal{R}_2)_*$.

By our Definition 3.1.2, $(\mathcal{R}_1)_* * (\mathcal{R}_2)_*$ has operator space structure induced from $(\mathcal{R}_1 * \mathcal{R}_2)^*$. Let $\mathcal{A}$ be a von Neumann algebra on Hilbert space $\mathcal{H}$ with predual $\mathcal{A}_*$. $\mathcal{A}_*$ has operator space structure by identifying $\mathcal{A}_* \cong \mathcal{T}(\mathcal{H})/\mathcal{A}^{\perp}$. By Proposition 4.2.2 in [21], $\mathcal{A}$ is the operator space dual of $\mathcal{A}_*$. Furthermore, by Proposition 3.2.1 in [21], the canonical inclusion $\mathcal{A}_* \hookrightarrow \mathcal{A}^*$ is completely isometric. It implies that $\mathcal{A}_*$ has operator space structure induced from $\mathcal{A}^*$. Hence, $(\mathcal{R}_1)_* * (\mathcal{R}_2)_*$ and $(\mathcal{R}_1 * \mathcal{R}_2)_*$ have the same operator space structure. □

Remark 3.1.4. *Effros asked about the existence of the free product of the preduals of von Neumann algebras of type II_1, but our result above answers the question affirmatively for general von Neumann algebras acting on separable Hilbert spaces.*

3.2 Local Lifting Property of the Free Product

In this section, we define the *freeness* for a family of subspaces of a dual operator space. We show that the (reduced) free product of operator spaces does not preserve the local lifting property (Theorem 3.2.4). It is proved that operator spaces

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with operator space local lifting property have certain property of completely isometrically embedding into their free product (Theorem 3.2.5 and Corollary 3.2.6).

We have known that the space $\mathcal{S}_1^* \odot \mathcal{S}_2^*$ in the construction of the (reduced) free product of operator spaces is a natural generalization of free product of von Neumann algebras. As a natural generalization of freeness of von Neumann subalgebras in a von Neumann algebra (see Section 2.1), we shall define *freeness* for subspaces of a dual operator space.

Definition 3.2.1. *Let $\{\mathcal{S}_\lambda : \lambda \in \Lambda\}$ be a family of W^*-closed subspaces of a dual operator spaces $\mathcal{S}$ acting on Hilbert spaces $\mathcal{H}$ with distinguished vectors ξ. We say $\mathcal{S}_\lambda$, $\lambda \in \Lambda$, are free in $\mathcal{S}$ with respect to vector state ω_ξ if*

1. $T_{i_1}T_{i_2}\cdots T_{i_n} \in \mathcal{S}, \forall T_{i_j} \in \mathcal{S}_{i_j}, i_j \in \Lambda, i_1 \neq i_2 \neq \cdots \neq i_n, n$ *in* $\mathbb{N}$;
2. $\omega_\xi(T_{i_1}T_{i_2}\cdots T_{i_n}) = 0,$ *if* $\omega_\xi(T_{i_j}) = 0, T_{i_j} \in \mathcal{S}_{i_j}, i_j \in \Lambda, i_1 \neq i_2 \neq \cdots \neq i_n, n$ *in* $\mathbb{N}$.

Generally, we say that a family $\{A_\lambda : A_\lambda \in \mathcal{S}, \lambda \in \Lambda\}$ is free in $\mathcal{S}$ if the W^*-closed subspaces $\mathcal{S}_\lambda$ generated by A_λ, λ in Λ, are free in $\mathcal{S}$.

Remark 3.2.2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be non-zero operator spaces. Let $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ be the dual operator spaces of $\mathcal{S}_1$ and $\mathcal{S}_2$ acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with distinguished vectors ξ_1 and ξ_2, respectively. Then $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ are free in $\mathcal{S}_1^* \odot \mathcal{S}_2^*$ with respect to vector state ω_ξ.*

Proposition 3.2 in [19] states that a dual operator space $\mathcal{S}$ is injective if and only if there are an injective von Neumann algebra $\mathcal{R}$, a projection $P \in \mathcal{R}$, and a linearly completely isometric and W^*-homeomorphic map $\varphi : \mathcal{S} \to P\mathcal{R}(I-P)$. Now we prove that, for given injective dual operator spaces $\mathcal{S}_1$ and $\mathcal{S}_2$, there exist realizations $(\varphi_1, \mathcal{H}_1, \xi_1)$ and $(\varphi_2, \mathcal{H}_2, \xi_2)$ of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, such that $\mathcal{S}_1 \odot \mathcal{S}_2$ is not injective.

Theorem 3.2.3. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be non-zero injective dual operator spaces. Then $\mathcal{S}_1 \odot \mathcal{S}_2$ is not injective for some realizations $(\varphi_1, \mathcal{H}_1, \xi_1)$ and $(\varphi_2, \mathcal{H}_2, \xi_2)$ of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively.*

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Proof. By Proposition 3.2 in [19], there are $\mathcal{H}_1$, $\mathcal{H}_2$, injective von Neumann algebras $\mathcal{R}_1 \subseteq \mathcal{B}(\mathcal{H}_1)$, $\mathcal{R}_2 \subseteq \mathcal{B}(\mathcal{H}_2)$, and projections $P_i \in \mathcal{R}_i, i = 1,2$, such that $\mathcal{S}_1 \cong P_1\mathcal{R}_1(I_1 - P_1), \mathcal{S}_2 \cong P_2\mathcal{R}_2(I_2 - P_2)$, where $\cong$ means "linearly completely isometric and W^*-homeomorphic". So, we can assume that $\mathcal{S}_1 = P_1\mathcal{R}_1(I_1 - P_1), \mathcal{S}_2 = P_2\mathcal{R}_2(I_2 - P_2)$. Take unit vectors $\xi_1 \in \mathbf{Im}(P_1)$ and $\xi_2 \in \mathbf{Im}(P_2)$ such that there is an operator $T_i \in \mathcal{R}_i$ satisfying $(I_i - P_i)T_iP_i\xi_i \neq 0, i = 1,2$, where $\mathbf{Im}(P)$ is the image of operator P. Let ξ_1 and ξ_2 be the distinguished vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then, $X_i\xi_i = 0$, for X_i in $\mathcal{S}_i, i = 1,2$. By the definition of reduced free product of von Neumann algebras (see [64]), there are natural representations $\lambda_i : B(\mathcal{H}_i) \to B(\mathcal{H})$, and $\lambda_i : \mathcal{R}_i \to \lambda_i(\mathcal{R}_i)$ is normal *-isomorphism, $i = 1,2$. Therefore, we can assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are injective von Neumann algebras in $\mathcal{B}(\mathcal{H})$, where $(\mathcal{H},\xi)$ is $(\mathcal{H}_1,\xi_1) * (\mathcal{H}_2,\xi_2)$, and $\mathcal{S}_1 = P_1\mathcal{R}_1(I_1 - P_1)$ and $\mathcal{S}_2 = P_2\mathcal{R}_2(I_2 - P_2)$ are free in $B(\mathcal{H})$. Thus, $\mathcal{S}_1 \odot \mathcal{S}_2$ is the W^*-closure of the linear span of

$$\{\alpha I : \alpha \in \mathbb{C}\} \cup \{P_{i_1}T_{i_1}(I - P_{i_1}) \cdots P_{i_m}T_{i_m}(I - P_{i_m}) : T_{i_j} \in \mathcal{R}_{i_j}, i_1 \neq \cdots \neq i_m,\},$$

where $i_j \in \{1,2\}$, $m \in \mathbb{N}$. It is easy to see that $\mathcal{S}_1 \odot \mathcal{S}_2$ is a unital operator sub-algebra in $B(\mathcal{H})$. Proposition 15.15 in [48] states that the injective envelope $\mathcal{I}(\mathcal{S}_1 \odot \mathcal{S}_2)$ of $\mathcal{S}_1 \odot \mathcal{S}_2$ is a C*-algebra and the natural inclusion from $\mathcal{S}_1 \odot \mathcal{S}_2$ into its envelope $\mathcal{I}(\mathcal{S}_1 \odot \mathcal{S}_2)$ is a complete isometry. Moreover,

$$\mathcal{I}(\mathcal{S}_1 \odot \mathcal{S}_2) = \mathcal{I}(\mathcal{S}_1 \odot \mathcal{S}_2 + (\mathcal{S}_1 \odot \mathcal{S}_2)^*),$$

where $(\mathcal{S}_1 \odot \mathcal{S}_2)^* = \{X^* : X \in \mathcal{S}_1 \odot \mathcal{S}_2\}$. Let $\Phi : \mathcal{S}_1 \odot \mathcal{S}_2 + (\mathcal{S}_1 \odot \mathcal{S}_2)^* \to \mathcal{I}(\mathcal{S}_1 \odot \mathcal{S}_2)$ be the natural inclusion. If $\mathcal{S}_1 \odot \mathcal{S}_2$ is injective, we have

$$\Phi(\mathcal{S}_1 \odot \mathcal{S}_2) = \Phi(\mathcal{S}_1 \odot \mathcal{S}_2 + (\mathcal{S}_1 \odot \mathcal{S}_2)^*) = \mathcal{I}(\mathcal{S}_1 \odot \mathcal{S}_2).$$

Therefore, $\mathcal{S}_1 \odot \mathcal{S}_2 = \mathcal{S}_1 \odot \mathcal{S}_2 + (\mathcal{S}_1 \odot \mathcal{S}_2)^*$, since Φ is injective. Thus, $\mathcal{S}_1 \odot \mathcal{S}_2$ is self-adjoint. On the other hand, we shall prove that $\mathcal{S}_1 \odot \mathcal{S}_2$ is not self-adjoint. In fact, we have known that $(I_1 - P_1)T_1P_1\xi_1 \neq 0$. So, $(I - P_1)T_1P_1\xi \neq 0$, where ξ is

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the distinguished vector in $\mathcal{H} = \mathcal{H}_1 * \mathcal{H}_2$. On the other hand, $P_1S_1(I - P_1)\xi = 0$, for S_1 in $\mathcal{R}_1$. Moreover,

$$P_2S_2(I - P_2)\xi = P_2S_2(I - P_2)(\xi_2 \otimes \xi) = 0, \forall S_2 \in \mathcal{R}_2.$$

Therefore, if $\mathcal{S}_1 \odot \mathcal{S}_2$ is self-adjoint, $(I_1 - P_1)T_1P_1 \in \mathcal{S}_1 \odot \mathcal{S}_2$. Thus, there is a net $\{X_\alpha\} \subset \mathcal{L}$ such that

$$X_\alpha \overset{w*}{\rightarrow} (I_1 - P_1)T_1P_1.$$

Then we have

$$\langle (I_1 - P_1)T_1P_1\xi, \eta\rangle = \lim_\alpha \langle X_\alpha \xi, \eta\rangle, \forall \eta \in \mathcal{H}. \tag{3.2.1}$$

By the discussion above, $X_\alpha \in \mathbb{C}I$ (otherwise, the right side of (3.2.1) is zero, so $(I_1 - P_1)T_1P_1\xi = 0$, which contradicts the choice of ξ). Thus, $0 \neq (I_1 - P_1)T_1P_1$ is in $\mathbb{C}I$. This is impossible. Therefore, $\mathcal{S} \odot \mathcal{S}_2$ is not injective. □

Recall that an operator space $\mathcal{S}$ is said to have the *operator space local lifting property* if for given an operator space Y, a closed subspace M of Y, $q : Y \rightarrow Y/M$ the quotient map, a complete contraction $\varphi : \mathcal{S} \rightarrow Y/M$, and each finite dimensional subspace $E \subseteq \mathcal{S}$ and $\varepsilon > 0$, there is a mapping $\varphi' : E \rightarrow Y$ such that $||\varphi'||_{cb} < 1+\varepsilon$ and $q \circ \varphi' = \varphi|_E$ (see [4] and [14]). Now we show that $\mathcal{S}_1 * \mathcal{S}_2$ does not have this property for some representations $(\varphi_1, \mathcal{H}_1, \xi_1)$ and $(\varphi_2, \mathcal{H}_2, \xi_2)$ of $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$, respectively, even if both $\mathcal{S}_1$ and $\mathcal{S}_2$ have this property.

Theorem 3.2.4. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be non-zero operator spaces with operator space local lifting property. Then there are realizations $(\varphi_1, \mathcal{H}_1, \xi_1)$ and $(\varphi_2, \mathcal{H}_2, \xi_2)$ of $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$, respectively, such that $(\mathcal{S}_1, \varphi_1, \mathcal{H}_1, \xi_1) * (\mathcal{S}_2, \varphi_2, \mathcal{H}_2, \xi_2)$ does not have operator space local lifting property.*

Proof. Proposition 3.2 in [19] states that an operator space has locally lifting property if and only if its dual space is injective. Hence, $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ are injective dual operator spaces. By Theorem 3.2.3, $\mathcal{S}_1^* \odot \mathcal{S}_2^*$ is not injective, for some representations $(\varphi_1, \mathcal{H}_1, \xi_1)$ and $(\varphi_2, \mathcal{H}_2, \xi_2)$. Now we show that $(\mathcal{S}_1 * \mathcal{S}_2)^* = \mathcal{S}_1^* \odot \mathcal{S}_2^*$. In fact,

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the predual $(\mathcal{S}_1^* \odot \mathcal{S}_2^*)_*$ of $\mathcal{S}_1^* \odot \mathcal{S}_2^*$ is $\mathcal{T}(\mathcal{H})/\mathcal{N}$, where

$$\mathcal{N} = \{T \in \mathcal{T}(\mathcal{H}) : tr(ST) = 0, \forall S \in \mathcal{S}_1^* \odot \mathcal{S}_2^*\},$$

by Proposition 4.2.2 in [21]. Note that

$$\mathcal{W} = ((\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2))\widetilde{\otimes}((\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)) \subset \mathcal{T}(\mathcal{H}), \mathcal{M} = \mathcal{N} \cap \mathcal{W},$$

there is an isometric injection $\psi : \mathcal{S}_1 * \mathcal{S}_2 \hookrightarrow (\mathcal{S}_1^* \odot \mathcal{S}_2^*)_*$. On the other hand, $\mathcal{W}$ is dense in $\mathcal{T}(\mathcal{H})$ with respect to the trace norm, therefore, $\mathcal{W}/\mathcal{M}$ is dense in $\mathcal{T}(\mathcal{H})/\mathcal{N}$. Hence, $\mathcal{S}_1 * \mathcal{S}_2 = \mathcal{T}(\mathcal{H})/\mathcal{N} = (\mathcal{S}_1^* \odot \mathcal{S}_2^*)_*$, which does not have the operator space local lifting property. □

Given unital C^*-algebras (or von Neumann algebras) $\mathcal{A}_1$ and $\mathcal{A}_2$, it is well known that there is a *-isomorphism from $\mathcal{A}_1$ (or $\mathcal{A}_2$) into the free product $\mathcal{A}_1 * \mathcal{A}_2$ (see [64]). Now we show that the free product of operator spaces has a similar property under certain conditions.

Theorem 3.2.5. *Let $(\mathcal{S}, \mathcal{H}_1, \xi_1)$ and $(\mathcal{S}_2, \mathcal{H}_2, \xi_2)$ be two tuples. Given $j = 1$ or 2, let $\mathcal{S}_j^0$ be the closure of $V_j = \{\varphi_j(x) : x = \sum_{i=1}^n x_i \otimes y_i \in \mathcal{H}_j^0 \widetilde{\otimes} \mathcal{H}_j^0, \sum_{i=1}^n \langle x_i, y_i \rangle = 0\}$ in $\mathcal{S}_j$, where $\varphi_j : \mathcal{H}_j \widetilde{\otimes} \mathcal{H}_j \to \mathcal{S}_j$ be the completely quotient map (see Proposition 3.1.1). If both $\mathcal{S}_1$ and $\mathcal{S}_2$ are non-zero operator spaces and $T_i \xi_i = 0$, for all T_i in $\mathcal{S}_i^*$ and $i = 1, 2$, then there is a complete isometry Φ_1 (or Φ_2) from $\mathcal{S}_1^0$ (or $\mathcal{S}_2^0$) into $\mathcal{S}_1 * \mathcal{S}_2$.*

Proof. Without loss of generality, we prove only that there is a complete isometry from $\mathcal{S}_1^0$ into $\mathcal{S}_1 * \mathcal{S}_2$. Define $\Phi_1 : V_1 \to \mathcal{S}_1 * \mathcal{S}_2$ as follows. For $\varphi_1(\sum_{i=1}^n x_i \otimes y_i) \in V_1$, we define $\Phi_1(\varphi_1(\sum_{i=1}^n x_i \otimes y_i)) = [\sum_{i=1}^n x_i \otimes y_i] \in \mathcal{S}_1 * \mathcal{S}_2$, where $[x]$ is the image of $x \in \mathcal{H}_1^0 \widetilde{\otimes} \mathcal{H}_1^0$ in $\mathcal{W}/\mathcal{M}$. We first show that this map is well-defined and one to one. It is obvious that it is one to one, i. e., $\varphi_1(\sum_{i=1}^n x_i \otimes y_i) = 0$ in $\mathcal{S}_1$, if $[\sum_{i=1}^n x_i \otimes y_i] = 0$ in $\mathcal{S}_1 * \mathcal{S}_2$. Conversely, suppose $\varphi_1(\sum_{i=1}^n x_i \otimes y_i) = 0$ in $\mathcal{S}_1$, now we show that $[\sum_{i=1}^n x_i \otimes y_i] = 0$ in $\mathcal{S}_1 * \mathcal{S}_2$. In fact, for $\sum_{i=1}^n x_i \otimes y_i \in \mathcal{H}_1^0 \otimes \mathcal{H}_1^0$, and $T_j \in \mathcal{S}_{i_j}^*$, $i_1 \neq i_2 \neq \cdots \neq i_m$, $i_j \in \{1, 2\}$, we have

$$\sum_{i=1}^n \langle \lambda_{i_1}(T_1) \cdots \lambda_{i_m}(T_m) x_i, y_i \rangle$$

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$$
\begin{aligned}
&= \sum_{i=1}^{n} \langle \lambda_{i_1}(T_1) \cdots \lambda_{i_{m-1}}(T_{m-1})(T_m x_i)^0, y \rangle \\
&+ \sum_{i=1}^{n} \langle T_m x_i, \xi_{i_m} \rangle \langle \lambda_{i_1}(T_1) \cdots \lambda_{i_{m-1}}(T_{m-1})\xi, y_i \rangle \\
&= 0,
\end{aligned}
$$

if $i_m = 1$, $m > 1$ and $m \in \mathbb{N}$ (since $T_i \xi_i = 0$, for all $T_i \in \mathcal{S}_i^*$, $i = 1, 2$), where $(Tx)^0 = Tx - \langle Tx, \xi_i \rangle \xi_i$, for $T \in \mathcal{S}_i^* \subseteq \mathcal{B}(\mathcal{H}_i)$, $x \in \mathcal{H}_i$, ξ_i is the distinguished vector and $i = 1, 2$. Similarly, we have

$$
\sum_{i=1}^{n} \langle \lambda_{i_1}(T_1) \cdots \lambda_{i_m}(T_m) x_i, y_i \rangle = 0,
$$

if $i_m \neq 1$ and $m \geq 1, m \in \mathbb{N}$. It follows that $\sum_{i=1}^{n} \langle T x_i, y_i \rangle = 0$, for all $X \in \mathcal{L}$, where $\mathcal{L}$ is the linear span of

$$
\mathbb{C}I \cup \{\lambda_{i_1}(T_1) \cdots \lambda_{i_m}(T_m) : T_j \in \mathcal{S}_{i_j}, i_1 \neq \cdots \neq i_m, i_j \in \{1, 2\}, m = 1, 2, \cdots\}.
$$

Hence, $\sum_{i=1}^{n} \langle T x_i, y_i \rangle = 0$, for all $T \in \mathcal{S}_1^* \odot \mathcal{S}_2^*$, the weak*-closure of $\mathcal{L}$, since $\sum_{i=1}^{n} x_i \otimes y_i$, as a linear functional on $\mathcal{L}$, is weak*-continuous. Hence, the map Ψ of V_1 into $\mathcal{S}_1 * \mathcal{S}_2$ is well-defined. Moreover, for $x = \varphi_1(\sum_{i=1}^{n} x_i \otimes y_i) \in V_i$, we have

$$
\begin{aligned}
\|[\sum_{i=1}^{n} x_i \otimes y_i]\|_{\mathcal{S}_1 * \mathcal{S}_2} &= \sup\{|\sum_{i=1}^{n} \langle T x_i, y_i \rangle| : T \in \mathcal{S}_1^* \odot \mathcal{S}_2^*, \|T\| \leq 1\} \\
&= \sup\{|\sum_{i=1}^{n} T x_i \otimes y_i| : T \in \mathcal{L}, \|T\| \leq 1\} \\
&= \sup\{|\sum_{i=1}^{n} T x_i \otimes y_i| : T \in \mathcal{S}_1^*, \|T\| \leq 1\} \\
&= \|\varphi_1(\sum_{i=1}^{n} x_i \otimes y_i)\|_{\mathcal{S}_1}.
\end{aligned}
$$

Hence, the mapping Φ of V_1 into $S_1 * S_2$ is an isometry. Therefore, Φ can be extended to be an isometry from $\mathcal{S}_1^0$ into $\mathcal{S}_1 * \mathcal{S}_2^*$. Now we show that Φ is a complete isometry. Proposition 3.2.1 in [21] says that the canonical embedding $\imath : \mathcal{S}_1 \to (\mathcal{S}_1)^{**}$ is completely isometric. Thus,

$$
M_n(\mathcal{S}_1^0) \subseteq M_n((\mathcal{S}_1)^{**}) \cong \mathcal{CB}(\mathcal{S}_1^*, M_n),
$$

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where $\mathcal{CB}(\mathcal{S}_1^*, M_n)$ is the space of all completely bounded maps from operator space $\mathcal{S}_1^*$ into the space M_n of all $n \times n$ complex matrices and $\cong$ means a linear isomorphism (see Section 3.2 in [21]), $n \in \mathbb{N}$. For $X \in M_n(\mathcal{S}_1^0)$, we have

$$\|X\|_{M_n(\mathcal{S}_1^0)} = \|X\|_{\mathcal{CB}(\mathcal{S}_1^*, M_n)} = \|X\|_{\mathcal{CB}(\lambda_1(\mathcal{S}_1^*), M_n)}.$$

On the other hand, we have proved that, for $\varphi_1(x) = \varphi_1(\sum_{i=1}^n x_i \otimes y_i) \in V_1$,

$$\sum_{i=1}^n \langle Tx_i, y_i \rangle = 0,$$

if $T \in \mathcal{S}_1^* \odot \mathcal{S}_2^* - \lambda_1(\mathcal{S}_1^*)$. Hence, we have

$$\begin{aligned} &\|\Phi_1(X)\|_{M_n(\mathcal{S}_1 * \mathcal{S}_2)} = \|\Phi_1(X)\|_{M_n((\mathcal{S}_1^* \odot \mathcal{S}_2^*)^*)} \\ =&\|\Phi_1(X)\|_{\mathcal{CB}(\mathcal{S}_1^* \odot \mathcal{S}_2^*, M_n)} = \|X\|_{\mathcal{CB}(\lambda_1(\mathcal{S}_1^*), M_n)} \\ =&\|X\|_{M_n(\mathcal{S}_1^0)}. \end{aligned}$$

Hence, $\Phi : \mathcal{S}_1^0 \longrightarrow \mathcal{S}_1 * \mathcal{S}_2$ is completely isometric. □

By theorems 3.4 and 3.5, we can get the following corollary, which provides a kind of examples of operator spaces that satisfy the conclusion of Theorem 3.5.

Corollary 3.2.6. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be non-zero operator spaces with the operator space local lifting property. Then there exist dual representations $(\varphi_1, \mathcal{H}_1, \xi_1)$ and $(\varphi_1, \mathcal{H}_1, \xi_1)$ of dual spaces $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$, respectively, such that there is a complete isometry from $\mathcal{S}_1^0$ (or $\mathcal{S}_2^0$) to $(\mathcal{S}_1, \mathcal{H}_1, \xi_1) * (\mathcal{S}_2, \mathcal{H}_2, \xi_2)$ (see Theorem 3.5 for $\mathcal{S}_1^0$ (or $\mathcal{S}_2^0$)).*

3.3 An example

Given C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, we can construct two reduced free products for them. One is the C*-algebra free product, the other is the operator space free product. A natural question is that whether the two reduced free products are the same operator space. Moreover, is the free product for operator spaces a

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generalization of free products for C*-algebras? In this section, we shall present an example to show that the C^*-algebra reduced free product of two C^*-algebras may be contractively isomorphic to a proper subspace of the operator space reduced free product of the two C^*-algebras. So, we see that the operator space reduced free product is "bigger" than the C^*-algebra reduced free product.

Theorem 3.3.1. *Let V_1 and V_2 be two dimensional unital C^*-algebras. Then there are a reduced free product $V_1 * V_2$ of C^*-algebras V_1 and V_2 and a reduced free product $\mathcal{V}_1 * \mathcal{V}_2$ of operator spaces V_1 and V_2 such that $V_1 * V_2$ is contractively isomorphic to a proper subspace of $\mathcal{V}_1 * \mathcal{V}_2$.*

Proof. Let V be the two dimensional C*-algebra. Without loss of generality, we can assume that $V_i = C_r^*(\mathbf{G}_i)$, the reduced group C*-algebra of group $\mathbf{G}_i$, where $i = 1, 2$, and $\mathbf{G}_1 = \{I, v_1\}$ and $\mathbf{G}_2 = \{I, v_2\}$ are two free copies of the group $\mathbb{Z}_2$. That is, $\{v_1, v_2\}$ is a free family of unitary operators of order 2. Then the reduced free product $V_1 * V_2$ of C*-algebras V_1 and V_2 is

$$span\{I, v_{i_1} \cdots v_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, j = 1, 2, \cdots\}^{-\|\cdot\|} \subseteq \mathcal{B}(l^2(\mathbb{Z}_2 * \mathbb{Z}_2)),$$

where $\mathbb{Z}_2 * \mathbb{Z}_2$ is the free product of group $\mathbb{Z}_2$ with itself. The dual space of V_1 (or V_2) is

$$l_2^1 = \{x = (x_1, x_2) : x_1, x_2 \in \mathbb{C}, ||x|| = |x_1| + |x_2|\}.$$

Since V is an abelian C*-algebra, by Proposition 3.3.1 in [21], V is $\min l_2^\infty$, where l_2^∞ is the two dimensional sequence space with l^∞ norm. Therefore, $V^* = \max l_2^1 = \min l_2^1 = span\{I, u\} \subset C(\mathbb{T})$, where $C(\mathbb{T})$ is the C*-algebra of all continuous functions on the unit disk $\mathbb{T}$ of the complex plane, I is the constant function $I(t) = 1, \forall t \in \mathbb{T}$, and u is the generator of $C(\mathbb{T})$ (see Section 3.3 in [21]). Let $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{T})$, we have

$$\begin{aligned} V_1^* \odot V_2^* &= span\{I, u_{i_1} \cdots u_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, j = 1, 2, \cdots\}^{-W^*} \\ &\subset \mathcal{L}_{\mathbb{F}_2}, \end{aligned}$$

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where u_1 and u_2 are the generators of free group $\mathbb{F}_2$, "$-W^*$" means the ultra-weak operator topological closure. By the definition of the reduced free product of operator spaces (see Section 3.1), the reduced free product $\mathcal{V}_1 * \mathcal{V}_2 = (V_1^* \odot V_2^*)_*$. Let

$$\begin{aligned} \mathcal{S} &= span\{u_1 \otimes u_1, u_{i_{k+1}} \otimes u_{i_1} \cdots u_{i_{k+1}} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, \\ j &= 1, 2, \cdots, k, k = 1, 2, \cdots\} \subset l^2(\mathbb{F}_2) \tilde{\otimes} l^2(\mathbb{F}_2), \end{aligned}$$

Moreover, let

$$\begin{aligned} \mathcal{M} &= (V_1^* \odot V_2^*)^{\perp} \\ &= \{\sum_{i=1}^{n} x_i \otimes y_i \in l^2(\mathbb{F}_2) \tilde{\otimes} l^2(\mathbb{F}_2) : \sum_{i=1}^{n} \langle Ax_i, y_i \rangle = 0, \forall A \in V_1^* \odot V_2^*\}. \end{aligned}$$

For $x = \sum \alpha_{i_1 \cdots i_k} u_{i_{k+1}} \otimes u_{i_1} \cdots u_{i_{k+1}} \in \mathcal{S} \cap \mathcal{M}$, $i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}$ and $A = u_{i_1} \cdots u_{i_k} \in V_1^* \odot V_2^*$, we have

$$A(x) = \alpha_{i_1 \cdots i_k} = 0.$$

Therefore, $x = 0$. Hence, we may identity $\mathcal{S}$ with its image in $l^2(\mathbb{F}_2) \tilde{\otimes} l^2(\mathbb{F}_2)/\mathcal{M}$, and regard $\mathcal{S}$ as a subspace of $\mathcal{V}_1 * \mathcal{V}_2 = (V_1^* \odot V_2^*)_*$.

Now we show that $\mathcal{S}$ is dense in $\mathcal{V}_1 * \mathcal{V}_2$. By Hahn-Banach Theorem, it is sufficient to show that for all $A \in V_1^* \odot V_2^*$, $A = 0$ if $A(x) = 0, \forall x \in \mathcal{S}$. In fact, let $\mathcal{W}_n$ be the closed subspace of $l^2(\mathbb{F}_2)$ generated by

$$\{e, u_{i_1}^{\delta_1} \cdots u_{i_k}^{\delta_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, \delta_j \in \mathbb{Z}, |\delta_j| \leq n, j = 1, 2, \cdots k, k \leq n\},$$

where e is the unit of group $\mathbb{F}_2$, and P_n be the projection from $l^2(\mathbb{F}_2)$ onto $\mathcal{W}_n$. Given $A \in V_1^* \odot V_2^*$ with $A|_{\mathcal{S}} = 0$, there is a net

$$B_\lambda \in span\{I, u_{i_1} \cdots u_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, j = 1, 2, \cdots\},$$

for $\lambda \in \Lambda$, such that $\lim_\lambda B_\lambda = A$ in the ultra-weak operator topology. Let $B_\lambda = \sum \beta^{\lambda}_{i_1 \cdots i_k} u_{i_1} \cdots u_{i_k}$. For $u_{i_{k+1}} \otimes u_{i_1} \cdots u_{i_{k+1}} \in \mathcal{S}$, we have

$$0 = A(u_{i_{k+1}} \otimes u_{i_1} \cdots u_{i_{k+1}}) = \lim_{\lambda} B_\lambda (u_{i_{k+1}} \otimes u_{i_1} \cdots u_{i_{k+1}}) = \lim_{\lambda} \beta^{\lambda}_{i_1 \cdots i_k}.$$

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Hence, given $n \in \mathbb{N}, \lim_\lambda \beta^\lambda_{i_1\cdots i_k} = 0$ uniformly, for $k \le n$. Now, for $n \in \mathbb{N}$, and $\gamma, \gamma' \in \mathcal{W}_n$, where

$$\gamma = \sum_{|\delta_j|\le n, l\le n} \gamma_{(j_1,\delta_1),\cdots,(j_l,\delta_l)} u_{j_1}^{\delta_1} \cdots u_{j_l}^{\delta_l},$$

$$\gamma' = \sum_{|\delta_j|\le n, l\le n} \gamma'_{(j_1,\delta_1),\cdots,(j_l,\delta_l)} u_{j_1}^{\delta_1} \cdots u_{j_l}^{\delta_l},$$

we have

$$\begin{aligned}
&|\langle P_n A P_n \gamma, \gamma' \rangle| \\
=&\lim_\lambda |\langle P_n B_\lambda P_n \gamma, \gamma' \rangle| \\
\le&\lim_\lambda | \sum_{k+l\le n, i_k\ne j_1} \beta^\lambda_{i_1\cdots i_k} \gamma_{(j_1,\delta_1),\cdots,(j_l,\delta_l)} \overline{\gamma'_{i_1\cdots i_k (j_1,\delta_1),\cdots,(j_l,\delta_l)}} \\
&+ \sum_{1\le l\le k\le n} \beta^\lambda_{i_1\cdots i_k} \gamma_{(i_k,-1),\cdots,(i_{k-l+1},-1)} \overline{\gamma'_{i_1\cdots i_{k-l}}} \\
&+ \sum_{\delta_0\ne -1, k-l+l'\le n, j_1\ne i_{k-l}, l\le k\le n} (\beta^\lambda_{i_1\cdots i_k} \gamma_{(i_k,-1),\cdots,(i_{k-l+1},-1),(i_{k-l},\delta_0),(j_1,\delta_1),\cdots,(j_{l'},\delta_{l'})}) \\
&\cdot (\overline{\gamma'_{i_1\cdots i_{k-l-1}(i_{k-l},\delta_0+1)(j_1,\delta_1),\cdots,(j_l,\delta_{l'})}})| \\
\le&\lim_\lambda \sup\{|\beta^\lambda_{i_1\cdots i_k}| 3n^{1/2} ||\gamma|| \cdot ||\gamma'|| : k \le 2n\} \\
=&0.
\end{aligned}$$

It flows that $P_n A P_n = 0, \forall n \in \mathbb{N}$. Note that $\lim_\lambda P_n = I$ in the strong operator topology. Therefore, $\forall \varepsilon > 0$ and $\gamma, \gamma' \in l^2(\mathbb{F}_2)$, there are a $n \in \mathbb{N}$ and $\gamma_n = P_n\gamma, \gamma'_n = P_n\gamma' \in \mathcal{W}_n$ such that $||\gamma - \gamma_n|| < \varepsilon, ||\gamma' - \gamma'_n|| < \varepsilon$. Hence,

$$\begin{aligned}
|\langle A\gamma, \gamma' \rangle| &= |\langle A(\gamma_n + (\gamma - \gamma_n)), \gamma'_n + (\gamma' - \gamma'_n) \rangle| \\
&\le |\langle A(\gamma - \gamma_n), (\gamma' - \gamma'_n) \rangle| + |\langle A\gamma_n, (\gamma' - \gamma'_n) \rangle| + |\langle A(\gamma - \gamma_n), \gamma' \rangle| \\
&\le ||A|| \cdot ||\gamma_n|| \cdot ||\gamma' - \gamma'_n|| + ||A|| \cdot ||\gamma'|| \cdot ||\gamma - \gamma_n|| + ||A||\varepsilon^2 \\
&\le \varepsilon ||A|| (||\gamma|| + ||\gamma'|| + \varepsilon).
\end{aligned}$$

It implies that $\langle A\gamma, \gamma' \rangle = 0$. So, $A = 0$. It follows that $\mathcal{S}$ is dense in $\mathcal{V}_1 * \mathcal{V}_2$.

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Let $\mathcal{S}' = span\{e, v_{i_1}\cdots v_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, j = 1,2,\cdots,k, k = 1,2,\cdots\}$. It is obvious that $\mathcal{S}'$ is a dense subspace of $V_1 * V_2$. Define a linear mapping $\Phi : \mathcal{S}' \to \mathcal{S}$ by $\Phi(e) = u_1 \otimes u_1, \Phi(v_{i_1}\cdots v_{i_k}) = u_{i_{k+1}} \otimes u_{i_1}\cdots u_{i_{k+1}}$, for any $i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, j = 1,2,\cdots,k, k = 1,2,\cdots$. Obviously, Φ is bijective. For $x = \sum \alpha_{i_1\cdots i_k} v_{i_1}\cdots v_{i_k} \in \mathcal{S}'$, by definitions,

$$\begin{aligned}
||x|| &= \sup\{|\langle x\gamma, \gamma'\rangle| : \gamma, \gamma' \in l^2(\mathbb{Z}_2 * \mathbb{Z}_2), ||\gamma|| \le 1, ||\gamma'|| \le 1\} \\
&= \sup\{|\sum \alpha_{i_1\cdots i_k}\beta_{i_1\cdots i_k}| : \beta_{i_1\cdots i_k} = \sum_{l=0, i_k\neq j_1}^{\infty} \gamma_{j_1\cdots j_l}\overline{\gamma'_{i_1\cdots i_k j_1\cdots j_l}} \\
&\quad + \sum_{l=1}^{k-1}\gamma_{i_k\cdots i_{k-l+1}}\overline{\gamma'_{i_1\cdots i_{k-l}}} + \sum_{l=0, i_1\neq j_1}^{\infty} \gamma_{i_k\cdots i_1 j_1\cdots j_l}\overline{\gamma'_{j_1\cdots j_l}}, \\
\gamma &= \sum \gamma_{j_1\cdots j_l} v_{j_1}\cdots v_{j_k}, \gamma' = \sum \gamma'_{j_1\cdots j_l} v_{j_1}\cdots v_{j_k} \in l^2(\mathbb{Z}_2 * \mathbb{Z}_2), \\
||\gamma|| &\le 1, ||\gamma'|| \le 1\},
\end{aligned}$$

and

$$\begin{aligned}
||\Phi(x)|| &= \sup\{|A(\Phi(x))| : A \in V_1^* \odot V_2^*, ||A|| \le 1\} \\
&= \sup\{|\sum \alpha_{i_1\cdots i_k}\langle Au_{i_{k+1}}, u_{i_1\cdots i_{k+1}}\rangle| : A \in V_1^* \odot V_2^*, ||A|| \le 1\}.
\end{aligned}$$

For $A = \sum \beta_{j_1\cdots j_l} u_{j_1}\cdots u_{j_l} \in \Gamma$, where Γ is the linear span of

$$\{u_{i_1}\cdots u_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, j = 1,2,\cdots,k = 1,2,\cdots\},$$

and $||A|| \le 1$, we have

$$A(\Phi(x)) = \sum \alpha_{i_1\cdots i_k}\beta_{i_1\cdots i_k}.$$

Note that

$$(\sum |\beta_{i_1\cdots i_k}|^2)^{\frac{1}{2}} = ||A||_{l^2(\mathbb{F}_2)} \le ||A|| \le 1.$$

Now, let $\gamma = e, \gamma' = \sum \overline{\beta_{i_1\cdots i_k}} v_{i_1}\cdots v_{i_k} \in l^2(\mathbb{Z}_2 * \mathbb{Z}_2)$, we have

$$\langle x\gamma, \gamma'\rangle = \sum \alpha_{i_1\cdots i_k}\beta_{i_1\cdots i_k}, ||\gamma|| = 1, ||\gamma'|| = (\sum |\beta_{i_1\cdots i_k}|^2)^{\frac{1}{2}} \le 1.$$

Hence,

$$\sup\{|A(\Phi(x))| \;:\; A \in span\{u_{i_1}\cdots u_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\},$$

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$$j = 1,2,\cdots,k=1,2,\cdots\},||A|| \leq 1\} \leq ||x||.$$

Note that there is a sequence $\{A_n\}$ of operators in $V_1^* \odot V_2^*$ with $||A_n|| \leq 1$ such that $\lim_{n\to\infty} |A_n(\Phi(x))| = ||\Phi(x)||$. Moreover, for $A \in V_1^* \odot V_2^* \subset \mathcal{L}_{F_2}, ||A|| \leq 1$, by Kaplansky density theorem, there is a net $\{B_\lambda : \lambda \in \Lambda\}$ of elements in $\mathbb{C}F_2$, the group algebra of group $\mathbb{F}_2$, such that $\lim_\lambda B_\lambda = A$ in the weak operator topology, and $||B_\lambda|| \leq 1, \forall \lambda \in \Lambda$. Therefore, $A(\Phi(x)) = \lim_\lambda B_\lambda(\Phi(x))$. Hence,

$$||\Phi(x)|| = \lim_{n\to\infty} ||B_n(\Phi(x))||,$$

for a sequence $\{B_n\}$ in $\mathbb{C}F_2$ with $||B_n|| \leq 1$. Let $B_n = \sum \beta^n_{(j_1^n,\delta_1^n)\cdots(j_l^n,\delta_l^n)} u_{j_1^n}^{\delta_1^n} \cdots u_{i_l^n}^{\delta_l^n}$, we have

$$B_n(\Phi(x)) = \sum \alpha_{i_1\cdots i_k} \beta^n_{i_1\cdots i_k},$$

and

$$(\sum |\beta_{i_1\cdots i_k}|^2)^{\frac{1}{2}} \leq (\sum |\beta^n_{(j_1^n,\delta_1^n)\cdots(j_l^n,\delta_l^n)}|^2)^{\frac{1}{2}} \leq ||B_n|| \leq 1.$$

It follows that

$$\begin{aligned}
||\Phi(x)|| &= \lim_{n\to\infty} ||B_n(\Phi(x))|| \\
&= \lim_{n\to\infty} ||\sum \alpha_{i_1\cdots i_k} \beta^n_{i_1\cdots i_k}| \\
&\leq \sup\{|A(\Phi(x))| : A \in span\{u_{i_1} \cdots u_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, \\
&\quad j = 1,2,\cdots,k=1,2,\cdots\}, ||A|| \leq 1\} \\
&\leq ||x||.
\end{aligned}$$

Hence, Φ is contractive, and we can extend Φ to be a contractive mapping from $V_1 * V_2$ into $\mathcal{V}_1 * \mathcal{V}_2$. Now we show that $\Phi : V_1 * V_2 \to \mathcal{V}_1 * \mathcal{V}_2$ is injective. For $x \in \mathcal{S}'$, define $||\Phi(x)||' = ||x||$. Let $\overline{\mathcal{S}}$ be the closure of $\mathcal{S}$ with respect to norm $||\cdot||'$, then $\overline{\mathcal{S}}$, as a Banach spaces, is isomorphic to $V_1 * V_2$. Now we prove that $\overline{\mathcal{S}} \subseteq \mathcal{V}_1 * \mathcal{V}_2$. For $y \in \overline{\mathcal{S}}$, there are $x_n \in \mathcal{S}'$ such that $\lim_{n\to\infty} ||y - \Phi(x_n)||' = 0$. Therefore, there is a $x \in V_1 * V_2$ such that $\lim_{n\to\infty} ||x - x_n|| = 0$. It implies that

$$||y - \Phi(x)|| = ||y - \Phi(x_n) - \Phi(x - x_n)||$$

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$$\begin{aligned}&\leq \|y-\Phi(x_n)\|+\|x-x_n\|\\ &\leq \|y-\Phi(x_n)\|'+\|x-x_n\|\\ &\rightarrow 0,\end{aligned}$$

as $n \to 0$. Hence, $\overline{\mathcal{S}} \subseteq \mathcal{V}_1 * \mathcal{V}_2$. Moreover,

$$\|y\|_{\mathcal{V}_1*\mathcal{V}_2} = \|\Phi(x)\|_{\mathcal{V}_1*\mathcal{V}_2} = \lim_{n\to\infty}\|\Phi(x_n)\| \leq \|x_n\| = \|x\|.$$

Hence, Φ is an injective contractive mapping from $V_1 * V_2$ into $\mathcal{V}_1 * \mathcal{V}_2$.

Now we show that $\Phi(V_1 * V_2) \neq \mathcal{V}_1 * \mathcal{V}_2$. Let $\mathcal{L}_{\mathbb{Z}_2*\mathbb{Z}_2}$ be the group von Neumann algebra of group $\mathbb{Z}_2 * \mathbb{Z}_2$, then $V_1 * V_2 \subsetneqq \mathcal{L}_{\mathbb{Z}_2*\mathbb{Z}_2}$. Now we consider to extend Φ to the space $\mathcal{L}_{\mathbb{Z}_2*\mathbb{Z}_2}$. By Theorem 6.7.2 in [34], for each $L \in \mathcal{L}_{\mathbb{Z}_2*\mathbb{Z}_2}$, there is an $x = \sum \alpha_{i_1\cdots i_k} v_{i_1}\cdots v_{i_k} \in l^2(\mathbb{Z}_2 * \mathbb{Z}_2)$ such that $L = L_x$. Let

$$y_n = \sum_{k\leq n} \alpha_{i_1\cdots i_k} u_{i_{k+1}} \otimes u_{i_1}\cdots u_{i_{k+1}} \in \Phi(V_1 * V_2), n \in \mathbb{N}.$$

We shall show that $\{y_n : n = 1,2,\cdots\}$ is a Cauchy sequence in $\mathcal{V}_1 * \mathcal{V}_2$. In fact, $\{x_n = \sum_{k\leq n}\alpha_{i_1\cdots i_k}v_{i_1}\cdots v_{i_k}\}_{n=0}^{\infty}$ is a Cauchy sequence in $l^2(\mathbb{Z}_2 * \mathbb{Z}_2)$, since $x = \lim_{n\to\infty} x_n$ in the norm of $l^2(\mathbb{Z}_2 * \mathbb{Z}_2)$. Hence, for $\forall\varepsilon > 0$, there is an N such that $(\sum_{k_1\leq k\leq k_2} |\alpha_{i_1\cdots i_k}|^2)^{\frac{1}{2}} < \varepsilon$, whenever $N < k_1 < k_2$. Now, for $k_1, k_2 \in \mathbb{N}$ with $N < k_1 < k_2$, and $B = \sum \beta_{j_1\cdots j_l} u_{j_1}\cdots u_{j_l} \in span\{I, u_{i_1}\cdots u_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, j = 1,2,\cdots\} \subset V_1^* \odot V_2^*$ with $||B|| \leq 1$, we have

$$\begin{aligned}|B(y_{k_2}-y_{k_2})| &= |B\sum_{k_1\leq k\leq k_2}\alpha_{i_1\cdots i_k}u_{i_{k+1}}\otimes u_{i_1}\cdots u_{i_k}u_{i_{k+1}}|\\ &= |\sum_{k_1\leq k\leq k_2}\alpha_{i_1\cdots i_k}\beta_{i_1\cdots i_k}|\\ &\leq (\sum_{k_1\leq k\leq k_2}|\alpha_{i_1\cdots i_k}|^2)^{\frac{1}{2}}(\sum_{k_1\leq k\leq k_2}|\beta_{i_1\cdots i_k}|^2)^{\frac{1}{2}}\\ &\leq \varepsilon||\sum\beta_{j_1\cdots j_l}u_{j_1}\cdots u_{j_l}||_{l^2}\\ &\leq \varepsilon||B|| \leq \varepsilon.\end{aligned}$$

It implies that

$$||y_{k_2} - y_{k_2}|| = \sup\{|B(y_{k_2} - y_{k_2})| : B \in span\{I, u_{i_1}\cdots u_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k,$$

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$$i_j \in \{1,2\}, j = 1,2,\cdots\}, ||B|| \leq 1\}$$
$$\leq \varepsilon.$$

So, we can define $\widetilde{\Phi}(x) = \lim_{n\to\infty} y_n \in \mathcal{V}_1 * \mathcal{V}_2$. For $x \in V_1 * V_2$, there is a sequence $\{x_n\}$ of elements in $span\{I, v_{i_1}\cdots v_{i_k} : i_1 \neq i_2 \neq \cdots \neq i_k, i_j \in \{1,2\}, j = 1,2,\cdots\}$ such that $\lim_n ||x - x_n||_{B(l^2(\mathbb{Z}_2 * \mathbb{Z}_2))} = 0$, so, $\lim_n ||x - x_n||_{l^2(\mathbb{Z}_2*\mathbb{Z}_2)} = 0$. Hence,

$$\Phi(x) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} \widetilde{\Phi}(x_n) = \widetilde{\Phi}(x).$$

Hence, $\widetilde{\Phi}$ is a generalization of Φ. Finally, we show that $\widetilde{\Phi}$ is injective. Suppose that there is an $x \in l^2(\mathbb{Z}_2 * \mathbb{Z}_2)$ such that $L_x \in \mathcal{L}_{\mathbb{Z}_2*\mathbb{Z}_2}$ and $\widetilde{\Phi}(L_x) = 0$. Let $x = \sum \alpha_{i_1\cdots i_k} v_{i_1}\cdots v_{i_k}$ and $y_n = \sum_{k\leq n} \alpha_{i_1\cdots i_k} u_{i_{k+1}} \otimes u_{i_1}\cdots u_{i_k}$, we have $\lim_{n\to\infty} ||y_n|| = 0$. Hence, for any $i_1^0 \neq i_2^0 \neq \cdots \neq i_k^0$, $B = u_{i_1^0}\cdots u_{i_k^0} \in V_1^* \odot V_2^*$, we have

$$0 = B(\widetilde{\Phi}(L_x)) = \lim_{n\to\infty} By_n = \alpha_{i_1^0\cdots i_k^0}.$$

It follows that $L_x = 0$, that is, $\widetilde{\Phi}$ is injective. Hence, we get

$$\Phi(V_1 * V_2) \subsetneq \widetilde{\Phi}(\mathcal{L}_{\mathbb{Z}_2*\mathbb{Z}_2}) \subseteq \mathcal{V}_1 * \mathcal{V}_2.$$

□

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Chapter 4

FREE MARKOV PROCESSES

This chapter is devoted to the study of free Markov processes.

This chapter is organized as follows. In Section 4.1, a notion of weak Markov processes in W^*-probability spaces is defined in an explicit way similar to that of classical Markov processes in probability theory (Definition 4.1.1). We give some sufficient and necessary conditions for a process of noncommutative random variables to have the weak Markov property (Theorem 4.1.3), which are Parallel to those for a stochastic process to have Markov property in classical probability. We show that weak Markov processes have certain transition functions. In the commutative case, having the transition functions is the same as having Markov property (Corollary 4.1.4).

Section 4.2 is devoted to the study of free Markov processes of (unbounded) random variables. We prove that processes with free additive (or multiplicative) increments are free Markov processes (Theorem 4.2.6 and Theorem 4.2.7). It is shown that every free Markov process of bounded self-adjoint operators in a W^*-probability space is a weak Markov process (Theorem 4.2.8).

Examples of Free Brownian motion were introduced and explored by R. Speicher [57] and P. Biane [8]. Together in [10] and [11], they studied the solutions to free stochastic differential equations driven by free Brownian motion. They prove that the solutions satisfy certain free Markov property with resect to some filtration. In Section 4.3, we shall consider similar equations driven by free Lévy

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processes (of bounded random variables). We prove the existence and uniqueness of the solution to this system of the equations (Theorem 4.3.6). We show also that the solution of the system of free stochastic differential equations is free Markov process (in general, of random variables with un-compactly supported distributions) (Theorem 4.3.8 and Theorem 4.3.9). The proof of our result relies on a free Burkholder-Gundy type inequality in L^2-norm (for the Lévy case) proved by M. Anshelevich [1]. A similar inequality in operator norm for stochastic integrals with respect to free Brownian motion was obtained in [10].

4.1 Weak Markov processes

An analogue of the notion of markov processes in non-commutative probability theory is the following notion of weak Markov process.

Definition 4.1.1. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space, $(X_t)_{t\geq 0}$ a family of self-adjoint operators in $\mathcal{A}$. Let $\mathcal{A}_{\leq t} = W^*\{X_s : s \leq t\}$, $\mathcal{A}_{=t} = W^*(X_t)$ and $\mathcal{A}_{\geq t} = W^*\{X_s : s \geq t\}$. We say $\{X_t : t \geq 0\}$ is a weak Markov process (or it has a weak Markov property) in $(\mathcal{A}, \tau)$, if*

$$\mathbf{E}_{=t}(AB) = \mathbf{E}_{=t}(A)\mathbf{E}_{=t}(B), \forall A \in \mathcal{A}_{\leq t}, B \in \mathcal{A}_{\geq t},$$

where $\mathbf{E}_{=t} : \mathcal{A} \to \mathcal{A}_{=t}$ is the trace preserving conditional expectation onto $\mathcal{A}_{=t}$.

The following result shows that, in the commutative case, weak Markov property in our Definition 4.1.1 is the Markov property in classical probability. We shall show in next section that free Markov processes in Voiculescu sense have the weak Markov property (Theorem 4.2.8).

Theorem 4.1.2. *A family $\{f_t : t \geq 0\}$ of self-adjoint elements in the abelian von Neumann algebra $\mathcal{A} = L^\infty(\Omega, \mathcal{F}, P)$ is a weak Markov process in sense of Definition 4.1.1 if and only if $\{f_t : t \geq 0\}$ is a Markov process in classical sense.*

Proof. Let $f \in \mathcal{A} = L^\infty(\Omega, \mathcal{F}, P)$ be a real valued random variable, then

$$W^*(f) \cong L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df),$$

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where $W^*(f)$ is the von Neumann subalgebra generated by f, $\mathcal{B}_{\sigma(f)}$ is the Borel algebra on $\sigma(f)$, and df (or μ_f) is the distribution of random variable f, and $\cong$ means $*$-isomorphism as von Neumann algebras. Let $\{f_t : t \leq 0\}$ be a random process of real valued random variables in $\mathcal{A}$. Let $\mathcal{F}_{=t} = \{f^{-1}(B) : B \in \mathcal{B}\}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. Then $\mathcal{F}_{=t}$ is a σ-subalgebra of $\mathcal{F}$. Define

$$\pi : L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df) \rightarrow L^\infty(\Omega, \mathcal{F}_{=t}, P)$$

such that $\pi(g) = g \circ f$, for $g \in L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df)$. It is obvious that $\pi(g) = g \circ f \in L^\infty(\Omega, \mathcal{F}_{=t}, P)$. Given, $g_1, g_2 \in L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df)$, $x \in \Omega$, and $\lambda_1, \lambda_2 \in \mathbb{C}$, we have

$$(\lambda_1 g_1 + \lambda_2 g_2) \circ f(x) = \lambda_1 g_1(f(x)) + \lambda_2 g_2(f(x)),$$

and

$$g_1(f(x)) \cdot g_2(f(x)) = (g_1 g_2)(f(x)), \overline{g_1(f(x))} = \overline{g}(f(x)).$$

Thus, π is a $*$-homomorphism. Moreover, the image of f (i. e. the spectrum of f) is the domain of elements in $L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df)$. Hence, π is injective. For any simple function $s = \Sigma_{i=1}^k \lambda_i \chi_{B_i} \in L^\infty(\Omega, \mathcal{F}_{=t}, P)$, let $g = \Sigma_{i=1}^k \lambda_i \chi_{f(B_i)}$. Then, $g \in L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df)$ and $s = g \circ f$. It implies that the image of π is dense in $L^\infty(\Omega, \mathcal{F}_{=t}, P)$. Hence, π is a *-isomorphism. To prove

$$\mathcal{A}_{\leq t} \cong L^\infty(\Omega, \mathcal{F}_{\leq t}, P), \mathcal{A}_{\geq t} = L^\infty(\Omega, \mathcal{F}_{\geq t}, P),$$

we first note that $\mathcal{A}_{\leq t}$ is generated, as a von Neumann algebra, by $\{X_s : s \leq t\}$, and we have proved that $W^*(X_s)$ is *-isomorphic to $L^\infty(\Omega, \mathcal{F}_{=s}, P)$. Thus, up to *-isomorphisms, we can assume that $\mathcal{A}_{\leq t}$ is the von Neumann algebra generated by elements in $L^\infty(\Omega, \mathcal{F}_{=s}, P)$, $s \leq t$, and it is enough to show that $L^\infty(\Omega, \mathcal{F}_{\leq t}, P)$ is generated by $L^\infty(\Omega, \mathcal{F}_{=s}, P)$, $s \leq t$. In fact, given a sequence $t \geq s_1 \geq s_2 \geq \cdots$, and $B_1, B_2, \cdots \in \mathcal{B}$, we have

$$\chi_{\cap_{i=1}^\infty f_{s_i}^{-1}(B_i)} = \lim_{n \to \infty} \chi_{f_{s_1}^{-1}} \cdots \chi_{f_{s_1}^{-1}} \in \mathcal{A}_{\leq t}.$$

Moreover, let $S_1 = f_{s_1}^{-1}(B_j)$, and

$$S_j = f_{s_j}^{-1}(B_j) - (\cup_{i=1}^{j-1} f_{s_i}^{-1}(B_i)), j = 2, 3, \cdots,$$

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then

$$\chi_{\cup_{i=1}^{\infty} f_{s_i}^{-1}(B_i)} = \chi_{\cup_{i=1}^{\infty} S_j} = \sum_{i=1}^{\infty} \chi_{S_i} \in \mathcal{A}_{\leq t}.$$

Hence, for $S \in \mathcal{F}_{\leq t}$, $\chi_S \in \mathcal{A}_{\leq t}$. Hence, $L^\infty(\Omega, \mathcal{F}_{\leq t}, P) \subseteq \mathcal{A}_{\leq t}$. Conversely, it is obvious that $\mathcal{A}_{\leq t} \subseteq L^\infty(\Omega, \mathcal{F}_{\leq t}, P)$ (up to *-isomorphism). Hence, $\mathcal{A}_{\leq t} \cong L^\infty(\Omega, \mathcal{F}_{\leq t}, P)$. Similarly, $\mathcal{A}_{\geq t} \cong L^\infty(\Omega, \mathcal{F}_{\geq t}, P)$. Suppose $\{f_t : t \geq 0\}$ is a weak Markov process in sense of Definition 4.1.1, for all $t \geq 0$, $A \in \mathcal{F}_{\leq t}, B \in \mathcal{F}_{\geq t}$, we have $\chi_A \in \mathcal{A}_{\leq t}, \chi_B \in \mathcal{A}_{\geq t}$. Hence,

$$P(AB|f_t) = \mathbf{E}_{=t}(\chi_A \chi_B) = \mathbf{E}_{=t}(\chi_A)\mathbf{E}_{=t}(\chi_B) = P(A|f_t)P(B|f_t).$$

It follows that random process $\{f_t : t \geq 0\}$ is a classical Markov process.

Conversely, suppose $\{f_t : t \geq 0\}$ is a classical Markov process, by the above discussion, $\mathbf{E}_{=t}(PQ) = \mathbf{E}_{=t}(P)\mathbf{E}_{=t}(Q), \forall t \geq 0$, where P, Q are projections in $\mathcal{A}_{\leq t}$ and $\mathcal{A}_{\geq t}$, respectively. Thus, for $\lambda_i, \lambda_i' \in \mathbf{C}$, $p_i \in \mathcal{A}_{\leq t}, q_i \in \mathcal{A}_{\geq t}$, and $X = \sum_{i=1}^n \lambda_i P_i$, $Y = \sum_{i=1}^n \lambda_i' Q_i$, we have

$$\mathbf{E}_{=t}(XY) = \sum_{i,j=1}^n \lambda_i \lambda_j' \mathbf{E}_{=t}(P_i Q_j) = \sum_{i,j=1}^n \lambda_i \lambda_j' \mathbf{E}_{=t}(P_i)\mathbf{E}_{=t}(Q_j) = \mathbf{E}_{=t}(x)\mathbf{E}_{=t}(Y).$$

Note that conditional expectation $\mathbf{E}_{=t}$ is norm continuous and the linear span of all projections is norm dense in a von Neumann algebra, so we have

$$\mathbf{E}_{=t}(AB) = \mathbf{E}_{=t}(A)\mathbf{E}_{=t}(B), \forall A \in \mathcal{A}_{\leq t}, B \in \mathcal{A}_{\geq t}.$$

It follows that $\{f_t : t \geq 0\}$ is a weak markov process in sense of Definition 4.1.1. □

The following result gives some sufficient and necessary conditions for a process to be a weak Markov process.

Theorem 4.1.3. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Let $(X_t)_{t\geq 0}$ be a family of self-adjoint operators in $\mathcal{A}$. Then the following are equivalent.*

1. *The process $\{X_t : t \geq 0\}$ is a weak Markov process.*

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2. *For all* $t \geq 0$, $\mathbf{E}_{\leq t}(A) = \mathbf{E}_{=t}(A), \forall A \in \mathcal{A}_{\geq t}$, *where* $\mathbf{E}_{\leq t} : \mathcal{A} \to \mathcal{A}_{\leq t}$ *is the trace preserving conditional expectation onto* $\mathcal{A}_{\leq t}$.

3. *For all* $t \geq 0$, $\mathbf{E}_{\geq t}(A) = \mathbf{E}_{=t}(A), \forall A \in \mathcal{A}_{\leq t}$, *where* $\mathbf{E}_{\geq t} : \mathcal{A} \to \mathcal{A}_{\geq t}$ *is the trace preserving conditional expectation onto* $\mathcal{A}_{\geq t}$.

4. *For all* $0 \leq s \leq t$, *let* $\mathcal{A}_{s,t} = W^*\{X_r : s \leq r \leq t\}$ *and* $\mathbf{E}_{s,t} : \mathcal{A}_{\leq t} \to \mathcal{A}_{\leq s}$ *be the trace preserving conditional expectation. Then,* $\mathbf{E}_{s,t}(\mathcal{A}_{s,t}) \subseteq \mathcal{A}_{=s}$.

Proof. (1) $\Rightarrow$ (2) Without loss of generality, we can assume that von Neumann algebra $\mathcal{A}$ acts on the Hilbert space $L^2(\mathcal{A}, \tau)$. Then, τ is the vector state associated to identity element I of $\mathcal{A}$. Thus, τ is continuous with respect to WOT (weak operator topology). Note that the linear span $\mathcal{L}$ of the set

$$\{X_{t_1} \cdots X_{t_n} : t_j \geq t, j = 1, 2, \cdots, n, n = 1, 2, \cdots.\}$$

is dense in $\mathcal{A}_{\geq t}$ with respect to WOT. If we can prove

$$\mathbf{E}_{\leq t}(X_{t_1} \cdots X_{t_n}) = \mathbf{E}_{=t}(X_{t_1} \cdots X_{t_n}), \forall t_j \geq t, j = 1, 2, \cdots, n, n = 1, 2, \cdots, \quad (4.1.1)$$

then, we have $\mathbf{E}_{\leq t}(X) = \mathbf{E}_{=t}(X), \forall X \in \mathcal{L}$. Moreover, for $A \in \mathcal{A}_{\geq t}$, there is a net $\{X_\lambda : \lambda \in \Lambda\}$ in $\mathcal{L}$ such that $\lim_\lambda X_\lambda = A$, where the limit is in WOT. Hence, for $B \in \mathcal{A}_{\leq t}$, we have

$$\begin{aligned}\tau(\mathbf{E}_{=t}(A)B) &= \lim_\lambda \tau(\mathbf{E}_{=t}(X_\lambda)B) \\ &= \lim_\lambda \tau(\mathbf{E}_{\leq t}(X_\lambda)B) = \lim_\lambda \tau(X_\lambda B) \\ &= \tau(\mathbf{E}_{\leq t}(A)B).\end{aligned}$$

Hence, it is sufficient to show (4.1.1). For $t_j \geq t, j = 1, 2, \cdots, n$ and $B \in \mathcal{A}_{\leq t}$, we have

$$\begin{aligned}\tau(X_{t_1} \cdots X_{t_n} B) &= \tau(\mathbf{E}_{=t}(X_{t_1} \cdots X_{t_n} B)) \\ &= \tau(\mathbf{E}_{=t}(X_{t_1} \cdots X_{t_n})\mathbf{E}_{=t}(B))\end{aligned}$$

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$$= \tau(\mathbf{E}_{=t}(X_{t_1} \cdots X_{t_n})B),$$

where the second equality holds because of Definition 4.1.1. Hence,

$$\mathbf{E}_{\leq t}(X_{t_1} \cdots X_{t_n}) = \mathbf{E}_{=t}(X_{t_1} \cdots X_{t_n}).$$

(2) $\Rightarrow$ (1) For any $A \in \mathcal{A}_{\leq t}, B \in \mathcal{A}_{\geq t}$ and $C \in \mathcal{A}_{=t}$, we have

$$\begin{aligned}\tau(ABC) &= \tau(CA\mathbf{E}_{\leq t}(B)) \\ &= \tau(CA\mathbf{E}_{=t}(B)) = \tau(A\mathbf{E}_{=t}(B)C) \\ &= \tau(\mathbf{E}_{=t}(A)\mathbf{E}_{=t}(B)C).\end{aligned}$$

Hence,

$$\mathbf{E}_{=t}(AB) = \mathbf{E}_{=t}(A)\mathbf{E}_{=t}(B).$$

The proof of the equivalence of (1) and (3) is the same as that of (1) $\Leftrightarrow$ (2).
(4) $\Rightarrow$ (2) It is enough to show that

$$\mathbf{E}_{\leq t}(X_{t_1} \cdots X_{t_n}) \in \mathcal{A}_{=t}, \forall t_j \geq t, j = 1, 2, \cdots, n, n \in \mathbb{N}.$$

Let $u = \max\{t_j : j = 1, 2 \cdots, n\}$, then $X_{t_1} \cdots X_{t_n} \in \mathcal{A}_{t,u}$. Hence, by (4),

$$\mathbf{E}_{\leq t}(X_{t_1} \cdots X_{t_n}) = \mathbf{E}_{s,t}(X_{t_1} \cdots X_{t_n}) \in \mathcal{A}_{=s}.$$

(2) $\Rightarrow$ (4) It is enough to show that $\mathbf{E}_{s,t}(X_{r_1} \cdots X_{r_n}) \in \mathcal{A}_{=s}$, for all $s \leq r_j \leq t$. Note that $X_{r_1} \cdots X_{r_n} \in \mathcal{A}_{\geq s} \cap \mathcal{A}_{\leq t}$, so $\mathbf{E}_{s,t}(X_{r_1} \cdots X_{r_n}) = \mathbf{E}_{\leq s}(X_{r_1} \cdots X_{r_n}) \in \mathcal{A}_{=s}$, by (2). □

Corollary 4.1.4. *Let $\{X_t : t \geq 0\}$ be a weak Markov process in W^*-probability space $(\mathcal{A}, \tau)$. Then the following statements hold.*

1. *there is an operator*

$$\mathcal{K}_{s,t} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}),$$

for $0 \leq s < t$, such that

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(a) $k_{s,t}(x,\cdot) : \Gamma \to k_{s,t}(x,\Gamma) = \mathcal{K}_{s,t}(\chi_\Gamma)(x)$ *is a probability measure on* $\mathcal{B}_{\sigma(X_s)}$ *for almost all* $x \in \sigma(X_s)$ *with respect to* dX_s,

(b) $k_{s,s}(x,F) = \chi_F(x)$,

(c) $\mathbf{E}_{\leq s}(\varphi(X_t)) = \mathcal{K}_{s,t}(\varphi)(X_s), \forall \varphi \in L^\infty(\mathbb{R})$.

2. *If* $\{X_t : t \geq 0\}$ *is commutative random process of operators in* $\mathcal{A}$ *(i. e.* $X_tX_s = X_sX_t$, *for all* $t, s \geq 0$*), and there is an operator*

$$\mathcal{K}_{s,t} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}),$$

for $0 \leq s < t$, *satisfies conditions (a), (b) and (c) above. Then,* $\{X_t : t \geq 0\}$ *is a weak Markov process in sense of Definition 4.1.1.*

Proof. By (4) in Theorem 4.1.3, $\mathbf{E}_{s,t}(\mathcal{A}_{=t}) \subseteq \mathcal{A}_{=s}$, for $0 \leq s < t$. Note that there is a *-isomorphism

$$\pi_t : \mathcal{A}_{=t} \to L^\infty(\sigma(X_t), \mathcal{B}_{\sigma(X_t)}, dX_t),$$

where dX_t is the distribution of X_t with respect to τ. For $0 \leq s < t$, define

$$\mathcal{K}_{s,t}(f)(x) = \pi_s \mathbf{E}_{s,t}(f(X_t))(x), \forall f \in L^\infty(\mathbb{R}), x \in \mathbb{R}.$$

Then, $\mathcal{K}_{s,t} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ and

$$\mathbf{E}_{\leq s}(f(X_t)) = \pi_s^{-1}(\mathcal{K}_{s,t}(f)) = \mathcal{K}_{s,t}(f)(X_s), \forall f \in L^\infty(\mathbb{R}).$$

This means that $\mathcal{K}_{s,t}$ satisfies condition (c). Now we show that $\mathcal{K}_{s,t}$ satisfies the properties (a) and (b). It is obvious that function $\mathcal{K}_{s,t}(f)(x)$ is measurable, since $\mathcal{K}_{s,t}(f) \in L^\infty(\sigma(X_s), \mathcal{B}_{\sigma(X_s)}, dX_s)$. For $0 \leq s \leq t, x \in \mathbb{R}$, a Borel set $F = \bigcup_{i \geq 1} F_i \in \mathcal{B}, F_i \cap F_j = \emptyset, \forall i \neq j, i, j = 1, 2, \cdots, \forall G \in \mathcal{B}_{\sigma(X_s)}$, and $k_{s,t}(x,F) = \mathcal{K}_{s,t}(\chi_F)(x)$, we have

$$\int_G \mathcal{K}_{s,t}(\chi_F) dX_s = \int_{\sigma(X_s)} (\mathcal{K}_{s,t}(\chi_F)\chi_G) dX_s = \tau(\mathbf{E}_{s,t}(\chi_F(X_t))\chi_G(X_s))$$

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$$= \tau(\chi_F(X_t)\chi_G(X_s)) = \tau(\sum_{i=1}^{\infty}\chi_{F_i}(X_t)\chi_G(X_s))$$
$$= \sum_{i=1}^{\infty}\tau(\chi_{F_i}(X_t)\chi_G(X_s)) = \sum_{i=1}^{\infty}\int_G \mathcal{K}_{s,t}(\chi_{F_i})dX_s$$
$$= \int_G(\sum_{i=1}^{\infty}\mathcal{K}_{s,t}(\chi_{F_i}))dX_s.$$

It follows that

$$k_{s,t}(F,x) = \sum_{i=1}^{\infty}k_{s,t}(F_i,x),$$

for almost all $x \in \sigma(X_s)$ with respect to dX_s. Moreover,

$$k_{s,t}(x,\sigma(X_s)) = \mathcal{K}_{s,t}\chi_{\sigma(X_s)}(x) = \pi_s\mathbf{E}_{s,t}(\chi_{\sigma(X_t)}(X_t))(x) = 1.$$

Hence, $k_{s,t}(x,\cdot)$ is a Probability measure on $\sigma(X_s)$, for almost all $x \in \sigma(X_s)$. This completes the proof of (a). (b) is obvious.

Conversely, by Property (c) of operator $\mathcal{K}_{s,t}$, we have $\mathbf{E}_{s,t}(\mathcal{A}_{=t}) \subseteq \mathcal{A}_{=s}$, for $0 \le s < t$. Now we show that $\mathbf{E}_{s,t}(\mathcal{A}_{s,t}) \subseteq \mathcal{A}_{=s}$, for $0 \le s < t$. Note that $\mathcal{A}_{s,t}$ is abelian, the linear span $\mathcal{L}$ of elements in $\{X_{r_1}\cdots X_{r_n} : s \le r_1 \le \cdots \le r_n \le t, n \in \mathbb{N}\}$ is dense in $\mathcal{A}_{s,t}$ with respect to WOT. Hence, it is sufficient to show that

$$\tau(X_{r_1}\cdots X_{r_n}B) = \tau(\mathbf{E}_{=s}(X_{r_1}\cdots X_{r_n})B),$$

for all B in $\mathcal{A}_{\le s}$. We shall prove it by induction in n. For $n = 1$, $\mathbf{E}_{s,t}(f(X_{t_1}))$ in $\mathcal{A}_{=s}$, for all $f \in L^\infty(\mathbb{R})$, since $\mathbf{E}_{s,t}(\mathcal{A}_{=t}) \subseteq \mathcal{A}_{=s}$. Suppose $\mathbf{E}_{s,t}(f_1(X_{t_1})\cdots f_n(X_{t_n}))$ in $\mathcal{A}_{=s}$, for all $f_1,\cdots,f_n$ in $L^\infty(\mathbb{R})$, $s \le t_1 \le \cdots \le t_n \le t$. Now for $f_1,\cdots,f_{n+1} \in L^\infty(\mathbb{R})$, $s \le t_1 \le \cdots \le t_{n+1} \le t$, and $B \in \mathcal{A}_{\le s}$, we have

$$\begin{aligned}\tau(f_1(X_{t_1})\cdots f_{n+1}(X_{t_{n+1}}))B) &= \tau(f_1(X_{t_1})\cdots f_n(X_{t_n})\mathbf{E}_{\le t_n}(f_{n+1}(X_{t_{n+1}}))B)\\ &= \tau(f_1(X_{t_1})\cdots(f_n(X_{t_n})\mathbf{E}_{\le t_n}(f_{n+1}(X_{t_{n+1}})))B)\\ &= \tau(\mathbf{E}_{=s}(f_1(X_{t_1})\cdots(f_n(X_{t_n})\mathbf{E}_{\le t_n}(f_{n+1}(X_{t_{n+1}}))))B)\\ &= \tau(\mathbf{E}_{=s}(f_1(X_{t_1})\cdots f_n(X_{t_n})f_{n+1}(X_{t_{n+1}}))B).\end{aligned}$$

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It implies that $E_{s,t}(f_1(X_{t_1})\cdots f_{n+1}(X_{t_{n+1}}))\in \mathcal{A}_{=s}$. we have proved (4) in Theorem 2.4. Hence, $\{X_t : t\geq 0\}$ is a weak Markov process in W^*-probability space $(\mathcal{A},\tau)$. □

Theorem 4.1.5. *Let $\{X_t : t \geq 0\}$ be a random process in W^*-probability space $(\mathcal{A},\tau)$. For $0\leq s<t$, let $\mathbf{E}_{s,t}:\mathcal{A}_{\leq t}\to\mathcal{A}_{\leq s}$ be the trace preserving conditional expectation, and $C^*_{s,t}$ be the untial C^*-algebra generated by $\{X_r : s\leq r\leq t\}$. If $E_{s,t}(C^*_{s,t})\subseteq C^*(X_s)$, for all $0\leq s\leq t$, then the following statements hold.*

1. *Let $C_0(\mathbb{R})$ be the C^*-algebra of all continuous functions on $\mathbb{R}$, such that the functions vanishes at infinity. For all $0\leq s\leq t$, there is a completely positive contraction $\Pi_{s,t}:C_0(\mathbb{R})\to C_0(\mathbb{R})$ such that*

$$\Pi_{s,t}:C(\sigma(X_t))\to C(\sigma(X_s)),$$

$$\Pi_{s,t}(1_{C(\sigma(X_t))})=1_{C(\sigma(X_s))},\forall 0\leq s<t,$$

$$\Pi_{s,u}=\Pi_{s,t}\Pi_{t,u},\forall 0\leq s\leq t\leq u,$$

and

$$\mathbf{E}_{s,t}(f(X_t))=\Pi_{s,t}(f)(X_s),\forall f\in C(\sigma(X_t)).$$

2. *$\{X_t : t\geq 0\}$ is a weak Markov process.*

Proof. It follows from the hypotheses that $\mathbf{E}_{s,t}(C^*(X_t))\subseteq C^*(X_s)$, for $0\leq s<t$. Let $\Pi_{s,t}=\pi_s\circ\mathbf{E}_{s,t}|_{C^*(X_t)}\circ\pi_t^{-1}$, where $\pi_t : C^*(X_t)\to C(\sigma(X_t))$ is the canonical *-isomorphism from abelian $C^*(X_t)$ onto the function algebra $C(\sigma(X_t))$, then $\Pi_{s,t}$ is identity preserving completely positive contraction from for $C^*(\sigma(X_t))$ into $C^*(\sigma(X_s))$. Moreover, define

$$\Pi_{s,t}(f)=\Pi_{s,t}(f\circ\chi_{\sigma(X_t)}),\forall f\in C_0(\mathbb{R}),0\leq s<t,$$

where $\chi_{\sigma(X_t)}$ is the characteristic function of set $\sigma(X_t)$. Then $\Pi_{s,t}:C_0(\mathbb{R})\to C_0(\mathbb{R})$ is a completely positive contraction. For $0\leq s<t<u$, $f\in C(\sigma(X_u))$, we have

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$$\Pi_{s,u}(f) = \pi_s \circ \mathbf{E}_{s,u}(f(X_u)) = \pi_s \circ \mathbf{E}_{s,t}(\mathbf{E}_{t,u} f(X_u))$$
$$= \pi_s \circ \mathbf{E}_{s,t} \circ \pi_t^{-1} \circ \pi_t \circ (\mathbf{E}_{t,u} f(X_u))$$
$$= \Pi_{s,t}\Pi_{t,u}(f),$$

and

$$\mathbf{E}_{s,t}(f(X_t)) = \mathbf{E}_{s,t} \circ \pi_t^{-1}(f) = \pi_s^{-1} \circ \Pi_{s,t}(f) = \Pi_{s,t}(f)(X_s).$$

Assume $\mathcal{A}$ acts on the Hilbert space $L^2(\mathcal{A}, \tau)$, then trace τ is the vector state associated with the identity operator in $\mathcal{A}$. Hence, τ is WOT continuous. To prove that $\{X_t : t \geq 0\}$ is a weak Markov process, by Theorem 4.1.3, it is enough to show that $\mathbf{E}_{s,t} : \mathcal{A}_{s,t} \to W^*(X_s)$. In fact, for $A \in \mathcal{A}_{s,t}$, there is a net $A_\lambda \in C^*_{s,t}$ such that $A_\lambda \to A$ with respect to WOT. Thus, for any $B \in \mathcal{A}_{\leq s}$, we have

$$\tau(AB) = \lim_\lambda \tau(A_\lambda B) = \lim_\lambda \tau(A_\lambda \mathbf{E}_{=s}(B))$$
$$= \tau(A\mathbf{E}_{=s}(B)) = \tau(\mathbf{E}_{=s}(A)\mathbf{E}_{=s}(B))$$
$$\tau(\mathbf{E}_{=s}(A)B).$$

Hence, $\mathbf{E}_{s,t}(A) = \mathbf{E}_{=s}(A) \in \mathcal{A}_{=s}$. By Theorem 4.1.3, process $\{X_t : t \geq 0\}$ has weak Markov property. □

4.2 Free Markov processes

In this section, we study free Markov processes of (unbounded) random variables in a W^*-probability space. We show that every process with free increments is a free Markov process, and every free Markov process is a weak Markov process in sense of Definition 4.1.1.

By [61] and [62], we have

Definition 4.2.1. *Let $\{X_t : t \geq 0\}$ be a family of (unbounded) operators in $\widetilde{\mathcal{A}}$. Let $\mathcal{A}_{\leq t}$ be the von Neumann subalgebra of $\mathcal{A}$ generated by $\{A : A \in W^*(X_s), 0 \leq$*

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$s \le t\}$, $\mathcal{A}_{\ge t}$ *be the von Neumann subalgebra generated by* $\{A : A \in W^*(X_s), s \ge t\}$ *and* $\mathcal{A}_{=t} = W^*(X_t)$, *for* $t \ge 0$. *We say that the random process* $\{X_t : t \ge 0\}$ *is a free Markov Process, if, for* $t \ge 0$, $\mathcal{A}_{\le t}$ *and* $\mathcal{A}_{\ge t}$ *are* $\mathcal{A}_{=t}$*-free.*

We generalized it to a more general case.

Definition 4.2.2. *Given* $t \to X_t = (X_{1,t}, \cdots, X_{k,t}) \in \widetilde{\mathcal{A}}^k$. *Let* $\mathcal{A}_{\le t}$, $\mathcal{A}_{=t}$, *respectively,* $\mathcal{A}_{\ge t}$ *be the von Neumann subalgebras of* $\mathcal{A}$ *generated by* $\{A \in W^*(X_{i,s}) : 0 \le s \le t, i = 1, 2, \cdots, k\}$, $\{A \in W^*(X_{i,t}) : i = 1, 2, \cdots, k\}$, *respectively,* $\{A : A \in W^*(X_{i,s}), s \ge t, i = 1, 2, \cdots, k\}$. *We say random process* $\{X_t : t \ge 0\}$ *is a free Markov process, if* $\mathcal{A}_{\le t}$ *and* $\mathcal{A}_{\ge t}$ *are* $\mathcal{A}_{=t}$*-free.*

In order to prove that every process with free increments is a free Markov process, we need the following

Definition 4.2.3. *([9], Definition 4.2)*

1. *A free additive increments process is a random process* $\{X_t : t \ge 0\}$ *of elements in* $\widetilde{\mathcal{A}}_{sa}$ *such that, for any sequence* $0 \le t_1 < t_2 < \cdots < t_n$, *the elements* $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ *of* $\widetilde{\mathcal{A}}_{sa}$ *form a free family.*

2. *A unitary process with (left) multiplicative free increments is a family* $\{U_t : t \ge 0\}$ *of unitary operators in* $(\mathcal{A}, \tau)$ *such that, for any* $0 \le t_1 \le \cdots \le t_n$, *the elements*

$$U_{t_1}, U_{t_2}U_{t_1}^{-1}, \cdots, U_{t_n}U_{t_{n-1}}^{-1}$$

form a free family in $(\mathcal{A}, \tau)$.

Lemma 4.2.4. *([61, Lemma 3.3]) Let* $1 \in \mathcal{D} \subseteq \mathcal{B} \subseteq \mathcal{A}_1$, $1 \in \mathcal{A}_2 \subseteq \mathcal{A}_1$ *be von Neumann subalgebras of finite von Neumann algebra* $\mathcal{A}$, *and* $\Omega \subseteq \mathcal{A}$ *be a subset such that* $\mathcal{A}_1$ *and* Ω *are -*$\mathcal{D}$ *free. Let* $1 \in \mathcal{C} \subseteq W^*(\mathcal{B} \cup \Omega)$ *be a von Neumann subalgebra. Then* $\mathcal{A}_2$ *and* $\mathcal{C}$ *are* $\mathcal{B}$*-free.*

Lemma 4.2.5. *Let* X, Y *be self-adjoint operators affiliated with a* W^**-probability space* $(\mathcal{A}, \tau)$. *Then* $W^*(X+Y) \subseteq W^*(X, Y)$, *where* $W^*(X, Y)$ *is the von Neumann algebra generated by* $W^*(X)$ *and* $W^*(Y)$.

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Theorem 4.2.6. *Let $\{X_t : t \geq 0\}$ be a free additive increments process affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. Then $\{X_t : t \geq 0\}$ is a free Markov process.*

Proof. Let $\mathcal{A}_{\leq t} = W^*\{X_s : s \leq t\}$, $\mathcal{A}_{=t} = W^*\{X_t\}$ and $\mathcal{A}_{>t} = W^*\{X_s - X_t : s \geq t\}$. Then, by Lemma 4.2.5, $\mathcal{A}_{\geq t} = W^*\{X_s : s \geq t\} \subseteq W^*(\mathcal{A}_{>t} \cup \mathcal{A}_{=t})$. Thus, to prove that $\{X_t : t \geq 0\}$ is a free Markov process, it is enough to show that $W^*(\mathcal{A}_{>t} \cup \mathcal{A}_{=t})$ and $\mathcal{A}_{\leq t}$ are $\mathcal{A}_{=t}$ free.

First, we show that $\mathcal{A}_{\leq t}$ and $\mathcal{A}_{>t}$ are free. Let $\overline{\mathcal{A}}_{\leq t}$ and $\overline{\mathcal{A}}_{>t}$ be the *-subalgebras generated by $\{f(X_s) : f \in BC(\mathbb{R}), s \leq t\}$ and $\{f(X_s - X_t) : f \in BC(\mathbb{R}), s > t\}$, respectively. Thus, by Proposition 2.5.7 in [64], it is enough to show that $\overline{\mathcal{A}}_{\leq t}$ and $\overline{\mathcal{A}}_{>t}$ are free. For $A_1, \cdots, A_n \in \overline{\mathcal{A}}_{\leq t}$ and $B_1, \cdots, B_n \in \overline{\mathcal{A}}_{>t}$ with $\tau(A_i) = \tau(B_i) = 0, 1 \leq i \leq n$, we have to show that

$$\tau(A_1 B_1 \cdots A_n B_n) = 0. \tag{4.2.1}$$

In fact, there are $0 \leq t_0 \leq t_1 \leq \cdots \leq t_m = t \leq s_1 \leq s_2 \leq \cdots \leq s_k$ and

$$f_0, f_1, \cdots, f_m, g_1, \cdots, g_k \in BC(\mathbb{R})$$

such that $A_1, \cdots, A_n$ are in the *-algebra $\mathcal{B}$ generated by

$$\{f_0(X_{t_0}), f_1(X_{t_1}), \cdots, f_n(X_{t_n})\}.$$

Lemma 4.2.5 implies that $\mathcal{B}$ is a subset of $\mathcal{C}_1 = W^*\{X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_m - X_{t_{m-1}}}\}$, and $B_1, \cdots, B_n$ are in $\mathcal{C}_2 = W^*\{X_{s_1} - X_t, X_{s_2} - X_{s_1}, \cdots, X_{s_k} - X_{s_{k-1}}\}$. But, by Definition 4.2.3, $\mathcal{C}_1$ and $\mathcal{C}_2$ are free. Hence, (4.2.1) holds true. Therefore, $\mathcal{A}_{\leq t}$ and $\mathcal{A}_{>t}$ are free. Let $\mathcal{D} = \mathbb{C}I$, $\mathcal{A}_1 = \mathcal{A}_{\leq t}$, $\mathcal{B} = \mathcal{A}_{=t}$ and $\Omega = \mathcal{A}_{>t}$. We have proved that then $\mathcal{A}_1$ and $W^*(\Omega \cup \mathcal{D}) = \Omega$ are free (i. e. $\mathcal{D}$ free). By Lemma 4.2.4, $\mathcal{A}_{\leq t} = \mathcal{A}_1$ and $W^*(\Omega \cup \mathcal{B}) = \mathcal{A}_{\geq t}$ are $\mathcal{A}_{=t}$-free. ☐

Theorem 4.2.7. *Let $\{U_t : t \geq 0\}$ be a unitary process with (left) multiplicative free increments. Then $\{U_t : t \geq 0\}$ is a free Markov process.*

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Proof. Let $\mathcal{A}_{\leq t} = W^*\{U_s : s \leq t\}$, $\mathcal{A}_{=t} = W^*\{U_t\}$ and $\mathcal{A}_{>t} = W^*\{U_sU_t^{-1} : s \geq t\}$. Then $\mathcal{A}_{\geq t} = W^*\{U_s : s \geq t\} = W^*(\mathcal{A}_{>t} \cup \mathcal{A}_{=t})$. Let $\overline{\mathcal{A}}_{\leq t}$ and $\overline{\mathcal{A}}_{>t}$ be the *-subalgebras generated by $\{X_s : s \leq t\}$ and $\{U_sU_t^{-1} : s \geq t\}$, respectively. To prove that the process $\{U_t : t \geq 0\}$ is a free Markov process, by Lemma 4.2.4 and Proposition 2.5.7 in [64], it is enough to show that $\overline{\mathcal{A}}_{\leq t}$ and $\overline{\mathcal{A}}_{>t}$ are free. For $A_1, \cdots, A_n \in \overline{\mathcal{A}}_{\leq t}$ and $B_1, \cdots, B_n \in \overline{\mathcal{A}}_{>t}$ with $\tau(A_i) = \tau(B_i) = 0, 1 \leq i \leq n$, we have to show that (4.2.1). In fact, there are $0 \leq t_0 \leq t_1 \leq \cdots \leq t_m = t \leq s_1 \leq s_2 \leq \cdots \leq s_k$ such that $A_1, \cdots, A_n$ are in the *-algebra generated by $\{U_{t_0}, U_{t_1}, \cdots, U_{t_n}\}$, which is a subset of $\mathcal{C}_1 = W^*\{U_{t_0}, U_{t_1}U_{t_0}^{-1}, \cdots, X_{t_mU_{t_{m-1}}}^{-1}\}$, and $B_1, \cdots, B_n$ are in $\mathcal{C}_2 = W^* = \{U_{s_1}U_t^{-1}, U_{s_2}U_{s_1}^{-1} \cdots, U_{s_k}U_{s_{k-1}}^{-1}\}$. But, by Definition 3.3, $\mathcal{C}_1$ and $\mathcal{C}_2$ are free. Hence, (4.2.1) holds true. Therefore, $\mathcal{A}_{\leq t}$ and $\mathcal{A}_{>t}$ are free. By Lemma 4.2.4, we finish the proof. □

Theorem 4.2.8. *Let $\{X_t : t \geq 0\}$ be a free Markov process of elements in $\mathcal{A}_{sa}$. Then, $\{X_t : t \geq 0\}$ is a weak Markov process in W^*-probability space $(\mathcal{A}, \tau)$.*

Proof. For any $t_0 \geq 0$, let $\mathcal{A}_{\leq t_0} = W^*\{X_t : t \leq t_0\}$, $\mathcal{A}_{=t_0} = W^*(X_{t_0})$ and $\mathcal{A}_{\geq t_0} = W^*\{X_t : t \geq t_0\}$. Let $\mathbf{E}_{t_0}$ be the trace-preserving conditional expectation on $\mathcal{A}_{=t_0}$. For $A \in \mathcal{A}_{\leq t_0}$ and $B \in \mathcal{A}_{\geq t_0}$, we have

$$\begin{aligned}
\mathbf{E}_{t_0}(AB) &= \mathbf{E}_{t_0}((A - \mathbf{E}_{t_0}(A) + \mathbf{E}_{t_0}(A))((B - \mathbf{E}_{t_0}(B) + \mathbf{E}_{t_0}(B))) \\
&= \mathbf{E}_{t_0}((A - \mathbf{E}_{t_0}(A))(B - \mathbf{E}_{t_0}(B))) + (\mathbf{E}_{t_0}(A)\mathbf{E}_{t_0}((B - \mathbf{E}_{t_0}(B))) \\
&+ \mathbf{E}_{t_0}((A - \mathbf{E}_{t_0}(A))\mathbf{E}_{t_0}(B)) + \mathbf{E}_{t_0}(A)\mathbf{E}_{t_0}(B) \\
&= (\mathbf{E}_{t_0}(A)\mathbf{E}_{t_0}((B - \mathbf{E}_{t_0}(B))) + \mathbf{E}_{t_0}((A - \mathbf{E}_{t_0}(A))\mathbf{E}_{t_0}(B)) \\
&+ \mathbf{E}_{t_0}(A)\mathbf{E}_{t_0}(B) \\
&= \mathbf{E}_{t_0}(A)\mathbf{E}_{t_0}(B),
\end{aligned}$$

where the third equality holds true because that free Markov property of X_t. □

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4.3 A Kind of Free SDEs

In this section, we study a kind of free stochastic differential equation (free SDE) (4.3.3) and the free Markov property of its solution. We generalize Biane and Speicher's results on free differential equations driven by free Brownian motion (see [11]) to those on free SDEs driven by free Levy processes. On the other hand, our results provide a method to get free Markov processes (of random variables with un-compactly supported distributions).

Let $(\mathcal{A}, \tau)$ be a filtered W^*-probability space with filtration $\{\mathcal{A}_t : t \geq 0\}$, in which $S_{1,t}, \cdots, S_{k,t}(t \geq 0)$, a k-dimensional $\mathcal{A}_t$-free Brownian motion is defined. Each $S_{i,t}$ is an $\mathcal{A}_t$-free Brownian motion, and $\{S_{1,t} : t \geq 0\}, \cdots, \{S_{k,t} : t \geq 0\}$ are free in $(\mathcal{A}, \tau)$. In [11], Biane and Speicher showed that

Theorem 4.3.1. *(Theorem 3.1, Proposition 3.3 in [11]) Let $Q_1, Q_2, \cdots, Q_k : \mathcal{A}_{sa}^k \to \mathcal{A}$ be k locally operator-valued Lipschitz functions (with respect to operator norm) such that each $Q_i : \mathcal{A}_{s,sa}^k \to \mathcal{A}_{s,sa}$ for all $s \geq 0$. If there exist constants $a, b \in \mathbb{R}$ and $a > 0$ such that*

$$\sum_{i=1}^{k}(Q_i(X_1, \cdots, X_k)X_i + X_iQ_i(X_1, \cdots, X_k) + 1) \leq a\sum_{i=1}^{k} X_i^2 + b, \qquad (4.3.1)$$

for all $X_1, \cdots, X_k \in \mathcal{A}_{sa}$. Then, given arbitrary initial conditions $X_{i,0} \in \mathcal{A}_0(i = 1, 2, \cdots, k)$, the system

$$dX_{i,t} = Q_i(X_{1,t}, \cdots, X_{k,t})dt + dS_{i,t}, i = 1, \cdots, k, t \geq 0 \qquad (4.3.2)$$

has a unique solution $X(t) = (X_{1,t}, \cdots, X_{k,t})$ for all $t \geq 0$. Furthermore, we have $X_{i,t} \in \mathcal{A}_t$ for all $i = 1, \cdots, k, t \geq 0$, the maps $t \to X_{i,t}$ are norm continuous. Moreover, let $\mathcal{B}_{\leq t} = W^*\{X_{i,0}, S_{i,s} : s \leq t, 1 \leq i \leq k\}$, $\mathcal{B}_{\geq t} = W^*\{X_{i,t}, S_{i,s} - S_{i,t} : s \geq t, 1 \leq i \leq k\}$ and $\mathbf{B}_{=t} = W^*\{X_{i,t} : 1 \leq i \leq k\}$, then $(\mathcal{B}_{\leq t}, \mathcal{B}_{=t}, \mathcal{B}_{\geq t})$ is a free Markovian triple (i. e., $\mathcal{B}_{\leq t}$ and $\mathcal{B}_{\geq t}$ are $\mathcal{B}_{=t}$-free).

It is obvious that (4.3.1) is equivalent to

$$\sum_{i=1}^{k}(Q_i(X_1, \cdots, X_k)X_i + X_iQ_i(X_1, \cdots, X_k)) \leq a\sum_{i=1}^{k} X_i^2 + b, \qquad (4.3.1)'$$

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for all $X_1, \cdots, X_k \in \mathcal{A}_{sa}$ and some real numbers a, b and $a > 0$. In this section, we consider a system similar to (4.3.2) as follows.

$$dX_i(t) = Q_i(X_{1,t}, \cdots, X_{k,t})dt + dS_{i,t}, i = 1, \cdots, k, t \geq 0, \tag{4.3.3}$$

where $\{S_{i,t} : t \geq 0\}$ $(i = 1, \cdots, k)$ are $\mathcal{A}_t$-free Lévy processes of elements in $\mathcal{A}_{sa}$ (By [1, Lemma 1], the function $t \to S_{i,t}$ is continuous in $L^n(\mathcal{A}, \tau)$, for all $n \in \mathbb{N}$), and $\{S_{1,t} : t \geq 0\}, \cdots, \{S_{k,t} : t \geq 0\}$ are free in $(\mathcal{A}, \tau)$. We shall prove that, under conditions similar to those in Theorem 4.3.1, the system (4.3.3) has a unique solution $X_t = (X_{1,t}, \cdots, X_{k,t}) \in L^2(\mathcal{A}, \tau)$. Moreover, we shall prove that $\{X_t : t \geq 0\}$ is a free Markov process.

Lemma 4.3.2. *For $1 \leq i \leq k$, let $Q_i : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ be a Lipschitz maps with respect to $\|\cdot\|_2$, such that $Q_i : \mathcal{A}_{s,sa}^k \to \mathcal{A}_{s,sa}$, for $i = 1, \cdots, k, s \geq 0$. Then, given arbitrary initial conditions $X_{i,0} \in \mathcal{A}_0$, $i = 1, 2, \cdots, k$, (4.3.3) has a unique solution $X_t = (X_{1,t}, \cdots, X_{k,t})$ for all $t \geq 0$. Furthermore, we have $X_{i,t} \in L^2(\mathcal{A}_{t,sa}, \tau)$ for all $i = 1, ..., k, t \geq 0$ and $t \to X_{i,t}$ is continuous with respect to $\|\cdot\|_2$.*

Proof. The solution $X_{i,t}$ to (4.3.3) is a process $t \to X_{i,t} \in L^2(\mathcal{A}_{sa}, \tau)$ such that

$$X_{i,t} = X_{i,0} + \int_0^t Q_i(X_{1,s}, \cdots, X_{k,s})ds + S_{i,s}, \forall t \geq 0, 1 \leq i \leq k. \tag{4.3.4}$$

We use Picard iteration method to get the solution. Since Q_i is a Lipschitz function, there exists $C > 0$ such that

$$\|Q_i(X_1, \cdots, X_k) - Q_i(Y_1, \cdots, Y_k)\|_2 \leq C \sum_{i=1}^k \|X_i - Y_i\|_2,$$

for all $X_i, Y_i \in \mathcal{A}_{sa}, 1 \leq i \leq k$. Take $T > 0$ such that $kCT < 1$. For $0 \leq t \leq T$, let $X_{i,t}^{(0)} = X_{i,0}, 1 \leq i \leq k$, and

$$X_{i,t}^{(n+1)} = X_{i,0} + \int_0^t Q_i(X_{1,s}^{(n)}, \cdots, X_{k,s}^{(n)})ds + S_{i,t}, n = 1, 2, \cdots. \tag{4.3.5}$$

Then, $X_{i,t}^{(0)} \in \mathcal{A}_{t,sa}$ and $X_{i,t}^{(0)} \in \mathcal{A}_{t,sa}$ is continuous with respect to $\|\cdot\|_2$, for $1 \leq i \leq k$. Assume $X_{i,t}^{(n)}(t) \in L^2(\mathcal{A}_{t,sa}, \tau)$ and $t \to X_{i,t}^{(n)} \in L^2(\mathcal{A}_{sa}, \tau)$ is continuous with respect to $\|\cdot\|_2$. Then, $Q_i(X_{1,s}^{(n)}, \cdots, X_{k,s}^{(n)}) \in L^2(\mathcal{A}_{s,sa}, \tau)$ and $s \to Q_i(X_{1,s}^{(n)}, \cdots, X_{k,s}^{(n)})$

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is continuous with respect to $\|\cdot\|_2$, since $Q_i : L^2(\mathcal{A},\tau)^k \to L^2(\mathcal{A},\tau)$ is continuous. It implies that $X_{i,t}^{(n+1)} = X_{i,0} + \int_0^t Q_i(X_{1,s}^{(n)},\cdots,X_{k,s}^{(n)})ds + S_{i,t}) \in L^2(\mathcal{A}_{t,sa},\tau)$ and $t \to X_{i,t}^{(n+1)} L^2(\mathcal{A}_{sa},\tau)$ is continuous. By induction, $X_{i,t}^{(n)} \in \mathcal{A}_{t,sa}$ and $t \to X_{i,t}^{(n)} \in L^2(\mathcal{A}_{sa},\tau)$ is continuous with respect to $\|\cdot\|_2$. Note that

$$\begin{aligned}
\|X_{i,t}^{(n+1)} - X_{i,t}^{(n)}\|_2 &= \|\int_0^t (Q_i(X_{1,s}^{(n)},\cdots,X_{k,s}^{(n)}) - Q_i(X_{1,s}^{(n-1)},\cdots,X_{k,s}^{(n-1)}))ds\|_2 \\
&\le \int_0^t \|Q_i(X_{1,s}^{(n)},\cdots,X_{k,s}^{(n)}) - Q_i(X_{1,s}^{(n-1)},\cdots,X_{k,s}^{(n-1)})\|_2 ds \\
&\le C\int_0^t \sum_{i=1}^k \|X_{i,s}^{(n)} - X_{i,s}^{(n-1)}\|_2 ds.
\end{aligned}$$

Let $\mathbf{D}_n = \sup_{0\le t\le T}\sum_{i=1}^k \|X_{i,t}^{(n)} - X_{i,t}^{(n-1)}\|_2$, we have

$$\mathbf{D}_n \le kTC\mathbf{D}_{n-1} \le \cdots \le (KTC)^{n-1}\mathbf{D}_1.$$

It follows that $\{X_{i,t}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence with respect to $\|\cdot\|_2$, since $0 < kTC < 1$. Therefore, there exist $X_{i,t} \in L^2(\mathcal{A}_{t,sa},\tau)$, for $0 \le t \le T, i = 1,2,\cdots,k$, such that $X_{i,t} = \lim_{n\to\infty} X_{i,t}^{(n)}$ where the limit is taken in the topology of norm $\|\cdot\|_2$. Note that $Q_i : L^2(\mathcal{A}_{sa},\tau)^k \to L^2(\mathcal{A}_{sa},\tau)$ is continuous with respect to $\|\cdot\|_2$. Let n approach ∞ in (4.3.5), we get (4.3.4). Hence, $X_t = (X_{1,t},\cdots,X_{k,t})$ is a solution to (4.3.3), and $X_{i,t} \in L^2(\mathcal{A}_{t,sa},\tau)$, for $0 \le t \le T$. Now we show that $t \to X_{i,t} \in L^2(\mathcal{A}_{sa},\tau)$ is continuous. For $0 \le s,t$, we have

$$\begin{aligned}
\|X_{i,s} - X_{i,s}\|_2 &\le \|X_{i,s} - X_{i,s}^{(n)}\|_2 + \|X_{i,s}^{(n)} - X_{i,t}^{(n)}\|_2 + \|X_{i,t} - X_{i,t}^{(n)}\|_2 \\
&= \lim_{m\to\infty}\|X_{i,s}^{(m)} - X_{i,s}^{(m)}\|_2 + \|X_{i,s}^{(n)} - X_{i,t}^{(n)}\|_2 + \lim_{m\to\infty}\|X_{i,t}^{(m)} - X_{i,t}^{(n)}\|_2 \\
&\le 2\sum_{m=n}^\infty (kCT)^m (KTC)^{n-1}\mathbf{D}_1 + \|X_{i,s}^{(n)} - X_{i,t}^{(n)}\|_2.
\end{aligned}$$

Since $\lim_{n\to\infty}\sum_{m=n}^\infty (kCT)^m (KTC)^{n-1}\mathbf{D}_1 = 0$, for $\epsilon > 0$, there exists n such that

$$\sum_{m=n}^\infty (kCT)^m (KTC)^{n-1}\mathbf{D}_1 < \epsilon/4.$$

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Note also that $t \mapsto X_{i,t}^{(n)}$ is continuous. For the above $\epsilon > 0$, and $t \in [0, T]$, there exists $\delta > 0$ such that $\|X_{i,s}^{(n)} - X_{i,t}^{(n)}\|_2 < \epsilon/2$, whenever, $|t-s| < \delta$. Hence, we have

$$\|X_{i,s} - X_{i,t}\|_2 \leq \epsilon,$$

whenever $|t - s| < \delta$. It follows that $t \to X_{i,t} \in L^2(\mathcal{A}_{sa}, \tau)$ is continuous. For $T < t \leq 2T$, (4.3.4) can be rewritten as

$$X_{i,t} = X_{i,T} + \int_T^t Q_i(X_{1,s}, \cdots, X_{k,s})ds + S_{i,t} - S_{i,T}.$$

Let $X_{i,t}^{(0)} = X_{i,T}$ and

$$X_{i,t}^{(n+1)} = X_{i,T} + \int_T^t Q_i(X_{1,s}^{(n)}, \cdots, X_{k,s}^{(n)})ds + S_{i,t} - S_{i,T}, n = 1, 2, \cdots.$$

As the above proof, we can prove that (4.3.3) has solution $X_t = (X_{1,t}, \cdots, X_{k,t})$, for $T < t \leq 2T$. Generally, for $t > 0$, there exists $n \in \mathbb{N}$ such that $nT < t \leq (n+1)T$. Thus, after doing the above process n times, we get a solution of (4.3.3). Hence, by the construction of $X_t = (X_{1,t}, \cdots, X_{k,t})$, $(X_{1,t}, \cdots, X_{k,t}) \in L^2(\mathcal{A}_{t,sa}, \tau)$ and $t \to X_{i,t}$ is continuous with respect to $\|\cdot\|_2$.

Uniqueness. Suppose there are two solutions $X_{i,t}$ and $Y_{i,t}$ in $L^2(\mathcal{A}_{sa}, \tau)$ $(1 \leq i \leq k)$. Then, we have

$$\sup_{0 \leq s \leq t} \sum_{i=1}^{k} \|X_{i,s} - Y_{i,s}\|_2 \leq kCt \sup_{0 \leq s \leq t} \sum_{i=1}^{k} \|X_{i,s} - Y_{i,s}\|_2.$$

By Bellman-Gronwall Inequality (Lemma 3.2 in [24]), $\sup_{0 \leq s \leq t} \sum_{i=1}^{k} \|X_{i,t} - Y_{i,t}\|_2 = 0$. Hence, $X_{i,t} = Y_{i,t}, \forall t \geq 0, 1 \leq i \leq k$. □

Lemma 4.3.3. *Let $Q : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ be a locally operator-valued Lipschitz function, and $h : [0, \infty) \to [0, 1]$ be a continuous function with the following property: there is a $R > 0$ such that $h|_{[0,R]} = 1$, $h|_{[2R,\infty)} = 0$ and there is a $C > 0$ such that $|h(t) - h(s)| \leq C|t - s|, \forall t, s \geq 0$. Let*

$$f(X_1, \cdots, X_k) = Q(X_1, \cdots, X_k)h(\sum_{i=1}^{k} \|X_i\|_2), \forall X_1, \cdots, X_k \in L^2(\mathcal{A}_{sa}, \tau).$$

Then f is a Lipschitz function.

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Proof. The proof is the same as that of Lemma 3.2 in [11]. □

Lemma 4.3.4. *Let $A \in L^2(\mathcal{A}_{sa}, \tau) \subseteq \widetilde{\mathcal{A}}_{sa}$. Then $A^2 \in L^1(\mathcal{A}_{sa}, \tau)$ and $\|A\|_2 = \tau(A^2)^{1/2}$.*

Proof. There exists a sequence $\{A_n : n \in \mathbb{N}\}$ of elements in $\mathcal{A}_{sa}$ such that $\lim_{n\to\infty} \|A - A_n\|_2 = 0$. Let $A^2 - A_n^2 = U_n^*|A^2 - A_n^2|$ be the polar decomposition of $A^2 - A_n^2$. Then we have

$$\begin{aligned}
\|A^2 - A_n^2\|_1 &= \tau(U_n(A^2 - A_n^2)) \\
&\leq |\tau(U_n A(A - A_n))| + |\tau((A - A_n)A_n U_n)| \\
&\leq (\|U_n A\|_2 + \|U_n A_n\|_2)\|A - A_n\|_2 \\
&\leq (\|A\|_2 + \|A_n\|_2)\|A - A_n\|_2 \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Hence, $A^2 \in L^1(\mathcal{A}, \tau)$. Moreover,

$$\begin{aligned}
\lim_{n\to\infty} |\tau(A^2 - A_n^2)| &= \lim_{n\to\infty} (|\tau((A - A_n)A)| + |\tau(A_n(A_n - A))|) \\
&\leq \lim_{n\to\infty} \|A + A_n\|_2(\|A\|_2 + \|A_n\|_2) \\
&= 0.
\end{aligned}$$

Thus,

$$\|A\|_2^2 = \lim_{n\to\infty} \|A_n\|_2^2 = \lim_{n\to\infty} \tau(A_n^2) = \tau(A^2).$$

□

To prove the existence of the solution of (4.3.3), we need the following lemma. First, we introduce some notions (see [1] for details).

Let $\mathcal{A}^{op}$ be the opposite algebra of $\mathcal{A}$ (i. e., the von Neumann algebra obtained by defining $A \cdot B = BA$, for A, B in $\mathcal{A}$ and preserving all other operations in $\mathcal{A}$). Given $0 \leq t_1 \leq \cdots \leq t_{n+1} < \infty$ and $A_1, B_1, \cdots, A_n, B_n \in \mathcal{A}$, the function $U(t) = \sum_{i=1}^n A_i \otimes B_i \chi_{[t_i, t_{i+1})}$ is called a *simple bi-process.* A simple bi-process $U(t)$ is *adapted with filtration* $\{\mathcal{A}_t : t \geq 0\}$, if $U(t) \in \mathcal{A}_t \otimes \mathcal{A}_t^{op}$, for all $t \geq 0$.

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The space of all $\mathcal{A}_t$-adapted simple bi-processes is denoted by $\mathcal{B}$. For $U(t) = \sum_{i=1}^n A_i \otimes B_i \chi_{[t_i,t_{i+1})} \in \mathcal{B}$, define

$$\int_0^\infty U(s)\sharp dS_s := \sum_{i=1}^n A_i(S_{t_{i+1}} - S_{t_i})B_i.$$

Denoted by m the multiplication map $\mathcal{A} \otimes \mathcal{A}^{op} \to \mathcal{A}$. Then $m(U(t)) = A_iB_i$, if $U(t) = \sum_{j=1}^n A_j \otimes B_j \chi_{[t_j,t_{j+1})}(t)$ and $t_i \le t < t_{i+1}$. Given $a > 0$, we may define a norm

$$\|U\|_{2,a}' = (\int_0^\infty \|U(s)\|_2^2 ds)^{1/2} + a\|\int_0^\infty m(U(s))ds\|_2,$$

for $U \in \mathcal{B}$. The completion of $\mathcal{B}$ with respect to $\|\cdot\|_{2,a}'$ is denoted by $\mathcal{B}_2^{2,a}$.

Lemma 4.3.5. *Let $t \to X_t$ be a continuous function in $L^2(\mathcal{A},\tau)$, $\{S_t : t \ge 0\}$ be an $\mathcal{A}_t$-free Lévy process of elements in $\mathcal{A}_{sa}$, and $r_1 = |\tau(S_1)|$. Then*

$$\max\{\|\int_0^t X_s dS_s\|_2, \|\int_0^t dS_s X_s\|_2\} \le \|X.\chi_{[0,t]}(\cdot)\|_{2,r_1}'.$$

Proof. By Proposition 6 in [1], for $X_t \in \mathcal{B}_2^{2,r_1}$, $\|\int_0^\infty X_s\sharp dS_s\|_2 \le \|X\|_{2,r_1}'$. Thus, it is enough to show that $X_s\chi_{[0,t]}(s) \in \mathcal{B}_2^{2,r_1}$, for all $t > 0$. In fact, for $n \in \mathbb{N}$, let $U_{n,s} = \sum_{i=1}^n X_{\frac{i}{n}t}\chi_{[\frac{(i-1)t}{n},\frac{it}{n})}(s)$. Then $U_n \in \mathcal{B}$ and

$$\begin{aligned}
&\|X.\chi_{[0,t]} - U_n\|_{2,r_1}' \\
&= (\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \|X_s - X_{\frac{it}{n}}\|_2^2 ds)^{1/2} + \|\sum_{i=1}^n \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} (X_s - X_{\frac{it}{n}})ds\|_2 \\
&\le \sum_{i=1}^n (\sup_{\frac{(i-1)t}{n} \le s \le \frac{it}{n}} \|X_s - X_{\frac{it}{n}}\|_2^2 \frac{t}{n})^{1/2} + \sum_{i=1}^n \sup_{\frac{(i-1)t}{n} \le s \le \frac{it}{n}} \|X_s - X_{\frac{it}{n}}\|_2 \frac{t}{n} \\
&\le \sum_{i=1}^n \sup_{0 \le s,s' \le t, |s-s'| \le \frac{t}{n}} \|X_s - X_s)\|_2(t^{1/2} + t) \\
&\to 0,
\end{aligned}$$

as $n \to \infty$, where we have used the fact that $s \to X_s$ is uniformly continuous as a function from $[0,t]$ into $L^2(\mathcal{A},\tau)$. Hence, $X\chi_{[0,t]}(\cdot) \in \mathcal{B}_2^{2,r_1}$. □

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Theorem 4.3.6. *Let $Q_i : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$, $(i = 1, \cdots, k)$ be k local Lipschitz mappings with respect to $\|\cdot\|_2$ such that $Q_i : \mathcal{A}_{s,sa}^k \to \mathcal{A}_{s,sa}$, for $i = 1, \cdots, k$, $s \geq 0$, and there exist constants $a, b > 0$ such that (4.3.1)′ holds. Then, given arbitrary initial conditions $X_{i,0} \in \mathcal{A}_0$ $(i = 1, 2, \cdots, k)$, the system (4.3.3) has a unique solution $X_t = (X_{1,t}, \cdots, X_{k,t})$ for $t \geq 0$. Furthermore, we have $X_{i,t}$ in $L^2(\mathcal{A}_{t,sa}, \tau)$ for $i = 1, \cdots, k, t \geq 0$, and $t \to X_{i,t}$ is continuous with respect to $\|\cdot\|_2$.*

Proof. For $R > 0$, let h_R be the function Lemma 4.3.3, and

$$f_i(X_1, \cdots, X_k) = Q_i(X_1, ..., X_k) h_R(\sum_{i=1}^{k} \|X_i\|_2),$$

for all $X_1, \cdots, X_k \in L^2(\mathcal{A}_{sa}, \tau)$ and $1 \leq i \leq k$. By lemmas 4.3.2, 4.3.3, the following system

$$X_{i,t} = X_{i,0} + \int_0^t f_i(X_{1,s}, \cdots, X_{k,s}) ds + S_{i,t}, 1 \leq i \leq k$$

has a unique solution $X_t^R = (X_{1,t}^R, \cdots, X_{k,t}^R)$. Note that, if $\sum_{i=1}^k \|X_{i,t}\|_2 \leq R$, we have $f_i = Q_i, 1 \leq i \leq k$. So, X_t^R is a solution to (4.3.3). Let $T_R = \inf\{t : \sum_{i=1}^k \|X_{i,t}^R\|_2 > R\}$, then X_t^R is a solution to (4.3.3), if $t < T_R$. Hence, we shall be done if we can prove that

$$\lim_{R \to \infty} T_R = \infty.$$

By [1, Corollary 12],

$$(S_{i,t})^2 = \int_0^t dS_{i,s} S_{i,s} + \int_0^t S_{i,s} dS_{i,s} + \Delta_{i,2}(t),$$

where $\Delta_{i,2}(t) = \lim_{N \to \infty} \sum_{j=1}^N (S_{i,\frac{j}{N}t} - S_{i,\frac{i-1}{N}t})^2$, the limit here is in operator norm (see Definition 3 in [1]). By Lemma 2 in [1], $\{\Delta_{i,k}(t) : t \geq 0\}$ is an $\mathcal{A}_t$-free Lévy process. Hence,

$$d(S_{i,t}^2) = dS_{i,t} S_{i,t} + S_{i,t} dS_{i,t} + d\Delta_{i,2}(t).$$

Let $X_t^R = (X_{1,t}^R, \cdots, X_{k,t}^R)$, we have

$$d((X_{i,t}^R)^2) = d(X_{i,0}^2 + X_{i,0} \int_0^t Q_i(X_s^R) ds) + X_{i,0} S_{i,t}$$

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$$+\int_0^t Q(X_s)dsX_{i,0} + (\int_0^t Q(X_s)ds)^2 + \int_0^t Q(X_s)dsS_{i,t}$$
$$+ S_{i,t}X_{i,0} + S_{i,t}\int_0^t Q(X_s)ds + (S_{i,t})^2)$$
$$= X_{i,0}dX_{i,t}^R + Q(X_t^R)dtX_{i,t}^R + \int_0^t Q(X-s)dsdX_{i,t}^R$$
$$+ dS_{i,t}X_{i,0} + dS_{i,t}\int_0^t Q(X_s)ds + S_{i,t}Q(X_t^R)dt + d(S_{i,t}^2)$$
$$= X_{i,t}^R dX_{i,t}^R + dX_{i,t}^R X_{i,t}^R + d\Delta_{i,2}(t).$$

Let $Z_t = (\sum_{i=1}^k (X_{i,t}^R)^2)^{1/2}$. Then,

$$d(e^{-at}Z_t^2) = -ae^{-at}(\sum_{i=1}^k (X_{i,t}^R)^2)$$
$$+ e^{-at}\sum_{i=1}^k (dX_{i,t}^R \cdot X_{i,t}^R + X_{i,t}^R \cdot dX_{i,t}^R + (d\Delta_{i,2}(t)))$$
$$= -ae^{-at}(\sum_{i=1}^k (X_{i,t}^R)^2) + e^{-at}\sum_{i=1}^k (f_i(X_{1,t}^R, \cdots, X_{k,t}^R))X_{i,t}^R$$
$$+ X_{i,t}^R f_i(X_{1,t}^R, \cdots, X_{k,t}^R)) + e^{-at}\sum_{i=1}^k (dS_{i,t}X_{i,t}^R + X_{i,t}^R dS_{i,t})$$
$$+ e^{-at}\sum_{i=1}^k (d\Delta_{i,2}(t)).$$

By Lemma 4.3.2, $t \to X_{i,t}^R$ is continuous with respect to $\|\cdot\|_2$. Therefore, $T_R > 0$, if R is big enough. Moreover, $\{t : \sum_{i=1}^k \|X_{i,t}^R\|_2 > R\}$ is open. So, for $t \le T_R$, we have X_t^R is a solution to (4.3). Hence, we have

$$Z_t^2 = e^{at}(Z_0^2 - a\int_0^t e^{-au}\sum_{i=1}^k (X_{i,u}^R)^2 du$$
$$+ \int_0^t e^{-au}(\sum_{i=1}^k f_i(X_{1,u}^R, \cdots, X_{k,u}^R)X_{i,u}^R + X_{i,u}^R f_i(X_{1,u}^R, \cdots, X_{k,u}^R))du$$
$$+ \int_0^t e^{-au}\sum_{i=1}^k (dS_{i,u}X_{i,u}^R + X_{i,u}^R dS_{i,u})du + e^{at}\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u)$$

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$$\leq e^{at}Z_0^2 + e^{at}\int_0^t be^{-au}du + e^{at}\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u)$$
$$+ e^{at}\int_0^t e^{-au}\sum_{i=0}^k (dS_{i,u}X_{i,u}^R + X_{i,u}^R dS_{i,u}),$$

where the inequality holds because of (4.3.1)$'$. Let $r = \max\{|\tau(S_{i,1}| : 1 \leq i \leq k\}$, we have

$$\tau(Z_t^2) \leq e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u))$$
$$+ e^{at}\sum_{i=1}^k |\tau(\int_0^t e^{-au}(dS_{i,u}X_{i,u}^R + X_{i,u}^R dS_{i,u}))|$$
$$\leq e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u))$$
$$+ e^{at}\sum_{i=1}^k \|\int_0^t e^{-au}dS_{i,u}X_{i,u}^R\|_2 + e^{at}\sum_{i=1}^k \|\int_0^t e^{-au}X_{i,u}^R dS_{i,u}\|_2$$
$$\leq e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u))$$
$$+ 2e^{at}\sum_{i=1}^k (\int_0^t \|X_{i,u}^R\|_2^2 e^{-2au}du)^{\frac{1}{2}} + 2re^{at}\sum_{i=1}^k \|\int_0^t e^{-au}X_{i,u}^R du\|_2$$
$$\leq e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_i(u))$$
$$+ 2e^{at}\sup_{0\leq u\leq t}\sum_{i=1}^k \|X_{i,u}^R\|_2((\int_0^t e^{-2au}du)^{\frac{1}{2}} + r\int_0^t e^{-au}du)$$
$$\leq e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u))$$
$$+ 2ke^{at}\sup_{0\leq u\leq t}(\sum_{i=1}^k \|X_{i,u}^R\|_2^2)^{\frac{1}{2}}((\int_0^t e^{-2au}du)^{\frac{1}{2}} + r\int_0^t e^{-au}du),$$

where the third inequality holds by Lemma 4.3.5. Let

$$\varphi(t) = \sup\{\tau(Z_u^2) : 0 \leq u \leq t\} = \sup_{0\leq u\leq t}\sum_{i=1}^k \|X_{i,u}^R\|_2^2,$$

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then we have

$$\varphi(t) \le e^{at}\varphi(0) + \frac{b}{a}(e^{at}-1) + e^{at}\tau(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u))$$
$$+ 2k[(\frac{e^{2at}-1}{2a})^{\frac{1}{2}} + \frac{r(e^{at}-1)}{a}]\varphi(t)^{\frac{1}{2}}.$$

Note that $\sum_{i=1}^k \|X^R_{i,T_R})\|_2 = R$, so $\max_{1\le i\le k}\|X^R_{i,T_R}\|_2 \ge R/k$. It follows that

$$\varphi(T_R)^{1/2} = (\sup_{0\le u\le T_R}\sum_{i=1}^k \|X^R_{i,u}\|_2^2)^{1/2} \ge R/k.$$

It implies that

$$R^2/k^2 \le \varphi(T_R)$$
$$\le e^{aT_R}\varphi(0) + \frac{b}{a}(e^{aT_R}-1) + e^{aT_R}\tau(\int_0^{T_R} e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u))$$
$$+ 2k[(\frac{e^{2aT_R}-1}{2a})^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a}]\varphi(T_R)^{\frac{1}{2}}.$$

Moreover,

$$\varphi(T_R) = \sup_{0\le u\le T_R}\sum_{1\le i\le k}\|X^R_{i,u}\|_2^2 \le \sup_{0\le u\le T_R}(\sum_{1\le i\le k}\|X^R_{i,u}\|_2)^2 \le R^2.$$

Hence, let $r_1' = \max \tau(\Delta_{i,2}(1)) : 1 \le i \le k$, we have

$$R/k^2 \le \frac{\varphi(0)}{R}e^{aT_R} + \frac{b(e^{aT_R}-1)}{aR}$$
$$+ e^{aT_R}\frac{1}{R}|\tau(\int_0^{T_R}\sum_{i=1}^k e^{-au}(d\Delta_{i,2}(u))| + 2k((\frac{e^{2aT_R}-1}{2a})^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a})$$
$$\le \frac{\varphi(0)}{R}e^{aT_R} + \frac{b(e^{aT_R}-1)}{aR} + e^{aT_R}\frac{1}{R}\sum_{i=1}^k \|\int_0^{T_R} e^{-au}d\Delta_{i,2}(u)\|_2$$
$$+ 2k((\frac{e^{2aT_R}-1}{2a})^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a})$$
$$\le \frac{\varphi(0)}{R}e^{aT_R} + \frac{b(e^{aT_R}-1)}{aR} + e^{aT_R}\frac{1}{R}\sum_{i=1}^k((\int_0^{T_R} e^{-2au}du)^{1/2}$$

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$$+ r_1' \int_0^{T_R} e^{-au} du) + 2k((\frac{e^{2aT_R}-1}{2a})^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a})$$

$$\leq \frac{\varphi(0)}{R} e^{aT_R} + \frac{b(e^{aT_R}-1)}{aR}$$

$$+ e^{aT_R} \frac{k}{R}((\frac{1}{2a})^{1/2} + r_1' a^{-1}) + 2k((\frac{e^{2aT_R}-1}{2a})^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a})$$

It is obvious that map $R \longrightarrow T_R$ is increasing. Thus, if $\lim_{R\to\infty} T_R \neq \infty$, the right hand side of the inequality above is upper bounded. On the other hand, the left hand side is upper unbounded as $R \longrightarrow \infty$. This gives rise of a contradiction. Hence, $\lim_{R\to\infty} T_R = \infty$. We finish the proof of the existence of solution to (4.3.3). Moreover, for $t \geq 0$, we can take $R > 0$ such that $t \leq T_R$, so, $X_{i,t} = X_{i,t}^R$. Hence, $X_{i,t} \in L^2(\mathcal{A}_{t,sa}, \tau)$, and $t \longrightarrow X_{i,t}$ is continuous with respect to $\|\cdot\|_2$, by Lemma 4.3.2.

Uniqueness. This result follows from the uniqueness of solutions to (4.3.4) (Lemma 4.3.2). □

We shall show that the solution X_t to (4.3.3) is a free Markov process in $L^2(\mathcal{A}, \tau)$. By the following well known result, X_t is a free Markov process in $\widetilde{\mathcal{A}}$.

Lemma 4.3.7. *Let $(\mathcal{A}, \tau)$ be a W^*-probability space, $L^2(\mathcal{A}, \tau)$ be the completion of $\mathcal{A}$ with respect to $\|\cdot\|_2$ and $L^2(\mathcal{A}_{sa}, \tau)$ be the completion of $\mathcal{A}_{sa}$ in $L^2(\mathcal{A}, \tau)$. Then, $L^2(\mathcal{A}, \tau) \subseteq \widetilde{\mathcal{A}}$ and $L^2(\mathcal{A}_{sa}, \tau) \subseteq \widetilde{\mathcal{A}}_{sa}$.*

Theorem 4.3.8. *Under the hypotheses of Theorem 4.3.6, and the condition that $Q : \mathcal{A}_{sa}^k \longrightarrow \mathcal{A}$ is polynomial of k non commutative unknown variables, the solution $X_t = (X_{1,t}, ..., X_{k,t})$ is a free Markov process.*

Proof. Let

$$\mathcal{B}_{\leq t} = W^*\{X_{i,0}, S_{i,s} : s \leq t, 1 \leq i \leq k\},$$

$$\mathcal{B}_{\geq t} = W^*\{X_{i,t}, S_{i,s} - S_{i,t} : s \geq t, 1 \leq i \leq k\},$$

$$\mathcal{C}_{\leq t} = W^*\{X_{i,s} : s \leq t, 1 \leq i \leq k\}, \mathcal{C}_{\geq t} = W^*\{X_{i,s} : s \geq t, 1 \leq i \leq k\},$$

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and

$$\mathcal{C}_{=t} = W^*\{X_{i,t} : 1 \leq i \leq k\}.$$

We want to show that

$$\mathcal{C}_{\leq t} \subseteq \mathcal{B}_{\leq t}, \mathcal{C}_{\geq t} \subseteq \mathcal{B}_{\geq t} \tag{4.3.6}$$

By the proofs of Lemma 4.3.2 and Theorem 4.3.6,

$$\lim_{n\to\infty} \|X_{i,t} - X_{i,t}^{(n)}\|_2 = 0, 1 \leq i \leq k,$$

where $X_{i,t}^{(0)} = X_{i,0} \in \mathcal{A}_0$, and $X_{i,t}^{(n)}$ $(n \geq 1)$ are defined by (4.3.5). Let $\mathcal{H}_{\leq t} = L^2(\mathcal{B}_{\leq t}, \tau)$. Then, $X_{0,t} \in \mathcal{H}_{\leq t}$. Let $f_i = Q_i h$ (see Lemma 4.3.3 for the definition of function h). Assume $X_{i,s}^{(n)} \in \mathcal{B}_{\leq t}, 1 \leq i \leq k, s \leq t$. Let $X_{i,s}^{(m,n)} \to X_{i,s}^{(n)}$ in norm $\|\cdot\|_2$, as $m \to \infty$, where $X_{i,s}^{(m,n)} \in (\mathcal{B}_{\leq t})_{sa}, 1 \leq i \leq k$. Then, $f_i(X_{1,s}^{(m,n)}(s), \cdots, X_{k,s}^{(m,n)}) \in \mathcal{B}_{\leq t}$, since Q_i is a polynomial. Note that $Q_i : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ is continuous with respect to $\|\cdot\|_2$. It implies that the $\|\cdot\|_2$ limit $f_i(X_{1,s}^{(n)}, \cdots, X_{k,s}^{(n)})$ of $f_i(X_{1,s}^{(m,n)}, \cdots, X_{k,s}^{(m,n)})$ is in $\mathcal{H}_{\leq t}$, for $s \leq t, 1 \leq i \leq k$. Hence,

$$X_{i,t}^{(n+1)} = X_{i,0} + \int_0^t f_i(X_{1,s}^{(n)}, \cdots, X_{k,s}^{(n)})ds + S_{i,t} \in \mathcal{H}_{\leq t}.$$

By induction, $X_{i,t}^{(n)} \in \mathcal{H}_{\leq t}$. Hence, $X_{i,t} = \lim_{n\to\infty}(X_{i,t}^{(n)}) \in \mathcal{H}_{\leq t}$. It follows that $\mathcal{C}_{\leq t} \subseteq \mathcal{B}_{\leq t}$. For $s \geq t$,

$$X_{i,s} = X_{i,t} + \int_t^s f_i(X_{1,u}, \cdots, X_{k,u})du + S_s - S_t.$$

By the above proof and the uniqueness of the solutions to (4.3.3), $\mathcal{C}_{\geq t} \subseteq \mathcal{B}_{\geq t}$. Now we show that $\mathcal{B}_{\leq t}$ and $\mathcal{B}_{\geq t}$ are $\mathcal{C}_{=t}$-free. Note that $W^*\{X_0, S_s : s \leq t\}$ and $W^*\{S_u - S_t : u \geq t\}$ are free in $(\mathcal{A}, \tau)$, and $\mathcal{C}_{=t} \subseteq W^*\{X_0, S_s : s \leq t\}$. By [11, Lemma 2.1], $\mathcal{B}_{\leq t}$ and $\mathcal{B}_{\geq t}$ are $\mathcal{C}_{=t}$-free. Therefore, $\mathcal{C}_{\leq t}$ and $\mathcal{C}_{\geq t}$ are $\mathcal{C}_{=t}$-free. By Definition 4.3.2, X_t is a free Markov process in $L^2(\mathcal{A}, \tau) \subseteq \widetilde{\mathcal{A}}$. □

For $k = 1$, we can get more general condition on Q so that the solution is a free Markov process.

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Theorem 4.3.9. *Under the hypotheses of Theorem 4.3.6, and the conditions that $k = 1$ and $Q : \mathbb{R} \to \mathbb{R}$ is Borel measurable, the solution X_t is a free Markov process.*

Proof. We use the same notation (with $k = 1$) as that in Theorem 4.3.8. Assume $X_{n,s} \in \mathcal{H}_{\leq t}$, then $f(X_{n,s}) \in \mathcal{B}_{\leq t}$, since $f = Qh$ is bounded measurable function. Hence,

$$X_{n+1,t} = X_0 + \int_0^t f(X_{n,s})ds + S_t \in \mathcal{H}_{\leq t}.$$

The rest of the proof is the same as that of Theorem 4.3.8. □

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Chapter 5

FREE ORNSTEIN-UHLENBECK PROCESSES

The main theme of this chapter is the study of free Ornstein-Uhlenbeck processes.

Barndorff-Nielsen and Thorbjornsen [4] established the stochastic integrals of continuous functions with respect to a free Levy process, the stochastic integral representation of free self-decomposable distribution (see Theorem 6.1, 6.5 in [4]), and remarked that a possible definition of a free OU-process driven by a free Lévy process can be given (but no further details were given). In Section 5.1, We show that free *OU* processes are solutions of a special kind of differential equations we studied in section 4.3 (Theorem 5.1.3). Furthermore, we show that, under certain condition ((5.1.3) below), a probability distribution on $\mathbb{R}$ is free self-decomposable if and only if it is the limit distribution of a free OU-process (Theorem 5.1.4). Moreover, it is showed that a probability measure on $\mathbb{R}$ is free self-decomposable if and only if it is the distribution of a stationary free OU process (Theorem 5.1.5). Section 5.2 is devoted to the study of periodic free OU processes. We show that a free OU process defined on the finite interval $[0, 1]$ can be extended periodically to a stationary process on the whole real line (Theorem 5.2.1). Moreover, we show that the class of the distributions of the stationary extensions is bigger strictly than the class of all free self-decomposable distributions, and the class is smaller

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strictly than that of all free infinitely divisible distributions on $\mathbb{R}$ (Theorem 5.2.3). At the end of this section, we give a characterization for a probability measure on $\mathbb{R}$ to be the stationary distribution of a periodic free OU process in terms of its Levy measure (Theorem 5.2.6). Finally, in Section 5.3, the notion of fractional free Brownian motion is introduced (Definition 5.3.2), and examples of free fractional Brownian motion are given. These examples are given in terms of creation and annihilation operators on full Fock spaces. (Theorem 5.3.4 and Remark 5.3.5). We show that the Langevin equation driven by fractional free Brownian motion has a unique solution. We call the solution a *fractional free OU process* (Theorem 5.3.8).

5.1 Free OU Processes

In this section, we consider a special case of (4.3.3). Let $k = 1$, $Q_1(X) = -\lambda X$, $\lambda > 0$ and $\{S_t : t \geq 0\}$ is $\mathcal{A}_t$-free Lévy process of operators in $\mathcal{A}_{sa}$. That is, we consider the following equation

$$X_t = X_0 - \lambda \int_0^t X_s ds + S_t, t \geq 0, \tag{5.1.1}$$

where self-adjoint operator $X_0 \in \mathcal{A}_0$. We call

$$X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-u)} dS_u, t \geq 0$$

a *free OU process*, where $\int_0^t e^{-\lambda(t-u)} dS_u$ is defined by Theorem 6.1 in [4] (Generally, we can define free OU process $\{X_t : t \geq 0\}$ by the formula above in the case that $\{S_t : t \geq 0\}$ is a free Levy process of self-adjoint operators in $\widetilde{\mathcal{A}}$, and X_0 is affiliated with $\mathcal{A}_0$). We show that the free OU process is the unique solution to (5.1.1) and the limit distribution of X_t, as $t \to \infty$, is free self-decomposable.

Lemma 5.1.1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. For $n \in \mathbb{N}$, and $a = t_{n,0} < t_{n,1} < \cdots < t_{n,k_n} = b$, a partition of $[a, b]$, let $f_n(t) = \sum_{i=1}^{k_n} a_{n,i} \chi_{[t_{n,i-1}, t_{n,i})}(t)$, $f_n(b) = f(b)$ be a step function such that $f_n(t) \rightrightarrows f(t)$ uniformly for $t \in [a, b]$. Then*

$$\lim_{n\to\infty} \| \int_a^b (f(t) - f_n(t)) dS_t \|_2 = 0.$$

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Proof. By Lemma 4.3.5, $f - f_n$ is in $\mathcal{B}_2^{2,a}$. Hence,

$$\begin{aligned}\|f_n - f\|_2 &\leq \|f_n - f\|_{L^2([a,b])} + |\tau(S_1)| \cdot \|f_n - f\|_{L^1([a,b])} \\ &\leq \|f_n - f\|_{L^\infty([a,b])}(b-a)(1 + |\tau(S_1)|) \\ &\to \infty,\end{aligned}$$

as $n \to \infty$, since $f_n \rightrightarrows f$ on $[a, b]$. □

The following lemma gives some kind of Fibini Theorem. Some ideas in the proof are from the proof of Proposition 35 in [23].

Lemma 5.1.2. *Let f and g be continuous functions on $[a, b]$,*

$$X = \int_a^b g(s) \int_a^s f(u) dS_u ds, \quad Y = \int_a^b f(u) \int_u^b g(s) ds dS_u.$$

Then $X = Y$.

Proof. Step I. We show that

$$\int_a^b u dS_u = bS_b - aS_a - \int_a^b S_s ds. \tag{5.1.2}$$

For $n \in \mathbb{N}$ and $t_{n,i} = a + \frac{i(b-a)}{2^n}$, $i = 0, \cdots, 2^n$, define $f_n(t) = \sum_{i=1}^{2^n} t_{n,i} \chi_{[t_{n,i-1}, t_{n,i})}$, $f_n(b) = f(b)$. Then $f_n(t) \rightrightarrows t$ uniformly for $t \in [a, b]$. By Lemma 5.1.1,

$$\int_a^b t dS_t = \lim_{n \to \infty} \int_a^b f_n(t) dS_t.$$

On the other hand,

$$\begin{aligned}\int_a^b f_n(t) dS_t &= \sum_{i=1}^{2^n} t_{n,i}(S_{t_{n,i}} - S_{t_{n,i-1}}) \\ &= \sum_{i=1}^{2^n-1} t_{n,i} S_{t_{n,i}} - \sum_{i=1}^{2^n-1} t_{n,i+1} S_{t_{n,i}} + bS_b - t_{n,1} S_a\end{aligned}$$

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$$= bS_b - t_{n,1}S_a - (\sum_{i=1}^{2^n-1} \frac{b-a}{2^n} S_{t_{n,i}} + \frac{b-a}{2^n} S_b) + \frac{b-a}{2^n} S_b$$

$$\to bS(b) - aS_a - \int_a^b S_t dt,$$

as $n \to \infty$. Hence, we get (5.1.2).

Step II. Let $f = \sum_{i=1}^k a_i \chi_{[t_{i-1},t_i)}$, $g = \sum_{j=1}^l b_j \chi_{[s_{j-1},t_j)}$ be step functions on $[a,b]$. Then $X = Y$.

The proof is the same as that of Proposition 35 in [54].

Step III. Let f be a continuous function, and $g = \sum_{i=1}^k b_i \chi_{[t_{i-1},t_i)}$ be a step function on $[a,b]$. Then $X = Y$.

For $n \in \mathbb{R}$, let $\{f_n : n \geq 1\}$ be a bounded step functions such that $f_n \rightrightarrows f$ on $[a,b]$, and let

$$X_n = \int_a^b g(s) \int_a^s f_n(u) dS_u ds, Y_n = \int_a^b f_n(u) \int_u^b g(s) ds dS_u.$$

By step II, $X_n = Y_n$. It is enough to show that

$$\lim_{n\to\infty} (\|X - X_n\|_2 + \|Y - Y_n\|_2) = 0.$$

In fact,

$$\begin{aligned}
\|(X - X_n)\|_2 &= \| \int_a^b g(s) \int_a^b \chi_{[a,s]}(u)(f_n(u) - f(u)) dS_u ds \|_2 \\
&\leq \int_a^b |g(s)| \cdot \| \int_a^b \chi_{[a,s]}(u)(f_n(u) - f(u)) dS_u \|_2 ds \\
&\leq \int_a^b |g(s)| (\|f_n - f\|_{L^2} + |\tau(S_1)| \|f_n - f\|_{L^1}) ds \\
&\leq \int_a^b |g(s)| ds \|f_n - f\|_{L^\infty} (b-a)(1 + |\tau(S_1)|) \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Similarly, $\lim_{n\to\infty} \|Y - Y_n\|_2 = 0$. Hence, $X = Y$.

Step IV. Both f and g are continuous functions on $[a,b]$. In this case, let $g_n = \sum_{i=1}^{k_n} b_{n,i} \chi_{[t_{n,i-1},t_{n,i})}$, $g_n(b) = g(b)$ such that $g_n \rightrightarrows f$ on $[a,b]$, and

$$X_n = \int_a^b g_n(s) \int_a^s f(u) dS_u ds, Y_n = \int_a^b f(u) \int_u^b g_n(s) ds dS_u.$$

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By Step III, $X_n = Y_n$. By the proof of Step III,

$$\lim_{n\to\infty} \|X - X_n\|_2 = 0, \lim_{n\to\infty} \|Y - Y_n\|_2 = 0.$$

Hence, $X = Y$. □

Theorem 5.1.3. *Let* $X_t = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-u)}dS_u$. *Then* $t \to X_t$ *is continuous with respect to* $\|\cdot\|_2$, *and* X_t *is the unique continuous solution to* (5.1.1).

Proof. It is obvious that $t \to X_t$ is continuous. Moreover,

$$\begin{aligned} -\lambda \int_0^t X_u du &= e^{-\lambda t}X_0 - X_0 - \lambda \int_0^t e^{-\lambda s} \int_0^s e^{\lambda u} dS_u ds \\ &= e^{-\lambda t}X_0 - X_0 - \lambda \int_0^t e^{\lambda u} \int_u^t e^{-\lambda s} ds dS_u \\ &= e^{-\lambda t}X_0 - X_0 + \int_0^t e^{-\lambda(t-u)} dS_u - S_t \\ &= X_t - X_0 - S_t, \end{aligned}$$

where the second equality holds because of Lemma 5.1.2.

Uniqueness. Suppose that (5.1.1) has another continuous solution Y_t. Let $Z_t = X_t - Y_t$, then $Z_t = -\lambda \int_0^t Z_u du$. By Bellman-Gronwell inequality, $Z_t = 0$, for $t \geq 0$. It follows that $X_t = Y_t$, for $t \geq 0$. □

Now we discuss the limit distribution of X_t. Let $\{S_t : t \geq 0\}$ be a free Lévy process of (unbounded) operators, (γ, σ) be the free generating pair of the process (see Section 2.1).

Theorem 5.1.4. *If the measure* σ *in the free generating pair* (γ, σ) *of Lévy process* $\{S_t : t \geq 0\}$ *of (unbounded) self-adjoint operators in* $\widetilde{\mathcal{A}}$ *satisfies*

$$\int_{|t|\geq 1} \log(1 + |t|)\sigma(dt) < \infty, \tag{5.1.3}$$

the limit distribution of X_t, *as* $t \to \infty$, *is* $\boxplus$ *self-decomposable.*

Conversely, if μ_0 *is a* $\boxplus$ *self-decomposable distribution on* $\mathbb{R}$, *there is a free OU process* $\{X_t | t \geq 0\}$ *such that the limit distribution of* X_t *is* μ_0.

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Proof. Note that $X_t = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-u)}dS_u$, so it is enough to show that the limit distribution of $\int_0^t e^{-\lambda(t-u)}dS_u$ is $\boxplus$ self-decomposable. Let $t_{n,j} = jt/n$, $j = 0, 1, ..., n$, $T_n = \sum_{j=1}^n e^{-\lambda(t-t_{n,j})}(S_{t_{n,j}} - S_{t_{n,j-1}})$, then $T_n \xrightarrow{\mathrm{P}} \int_0^t e^{-\lambda(t-u)}dS_u$, by Theorem 6.1 in [4]. On the other hand,

$$\begin{aligned}
T_n &= \sum_{j=1}^n e^{-\lambda(t-t_j)}(S_{t-(t-t_{n,j})} - S_{t-(t-t_{n,j-1})}) \\
&= \sum_{j=1}^n e^{-\lambda r_{n,n-j}}(S_{t-r_{n,n-j}} - S_{t-r_{n,n-j+1}}) \\
&\stackrel{\mathrm{d}}{=} \sum_{j=1}^n e^{-\lambda r_{n,n-j}}(S_{r_{n,n-j+1}} - S_{r_{n,n-j}}) \\
&= \sum_{i=1}^n e^{-\lambda r_{n,i-1}}(S_{r_{n,i}} - S_{r_{n,i-1}}) \xrightarrow{\mathrm{P}} \int_0^t e^{-\lambda u}dS_u.
\end{aligned}$$

Hence, we have

$$\int_0^t e^{-\lambda(t-u)}dS_u \stackrel{\mathrm{d}}{=} \int_0^t e^{-\lambda u}dS_u = \int_0^{t\lambda} e^{-u}dS_{u/\lambda}.$$

Let $\widetilde{S}_t = S_{t/\lambda}, \forall t \geq 0$. It is obvious that $\widetilde{S}_t$ is a $\mathcal{A}_t$-free Lévy process. Let $\phi_{\mu_1}(z)$ be the Voiculescu transform of $\mu(S_1)$. By [4], $\phi_{\mu(S_t)}(t) = t\phi_{\mu_1}(z)$. Let (γ, σ) be the free generating pair of μ_1, then $(t\gamma, t\sigma)$ is the free generating pair of $\mu(S_t)$. Hence, $\mu(\widetilde{S}_1) = \mu(S_{\frac{1}{\lambda}})$ has free generating pair $(\frac{1}{\lambda}\gamma, \frac{1}{\lambda}\sigma)$. It follows that the finite measure $\frac{1}{\lambda}\sigma$ in $(\frac{1}{\lambda}\gamma, \frac{1}{\lambda}\sigma)$ satisfies (5.1.3). By Theorem 6.5 in [4], there is a self-adjoint operator $X \in \widetilde{\mathcal{A}}$ such that

$$\int_0^t e^{-\lambda(t-u)}dS_u \stackrel{\mathrm{d}}{=} \int_0^{t\lambda} e^{-u}dS_{u/\lambda} \xrightarrow{\mathrm{d}} X,$$

as $t \to \infty$ and X has a $\boxplus$ self-decomposable distribution.

Suppose μ_0 is a free self-decomposable distribution on $\mathbb{R}$. By Theorem 6.5 in [4], there is free Levy process S_t satisfying (5.1.3) and $\mu(\int_0^\infty e^{-t}dS_t) = \mu_0$. Let

$$X_t = e^{-t}\int_0^t e^s dS_s, t \geq 0.$$

By the proof above, the limit distribution of X_t, as $t \to \infty$, is μ_0. □

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Theorem 5.1.5. *A probability measure μ on $\mathbb{R}$ is $\boxplus$ self-decomposable if and only if it is the distribution of a stationary free OU process.*

Proof. In [4], Barndorff-Nielsen and Thorbjornsen showed that a probability distribution μ on $\mathbb{R}$ is $\boxplus$ self-decomposable if and only if there is a free Levy process $\{Z'_t : t \geq 0\}$ of self-adjoint (probably unbounded) operators affiliated with a W^*-probability space $(\mathcal{B}, \tau_1)$ satisfying

$$\int_{|t|\geq 1} \log(1+|t|)\sigma(dt) < \infty,$$

where (γ, σ) is the free generation pair of μ, such that $\mu = \mu(\int_0^\infty e^{-t} dZ'_t)$. Let $\widetilde{Z}_t = Z'_{ct}$, for $t \geq 0$, and $\widetilde{Z}_t = -\widetilde{Z}_{-t}$, for $t < 0$, then $\{\widetilde{Z}_t : t \geq 0\}$ is a free Levy process. Let $\{\overline{Z}_t : t \geq 0\}$ be a free copy of $\{\widetilde{Z}_t : t \geq 0\}$ (i. e. $\{\overline{Z}_t : t \geq 0\}$ and $\{\widetilde{Z}_t : t \geq 0\}$ are free), and let

$$Z_t = \overline{Z}_t, \forall t \geq 0; Z_t = -\overline{Z}_{-t}, \forall t < 0.$$

Given $a > 0$, a continuous function f on $[-a, a]$, a partition $T : -a = t_0 < t_1 < \cdots < t_n = 0$ of $[-a, 0]$ and ξ_i in $[t_{i-1}, t_i]$, for $i = 1, 2, \cdots, n$. Let $\|T\| = \max\{t_i - t_{i-1} : i = 1, 2, \cdots, n\}$. We have

$$\begin{aligned}
\int_0^a f(s-a) dZ_s &= \int_{-a}^0 f(s) dZ_{s+a} = \lim_{\|T\|\to 0} \sum_{i=1}^n f(\xi_i)(Z_{a+t_i} - Z_{a+t_{i-1}}) \\
&\overset{\mathrm{d}}{=} \lim_{\|T\|\to 0} \sum_{i=1}^n f(\xi_i)(Z_{t_i - t_{i-1}}) \overset{\mathrm{d}}{=} \lim_{\|T\|\to 0} \sum_{i=1}^n f(\xi_i)(Z_{-t_{i-1}} - Z_{-t_i}) \\
&= \lim_{\|T\|\to 0} \sum_{i=1}^n f(\xi_i)(Z_{t_i} - Z_{t_{i-1}}) \\
&= \int_{-a}^0 f(s) dZ_s = \lim_{\|T\|\to 0} \sum_{i=1}^n f(-(-\xi_i))(Z_{-t_{i-1}} - Z_{-t_i}) \\
&= \int_0^a f(-s) dZ_s.
\end{aligned}$$

Let $X_0 = \int_{-\infty}^0 e^{cs} d\widetilde{Z}_s$, then

$$X_0 = \lim_{a\to\infty} \int_{-a}^0 e^{cs} d\widetilde{Z}_s = \lim_{a\to\infty} \int_0^a e^{-cs} d\widetilde{Z}_s$$

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$$= \int_0^\infty e^{-cs} d\widetilde{Z}_s = \int_0^\infty e^{-t} dZ'_t$$

has distribution μ. Define

$$X_t = e^{-ct}(X_0 + \int_0^t e^{cs} dZ_s), t \geq 0.$$

Then $\{X_t : t \geq 0\}$ is a free OU process. Now we show that it is stationary. For $t > 0$, we have

$$\begin{aligned} X_t &= e^{-ct} \int_{-\infty}^0 e^{cs} d\widetilde{Z}_s + e^{-ct} \int_0^t e^{cs} dZ_s \\ &= \int_{-\infty}^{-t} e^{cs} d\widetilde{Z}_{s+t} + \int_{-t}^0 e^{cs} dZ_{s+t} \stackrel{\mathrm{d}}{=} \int_{-\infty}^{-t} e^{cs} d\widetilde{Z}_s + \int_{-t}^0 e^{cs} dZ_s \\ &\stackrel{\mathrm{d}}{=} \int_{-\infty}^{-t} e^{cs} d\widetilde{Z}_s + \int_{-t}^0 e^{cs} d\widetilde{Z}_s = \int_{-\infty}^0 e^{cs} d\widetilde{Z}_s = X_0. \end{aligned}$$

Conversely, if $X_t, t \geq 0$ is a stationary free OU process, we proved in the preceding theorem that the limit distribution of a free OU process is free self-decomposable, if the limit exists. Hence, the limit distribution of X_t, as $t \to \infty$, is the distribution X_t, since it is stationary. Hence, the distribution of X_t, for all $t \geq 0$, is free self-decomposable. □

Remark 5.1.6. *From the theorem above, we see that the set $\mathcal{SD}(\boxplus)$ of all $\boxplus$ self-decomposable distributions on $\mathbb{R}$ can be described as $\mathcal{SD}(\boxplus) = \{$ the distributions of stationary free OU processes $\}$.*

5.2 Periodic Free OU Processes

In this section, we consider free OU processes in an interval. We extend them periodically to the whole real line. We show that the periodic free OU process is stationary. We also give a characterization of the stationary distribution in terms of its Levy measure.

Given a free Levy process $\{Z_t : t \geq 0\}$, consider the following Langevin equation

$$dX_t = -cX_t dt + dZ_t, t \in [0,1], X_0 = X_1. \tag{5.2.1}$$

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It has a unique solution

$$X_t = e^{-ct}(X_0 + \int_0^t e^{cs} dZ_s), t \in [0,1],$$

where $X_0 = X_1 = \frac{1}{e^c - 1}\int_0^1 e^{cs} dZ_s$. Let $X_{t+k} = X_t$, for $t \in [0,1]$ and $k \in \mathbb{Z}$. Then $\{X_t \in \mathbb{R}\}$ is a periodic process. So we call $\{X_t : t \in [0,1]\}$ a periodic free OU process.

Theorem 5.2.1. *The process $\{X_t : t \in \mathbb{R}\}$ defined by $X_{t+k} = X_t$, for $t \in [0,1]$ and $k \in \mathbb{Z}$, is a stationary process (i. e., $X_t \stackrel{d}{=} X_0$).*

Proof. For $t^0 \in [0,1]$, we construct a new process $\{Z_t^{t^0} : t \in [0,1]\}$ as follows. For $t \in [0,1]$,

$$Z_t^{t^0} = \begin{cases} Z_{t+t^0} - Z_{t^0}, & \text{if } t + t^0 \le 1; \\ Z_{1-t^0}^{t^0} + Z_{t+t^0-1}, & \text{if } t + t^0 > 1. \end{cases}$$

Now we show that $\{Z_t^{t^0} : t \in [0,1]\}$ is a free Levy process. Clearly, $Z_0^{t^0} = 0$. For $0 \le t_1 \le 1 - t^0 \le t_2 \le 1$, we have $Z_{t_2+t^0-1} = Z_{t_2}^{t^0} - Z_{1-t^0}^{t^0}, Z_{t^0} - Z_{t_2+t^0-1}, Z_{t_1}^{t^0} = Z_{t_1+t^0} - Z_{t^0}, Z_1 - Z_{t_1+t^0} = Z_{1-t^0}^{t^0} - Z_{t_1}^{t^0}$ are free. Thus, $Z_{t_2}^{t^0} - Z_{t_1}^{t^0} = Z_{t_2}^{t^0} - Z_{1-t^0}^{t^0} + Z_{1-t^0}^{t^0} - Z_{t_1}^{t^0}$ and $Z_{t_1}^{t^0} = Z_{t_1+t_0} - Z_{t^0}$ are free. Hence, $\{Z_t^{t^0} : t \in [0,1]\}$ has free increments. Moreover, for $0 \le t_1 \le 1 - t^0 \le t_2 \le 1$, we have

$$\begin{aligned}
\mu(Z_{t_2}^{t^0} - Z_{t_1}^{t^0}) &= \mu(Z_{t_2+t^0-1} + Z_1 - Z_{t_1+t^0}) = \mu(Z_{t_2+t^0-1}) \boxplus \mu(Z_1 - Z_{t_1+t^0}) \\
&= \mu(Z_{t_2+t^0-1}) \boxplus \mu(Z_{1-t_1-t^0}) = \mu(Z_{t_2} - Z_{1-t^0}) \boxplus \mu(Z_{1-t^0} - Z_{t_1}) \\
&= \mu(Z_{t_2} - Z_{t_1}) = \mu(Z_{t_2-t_1}).
\end{aligned}$$

Hence, $\{Z_t^{t^0} : t \in [0,1]\}$ has stationary increments. Moreover, let $t_1 = 0$, we get $Z_{t_2}^{t^0} \stackrel{d}{=} Z_{t_2}$, for $1 - t^0 \le t_2 \le 1$. It is obvious that $Z_t^{t^0} = Z_{t+t_0} - Z_{t^0} \stackrel{d}{=} Z_t$, for $\le t \le 1 - t^0$. Hence, $Z_t^{t^0} \stackrel{d}{=} Z_t$, for $t \in [0,1]$. It implies that $\{Z_t^{t^0} : t \ge 0\}$ is a free Levy process. Now we show that

$$X_{t+t^0} = \frac{1}{1 - e^{-c}} \int_0^1 e^{-c(t-s) \bmod 1} dZ_s^{t^0}, t, t^0 \in [0,1],$$

where

$$x \bmod 1 = \begin{cases} x + 1, & \text{if } -1 \le x < 0, \\ x, & \text{if } x \in [0,1]. \end{cases}$$

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For $0 \leq t \leq 1-t^0$, we have

$$\begin{aligned}
(1-e^{-c})X_{t+t^0} &= \int_0^1 e^{-c(t^0+t)}e^{-c+cs}dZ_s + \int_0^{t+t^0} e^{-c(t+t^0)}e^{cs}dZ_s \\
&- \int_0^{t+t^0} e^{-c(t+t^0)}e^{-c}e^{cs}dZ_s \\
&= \int_0^{t+t^0} e^{-c(t+t^0)}e^{cs}dZ_s + \int_{t+t^0}^1 e^{-c(t+t^0)}e^{-c}e^{cs}dZ_s \\
&= \int_t^{1-t^0} e^{-c(t+t^0)}e^{-c}e^{c(s+t^0)}dZ_s^{t^0} + \int_0^{t^0} e^{-c(t+t^0)}e^{cs}dZ_s \\
&+ \int_{t^0}^{t+t^0} e^{-c(t+t^0)}e^{cs}dZ_s \\
&= \int_t^{1-t^0} e^{-ct}e^{-c}e^{cs}dZ_s^{t^0} + \int_{1-t^0}^1 e^{-c(t^0+t)}e^{c(s-(1-t^0))}dZ_{z-(1-t^0)} \\
&+ \int_0^t e^{-c(t+t^0)}e^{c(s+t^0)}dZ_s^{t^0} \\
&= \int_t^{1-t^0} e^{-ct}e^{-c}e^{cs}dZ_s^{t^0} + \int_{1-t^0}^1 e^{-ct}e^{-c}e^{cs}dZ_s^{t^0} + \int_0^t e^{-ct}e^{cs}dZ_s^{t^0} \\
&= \int_t^1 e^{-ct}e^{-c}e^{cs}dZ_s^{t^0} + \int_0^t e^{-ct}e^{cs}dZ_s^{t^0} \\
&= \int_0^1 e^{-c(t-s) \bmod 1}dZ_s^{t^0}.
\end{aligned}$$

For $t+t^0 > 1$, we have

$$\begin{aligned}
&\int_0^1 e^{-c(t-s) \bmod 1}dZ_s^{t^0} \\
=&\int_0^{1-t^0} e^{-c(t-s)}dZ_{s+t^0} + \int_{1-t^0}^t e^{-c(t-s)}dZ_{s-(1-t^0)} + \int_t^1 e^{-c(t-s)}e^{-c}dZ_{s-(1-t^0)} \\
=&e^{-ct}\left(\int_{t^0}^1 e^{c(s-t^0)}dZ_s + \int_0^{t-(1-t^0)} e^{c(s+(1-t^0))}dZ_s + e^{-c}\int_{t-(1-t^0)}^{t^0} e^{cs}e^{c(1-t^0)}dZ_s\right) \\
=&e^{-c(t+t^0)}\left(\int_{t^0}^1 e^{cs}dZ_s + \int_{t-(1-t^0)}^{t^0} e^{cs}dZ_s + e^c\int_0^{t-(1-t^0)} e^{cs}dZ_s\right) \\
=&e^{-c(t+t^0)}\int_{t-(1-t^0)}^1 e^{cs}dZ_s + e^{-c(t+t^0-1)}\int_0^{t-(1-t^0)} e^{cs}dZ_s.
\end{aligned}$$

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On the other hand,

$$(1-e^{-c})X_{t+t^0} = (1-e^{-c})X_{t+t^0-1} = e^{-c(t+t^0)}\left(\int_{t+t^0-1}^{1} e^{cs}dZ_s + e^{c}\int_{0}^{t+t^0-1} e^{cs}dZ_s\right).$$

Hence,

$$X_{t+t^0} = \frac{1}{1-e^{-c}}\int_0^1 e^{-c(t-s) \bmod 1}dZ_s^{t^0}, \forall t, t^0 \in [0,1].$$

It follows that

$$\begin{aligned}\mu(X_{t+t^0}) &= \mu\left(\frac{1}{1-e^{-c}}\int_0^1 e^{-c(t-s) \bmod 1}dZ_s^{t^0}\right)\\ &= \mu\left(\frac{1}{1-e^{-c}}\int_0^1 e^{-c(t-s) \bmod 1}dZ_s\right)\\ &= \mu(X_t),\end{aligned}$$

$\forall t, t^0 \in [0,1]$. Let $t=0$, we get $X_{t^0} \stackrel{d}{=} X_0$, for all $t^0 \in [0,1]$. Hence, $\{X_t : t \in \mathbb{R}\}$ is a stationary process. □

Let $c > 0$, $\{Z_t : t \in [0,1]\}$ be a free Levy process and $\{X_t : t \in [0,1]\}$ be the periodic free OU process determined by c and $\{Z_t : t \in [0,1]\}$ (i. e., X_t is the solution to (5.2.1)). We call $\mu(X_0) = \mu(\frac{1}{e^c-1}\int_0^1 e^{cs}dZ_s)$ *the stationary distribution* of periodic free OU process $\{X_t : t \in [0,1]\}$. Note that

$$\mu\left(\frac{1}{e^c-1}\int_0^1 e^{cs}dZ_s\right) = \mu\left(\int_0^1 e^{cs}dZ_s'\right),$$

where $Z_s' = \frac{1}{e^c-1}Z_s$ is a free Levy process. So we have the following proposition.

Proposition 5.2.2. *For $c \neq 0$, let $\mathcal{I}(c)$ be the set of all $\mu(\int_0^1 e^{cs}dZ_s)$, where $\{Z_t, t \in [0,1]\}$, is a free Levy process. Then, given $c > 0$, $\mathcal{I}(c)$ is the set of all stationary distribution of periodic free OU processes determined by c and a free Levy process.*

Given a distribution $\mu_0 \in \mathcal{ID}(\boxplus)$, by [5] or [4], there is a unique $\mu' \in \mathcal{ID}(*)$, such that $\Lambda(\mu') = \mu$, where Λ is the bijection between $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$. By Theorem 7.10 in [55], there is a bijective correspondence between infinitely divisible

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laws and Levy processes in law, if we identify all Levy process with the same margin distributions. Hence, there is a classical Levy process $\widetilde{Z}_t$ such that $\mu(\widetilde{Z}_1) = \mu'$. By [4], there is a free Levy process Z_t such that $\mu(Z_1) = \mu_0$ and $\mu(Z_t) = \Lambda(\mu(\widetilde{Z}_t))$, for all $t \geq 0$. Thus, given, $c > 0$, we can define a map Φ_c from $\mathcal{ID}(\boxplus)$ into $\mathcal{I}(c)$ such that $\Phi_c(\mu_0) = \mu(\frac{1}{e^c-1}\int_0^1 e^{cs}dZ_s)$ in $\mathcal{I}(c)$, for $\mu_0 = \mu(Z_1)$ in $\mathcal{ID}(\boxplus)$. We shall show soon that Φ_c is bijective.

Theorem 5.2.3. *For $c > 0$, we have*

1. *If μ_1 and μ_2 are in $\mathcal{I}(c)$, $\mu_1 \boxplus \mu_2$ is in $\mathcal{I}(c)$.*
2. *Let a, b be in $\mathbb{R}$ and $\{Z_t : t \in [0,1]\}$ be a free Levy process, then $\mu(a + b\int_0^1 e^{cs}dZ_s)$ is in $\mathcal{I}(c)$.*
3. *$SD(\boxplus) \subsetneqq \mathcal{I}(c)$ and $SD(\boxplus) = \bigcap_{n\geq 1}\mathcal{I}(c_n)$, if $0 \neq c_n \in \mathbb{R}$ and $\lim_{n\to\infty} c_n = \infty$.*
4. *$\mathcal{I}(nc) \subseteq \mathcal{I}(c)$, for all $n \in \mathbb{N}$. $\mathcal{I}(-c) = \mathcal{I}(c)$.*
5. *$\Phi_c : \mathcal{ID}(\boxplus) \to \mathcal{I}(c)$ is one to one.*

Proof. (1). Let μ_1 and μ_2 be in $\mathcal{I}(c)$. We may choose two free Levy processes $\{Z_t^1 : t \in [0,1]\}$ and $\{Z_t^2 : t \in [0,1]\}$ such that

$$\mu_1 = \mu(\int_0^1 e^{cs}dZ_s^1), \mu_2 = \mu(\int_0^1 e^{cs}dZ_z^2),$$

and $\{Z_t^1 : t \in [0,1]\}$ and $\{Z_t^2 : t \in [0,1]\}$ are free. By the definition of integral $\int_0^1 e^{cs}dZ_s^j$ $(j = 1,2)$ (see Theorem 6.1 in [4]), $W^*(\int_0^1 e^{cs}dZ_s^j) \subseteq W^*(Z_t^j : t \in [0,1])$, $j = 1,2$. Hence, $\mu_1 \boxplus \mu_2 = \mu(\int_0^1 e^{cs}d(Z_s^1 + Z_s^2))$. To show that $\mu_1 \boxplus \mu_2 \in \mathcal{I}(c)$, it is enough to show that $\{Z_t^1 + Z_t^2 : t \in [0,1]\}$ is a free Levy process. For $0 \leq t_1 < t_2 \leq 1$, we have

$$\begin{aligned}\mu(Z_{t_2}^1 + Z_{t_2}^2 - (Z_{t_1}^1 + Z_{t_1}^2)) &= \mu((Z_{t_2}^1 - Z_{t_1}^1) + (Z_{t_2}^2 - Z_{t_1}^2)) \\ &= \mu(Z_{t_2-t_1}^1) \boxplus \mu(Z_{t_2-t_1}^2) \\ &= \mu(Z_{t_2-t_1}^1 + Z_{t_2-t_1}^2).\end{aligned}$$

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Thus, $Z_t^1 + Z_t^2$ has stationary-increments. Now we show that, for $0 \leq t_1 < \cdots t_n \leq 1$, $(Z_{t_n}^1 + Z_{t_n}^2) - (Z_{t_{n-1}}^1 + Z_{t_{n-1}}^2), \cdots, (Z_{t_2}^1 + Z_{t_2}^2) - (Z_{t_1}^1 + Z_{t_1}^2)$ and $Z_{t_1}^1 + Z_{t_1}^2$ are free. In fact, let $\mathcal{A}_{i,j} = W^*(Z_{t_i}^j - Z_{t_{i-1}}^j)$, for $i = 2, 3, \cdots, n$ and $j = 1, 2$, $\mathcal{A}_{1,j} = W^*(Z_1^j)$, $j = 1, 2$. Then, the von Neumann algebra

$$\begin{aligned}
&W^*(\mathcal{A}_{i,j} : i = 1, 2, \cdots, n, j = 1, 2) \\
=&W^*(W^*(\mathcal{A}_{i,1} : i = 1, 2, \cdots, n), W^*(\mathcal{A}_{i,2} : i = 1, 2, \cdots, n)) \\
=&W^*(\mathcal{A}_{i,1} : i = 1, 2, \cdots, n) * W^*(\mathcal{A}_{i,2} : i = 1, 2, \cdots, n) \\
=& *_{i=1,2,\cdots,n; j=1,2} \mathcal{A}_{i,j}.
\end{aligned}$$

Hence, $\mathcal{A}_{1,1}, \cdots, \mathcal{A}_{n,1}, \mathcal{A}_{1,2}, \cdots, \mathcal{A}_{n,2}$ are free. It follows that $(Z_{t_n}^1 + Z_{t_n}^2) - (Z_{t_{n-1}}^1 + Z_{t_{n-1}}^2) \in W^*(\mathcal{A}_{n,1}, \mathcal{A}_{n,2}), \cdots, (Z_{t_2}^1 + Z_{t_2}^2) - (Z_{t_1}^1 + Z_{t_1}^2) \in W^*(\mathcal{A}_{n,1}, \mathcal{A}_{n,2})$ and $Z_{t_1}^1 + Z_{t_1}^2 \in W^*(\mathcal{A}_{1,1}, \mathcal{A}_{1,2})$ are free. Moreover, $Z_t^1 + Z_t^2 \xrightarrow{\text{p}} 0$, as $t \to 0$, since addition of two elements in $\widetilde{A}$ is continuous with respect to the measure topology on $\widetilde{A}$ (see [4]). Hence, $Z_t^1 + Z_t^2$ is a free Levy process, and $\mu_1 \boxplus \mu_2 \in \mathcal{I}(c)$.

(2). Given $a \in \mathbb{R}$, let $Z_t^a = \frac{cat}{e^c - 1} I$, for $t \geq 0$. Then Z_t is a trivial free Levy process, and $\int_0^1 e^{cs} dZ_s = aI$. It follows that $\mu(aI) = \delta_a \in \mathcal{I}(c)$. Generally, for $0 \neq b, a \in \mathbb{R}$,

$$\mu(aI + b\int_0^1 e^{cs} dZ_s) = \mu(\int_0^1 e^{cs} d(Z_s^a + bZ_s)).$$

It is obvious that $(Z_s^a + bZ_s)$ is a free Levy process. Hence, $\mu(aI + b\int_0^1 e^{cs} dZ_s) \in \mathcal{I}(c)$.

(3), (4) and (5). Let

$$\widetilde{\mathcal{I}}(c) = \{\mu(\int_0^1 e^{cs} d\widetilde{Z}_s) : \{\widetilde{Z}_t : t \in [0, 1]\} \text{ is a classical Levy process}\}.$$

We show first that $\Lambda(\widetilde{\mathcal{I}}(c)) = \mathcal{I}(c)$. In fact, for Levy process $\widetilde{Z}_t$ $(t \in [0, 1])$, $\widetilde{\mu} = \mu(\int_0^1 e^{cs} d\widetilde{Z}_s)$ in $\widetilde{\mathcal{I}}(c)$, there is a Levy process $\widetilde{Z}_t'$ $(t \geq 0)$ such that $\widetilde{Z}_t' = \widetilde{Z}_t$, for $t \in [0, 1]$. Hence, by Theorem 5.4 and Corollary 6.2 in [4], there exists a free Levy process Z_t $(t \geq 0)$ affiliated with a W^*-probability space $(\mathcal{A}, \tau)$ such that $\mu(Z_t) = \Lambda(\mu(\widetilde{Z}_t'))$, for $t \geq 0$ and $\mu(\int_0^1 e^{cs} dZ_s) = \Lambda(\mu(\int_0^1 e^{cs} d\widetilde{Z}_s))$. It follows that

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$\Lambda : \widetilde{\mathcal{I}}(c) \to \mathcal{I}(c)$. Moreover, by Theorem 5.4 in [4], the map Λ from $\widetilde{\mathcal{I}}(c)$ to $\mathcal{I}(c)$ is onto. Note that $\Lambda : \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is bijective. Hence, $\Lambda : \widetilde{\mathcal{I}}(c) \to \mathcal{I}(c)$ is bijective. Theorem 4.8 in [4] showed that $\Lambda(SD(*)) = SD(\boxplus)$, and Theorem 3.1 in [50] showed that

$$SD(*) \subsetneqq \widetilde{\mathcal{I}}(c), \bigcap_{n\geq 1} \widetilde{\mathcal{I}}(c_n) = SD(*),$$

for $0 < c_n \in \mathbb{R}$, and $\lim_{n\to\infty} c_n = \infty$. It follows that

$$SD(\boxplus) = \Lambda(SD(*)) \subsetneqq \Lambda(\widetilde{\mathcal{I}}(c)) = \mathcal{I}(c),$$

$$\bigcap_n \mathcal{I}(c_n) = \bigcap_n \Lambda(\widetilde{\mathcal{I}}(c_n)) = \Lambda(\bigcap_n \widetilde{\mathcal{I}}(c_n)) = \Lambda(SD(*)) = SD(\boxplus),$$

and

$$\mathcal{I}(nc) \subseteq \mathcal{I}(c), \forall n \in \mathbb{N}, \mathcal{I}(-c) = \mathcal{I}(c).$$

Let $\widetilde{\Phi}_c : \mathcal{ID}(*) \to \widetilde{\mathcal{I}}(c)$ be a map defined by $\widetilde{\Phi}_c(\mu) = \mu(\int_0^1 e^{cs} d\widetilde{Z}_s) \in \widetilde{\mathcal{I}}(c)$, for $\mu \in ID(*)$, where $\widetilde{Z}_t$ is the Levy process determined by μ. For $\mu_1, \mu_2 \in \mathcal{ID}(\boxplus)$, there are measures $\mu_1', \mu_2' \in \mathcal{ID}(*)$ such that $\Lambda(\mu_i') = \mu_i$, and $\Phi_c(\mu_i) = \Lambda(\mu(\int_0^1 e^{cs} s\widetilde{Z}_s^i))$, where $\widetilde{Z}_t^i$ is the Levy process determined by μ_i', $i = 1, 2$. Thus, if $\Phi_c(\mu_1) = \Phi_c(\mu_2)$, $\Lambda(\mu(\int_0^1 e^{cs} d\widetilde{Z}_s^1)) = \Lambda(\mu(\int_0^1 e^{cs} d\widetilde{Z}_s^2))$. It follows that $\widetilde{\Phi}_c(\mu_1') = \widetilde{\Phi}_c(\mu_2')$. By Corollary 2.8 in [50], $\widetilde{\Phi}_c$ is one to one. So, $\mu_1 = \Lambda(\mu_1') = \Lambda(\mu_2') = \mu_2$. Hence, Φ_c is one to one. □

Lemma 5.2.4. *Let v be a Levy measure on $\mathbb{R}$. Then*

1. *v has a polar decomposition, i. e., there are non-negative real numbers λ_1, λ_{-1} and finite measures v_1 and v_{-1} on $(0, \infty)$ such that*

$$\int_0^\infty \min\{1, u^2\} dv_j(du) < \infty, j = 1, -1$$

and

$$v(B) = \lambda_1 v_1(B) + \lambda_{-1} v_{-1}(B), \text{ for all Borel set } B \subseteq \mathbb{R}.$$

We define a measure λ on the set $S = \{1, -1\}$ by $\lambda(1) = \lambda_1, \lambda(-1) = \lambda_{-1}$. Then

$$v(B) = \int_S \lambda(d\xi) \int_0^\infty \chi_B(u\xi) v_\xi(du), \text{ for Borel set } B \subseteq \mathbb{R}.$$

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2. *if* $\lambda_1\lambda_{-1} \neq 0$, *then*

   (a) v *has a nonnegative density function* $\frac{k(u)}{u}$ $(u \neq 0)$ *if and only if* v_1 *has a non-negative density function* $\frac{k_1(u)}{u}$ *with* $k_1(u) = \frac{1}{\lambda_1}k(u)$, *and* v_{-1} *has a non-negative density function* $\frac{k_{-1}(u)}{u}$ *with* $k_1(u) = \frac{1}{\lambda_{-1}}k(-u)$.

   (b) $G_c(u) = \sum_1^\infty k(e^{jc}u)$ *is increasing for* $u < 0$ *and decreasing for* $u > 0$ *if and only if both* $G_{1,c}(u) = \sum_1^\infty k_1(e^{jc}u)$ *and* $G_{-1,c}(u) = \sum_1^\infty k_{-1}(e^{jc}u)$ *are decreasing for* $u > 0$

Proof. Let v be a Levy measure on $\mathbb{R}$ and

$$\lambda_1 = \int_0^1 u^2 v(du) + v([1,\infty)), \lambda_{-1} = \int_{-1}^0 u^2 v(du) + v((-\infty,-1]).$$

For all B in $\mathcal{B}([0,\infty))$, the Borel σ-algebra on $\mathbb{R}$, let

$$v_1(B) = \begin{cases} \frac{1}{\lambda_1}v(B), & \text{if} \lambda_1 \neq 0 \\ 3\int_{B\cap[0,1]} u^2 du, & \text{if } \lambda_1 = 0. \end{cases}$$

Let

$$v_{-1}(B) = \begin{cases} \frac{1}{\lambda_{-1}}v(-B), & \text{if } \lambda_{-1} \neq 0 \\ 3\int_{B\cap[0,1]} u^2 du, & \text{if } \lambda_{-1} = 0. \end{cases}$$

Then,

$$\int_0^\infty \min\{1,u^2\}v_1(du) = \int_0^\infty \min\{1,u^2\}v_{-1}(du) = 1.$$

Moreover,

$$v(B) = v(B\cap(0,\infty)) + v(B\cap(-\infty,0)) = \lambda_{-1}v_{-1}(B\cap(-\infty,0)) + \lambda_1 v_1(B\cap(0,\infty)).$$

This is the polar decomposition of measure μ.

Suppose $\lambda_1 \neq 0$ and $\lambda_{-1} \neq 0$. Suppose v has a non-negative density function $\frac{k(u)}{u}$, i. e., $v(B) = \int_{B-\{0\}} \frac{k(u)}{u}du$, for every Borel set $B \subseteq \mathbb{R}$. Then, for $B \in \mathcal{B}((0,\infty))$, we have

$$v_1(B) = \int_B \frac{1}{\lambda_1}\frac{k(u)}{u}du.$$

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Hence, v_1 has density function $\frac{1}{\lambda_1}\frac{k(u)}{u}$. Similarly, v_{-1} has density function $\frac{1}{\lambda_{-1}}\frac{k(-u)}{u}$, Conversely, if v_1 has non-negative density function $\frac{k_1(u)}{u}$, and v_{-1} has non-negative density function $\frac{k_{-1}(u)}{u}$, we have, for a Borel set $B \subseteq \mathbb{R}$,

$$\begin{aligned} v(B) &= \lambda_1 v_1(B \cap (0,\infty)) + \lambda_{-1} v_{-1}(-B \cap (0,\infty)) \\ &= \lambda_1 \int_{B\cap(0,\infty)} \frac{k_1(u)}{u} du + \lambda_{-1} \int_{B\cap(-\infty,0)} \frac{k_{-1}(-u)}{u} du \\ &= \int_B (\chi_{(0,\infty)}\lambda_1 \frac{k_1(u)}{u} + \chi_{(-\infty,0)}\lambda_{-1}\frac{k_{-1}(-u)}{u})du. \end{aligned}$$

Let $k(t) = \chi_{(0,\infty)}\lambda_1 k_1(u) + \chi_{(-\infty,0)}\lambda_{-1}k_{-1}(-u)$, we see that v has density function $\frac{k(u)}{u}$, $k_1(u) = \frac{1}{\lambda_1}k(u)\chi_{(0,\infty)}(u)$ and $k_{-1}(u) = \frac{1}{\lambda_{-1}}k(-u)\chi_{(-\infty,0)}(u)$.

Let

$$G_{1,c}(u) = \sum_{j=1}^{\infty} k_1(e^{jc}u), G_{-1,c}(u) = \sum_{j=1}^{\infty} k_{-1}(e^{jc}u).$$

Then $G_{1,c} = \frac{1}{\lambda_1}\sum_{j=1}^{\infty} k(e^{jc}u)$ is decreasing for $u > 0$ if and only if $G_c(u) = \sum_{j=1}^{\infty} k(e^{jc}u)$ is decreasing for $u > 0$, and $G_{-1,c} = \frac{1}{\lambda_{-1}} = \sum_{j=1}^{\infty} k(-e^{jc}u)$ is decreasing for $u > 0$ if and only if $G_c(u)$ is increasing for $u < 0$. □

Remark 5.2.5. *The result that every Levy measure on $\mathbb{R}^d$ ($d \geq 1$) has polar decomposition was give in Proposition 2.6 of [50], but there were no precise proofs given. So we give a constructive proof of the result for d = 1 in the Lemma above.*

The following theorem give a characterization of measures in $\mathcal{I}(c)$.

Theorem 5.2.6. *For $c > 0$, a free infinitely divisible measure μ is in $\mathcal{I}(c)$ if and only if the Levy measure v of μ has a non-negative density function $\frac{k(u)}{u}$, and there is a function $G_c(u)$ such that $G_c(u)$ is decreasing, for $u > 0$, $G_c(u)$ is increasing, for $u < 0$, and*

$$G_c(u) = \sum_{j=1}^{\infty} k(e^{jc}u),$$

for almost all $u \in (0,\infty)$ with respect to the Lebesgue measure on $(0,\infty)$.

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Proof. Suppose $\lambda_{-1}\lambda_1 \neq 0$. A measure $u \in \mathcal{I}(c)$ if and only if $\Lambda^{-1}(u) \in \widetilde{\mathcal{I}}(c)$. Let (γ', A, v) be the generating triple of $\Lambda^{-1}(u)$, where v is the Levy measure of u (and $\Lambda^{-1}(u)$). Let

$$v = \lambda_1 v_1 + \lambda_{-1} v_{-1}$$

be the polar decomposition of v. [50] Proposition 2.7 showed that $\Lambda^{-1}(u) \in \widetilde{I}(c)$ if and only if v_1 and v_{-1} has non-negative density functions $\frac{k_1(u)}{u}$ and $\frac{k_{-1}(u)}{u}$, respectively, and there are decreasing functions

$$G_{1,c}(u) \stackrel{a}{=} \sum_{j=1}^{\infty} k_1(e^{cj}u), G_{-1,c}(u) \stackrel{a}{=} \sum_{j=1}^{\infty} k_{-1}(e^{cj}u), \forall u > 0,$$

where $\stackrel{a}{=}$ means $=$ for almost all $u > 0$ with respect to the Lebesgue measure on $(0, \infty)$. By Lemma 5.2.4, this is the case if and only if the Levy measure v of μ has a non-negative density function $\frac{k(u)}{u}$, and there is a function $G_c(u)$ such that $G_c(u)$ is decreasing, for $u > 0$, $G_c(u)$ is increasing, for $u < 0$, and

$$G_c(u) = \sum_{j=1}^{\infty} k(e^{jc}u),$$

for almost all $u \in (0, \infty)$ with respect to the Lebesgue measure on $(0, \infty)$.

If $\lambda_1 = 0$, but $\lambda_{-1} \neq 0$, then

$$\lambda_1 = \int_0^1 u^2 \frac{k(u)}{u} du + \int_1^{\infty} \frac{k(u)}{u} du = 0.$$

It implies that $k(u) = 0$ for almost all $u > 0$, and $G_c(u) = \sum_{i=1}^{\infty} k(e^{cj}u) \stackrel{a}{=} 0$, for $u > 0$. In this case, $v(B) = \lambda_{-1} v_{-1}(-B \cap (0, \infty))$. By Lemma 5.2.4, v_{-1} has a non-negative density function $\frac{k_{-1}(u)}{u}$ and there is a decreasing function $G_{-1,c}(u) \stackrel{a}{=} \sum_{j=1}^{\infty} k_{-1}(e^{jc}u)$, for $u > 0$ if and only if v has a non-negative density function $\frac{k(u)}{u}$ and there is an increasing function $G_c(u)$ for $u < 0$ such that $G_c(u) \stackrel{a}{=} \sum_{i=1}^{\infty} k(e^{cj}u)$, for $u > 0$ (In fact, we can let $k(-u) = k_{-1}(u)$, for $u > 0$). Hence, in this case, we have proved the result.

Similarly, we can prove the result in the case of $\lambda_1 \neq 0$ and $\lambda_{-1} = 0$.

Finally, when $\lambda_1 = \lambda_{-1} = 0$, $v = 0$, the result is trivial. □

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5.3 Fractional Free OU Processes

In this section, we introduce the notion of *fractional free Brownian motion.* We show that the corresponding Langevin equation has a unique solution, which is called a *fractional free OU process.*

Recall that a stochastic process $\{X_t : t \in I \subseteq \mathbb{R}\}$ is *Gaussian* if for $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, $\sum_{i=1}^{n} \lambda_i X_{t_i}$ has a normal distribution (see [30]). Similarly, We can define the analogue in free probability.

Definition 5.3.1. *A family $\{X_t : t \in I \subseteq \mathbb{R}\}$ of self-adjoint operators in a W^*-probability space $(\mathcal{A}, \tau)$ is called a semicircle process, if for $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, $\sum_{i=1}^{n} \lambda_i X_{t_i}$ has a semicircle distribution. A semicircle process is centered, if $\tau(X_t) = 0$, for $t \in I$.*

Now we are in a position to give the definition of fractional free Brownian motion.

Definition 5.3.2. *A centered semicircle process $\{X_t : t \in I \subseteq \mathbb{R}\}$ is a fractional free Brownian motion with parameter $H \in (0, 1]$, if*

$$\tau(X_t X_s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \forall s, t \in I.$$

Theorem 5.3.3. *Free Brownian motion is fractional free Brownian motion with parameter $H = \frac{1}{2}$.*

Proof. Let $\{B_t : t \geq 0\}$ be free Brownian motion. First, we show that $\{X_t : t \geq 0\}$ is a semicircle process. For $\lambda \in \mathbb{R}$ and $t > 0$, we have λX_t has distribution $\mathcal{D}_\lambda \mu(X_t)$, which is a semicircle law. by [64]. Hence, it is sufficient to show that $X_{t_1} + \cdots + X_{t_n}$ has semicircle distribution, for $0 < t_1, \cdots < t_n$. Note that

$$\begin{aligned}
&X_{t_1} + \cdots + X_{t_n} \\
&= X_{t_n} - X_{t_{n-1}} + 2X_{t_{n-1}} + \cdots + X_{t_1} \\
&= (X_{t_n} - X_{t_{n-1}}) + 2(X_{t_{n-1}} - X_{t_{n-2}}) + 3X_{t_{n-3}} + \cdots + X_{t_1}
\end{aligned}$$

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$$= \cdots$$

$$= (X_{t_n} - X_{t_{n-1}}) + 2(X_{t_{n-1}} - X_{t_{n-2}}) + \cdots + (n-1)(X_{t_2} - X_{t_1}) + nX_{t_1}.$$

Note that $(X_{t_n} - X_{t_{n-1}}), 2(X_{t_{n-1}} - X_{t_{n-2}}), 3(X_{t_{n-2}} - X_{t_{n-3}}), \cdots, (n-1)(X_{t_2} - X_{t_1}), nX_{t_1}$ are free. Hence, by induction, it is enough to show that $X+Y$ has semicircle distribution, if X and Y form a semicircle family (i. e. X and Y are free and they have semicircle distributions). Note that A self-adjoint operator $X \in \mathcal{A}$ with $\tau(X) = 0$ is a semicircle element if and only if the R-transform $R_{\mu(X)}(z) = \frac{r^2}{4}z$, for some $r > 0$ (see [64]). If $R_{\mu(X)} = \frac{r_1}{4}z$ and $R_{\mu(Y)} = \frac{r_2^2}{4}z$. Then

$$R_{\mu(X+Y)}(z) = R_{\mu(X)}(z) + R_{\mu(Y)}(z) = \frac{\sqrt{(r_1^2 + r_2^2)^2}}{4}z.$$

It follows that $X+Y$ is a semicircle element. Hence, we have shown that $\{X_t : t \geq 0\}$ is a centered semicircle process. Moreover, for $t > s > 0$, we have

$$\tau(X_t X_s) = \tau((X_t - X_s)X_s + X_s^2) = (t-s).$$

Hence, $\{X_t : t \geq 0\}$ is a fractional free Brownian motion with parameter $H = \frac{1}{2}$. □

Theorem 5.3.4. *Let $\{B_t^{(1)} \in \mathcal{A}_{sa} : t \geq 0\}$ be free Brownian motion, $\{B_t^{(2)} : t \geq 0\}$ be a free copy of $\{B_t^{(1)} : t \geq 0\}$ (i. e., $\{B_t^{(2)} : t \geq 0\}$ is a free Brownian motion, and $\{B_t^{(2)} : t \geq 0\}$ and $\{B_t^{(1)} : t \geq 0\}$ are free). Define*

$$B_t = \begin{cases} B_t^{(1)}, & \text{if } t \geq 0, \\ B_{-t}^{(2)}, & \text{if } t < 0. \end{cases}$$

Then

1. *for $f \in L^2(\mathbb{R})$, we have $\tau((\int_{\mathbb{R}} f(t)dB_t)^2) = \int_{\mathbb{R}} |f(t)|^2 dt$;*
2. *for $0 < H < 1$, let $C_H = (\int_{-\infty}^0 ((1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}})^2 du + \frac{1}{2H})^{-1/2}$, and*

$$X_t = C_H\Big(\int_{-\infty}^0 ((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}})dB_u + \int_0^t (t-u)^{H-1/2})dB_u.$$

We have $\{X_t : t \in \mathbb{R}\}$ is a fractional free Brownian motion with parameter H.

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Proof. Result (1) follows from Proposition 6 in [1].

(2). Since $\int_{-\infty}^{0}((t-u)^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}})dB_u$ and $\int_0^t(t-u)^{H-1/2})dB_u$ are free, and

$$\tau(\int_{-\infty}^{0}((t-u)^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}})dB_u) = \int_{-\infty}^{0}((t-u)^{H-\frac{1}{2}}-(-u)^{h-\frac{1}{2}})d\tau(B_u)$$
$$= 0 = \tau(\int_0^t(t-u)^{H-1/2})dB_u),$$

we have

$$\begin{aligned}
\tau(X_t^2) &= C_H^2(\tau((\int_{-\infty}^{0}((t-u)^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}})dB_u)^2)+\tau((\int_0^t(t-u)^{H-1/2})dB_u)^2)) \\
&= C_H^2(\int_{-\infty}^{0}(t-u)^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}})^2du+\int_0^t(t-u)^{2H-1}du) \\
&= C_H^2t^{2H}(\int_{-\infty}^{0}(1-u)^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}})^2du+\frac{1}{2H}) \\
&= t^{2H}.
\end{aligned}$$

Moreover, for $h, t \in \mathbb{R}$, we have

$$\begin{aligned}
&\tau((X_{t+h}-X_h)^2) \\
&= C_H^2\tau((\int_{-\infty}^{0}((t+h-u)^{H-\frac{1}{2}}-(h-u)^{H-\frac{1}{2}})dB_u \\
&+\int_0^{t+h}(t+h-u)^{H-1/2}dB_u-\int_0^h(h-u)^{H-1/2}dB_u)^2) \\
&= C_H^2\tau((\int_{-\infty}^{0}((t-(u-h))^{H-\frac{1}{2}}-(-(u-h))^{H-\frac{1}{2}})dB_u \\
&+\int_0^h((t-(u-h))^{H-\frac{1}{2}}-(-(u-h))^{H-\frac{1}{2}})dB_u \\
&+\int_h^{t+h}(t-(u-h))^{H-\frac{1}{2}}dB_u)^2) \\
&= C_H^2\tau((\int_{-\infty}^{0}((t-u)^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}})dB_{u+h}+\int_0^t(t-u)^{H-\frac{1}{2}}dB_{u+h})^2) \\
&= C_H^2\tau((\int_{-\infty}^{0}((t-u)^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}})dB_{u+h})^2 \\
&+(\int_0^t(t-u)^{H-\frac{1}{2}}dB_{u+h})^2)
\end{aligned}$$

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$$= C_H^2 \Big(\int_{-\infty}^{0} ((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}})^2 du + \int_0^t ((t-u)^{H-\frac{1}{2}})^2 du \Big)$$
$$= t^{2H}.$$

It follows that

$$\tau(X_s X_t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

It is obvious that $\tau(X_t) = 0$, for $t \in \mathbb{R}$. Note that $\{B_t : t \in \mathbb{R}\}$ is a semicircle process. So, for real-valued step function $s = \sum_{i=1}^{n} \alpha_i \chi_{[t_{i-1}, t_i)}$, where $-\infty < a = t_0 < t_1 < \cdots < t_n = b < \infty$, we have

$$\int_a^b s(t) dB_t = \sum_{i=1}^{n} \alpha_i (B_{t_i} - B_{t_i})$$

is a semicircle element. Generally, let f is a continuous function an interval $[a, b]$, then there exits a sequence (f_n) of step functions such that $\lim_{n\to\infty} \| \int_a^b f(t) dB_t - \int_a^b f_n(t) dB_t \| = 0$. Let $\tau((\int_a^b f_n(t) dB_t)^2) = \frac{r_n^2}{4}$, then

$$\tau((\int_a^b f_n(t) dB_t)^k) = \begin{cases} 0, & \text{if } k = 2m+1, \\ \frac{2m!}{m!(m+1)!}(\frac{r_n^2}{4})^m, & \text{if } k = 2m, m \geq 0 \end{cases}$$

(see (1.8) in [31]). It implies that

$$\tau((\int_a^b f(t) dB_t)^k) = \lim_{n\to\infty} \tau(\int_a^b f_n(t) dB_t) = \begin{cases} 0, & \text{if } k = 2m+1, \\ \frac{2m!}{m!(m+1)!}(\frac{r^2}{4})^m, & \text{if } k = 2m, m \geq 0, \end{cases}$$

where $r = \lim_{n\to\infty} r_n$. Hence, $\int_a^b f(t) dB_t$ is a semicircle element. It follows that X_t is a semicircle element, for $t \in \mathbb{R}$.

Generally, for $-\infty < t_1 < t_2 < \cdots < t_n < \infty$, Let $T_u = \sum_{i=1}^{n} t_i^{H-\frac{1}{2}} B_{t_i u}$. Then, for a step function $s = \sum_{j=1}^{m} \alpha_j \chi_{[s_{j-1}, s_j}$, where $a = s_0 < s_1 < \cdots < s_m = b$, we have

$$\int_a^b s(u) dS_u = \sum_{j=1}^{m} \alpha_j (T_{s_j} - T_{s_j}) = \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha_j t_i^{H-\frac{1}{2}} (B_{s_j t_i} - B_{s_{j-1} t_i})$$

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is a semicircle element, since B_t is a semicircle process. For a continuous function f on $[a,b]$, there is a sequence s_n of real-valued step functions on $[a,b]$ such that

$$\lim_{n\to\infty} \| \int_a^b f(u)dT_u - \int_a^b f_n(u)dT_u \| = 0.$$

Hence, by the proof above, $\int_a^b f(u)dT_u$ is a semicircle element. Now we show that (X_t) is a semicircle process. By an elementary computation, we have that

$$X_t = C_H\left(\int_{-\infty}^0 ((1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}})dt^{H-\frac{1}{2}}B_{tu} + \int_0^1 (1-u)^{H-\frac{1}{2}}dt^{H-\frac{1}{2}}B_{tu}\right).$$

Hence,

$$X_{t_1} + \cdots + X_{t_n} = C_H\left(\int_{-\infty}^0 ((1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}})dT_u + \int_0^1 (1-u)^{H-\frac{1}{2}}dT_u\right)$$

is a semicircle element. It follows that $\{X_t : t \geq 0\}$ is a fractional free Brownian motion. □

Remark 5.3.5. *From the Theorem above, we have*

1. *For $H = 1$, let X_1 be a standard semicircle element (i. e., X_1 is a semicircle element with $\tau(X_1) = 0$ and $\tau(X_1^2)=1$). Let $X_t = tX_1$, for $t \in \mathbb{R}$, then $\tau(X_tX_s) = |ts|$. Hence, (X_t) is a fractional free Brownian motion with parameter $H = 1$.*

2. *Biane and Speicher gave an example of free Brownian motion, which comes from creation and annihilation operators on a full Fock space, in [10]. From Theorem 5.3.4 and (1) in Remark 5.3.5, we can construct examples of fractional free Brownian motion for every $H \in (0,1]$.*

Recall that A family $\{X_t : t \geq 0\}$ of self-adjoint operators affiliated with a W^*-probability space is called a *free self-similar process*, if for any $c > 0$, there exists $b > 0$ such that

$$\mu(X_{ct}) = \mu(bX_t)), \forall t \geq 0.$$

$\{X_t : t \geq 0\}$ is a *free H self-similar process*, if there exists a $0 < H$ such that $b = c^H$, for all $c > 0$ (see [23]). We now give an abstract characterization of fractional free Brownian motion.

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Theorem 5.3.6. *A centered semicircle process $\{X_t : t \in \mathbb{R}\}$ of self-adjoint operators in $\mathcal{A}$ is a fractional free Brownian motion with parameter $H \in (0,1]$ if and only if it is free H self-similar, it has stationary increments and $\tau(X_1^2) = 1$.*

Proof. If $\{X_t : t \geq 0\}$ is a fractional free Brownian motion. By definition, we have $\tau(X_1^2) = 1$ and $\tau(X_0^2) = 0$. Thus, $X_0 = 0$. Moreover,

$$\tau((X_t - X_s)^2) = \tau(X_t^2 + X_s^2 - X_sX_t - X_tX_s) = |t-s|^{2H}, \forall t, s \in \mathbb{R}.$$

Note that $X_t - X_s$ is a semicircle element, thus, $\mu(X_t - X_s) = \mu(X_{t-s})$. Hence, $\{X_t : t \geq 0\}$ has stationary increments. For $t \in \mathbb{R}$ and $c > 0$, we have

$$\tau(X_{ct}^2) = c^{2H}t^{2H} = \frac{r_{ct}^2}{4}.$$

It follows that $r_{ct} = 2C^H|t|^H$, while $r_t = 2|t|^H$, where r_t is the spectral radius of semicircle element X_t. Note that

$$\tau(X_{ct}^k) = \begin{cases} 0, & \text{if } k = 2m+1, \\ \frac{2m!}{m!(m+1)!}c^{2mH}(\frac{r_t^2}{4})^m, & \text{if } k = 2m, \text{ for } m \geq 0 \end{cases}$$

Hence, $\tau(X_{ct}^k) = (c^H)^k\tau(X_t^k) = \tau((c^HX_t)^k)$. It follows that $\mu(X_{ct}) = \mu(c^HX_t)$. Hence, $\{X_t : t \geq 0\}$ is an H self-similar process.

Conversely, suppose $\{X_t : t \in \mathbb{R}\}$ is centered semicircle process with $\tau(X_1^2) = 1$. Suppose that this process has stationary increments and it is H self-similar. Then $\tau(X_t^2) = \tau((t^HX_1)^2) = t^{2H}$. It follows that

$$\begin{aligned} \tau(X_sX_t) &= \frac{1}{2}\tau(X_t^2 + X_s^2 - (X_t - X_s)^2) \\ &= \frac{1}{2}\tau(X_t^2 + X_s^2 - (X_{t-s})^2) \\ &= \frac{1}{2}(|t|^{2H} + |s|^{2H} - (|t-s|^{2H}). \end{aligned}$$

Hence, $\{X_t : t \geq 0\}$ is a fractional free Brownian motion. □

Let f be a $\mathcal{A}$-valued continuous function on $[a,b]$, we can define the integral $\int_a^b f(t)dt$ as follows. Given a partition $T_n : a = t_0 < t_1 < \cdots < t_n = b$ with norm

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$\|T_n\| = \max\{t_i - t_{i-1} : i = 1, 2, \cdots, n\}$ and intermediates $t_0 \leq \xi_1 \leq t_1 \leq \xi_2 \leq \cdots \leq t_{n-1} \leq \xi_n \leq t_n$, we have a Riemann sum $R_{T_n} = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$. It is well known that $\{R_{T_n} : n = 1, 2, \cdots\}$ is a Cauchy sequence in operator norm of $\mathcal{A}$, as $\|T_n\| \to 0$. We define $\int_a^b f(t)dt$ as the operator norm limit of $\{R_{T_n} : n = 1, 2, \cdots\}$.

Let $f : [a, b] \to \mathbb{R}$ be a function and $\{B_t : t \in \mathbb{R}\}$ be a fractional free Brownian motion with parameter $H \in (0, 1]$. Define Riemann sum

$$R_{T_n} = \sum_{i=1}^{n} f(\xi_i)(B_{t_i} - B_{t_{i-1}}),$$

for a partition $T_n : a = t_0 < t_1 < \cdots < t_n = b$ with norm

$$\|T_n\| = \max\{t_i - t_{i-1} : i = 1, 2, \cdots, n\}$$

and intermediates $t_0 \leq \xi_1 \leq t_1 \leq \xi_2 \leq \cdots \leq t_{n-1} \leq \xi_n \leq t_n$. We define $\int_a^b f(t)dB_t$ as the operator norm limit of Riemann sum $R_{T_n} = \sum_{i=1}^{n} f(\xi_i)(B_{t_i} - B_{t_{i-1}})$, as $\|T_n\| \to 0$, if this limit exists.

Theorem 5.3.7. *Let $\{B_t : t \in \mathbb{R}\}$ be a fractional free Brownian motion with parameter $H \in (0, 1]$, and $f : [a, b] \to \mathbb{R}$ be continuously differentiable function. Then, $\int_a^b f(t)dB_t$ exists and*

$$\int_a^b f(t)dB_t = f(b)B_b - f(a)B_a - \int_a^b f'(t)B_t dt.$$

Proof. Since $\{B_t : t \in \mathbb{R}\}$ is a semicircle process, $\tau((B_t - B_s)^2) = |t - s|^{2H} = \frac{r^2}{4}$, where r is the spectral radius of $B_t - B_s$ (it is also $\|B_t - B_s\|$). It follows that $\|B_t - B_s\| = 2|t - s|^{2H}$. Hence, $t \to B_t$ is norm continuous, and $\int_a^b f'(t)B_t dt$ exists. Let $T : a = t_0 < \cdots < t_n = b$ be a partition of $[a, b]$ with intermediates $t_0 \leq \xi_1 \leq t_1 \leq \xi_2 \leq t_2 \leq \cdots \leq t_{n-1} \leq \xi_n \leq t_n$.Then we have

$$\sum_{i=1}^{n} f(\xi_i)(B_{t_i} - B_{t_{i-1}}) = \sum_{i=1}^{n} f(\xi_i)B_{t_i} - \sum_{i=1}^{n} f(\xi_i)B_{t_{i-1}}$$
$$= \sum_{i=2}^{n+1} f(\xi_{i-1})B_{t_{i-1}} - \sum_{i=1}^{n} f(\xi_i)B_{t_{i-1}}$$

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$$
\begin{aligned}
&= f(\xi_n)B_b - \sum_{i=2}^{n}(f(\xi_i) - f(\xi_{i-1}))B_{t_{i-1}} - f(\xi_1)B_a \\
&= f(\xi_n)B_b - f(\xi_1)B_a + (f(\xi_1) - f(a))B_a + (f(b) - f(\xi_n))B_b \\
&- [(f(\xi_1) - f(a))B_a + (f(b) - f(\xi_n))B_b + \sum_{i=2}^{n}(f(\xi_i) - f(\xi_{i-1}))B_{t_{i-1}}] \\
&= f(b)B_b - f(a)B_a - \Gamma_{T'},
\end{aligned}
$$

where $\Gamma_{T'}$ is the Riemann sum of Riemann-Stieljies integral $\int_a^b B_t df(t)$ with respect to partition $T' : a = \xi_0 < \xi_1 < \cdots < \xi_n \leq \xi_{n+1} = b$ with intermediates $\xi_0 = t_0 \leq \xi_1 \leq t_1 \leq \cdots \leq t_{n-1} \leq \xi_n \leq t_n = \xi_{n+1}$. Note that $\|T'\| \to 0$ as $\|T\| \to 0$, and $\int_a^b B_t df(t) = \int_a^b f'(t)B_t dt$ exists. Hence, let $\|T\| \to 0$, we have

$$\int_a^b f(t)dB_t = f(b)B_b - f(a)B_a - \int_a^b f'(t)B_t dt.$$

□

Theorem 5.3.8. *Let $\{B_t : t \in \mathbb{R}\}$ be a fractional free Brownian motion with parameter $H \in (0,1]$, $\lambda, \sigma > 0$. Then*

1. *the following Langevin equation*

$$X_t = X_0 - \lambda \int_0^t X_s ds + \sigma B_t, t \geq 0 \tag{5.3.1}$$

   with $X_0 \in \mathcal{A}_{sa}$ has a unique solution

$$X_t = e^{-\lambda t}X_0 + \sigma e^{-\lambda t}\int_0^t e^{\lambda s}dB_s, t \geq 0,$$

   which we call a fractional free OU process.

2. *Let Y_0 be $\int_{-\infty}^0 e^{\lambda s}dB_s = \lim_{A\to\infty}\int_{-A}^0 e^{\lambda s}dB_s$, then $Y_t = \sigma\int_{-\infty}^t e^{-\lambda(t-u)}dB_u$, for $t \geq 0$, is a stationary solution to (5.3.1). We call $Y(t)$ defined above a stationary fractional free OU process.*

Proof. (1). For $t > 0$, we have

$$\lambda\int_0^t X_s ds = \int_0^t \lambda e^{-\lambda s}ds X_0 + \lambda\sigma\int_0^t e^{-\lambda s}\int_0^s e^{\lambda r}dB_r$$

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$$= (1-e^{-\lambda t})X_0 + \lambda\sigma \int_0^t e^{-\lambda s}(e^{\lambda s}B_s - \lambda \int_0^s e^{\lambda r}B_r dr)ds)$$
$$= (1-e^{-\lambda t})X_0 + \lambda\sigma \int_0^t B_s ds - \lambda^2\sigma \int_0^t e^{-\lambda s} \int_0^s e^{\lambda r}B_r dr ds$$
$$= (1-e^{-\lambda t})X_0 + \lambda\sigma \int_0^t B_s ds - \lambda\sigma \int_0^t e^{\lambda r}B_r(e^{-\lambda r} - e^{-\lambda t})dr$$
$$= (1-e^{-\lambda t})X_0 + \lambda\sigma e^{-\lambda t} \int_0^t e^{\lambda r}B_r dr$$
$$= (1-e^{-\lambda t})X_0 + \sigma e^{-\lambda t}(e^{\lambda t}B_t - \int_0^t e^{\lambda r}dB_r)$$
$$= X_0 + \sigma B_t - e^{-\lambda t}X_0 - \sigma e^{-\lambda t} \int_0^t e^{\lambda r}dB_r$$
$$= X_0 - X_t + \sigma B_t.$$

Hence, X_t is a solution to (5.3.1). The uniqueness of solutions to (5.3.1) is clear.

(2). Now we show that $\int_{-\infty}^0 e^{\lambda s}dB_s = \lim_{A\to\infty} \int_{-A}^0 e^{\lambda s}dB_s$ exists. In fact,

$$\lim_{A\to\infty} \int_{-A}^0 e^{\lambda s}dB_s = \lambda \lim_{A\to\infty} \int_0^A e^{-\lambda s}B_{-s}ds.$$

Note that, for $A' > A > 0$,

$$\| \int_A^{A'} e^{-\lambda t}B_{-t}dt\| \le \int_A^{A'} e^{-\lambda t}\|X_{-t}\|dt = 2\int_A^{A'} e^{-\lambda t}t^H dt \to 0,$$

as $A \to \infty$. Hence,

$$Y_0 := \int_{-\infty}^0 e^{\lambda s}dB_s = \lim_{A\to\infty} \int_{-A}^0 e^{\lambda s}dB_s$$

exists. By (1), $Y_t = \sigma \int_{-\infty}^t e^{-\lambda(t-u)}dB_u$ is a solution to (4.1) with initial value Y_0. Now we show that $Y_t \stackrel{d}{=} Y_0$. Note that $Y_t = \int_{-\infty}^0 e^{\lambda r}dB_{t+r}$. So it is enough to show that

$$\int_{-\infty}^0 e^{\lambda r}dB_{t+r} \stackrel{d}{=} \int_{-\infty}^0 e^{-\lambda(t-s)}dB_s, \tag{5.3.2}$$

for $t \in \mathbb{R}$. First, we show that

$$\int_a^b s(r)dB_{t+r} \stackrel{d}{=} \int_a^b s(r)dB_r, \tag{5.3.3}$$

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for real-valued step function $s(r) = \sum_{i=1}^{n} \alpha_i \chi_{[r_{i-1}, r_i)}(r)$, and $a, b, t \in \mathbb{R}$. Since

$$\int_a^b s(r) dB_{t+r} = \sum_{i=1}^{n} \alpha_i (B_{t+r_i} - B_{t+r_i})$$

is a semicircle element, and

$$\begin{aligned}\tau((\int_a^b s(r) dB_{t+r})^2) &= \sum_{i,j=1}^{n} \alpha_i \alpha_j \tau((B_{r_i+t} - B_{r_{i-1}+t})(B_{r_j+t} - B_{r_{j-1}+t})) \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j (|r_i - r_{j-1}|^{2H} \\ &+ |r_{i-1} - r_j|^{2H} - |r_{i-1} - r_{j-1}|^{2H} - |r_i - r_j|^{2H}) \\ &= \tau((\int_a^b s(r) dB_r)^2),\end{aligned}$$

(5.3.3) holds true. By taking limits, we get (5.3.2). Hence, Y_t is a stationary process.

□

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