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Making connections: Compartmentalization in pre -calculus students' understanding of functions

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MAKING CONNECTIONS: COMPARTMENTALIZATION IN PRE-CALCULUS STUDENTS' UNDERSTANDING OF FUNCTIONS

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ABSTRACT

MAKING CONNECTIONS: COMPARTMENTALIZATION IN
PRE-CALCULUS STUDENTS UNDERSTANDING OF FUNCTIONS

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Students develop knowledge constructs that they build into concepts through their experiences. Students demonstrate compartmentalization when they understand a construct or concept within one representation, but not another, or when they do not connect mathematically related ideas. For instance, a student may understand $f(x)$ to mean plug x into the function within a symbolic representation, but the same student may understand $f(x)$ to mean f times x within a tabular representation. A student with these understandings has a compartmentalized understanding of function notation.

A two-month study was conducted with a class of pre-calculus students enrolled in a parochial high school. The class was observed and a subset of students (n=7) were given a series of tasks in an interview setting in order to determine their understanding of functions and in particular periodicity within the three representations: equations, graphs, and tables. The researcher studied compartmentalization in the students' understanding.

Three of the seven students showed compartmentalization. All three had a compartmentalized understanding of function notation within the tabular representation. In addition, two had compartmentalization within representations in their understanding of periodicity. Students with compartmentalization in their understanding, had the greatest difficulty in solving the interview tasks. Furthermore, those students who could not translate between representations had an automatic compartmentalization in their understanding and lacked flexibility in problem-solving.

All seven of the students preferred the symbolic representation. The students used this representation overwhelmingly in their classwork and homework. Six of the seven students attempted to find equations for the functions in the interview tasks before trying any other solution strategy. However, only one student was able to solve the interview tasks in this representation.

Some interesting conceptions of periodicity emerged in the students' understandings. The students used symmetry, familiarity, and continuity to determine whether a function was periodic. The students did not work from a conventional definition of period. Instead, they constructed their own definition of periodicity by generalizing sinusoids and other familiar functions. The generalizations that the students made were often inconsistent with the conventional definitions. These unconventional understandings imply that they need experiences with more than just sinusoids.

1: Rationale and Overview

Rationale

Expectations for students understanding of functions. Functions can be a unifying concept in high school and college mathematics. The focus on functions allows the mathematics curriculum to be organized around major ideas and not just abstract algebraic manipulations (Yerushalmy, 1997). The function concept is a basic building block for calculus and one that is commonly misunderstood from Algebra through undergraduate mathematics (Dugdale, 1990; Eisenberg, 1992; Knuth, 1995; Leinhardt, Zaslavsky, and Stein, 1990; Martínez-Cruz, 1995; Monk, 1992; Sfard, 1989; Vinner, 1989). The *Principles and Standards for School Mathematics* (NCTM, 2000) suggest that high school students should be able to:

- understand relations and functions and select, convert flexibly among, and use various representations for them;
- understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more complicated symbolic expressions;
- understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions. (p. 296)

Because understanding of functions and flexibility in using representations is expected from high school students, it is informative to describe students' understanding of functions and their ability to use representations.

Translating between representations and transfer. In order for students to demonstrate the understanding called for in the *Principles and Standards for School Mathematics* (NCTM, 2000), they must be able to translate between representations of functions. This requires not only that they be able to plot points or find an equation representing a function presented graphically, but also that they be able to transfer other information that they have learned from one representation to another.

For example, consider the translation of $f(x) = (x-1)(x-5)$, from equation to graph. A pre-calculus student might notice that this equation represents a parabola. He or she may notice that $f(1) = f(5) = 0$ and that $f(0) = 5$. Each of these ideas can be found within the symbolic representation. When translating the equation from symbolic to graphical form, the student should be able to retain the knowledge gleaned from the symbolic form. That is, the student should be able to graph the x- and y- intercepts and should be able to remember that the function is a parabola and picture the shape that it will form in the graph. When viewing this function in the graphical representation other information comes to the surface. The student may notice that the parabola opens upward, that the vertex is located halfway between the x-intercepts at $x = 3$ and use this information to find the coordinates of the vertex. The student might notice that the function has symmetry about the line $x = 3$. Also, the student may notice that the graph is congruent to the graph of $g(x) = x^2$, but has been shifted 3 units to the right and 4 units down. If this same student translated back into equation form, he or she should be able to write the equation in the form $f(x) = (x-3)^2 - 4$ by using the new information that was apparent in the graphical representation. This student should be able to explain how these two different equations represent the same graph.

It is not enough to be able to plot points. A student with a well-developed understanding of functions should be able to translate functions from one representation to another, gaining information from each different representation and transferring that information from one representation to another. In other words, knowledge gained from one representation of function should be available to the students when they translate the function into another representation.

Compartmentalization in students' understanding of functions. Several studies (Eisenberg and Dreyfus, 1994; Martínez-Cruz, 1995; Moschovich and Arcavi, 1993) have indicated that students have difficulty translating between representations. Even when students can translate from one representation to another, the knowledge gained from the first representation is not necessarily available to the students in the second representation. For example, Eisenberg and Dreyfus found that although students were able to talk about the slope of a line within equations and graphs, the students did not relate the slope of the equation to the steepness of the line in the graph. In fact, students often treated different representations of a function as independent. I found this difficulty interesting in that it appeared to be a key idea that cropped up again and again in the literature on students' understanding. I gave this phenomenon a name, compartmentalization, and began to think about how I could study it. (See chapter 2 for a more detailed definition of compartmentalization).

Why study periodicity and algebraic combination of functions? Studying compartmentalization is different from studying students' understanding of a concept, because compartmentalization happens when a student has not made connections between representations or between concepts. However, compartmentalization does not necessarily have to be tied to any particular concept such as slope. It involves the study of the connections between ideas or representations and not the ideas or representations themselves. However, because the connections are made between concepts or representations, the connections cannot be studied apart from the concepts.

In order to further explore this notion of compartmentalization, the content area of functions and their representations (graph, equation, and table) were chosen. In particular, the ideas of periodicity and the algebraic combination of two functions were selected as the backdrop for the study of connections and compartmentalization. Periodicity was chosen because it is most apparent in the graphical representation. That is, the repeating nature of a periodic function is explicit in the graphical representation. This lead me to suspect that students might have a stronger understanding of periodicity within a graphical representation. The algebraic combination of two functions was chosen based on the results of a pilot study completed in 1998 that indicated the students would have the most experience with this topic within a symbolic representation. The study of these two ideas might shed some light on compartmentalization in the students' understanding of functions and, in the process, contribute to the research on students' understanding of functions. While compartmentalization has surfaced in several studies, it has not been researched explicitly. The study of compartmentalization might improve our knowledge about student's learning of functions.

The research problem. Does compartmentalization depend on content or representation? Or, do students compartmentalize their understanding across concepts or across representations or both? These were the questions that motivated this study. It was from these questions that the research questions (see Chapter 3) were developed and a direction of study was determined. The intent of this study is to determine whether compartmentalization exists in students' understandings of functions and periodicity. Particular attention has been paid to students' ability to understand a mathematical concept within multiple representations and to make connections between representations of the same concept and between different concepts.

The sample and setting. The sample consisted of 7 students enrolled in a pre-calculus class at a parochial high school in a small town in New Hampshire. The teacher was chosen by the principal because of her experience and exemplary teaching. The class was observed over a period of 2 months in the spring semester. Student work was collected over this period. In addition, the students participated in a series ofweekly interviews.

Overview

The literature review situates the research within the larger context of the existing research in students' understanding of functions. The research questions are presented after the literature review and before the theoretical and methodological frameworks in order to highlight the fact that these questions emerged out of the researchers' reflection upon existing research. The frameworks provide the reader with a basis from which to understand and interpret the research. The theoretical framework is a discussion of how students come to understand mathematics. This theoretical framework and the concepts defined in the theoretical section of the literature review give the researcher a way to communicate a "snap-shot" of a student's understanding of function.

A discussion of the research begins in Chapter 5 with a description of the sample and study. This is followed by an analysis of the data in subsequent chapters. Three types of analysis were performed, each highlighting a different component to students' understanding of functions. The first component is a culture piece in which the researcher presents a description of the culture of the classroom. This piece is offered for two reasons: to ground the students' understanding in the culture in which it is operating and to provide a framework from which their understanding can be interpreted and generalized. The culture piece is followed by a statistical analysis of the students' classwork. This analysis highlights the students' preferences for using representations while solving problems. The individual students' responses to the clinical interviews are analyzed and summarized in concept maps. Finally, the research questions are re-addressed providing support for the theoretical framework. The relationships to previous research and some new results particularly in the area of compartmentalization and students' understanding of periodic functions are highlighted.

2: Literature Review

Literature on students' understanding of functions is plentiful. Studies have shown that students have difficulty understanding the definition of function (Martínez-Cruz, 1995; Vinner, 1989). Students also have trouble understanding the representations that are used to represent functions such as graphs, tables, and equations (Lesh, Post, and Behr, 1993; Moschovich and Arcavi, 1993). In addition, it is difficult for students to gain an understanding of functions and flexibility in solving problems with functions (Eisenberg, 1992; Eisenberg and Dreyfus, 1994). As students learn the concept of function, they do not connect new ideas to previously learned ideas (Eisenberg and Dreyfus; Martínez-Cruz; Moschovich and Arcavi). In addition, students may not be able to translate from one representation to another.

Definitions

Concept image. Vinner (1983) defines concept image as a "set of properties together with the mental picture [of the concept.]" The mental picture consists of "the set of all pictures that have ever been associated with function in [the student's] mind" (p. 293). For example, the set of properties of functions might include: "you plug a number into it" or "it passes the vertical line test." The mental picture might include "$f(x)$" or the graph of the function $f(x) = x^2$. It includes things such as memories of experiences, examples, and prototypical problems with or without their solutions. The concept image also contains connections between the elements of the concept image and between concept images of different concepts.

Many people assume that once a concept is well defined, students will be able to utilize the definition in solving problems. However, Vinner (1983) found that although a student may be able to recite a definition, the concept image is what is accessed when a student is solving a problem.

> In order to handle concepts one needs a concept image and not a concept definition. Concept definitions (where the concept was introduced by means of a definition) will remain inactive or even will be forgotten. In thinking, almost always the concept image will be evoked. (p. 293)

Since it is the concept image that is used when solving problems, it is important that students develop a rich and accurate concept image.

Representations. Representations can be internal (i.e., a mental image) or external (i.e., a picture or graph). Lesh et al. (1993) submit that there are five types of representation systems:

- experience-based "scripts"–in which knowledge is organized around "real world" events that serve as general contexts for interpreting and solving other kinds of problem situations:
- manipulatable models–like Cuisenaire rods, arithmetic blocks, fraction bars, number lines, etc., in which the "elements" in the system have little meaning

per se, but the "built in" relationships and operations fit many everyday situations;

- pictures or diagrams–static figural models that, like manipulatable models, can be internalized as "images";
- spoken languages–including specialized sub-languages related to domains like logic, etc.;
- written symbols–which, like spoken languages, can involve specialized sentences and phrases $(x+3=7)$ as well as normal English sentences and phrases (p. 34)

Representations serve to denote or exemplify a concrete object or a construct such as the concept of function.

The mental pictures that are a part of a student's concept image also are representations. Since they reflect what the student has encountered, the student's concept image will likely contain common representations such as graphs, tables, and equations as well as other types of representations as reflected in the list above. Since the concept image is what a student accesses while solving a mathematical task, a researcher can gain information about a student's concept image, by watching how a student represents ideas on paper and asking questions about a student's thinking while in the midst of solving a mathematical task. In order to communicate his thinking he will have to use representations. These representations will reflect what is contained in the concept image.

Representations may be analogical or metaphorical (Perkins and Unger, 1994). For example, the graph of $f(x)=\sin x$, is an analogical representation, but thinking of the graph of $f(x) = \sin x$ as a wave is a metaphorical representation. In any case, representations are a part of, and contribute to, a student's concept image. For this reason, they are important to consider whenever one is trying to describe a student's understanding.

Compartmentalization. Although many studies have shown compartmentalization in students' understandings of functions (Eisenberg and Dreyfus, 1994; Martínez-Cruz, 1995; Moschovich and Arcavi, 1993), none have used this term to describe their findings. There are two types of compartmentalization: within representations and within concepts. Each is illuminated by a lack of connections between objects in a student's concept image. However, each type is quite different in nature.

A student's understanding is compartmentalized within representations if he or she is unable to connect two representations of the same concept. That is, if a student has an image of a concept in two different representations, but does not connect them, his or her understanding of the concept is compartmentalized within representations. For example, a student who is able to give the period when a function is presented graphically, but is unable to give the period when the function is presented in a table has compartmentalized the concept of period within the graphical representation. A student's understanding is compartmentalized within concepts if he or she is unable to connect two related concepts together. This type of compartmentalization is often encouraged in textbooks where each chapter is a separate entity and does not connect to previous or future chapters. An example of this type of compartmentalization follows: a student who is familiar with both quadratic functions and trigonometric functions is able to describe how the transformations $f(x+3)$ and $f(x)+3$ would affect the graph of $f(x)=x^2$, but is not able to describe the same transformations on the graph of $f(x)=\sin x$. In this case, the student has compartmentalized his or her understanding of transformation within the type of function.

The current study focuses mainly on compartmentalization within representations. This type of compartmentalization is particularly troublesome when solving non-routine problems where it is helpful to be able to view the problem from different perspectives. If a student is only able to view the problem in one representation at a time, or is unable to translate what has already been learned from one representation to a new representation, he or she loses flexibility in problem solving.

Students' Understanding of Functions and Their Representations

Students' difficulties with graphs. The graphical representation has attracted much study (Monk, 1992; Janvier, 1981; Preece and Janvier, 1992; Goldenberg, 1988; Moschovich and Arcavi, 1993). Graphs are one of the richest representations normally used in school mathematics. With a graph, students can see global features (such as periodicity) of a function as well as local ones (such as individual points). However, some studies have shown that students have particular difficulty in understanding the graphical representation.

Students often mix up physical features of a context with the features of a graph (Monk, 1992; Janvier 1981; Preece and Janvier 1992). For example, when students are asked to graph the distance vs. time of a person traveling up and then down a hill on a bicycle, the students draw the contour of the hill. They interpret the graph as the path the bicycle travels rather than the relationship between the distance traveled and the time. In addition, Janvier (1981) found that many students switch back and forth between a correct mathematical interpretation of points and an incorrect pictorial interpretation in context.

Students also have trouble interpreting graphs. "When students begin to experiment with graphs, they often lose what little sense they had of the meaning of the variables in a rule describing a function. Many difficulties seem to arise from the problem of decoding the graphic notation itself"(Goldenberg, Lewis, and O'Keefe, 1992). For example, Goldenberg (1988) found that students solved simultaneous linear equations successfully using tabular and graphical methods until they were introduced to the slope-intercept graphing method. This graphing method caused students to objectify the line so that they could no longer visualize the points that made up the line. When asked to find the intersection point, the students only were able to gesture towards the intersection and no longer were able to report the coordinates.

Students also encounter some additional pitfalls when working with graphing technology. Students may have trouble understanding the pixels on the computer or calculator screen. This is true in studying linear functions as well as more complex functions. Moschovich and Arcavi (1993) reported that when asked to compare two lines with different slopes, students noticed the jaggedness of the lines rather than the steepness. In addition, Goldenberg et al. (1992) noticed that students had trouble understanding the jaggedness of the graphs as they magnified the graph. The students even had a tendency to believe they were looking at a different graph altogether (Goldenberg, 1988). Although graphs and graphing technologies have been shown to be helpful to students in developing their understanding of functions (Dugdale, 1990; Eisenberg and Dreyfus, 1994; Guttenberger, 1992; Leinhardt et al., 1990; Ruthven, 1990; Slavit, 1995), they must be used carefully and thoughtfully.

Benefits of graphing technology. Graphing technology may sometimes help students to connect graphical and symbolic representations of functions (Dugdale, 1990; Eisenberg and Dreyfus, 1994; Ruthven, 1990). Student's ability to visualize features of functions represented symbolically may be improved significantly by utilizing the strengths of graphing technology (Eisenberg and Dreyfus, 1994; Ruthven, 1990).

Graphing technology may be used to graph many functions quickly. This allows the students to make generalizations about families of functions presented symbolically (Eisenberg and Dreyfus) and to see specific relationships between particular symbolic and graphic representations (Ruthven). For example, Eisenberg and Dreyfus (1994) found that after using graphing technology, students' ability to visualize transformations of functions improved significantly. The students showed the most improvement when working with familiar functions. For instance students were able to identify complex transformations (e.g., $a[f(x-h)]+k$) on the function $f(x)=x^2$. However, students were only able to identify simple transformations (e.g., $f(x)+h$) on other less familiar functions. Familiarity with the function was a good indicator of whether or not students were able to visualize transformations on it.

Evidence of compartmentalization. Students have difficulty translating from one representation to another (Moschovich and Arcavi, 1993; Martínez-Cruz, 1995; Janvier, 1981; Goldenberg et al., 1992). Sometimes they even treat different representations of the same function as essentially independent (Knuth, 1995). This greatly reduces the power and flexibility of the knowledge contained in the concept image because only one mathematical representation can be accessed in a given situation. If students are unable to translate between representations, their understanding of functions is automatically compartmentalized within representations because they are unable to connect their understanding from one representation to another.

Moschovich and Arcavi (1993) found students were unable to see fundamental connections between representations of functions. For example, the authors found that many students treated algebraic and graphical representations of linear equations as if they were independent. Students did not refer to graphical properties of slope when comparing equations with different values of *m*, even though they used the term slope in both representations separately. A similar lack of understanding was found in the interpretation of the *y*-intercept. Students could point to the *y*-intercept on the graph and locate the *y*-intercept in the equation, but often did not connect the two perspectives together. For instance, many students did not know that the point $(0, b)$ lies on the graph of the equation $y = mx + b$.

Sometimes students are able to translate from one representation to another, but still exhibit compartmentalization within representations. Martínez-Cruz (1995) found that several students had inconsistent ideas about the definition of function. The student's "concept image may be incoherent, contain conflictive parts with the concept image itself or with the concept definition, or contain potential seeds for future conflict" (p. 278).

Most of the students in Martínez-Cruz's study were able to identify graphs of functions, but had not made strong connections between the definition and algebraic equations and tables. In addition, although the definition of function had been made explicit verbally, the students utilized the definition in its graphical representation only. For example, a student was likely to be able to use the vertical line test, but did not connect this test with the concept of univalence: the property that each input corresponds to a unique output. This was surprising because the vertical line test is a graphical representation of univalence. The students had compartmentalized understanding of the definition of functions. They were able to utilize the definition only from a graphical representation and did not connect the information gained from the graph to the other representational forms of the definition.

Returning to a previous example, the students studied by Eisenberg and Dreyfus (1994) showed compartmentalization both within concepts and within representations. The students were able to visualize function transformations on familiar functions far better than on less familiar functions. Eisenberg and Dreyfus suggested that the students had developed a rich concept image of the function $f(x) = x^2$ so that the graphical representation was at least as strong as the algebraic representation. Thus students made connections between graphic and algebraic representations of transformations on the standard parabola. These connections allowed students to visualize transformations. Other less familiar functions were situated in an algebraic representation in student's concept image. Thus, the graphical representation was not connected to allow for visualization of transformations on them.

In summary, the concept of function, although important in mathematics, is difficult for students to understand. Although the representations of functions, graphs, tables, and equations, are designed to represent functions in a concise, illuminating fashion, students have difficulty understanding different representations and connecting them in their concept images (Moschovich and Arcavi, 1993; Martínez-Cruz, 1995; Janvier, 1981; Goldenberg et al., 1992). The graphical representation seems to present the most difficulty for students. Students have difficulty interpreting graphs and connecting graphical representations with algebraic representations (Monk, 1992; Janvier, 1981; Preece and Janvier 1992; Goldenberg et al., 1992; Goldenberg, 1988; Moschovich and Arcavi, 1993). Although graphing technology can help, difficulties persist (Dugdale, 1990; Eisenberg and Dreyfus, 1994; Ruthven, 1990). Students tend to compartmentalize their understanding of function within representations. As long as compartmentalization is present in their understanding, the students will not have the rich understanding of functions called for in the *Principles and Standards* (NCTM, 2000).

Periodic Functions and Algebraic Combination of Functions

Studies on students' learning of sine and cosine. Guttenberger (1992) performed a teaching experiment to study the concepts of amplitude and period of sine and cosine functions. The experiment was carried out on 31 Mexican high school students who had no previous experiences with trigonometry. An experimental group of eight students was given lessons on a computer, while a control group of 23 students attended lectures. The computer instruction emphasized exploration and guided discovery.

Students were tested on their ability to "relate graphical and algebraic representations of trigonometric functions and interpret graphical or algebraic representations of trigonometric functions" (p. 109). specifically, they were asked to graph equations and find equations for graphs of functions in the form $y = a \sin bx$ and $y = a \cos bx$. They were asked to interpret a and b in the equation, find the amplitude and period in the graph, and to relate the two.

Students given the computer discovery lessons outperformed those in the lecture on both a post-test and a retention test. This suggests that the use of the computer along with discovery tasks "had a positive effect on the formation of concept images and construction of concepts for students in the computer group."

DaCosta and Magina (1992) also performed a teaching experiment with 32 Brazilian high school students aimed at determining whether laboratory and computer activities would improve students understanding of sine and cosine.

The control group sat through classroom lectures while the experimental group was given laboratory activities. Computer activities in Cabrí Géomètre and Graphmatica were used to allow students to draw a lot of figures quickly and precisely and to give them immediate feedback. For the laboratory activities, three apparatuses were created. The first was a sand pendulum that swung back and forth on a moving platform and released a thin trail of sand. The students started the pendulum moving and then rolled the platform across the room in a single direction. The second apparatus was a circular piece of wood with a crank and a laser pointer attached. As one student turned the crank, another student pulled on a roll of paper while a third student marked the path of the laser on the paper. The third apparatus was a watch with the second hand removed and replaced with a small stick. Students directed a light at the stick creating a shadow on a paper. The students then recorded the projection of the stick on the paper as it moved around the watch face.

The students were tested three times - before, during and after the experimental conditions. The results showed that students who had the physical activities prior to the computer activities showed the greatest amount of improvement over the three tests. The authors concluded that this experimental condition was optimal.

Dugdale (1990) compared students in two different classroom settings. In the first classroom, students studied trigonometric functions and identities in a mostly algebraic way supplemented by a graphing utility. In this class, students were encouraged to relate graphs to the identities proven by algebraic means, but they were not asked to reason about the graphs. In the second classroom, the curriculum was designed around a graphing utility. Trigonometric functions and identities were introduced graphically. Students were asked to reason about these graphs by either revising functions to change a graph or by using graphs to predict the shapes of other graphs. This curriculum was designed to allow students to build experiences with global features of graphs, such as periodicity, and to make connections between algebraic and graphical representations.

The students in the graph-centered class performed better on a trigonometric-identities test. They were able to relate functions to their graphical representations and exhibited more variety in their approaches to standard identity problems. They were also able to predict the shape of new graphs created by combining periodic functions.

Composition and algebraic combination of functions. Ayers, Davis, Dubinsky and Lewin (1988) studied the affect of teaching with computer functions on students' understanding of composition through a teaching experiment. One class was given paper-and-pencil exercises on function composition while another was given computer exercises in UNIX. Examples of a paper and pencil exercise and a computer exercise are shown in Table 1. They designed a test to assess rich understanding of composition of functions.

The researchers found that the students in the computer class scored significantly higher on the test than did the control class. The authors concluded that the computer group had a better opportunity to reflect upon and abstract functions than the control group. Thus, the computer group had developed richer concept images.

Table 1: Examples of Composition Tasks

Paper-and-pencil	Computer
Q. Each of the following functions is considered to act on some text written in English characters: F interchanges the letters 'o' and 'w'. G counts all occurrences of the string 'two'. What is the effect of applying these two functions in sequence, first F and then G?	Q. Create a new command by piping the command #lin (which the student knows to count the number of lines in a text) with the command text the (which the student knows to return those lines of a text containing the string 'the') so that the latter is performed first. Describe the new command.
A. The student is expected to answer that the effect is to count all occurrences of the string 'tow'	A. The student is then expected to answer that the new command, name, counts the number of lines in a text that contain the string the.

Dugdale et al. (1992) describe a two-year teaching experiment where students in Advanced Algebra and Analytic Geometry were introduced to polynomial functions using graphing technology. In this approach, polynomials were interpreted as sums of monomials. For instance students were given the graphs of $y = -2x^3$ and $y = 2x$ and asked to predict the graph of their sum $y = -2x^3 + 2x$.

Before the approach was started with the students, they were able to relate equations of polynomials to their graphs only by memorized rules for relating polynomial functions to their graphical representations. In cases where the available rules offered no help, students resorted to random guessing. By the end of the two years, the authors found that students had developed a much richer understanding of functional behavior, visualized functional relationships between equations better, and used graphical representations more ably in their problem solving.

In summary, the literature on algebraic operations, composition of functions, and periodicity is limited. These concepts have not been the focus of many studies. Most of the information researchers have gleaned on these topics have come on the periphery of studies on students' understanding of functions. In the next chapter, the research questions are presented. It is important to note that these research questions emerged out of a reading of the research presented in this chapter.

3: Research Questions

Research Questions

Does compartmentalization depend on content or representation? Or, do students compartmentalize their understanding between concepts, within representations or both? These were the questions that motivated this study. It was from these questions that the more specific research questions were developed and a direction of study was determined. The intent of this study is to determine whether compartmentalization exists in students' understandings of functions and periodicity. Particular attention has been paid to students' ability to understand a mathematical concept within multiple representations and to make connections between representations of the same concept and between different concepts.

Periodicity and operations on functions have been chosen in this study because they are rather different ideas that could easily be situated in a particular representation in a student's concept image of function. The repeating nature of periodic functions is clear when viewing them, on an appropriate scale, in a graphical representation. This repeating nature could be seen in a table if the input-values were chosen carefully, but is more difficult to see in an equation. However, the definition of periodicity is mathematically presented in a symbolic representation $f(x+p)=f(x)$. The concept of periodicity has an interesting property in terms of representation. It was my conjecture that the students' concept image of periodicity would be situated mostly in a graphical representation and therefore students would have greater difficulty solving periodicity problems when presented using tables and equations. However, given American students' heavy reliance on the algebraic representation (Knuth, 1995; Ruthven, 1990), and the data from a pilot study conducted in 1998, I suspected that their concept images of operations on functions would be more likely to be situated in an algebraic representation. If this were the case, the students would have a more difficult time with tasks involving operations on functions in a tabular or graphical form. Operations on functions and periodicity were chosen in order to find out if compartmentalization occurred consistently in different concepts or varied from concept to concept. A list of research questions follows:

- Is students' knowledge of functions and periodicity compartmentalized by a particular representation?
- Do students have a particular representation from which they solve most problems regardless of the content or do they choose a particular representation depending on the content (operations on functions, periodicity)?
- Are students able to solve problems presented in all three representations? Does this facility depend upon the content of the problem?

- Are students able to attempt to re-solve a problem utilizing a different representation when prompted? Are they able to re-solve the problem correctly?

The next chapter sets the stage for the current study by outlining the researcher's perspective on students' learning. Recall that the students in this study were enrolled in a high school pre-calculus class. They were juniors and seniors who had completed a sequence consisting of pre-algebra, algebra I and II, and geometry, not necessarily in that order (see Chapter 5 for a more complete description).

4: Theoretical and Methodological Frameworks

There are three main themes in the theoretical framework. First, students learn mathematics both constructively and socially. That is, students create mathematical meaning through their interaction with the environment and, also, through interaction with others (Cobb, Yackel, and Wood, 1992; Simon, 1995; vonGlassersfeld, 1990). Secondly, the knowledge that is built is organized to allow the student to efficiently access what has been learned. Finally, as the student organizes his or her knowledge, inconsistencies arise. The student learns by confronting these inconsistencies (Groen and Kieran, 1984; Gagne, Yekovich-Walker, and Yekovich, 1993; Leinhardt et al. 1990).

Theoretical Framework

It is important to ground research in a theoretical framework. Such a framework gives the researcher a starting point from which to generate questions and determine how these questions can be addressed. It also allows the reader to understand where the researcher is coming from and why he or she chose to answer the research questions in a particular way.

Constructivism. Constructivism provides a framework for individual learning. The individual is continually interacting with his or her environment, encountering new experiences that are used to develop constructs which connect together to form a concept image. The collection of mathematical concept images forms a student's body of mathematical knowledge. The learner is constantly editing and amending these concept images as he or she learns. Thus, the student's body of knowledge is built progressively through a constructive process (Cobb et al., 1992; Simon, 1995; vonGlassersfeld, 1990).

> Constructivism derives from a philosophical position that we as human beings have no access to an objective reality, that is, a reality independent of our way of knowing it. Rather, we construct our knowledge of our world from our perceptions and experiences, which are mediated, through our previous knowledge (Simon 1995, p. 115).

The individual is able to understand a new experience to the extent that the experience fits into an existing concept image. If a new experience causes a contradiction, the individual must find a way to deal with the new experience. This may cause the individual to revise an old concept image, to replace an old concept image with a new concept image, or to view the new experience as an exception and therefore retain the old concept image along with a new construct. After the new construct has been formed, the student's concept image reflects the new situation. (Simon, 1995; Strike and Posner, 1985)

Reflective abstraction. At the root of constructivism is the concept of reflective abstraction (Groen and Kieran, 1984) that states that learning is a process in which the pupil reflects upon and generalizes previous knowledge so that it may be used in new situations. This process allows a student to reference previous knowledge in order to construct new knowledge by observing his or her own thoughts and abstracting from them. Reflective abstraction is the way in which our experiences are transferred into abstract mathematical ideas.

The older a student becomes and the more mathematics he or she is exposed to, the more personal mathematical knowledge is accumulated from which he or she can build and expand. Not all of this knowledge is mathematically correct. Any new knowledge must be reconciled with this previous knowledge. Therefore, older students have many preconditions on their learning. New experiences must find a place in a student's set of concept images. If these new experiences don't seem to fit, it may be much easier to isolate or compartmentalize the new experience from pre-existing constructions thereby limiting the previous knowledge, which can be applied to the novel experience.

> In their ideal form, conceptions are highly interactive with supportive intuitions. But they also can be actively split from intuitions when they cannot easily fit into the intuitive structure; hence it is easier for the student to keep two separate islands of knowledge (Leinhardt et al., 1990, p. 6).

As long as the constructions remain isolated, there may be inconsistencies between them. The learner may be unaware that inconsistencies exist. The two contradictory constructions will remain intact until the student notices and confronts the inconsistencies.

Social Interaction. Learning occurs as a student interacts with the environment. Other people are also a part of the environment. Since each person's experiences are unique, constructivism cannot account for the fact that many different people appear to construct the same knowledge or similar knowledge. And, in spite of our unique experiences, we are able to communicate with each other. Therefore there needs to be a theory that incorporates social interaction.

Cobb, Yackel, and Wood (1992) explain that we are able to communicate with each other not because we share the same constructions or the same knowledge, but because we are able to subjectively interpret others' words from our own experiences. Therefore, a certain amount of knowledge is "taken-as-shared" by the individuals who are communicating.

> Taken-as-shared indicates that members of the classroom community, having no direct access to each others' understanding, achieve a sense that some aspects of knowledge are shared, but have no way of knowing whether the ideas are in fact shared (Simon 1995, p. 116).

Taken-as-shared allows an individual to communicate what he or she has learned to others. It also allows the listener to incorporate others' experiences into his or her own. Therefore, learning is a social experience as well as an individual experience.

Automation. In addition to constructing and sharing knowledge, a large part of a student's learning process consists of making commonly used constructs easier to access. High school students, in particular, encounter problems which require the utilization of several constructs concurrently.

> Because of the limited capacity of working memory, it is difficult to perform several mental tasks at once . . . One way in which people can do more than one mental task at the same time is by "automating" one of the tasks. That is practicing a task until it can be performed with a minimum of awareness (Gagne et al., 1993, p. 45).

Automation (Gagne et al.; Heibert, 1990) occurs when a student becomes so familiar with a construct that he or she no longer needs to concentrate on the construct in order to generalize it to a new situation. If commonly used tools, such as basic algebraic manipulations, are automated, a student is free to focus attention on the novelty of an experience.

> Efficient execution of a procedure is the key to understanding the cognitive benefits of automatization. The more efficiently a procedure is executed, the less mental effort is required. If a procedure is fully automated, it can be run off without thinking, without any mental effort (Heibert, p. 34).

The learning of a new concept can be blocked by a deficiency in prerequisite tools. That is, a student who is only focusing on previously learned constructs may not be able to assimilate a new experience. In order to construct knowledge when faced with a complex experience, a student must be able to focus attention on the novelty of the experience. Automation of commonly used tools allows a student to utilize those tools while still applying reflective abstraction to another construct. It may allow a student to access compartmentalized constructs concurrently (Gagne et al., 1993).

In addition, if the concept image is not rich with constructs and connections before a tool is automated, automation may contribute to problems with understanding. The fact that the student does not have to think about the automated tool means that the understanding of the tool is not accessed while experiencing the new situation. This can limit the connections made and therefore encourage both compartmentalization and the acceptance of contradictory constructs. Therefore, it is very important that the student have multiple constructs and connections before a tool is automated.

Connections. In addition to the actual construction of knowledge, the learner makes connections between different constructions. As before, a new construct may be added to existing construct, but it also may be a connector between two existing constructs which were previously isolated. By connecting isolated constructs, a student is able to confront inconsistencies between the constructs (Leinhardt et al., 1990). As an individual makes more connections, his or her understanding becomes more robust and an individual concept is less likely to be misunderstood.

It may be that connections are made inappropriately just as knowledge can be constructed inappropriately. In this case, the inappropriate connection will remain until a contradiction is discovered by the individual or until the connection is no longer deemed useful. At this point, the connection is either disregarded or replaced with a more appropriate connection or construction (Strike and Posner, 1985).

As a student gets older, connections become a more important factor in learning (Leinhardt et al., 1990). As mathematical knowledge increases, it becomes more troublesome to access stored information. By the time a student enters pre-calculus, he or she has several different representations from which to view a problem. He or she will have had some exposure to geometric representations and algebraic representations, for instance. If these concepts are not connected in some way, then the student will find it difficult to draw from both representational systems concurrently. Making connections between different representations helps students to develop a deeper understanding of each representation and the concept being represented.

Representations. Mathematics is presented through various representations. A problem in pre-calculus may be presented in algebraic form accompanied by a diagram or graph. These different representations may be problematic for the student who has made little or no connection between algebraic and geometric representations.

> The formal notations and carefully defined concepts of mathematics produce rather clear problem spaces. It is notable, however, that these benefits only accrue when the inquirer can easily translate back and forth between the more formalized representations of the discipline and the things and phenomena to which they refer. However, a student able to access both representations concurrently will have more tools available with which to understand and solve the problem (Perkins and Unger, 1994, p. 10).

As more connections are made, more knowledge becomes available for the student to use in an efficient manner. A student with fewer connections may have developed helpful tools, but they may be inaccessible concurrently.

Graphs, diagrams, and pictures are particularly suited for studying connections because of their representational qualities. A graph represents mathematical relationships and interactions. In order to draw or understand a graph, students must make connections between the picture and the situation it represents. If the connections are not present it is particularly problematic for students to draw graphs or interpret them.

In summary, students create mathematical meaning through their interaction with the environment and also through interaction with others (Cobb et al., 1992; Simon, 1995; vonGlassersfeld, 1990). The student translates mathematical experiences into abstract mathematical ideas through the process of reflective abstraction (Groen and Kieran, 1984). Students organize their understanding through the processes of automation (Gagne et. al., 1993, Heibert, 1990) and building connections (Leinhardt et. al., 1990). The knowledge that is built is organized to allow the student to efficiently access what has been learned. As the student organizes his or her knowledge, inconsistencies arise. The student learns by confronting these inconsistencies and revising constructs to eliminate them (Groen and Kieran, 1984; Gagne et al., 1993; Leinhardt et al., 1990).

Methodological Framework

Because a student's understanding is built from his or her experiences, it is difficult to view that understanding. A researcher can assess constructs and connections from a student's understanding by getting a picture of what some of the experiences might have been, developing a set of tasks that will challenge the student to think, and getting the student to think out loud and explain his thinking (Confrey, 1981; Boostrom 1994). These can be translated into three methods of qualitative research, thick description, the clinical interview, and concept mapping. Thick description is a method of describing the culture of the classroom. It gives a picture of the context in which the content was learned. Clinical interviewing is a method for uncovering the student's thinking when solving a problem. Concept mapping done by the researcher or the students provides a diagram of some of the constructs in the student's concept image as well as the connections he or she has made between constructs.

Thick description. Geertz (1973) explained that the culture is "whatever it is one has to know or believe in order to operate in a manner acceptable to its members." Culture plays an important role in an individual's actions and reactions within the group. The culture created by the teacher and students in a particular classroom will affect the learning that occurs there.

In addition, the mathematics influences the culture of a classroom in which it is being taught. Social and mathematical interaction in a given classroom, or the lack thereof, can help or hinder learning (Nickson, 1992). As learning is a social and constructive process, it is important to ground a student's understanding in the culture in which it is operating. Thick description is the description of the culture of the classroom, explained in depth from the perspective of the participants. It is the uncovering and explanation of cultural norms.

The clinical interview. A researcher can gain knowledge about a student's understanding of mathematics by viewing student's work. However, when a problem is solved incorrectly, it is very difficult to determine what the student was thinking when he or she attempted to solve the problem without interacting with the student. (Confrey, 1981) Such interaction is often most fruitful when the student solves a problem incorrectly or expresses an incorrect mathematical idea.

> Moments of controversy, conflict, confusion and change often reveal individuals' most fundamental, determinedly-held beliefs, and allow for inspection of the processes used in forming those beliefs (Confrey, 1981 p. 12).

The clinical inteview gives the researcher more opportunity to interact with the student and have the student clarify and elaborate his or her ideas. Therefore this method provides the researcher the opportunity not only to examine students understandings, but also identify the rationale behind those understandings. (Taylor and Bogdan, 1984).

Table 2: **Example Interview Questions by Content and Representation**

	Graph	Table	Equation
Operations on Functions and notation	Given a graph of $f(x)$ find $f(a)$ (given on graph)	Given the tables of $f(x)$ and $g(x)$ find $(f+g)(x)$, $(fg)(x)$ or $(f \circ g)(x)$.	Given equations of $f(x)$ and $g(x)$ write the equation of $(f+g)(x)$, $(fg)(x)$ or $(f \circ g)(x)$.
Periodicity	Determine if a graph could represent a periodic function and give the smallest possible period	Given $f(x)$ with period p in a table, fill in the missing values	Determine if the combination (i.e. $f+g$, fg, $f \circ g$) of two functions in equation form creates a periodic function

Development of Tasks. A set of 35 tasks was developed for use in the clinical interviews. These tasks were designed to challenge the students in that they were different, for the most part, from questions which were asked in the book or in class. They were informed by a pilot-study conducted in 1998 and by previous research on students understanding of functions (Ayers, et al., 1988; Berk, Gerson, & Portnoy, 1999; Dubinsky, 1992; Eisenberg and Dreyfus, 1994; Moschovich and Arcavi, 1993). These tasks were designed to cover the content areas of periodicity and algebraic operations on two functions in each of the representations: graphs, tables, and equations. Table 2, shows examples of the tasks. A full matrix of tasks can be found in Table 3 in Chapter 5.

Concept Maps. Concept maps are a way of representing a student's concept image. A concept map consists of concepts (usually circled) connected by lines. There are different ways of constructing and using concept maps (Williams, 1998). Concept maps can be organized in a hierarchy (Bartels, 1995). An example of a hierarchical concept map using the concepts function, polynomial, trigonometric, periodic, $f(x) = x^2$, exponential, logarithmic, $y = a\sin x$ is in Figure 1.

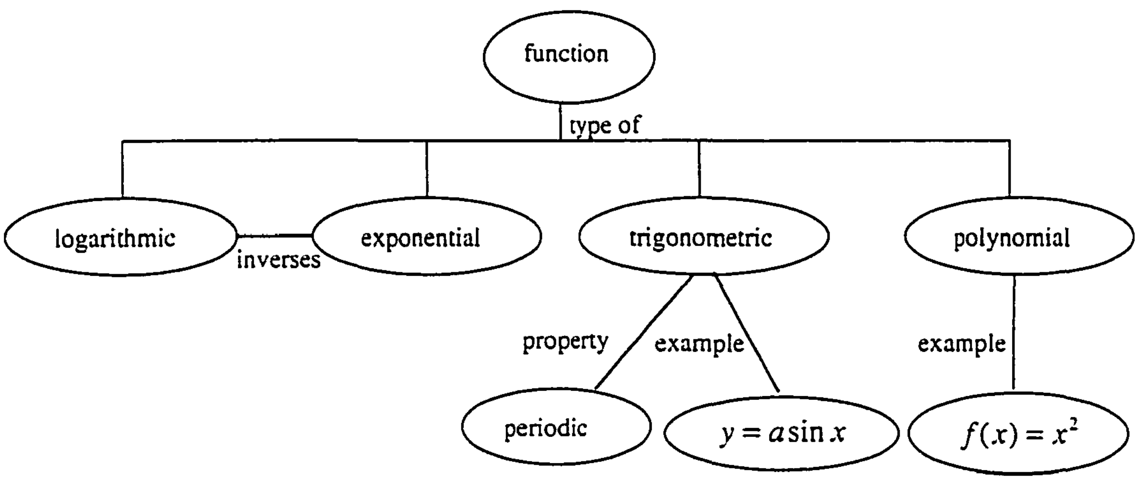


Figure 1: A Hierarchical Concept Map

Other researchers have used "spider maps" (Martinez Cruz, 1995; Williams, 1998) which consist of maps with a general concept in the center with other concepts organized around it resembling a spider's legs. An example of a "spider map" can be found in Figure 2.

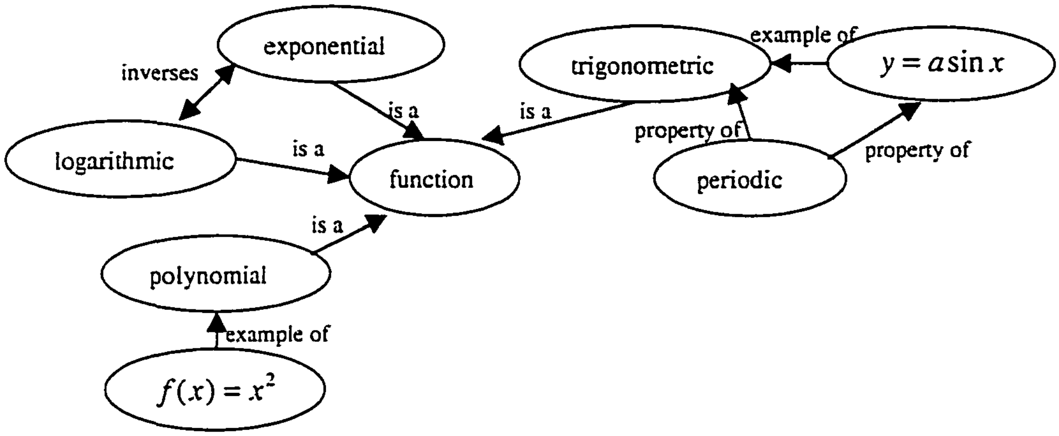


Figure 2: A "Spider Map"

It is informative to have the students create concept maps because they must illustrate and name the connections explicitly (Bartels, 1995; Williams 1998). The concept map is also a nice representation for presenting parts of a student's concept image. In this study, concept maps are used in two ways. Students were asked to sort a set of 27 cards into groups and then label each of the groups. They completed this task as many times as they could within a 30 minute period. The researcher later took the students groupings and created concept maps using the cards as the concepts and labels as connectors between concepts. The students' concept maps are a combination of "spider maps" and hierarchical maps. In addition, the researcher used the data collected from the interviews and classroom observations to create concept maps for the students.

These methods are used in the current study to examine the student's constructed knowledge. The classroom description helps to ground the students' understanding in the culture in which it was operating. Clinical interviews helped the researcher to uncover students' constructed knowledge and the connections they made in the area of functions, operations on functions and composition, and periodic functions. Concept mapping allows students to make connections they have made between constructs explicit. Also it allows the researcher to summarize the students understandings.

5: Methods

Sample and Setting

The sample consisted of students enrolled in a pre-calculus class at a parochial high school in a small town in the Seacoast area of New Hampshire. The teacher, Mrs. Brown, was a middle-aged teacher with a master's degree in mathematics teaching. She taught advanced placement calculus, pre-calculus, and honors pre-calculus.

Initially, a different teacher was selected for a pilot study to begin in the Fall of 1997. Arrangements were made with the school administration for the researcher to be on campus through parts of 1997 and 1998 for both the pilot study and the dissertation study. Over the summer of 1997 a new principal was hired. He moved the teacher from pre-calculus to algebra II. The principal then suggested Mrs. Brown as a replacement to participate in the current study because of her experience and exemplary teaching.

There were 19 students in the class, 5 were juniors and 14 were seniors. Mrs. Brown used the Houghton Mifflin, Advanced Mathematics text (Brown, 1992) extensively and supplemented with her own labs, essays, exams and quizzes.

The data collected came from 4 sources: classroom observations, interviews of students, a card-sorting activity, and student work assigned and collected by the teacher.

During classroom observations, the researcher observed the entire class of 19 students. Student work was collected from a subset of 9 students who volunteered to participate in this study. Two students withdrew from the study and therefore only the data from 7 students was analyzed. Interviews were carried out with the 7 students: 1 senior girl, 3 senior boys, 1 junior girl, and 2 junior boys. Each student was interviewed 6-7 times over a two-month period and completed as many of the tasks as he or she could in each interview. The card-sorting task took place during the final interview.

Instruments

The researcher developed a set of 39 tasks for use in the study. These questions were informed by and adapted from previous research (Ayers, et al., 1988; Berk, Gerson, & Portnoy, 1999; Dubinsky, 1992; Eisenberg and Dreyfus, 1994; Moschovich and Arcavi, 1993). The interview tasks covered two content areas: operations on functions/notation and periodic functions, and three representations: graphs, tables, and equations. The operations/notation tasks consisted of providing students with the opportunity to show an understanding of function notation, perform arithmetic operations on functions, and compose functions. The tasks were designed to give ample opportunity for the students to solve problems in each content area and representation. Table 3 shows the full matrix of tasks.

Table 3: Matrix of Interview Tasks

	Graph	Table	Equation
Operations on Functions and Notation	Given a graph of $f(x)$ find $f(a)$. Given a graph of $f(x)$ describe its properties. Given the graphs of $f(x)$ and $g(x)$ find $(f+g)(a)$, $(fg)(a)$, $(f \circ g)(a)$. Given the graphs of $f(x)$ and $g(x)$ draw the graph of $(f+g)(x)$, $(fg)(x)$, and $(f \circ g)(x)$.	Given a table of $f(x)$ find $f(a)$. Given a table of $f(x)$ describe the pattern. Given the tables of $f(x)$ and $g(x)$ find $(f+g)(a)$, $(fg)(a)$, $(f \circ g)(a)$. Given the tables of $f(x)$ and $g(x)$ create a table for $(f+g)(x)$, $(fg)(x)$, and $(f \circ g)(x)$.	Given an equation of $f(x)$ find $f(a)$. Given equations of $f(x)$ and $g(x)$ find $(f+g)(a)$, $(fg)(a)$, $(f \circ g)(a)$. Given equations of $f(x)$ and $g(x)$ find $(f+g)(x)$, $(fg)(x)$, and $(f \circ g)(x)$.
Periodic Functions	Given $f(x)$ in a graph with period p, find the value of $f(a)$ not represented on the graph. Given a graph of $f(x)$ with period p, draw a particular part of the graph not represented. Determine the period of a function represented graphically. Determine if a graph could represent a periodic function and give a possible period.	Given $f(x)$ with period p in a table, fill in the missing values. Given $f(x)$ with period p in a table, find the value of $f(a)$ not given in the table. Determine the period of a function represented in a table. Determine if a table could represent a periodic function and give a possible period.	Given $f(x)$ with period p in an equation, determine the value of $f(a)$. Determine the period of a function represented in an equation. Determine if an equation is periodic Determine if the combination (i.e. $f+g$, fg, $f \circ g$) of two functions in equation form creates a periodic function.

Interview tasks were a combination of brainstorming questions and mathematical questions. In the brainstorming questions students were asked to write down everything that they could think of related to a given topic. The brainstorming questions were used to develop the cards for the card sorting activity. The tasks were designed to allow for multiple solution strategies, and to encourage the use of multiple representations. The interview tasks also provided opportunities for the students to translate between representations. Because the class observations showed that the students used equations most of the time (see chapter 7), most of the graphs and tables presented in the interview tasks represented functions that were not covered in class. Therefore it would be difficult for the students to solve them using equations. This guaranteed that the students would have to solve at least some of the tasks using tables or graphs. However, some of the functions represented familiar functions. These were added to provide more information on each student's ability to translate between representations.

When analyzing the interview tasks, the researcher looked for instances of translation between representations, interpretation of representations, interpretation of notation, and consistent right or wrong answers that would demonstrate understanding of one of the content areas or representations.

Analysis of interview questions. Each interview task fits into the matrix shown in Table 3. The questions were designed to give students ample opportunity to demonstrate their understanding of each content area and representation. A certain amount of overlap was built in. The tasks were open-ended, so the analysis of the interview data depended to a certain extent upon the way each student answered a given task. However, each task was designed to probe certain aspects of the students' understandings. Each question is laid out below along with an explanation of what aspect of the students' understanding the question was designed to explore. Answers are provided for each task. These answers do not reflect the students solutions, but merely provide the reader with the outcome of a correct solution.

The function represented in the table below is periodic. It has a period of 5. Fill in the missing values.

x	$h(x)$
-3	2
-1	-1
1/2	0
1	3
4	
6	

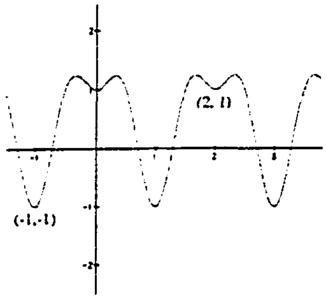
Answer: The student should fill in -1 and 3.

Figure 3: Interview Task 1

The first interview task, in Figure 3, is designed to examine a student's understanding of period in a tabular representation. The function is 'unfamiliar' to the students because it has not been covered in class or in the text. In addition, it is not easily converted to an equation. The notation $h(x)$ is used to test students' understanding of function notation.

Task 2, is designed to probe a student's understanding of period in a graphical representation (see Figure 4). The function is 'unfamiliar', but does share some features with sines and cosines. For instance it is continuous and has symmetrical peaks and symmetrical valleys. Function notation is used again, but this time with the more familiar $f(x)$.

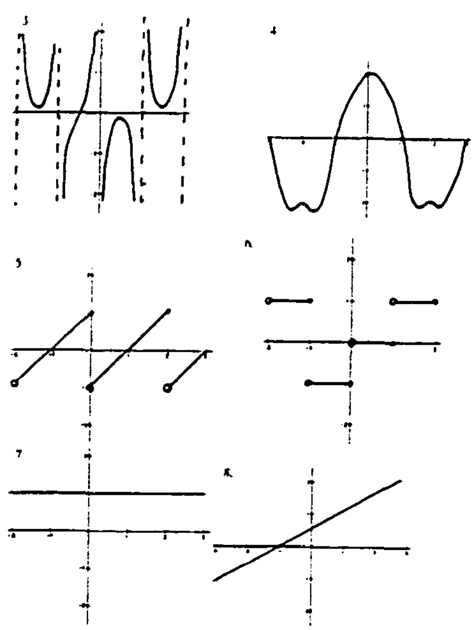
The graph below represents $f(x)$. If it has a period of 2, what is $f(6)$?



Answer. $f(6) = 1$

Figure 4: Interview Task 2

Which of the following could be periodic. What could the period be? If it could be periodic, describe or draw more of the graph. If it could not be periodic, explain.



Answer. Each of the graphs in tasks 3-8 could represent a periodic function. Students should continue the graphs with a possible period, and state a possible period. The smallest possible periods for tasks 3-6 would be as follows: 3, 4, 2, and 3 respectively. In 7 and 8 either the period is greater than 6 or the functions are not periodic.

Figure 5: Interview Tasks 3-8

Tasks 3-8 also are designed to scrutinize students' understandings of periodic functions in a graphical representation (see Figure 5). Some of the graphs are familiar while others are not. In addition, some graphs are familiar to the students, but not in the content area of periodic functions. For example, students were familiar with step functions, but not in the context of periodic functions.

Determine if each the following tables could represent a periodic function. What could the period be? If it could not be a periodic function, explain.

x	$f(x)$
-1	5
0	4
1	3
2	2
3	1
4	0

x	$f(x)$
-1	1
0	2
1	0
2	-1
3	1
4	2

x	$f(x)$
-2	-1
0	0
1/2	1
2	0
4	-1
5	0

x	$f(x)$
-2	0
-1	1
0	0
2	0
4	0
5	1

Answers. Every table could be periodic. However, it was also expected that for the tables which did not start to repeat students could make a valid argument to support the answer that it was not periodic.

Figure 6: Interview Tasks 9-12

Interview Tasks 9-12 are designed to identify students' understandings of periodicity in a tabular representation (see Figure 6). Some of the tables represent familiar functions and others represent non-familiar functions. For example, the data points in Task 9 have a linear relationship. In Task 10 the x-values are evenly spaced and the last two entries in the table are the same as the first two. When graphed, the function in Task 10 looks like a non-symmetrical wave. Task 11 is similar to Task 10 except the x-values are not evenly spaced. And finally Task 12 gives a function that can be interpreted in a number of ways since the data points for 1 and 3 are missing.

Figure 7 shows Interview Tasks 13-18. These tasks are designed to examine the students' understandings of algebraically combining two functions within the tabular representation. The first function is almost a quadratic and the second function is unfamiliar.

Use the tables for the functions f and g to find the values of:
$(f+g)(1)$, $(fg)(1)$, and $(f \circ g)(1)$

x	$f(x)$
-1	6
0	3
1	2
2	5

Table for f

x	$g(x)$
-1	2
0	0
1	-1
2	-1

Table for g

Use the tables for the functions f and g, to draw tables for: $(f+g)(x)$, $(fg)(x)$, and $(f \circ g)(x)$

<u>Answer.</u> $(f+g)(1)=1$, $(fg)(1)=-2$, $(f \circ g)(1)=6$.

x	$(f+g)(x)$	$(fg)(x)$	$(f \circ g)(x)$
-1	8	12	5
0	3	0	3
1	1	-2	6
2	4	-5	6

<u>Figure 7</u>: Interview Tasks 13-18

The following are the graphs of two functions f and g. Draw the graphs of $f+g$, fg, and $f \circ g$.

Answers.

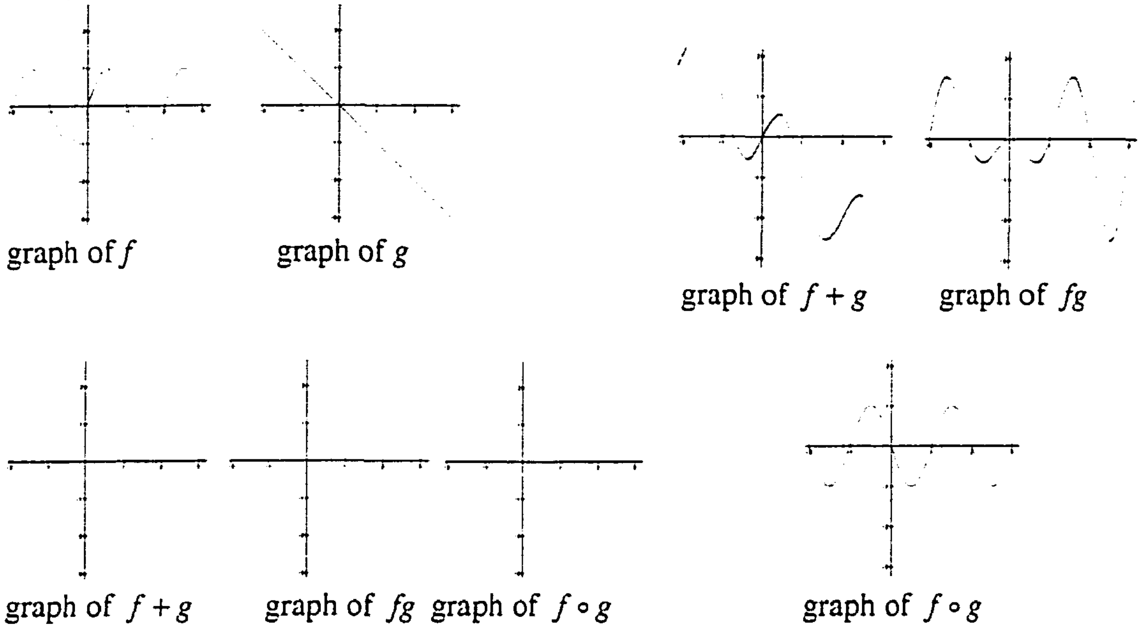


Figure 8: Interview Tasks 19-21

Interview Tasks 19-28 can be found in Figure 8. These tasks probe students' understanding of combining functions in the graphical representation. Both graphs represent familiar functions.

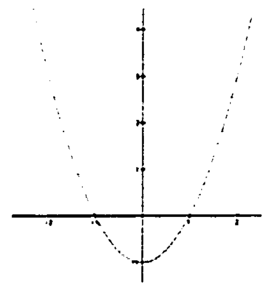
Figure 9 show Interview Tasks 22-24. These tasks are presented in mixed representations. This task was developed to ensure that students would have to translate between representations. It was designed to illuminate students understanding of combining functions. Each of the functions is familiar.

Given the functions f, g, and h below Write an equation, draw a graph, or make a table to show your results:
$(f+g)(x)$
$(f+h)(x)$
$(g+h)(x)$

x	$f(x)$
-2	2
-1	2
0	2
1	2
2	2

$g(x) = \frac{1}{2}x + 1$

the graph of $h(x)$



Answers.
$(f+g)(x) = (1/2)x + 3$
$(f+h)(x)$ would be the graph of $h(x)$ shifted up 2 units.
$(g+h)(x) = 1/2x + x^2$

Figure 9: Interview Tasks 22-24

Tasks 25-33 are presented in Figure 10. These tasks examine students' understandings of function notation, combining functions, and periodic functions within an algebraic representation.

Interview task 34 is designed to probe students' understanding of functions. Even though it is presented in a representation neutral form, the researcher believed that the students would use their graphing calculator. Therefore it is expected that this task will scrutinize students understanding of periodic functions within a graphical representation.

Given the functions $f(x) = x^2 - 5$ and $g(x) = \sin(\pi x)$ find the following:	Answers: It was expected that the students would write the following answers:
$f(3)$	4
$g\left(\frac{1}{2}\right)$	1
$g(k)$ for k = 0, 1, 2, 3, ...	0
$(f+g)(x)$	$x^2 - 5 + \sin(\pi x)$
$(f+g)(3)$	4
$(fg)(0)$	0
$(fg)(x)$	$(x^2 - 5)(\sin(\pi x))$
$(f \circ g)(1)$	–5
$(g \circ f)(x)$	$\sin(\pi(x^2 - 5))$

Figure 10: Interview Tasks 25-33

sinx, cosx, and tanx are periodic functions. Is it possible to get new periodic functions by combining 2 periodic functions (adding, subtracting, multiplying, dividing, and composing)? How about if you combine a periodic function with a non-periodic function?	Answer. Students are expected to methodically test different combinations of functions and make hypotheses for which ones will be periodic.

Figure 11: Interview Task 34

Brainstorming Questions. The brainstorming questions (see Figure 12) provided a way to observe those constructs, concepts, pictures, problems, examples, or non-examples each student felt were significant and linked to a given concept. The answers formed a basis for examining the student's prototypical examples and non-examples. They also illuminated misconceptions.

1. Brainstorm with your partner about the following idea. What are the most important ideas in this topic and what other things are connected to it. FUNCTIONS

2. A function is a relationship between two variables x and y. Each x value is related to only one y value. Its graph passes the vertical line test. Draw 5 graphs of functions that you have seen before. Also write 5 equations.

3. Brainstorm with your partner about the following idea. What are the most important ideas in this topic and what other things are connected to it. PERIODIC FUNCTION

<u>Figure 12</u>: Brainstorming Tasks

The brainstorming tasks formed the basis of the card sorting activity. Anything that was on more than one student's paper was included in the card sorting activity. In addition, the researcher added a few cards in order to have representatives of a variety of functions, ideas, and representations. The list of words and phrases on the cards is in Figure 13.

Students were asked to sort the cards into groups. Once the cards were sorted, the students were asked to explain how the cards in each group were connected. Each group was asked to sort the cards in two or three different ways as time permitted. The card sorting activity was designed to make it quicker and easier for students to make concept maps. As they sort the cards, students label the connections. This is a dynamic way of making a concept map and allows for the students to be more flexible with the connections and groupings they make than they would be when drawing a concept map on paper. For instance, when drawing a concept map, a student would be required to fix the location of each node and then draw connections between them. This does not allow the students to visualize other ways of grouping or connecting the nodes. The card sort allows the students to organize the nodes fairly quickly and then reorganize them numerous times.

---

equation

period

table *

slope

x

X	$f(x)$ *
0	0
1	1
2	4
3	9
4	16

y

graph *

transformation *

function *

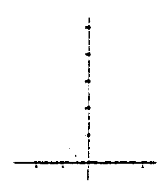
$f(x)$

sinx

it has a pattern

periodic

logs



vertical line test

there is only one y-value for each x-value

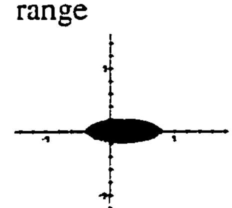
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

ellipse

used to solve a problem

$f(x)=x^2$

range



$y = mx + b$

amplitude*

domain*

Figure 13: Card Sorting Task

* indicates cards that were added by the researcher.

Table 4 shows the tasks with which a student might demonstrate an understanding of content and representation. Peripheral tasks are tasks that might give the researcher some information about the students' understanding of the concept, but indirectly. For instance, in problem 1, the table does not represent a function that can be written as an equation easily. Therefore, the students should not be able to translate it into an equation. However, if the students attempt to do so, their methods will reveal something about their understanding of translation from table to equations.

Table 4: Tasks that Probe Each Concept

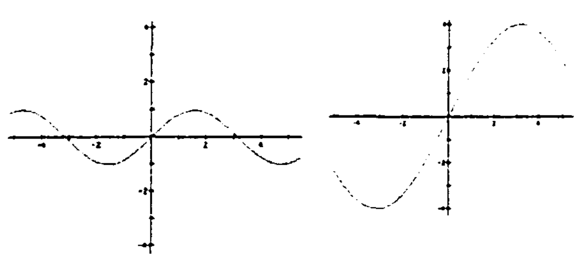
Concept	Tasks	Peripheral Tasks
function notation	1, 2, 25, 26, 27	9-12, 13-18, 22-24
periodic functions (table)	1, 9-12	
periodic functions (graph)	2, 3-8, 32	1, 9-12
periodic functions (equations)	26, 27, 32	1
combining functions (table)	13-18, 22, 23	19-21, 22-24
combining functions (graph)	19-21, 23, 24	32
combining functions (equations)	22, 24, 28-31,32	19-21, 22-24
translation (table to graph)	1, 9-12, 13-18, 22, 23	
translation (equation to graph)	22, 24	
translation (table to equation)	13-18, 22, 23	1, 9-12
translation (graph to equation)	19-21, 23, 24	2, 13-18
translation (equation to table)	22, 24	
translation (graph to table)	19-21, 23 24	

In addition to the researcher-developed instruments, samples of students' work done strictly as a part of the class were collected. This work included 11 homework assignments (115 problems), 4 quizzes (63 problems), 2 tests (56 problems), 2 worksheets (24 problems), 1 lab (16 problems) and 1 essay (7 problems). These problems covered the content area of trigonometry and its applications. The collected work was coded in two ways. First each problem was given a code based upon it's content and the representation it was presented in. The coding categories for content included: function, periodic, transformation and others. The categories for representation included: symbolic, graphical, tabular, pictorial, and verbal.

Each problem received as many of the codes as applied. For example, the problem: "Graph the functions $y = \sin x$ and $y = 4\sin(x/2)$. State the difference between the two graphs" was given the following content codes: function, periodic, and transformation because the equations both represent functions, periodicity is key to the problem, and the students are asked to describe a transformation. The problem was given the representation codes: verbal and symbolic because the statement of the problem includes equations and words.

---

a=-4, b=1/2, p = 2π/b, p=4π;



the second graph goes between -4 and 4 and is flipped instead of going between−1 and 1, it is flipped and has a longer period.

Figure 14: A Sample Solution

In addition, each student's solution was coded by the representation(s) used in the solution. For example, consider the solution to the example problem above (see Figure

14). The student used symbols, graphs, and a verbal description in the solution of this problem. The coding allowed the researcher to calculate frequencies of representations presented in the statement of the problem and used in the solution of the problems. In addition it allowed the researcher to establish each student's preferences based on content and representation.

Procedure

The researcher observed the class two or three times per week over a period of six weeks during the spring of 1999. Class notes and observations of student-teacher interactions were kept in a journal. These observations focused on placement of authority both by students and the teacher, mathematical interactions among students and between students and the teacher, and how the graphing calculator was used. The goal for these observation notes was to develop a thick description (Geertz, 1973) of the culture of the classroom – that is, to explain the ordinary and everyday workings of the classroom by observing student interactions with each other and with their teacher.

Pearson's χ^2 Test of Independence was applied to the collected problems to test for dependence between the content area of each problem (functions and or periodic functions) and the representation students used to answer the question. Analyzing dependence or independence allows the researcher to determine whether students preferred a particular representation over another given the content of the problem. This test applies to frequencies calculated on nominal data. First, contingency tables were constructed for content area and representation. For example, if a student answered 100 problems with 50 being in the content area of functions, and used a symbolic representation on 40 of the function problems and 75 times all together, the contingency table would be constructed as in Table 5.

Table 5: Example Contingency Table for Pearson's χ^2 Test of Independence.

	Functions	Non-functions	
Symbolic	40	35	75
Non-symbolic	10	15	25
	50	50	

Six contingency tables were constructed for each student representing all of the combinations of graphs, equations, and tables with functions and periodic functions (see Table 6). These contingency tables were used as inputs for Pearson's Test and were calculated using SPSS.

Pearson's χ^2 Test becomes inaccurate whenever expected values are less than five. Because the expected values for the tabular representation were less than 5 in the contingency tables a meaningful statistical test could not be applied. Therefore, only symbolic and graphical represenations were tested.

Table 6: Combinations of Content and Representation used for Contingency Tables

	functions	periodic functions
graphs	functions and graphs	periodic functions and graphs
equations	functions and equations	periodic functions and equations
tables	functions and tables	periodic functions and tables

The researcher conducted clinical interviews (Confrey, 1981) with single students or pairs of students over the same time period as the classroom observations. This was done in order to allow the researcher to conduct as many interviews as possible and to make the students as comfortable as possible. The researcher set up the interviews during students' free periods. In several of the interviews, pairs of students were scheduled, but only one student showed up. One student, Steven preferred to work by himself before school. In the interview tasks the students worked independently. The researcher presented tasks to the students and allowed them to work at their own pace. Once a student finished a task and explained his or her solution, a new task was given to the student. Some students were able to complete only one task during an interview while other students were able to complete as many as five tasks. The researcher watched the students work and asked questions alternating between students. Because the students worked at their own pace, most of the time each student of the pair was working on different problems. Some of the students collaborated on the brainstorming tasks and the card-sorting task. During the interviews, students had access to their class notes, a graphing calculator, and their "crib sheets," which included a trig-wheel and notes on transformations. Students were asked to explain their thinking both verbally and in writing. In addition, the researcher interacted with the students by asking questions. Students were given up to 35 tasks to solve in seven interviews.

When a student was unable to start or continue with a solution, the interviewer prompted him or her to think differently about the problem. The interviewer would begin with prompts such as: "Is there another way you could look at the problem?" or "What information would make this problem easier?" If the student was unable to continue, the interviewer would suggest a new way of looking at the problem. "What if $h(x)$ was replaced with $f(x)$?" or "What would happen if you graphed the problem?" If the student continued to struggle, the interviewer would suggest a simpler problem to solve. Finally, if the student still was unable to solve the problem the interviewer would walk through the solution with the student and then ask the student to explain the solution in his or her own words.

After a student completed the task, the interviewer would ask if he or she could see another way to solve the problem. Where applicable, the interviewer would ask the students if they could solve the problem using a particular representation.

During the last interview, the students were asked to sort a set of 28 cards into groups in as many different ways as they could in 30 minutes. The card sorting activity was used to allow the students to make their own concept maps by manipulating cards into groups and then stating their connections. The cards were used as the nodes of the concept maps, and the titles that students gave for each grouping served as connections. An initial concept map was formed by combining the groupings from each of the card-sorts for each group of students. These maps were then compared with student's responses on interview tasks. Pairs of students were given a set of 28 3x5 index cards with words, graphs, equations and tables as illustrated in Figure 13. In this way, the researcher was able to have the students explicitly state the information needed to draw a concept map without spending extra time actually drawing them. The researcher took the groups and connections from each card sort and combined them into concept maps for each of the groups of students. This provides more insight into the connections that students have made between constructs and representations.

In summary, the study was conducted with seven students in a single pre-calculus class. Over a two-month period classroom observations were made two to three times per week. Two months of homework, classwork, quizzes, tests, and labs on trigonometric functions and their applications was collected. The students were interviewed once a week. During these interviews students were given brainstorming tasks and mathematical tasks which were designed to elicit aspect of their knowledge of function notation, operations on functions, and periodicity in each of three representations graphs, tables and equations.

6: Description of the Classroom Culture

The researcher observed the class two to three times per week over the course of the study. The intent of these observations was to describe the everyday workings of the classroom in detail. This is intended to be an overall description of the culture of the classroom and not a description of individuals. A majority of the students in the class did not volunteer to participate in the study. While they gave permission for the researcher to observe them in class, the researcher felt that the best way to protect their identity while still gaining the cultural information was to identify them only with numbers. Note that each dialogue is renumbered. That is, student 1 represents a student in one dialogue, but does not represent the same student in a different dialogue. A pseudonym is used for the teacher in order to protect her identity.

Description of the Class

Mrs. Brown's classroom was the school's computer lab. Computer workstations surrounded the classroom on three sides. On the fourth wall hung a large whiteboard. There was a large seminar table in the center of the room surrounded by chairs. One short end of the table was just in front of the whiteboard. Mrs. Brown would stand and present notes at the whiteboard and roam around the table helping students individually.

The students sat at the long sides of the seminar table. At the beginning of each class, they went to the whiteboard and put up their homework problems. Some of the students sat at the table correcting their homework or finishing a last problem before Mrs. Brown could see them. Mrs. Brown checked to be sure that all of the problems had been put on the board. If they had not, she would call on students to do the remaining problems.

Each day, one student was chosen to be "the corrector." This student would compare the answers in the instructor's copy of the text with the answers his or her classmates had put on the board. If the classmate's answer differed from the answer key, the answer was crossed out and the book's answer was written beside it. If the answers were the same, "the corrector" would mark it with a "C".

Placement of Authority

Mrs. Brown implicitly encouraged the students to place authority in the textbook answer key. When students asked questions about the homework, Mrs. Brown would refer them to the answer key. For example, in the following dialogue a student and Mrs. Brown discuss a homework problem.

> **Student 1**: Mrs. B., the answers I got for 8 are different [from the answers in the instructor edition] is that OK?
> **Mrs. B.**: {*without looking at his paper*} No they are wrong.
> **Student 1**: But they add up to 180.
> **Mrs. B.**: They would add up to 180 if you did it your way, which is wrong.

The student expressed that he believed that his answer was correct because he checked his answer. Mrs. Brown tells him his answer is wrong without looking at his paper. She sent the student and the rest of the class the message that the textbook answer was right and any other answer, no matter how it was supported, was wrong.

The students took their cues from Mrs. Brown and also used the answer key as the authority on the homework. However, some of the students were not able to evaluate answers that were in different forms. For example, in the following dialogue, one student is at the board correcting the others' homework when he questions another students' answers.

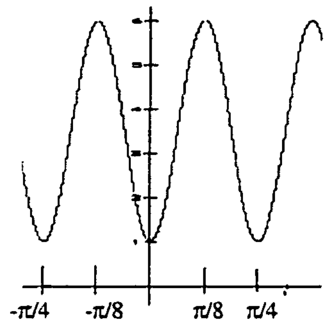
> **Corrector**: Did you make up these answers? {*Points to* (1) .5, (2) .866, (3) .577, (4) 1.732}
> **Student 1**: How could I make up those numbers?
> **Student 2**: I got the same thing.
> **Student 1**: See [Student 2] got the same thing. I didn't make them up.
> {*Corrector dramatically crosses out the answers and writes in the book's answers on the board* (1) 1/2, (2) $\sqrt{3}/2$, (3) $\sqrt{2}/2$, (4) $\sqrt{3}$ }

The corrector did not see that the answers supplied by the student were just decimal approximations. Mrs. Brown did not accept decimal approximations and would have marked them incorrect just as the corrector did. However, the corrector clearly did not recognize that the answers supplied by student 1 were related to the answers in the answer key.

Once all of the problems were on the board and corrected, the students had a chance to ask questions. During this period, when students asked questions it was usually to clarify an answer or method used on the board. However, occasionally a student would ask a deeper question or ask Mrs. Brown to re-teach a principle or method. Mrs. Brown would react to this type of question with a concise procedural answer, and gloss over any misunderstandings that the student continued to express. In this way, Mrs. Brown communicated to the students that the question period immediately before the quiz was designed to answer clarifying questions, not to re-teach a principle or method. After the question period Mrs. Brown gave the students a quiz, which she would read word-for-word from the homework.

Interaction in the Classroom

After all the quiz papers were collected, it was time for the students to take notes. At this time, Mrs. Brown gave examples similar to the homework problems and questioned the students. Many of the students were actively involved in discussion while they took notes. For example, see the dialogue following Figure 15. In this dialogue, the teacher asked the students for the answers at each step of the problem. The students answered her and sometimes interacted with each other as the teacher wrote what they said on the board.



<u>Figure 15</u>: Graph that Mrs. Brown Drew on the Board

In the following dialogue Mrs. Brown attempts to get the students to translate a graph (see Figure 15) into two different equations: a sine curve and a cosine curve.

Mrs. B.: What is the period?
Student 1: π/4
Mrs. B.: Everyone agree?
Student 2: I think it's a little past π/4.
Mrs. B.: If p is $\pi/4$ what is B? {*Three students guess the wrong value. She writes on the board* $p = \pi/4$ $2\pi/B = \pi/4$ $2\pi = B\pi/4$ $8\pi = B\pi$ $B = 8$.} You've got to take the period you see and set it equal to 2π/B and solve.

Mrs. Brown chose to ignore student 2 who didn't agree and the students with wrong guesses and told the students a procedure to use in calculating the period. It was the same formula that they had been using for a few days. She continued.

Mrs. B.: How do I find the amplitude?
Student 3: Highest lowest divided by 2

Mrs. Brown translated what the boy said into numerical values and wrote (6-1)/2 = 5/2 on the board even though the student did not say to subtract. She paused and then wrote the general form of the equation, $y - k = A \sin B(x - h)$, on the board.

> **Mrs. B.**: Now what do you think *h* and *k* are?
> **Student 2**: Zero and 3.5
> **Mrs. B.**: {*pointing to the graph*} When you shift it's actually half way between 0 and $\pi / 4$.
> **Student 4**: Isn't it a quarter of the way?
> **Mrs. B.**: I can tell you h is not zero.
> **Student 2**: If it's a quarter of the cycle its $\pi / 16$. If it's half, then its $\pi / 8$.
> **Student 4**: The point is 1/4 because that is just 1/4 of the cycle.

It is interesting that the students talked about the problem graphically, but Mrs. Brown talked about it graphically and algebraically. She used the terms "*B*", "*h*" and "*k*" which are part of the equation form. When she used the terms "period" and "amplitude," it is impossible to tell if she is talking about the algebraic representation or the graphical representation as these terms apply to both. However, when she talks about the shift she is definitely referring to the graph.

Student 4 knew that the shift was not halfway between 0 and $\pi / 4$. He tried to convince the others that Mrs. Brown's suggestion was wrong. Mrs. Brown either made a mistake, or was using "half way" in a colloquial manner to mean "somewhere" in between the two values. Either way, the two students were able to figure out the horizontal shift and the students did not seem bothered by Mrs. Brown's mistake. The dialogue continued.

Mrs. B.: So what's the equation?

Student 3: $y - 3.5 = (5/2)\sin 8[x - (\pi/16)]$.

Student 5: Why is there an 8?

Mrs. B.: Because B is 8 and that's where it goes. Get out your trusty little calculator and check it. Make sure your range is right and you are in radians.

Student 5 had lost sight of the first value that they had calculated. He had been following along with the individual steps, but was not thinking about the equation all the way through. This is not surprising since they had only been practicing the pieces. This was the first time they had written equations for a graph.

Mrs. Brown's answer to Student 5's question was also interesting. She did not refer him back to the graph on the board or attempt to explain the 8 in terms of the graph. She referred him to the general equation. But, then she encouraged him to check if the equation was correct by graphing it on his calculator. Doing what Mrs. Brown suggested would not have given student 5 an understanding of which part of the graph related to the 8. It did, however, give him a chance to see that the equation was correct by comparing the two graphs.

Mrs. Brown often encouraged the students to check their answers with their graphing calculators. When disagreements about an answer emerged during a discussion, Mrs. Brown would refer the students to their graphing calculators. She encouraged them to check their answers and their classmates' answers by graphing equations, or by comparing two different arithmetic expressions.

In the following dialogue, the class is solving the problem:

$$\cos 30° \cos 45° - \sin 30° \sin 45°.$$

The students prompt Mrs. Brown to write $(\sqrt{6} - \sqrt{2})/4$ on the board.

Mrs. B.: Can I simplify that down?
Student 1: Yeah, $\sqrt{4}$.
Mrs. B.: How can I get the square root of 4?
Student 1: Subtract them.
Mrs. B.: OK take out your calculator and find $\sqrt{6} - \sqrt{2}$. You don't get 2 do you?
Student 1: No.
Mrs. B.: So you can't subtract them if the number under the radical is different. What if I have $2\sqrt{2} - \sqrt{2}$?
Student 1: Yeah.
Mrs. B.: Right, you can take out root 2 and get 2 minus 1, which is root 2.

Mrs. Brown tried to help the student to see that her generalization $\sqrt{6} - \sqrt{2} = \sqrt{4}$ was incorrect. It is not clear whether the student understood. Her answers are ambiguous, and she may or may not have known why Mrs. Brown was asking her to compare $\sqrt{6} - \sqrt{2}$ with 2 since she had supplied the answer $\sqrt{4}$. The student likely knew that $\sqrt{4} = 2$, but she may not have made this connection during her interaction with the teacher. In her last statement, Mrs. Brown refers to the following steps $2\sqrt{2} - \sqrt{2} = \sqrt{2}(2-1) = \sqrt{2}$. Mrs. Brown assumed that the students understood her verbal explanation. However, she did not show each step on the board and, therefore, students may have misunderstood her last statement.

Mrs. Brown often gave classwork to the students after presenting the notes for the day. She encouraged the students to consult with their neighbors and often asked them to work in groups. The students interacted with each other in three ways: asking for help, telling the answer, and socializing. When students conversed about mathematics, the most common pattern that they followed was for one student to ask for help and the other to show or tell the answer. Once the answer was given, the students sometimes continued to explain the answer. Much of the time, the interaction ended after the answer was given. This is not unlike Mrs. Brown's responses.

Mrs. Brown had a good rapport with her students. Students often expressed a lack of understanding and requested help. This type of interaction occurred during note taking or classwork. The students wanted to understand the material. They had confidence that Mrs. Brown could answer their questions and would keep explaining until they understood even if it meant working outside of class. There was a helpful spirit in the class, and students did not feel uncomfortable in expressing confusion in front of each other or in front of Mrs. Brown. For example, in the following dialogue, a student asked Mrs. B. to explain the answer $\pi/2 + \pi k$.

Student 1: Mrs. B, in problems like $\cos x = 0$, what is the k for?
Mrs. B.: It is a multiple of how many times you go around the circle
Student 1: I still don't understand
Mrs. B.: You have to assume that you can go around the circle a bunch of times so if I say you go around 3 times you have to add 6π. So it's just a multiple.

In this dialogue, student 1 asked Mrs. Brown to explain an answer that did not make sense to her. Mrs. Brown offered a short answer, but the student was still confused. Instead of giving up, she confessed that she was still confused. This confession shows that the student felt that Mrs. Brown could explain the problem to her. Also, it shows that she felt supported enough in the classroom to be able to express a lack of understanding without the fear of being ridiculed.

The students considered Mrs. Brown to be a difficult teacher. She did not allow them to make excuses. She was firm, but fair. She also used humor to lighten-up the mood in the class. Several students expressed that they thought Mrs. Brown was the "best math teacher" in the school. Each year a few students would switch into her classes partway through the semester because of this perception. The students also believed that they were learning more than the other pre-calculus class at the school. In the following dialogue, the students were preparing to take a quiz. Mrs. Brown used a colored sheet of paper for each daily quiz in a week. In this week, the colored sheet was an orange-yellow color. Mrs. Brown makes a joke and then the students chide her for being too difficult.

> **Mrs. Brown**: OK everyone, get out your cheese colored sheets.
> **Class**: {in unison} Cheese! {laughter}
> **Student 1**: Can this be an honors class? We are doing like twice the work as the other senior class.
> **Student 2**: Yeah, we do like 4 labs a week and they don't do any.

The student's complaint was a gross exaggeration. In the two months of observation, the students completed only two labs. However, the students did perceive that they were doing more work than the other pre-calculus class.

Use of Representations

Classwork from nine students was collected. This classwork was coded in terms of the representation(s) given in the problem statement and the representation(s) used by the students to answer the question. The coded questions were then tallied by representation.

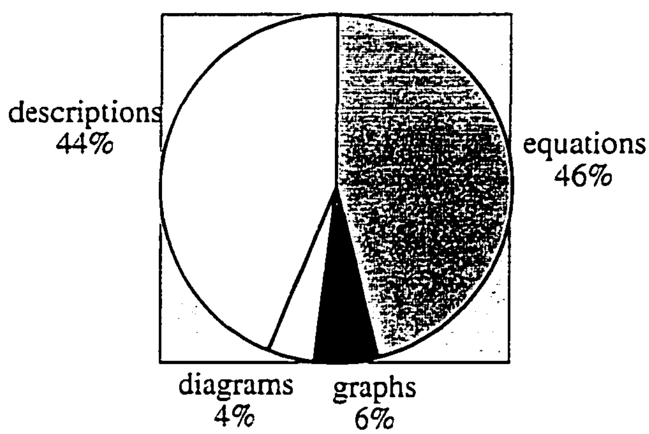


Figure 16: Mrs. Brown's Use of Representations in Assigned Work

Mrs. Brown showed a clear preference for equations over graphs and tables. Of the 273 questions that were collected as a part of the class, 127 questions were presented in an algebraic representation (46%) and 16 questions were presented in a graphical representation (6%). None of the questions were presented in a tabular representation on homework, classwork, labs, quizzes, or tests in the two-month observation period. Most of the remaining problems were verbal descriptions, or word problems that asked for a numerical solution (See Figure 16).

Although most of the questions Mrs. Brown assigned were in the form of equations, she often suggested that the students view a graphical representation during note taking. Mrs. Brown often asked students to graph an equation on their graphing calculator. Although this was an integral part of note taking, she only drew the graph on the board when she was explaining a graphing technique or when the homework specifically required the students to draw a graph. The students did not copy a graph from their graphing calculators into their notebooks unless Mrs. Brown drew it on the board, so their notes contained very few graphs. Even when Mrs. Brown used a graphical representation in the notes, the focus was on forming equations from the graph.

> **Mrs. Brown**: Tonight you will be doing some graphing and some word problems. I need a graph of the sine function and the cosine function.
> {Two students come to the board and draw the requested graphs from their graphing calculators}
> **Mrs. Brown**: How long is the period?
> **Student 3**: 2π
> **Mrs. Brown**: The amplitude is {She writes on the board, explaining, $\text{Amp} = (\max - \min)/2 = [1-(-1)]/2 = 1$ }Is it the same for cosine?

In this interaction, Mrs. Brown used the equation $(\max - \min)/2$ rather than using a graphical explanation such as half the distance between the peak and the valley.

During class, Mrs. Brown implicitly communicated to her students three things involving the graphical representation. First, she wanted them to be able to graph equations and write labels on the axes. She did not expect them to do this by hand, but on the graphing calculator. Since the students were required to label the axes, they had to have an understanding of the graphing window and the horizontal and vertical scales. Second, she expected the students to be able to find an equation for a graph. She taught the students how to do this in a procedural way and did not require them to know how the parts of the equation connected to the graph except in a rudimentary way. For instance, the students knew that to get B, in the general equation $y - k = A\sin B(x - h)$, they had to find the period and plug it into the formula $p = 2\pi / B$. However, they were not expected to understand why this formula worked. Third, Mrs. Brown expected her students to be able to compare graphs using the graphing calculator. She wanted them to be able to check the correctness of solutions both by comparing their graphs with those in the answer key and by graphing an equation and comparing the result with the original constraints of a graphical problem. This also required knowledge of the graphing window and x and y scales.

The students' use of representations was similar to that of the teacher. The most used representation, in solving problems, was equations (49%). Students used graphs to solve problems 9% of the time and never used tables. This was to be expected given that tables were neither taught in class nor presented in the book except in the first review chapter and in the section where functions were introduced.

Of the problems collected from the nine students 119 (44%) explicitly stated a representation for the answer. For example a question such as: "Graph the function $f(x)=5\sin(x-\pi)$" specifies that the answer should be represented with a graph. Of the 154 problems which did not specify a graphical, tabular, or symbolic representation for the answer, 128 (83%) had numerical answers and 26 (17%) had verbal answers. Of these problems students used a method other than expressions/equations only 25 out of a possible 154 times (16%). Each of these times a graph was used.

In summary, the following themes were discussed in the descriptive analysis. Mrs. Brown implicitly encouraged her students to place authority in the textbook answer key and in their graphing calculators. Mrs. Brown encouraged interaction in the classroom by questioning students, by asking students to work in groups, and by creating along with her students a supportive atmosphere where students felt comfortable to express any lack of understanding. Mrs. Brown through notes and assigned problems encouraged students to use algebraic representations whenever possible. Although she encouraged them to utilize their graphing calculator to check their answers, she did not encourage them to draw graphs or use them to solve problems.

Although it is difficult to see a direct relationship to each aspect of the classroom culture and the students' responses to the interview tasks, the nature of learning is that students build their understanding from their experiences both mathematical and social. Therefore a description of the culture of the classroom acts as a framework for interpreting and generalizing the research.

7: Analysis of Classwork

In this chapter, the data from the classwork is analyzed. The class as a whole is no longer the unit of analysis. Instead the analysis moves to a description of individual students and their use of representations. In Chapters 8 and 9 the description of individuals continues with an analysis of the data from the clinical interviews. As the analysis unfolds, individual student's understandings of functions and representations emerge from the data. Although classwork for nine students was collected, only the data from the seven students interviewed is presented.

The chapter begins with a short description of each student. Next follows an analysis of the classwork. This includes the frequencies with which each student used the various representations and the results from Pearson's χ^2 Test for independence. This statistical analysis, carried out for each student with the problems collected during the observation period, shows student's preferences, if they exist, for use of representations in their solutions of problems in the content area of functions.

The Students.

David. David was a confident senior with a good sense of humor. He socialized a lot in class, but also asked deep questions and helped his classmates. He usually received A's on classwork and tests.

Ike. Ike was a talkative senior who loved being at the center of attention. It did not matter to him whether the attention was positive or negative. He was the class clown. He often missed instructions and often seemed distracted. However, when he was working in a group with another boy, usually David, he was able to focus more on the problem at hand. He was not at the top of his class, but he did well on tests. He scores remained in the B/C range.

Beth. Beth was a popular senior. She had a lot of interests and was planning to attend college. She was taking mathematics because she had to in order to get into college. She felt that mathematics was her weakest subject.

Steven. Steven was a senior who was really struggling with the concepts of pre-calculus. He believed math was "very difficult." Steven had trouble sitting still. He often stretched out in his chair, so that his head was hanging over the back, or laid his head on the table. And although Mrs. Brown did not allow him to leave class, he often asked to go to the bathroom. He had a hard time staying focused on the lesson or on a problem he was solving.

Matthew. Matthew was a junior who intended to take calculus the following year. He was getting an A in the class. He worked problems slowly and methodically. He was very careful about his drawings and explanations. In a group he often was quiet until the others had started the problem. Then he would bring up thoughtful suggestions and new directions for solving the problem.

Rachel. Rachel was a confident junior who intended to take calculus the following year. She consistently scored A's in pre-calculus and said that she enjoyed mathematics. She participated regularly in class discussions.

Michael. Michael was a quiet student who preferred to work on his own. He worked very quickly and used his graphing calculator extensively. When working in a group, he tended to monopolize the work. He worked more quickly than his classmates and would do the classwork for the group.

Statistical Analysis of Classwork

David's use of representations. David solved 276 problems of classwork in the course of the two-month observation period. His use of each representation is presented in Figure 17.

Of the 276 classwork problems David solved, he used equations to solve the problem 127 times. Eighty-eight (88) of the problems solved with equations dealt with functions. Of the problems dealing with functions, David solved 101 in a representation other than equations. David's preference to use equations did not depend on the content of the problem. The proportion of David's use of equations to other representations to solve a problem was not significantly different whether the problem dealt with functions or not. David used equations to solve a problem dealing with functions 46% of the time.

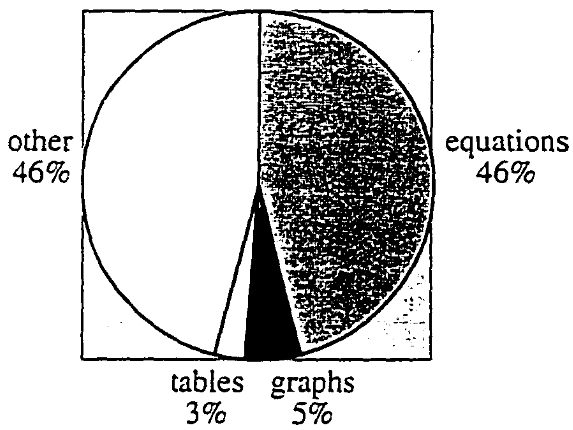


Figure 17: David's Use of Representations in Classwork

David was more likely to draw a diagram or show no work in problems dealing with periodic functions. There were 163 problems dealing with periodic functions. Of these David chose to use equations 64 times. David's use of equations on periodic functions differed from his use of other representations for periodic functions $\chi^2(1, \underline{N} = 276) = 6.66$, $p < .05$. David used a representation other than equations on periodic functions 61% of the time, but in other types of problems he used a representation other than equations only 44% of the time. However, David used a graph only 7% of the time and a table only 1% of the time.

David used a graphical representation to solve only 14 problems. Of these, 11 dealt with periodic functions. David used a representation other than a graph to solve problems dealing with periodic functions 152 times. David's use of graphs was not significantly different with periodic functions or non-periodic functions. He used graphs to solve problems overall, only 5% of the time. David did not use graphs or tables enough to perform meaningful statistical analysis on their use with functions.

In summary, David used equations to solve 41% of the problems, graphs to solve 7% of the problems and he used tables less than 1% of the time. David's preference to use equations did not depend on the content of the problem. David was more likely to draw a diagram or show no work with periodic functions. Most of the problems which David used graphs to solve (79%) involved periodic functions. However David did not show a preference for using graphs to solve periodic functions overall. The χ^2 values for David's use of representation vs. content area is summarized in Table 7.

Table 7: David's χ^2 Values for Representation vs. Content Area

	Function	Periodic
Equation	0.02	6.66*
Graph	NA	1.55

χ^2 (1, N = 154), $p < .05$

* indicates independence between the use of representation and content area

Ike's use of representations. Ike solved 174 problems of classwork in the course of the two-month observation period. A pie chart representing Ike's overall use of representations is shown in Figure 18.

Of these 174 problems, Ike used equations to solve problems 64 times. Of the problems solved with equations 39 dealt with functions. Of the problems dealing with functions, Ike solved 71 in a representation other than equations. Ike did not show a preference to use equations in one content area over another. The proportion of Ike's use of equations to other representations to solve a problem was not significantly different whether the problem dealt with functions or not. Ike used equations to solve a problem 37% of the time.

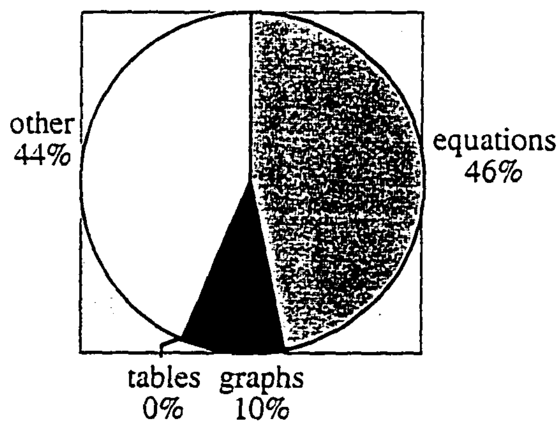


Figure 18: Ike's Use of Representations in Classwork

There were 90 problems dealing with periodic functions. Of these Ike chose to use equations 22 times. Ike was more likely to draw a diagram or show no work on problems dealing with periodic functions. Ike's use of equations on periodic functions differed from his use of other representations for periodic functions $\chi^2(1, \underline{N} = 174) = 11.13$, $p < .05$. Ike used a representation other than equations on periodic functions 76% of the time, but in other types of problems he used a representation other than equations only 50% of the time. Ike's use of representations on periodic functions was similar to David's use. Ike only used a graph 13% of the time and never used a table. Ike used a graphical representation to solve only 17 problems. Of these, 12 dealt with periodic functions. This shows that when Ike chose to use a graph, which was seldom, he most likely was dealing with a periodic function.

Ike used a representation other than a graph to solve problems dealing with periodic functions 78 times. Ike's use of graphs was not significantly different with periodic functions or non-periodic functions. Ike did not use tables enough to perform meaningful statistical analysis on their use in problems dealing with functions.

In summary, Ike used equations in the same proportion regardless of whether the problem dealt with functions or non-functions. However, he did show a preference to use diagrams or to show no work on periodic functions problems. Although Ike did not show a preference to use graphs to solve problems involving periodic functions, when Ike was using a graph to solve a problem, it involved a periodic function 71% of the time. The χ^2 values for Ike's use of representation vs. content area are summarized in Table 8.

Table 8: **Ike's χ^2 Values for Representation vs. Content Area**

	Function	Periodic
Equation	0.10	11.13*
Graph	0.44	1.913

χ^2 (1, N = 174), p < .05

* indicates independence between the use of representation and content area

Beth's use of representations. Beth solved 260 classwork problems. She used equations to solve 137 problems. A pie chart showing Beth's use of representations is in Figure 19. Ninety-seven (97) of the problems in which she used equations, dealt with functions. She used some other representation on 88 problems that dealt with functions. Beth's preference to use equations did not depend upon the content of the problem. The proportion of Beth's use of equations to other representations was not significantly different whether the problem dealt with functions or not. Beth used equations 67 times with periodic functions and used another representation 92 times with periodic functions. Beth's use of equations was significantly different for periodic and non-periodic functions χ^2 (1, N = 260) = 17.22, p < .05. She used another representation on periodic functions 58% of the time, but used another representation on non-periodic functions only 31% of the time. Beth was more likely to use a representation other than equations with periodic functions.

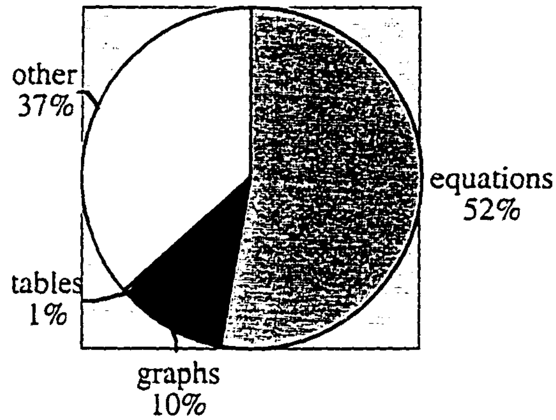


Figure 19: Beth's Use of Representations in Classwork

Beth used graphs in solving problems with functions 20 times and some other representation 165 times. Beth used a graph to solve a problem dealing with periodic functions 22 times and used another representation 137 times. The proportion of Beth's use of graphs to other representations was not significantly different whether the problem dealt with functions or non-functions. However, her use of graphs with periodic functions differed from her use of other representations with periodic functions. She was more likely (85%) to use a graph on a periodic function than on a non-periodic function (15%).

In summary, Beth used equations and graphs in the same proportion whether dealing with a function or non-function. However, she showed a preference for using graphs with periodic functions. The χ^2 values for Beth's use of representation vs. content area are summarized in Table 9.

Table 9: Beth's χ^2 Values for Representation vs. Content Area

	Function	Periodic
Equation	0	17.22*
Graph	0.21	5.64*

χ^2 (1, N = 260), $p < .05$
* indicates independence between the representation and content area

Matthew's use of Representations. Matthew solved 212 classwork problems. Matthew used equations to solve 105 problems. See Figure 20 for a summary of Matthew's use of representations. Of these, 66 dealt with functions. In addition, he used a representation other than equations 72 times. Matthew's choice to use equations in the solution of a problem did not depend on whether the problem dealt with functions or non-functions. The proportion of Matthew's use of equations to other representations was not significantly different whether the problem dealt with functions or non-functions. He used equations 42% of the time.

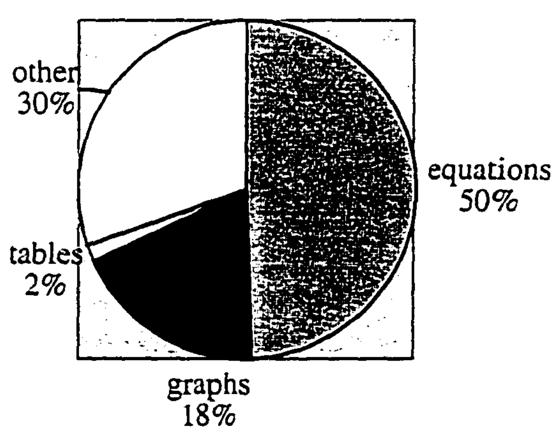


Figure 20: Matthew's Use of Representations in Classwork

Matthew preferred to use a representation other than equations on periodic functions as opposed to non-periodic functions. Matthew used equations with periodic functions 48 times. He used a representation other than equations with periodic functions 73 times. Matthew's use of equations with periodic functions was different in proportion to his use of equations with non-periodic functions χ^2 (1, N = 212) = 10.06, p < .05. He used equations about the same amount of time with periodic (48 times) and non-periodic functions (57 times). However, he used a representation other than an equation 68% of the time with periodic functions. He used graphs 27% of the time with periodic functions and did not show his work on 73% of the problems.

Matthew preferred to use graphs with functions as opposed to non-functions. Matthew used graphs to solve problems dealing with functions 35 times and another representation 103 times. The proportion of Matthew's use of graphs to another representation was significantly different whether the problem dealt with functions or non-functions. χ^2 (1, N = 212) = 11.48, p < .05. He used graphs with non-functions only 10% of the time. However he used a representation other than graphs with non-functions 40% of the time.

This difference in proportion did not hold for periodic functions. Matthew used graphs 26 times with periodic functions and used a representation other than graphs with periodic functions 95 times. Matthew used graphs in about the same proportion whether dealing with periodic functions or non-periodic functions.

Table 10: Matthew's χ^2 Values for Content vs. Representation

	Function	Periodic
Equation	0.26	10.06*
Graph	NA	NA

χ^2 (1, N = 212), p < .05

* indicates independence between the representation and content area

In summary, Matthew used equations 42% of the time, but preferred to use graphs and numerical answers for periodic functions. He also showed a preference for using non-graphical representations with functions. In Matthew's case, this reflects his failure to show his work. A summary of Matthew's χ^2 values is provided in Table 10.

Steven's use of representations. Steven solved 218 classwork problems. His use of representations is summarized in the pie chart in Figure 21. His failure to show work is evident in that he used no representation in a majority of problems. This may indicate that he solved the problems mentally, on his graphing calculator, or on a separate sheet of paper.

Steven used equations 89 times. He used equations with functions 57 times and a representation other than equations with functions 71 times. The proportion of Steven's use of equations to other representations was not significantly different whether the problem dealt with functions or non-functions.

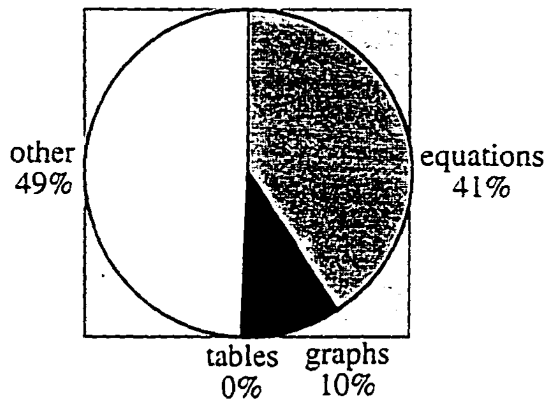


Figure 21: Steven's Use of Representations in Classwork

Steven showed a preference to use equations on non-periodic functions as opposed to periodic functions. Steven used equations 25 times with periodic functions and a representation other than equations 69 times. The proportion of Steven's use of equations to other representations was significantly different whether the problem dealt with periodic functions or non-periodic functions χ^2 (1, N = 218) = 12.83, $p < .05$. Steven used equations only 28% of the time with periodic functions as opposed to non-periodic functions. However, he used a representation other than equations 53% of the time on periodic functions as opposed to non-periodic functions.

Steven did not solve enough problems with graphs in the content areas of non-functions and non-periodic functions to carry out meaningful statistical analysis with Pearson's χ^2 test of independence. In Table 11 Steven's χ^2 values are summarized.

Table 11: Steven's χ^2 Values for Representation vs. Content Area

	Function	Periodic
Equation	1.41	12.84*
Graph	NA	NA

χ^2 (1, N = 218), $p < .05$

* indicates independence between the representation and content area

Rachel's use of representations. Rachel solved 292 problems in the classwork. Her use of representations in the classwork is summarized in Figure 22. Rachel preferred to use equations with problems involving functions. Of the 292 problems, Rachel used equations in 130.

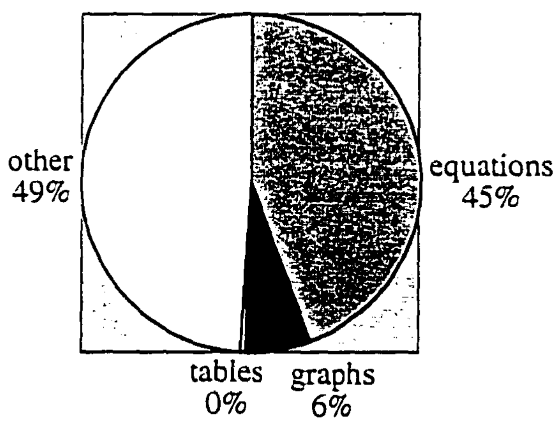


Figure 22: Rachel's Use of Representations on Classwork

Rachel used equations to solve problems with functions 91 times and a representation other than functions 94 times. The proportion of Rachel's use of equations to other representations was significantly different whether the problem dealt with functions or non-functions χ^2 (1, N = 292) = 3.95, p < .05. Rachel used equations 75% of the time with functions, and she used non-equations only 58% of the time.

Rachel did not show the same preference for periodic functions. The proportion of Rachel's use of equations to other representations was not significantly different whether the problem dealt with functions or non-functions. Rachel did not use graphs at all on non-functions and only three times on non-periodic functions. These frequencies were too small to perform a meaningful statistical analysis. However, her responses do show that when she has used a graph to solve the problem, 83% of the time it was a periodic function. She did not show a preference for using a graph with periodic functions, but most of the problems where she used a graph were periodic. Rachel's χ^2 values are summarized in Table 12.

Table 12: **Rachel's χ^2 Values for Representation vs. Content Area**

	Function	Periodic
Equation	3.95*	3.13
Graph	NA	NA

χ^2 (1, N = 218), p < .05

* indicates independence between the representation and content area

Michael's use of representations. Michael solved 215 classwork problems. His use of representations regardless of content is shown in Figure 23. Michael used equations 113 times. On problems involving functions, Michael used equations 69 times and another representation 61 times. The proportion of Michael's use of equations to other representations was not significantly different whether the problem dealt with functions or non-functions.

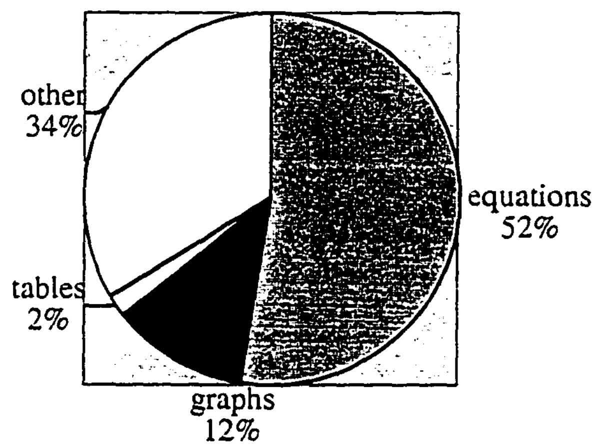


Figure 23: Michael's Use of Representations in Classwork

Michael used equations on periodic functions 40 times and on non-periodic functions 60 times. Michael used a representation other than equations on periodic functions 59% of the time. Michael's use of equations on periodic functions differed from his use of other representations for periodic functions χ^2 (1, N = 292)=10.90, $p < .05$. Michael preferred using equations on problems dealing with non-periodic functions and a representation other than equations for periodic functions. Michael did not use graphs enough in non-functions problems for meaningful statistical analysis. However, he did use a representation other than graphs 84% of the time in functions problems. This shows that he often wrote numerical answers without showing work. Michael's use of graphs with periodic functions was in the same proportion as his use of representations other than graphs with periodic functions. He used graphs about the same amount of time on periodic and non-periodic functions. Michael's χ^2 values are summarized in Table 13.

Table 13: Michael's χ^2 Values for Representation vs. Content Area

	Function	Periodic
Equation	0.0024*	10.9
Graph	NA	0.13*

χ^2 (1, N = 154), p < .05

* indicates dependence between the representation and content area

In summary, most problems where David, Ike, Beth and Rachel used graphs were periodic functions. However, none of the students actually showed a preference to use graphs to solve periodic functions. Steven and Michael preferred using equations on problems dealing with non-periodic functions. David, Matthew, and Michael preferred using representations other than equations for periodic functions, as opposed to non-periodic functions. Rachel preferred to use equations with problems involving functions, as opposed to non-functions. Finally, Matthew preferred to use graphs with functions, as opposed to non-functions. In general the students preferred to use equations and when they did use graphs it was most likely in the context of periodic functions.

8: Analysis of the Card-Sort

The card-sorts and ensuing concept maps gave the researcher a way of looking at the students' conceptions of functions. The card-sorting task required the students to make explicit the connections they saw between different constructs. This allowed the researcher access to the connections the students were aware of. The combination of the clinical interviews and the card-sorting tasks allowed the researcher to uncover students explicit and implicit understandings. The analysis and concept maps provide a summary of the connections students made between constructs in their understanding of functions.

Students completed the card-sorting activity during the final clinical interview, but generated the cards during the first interview when each was given several brainstorming tasks. In order to have some standardization of the nodes in the concept maps, the researcher took words, phrases, symbols, tables, and pictures from each student's brainstorming tasks. Ideas that two or more students mentioned were included. In order to make sure that periodicity and the three representations: graphs, equations, and tables were included; and to make sure that students could make a variety of connections between the cards, the researcher added some cards (See chapter 5, Figure 13). Unlike the other clinical interviews, the researcher took an inactive role in the card-sorting activity, recording what the students said and did, but not asking questions or probing. Thus, the groups, categories, and connections belong completely to the students as if they had drawn their own concept map. The activity, therefore, indicated connections that students were aware of rather than the researcher's biases.

Six of the seven students completed the card-sorting activity. Steven was the exception. Steven was extremely uncomfortable doing this task and left after only 5 minutes. Therefore the discussion of Steven's understanding is based solely upon his interviews and classwork. Rachel and Michael completed the task alone while Beth and Matthew, and David and Ike worked in teams.

The discussion of each card sort is presented thematically. These themes come from the concept maps that the students developed during the card-sorting task. Each of these concept maps is a conglomeration of two or three different sorts of the cards. The connections between the nodes in each concept map reflect cards that the students placed in a group together. The labels on these connections are the labels that students used for each group of cards. Patterns that emerged in the way the students grouped their cards and labeled their groups are reported as themes.

Ike and David's Card-sorting Activity.

Ike and David worked together on the card sorting activity. They were very good friends who often worked together in and outside of class. Both David and Ike were seniors. Ike often missed instructions and seemed distracted in class. However, when he was working in a group with David, he was able to focus more on the problem at hand. Ike and David talked to each other throughout the card sorting activity. Their concept map is shown in Figure 24.

"Function" not connected with its definitions. Ike and David connected the familiar equation "$y = mx + b$" to both the "vertical line test" and "there is only one y-value for each x-value." This indicates that Ike and David understand that this function has both properties. However, while they chose to connect the concept "there is only one y-value for each x-value" with "function," they did not connect either of these to "the vertical line test" in any of their sorts. It is unclear whether they recognized these ideas as being different representations of the same concept, or whether they understand the similarity only when talking about familiar functions. In addition, in every card sort Ike and David put non-functions together with functions.

Table not connected with other representations. While Ike and David labeled "$f(x)$", "y", "equation", and "graph" as basics for equations, it is interesting that they did not include tables. During the card-sorting activity Ike and David had a conversation about tables.

David: You can determine the domain and range from a table.
Ike: You can't because you can't tell where it ends. You can tell where it begins, but not where it ends.
David: Or else when it is a terminating graph.

They discussed the incompleteness of the tabular representation. That is, unless the function is only made up of a finite number of discrete points, a table does not give information about the whole function, only parts of it.

Unconventional understandings of amplitude. In the following dialogue, both students reveal that they have misconceptions about the concept of amplitude. Ike reveals an interesting idea about how amplitude is connected with the vertical line test and David believes that every graph has an amplitude.

> **David:** Where's the parabola, 'cause that's the graph of it. We could say any graph has amplitude and period. Or an ellipse is one that has amplitude and period.
>
> **Ike:** If a graph has an amplitude it automatically passes the vertical line test so if it doesn't have an amplitude, it doesn't pass.

It is likely that Ike is thinking about a sinusoidal function that has amplitude. Because amplitude was introduced only in the context of periodic functions, this is a reasonable conception for Ike to have developed. Perhaps David is confusing the ideas of major and minor axes with the concept of amplitude. He is trying to draw together two different ideas that he has learned separately. While this particular connection is not mathematically correct, it does show that David tries to connect things in his concept image.

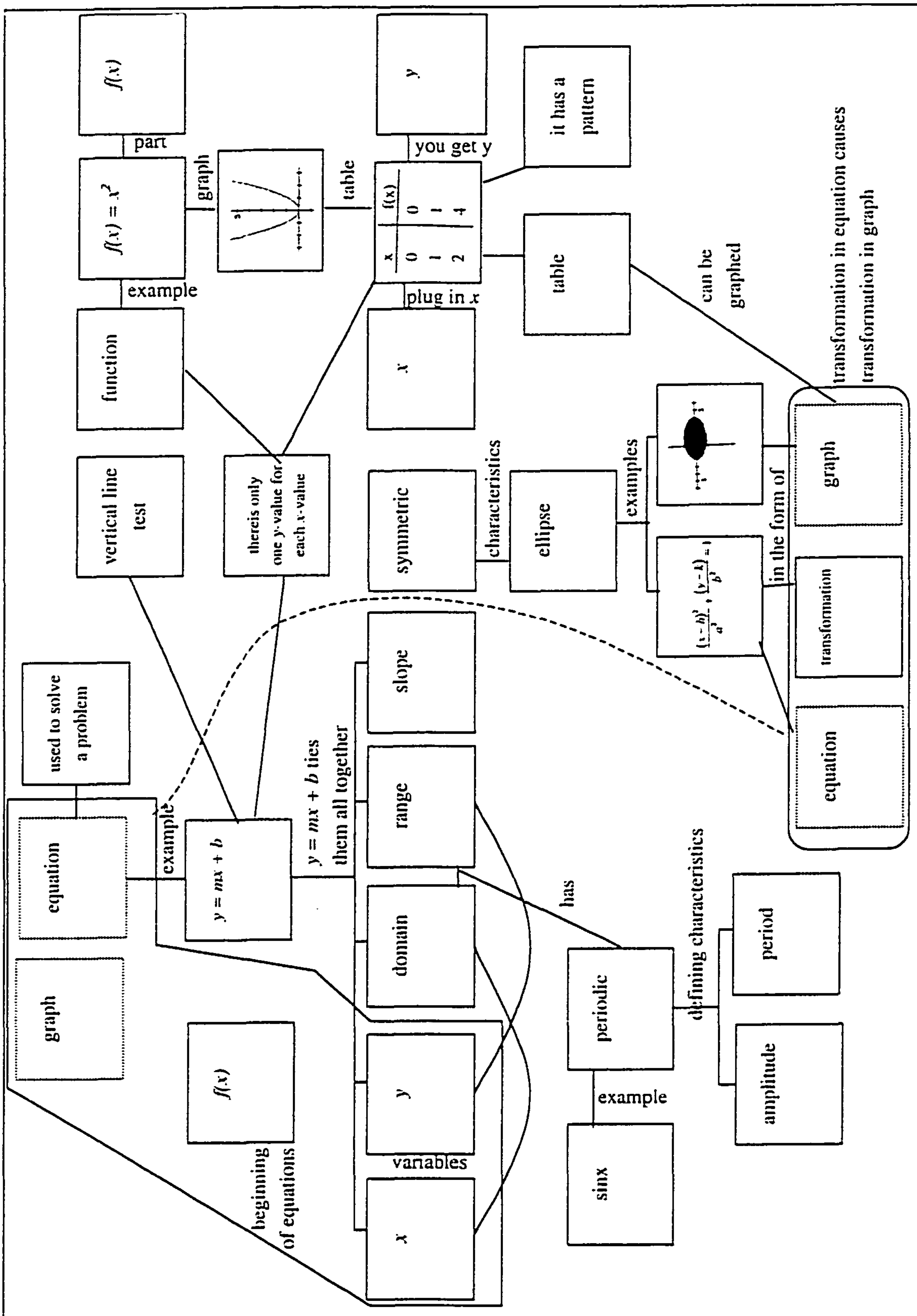


Figure 24: Card Sort Concept Map for Ike and David

An ellipse has period. In the same dialogue, above, David also reveals that his conception of period is not consistent with the conventional definition that $f(x+p)=f(x)$. He believed that an ellipse has amplitude and period. This is also interesting in that it is not consistent with his answers to other interview tasks such as Tasks 3-8 (see David's Interviews). Perhaps this statement reveals more about David's conception of period than was uncovered in the interview tasks.

David saw processes. During the Ike and David's card-sorting activity, David organized some of the cards around processes. He was the only student to sort the cards in this way. For instance, one of David's categories was "transformation in equation causes transformation in graph." Another example of David's process oriented groupings is evident as he talks to Ike during the activity, "'x', '$f(x)=x^2$' and the table [go together] because if you plug in x in the table this is the function."

Math is all connected. Ike shows a desire to make connections in the following excerpt from their conversation.

> **Ike:** Do you want to do random piles and try to find what connects them.
> **David:** Uh?
> **Ike:** Because math is all connected.

It is unclear what types of connections Ike sees in mathematics. When he sorted the cards, he did not put them in random piles. He and David worked methodically to put the cards into groups that made sense to them.

Rachel's Card-sorting Activity

Rachel was scheduled to do the card sort with Steven. However, Steven left the interview after only a few minutes. Therefore, Rachel worked alone on this activity.

A few deeper connections. In Rachel's card sort, she did a surface grouping and a deeper grouping (see Figure 26). In the surface grouping, she sorted the cards by representation as did Beth and Matthew. For instance, Rachel grouped the tabular representation of the squaring function with "table" and the graphs of the ellipse and parabola with "graph". In the deeper grouping, Rachel organized the cards into two groups. One was centered on graphs and the other around functions. She connected "table" and "equation" with "graph" saying each "determines a graph." In her function group, Rachel organized the cards into components and examples. Rachel used "graph" as a central card. She connected "table" and "equation" with the link "determines a graph." Then she connected examples of tables, graphs, and equations.

Definition of function connected with the table. Rachel connected the cards, "function", "there is only one y value for each x value", and the table representing the squaring function. In the text, tables were used only in the section introducing functions. Therefore it is not surprising that Rachel made this connection.

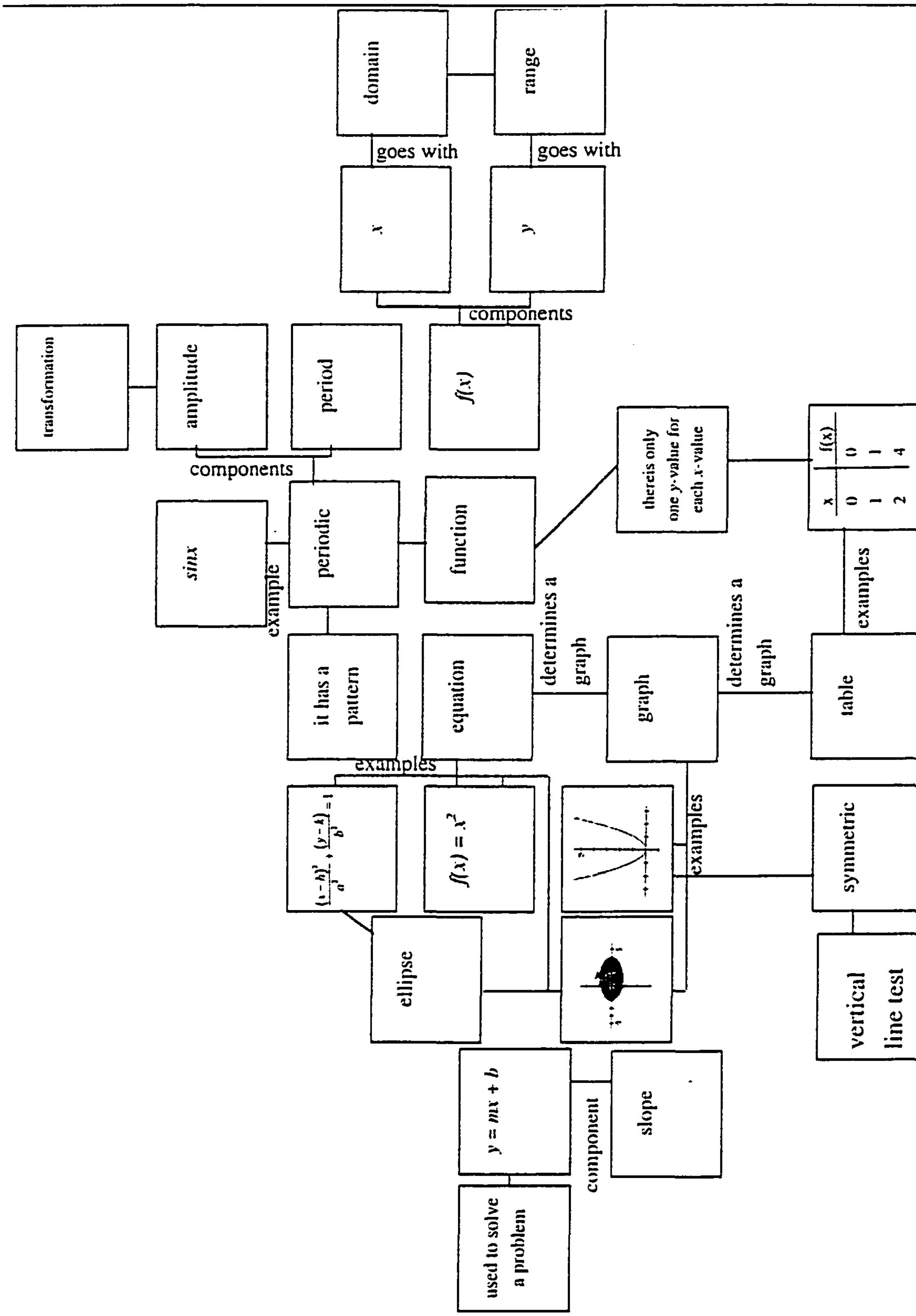


Figure 25: Card Sort Concept Map for Rachel

Functions separated from non-functions. Rachel's two groups could be titled "representations" and "functions." In the functions group she put cards related to functions and periodic functions. In the representations groups she put examples of each representation. This allowed her to put "ellipse" in a separate group from functions.

Unconventional understanding of amplitude. Rachel had trouble placing "$\sin x$". She put it with "period", "amplitude", "logs", and "transformations" in one card sort and in the miscellaneous pile in the other. Although, they had most recently worked with trigonometric functions, Rachel put "amplitude" with the ellipse.

Beth and Matthew's Card-sorting Activity

Beth and Matthew also worked together on the card sorting activity. Beth and Matthew were not as used to working together as Ike and David were. They sat in different parts of the room during class. So although they knew each other, they had not developed the rapport that Ike and David had. This pair was a lot less talkative during the card sort activity. Their concept map is in Figure 25.

Cards connected by surface features. Beth and Matthew did not include "table" in the group of things you can graph: "equation", "logs" and "ellipse". "Table" was set apart in it's own group connected only because it gives you values for an equation. Like Ike and David, Beth and Matthew classified many cards together by their surface features such as in which representation they were presented.

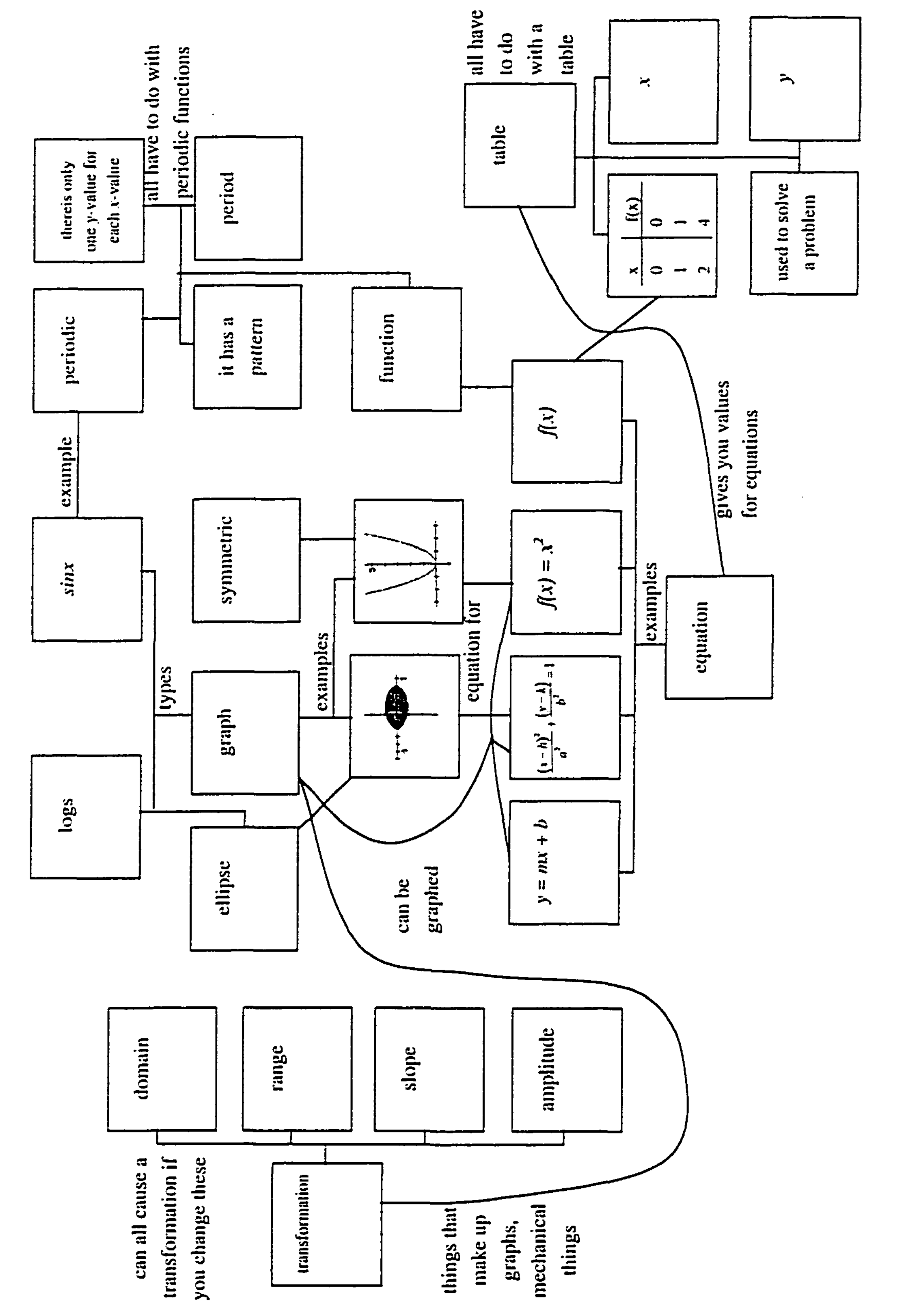


Figure 26: Card Sort Concept Map for Beth and Matthew

A few deep connections. Most of the connections Beth and Matthew made were based on surface features. For instance, they put each of the graphs with the "graph" card. Many of their labels revolved around the idea of example. In others they don't specify the connection they just say they "have to do with" one particular idea. However, a few of their groups and connections were deeper. One such group was "transformation", "amplitude", "range", "domain", and "slope" which Beth and Matthew put together in both sorts, but with two different labels. In one they put everything under "transformation" and said, "can all cause a transformation if you change these." In the other there was not a leading card and they labeled it "things that makeup graphs, mechanical things. Both of the labels were Matthew's. Both of these labels go beyond examples. They give a different type of relationship to the cards. The label, "can all cause a transformation if you change these," talks about the process of transformation rather than just examples of transformations. The label, "things that make up graphs, mechanical things," is Matthew's attempt to explain how the cards relate to graphs. Both of these labels are deeper than the other labels in Beth and Matthew's card sort. Matthew supplied both labels.

Functions not connected with the vertical line test. In both card sorts, Beth and Matthew put "function" with "there is only one y-value for each x-value." However, like Ike and David, they did not connect this with "vertical line test." Furthermore, they did not categorize functions and non-functions differently.

Michael's Card-sorting Activity

Michael was a quiet student who preferred to work on his own. When working in a group, he tended to monopolize the work. For this reason, he was scheduled to do the card sort by himself. His card sorting concept map is in Figure 27.

Tables not connected with equations or graphs. Like several of the other students, Michael did not connect "table" with "equation" or "graph". Michael did, however connect "table" with "function." Michael also connected "table" with "domain" and "range". Although Michael put "$f(x) = x^2$" with it's graph, Michael did not connect the table of the same function.

Function connected with its definitions. In all three card-sorts Michael put "function", "vertical line test", and "there is only one y-value for each x-value" together. Michael was the only student that explicitly stated that the vertical line test was the same as "there is only one y-value for each x-value."

A few deeper connections. Michael used surface features to organize some of the cards, but he also made deeper connections. For instance, Michael used symmetry to place the parabola equation with its graph and the ellipse equation with its graph. He also connected "symmetric" with "$\sin x$". In all three card-sorts, Michael put "$\sin x$", "periodic", "period", "amplitude", and it has a pattern together.

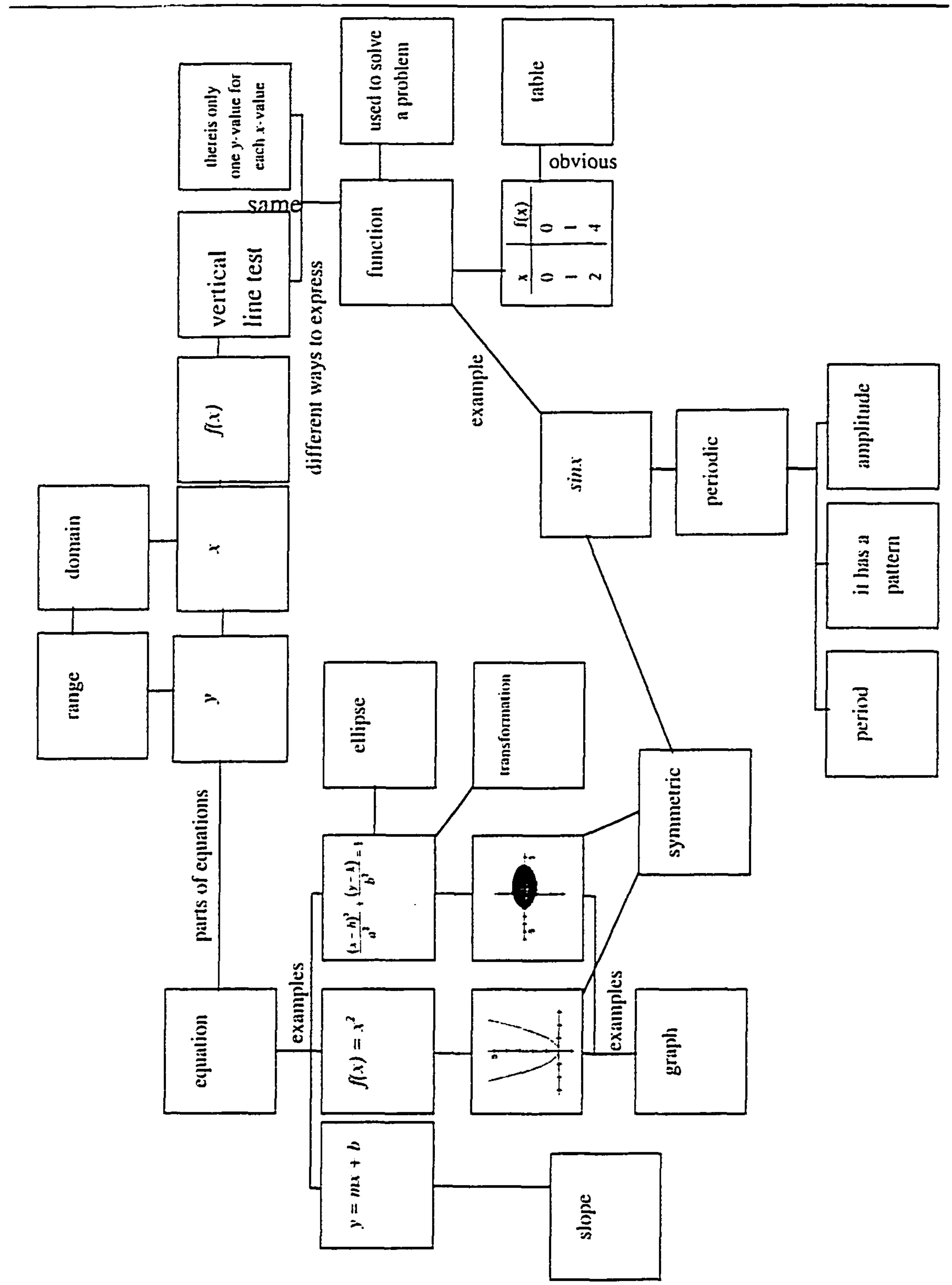


Figure 27: Card Sort Concept Map for Michael

In summary, although each of the concept maps was quite different, some common themes emerged. Each of the students made more connections and deeper connections with equations and graphs than they did with tables. The card, "there is only one y-value for each x-value" was placed with the table for $f(x) = x^2$ or "$y = mx + b$" by Ike and David and Rachel. It is possible that the two examples are prototypical functions in the students' concept images. All of the students used surface features such as representation to organize the cards. However, Michael, Rachel, and Ike and David seemed to make quite a few deeper connections as well. Michael was the only student who identified that the card, "there is only one y-value for each x-value" was the same as the vertical line test. Rachel and Michael were the only students who separated functions from non-functions. The concept maps by Michael and Rachel had more connections and deeper connections than the other students' maps.

9: Analysis of Interviews

Clinical interviews gave the researcher a window into the students' conceptions of functions. As the students worked on the tasks, they communicated with the researcher both verbally and on paper. These communications became the basis upon which the researcher determined what the students were thinking. Although the student occasionally made explicit statements about their understanding, most of the time their understandings were communicated implicitly through their attempts to solve the interview tasks. Thus, the interviews allowed the researcher to gain access to understandings of which the students were not necessarily aware. This makes the data in this section particularly interesting, but leaves the analysis particularly open to the researcher's biases.

The researcher guarded against bias during the interview process by having students explain their own answers instead of leaving it up to the researcher's interpretation. In addition, the researcher looked for patterns in each student's responses. These patterns were formed into emergent themes that were checked for consistency with the student's other responses.

In order to counteract the researcher's bias in the analysis, each student's responses to the interview tasks were analyzed as a set. The researcher went through each task listing understanding that could be shown by the student's response(s). When a particular understanding was shown in more than one task, it was taken as a theme that emerged from the data. In addition, if a student explained his or her thinking well on a particular task and this understanding deviated from accepted mathematical definitions or convention or if it differed from the understanding expressed by other students, it was included among the themes.

David's Interviews.

David was a confident senior with a good sense of humor. He was flexible in problem solving and did not give up easily. Although he was doing very well in the class, he demonstrated some interesting misconceptions about periodic functions and function notation.

Linear expectations with unfamiliar functions. When David was presented with an unfamiliar function represented in a table, he tried to find a linear equation to represent it. He did not check whether the function was linear by graphing or calculating ratios of differences. Instead he tried to find a multiplicative relationship between the vales for x and y. However, once this strategy failed, he plotted points correctly and showed no expectation that the graph should be a straight line. This shows that David did not necessarily believe that all tables represent linear functions, but instead that finding a linear relationship between the data points in a table was the way that David preferred to find an equation for a function represented in a table.

David demonstrated his method for translating between tables and equations in Tasks 1 and 13. He began Task 1 by attempting to find an equation for $h(x)$ (see Figure 28). He focused on the first data point (-3, 2), trying to find a linear relationship.

The function represented in the table below is periodic
It has a period of 5. Fill in the missing values.

x	h(x)
-3	2
-1	-1
1/2	0
1	3
4	3
6	7

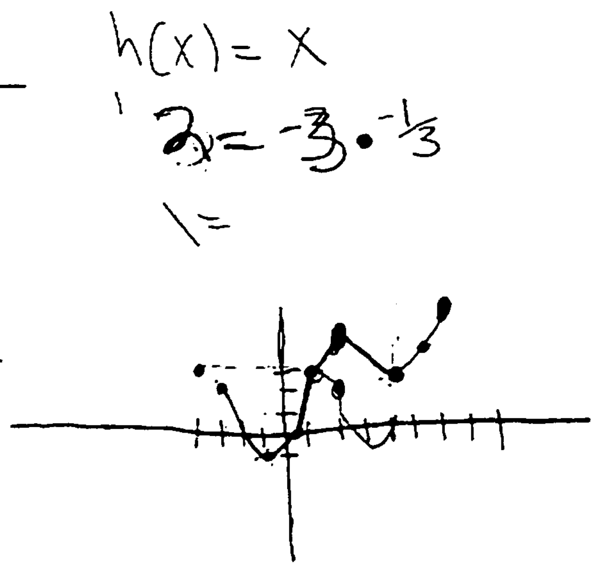


Figure 28: David's Answer to Task 1

David: {wrote $h(x) = x,\ 2 = -3 \cdot -1/3,\ x =$} Do you want me to find an equation?

Interviewer: You could try that. It looks like that is what you have been trying and it isn't working for you.

David: Yeah. I can't get it.

Interviewer: Can you think of another way to find the answer?

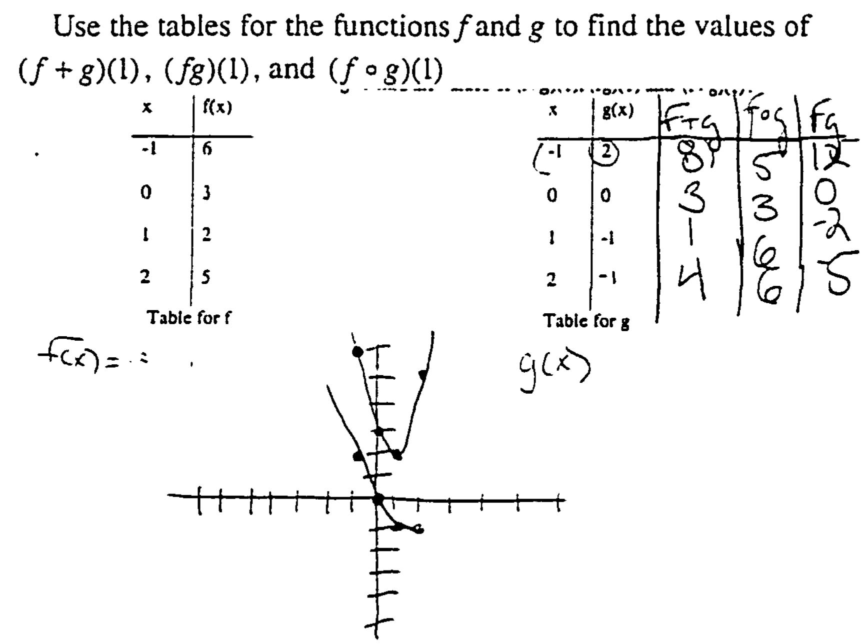
David: Um. I don't know because I want to plug in 4 and 6.

Interviewer: Do you think a graph might help?

David: I could try.

Once David plotted the points he had no difficulty in connecting the points with a curve. He had already given up on the idea that there was a linear relationship between the points.

David began Task 13 by trying to find an equation for f (see Figure 29). He wrote $f(x) = -3x +$ and then gave up on this strategy.



Use the tables for the functions f and g to find the values of $(f+g)(1)$, $(fg)(1)$, and $(f \circ g)(1)$

x	f(x)
-1	6
0	3
1	2
2	5

Table for f

x	g(x)
-1	2
0	0
1	-1
2	-1

Table for g

Figure 29: David's Answer to Tasks 13-15

Interviewer: Can you tell me what you have done so far?
David: I tried to find an equation for this table, but I couldn't.
Interviewer: What made it difficult?
David: I found the slope, but I can't find any numbers that work all the way down.
Interviewer: Ok, do you have another idea for how to solve it?
David: I'm going to graph it.

By using the word slope, he showed that he was expecting the values in the table to have a linear relationship. However, when he couldn't "find any numbers that work all the way down," he was not able to continue with this method.

David also showed an expectation of a linear relationship between x and $f(x)$ in Task 2. In this task, he was given an unfamiliar graph of a non-linear periodic function with period 2. He was asked to find $f(6)$, which was not represented on the graph. David had trouble understanding the meaning of $f(6)$. In the dialogue below, David is working on task 2. The graph and period of $f(x)$ is given and he has been asked to find $f(6)$.

David: So this is $f(x)$ this graph here.
Interviewer: Yeah.
David: I don't know what x would be. Like what is $6x$?
Interviewer: I'm not sure what you mean.
David: How can I find *x?*
Interviewer: Are you trying to find the equation for *f?*
David: Yeah.
Interviewer: Assuming you don't know what the equation is, could you figure out what it looks like to the right?
David: I'm not sure.
Interviewer: Does it make it easier if I tell you to find the y-value when $x = 6$?
David: Oh, so I could just draw the graph out here and then see where it is at 6.
Interviewer: Yep.

As soon as the interviewer translated the statement $f(6)$, David was able to find the solution. When David asked, "what is $6x$? He seemed to be looking for a linear relationship between $f(x)$ and x.

When David tried to find equations for unfamiliar functions, he looked for linear relationships. The idea that the equation of an unknown function should have a linear term (i.e ax) seemed particularly strong in David's concept image. However, he did not have a consistent strategy for finding the coefficient of this term. In Task 1 he used the point $(-3,2)$ for x and $h(x)$ and tried to find the coefficient. In Task 2 he interpreted $f(6)$ to be $6x$. And in Task 13, David calculated "the slope" between the first two given data points in the table.

Compartmentalization in function notation. David demonstrated a weak understanding of function notation in Task 2 and Tasks 13-18. Although he was able to overcome his confusion in each task, he needed prompting from the interviewer to understand the notation. In Task 2 David did not understand $f(6)$ in the context of the graph. He first interpreted $f(6)$ to mean that the equation contained $6x$. Once the interviewer interpreted $f(6)$ in the context of the graph, David was able to solve the problem.

Although David was able to generate correct tables for $(f+g)(1)$, $(fg)(x)$, and $(f \circ g)(1)$, in Tasks 13–18 (see Figure 29), he became confused when writing the values of $(f+g)(1)$, $(fg)(1)$, and $(f \circ g)(1)$. In each of these he listed all of the output values in each table. For instance, David wrote $(f+g)(1) = 8,\ 3,\ 1,\ 4$. When the interviewer questioned him about his answer, he quickly circled the correct answer. Although he was able to correct himself, this answer still supports the statement that David has some difficulty with function notation.

Ease in finding equations for familiar functions. In many of the tasks he began by looking for an equation. In most of the problems the functions given were unfamiliar. David was not able to get an equation of an unfamiliar function. However, he was able to do so quickly in the tasks where the functions given were familiar. For example, in Tasks 19-21, David immediately recognized the graph of f as a sine curve and the graph of g as $-x$. Although he made the mistake of interpreting $f(x) = \sin 4\pi x$ as $f(x) = \sin x$, he was still able to recognize the shape of the graph as a sinusoid.

Difficulty translating between tables and other representations. David had difficulty translating from a table to symbolic form as revealed in Tasks 1 and 13-18. Although these tables were not generated by an equation, his attempts to find an equation showed that he was expecting linear functions even when the table values were not linear. In tasks 19-21, David also showed that he had difficulty translating the equation $(f + g)(x) = -x + \sin x$ into a useful tabular representation (See Table 14).

Interviewer: What did you do?
David: Graphed it. It's $-x + \sin x$. Then tried to draw the table to support it, but it was difficult. I used values that work well with sine and then I added them together.
Interviewer: Were you able to make the table work?
David: Yeah because at when you plug in π you get $-\pi$ and 0 and you add those together. But it was a lot harder than the graph.
{*David's table worked for him, but he ignored the labels once he wrote them*

in. In his explanation column 1 was x, column 2 was $-x$, *column 3 was* $\sin x$, *and column 4 was* $-x+\sin x$. *He had difficulty translating the information from the graph to the table*}

Interviewer: Do you think you could have done it with the table only?

David: Maybe, but it was confusing. It was way easier to graph it.

Table 14: David's Answers to Tasks 19-21

x	$f(x)$	$g(x)$

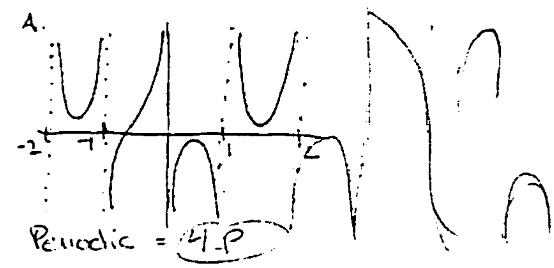
$f(x)$	$g(x)$	x
-1	-1	1
0		
1		

$f(x)$	$g(x)$	x	
π	$-\pi$	0	$-\pi$
2π	-2π	0	2π

Unconventional understanding of periodicity. In Tasks 3, 6, and 10 David revealed that symmetry is a part of his concept image of periodic functions. In these tasks, David was given graphs of functions and asked if they could be periodic. In Task 3, he reflected the graph through the x-axis and then shifted it to the right (see Figure 30). "It's like the tangent. I figured it would flip." He did not notice that the graph had already started to repeat and said that the period was 4.

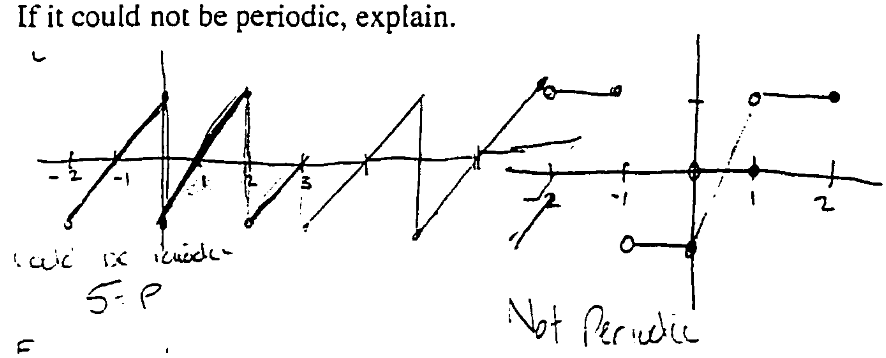
David's answers on tasks 5 and 6 were also interesting (see Figure 31). Both graphs were piecewise-defined functions. Both started to repeat in the last unit shown on the graph. David took the graph in task 5 and repeated the entire thing rather than continuing where it started to repeat. However in task 6, David believed that the function was not periodic.

Which of the following could be periodic. What could the period be? If it could be periodic, describe or draw more of the graph. If it could not be periodic, explain.



<u>Figure 30</u>: David's Answer to Task 3

Which of the following could be periodic. What could the period be? If it could be periodic, describe or draw more of the graph. If it could not be periodic, explain.



<u>Figure 31</u>: David's Answer to Tasks 5-6

In the following dialogue, the interviewer attempted to determine why David didn't think the function in Task 6 could be periodic.

Interviewer: If I gave you this graph, would it be periodic? (See Figure 32)

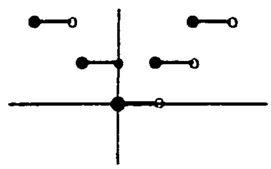


Figure 32: Interviewer's Graph for David's Task 6

David: I think so because it looks like a wave.

Interviewer: So the problem with number 6 is that it doesn't look the same on this side as the other side?

David: Yeah.

Once again, David revealed his expectation of symmetry in periodic graphs when he did not plot the points correctly in Task 10 (See Figure 33). Notice that the third data point (1, 0) is not plotted carefully. When David continued the graph, he interpreted this third point as (1, 1) which makes the graph symmetrical.

Determine if each of the following tables could represent a periodic function. What could the period be? If it could not be a periodic function, explain.

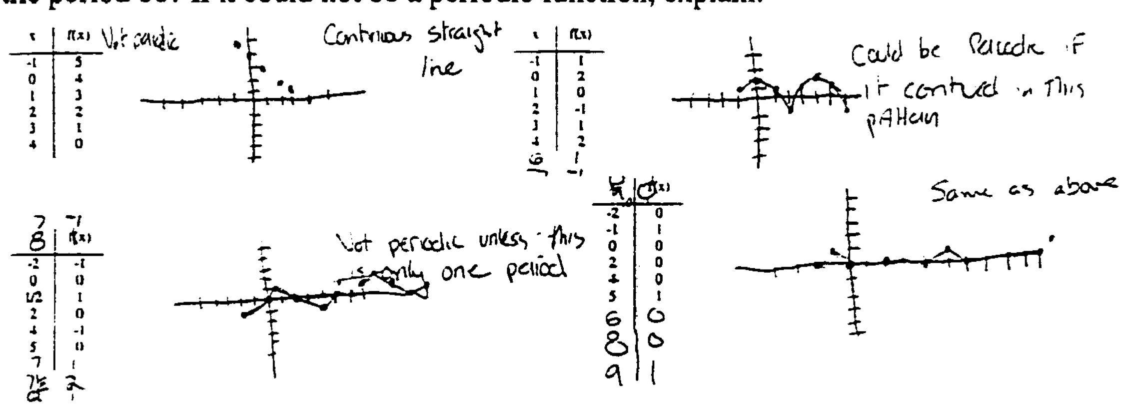


Figure 33: David's Answers for Tasks 9-12

David's answer to Tasks 1 and 11 show that he has an interesting conception of how a periodic graph repeats (see Figures 28 and 33). David had difficulty continuing the graph beyond the values given in the table. He understood that the shape needed to repeat, but he did not understand that the *y*-values of a periodic function remain the same as the *x*- values shift over by one period. He was confused by the fact that the starting and ending points were different in the graphs in Tasks 1 and 11. He began the second period where the first left off. "At first I didn't see any pattern in it but then I thought that this could be the beginning of it and it could just continue up from here." This results in shifting the pattern upward as the *x*-values shift over by one period.

<u>A preference for equations except when determining periodicity.</u> David's first solution strategy was to translate a given representation into an equation in all but Tasks 3 – 8 where he was asked to determine whether graphs represented periodic functions, and tasks 9 –12 where he was asked to determine whether the tables represented periodic functions. David showed a preference for using graphs to determine periodicity. The defining characteristics for periodicity seem to be situated in the graphical representation. For example, graphical ideas of symmetry and beginning where the last point left off are constructs in David's concept image of periodicity.

<u>Flexibility in problem solving</u>. Although his first attempts were often not fruitful, David was able to start afresh with a different representation after a prompting such as "Is

there another way you could do it?" When given a table or graph, David began by trying to find an equation. Although he preferred to use equations, David showed that he was able to translate between representations, especially tables to graphs and graphs to equations when familiar. That is, when David recognized a function as one he had seen before, he was more likely to be able to translate it into other representations.

While David was solving Tasks 13 – 18, he used a number of strategies. First, he attempted to find an equation to represent the table. When that was not fruitful, he plotted points in order to find an equation. Once again, he abandoned this solution strategy when the two graphs did not look familiar. However, the act of plotting the points seemed to trigger another solution strategy.

> **David**: {*after plotting the points*} I still don't see what these are. Wait! All I have to do is add these together right?
> **Interviewer**: Show me what you mean.
> {*David drew the table shown in Figure 29*}

Before David plotted points, he was unable to use the values in the table to solve the problem. The act of plotting the points allowed David to see the values in the table in a different light. He saw that the values in the table represented points on the graph. He realized that in order to add the two functions together he could just add the $f(x)$-values represented in the table. This realization did not occur to David until he had tried to interpret the addition of two functions graphically.

David's Concept Image

In the interview tasks, David attempted to use equations on every problem except those that involved determining whether a function was periodic. In order to determine whether a function was periodic, David used a graph. His preferences on the interview tasks were similar to those found in the classwork. For instance, when David used a graph to solve a problem in the classwork, 79% of the time the problem dealt with periodic functions. In addition, when David showed his work on classwork, he most often used equations. David used equations to solve the classwork 46% of the time and graphs only 5% of the time.

David's concept image of periodic functions contains sinusoids as the prototypical functions (see Figure 34). He has generalized that the "peaks and valleys" of a periodic function must be symmetrical and that the graph repeats from wherever the last point left off, even if it is not the same as the starting point. He also made the statement that "an ellipse is [a graph] that has amplitude and period."

David was able to solve tasks dealing with combining functions in each representation. Although his first step in searching for a solution was to try to find an equation, he was able to find solutions within other representations. His knowledge of combining functions was not compartmentalized.

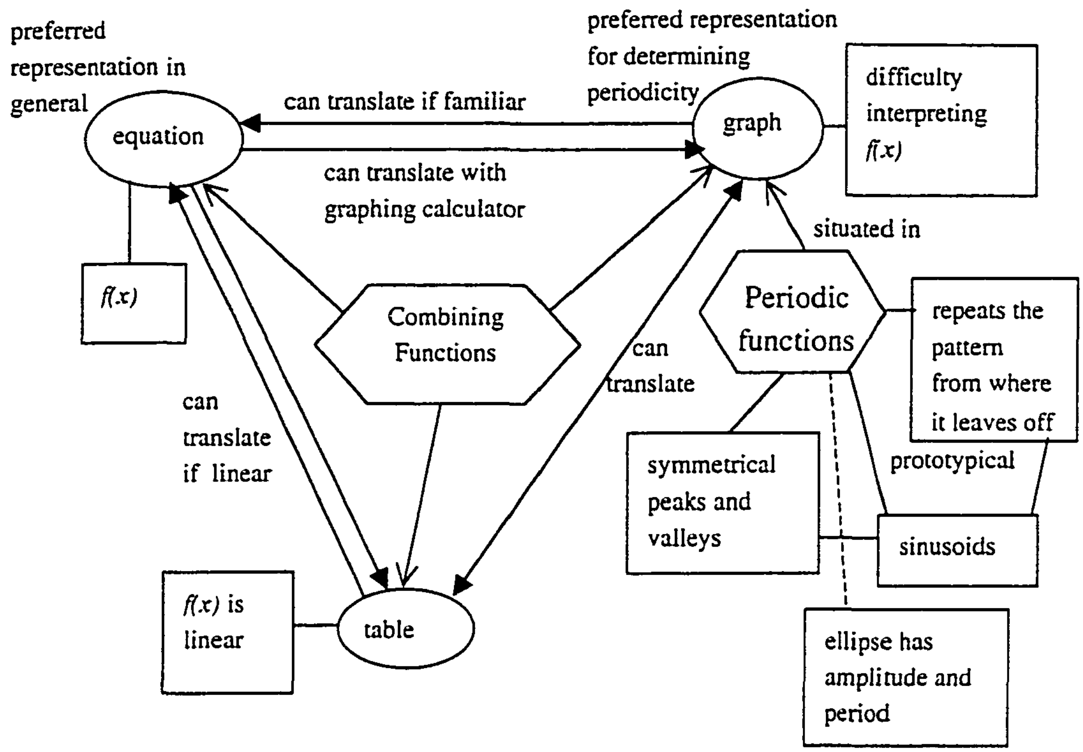


Figure 34: David's Concept Map

David had difficulty in interpreting function notation in tabular and graphical representations. Therefore his understanding of function notation was compartmentalized within the graphical representation. In addition he did not explicitly connect function with its definitions in the card-sorting activity.

David was able to translate easily between graphs and tables, between equations and graphs (with his graphing calculator) and between graphs and equations or tables if the function is familiar. David was able to translate between tables and equations if the function was linear. In spite of his ability to translate, David did not see an explicit connection between tables and the other representations.

Rachel's Interviews

Rachel was a confident junior who intended to take calculus the following year. She said that she enjoyed mathematics. She was willing to explain her thinking as she went along.

Compartmentalization in function notation. Rachel completed Task 1 correctly. However, she had trouble beginning the task. She did not understand what the table represented. She interpreted $h(x)$ to be h times x. However, when the interviewer suggested that she replace h with f, she was able to proceed using the function notation correctly.

> **Rachel**: First thing I'm wondering is the period h? I see $h(x)$ and I'm thinking *2*. We're doing cosine now and I'm thinking of it on a timeline. I get down to *4* and I don't have a value for h and I'm clueless.
>
> **Interviewer**: If I change $h(x)$ to $f(x)$, would that make a difference?
>
> **Rachel**: {Writes $f(x) = 2 \quad x = -3$} I think I'm stuck because I still can't get the equation.

In all of the other tasks, $h(x)$ was only given if $f(x)$ and $g(x)$ were also given. In these tasks, Rachel was able to use the function notation $h(x)$ correctly. This implies that her understanding of function notation was compartmentalized. That is, when Rachel came upon the function notation $h(x)$ apart from the symbol $f(x)$ she was unable to interpret it as function notation. However, when $h(x)$ appeared in the context with $f(x)$ she was able to interpret it as function notation. Therefore her understanding of $h(x)$ as a function was only accessible from her understanding of $f(x)$.

Preference for graphs to determine periodicity. In Tasks 9-12 Rachel drew the graphs, but then continued the tables without referring to the graph, because "it is easier to just follow the pattern [in the table]" (see Figure 35). Although Rachel wasn't immediately able to see a periodic pattern in the table, she was able to extend the graph using the given period. This allowed her to see the pattern in the table. This showed that Rachel's understanding of period was situated in the graphical representation. But, she was able to connect the two representations together to transfer the information learned between the two representations. Rachel was not bothered by the lack of familiarity with the graphs. That is, she was able to use her definition of period to determine if familiar and non-familiar graphs were periodic.

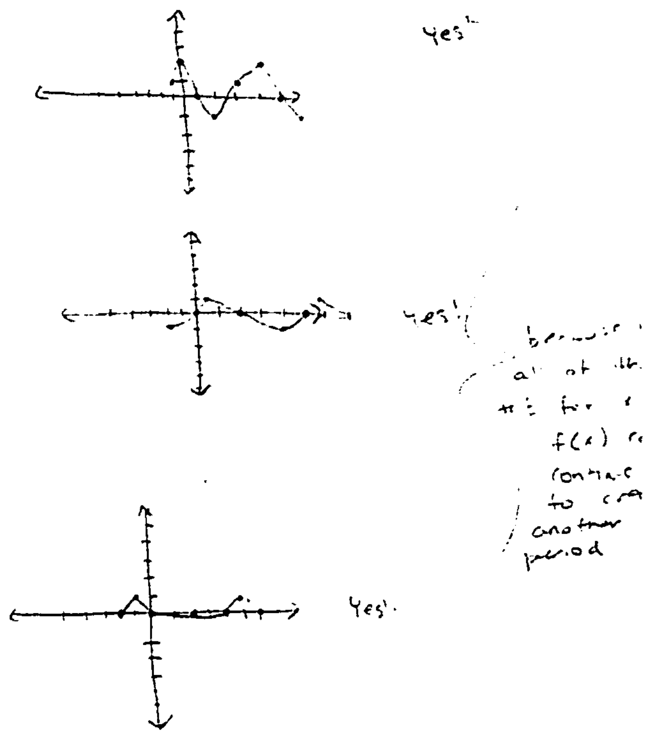
Flexibility with combining functions. Rachel was the only student who did not try to graph the tables first. In fact, Rachel was able to complete all of the combining functions problems in the representation given. This showed that her understanding of combination of functions was not compartmentalized within representations.

Determine if each of the following tables could represent a periodic function. What could the period be? If it could not be a periodic function, explain.

x	f(x)
-1	1
0	2
1	0
2	-1
3	1
4	2
5	0
6	-1

x	f(x)
-2	-1
0	0
1/2	1
2	0
4	-1
5	0
5½	1
7	0

x	f(x)
-2	0
-1	1
0	0
2	0
4	0
5	1
6	0
8	0
10	0



because all of the #'s for x and f(x) continue to another period.

Figure 35: Rachel's Answers to Tasks 9-12

A translation of Rachel's comment is provided, in italics, for readability

In Tasks 13-18 Rachel solved the problems in the tabular representation (see Figure 36). She made intermediate tables where she could put values for both f and g and then carried out the requested operations. Rachel did not have time to finish the problem, but she was able to explain how she would solve it correctly.

Interviewer: We are almost out of time. Can you explain how you would finish this problem?

Rachel: I would find g of x and then I would plug it into f and find the value from the table. Then I would do that for every x value and I would be done.

Use the tables for the functions *f* and *g*, to draw tables *for* $(f+g)(x)$, $(fg)(x)$, and $(f \circ g)(x)$.

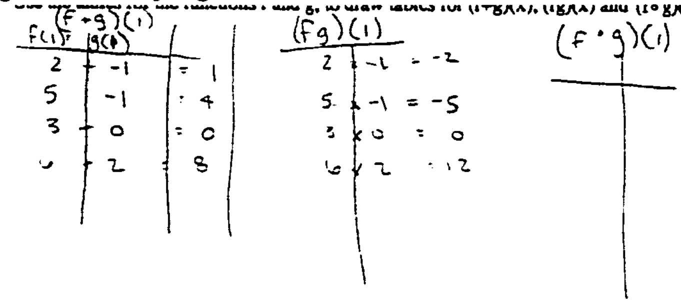


Figure 36: Rachel's Answers to Tasks 16-18

Rachel's Concept Image.

Rachel's concept map is presented in Figure 37. Rachel showed a preference to use equations. She used equations 45% of the time and graphs only 6% of the time in his classwork. Rachel's understanding of periodicity seemed to be situated in the graphical representation. Although she did not use graphs very often in classwork, this is consistent with her use of representations in the classwork. When Rachel used a graph to solve a problem in the classwork, 83% of the time the problem dealt with periodic functions.

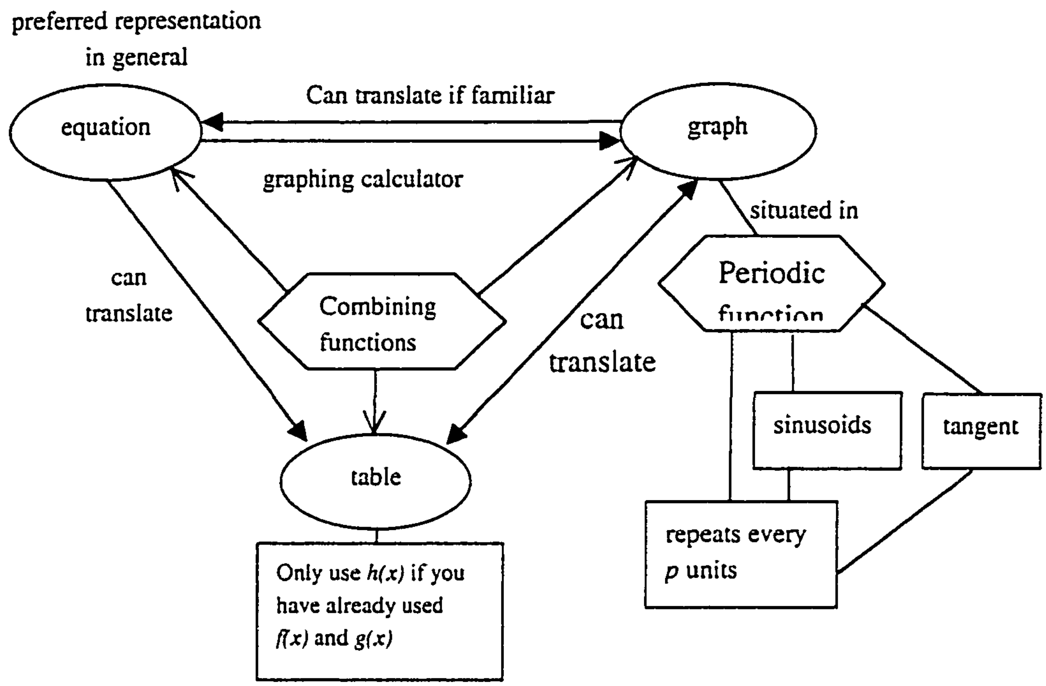


Figure 37: Rachel's Concept Map

Rachel's concept map has connections between all of the representations. Rachel was one of the few students who could consistently transfer information from one representation to another. Rachel's understandings of functions and periodicity were not compartmentalized.

Rachel had difficulty interpreting $h(x)$ in a table. She interpreted $h(x)$ as h times x. Once $h(x)$ was replaced with $f(x)$, she recognized $f(x)$ as a function. In other tasks $h(x)$ was presented along with $f(x)$ and $g(x)$. In these tasks Rachel interpreted the $h(x)$ correctly. This shows that Rachel expected to see $h(x)$ only if $f(x)$ and $g(x)$ had already been used.

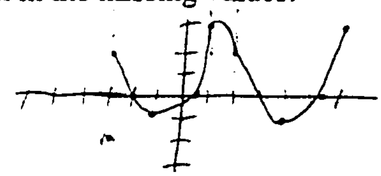
Matthew's Interviews

Matthew was a junior who intended to take calculus the following year. He worked problems slowly and methodically. He was very careful about his drawings and explanations. He wrote things down, reconsidered, erased and re-wrote. He did not feel comfortable talking while he worked, but did explain his solutions after he finished working the problems. He was willing to ask questions when he was unsure about how to proceed. In a group he often was quiet until the others had started the problem. Then he would bring up thoughtful suggestions and new directions for solving the problem

Ease in translating between graphs and tables. Matthew was able to translate between graphs and tables and also transfer knowledge from one representation to the other. He demonstrated his connections in Task 1 (see Figure 38). Matthew solved the problem within the graphical representation. He then transferred the graphical information back to the table by finding $h(4)$ and $h(6)$ on his graph and putting the values in the table. This shows that Matthew was able to translate between graphs and tables. Once he was finished, Matthew was able to explain how the table could have been used to fill in the missing values, without graphing or finding an equation." Well, it starts again in 5 units. So you could subtract 5 from these [4 and 6] and find the y-value." This shows that he was able to transfer the knowledge he learned about how to solve the problem in a graphical representation to the tabular representation. Matthew has made a strong connection between graphs and tables.

The function represented in the table below is periodic.
It has a period of 5. Fill in the missing values.

x	h(x)
-3	2
-1	-1
1/2	0
1	3
4	-1
6	3



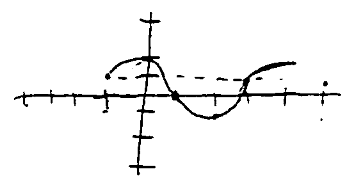
y = 20

Figure 38: Matthew's Answer to Task 1

Unconventional understanding of periodicity. Matthew revealed that he expected each peak to be symmetrical from left-to-right in his answer to Task 10 (see Figure 39). The function in Task 10 does not have symmetrical peaks and valleys. He wrote, "next value = (5, 1)." Matthew's chosen value makes the peak between $x = 3$ and $x = 5$ symmetrical even though the given peak between $x = -1$ and $x = 1$ is not symmetrical. The graph of the function in Task 10 resembled a sinusoid. It is likely that Matthew generalized the symmetrical peaks and valleys of sinusoids to this function.

Determine if each of the following tables could represent a periodic function. What could the period be? If it could not be a periodic function, explain.

x	f(x)
-1	1
0	2
1	0
2	-1
3	1
4	2



yes this could be a period.

next value = (5, 1)

Figure 39: Matthew's Answer to Task 10

Matthew believed that a graph had to look familiar (i.e. like $\sin x$, $\cos x$, or $\tan x$) in order to be identified as periodic. In Tasks 5 and 6 (see Figure 40). Matthew said that the discontinuous functions could not represent a perioidic function "because they don't look periodic." The functions which did look periodic to Matthew were continuous with symmetric peaks and valleys, or looked like a tangent function, as in Task 3.

Which of the following could be periodic. What could the period be? If it could be periodic, describe or draw more of the graph. If it could not be periodic, explain.

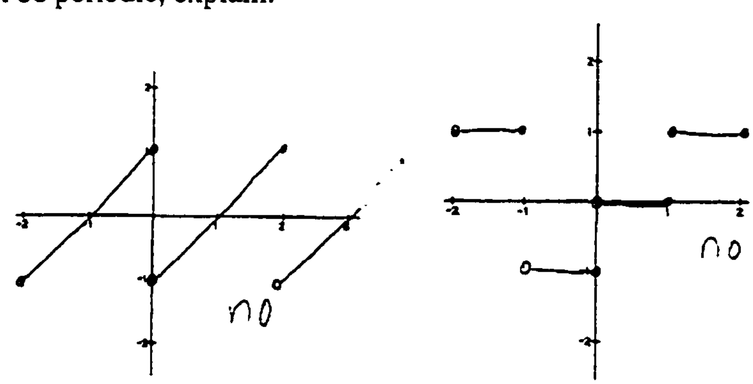


Figure 40: Matthew's Answer to Tasks 5 and 6

Flexibility in problem solving. Although Matthew's first attempt at solving a problem often did not provide him with the solution, he was able to translate the problem into another representation and solve it. This ability to view tasks from different representations shows flexibility in problem solving. It is evident in Tasks 1 and 13-18.

Matthew plotted points in order to determine an equation to represent each table in Tasks 13 - 18. He abandoned this solution strategy when the two graphs did not look familiar. Instead he used the tabular representation.

> **Matthew**: Is there any way to tell if the graph [on the calculator] is the same [as the one on the paper]? ... If I can find the equation, I can find $f(1)$ and carry it through.
> **Interviewer**: You could trace along and see if the values match up.
> **Matthew**: {*after trying*}. This is different. I don't see any way to find $f(1)$. Oh, it's right here {*pointing to the table*} $f(1)$ is 2.

Matthew had an epiphany. Up until this point he was thinking that he needed to find an equation in order to get values of *f*. Once he realized the values were given in the table, he proceeded to solve the problem in the tabular representation.

> **Matthew**: Can I do it in different ways? Like could I do $f(-1)$ and $g(2)$.
> **Interviewer**: That would be $f(-1)+g(2)$. I am looking for the value at 1.
> **Matthew**: Oh, I hate that . . . when it looks so difficult and it ends up being so easy. So all I have to do is add these two values right?
> **Interviewer**: Yeah.
> **Matthew**: {*for* $(f \circ g)(x)$ *he fills in the value for 1 and then pauses*} So for this one I have to find $g(-1)$ which is -1 . . . I mean two and then $f(2)$ is 5.
> **Interviewer**: Yeah.

This example was typical of Matthew's problem solving. He failed to solve the problem in his first attempt and then translated the function into a different representation from which he was able to solve the problem.

Matthew's Concept Image

Matthew's understanding of periodic functions and operations on functions is summarized in his concept map in Figure 41. Matthew showed a preference to use equations no matter the content of the problem. He used equations 50% of the time and graphs 18% of the time in his classwork. When compared with the rest of the students in the study, Matthew used graphs about twice as often. Matthew preferred to use graphs with functions as opposed to non-functions.

Matthew demonstrated a facility with translating between graphs and tables. In addition he showed that could translate a graph to an equation, if it represented a familiar function. He believed that every function has an equation.

Matthew's understanding of periodic functions was situated in the graphical representation, however, he was able to translate between graphs and other representations so he did not exhibit compartmentalization within representations. Matthew expected the "peaks and valleys" of a periodic function to be symmetrical.

Matthew also expressed that periodic functions should look like waves. Sinusoids were a strong part of Matthew's concept image of periodic functions.

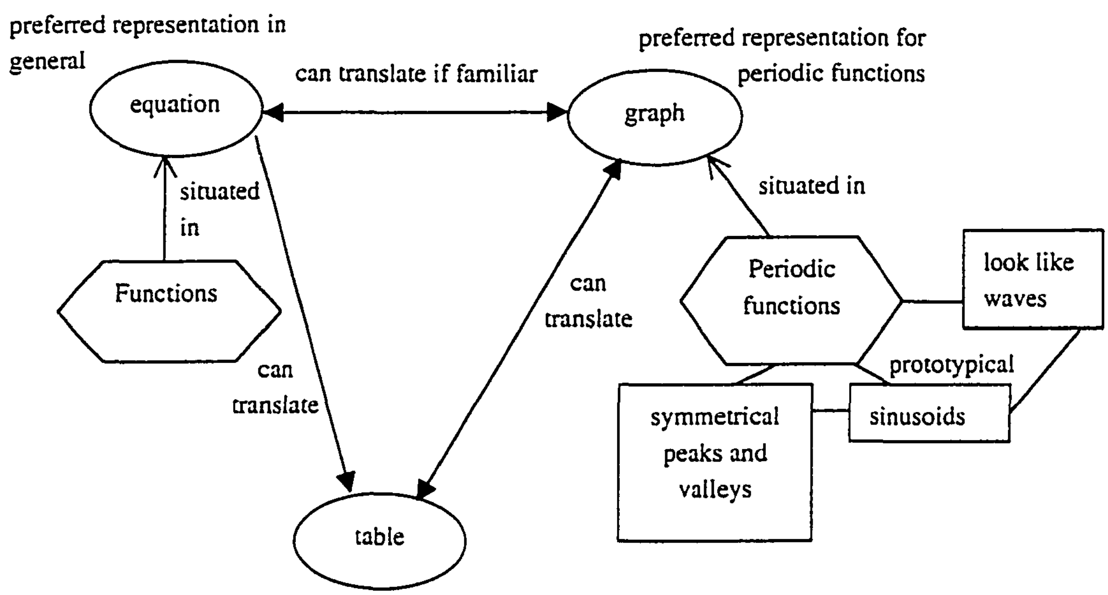


Figure 41: Matthew's Concept Map

Ike's Interviews

Ike was the class clown. He worked well in a group. He took the interview tasks seriously when he came to the interviews, but he did not show up to several of his interviews. There was therefore not enough data to discuss his understanding of combining functions. He was only able to finish the periodicity tasks and the card-sorting task.

Understood function notation in a tabular representation. In Task 1, a table was given with values for x and $h(x)$. Ike was one of the few students who understood the $h(x)$ notation right away (See Figure 42). He wrote $h(-3) = 2$ and $h(-1) = -1$ taking the values from the table.

The function represented in the table below is periodic. It has a period of 5
Fill in the missing values.

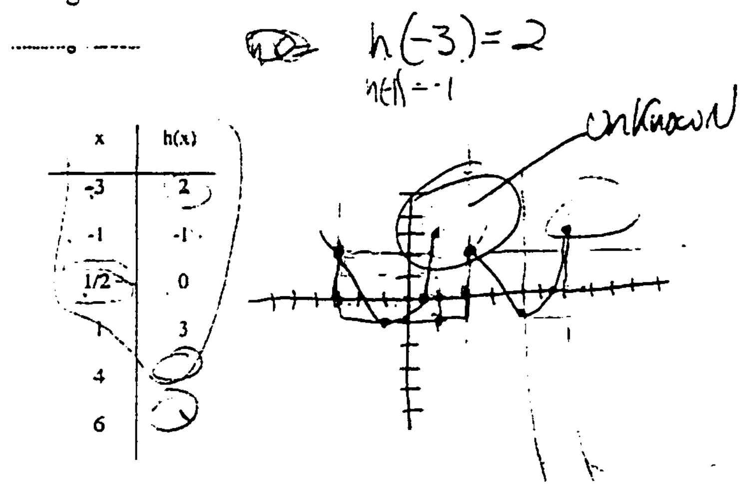


Figure 42: Ike's Answer to Task 1

Interviewer: Are you stuck?
Ike: I'm not sure
Interviewer: What have you been trying?
Ike: I guess …well I don't see any way to make this work?
Interviewer: Are you trying to find an equation?
Ike: Yeah, but I can't find the pattern.
Interviewer: What would you do if you had an equation?
Ike: I would just plug-in 4 and 6.
Interviewer: Can you think of a way to do it without using an equation.
Ike: Um. I guess I could graph it.
Interviewer: OK, give that a try.

Graphs "come from" equations. As Ike continued Task 1, he was uncomfortable extending his graph because he believed that the graph came from an equation that he couldn't determine. He didn't want to extend it without knowing what the values "actually" were.

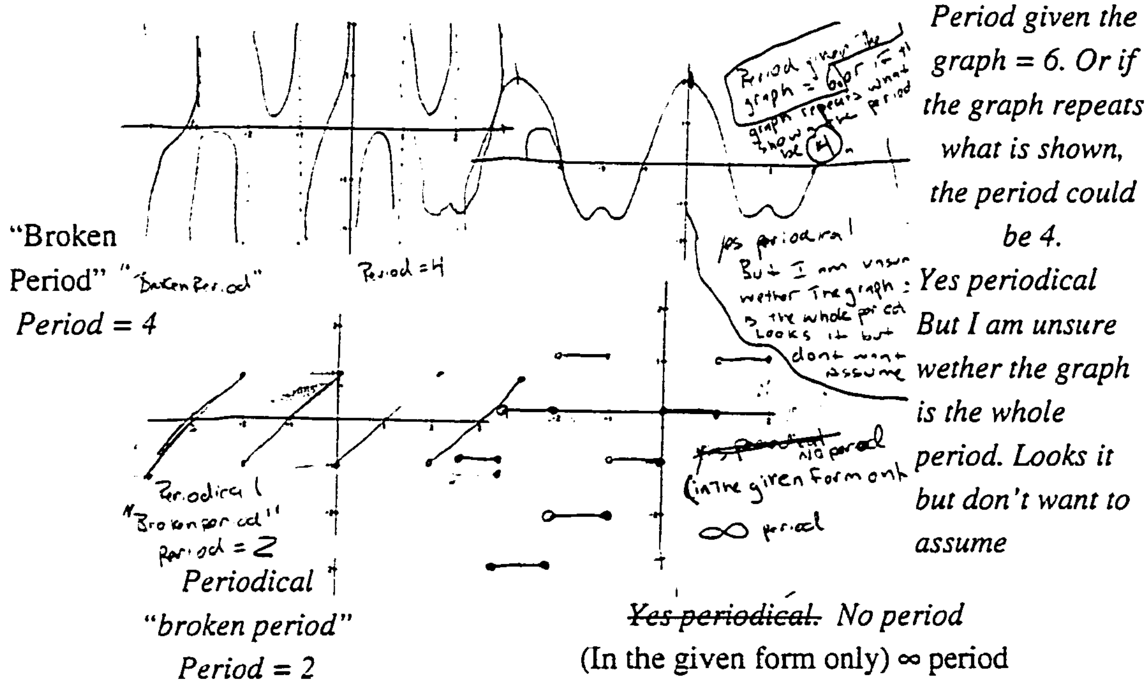
> **Interviewer**: Why did you circle that part of the graph?
> **Ike**: Because I can't tell where the points are. It's unknown because I can't find the equation.

It is interesting that Ike connected the points given in the table, but did not connect the points in-between the last point on the table and the point where it began to repeat. This shows that he is comfortable interpolating values from a table, but not extrapolating.

Lack of connection between x- and y-values on a graph. In Task 1, although he was able to extend the graph out to the x-values requested in the table, he was not able to determine $h(x)$ values. He drew vertical lines at $x = 4$ and $x = 6$, but he couldn't determine the y-values even though he had just plotted them. He translated the points from the table to the graph, but was not able to transfer the new information learned from the graph back to the table.

Unconventional understandings of periodicity. The prototypical images of periodic functions in Ike's concept image were continuous, probably sine and cosine, witnessed by his confusion on the piecewise defined functions and lack of confusion on the continuous functions in Tasks 3-8. (See Figure 43) Ike also demonstrated an expectation for continuity in Task 1 (See Figure 42). In this task, instead of leaving a blank space at the end of the plotted points, he circled the area and labeled it, "unknown." He expected that the line continued in the "unknown" area rather than stopping and starting again.

Which of the following could be periodic. What could the period be? If it could not be periodic, describe or draw more of the graph. If it could not be periodic, explain.



<u>Figure 43</u>: Ike's Answers to Tasks 3-6

For readability, Ike's explanations are typed in italics.

In order to talk with the interviewer about the discontinuous functions (Tasks 3, 5, and 6), Ike came up with the notion of a "broken period."

Ike: So you just want me to extend the graphs?

Interviewer: Uh huh.

Ike: {*working on task 5*} Does this line keep going up or does it stop here?

Interviewer: It could be either one.

Ike: So that's why you left it. {*struggling*} It looks like one whole period, but I can't tell if it goes farther or not. If it goes up like this, then does it start over? {*On the step function, Task 6, he changed his mind thinking that it couldn't be periodic at first.*}

Ike: Can you have a broken period?

Interviewer: I'm not sure what you mean.

Ike: Well on these curved lines, do they attach somewhere or is it broken {*pointing to the asymptotes in Task 3.*}

Interviewer: It's broken

Ike: Can it still be periodic?

Interviewer: Think about the tangent.

Ike: OK. {*He changed his answer to 6 showing it could be lines of steps. He thought the period could be infinity.*}

Ike: Can a straight line have a period?

Interviewer: What do you think?

Ike: Well it repeats the same thing all the time.

Interviewer: Then what do you think is the period?

Ike: Well it could be infinity, or maybe zero.

Interviewer: Why those?

Ike: 'Cause it repeats all the time.

Once he thought about the tangent, he was comfortable labeling the graphs in Tasks 3 and 5 as periodic. However, in Ike's concept image of periodic functions, he did not have a conception of a "broken period." This shows that Ike's conception of periodic functions probably consists of sinusoids and does not include tangents. However, once the interviewer brought up the tangent it allowed Ike to think of a function that was periodic and had a broken period. This shows that Ike knew that the tangent was periodic, but that this information was not strongly connected with his concept image of periodicity. Ike's concept image of periodic functions was probably compartmentalized.

Ike's answer to Task 6 was interesting (see Figure 43). He felt that the graph was not periodic unless it looked more like a traditional step function. He extended the "steps" in parallel groups. Then he wrote, "in the given form only ∞ period." When Ike was questioned about his answer, he said that it had to be "going on forever" in order to be periodic.

When asked to show one period of the function he traced his finger up one of the rows of steps and said, "this is one period" and then he traced up the other and said, "this is another one." He created a repeating pattern that is "going on forever," but he disregarded the idea that a periodicity was only defined for functions.

He saw the original graph of Task 6 as stopping unless it was extended like a step function. He was trying to reconcile what he knew about periodic functions, that they "go on forever" with the unfamiliarity of the step-like-function. He felt the need to make the graph familiar by extending the steps upward. This allowed him to create a more familiar looking function that repeated. His expectations of familiarity are also demonstrated in Tasks 2, 3, and 4. In these tasks, he felt comfortable extending the graphs because "they looked like periodic functions."

Ike's Concept Image

Ike's concept map is in Figure 44. Ike showed a preference for the use of equations. He used equations 46% of the time and graphs only 10% of the time in his classwork. When Ike used a graph to solve a problem in the classwork, 71% of the time the problem dealt with periodic functions. He did not, however, show a preference for using any particular representation to solve problems dealing with periodic functions.

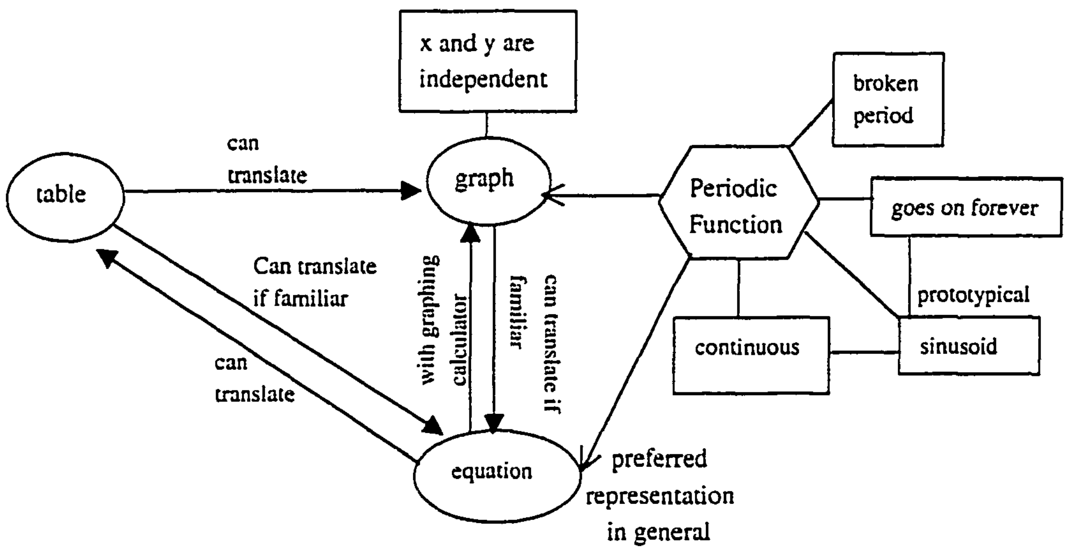


Figure 44: Ike's Concept Map

Ike was able to translate information from an equation to a graph (using the graphing calculator) or table, or from a graph or table to an equation if familiar. However, he could not translate from a graph to a table even when he had plotted the points himself. He did not seem to connect the *x*- and *y*-values on a graph. Ike did not explicitly connect tables with the other representations. Ike believed that all functions could be represented by equations. Ike did not explicitly connect function with its definitions.

Ike's prototypical periodic functions were sine and cosine. He generalized that periodic functions were continuous. However, he was able to construct the idea of a "broken period" to deal with discontinuous functions. Ike had the idea that periodic functions "go on forever." He also revealed that he believed a graph could be periodic even if it did not represent a function.

Beth's Interviews.

Beth was a popular senior. She felt that mathematics was her weakest subject. Beth seemed comfortable in the interviews, but often became confused. She tried to make sense of the tasks and her answers in spite of her confusion and tried to do each problem without help if she could. When she did ask for help, she wanted to understand. She didn't want just an answer.

Compartmentalization in understanding of function notation. Beth had trouble interpreting function notation in the context of a table or graph. She demonstrated this in Tasks 1, 2, 13–18, 22-24, and 28. In Task 1 Beth was given a tabular representation of $h(x)$. She did not understand what the table represented. "At first I thought it was x times h, but that didn't work, so then I thought the period of 5 is $5h$, but that didn't work." Beth struggled to understand what $h(x)$ could mean in the tabular representation. Beth's attempts show that she was trying to find an equation in h and x that would represent the data points from the table. When she said, "I thought the period of 5 is $5h$," she was confusing the idea of slope with period. Beth did not have a reference in her concept image for $h(x)$, she was trying to connect it to something but was not successful. However, when $h(x)$ was replaced with y, she was able to make sense of the table.

> **Interviewer**: Would it make more sense if you replaced h of x with f of x?
> **Beth:** Not really.
> **Interviewer**: How about if you replace it with y?
> **Beth**: Ok, I get it. This is the x and this is the y. {pointing to the left column and then the right} I'm stuck. How do I know what y is?

Even though Beth had made sense of the table, she was still unable to proceed without an equation.

Beth also showed a lack of understanding of function notation on Task 2. In this task, Beth was given a graph and period of $f(x)$ and was asked to find $f(6)$. Beth extended the graph to $x = 7$, but was unable to continue without help (See Figure 45). Beth did not understand the meaning of $f(6)$ in the context of the graph. Once the interviewer translated $f(6)$ into the context of the graph, Beth was able to proceed.

> **Beth**: I don't get what you are asking for.
> **Interviewer**: I want you to find the value of the function at $x = 6$.
> **Beth**: Ok, It is right here. {*she circles the value on the graph*}

The graph below represents $f(x)$. If it has a period of 2, what is $f(6)$?

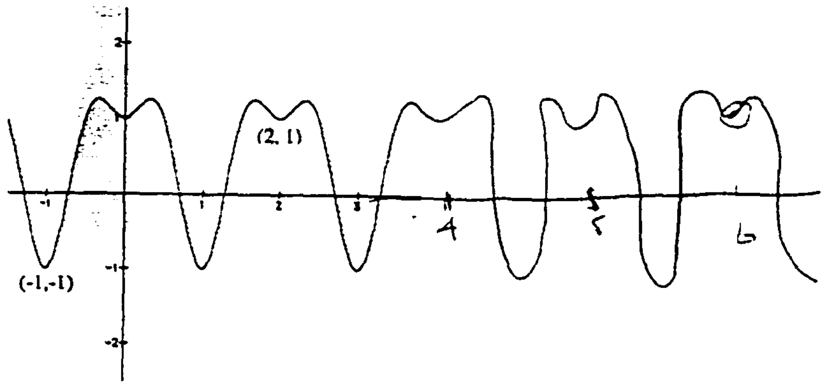
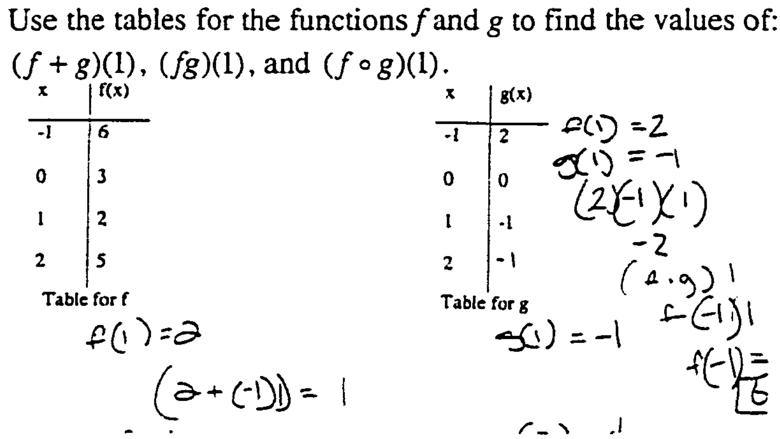


Figure 45: Beth's Answer to Task 2

Beth further showed a misunderstanding of function notation by consistently interpreting the notation $f(x)$ to be *f* times *x* when in the context of combining functions. That is she wrote algebraic statements such as: $(f+g)(x) = [f(x)+g(x)]\cdot(x)$. After reading Tasks 13-18, Beth did not understand what was meant by $(f+g)(x)$ (see Figure 46). She asked the interviewer to explain. The interviewer worked through $(f+g)(2)$ with her.

Interviewer: $(f+g)(2)$ just means $f(2)+g(2)$.
Beth: Ok.
Interviewer: So we need to find $f(2)$ and $g(2)$ in the table. So we look for $x = 2$ and find the *f*-value that goes with it. {pointing to table} So $f(2)=3$ and $g(2)=-1$. So $(f+g)(2)$ is $3+(-1)$ which is 2.
Beth: {*trying to find* $(f+g)(1)$ } So $f(1)=-1$ and 6.
Interviewer: No, It's like this one you did before {*pointing to her scrap paper*}. You wanted $(f+g)(2)$ so you found $f(2)$ and $g(2)$.
Beth: 1 points to 2 in this one and -1 in this one. So I add the 2 and -1?
Interviewer: Uh huh.
Beth: Then do I multiply by 1?

Interviewer: No, it's f of 1 and g of 1.


Use the tables for the functions f and g to find the values of:
$(f+g)(1)$, $(fg)(1)$, and $(f\circ g)(1)$.

x	f(x)
-1	6
0	3
1	2
2	5

Table for f

x	g(x)
-1	2
0	0
1	-1
2	-1

Table for g

Figure 46: Beth's Answer to Task 13

The interviewer's statement that you don't multiply by 1 because "it's f of 1" did not seem to have meaning for Beth. She did not understand how $f(1)$ was represented in the table. And she did not understand what $f(1)$ meant. Her first inclination was that $f(1)$ was the first two values, -1 and 6, in the first table. When the interviewer referred her back to a work done for $(f+g)(2)$, she was able to locate $f(1)$ and $g(1)$ with the correct values in the table, but she just ignored the x-values rather than giving them meaning. $f(1)$ to Beth meant to find the $f(x)$ value in the table that had 1 next to it and multiply it by 1. This is illustrated in the work for Task 13 (see Figure 46).

Again, Beth shows the same conception of $f(x)$ as f times x is shown again in her answer to Task 28 (See Figure 47). In each of these she multiplied by the x-value. Each of these problems involved combining functions given in an algebraic representation.

Unconventional use of notation in combining functions. Beth was the only student who calculated $\sin[(1/2)\pi]$ correctly in Task 27 (see Figure 47). Although she demonstrated that she could correctly use function notation in an algebraic representation in Tasks 25- 37, she was unable to use it correctly when combining functions. This showed that her ability to use function notation depended not only upon the representation the problem was presented in, but on the content of the problem as well. Beth was familiar with function notation within an algebraic representation, but was not familiar with combining functions within any of the representations. In more familiar content areas her knowledge of function notation was accessible, but it was not accessible in the less familiar setting of performing operations on functions.

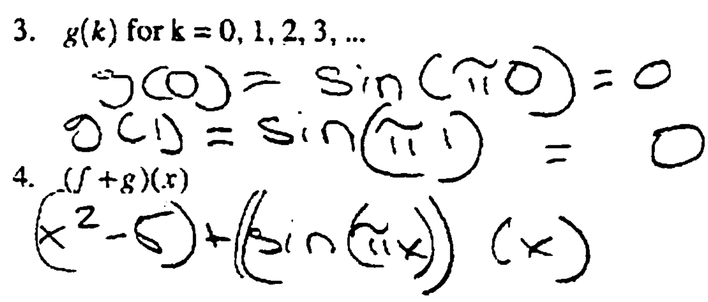


Figure 47: Beth's Answers to Tasks 27 and 28

Beth demonstrated a similar understanding of function notation in all three representations when combining functions. That is, she interpreted $(f+g)(x)$ to be $f+g$

times x. This implies that her knowledge of combining equations is not compartmentalized within representations.

Difficulty with graphical representation. Beth had difficulty interpreting points on the graph. She did not seem to be able to connect the x-values and y-values together. Although Beth understood Task 1 once $h(x)$ was replaced with y, she was unable to begin a solution (see Figure 48). She was unable to write anything until the interviewer suggested plotting the points.

The function represented in the table below is periodic. It has a period of 5. Fill in the missing values.

x	h(x)
-3	2
-1	-1
1/2	0
1	3
4	- 1
6	3

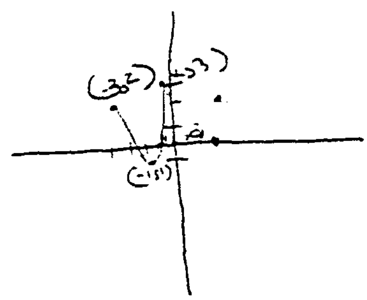


Figure 48: Beth's Answer to Task 1

Interviewer: What do you wish you knew in order to solve the problem?
Beth: The equation
Interviewer: What would you do if you had the equation?
Beth: I would plug in 4 and 6 and get the answer.
Interviewer: Can you think of another way to solve the problem?
Beth: What do you mean?
Interviewer: Do you think drawing the graph might help?
Beth: I could try. {Beth starts plotting the points, but she keeps getting confused between x's and y's. She seemed to want to combine or take apart numbers at random.}

After being prompted to draw a graph, Beth had trouble plotting the points. Her tick marks were not evenly spaced. She plotted the points almost as if the given x-values were evenly spaced. She plotted (1/2, 0) and (1, 3) to the left of the vertical axis. Even after the interviewer suggested that some of her points were plotted incorrectly, she did not correct her graph. In the area to the right of the last plotted point, she picked points that would continue the shape as she drew it, without referencing the x-values. Because she had no reference for the x-values, she could not fill in the missing values. Note that in her work the values are filled in. This didn't happen until the end of the interview when Beth asked the interviewer to solve the problem. It was during the interviewer's solution that Beth filled in the table.

Beth's answers to the interview tasks revealed that her concept image of function was compartmentalized within representations. She was unable to connect the points in her graph with the values in the table, even though she had just plotted them. She was able to see that there were points on the graph beyond the data points from the table, however she could not reference their location with x- and y- values. Only one representation was available to her at any given time.

As Beth struggled to complete Task 2, she again revealed a difficulty interpreting points on the given graph.

Interviewer: So what is $f(6)$?
Beth: It's right here.

Interviewer: Can you write down the value?
Beth: I don't know what you mean.
Interviewer: The problem asks for *f(6)*. Could you write *f(6)*=something. That is can you find the value of *f* at *6*?
Beth: It's right here.

The interviewer asked her the value of the function at $x=6$ and she circled the value on the graph. However, she did not find a numerical answer. This shows that she did not see the connection between the x-values, y-values and the graph. In addition, Beth did not understand that $f(6)$ had a numerical value.

Beth may have objectified the graph to the point that she was unable to view it as a collection of points. However, given her answer to Task 1, it is also likely that she did not know how to reference a point on the graph. Although she was able to plot points somewhat accurately, Beth was not able to do the reverse operation. Beth did not seem to have a rich concept image of the graphical representation. She had difficulty interpreting the information represented in the graph.

In addition, in Task 2, Beth was unable to extend the graph consistently (see Figure 45). Her tick marks were wider than the tick marks on the given portion of the graph and although her curve basically followed the same pattern as the given portion of the graph, it was not the same size.

<u>Evenly spaced x-values in tables.</u> In Tasks 9 - 12, Beth did not pay close attention to the values on the x-axis. She tended to make the x-values evenly spaced on the graph

even when the x-values in the table were not evenly spaced. This gave her units of unequal length. If a graph looked like a familiar periodic function (i.e. a sinusoid) she deemed it periodic.

> **Interviewer**: Why do you think these are periodic?
> **Beth**: Because it looks periodic.
> **Interviewer**: Because it looks familiar?
> **Beth**: Yeah.

Ease in finding equations for familiar functions. In tasks 22-24 Beth, showed a greater understanding of the graphical representation than she had previously (see Figure 49). She was able to find an equation for the table and the graph in Tasks 22-24. She found the equation for the function *f*, represented in a table, by first graphing the function and noticing that the points formed the horizontal line $y = 2$.

> **Beth**: I don't know how to do it because I don't know how to get this [*f*] to look like *g*.
> **Interviewer**: So you would like an equation for *f*.
> **Beth**: Yes.
> **Interviewer**: Can you think of a way to get an equation?
> **Beth**: Can I draw the graph?
> **Interviewer**: Yes.
> {*She starts by mixing up x and y. Once corrected she draws it.*}
> **Interviewer**: Does that look like anything?
> **Beth**: A straight line.
> **Interviewer**: So what would the equation be?
> **Beth**: $y = 2$?

She found the equation for the graph of *g* by recognizing it as the common parabola $y = x^2$ only shifted down 1 unit. Therefore, despite her difficulties in plotting points and understanding the graphical representation, Beth was able to translate between a table and an equation and between a graph and an equation when the function was familiar.

Given the functions, *f*, *g*, and below, find $(f+g)(x)$, $(f+h)(x)$, and $(g+h)(x)$. Write an equation, draw a graph or make a table to show your results.

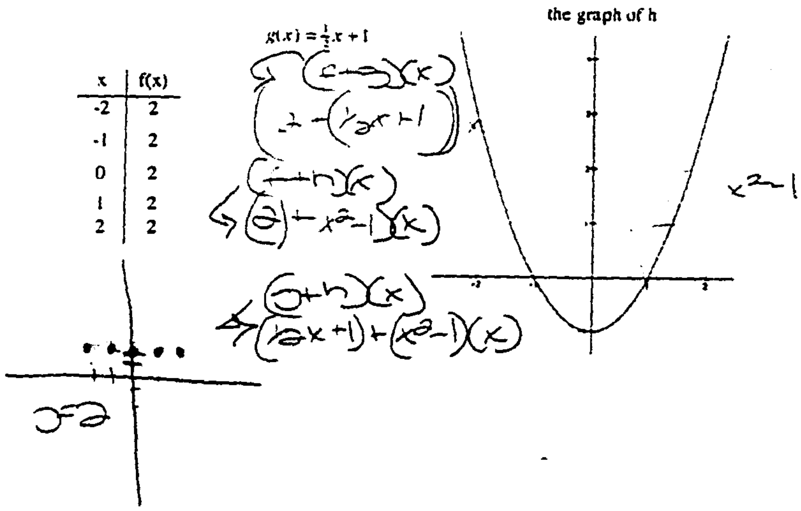


Figure 49: Beth's Answer to Tasks 22-24

Compartmentalization in periodic functions. Beth showed a different understanding of periodicity with familiar functions than she did with unfamiliar functions. Beth's answers revealed some understanding of periodicity in the graphical representation. Beth was able to extend graphs to show more than one period in Tasks 2, 3 and 5 (See Figure 50). In all three tasks she was able to see a pattern and repeat it appropriately. In addition, in Task 3 Beth was able to give the smallest possible period.

Which of the following could be periodic. What could the period be? If it could be periodic, describe or draw more of the graph. If it could not be periodic, explain.

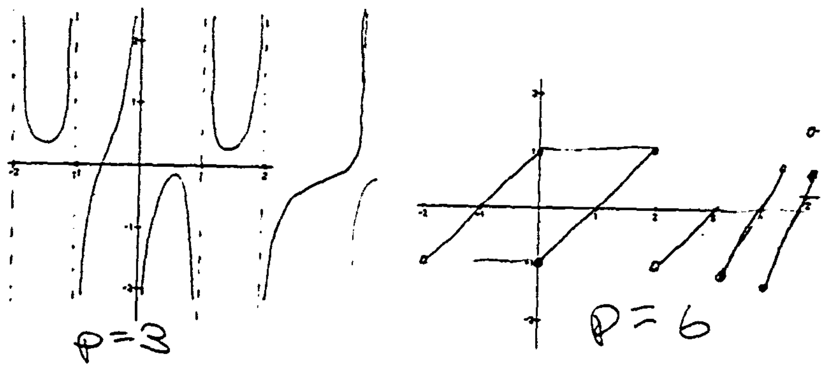


Figure 50: Beth's Answers to Tasks 3 and 5

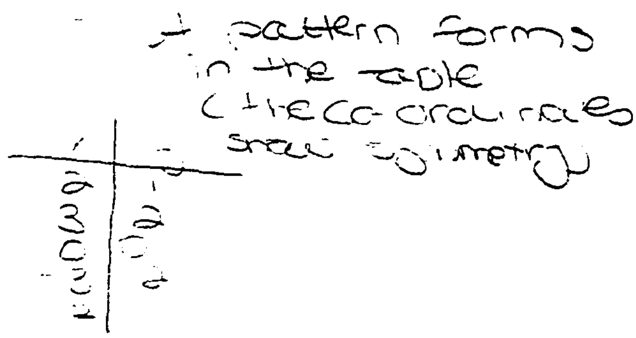
Beth was unable to start Task 2 on her own. The interviewer asked her to think about a more familiar problem. The dialogue follows.

Beth: How can I find out the period?
Interviewer: It is in the problem, the period is 2.
Beth: Yeah, but I don't see how that helps.
Interviewer: How about if we look at something familiar. Here is the graph of $\sin x$. What is the period?
Beth: 2π.
Interviewer: Ok, can you show that on the graph? {*Beth points to the graph at 0 and* 2π.} Is there anywhere else you can show the period? {*She points just beyond the two peaks at* $\pi/2$ *and* $5\pi/2$.} Does that help you to solve the problem?
Beth: I think so.

Beth showed that she could identify the period of the graph of $f(x) = \sin x$ even though she was confused by the idea in the unfamiliar graph presented in Task 2.

Beth also knew that a pattern would form in a tabular representation of a periodic function. However, her answer to the brainstorming question on periodic functions that she had completed in her first interview shows that she expected symmetry, about the *y*-axis in the table rather than a repeating pattern with respect to the *y*-axis (see Figure 51). This answer is inconsistent with the understanding she showed in the graphical representation. In Task 3, she did not require the graph to be symmetrical about the *y*-axis, nor about the asymptote. Instead she shifted the graph to the right an appropriate number of units. This shows Beth's concept image of periodicity is compartmentalized within representations. The knowledge Beth has about periodicity is different depending on the representation she is using.

Brainstorm with your partner about the following idea.
What are the most important ideas in this topic and what other things are connected to it? PERIODIC FUNCTION



<u>Figure 51</u>: Beth's Answer to Brainstorming Task 3

Beth's Concept Image

Beth preferred to use equations in both the classwork and the interview tasks. (See Figure 52) She used equations to solve classwork problems 52% of the time and used graphs only 10% of the time. When Beth used a graph to solve a problem in the classwork, 85% of the time the problem dealt with periodic functions. She did not, however, show a preference to use any particular representation to solve problems dealing with periodic functions.

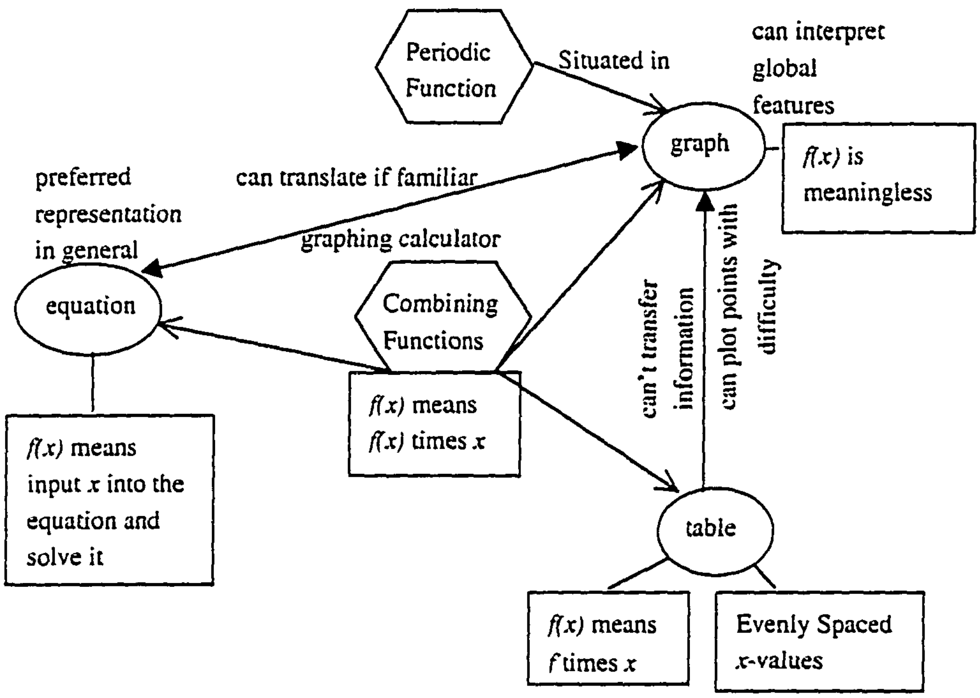


Figure 52: Beth's Concept Map

Beth's understanding of function notation was only accessible to her in the symbolic representation. She was not able to interpret $f(a)$ in a graphical or a tabular representation. In addition, Beth interpreted function notation correctly when solving algebraic tasks involving one function, but interpreted it to mean f times x when the task involved operations on functions. Not only was her understanding of function notation situated in the algebraic representation, but it was also interpreted correctly only in a familiar setting. Beth's concept image of function was compartmentalized within representations.

Beth also revealed that she was unable to interpret local features of a graph, such as the value of f at x. However, she was able to interpret global features such as periodicity in a graphical representation. Beth was able to interpret periodicity in an algebraic representation. Beth had more connections in her concept image of periodicity than in her concept image of function notation. However, the misconceptions and limitations on her understanding of function notation limited her understanding of periodicity as well.

Beth showed a lack of understanding of function notation through many of the problems. The more familiar the situation was, the less likely she was to make a mistake with the notation. This was probably due to the fact that the familiar problems were routine and automated to a certain degree and therefore did not take up as many cognitive resources.

When Beth attempted to translate from a table to a graph she interpreted the x-values as being evenly spaced even when they were not. Once she plotted points and connected them with a line, she was not able to interpret the line as a collection of points. That is, she was unable to find the value of the function at a certain x-value. Beth was also unable to transfer ideas such as periodicity from one representation to another. If the function was familiar, she was able to translate from a graph to an equation. In addition, if a familiar function was given in a table with evenly spaced x-values, Beth was able to translate it into a graph.

Steven's Interviews

Steven was a senior who was really struggling with the concepts of pre-calculus. Even with help, he was non-committal and required affirmation at every step. Steven had trouble sitting still. He had a hard time staying focused on the lesson or on a problem he was solving.

Steven had switched classes at the beginning of the 3rd quarter because he was struggling to understand the other pre-calculus teacher. He had been in Mrs. Brown's class for a month when the observation period started. He seemed to lack much of the pre-requisite knowledge for 3rd quarter pre-calculus as witnessed in both his poor performance on the collected classwork and on the interview tasks. It was difficult to tell whether this difficulty stemmed back to previous mathematics classes or was the result of not understanding the first two quarters of pre-calculus in the other class.

Lack of understanding of function notation. Steven was not able to demonstrate an understanding of function notation. In Task 1, Steven acted confused and constantly asked the interviewer for approval of his work (see Figure 53). With much prompting from the interviewer, he was finally able to plot the points correctly and extend the graph, by drawing in another cycle of the period. However, he was never able to identify the points on the graph that would correspond with the missing values in the table. His first understanding of the notation was that $h(x)$ meant *h* times *x*.

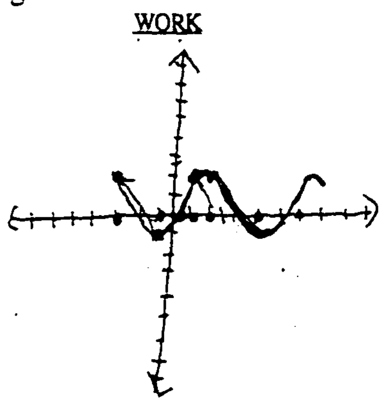
> **Steven**: So it's *h* times *x*?
> **Interviewer**: It might be easier to think of it as $f(x)$.
> **Steven**: So its –3 times something to get 2?
> **Interviewer**: It doesn't have to be times, it could be log or cosine.

Steven was confused by the notation, $h(x)$. Replacing the *h* with an *f* did not help him make sense of the problem.

Lack of connection between *x*- and *y*-values on a graph. Steven is not able to reference points on a graph with the values along the *x*-axis. In Task 1, although he drew the tick marks along the *x*-axis, he did not refer to them when extending the graph (see Figure 53). He just tried to repeat the shape visually. Even though he had extended the graph over his tick marks for $x = 4$ and $x = 6$, he did not reference these tick marks. He only knew that the new part of the graph was "somewhere over here." He knew that he needed to add 5 to something in the table, but he could not figure out what.

The function represented in the table below is periodic. It has a period of 5. Fill in the missing values.

x	-3(x) h(x)
-3	2
-1	-1
1/2	0
1	3
4	
6	



<u>Figure 53</u>: Steven's Answer to Task 1

> **Interviewer:** Can you use the graph you have just drawn to answer the question?
> **Steven:** Oh, so I can just add 5 to these, right?
> **Interviewer:** Uh huh.
> **Steven:** {sits with a puzzled look on his face for a few moments} I can't do it.

Once he plotted the points correctly, he was unable to extend the graph using tick marks on the x-axis, or by translating the points he had already drawn 5 units to the right. Once the line was drawn in, there were no points to reference, so he could only talk in general terms.

Steven was not able to reference the graph with the x-values in Task 2 as well. In Task 2, Steven struggled to extend the graph so that he could find the value at $x = 6$ (see Figure 54). He was able to draw three more cycles of the period, but had a difficult time referencing the values on the x-axis. He labeled the axis and graph in two different ways, neither of which was correct. He drew tick marks that had different spacing than the pre-existing tick marks, labeling on the x-axis each valley. Then he labeled the peaks and valleys consecutively beginning with 3 one unit past the 3 that was already labeled on the x-axis.

> **Interviewer**: Can you tell me what you have done so far?
> **Steven**: I'm trying to label the graph. I don't know where to put it.
> **Interviewer**: I'm not sure what you mean.
> **Steven**: I just don't know how to do it.
> **Interviewer**: Ok, let me see if I can help you. If you look at the points that are labeled on the graph and follow the pattern, then you can label the other points.
> {*The interviewer interpreted Steven's confusion in labeling the points as a difficulty in relating the x and y coordinates and thought that labeling the points with ordered pairs might help. She began to label the peaks and valleys on the original graph with ordered pairs*}
> OK, This one is (2, 1) and this one crosses the y-axis at the same height so it must be (0, 1). Then this one down here is (-1, -1) and if you look at the values on the x-axis and how far down it goes you can get (1, -1) and (3, -1) (see Figure 54).

The graph below represents $f(x)$. If it has a period of 2, what is $f(6)$?

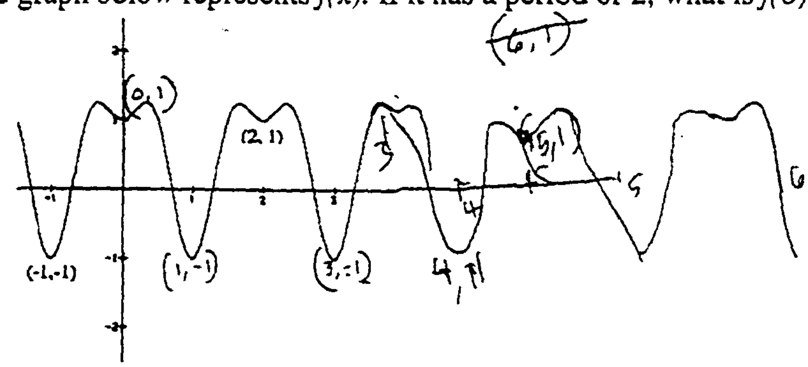


Figure 54: Steven's Answer to Task 2

Just before the end of the interview, the researcher helped Steven to count out the coordinates of the peaks and valleys one by one, ending at $x = 6$. Steven wrote (6, 1) on his paper and immediately crossed it out.

Steven's inconsistency and inability to follow an evenly spaced pattern, reveals that he did not understand the Cartesian coordinate system. He did not expect tick marks to be evenly spaced; he didn't connect the x-values with the y-values. He tried to follow a numerical pattern, but there is some randomness to his selection as shown by his various labels on the graph and on the x-axis. This lack of understanding is also revealed in the way that he plotted the points in Task 1, by plotting each coordinate separately along the x-axis.

Unconventional understanding of Periodicity. In the course of solving Task 1, Steven demonstrated that he believed that a periodic graph "makes something" and then starts over. He was able to verbally express that since the period is 5 the repeat is "5 away." But he was unable to determine the exact point at which the graph started to repeat.

> {*continuing Task 1*}
> **Steven**: I don't get it. How does it have a period of 5?
> **Interviewer**: {draws two cycles of $y = \sin x$ } What is the period of this one?
> **Steven**: The period is the time it takes to make something.
> **Interviewer**: Does that help you solve the problem?
> **Steven**: I could probably graph it. Is this a parabola? {He starts to plot points}

At this point the interviewer chose to continue with the problem to see if this could reveal more about Steven's understanding of periodicity.

> {*continuing*}
> **Interviewer:** OK, now can you tell what would be happening out here if you know the period is 5.
> **Steven:** Oh, if it is 5 away, then this can just start over. Does it start over right here?
> **Interviewer:** Yes.
> **Steven:** OK, so I can draw this part and it will start over somewhere over here?

Steven demonstrated that he understands that periodic functions repeat. He understands that the repeat "is 5 away." But he is not able to determine exactly where the repeat starts. He recognizes that the graph starts over and is able to draw another cycle on the axes.

In interview tasks 9-12, Steven graphed each of the data points in each table (see Figure 55). He then determined which graphs could represent periodic functions by familiarity.

> **Interviewer**: Can you tell me why these could be periodic and these can't
> **Steven**: They just look periodic.

It is interesting to note, which graphs "looked periodic" to Steven. He extended the linear function by curving the line back up to show a curved V-shape. This he said could be periodic and he used the symmetry of the data points to find the next value. Steven also thought that the table in Task 11 could be periodic because the graph "looked periodic." Given the wave-like features of his graph, he was probably comparing it with sin x. He did not notice that the distance between the first two data points was 2 units instead of one. He continued the graph and table as if the last two points were repeating the pattern.

Determine if each of the following tables could represent a periodic function. What could the period be? If it could not be a periodic function, explain.

x	f(x)
-1	5
0	4
1	3
2	2
3	1
4	0
5	1

x	f(x)
-2	-1
0	0
1/2	1
2	0
4	-1
5	0
6	1

x	f(x)
-2	0
-1	1
0	0
2	0
4	0
5	1

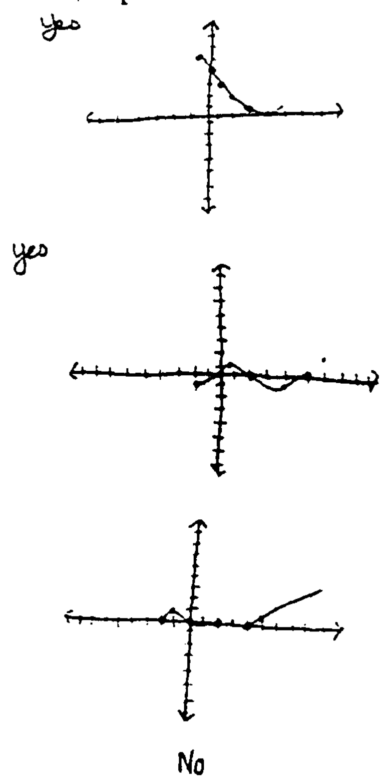


Figure 55: Steven's Answers for Tasks 9, 11, and 12

Steven believed that the other two functions were not periodic. In Task 10 (not shown) he thought that the graph looked strange and "smushed". The lack of symmetry in the peaks and valleys of the graph bothered him. In number four, he drew the data points and connected them with straight lines. The last two points he connected with a straight line and said that the graph didn't look periodic.

His question about the parabola on Task 1 is interesting when compared with his answer to Task 9 (see Figures 52 and 54). It is not clear whether Steven has a parabola in his concept image of periodic functions, or if a parabola is something that he is comfortable graphing, or if some other trigger made him think of a parabola. Given his answer to Task 9, it may be the case that Steven sees one peak or valley of the sine or cosine curve as equivalent to a parabola.

Steven's answers show that his concept image of periodic functions includes sinusoids as familiar as well as possibly parabolas. His answers on Tasks 1 and 2 also show that he has generalized that there are symmetries in periodic functions in the peaks and valleys.

Steven's Concept Image

Steven's concept map is in Figure 56. Steven preferred to use equations regardless of the content of the problem. He used equations 41% of the time and used graphs 10% of the time in the classwork. Steven also showed that he preferred to use equations with non-periodic functions more than with periodic functions. He did not show any other preferences.

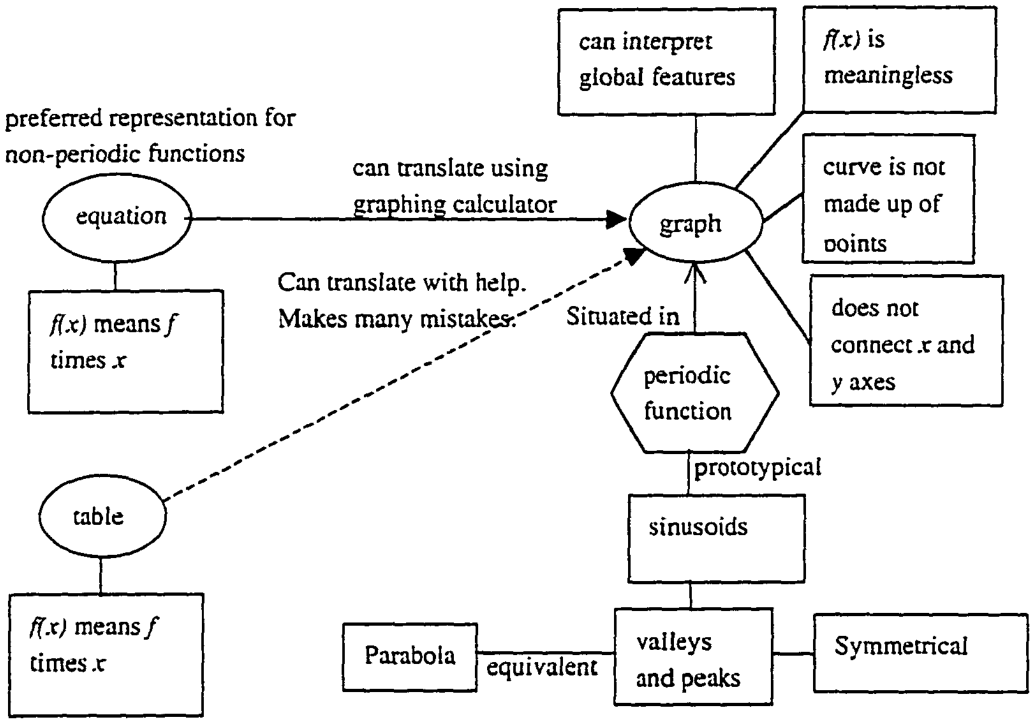


Figure 56: Steven's Concept Map

Steven had difficulties in interpreting function notation no matter what representation was presented. He interpreted $f(x)$ in a table or equation as f times x. He was unable to interpret $f(x)$ in the context of a graph.

Although Steven was able to describe global features of a graph, he did not connect x-values and y-values. He was unable to reference the values on a curve with the values on the x-axis and y-axis.

Steven classified a function as periodic if it looked familiar. He expected periodic functions to have symmetrical peaks and valleys. He also connected parabolas to the shape of the peaks and valleys. He described the period as the time it takes to make something.

Michael's Interviews

Michael was a quiet student who preferred to work on his own. He worked very quickly and used his graphing calculator extensively. When working in a group, he tended to monopolize the work. He worked more quickly than his classmates and would do the classwork for the group.

Ease in finding equations for functions. Michael preferred to use equations that he programmed in his graphing calculator to solve problems. He was able to translated tables and graphs into approximate or exact equations. Although his equations were not always correct (as in Tasks 13-15), he was able to use educated guessing to get the correct answer. This was particularly true when the answer was in graphical form. For example, in Tasks 19-21, Michael approximated the graph of f to be $f(x) = \sin x$. Even though this was not the correct equation, the shape was similar. He used this function and his graphing calculator to find the shape of the various combinations of f and g, and then drew them in on the axes provided.

Use of graphing calculator to estimate functions. In Tasks 13 – 15, two functions, f and g, were given in tabular form. The students were asked to find $(f+g)(1)$, $(fg)(1)$, and $(f \circ g)(1)$ Michael plotted the points in order to determine an equation to represent each table. Although the graphs weren't completely familiar, Michael was able to come up with equations that fit the data in the tables except at one point. "This first one looks like a parabola. It might be, but it might not be. But it looks like one, so I put the parabola in the calculator. The other one is the same -1 times x. So, I just put both in the calculator and tried it." He interpreted $f(x)$ as the parabola $f(x)=(x-1)^2+2$ (see Table 15) This function satisfied every data point in the table except for (2, 5). He wrote the equation $g(x)=-x$. This equation satisfied every data point in the table except (-1, 2). He used these functions to find $(f+g)(1)$, $(fg)(1)$, and $(f\circ g)(1)$. Since both equations were correct at 1, he was able to get the correct answer using this method.

Table 15: Tables for Tasks 13-15

x	$f(x)$
-1	6
0	3
1	2
2	5

x	$g(x)$
-1	2
0	0
1	-1
2	-2

Michael used his functions again on tasks 15-18 which asked for tables representing $(f+g)(x)$, $(fg)(x)$ and $(f\circ g)(x)$. Since Michael's equations didn't quite fit the data, his answers were incorrect at $x=-1$ and $x=2$.

Conventional understanding of periodicity. Michael was able to see both piecewise-defined functions in Tasks 5 and 6 as periodic and give the smallest possible period (see Figure 57). Michael was the only student who did not struggle with defining discontinuous graphs as periodic.

Which of the following could be periodic? What could the period be? If it could be periodic, describe or draw more of the graph. If it could not be periodic, explain.

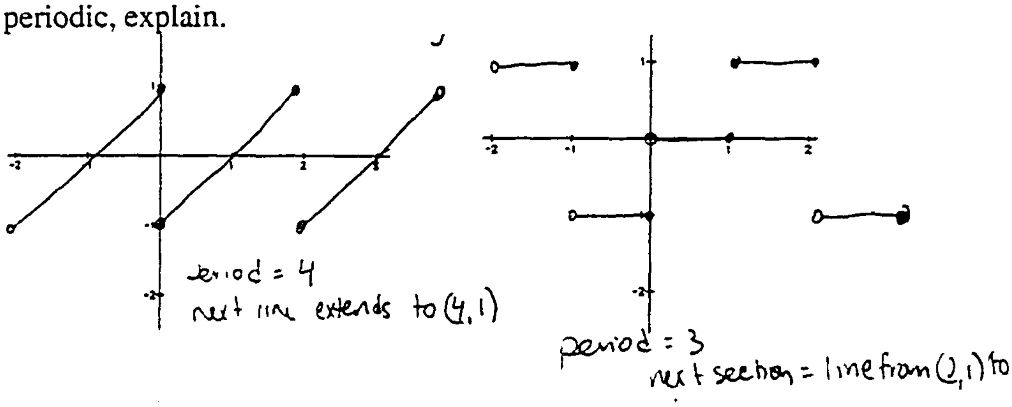


Figure 57: Michael's Answers to Tasks 5 and 6

Michael's answer to Task 12 was interesting in that it was different from all of his classmate's answers. (See Figure 58) Although others answered that this function could be periodic, Michael connected the data points with a sinusoidal wave rather than just connecting the dots with line segments. This showed that he recognize the pattern in the data points as being similar to a familiar sinusoidal function. However, the graph he drew does not continue exactly as a sinusoid would. This shows that his concept image of periodicity allows for unfamiliar looking graphs as well as familiar ones.

In Task 32, students were asked to combine periodic functions together to see if they could make other periodic functions. Part of Michael's answer is in Figure 58. In his experimentation, Michael found several functions that were symmetric and "went up and down." He was not sure if these functions were periodic or not, but he did recognize that they were symmetric. He said, "they look like periodic functions but they are symmetrical. I'm not sure if they are periodic because they don't exactly repeat. After thinking for a short time, he said, "they should repeat exactly if they are periodic." Although these functions shared some common features with Michael's prototypical periodic functions, he decided that they were not periodic.

$\sin x$, $\cos x$, and $\tan x$ are periodic functions. Is it possible to get new periodic functions by combining 2 periodic functions (adding, subtracting, multiplying, dividing, and composing)? How about if you combine a periodic function with a non-periodic function?

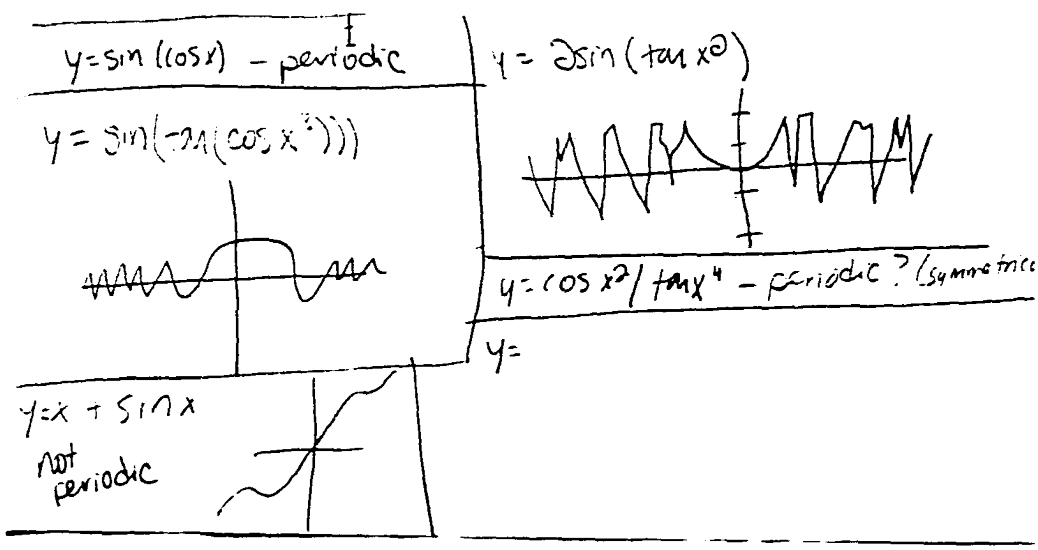
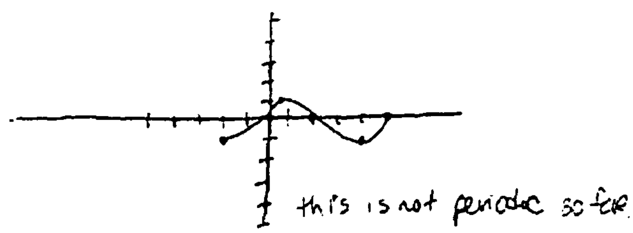


Figure 58: Michael's Answer to Task 32

Use of graphical representation to determine periodicity. In Tasks 9-12 Michael translated each of the tables to a graph in order to determine whether they were periodic or not (See Figure 59). For those that started to repeat he was able to determine the smallest possible period and extend the graph appropriately.

Determine if each of the following tables could represent a periodic function. What could the period be? If it could not be a periodic function explain.

x	f(x)
-2	-1
0	0
1/2	1
2	0
4	-1
5	0



x	f(x)
-2	0
-1	1
0	0
2	0
4	0
5	1
6	0
9	0

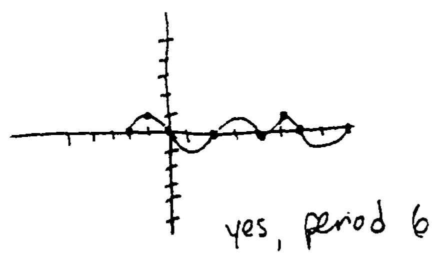


Figure 59: Michael's Answers to Tasks 11 and 12

Flexibility in problem solving. Michael showed that he could re-solve problems using different representations in Tasks 1 and 15-18. In the first case he was able to transfer the knowledge he learned from the graphical representation to the tabular representation. In Tasks 15-18 he was able to transfer information learned in symbolic and graphical representations to a tabular representation.

Michael completed Task 1 correctly by plotting the points and extending the graph (see Figure 60). In addition, Michael was able to explain how he could have used the table to fill-in the missing values. "The period is 5. So if you count back 5 from 4 and 6 you get the y. See if you count back 5 from 4 you get −1 and −1 goes with −1. So 4 goes with −1. And 6 goes with 3 because 1 is 3."

The function represented in the table below is periodic. It has a period of 5. Fill in the missing values.

x	h(x)
-3	1
-1	-1
1/2	0
1	3
4	−1
6	3

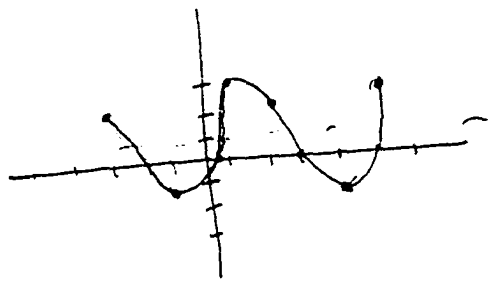


Figure 60: Michael's Answer to Task 1

Michael was able to explain how he could have done Tasks 15–18 in the tabular representation after he solved them symbolically.

Interviewer: Can you think of a way to do this without equations?
Michael: Oh, you could just use the values in the tables.
Interviewer: How would you do that?
Michael: You just add and multiple them together. Like $(f+g)(1)$ would be $2+(-1)$ . . . so 1.

Although Michael recognized, that he could solve the problem using the tables, he had a strong preference for solving problems using equations and his graphing calculator.

Michael's Concept Image

Michael's concept map is in Figure 61. Michael was able to translate between all three representations. He translated most problems into equations and used his graphing calculator to either graph or plug-in values as needed. This is consistent with the preferences he showed in his classwork. Michael preferred using equations in general and on problems dealing with non-periodic functions. He used equations 50% of the time and Graphs 18% of the time. Michael used graphs almost twice as often as his classmates did.

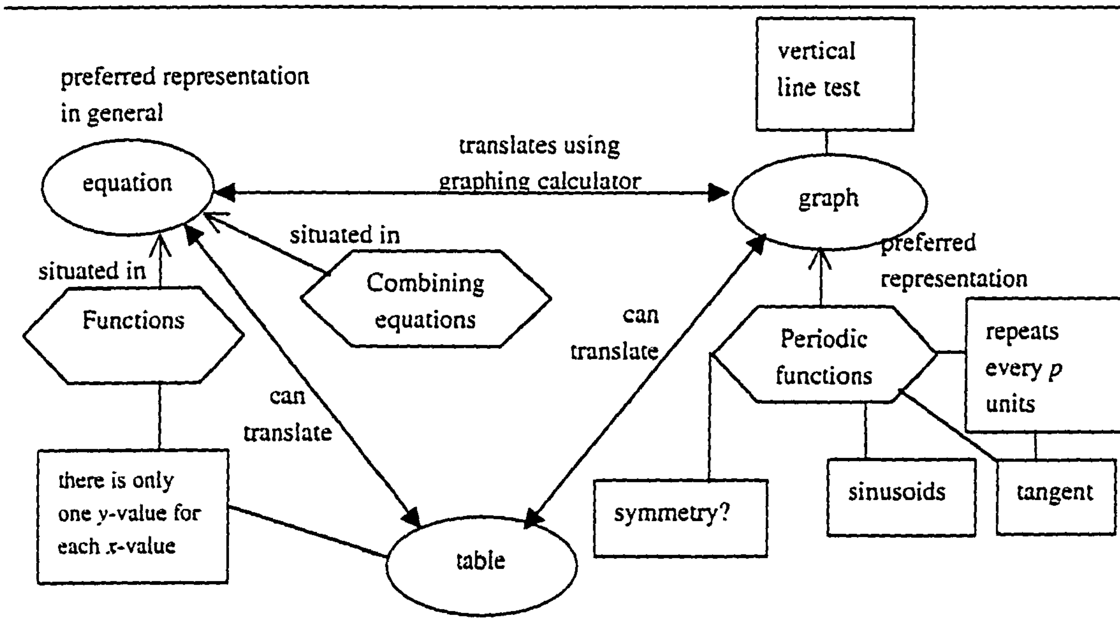


Figure 61: Michael's Concept Map

Michael's understanding of periodic functions was consistent with the conventional understanding. He believed that periodic functions repeat every p units, where p is the period. His understanding allowed for continuous or discontinuous and familiar or unfamiliar function to be periodic.

Although he preferred solving periodic functions problems in a graphical representation, Michael showed, in the clinical interviews that he could interpret period in tables and equations as well.

Michael's understanding of functions and operations on functions were situated in the equation representation. Michael showed a very strong preference for using equations to solve these types of problems even when equations did not get him quite the right answer. However, he was still able to explain how other representations could be used so his understanding was not compartmentalized. In addition, Michael was the only student who recognized the definition of function in both representations and explicitly connected them together.

Michael was one of only two students who were able to see piecewise-defined functions as periodic and give the smallest possible period. Although these graphs challenged most of the other students, Michael did not struggle with them.

In summary, the data collected from classroom observations, the card sorting task, classwork and clinical interviews have been presented and analyzed. Each of these types of data has contributed to the description of the students' understandings of functions and periodicity.

10: Conclusion

Analysis of individual students tells a story about their understanding of functions. However, there is more to tell than just what individuals understood. Where themes emerge, so does a story of the class and their understandings. The culture of the classroom, the student's preferences for representations, and their individual understandings come together to answer the original research questions.

Some Common Themes

Weakest connections with tables. Of the three representations: graphs, tables, and equations, the students made the fewest and weakest connections with tables. In the card sorting activity, all of the students used surface features when connecting "table" with other cards. They all connected "table" with the table representing the squaring function. Beth and Matthew connected table only with equations saying, "a table gives you values for equations." Although Ike and David connected all three representations in their concept map, they did not give each equal treatment. Ike and David connected graph and equation together each time they sorted the cards. They labeled these connections with deeper connections such as "transformation in equation causes transformation in graph." On the other hand, they connected table with graph only once with the surface connection, "can be graphed." Only Rachel connected "table" with "equation" and "graph" and treated all three representations equally in her concept map.

Three of the seven students, David, Beth, and Steven showed unconventional understandings of functions and function notation within the tabular representation. David was unable to translate tables into equations unless they represented linear functions or were familiar. Beth's idea of how a periodic function would appear in a table was inconsistent with the understanding she showed in the graphical representation. In addition Beth and Steven were unable to interpret function notation in the context of a table.

Rachel was the only student who chose to solve problems within the tabular representation without first trying to translate into another representation. Matthew, Michael and Rachel were the only students who were able to demonstrate that they could solve problems within the tabular representation.

The students rarely used tables in solving problems in their classwork. The teacher never used tables in class or on assignments. The students in this class did not have a lot of experiences with the tabular representation and therefore did not make as many connections with this representation.

Weak connections between function and its definitions. Only one student, Michael, connected "function" with its situated definitions: the graphical "vertical line test" and the concept of univalence, "there is only one y-value for each x-value" and said that they were the same. Beth, Matthew and Rachel had unconventional understandings of the vertical line test. Rachel connected "there is only one y-value for each x-value" with "function" and with the table of the squaring function. But Beth and Matthew only connected the concept of univalence with periodic functions. Ike and David connected both definitions with "$y = mx + b$." However, they only connected the concept of univalence with "function." Only Ike and David distinguished between functions and non-functions.

Periodicity situated in graphical representation. Six students, David, Rachel, Matthew, Beth, Steven, and Michael situated their understanding of periodic function in the graphical representation. Ike was more flexible and was able to view periodicity in graphical and tabular representations.

Combining equations sometimes situated symbolic representation. Only Michael and Matthew situated their understandings of combining equations in the algebraic representation. None of the other students showed a preference for a particular representation. David and Rachel were able to solve tasks involving combining equations equally well in each representation. Beth and Steven were not able to solve any of the combining equations tasks in any representation.

The students generalized sinusoids. Four students, David, Ike, Matthew, and Steven made generalizations about periodicity from sinusoids that lead to unconventional definitions of period. These included: a periodic function has symmetrical peaks and valleys, is continuous, "repeats the pattern from where it leaves off," and "looks like a wave". All of the students were able to consider familiar looking discontinuous graphs as periodic, (see Task 3 in Appendix A). However, only 2 of the seven students were able to consider non-familiar discontinuous functions as periodic.

Difficulties with graphs and function notation. Two students, Steven and Beth, did not connect the values on the axes with the curve. They were unable to reference a point on the graph even after they had drawn the graph by plotting points. David, Beth, and Steven all had difficulty interpreting $f(x)$ in a graphical representation. Beth interpreted $f(x)$ to mean f times x in the tabular representation and with combining functions in any representation. Steven interpreted $f(x)$ to mean f times x in both the symbolic and tabular representations.

Revisiting the Classroom Culture

The students created their understandings from their experiences. The most visible example of this is the lack of connections students made with the tabular representation. The teacher did not provide the students with experiences with tables. Consequently, her students made few connections with the tabular representation.

The way the teacher interacted with the students also affected how they interacted with the interviewer. Students would ask for help when they were stuck just as they did in the class. They were not afraid to show their lack of understanding to the teacher or to the researcher.

Students had trouble expressing their understandings. The mathematical interactions that the students were involved in every day were shallow in nature. Usually, the teacher asked for numerical answers and rarely asked the students to explain their thinking. In addition, although the teacher sometimes went into depth when answering a student's question, often she would offer the students a way of checking the answer without gaining understanding.

Revisiting the Research Questions

Question 1. **Is students' knowledge of functions, periodicity, and combination of functions compartmentalized within a particular representation?**

Of the seven students interviewed only three students, Beth, Steven, and David, exhibited compartmentalization within a particular representation. The four other students interviewed did not exhibit compartmentalization within a particular representation. Therefore, while compartmentalization was possible, it was not exhibited in a majority of the students' understandings.

Beth and Steven, exhibited a weak understanding of functions in general and did not perform well on collected work. Beth was a C/D student in pre-calculus and Steven was failing the course. The third student, David, was an A student. All three had a compartmentalized understanding of function notation. All three had a different understanding of function notation in a table than in another representation.

Although David was able to interpret function notation correctly with equations and tables, he was unable to interpret function notation as it applied to a graph. Beth was unable to interpret function notation in the context of a table or graph. Steven interpreted $f(x)$ to mean f times x in the symbolic and tabular representations, but was unable to interpret it in the graphical representation.

Beth and Steven had compartmentalization within representations in their understanding of periodicity. Beth showed inconsistent understanding of periodicity depending on the representation she was using. Steven had a compartmentalized understanding of periodic functions because his understanding was situated in the graphical representation and he was unable to translate it into another representation.

In general, if a student's understanding of functions was compartmentalized within representations it seemed to be weakest in the graphical and tabular representations. Given the teacher's and students' overwhelming use of the symbolic representation, compartmentalization may be related to the student's lack of experience with certain representations.

Question 2. **Do students have a particular representation from which they solve most problems regardless of the content or do they choose a particular representation depending on the content (periodicity or combination of functions)?**

All of the students preferred the symbolic representation. The students used this representation overwhelmingly in their classwork and homework. In addition, most of the students attempted to find equations for the functions in the interview tasks before trying any other solution strategy. However, only Michael was able to solve most of the interview tasks in this representation. He translated every graph he could into an equation and every table into a graph and then an equation. He used his graphing calculator to help him do this. Other students tried to do the same, but were unsuccessful. Michael was successful most of the time.

Question 3. **Are students able to solve problems presented in all three representations? Does this facility depend upon the content of the problem?**

The facility to solve problems presented in all three representations seemed to relate to the students' ability to translate between representations and the existence of compartmentalization in their understanding. Three of the students, Ike, David and Beth had difficulty with problems presented in a particular representation. In addition David and Beth's facility with representations depended upon the content of the problem. The four remaining students' ability to solve problems were not related to the representation or the content.

Matthew, Michael and Rachel were able to solve problems presented in any representation regardless of the content. Steven had difficulty solving any of the problems. His lack of understanding extended to all three representations and all of the content areas studied.

Ike had some difficulty with the graphical representation. He was unable to reference points on a graph with coordinates, but seemed to see the x- and y- values as independent. This limited his ability to solve graphically presented problems where locating or referencing particular points was necessary. This was true regardless of the content.

David had trouble with graphical representations of periodicity. He did not have the same problem with algebraic representations. This may have been due to the fact that he had a misconception about what it means for the values of a function to repeat. In the graphical representation he expected some periodic functions to be symmetric and some to repeat from a different starting point. He carried these same misconceptions when working with equations, but the misconception did not surface because symmetry and starting points are not explicit in the symbolic representation. David also had difficulty interpreting function notation in the graphical representation. David's difficulties with the graphical representation affected his ability to solve problems presented graphically. However, David was able to solve tasks dealing with combining functions correctly in all three representations. David's facility to solve problems in all three representations did depend upon the content of the problem.

Beth interpreted function notation incorrectly in graphical and tabular representations. She had difficulty translating from a table to the other representations. Therefore she was limited in her ability to solve any problem presented in a table. In addition, Beth was limited in her ability to solve problems presented graphically where she had to use function notation in the solution. Beth was unable to solve any of the combining functions tasks correctly, but misinterpreted them in the same way in all three representations. Therefore Beth's ability to solve problems depended both on the representation the problem was presented in and the content.

In general the ability to solve problems presented in any representation had a lot to do with the student's ability to translate from one representation to another. Students who were able to translate between representations were also able to solve problems in each representation. However, when students had compartmentalization within representations then the content of the problem did have a bearing on whether they were able to solve a particular problem correctly.

Question 4. **Are students able to attempt to re-solve a problem utilizing a different representation when prompted? Are they able to re-solve the problem correctly?**

Most of the students had difficulty re-solving problems utilizing a different representation. Four of the students were able to utilize different representations on some problems, but not others. Two of the students were unable to solve the problems within any representation without being prompted by the researcher. The ability to solve a problem within different representations seems to involve more than just a lack of compartmentalization or the ability to translate between representations.

Beth and Steven were not able to solve most problems correctly without guidance from the interviewer. Although Beth was not able to re-solve problems on her own, she was able to follow explanations of different ways of solving problems and seemed genuinely interested. Usually it took a long time for her to solve a problem and by the time she was done, she wasn't willing to begin again. Steven had a very short attention span and tended to give up easily. He usually excused himself from the interview immediately after solving a problem or the moment the time ran out. His understanding of functions was very weak and it is not likely that he could re-solve the tasks in a different way.

Michael was unable to solve problems utilizing a different representation. He did understand explanations of other methods. However, the explanations did not affect his solution methods on other problems. He always used the same method of trying to find the equation. David, Matthew and Rachel were able to resolve some questions and could understand explanations of other solution methods. Their ability to re-solve problems was not related to the content. David was creative and adaptable. If he became stuck, he would almost immediately try another way. With prompting, he was able to re-solve problems utilizing a different representation.

Ike worked quickly and was not very forthcoming with answers to the interviewer's questions. It was not clear whether he could resolve the problems or not.

In summary, while compartmentalization was only found in three students it was interesting to note that compartmentalization allowed the students to harbor misconceptions within one representation that were not present in another representation. These misconceptions were most often found within representations that were less familiar to the students, in this case, graphs and tables.

The ability to translate between representations seemed to eliminate compartmentalization because it allowed student to solve a problem within a representation in which they were comfortable. Translation between representations seemed to be related to better problem-solving skills. However, the ability to translate between representations did not necessarily coincide with the ability to transfer information from one representation to another. Transferring seemed to be a higher order skill than translation. Transferring was the skill that was needed in order for students to be able to solve a problem in different ways, utilizing different representations.

It was surprising to me that there was not more compartmentalization in the students' understanding of functions. While my findings are consistent with other research on student's understanding of functions, I suspected that my focus on compartmentalization would uncover more, especially when the teacher and students were using the symbolic representation so overwhelmingly in their problem solving. Although compartmentalization caused three of the students in this study to answer problems incorrectly even though they expressed the ability to solve similar problems within a different representation, compartmentalization was not as widespread amongst the students in the study as I had suspected it would be.

Contribution to Research

Some of the results found in this dissertation study were similar to results found in other studies. For instance, students revealed that they defined concepts not from a mathematical definition, but from generalizations they derived from their experience in working with functions. For example, although there is no mention of symmetry in the definition of periodicity, several students revealed that they expected symmetry similar to the symmetries that exist in the periodic functions, sine and cosine.

In addition, although students had access to a rigorous definition function and periodicity in their notes and text, they overwhelmingly used their concept image to answer the questions. This is consistent with Vinner's work (1983) that the concept definition is not accessed when students encounter problems; instead students rely on their concept image.

Beth and Ike seemed to objectify the graph of a function to the extent that they could no longer see it as a collection of ordered pairs even though they had plotted the points themselves. This is consistent with the findings of Goldenberg (1988) where the students who could solve simultaneous linear equations in a graphical representation until they were taught to graph functions using the slope-intercept method.

Similar results to those of Martínez-Cruz (1995) were also found. That is, some students had incomplete connections in some of the subject areas and were unable to make a connection between an idea presented in two different representations. In fact, many students showed the same lack of connection between the vertical-line-test and univalence that Martínez-Cruz found.

This dissertation study also extends previous research. Analysis of the data collected revealed that the existence and extent of compartmentalization by representation depended on the content area rather than extending across content areas. For example, students who were not able to use a table to solve a periodicity task, were able to use a table in a combining functions task. Another example is Beth's understanding of function notation. Beth showed a lack of facility with function notation in the content area of combining functions no matter what representation. However, she was able to correctly interpret and use function notation in the more familiar content area of periodic functions. Even in this content area, function notation only made sense to her in the algebraic representation.

Interestingly enough, even if a student's understanding of a particular concept was situated in one particular representation, this did not necessarily mean that their concept image was compartmentalized. For instance, Rachel's understanding of periodicity was situated in the graphical representation. In addition she was able to translate in between graphs and tables. Therefore, when she was given a table and asked about its periodicity, she was able to plot points, determine the periodicity and translate the pattern back into the table. Although a causal relationship was not found between the flexible use of representations and students' problem solving skills, the results of this study do point to that possibility. If a relationship exists, then flexibility in the use of representations is then even more desirable for our high school students to achieve because it can overcome compartmentalization.

Students preferred to use equations even when another representation was given. Only when they were unable to translate from the given representation to an equation, were students willing to try another representation. In addition in almost every task in which a tabular representation was given, the students first translated the table into a graph and then solved the problem.

This research supports the thoughtful use of multiple representations in teaching functions, periodic functions, and combining functions. As student gain more facility with using multiple representations they are not only more flexible problem solvers, but also compartmentalization becomes less of a problem. When students are able to translate from one representation to another, they can navigate around understandings that are situated in a particular representation. For example, David's knowledge of periodic functions was situated in the graphical representation. However, David was able to translate tables and equations into graphs and by doing so was able to access his understanding of periodic functions and solve a task given in any representation.

In order for students to understand periodicity, they need to have more prototypical examples than just sine and cosine. The use of these two functions as the prototypical periodic functions in this pre-calculus class lead to students developing expectations of both symmetry and continuity. Although the tangent and inverse trigonometric functions were covered in class, they were only rarely presented in a graphical representation. Since the understanding of periodicity for many of the students was situated in the graphical representation, they needed more graphical examples in order to generalize a more appropriate working definition for periodicity in their concept images.

Limitations

Students volunteered to be in the study. Therefore, this may not be a representative sample of the class. However, the students did represent a range in ability and a range of grade-levels.

Students used their graphing calculators extensively in class. It was customary for the students to reference their graphing calculator without indicating that they did so. Therefore, the information I had on the students' use of graphs in the classwork may have been incomplete. Students may have been referencing graphs on their graphing calculators much more than I have accounted for from looking at their work.

I have attempted to present a snapshot of the students' concept images of functions and periodicity. This is a difficult proposition because the students may not know what is in their concept image and may have had difficulty articulating what they know. While some students were very willing and able to share their thinking with me, others were more reticent. The descriptions of the concept images that I have developed are my best guess. Although I tried to give the students multiple ways to demonstrate their understanding, it is quite likely that the students understood more than they were able to articulate in the framework of the given questions in some cases, and less than I give them credit for in other cases. The concept maps show what I was able to uncover, but do not give the whole picture of the students' understandings.

In addition, for the majority of the data, I was the only person to analyze the data. Although I instituted safeguards against introducing researcher bias, it is possible that in the coding or analyzing of the data such bias was introduced.

Finally the clinical interview calls for the interviewer to flexibly question a student's answers in order to reveal his or her understanding. Sometimes a researcher must step away from planned interview questions some of the time, in order to gain more understanding about a student's response. The amount of information the researcher is able to glean from the student is not only dependent upon the student's willingness to explain his or her thinking, but on the researcher's ability to ask the "right" question. At times, when I was analyzing the data, I found myself wishing that I could go back and ask a student more questions about a particular response. This is a limitation of this method. I was only able to probe misunderstandings that I noticed and I was only able to ask the questions I could think of during the interview. Despite this fact, I believe the method of clinical interviewing was employed successfully in this study.

Implications for Future Research.

We have seen that students who had compartmentalization in their understanding, had the most difficulty in solving the interview tasks. Furthermore, those students who could not translate between representations had an automatic compartmentalization in their understanding. Flexibility in problem solving comes from the ability to look at a problem from different perspectives. It would interesting to conduct a similar study in a class which utilizes the three representations equally rather than relying only upon the symbolic representation. Would compartmentalization be less prevalent in such a class. The current study suggests that more experiences with different representations and practice in translating between representations might reduce compartmentalization in students' understandings.

Several interesting misconceptions of periodicity were uncovered in this study. For instance, several students used symmetry, continuity, and familiarity as defining characteristics of periodicity. In addition, Steven's statements connecting parabolas with sine and cosine is an intriguing mystery. If indeed he believed that each peak and valley was congruent in some way to a parabola, do others have the same conception? Similarly, consider Ike's solution to the step function problem (Task 6). He used his concept of periodicity, that it must "go on forever" to create a non-function that had "∞ period," but looked familiar to him. This raises the question: "What affect does a student's concept image have on his ability to generalize when faced with a non-routine problem?"

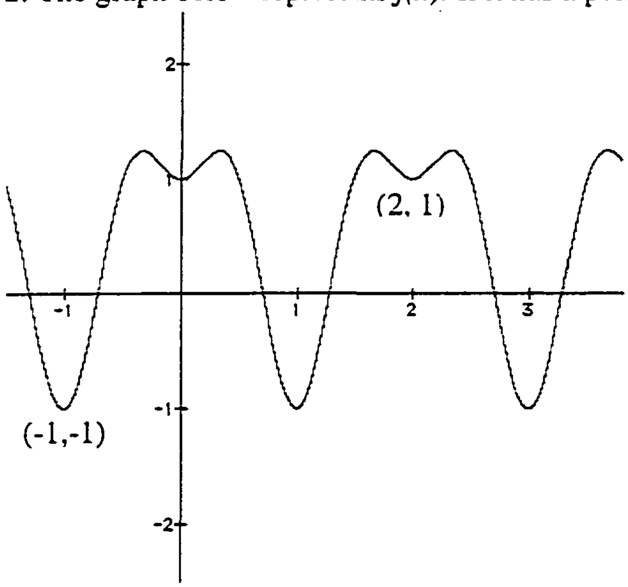
These and other questions would be interesting to explore. The results would give us information about the types of experiences that might be useful in helping students to create richer concept images without compartmentalization. The study of student's understanding of periodicity would help us to understand the types of generalizations that students make in light of the experiences they have. The continued study of students' understanding can only help to inform teaching practice and develop curriculum that supports deeper understandings.

Appendix A: Interview Tasks

1.The function represented in the table below is periodic. It has a period of 5. Fill in the missing values.

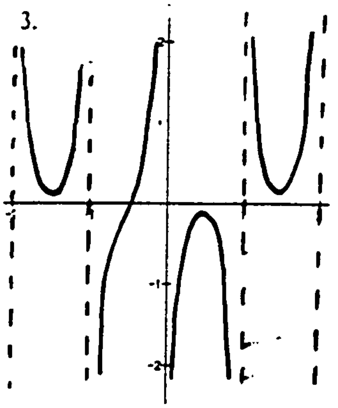
x	*h(x)*
-3	2
-1	-1
1/2	0
1	3
4	
6	

2. The graph below represents *f(x)*. If it has a period of 2, what is *f(6)*?



Which of the following could be periodic. What could the period be? If it could be periodic, describe or draw more of the graph. If it could not be periodic, explain.

3.



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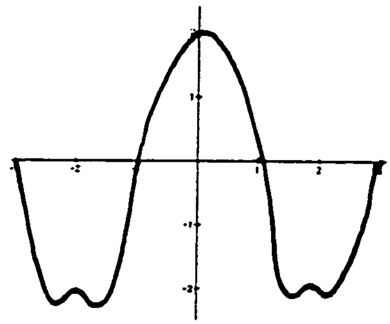

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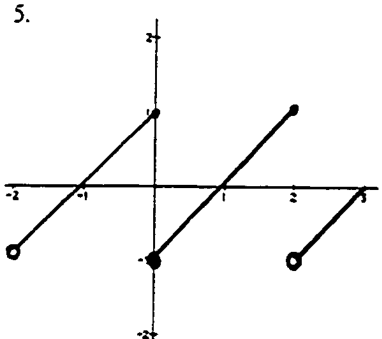
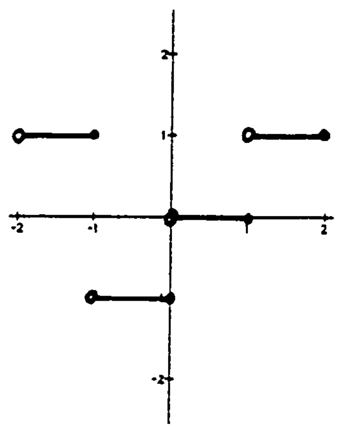


6.



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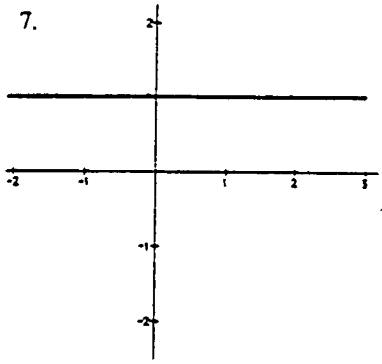

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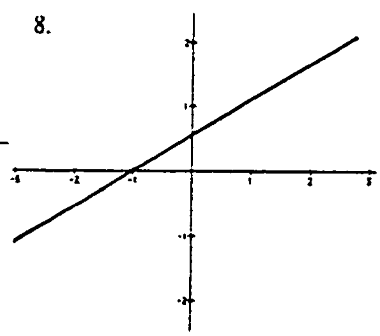


Determine if each the following tables could represent a periodic function. What could the period be? If it could not be a periodic function, explain.

9.

x	$f(x)$
-1	5
0	4
1	3
2	2
3	1
4	0

10.

x	$f(x)$
-1	1
0	2
1	0
2	-1
3	1
4	2

11.

x	$f(x)$
-2	-1
0	0
1/2	1
2	0
4	-1
5	0

12.

x	$f(x)$
-2	0
-1	1
0	0
2	0
4	0
5	1

Use the tables for the functions f and g to find the values of:

13. $(f+g)(1)$
14. $(fg)(1)$
15. $(f\circ g)(1)$

x	$f(x)$
-1	6
0	3
1	2
2	5

Table for *f*

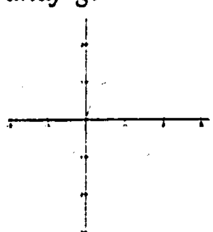
x	$g(x)$
-1	2
0	0
1	-1
2	-1

Table for *g*

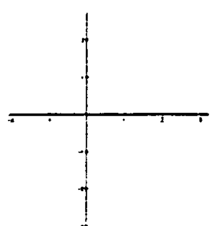
Use the tables for the functions f and g, to draw tables for

16. (f+g)(x)
17. (fg)(x)
18. (f∘g)(x)

The following are the graphs of two functions *f* and *g*. Draw the graphs of *f+g*, *fg*, and *f∘g*.

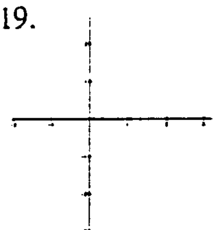


graph of *f*



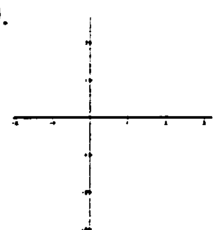
graph of *g*

19.



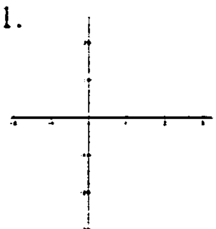
graph of *f+g*

20.



graph of *fg*

21.



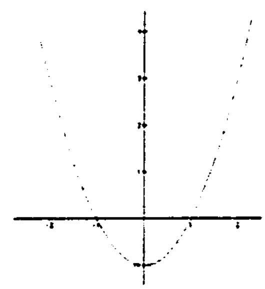
graph of f∘g

Given the functions f, g, and h below, write an equation, draw a graph, or make a table to show your results.

22. *(f+g)(x)*
23. *(f+h)(x)*
24. *(g+h)(x)*

the graph of *h*

$g(x) = \frac{1}{2}x + 1$



x	$f(x)$
-2	2
-1	2
0	2
1	2
2	2

Given the functions $f(x) = x^2 - 5$
and $g(x) = \sin(\pi x)$ find the following:

25. $f(3)$
26. $g\left(\frac{1}{2}\right)$
27. $g(k)$ for k = 0, 1, 2, 3, ...
28. $(f + g)(x)$
29. $(f + g)(3)$
30. $(fg)(0)$
31. $(fg)(x)$

32. $\sin x$, $\cos x$, and $\tan x$ are periodic functions. Is it possible to get new periodic functions by combining 2 periodic functions (adding, subtracting, multiplying, dividing, and composing)? How about if you combine a periodic function with a non-periodic function?

1. Brainstorm with your partner about the following idea. What are the most important ideas in this topic and what other things are connected to it.
FUNCTIONS

2. A function is a relationship between two variables (x and y) each x value is related to only one y value. Its graph passes the vertical line test. Draw 5 graphs of functions that you have seen before. Also write 5 equations.

3. Brainstorm with your partner about the following idea. What are the most important ideas in this topic and what other things are connected to it.
PERIODIC FUNCTION

Appendix B: IRB Approval

UNIVERSITY OF NEW HAMPSHIRE

Office of Sponsored Research
Service Building
51 College Road
Durham, New Hampshire 03824-3585
(603) 862-3564 FAX

LAST NAME	Gerson	FIRST NAME	Hope
DEPT	Mathematics - Kingsbury Hall	APP'L DATE	7/24/98
OFF-CAMPUS ADDRESS (if applicable)		IRB #	2012
		REVIEW LEVEL	FULL
PROJECT TITLE	Making Connections: Students' Understanding of Functions		

The Institutional Review Board for the Protection of Human Subjects in Research reviewed your response to its concerns, and approved the protocol for your project.

Approval is granted for one year from the approval date above. At the end of the approval period you will be asked to submit a project report with regard to the involvement of human subjects. If your project is still active, you may apply for extension of IRB approval through this office.

The protection of human subjects in your study is an ongoing process for which you hold primary responsibility. (Please refer to the Assurance of Compliance and the Belmont Report, enclosed.)

Changes in your protocol must be submitted to the IRB for review and approval prior to their implementation. If you have questions or concerns about your project or this approval, please feel free to contact me directly at 862-2003.

Please refer to the IRB # above in all correspondence related to this project. The IRB wishes you success with your research.

For the IRB,

Kara L. Eddy
Regulatory Compliance Officer
Office of Sponsored Research

cc: File
Graham, Karen - Mathematics

Modification entered: 10/7/98

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