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Topological vector spaces

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by

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Abstract

This major paper is a report on author's study of some topics on topological vector spaces. We prove a well-known Hahn-Banach theorem and some important consequences, including several separation and extension theorems. We study the weak topology on a topological vector space X and the weak-star topology on the dual space X^* of X. We also prove the Banach-Alaoglu theorem. Consequently, we characterize the closed convex hull and the closed linear span for sets in X and X^*, identify the dual of a subspace of X with the quotient of its annihilator, and obtain the Goldstine theorem as well as some characterizations of reflexive normed spaces.

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Contents

CHAPTER 1

Introduction

This major paper is a report on author's study of some materials on topological vector spaces contained in the references, mainly in [**2**, **3**]. We begin the paper with some basic definitions and properties on topological spaces.

Chapter 3 contains some basic results for topological vector spaces, in particular, for those topological vector spaces where the topology is induced by seminorms. We discuss quotient spaces and linear functionals of topological vector spaces. We prove a well-known Hahn-Banach theorem for real linear spaces and prove some consequences of this Hahn-Banach theorem, which include several separation theorems and an extension theorem for topological vector spaces.

In Chapter 4, we consider the weak topology on a topological vector space X and the weak-star topology on its dual space X^*, and give some of their basic properties. We study dual spaces over these topologies, and characterize the closed convex balanced hull and the closed linear span for sets in X and X^*. We identify the dual of a quotient space with the annihilator of the subspace, and identify the dual of a subspace of a locally convex topological vector space with the quotient space of the annihilator of the subspace.

In Chapter 5, we prove Banach-Alaoglu theorem and Goldstine theorem, which say that for a normed space X, the closed unit ball of X^* is weak-star compact and the closed unit ball of X^{**} is the weak-star closure in X^{**} of the canonical image of the closed unit ball of X. We give a number of characterizations of reflexive normed spaces. We prove that every reflexive space is weakly sequentially complete, and show that the converse is not true by checking the non-reflexivity of the weakly sequentially complete space ℓ_1. We end the paper by proving that X is separable if and only if the closed unit ball of X^* is weak-star metrizable.

CHAPTER 2

Basics of topological spaces

In this chapter, we give some basic definitions and properties on topological spaces mainly involving convergence, continuity and compactness, which will be used in later chapters. The main reference for this chapter is [**5**].

DEFINITION 2.1. Let X be a topological space, and let $x \in X$. A subset N of X is called a neighborhood of x if there exists an open subset U of X with $x \in U \subseteq N$. The collection of all neighborhoods of x is denoted by $\mathcal{N}_x$.

DEFINITION 2.2. Let X be a topological space.

(a) A base for the topology on X is a collection B of open sets in X such that each open set in X is a union of sets in B.

(b) A subbase for the topology on X is a collection S of open sets in X such that the collection of all finite intersections of sets in S is a base for the topology.

DEFINITION 2.3. Let X be a topological space and let $x \in X$. A net $(x_\alpha)_{\alpha \in A}$ in X is said to converge to x if for each $N \in \mathcal{N}_x$, there exists $\alpha_N \in A$ such that $x_\alpha \in N$ for all $\alpha \in A$ with $\alpha \succcurlyeq \alpha_N$.

DEFINITION 2.4. Let X and Y be topological spaces, and let $x_0 \in X$. A function $f : X \to Y$ is said to be continuous at x_0 if $f^{-1}(N) \in \mathcal{N}_{x_0}$ for each $N \in \mathcal{N}_{f(x_0)}$. If f is continuous at every point of X, we simply call f continuous.

PROPOSITION 2.1. *Let X and Y be topological spaces, let $f : X \to Y$, and let $x_0 \in X$. Then the following statements are equivalent.*

(i) *f is continuous at x_0.*

(ii) *If (x_λ) is a net in X with $x_\lambda \to x_0$, then $f(x_\lambda) \to f(x_0)$ in Y.*

DEFINITION 2.5. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Let $X = \prod_{i\in I} X_i$. The product topology on X is the coarest topology on X making all the coordinate projections $\pi_i : X \to X_i$ $(i \in I)$ continuous. We call X the product space of $\{X_i\}_{i\in I}$.

PROPOSITION 2.2. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces and let X be the product space of $\{X_i\}_{i\in I}$. Let $(f_\alpha)_{\alpha\in\Lambda}$ be a net in X and $f \in X$. Then $f_\alpha \to f$ in X if and only if $f_\alpha(i) \to f(i)$ in X_i for all $i \in I$.*

DEFINITION 2.6. Let X be a topological space and let $S \subseteq X$. The closure of S is defined as $\overline{S} = \bigcap\{E : S \subseteq E \text{ and } E \text{ is a closed subset of } X\}$. Therefore, $\overline{S}$ is the smallest closed subset of X containing S.

PROPOSITION 2.3. *Let X be a topological space and let $S \subseteq X$. Then S is closed in X if and only if $S = \overline{S}$.*

PROPOSITION 2.4. *Let X be a topological space and let $S \subseteq X$. Then*

$$\overline{S} = \{x \in X : x \text{ is a limit of a net in } S\}.$$

In particular, $\overline{S} = \{x \in X : x \text{ is a limit of a sequence in } S\}$ if X is first countable.

COROLLARY 2.5. *Let X be a topological space and let $S \subseteq X$. Then S is closed in X if and only if every net in S that converges in X has its limit in S.*

PROPOSITION 2.6. *Let X be a metric space and let Y be a subspace of X such that Y is complete. Then Y is closed in X.*

DEFINITION 2.7. Let X be a topological space. A subset D of X is said to be dense in X if $\overline{D} = X$. X is called separable if it contains a countable dense subset.

PROPOSITION 2.7. *Let X be a separable metric space and let Y be a subspace of X. Then Y is separable.*

PROPOSITION 2.8. *Let X be a topological space, let $x \in X$, and let $(x_\alpha)_{\alpha\in A}$ be a net in X with $x_\alpha \to x$. Then each subnet of $(x_\alpha)_{\alpha\in A}$ converges to x.*

DEFINITION 2.8. Let X be a topological space and let $(x_\alpha)_{\alpha\in A}$ be a net in X. A point x in X is called an accumulation point of $(x_\alpha)_{\alpha\in A}$ if for all $\alpha \in A$ and $N \in \mathcal{N}_x$, there exists $\beta \in A$ such that $\beta \succcurlyeq \alpha$ and $x_\beta \in N$.

PROPOSITION 2.9. *Let X be a topological space and let $S \subseteq X$. Then*

$$\overline{S} = \{x : x \text{ is an accumulation point of a net in } S\}.$$

THEOREM 2.10. *Let X be a topological space. Then the following statements are equivalent.*

(i) *X is compact.*

(ii) *Every net in X has an accumulation point.*

(iii) *Every net in X has a convergent subnet.*

PROPOSITION 2.11. *Let X be a topological space and let $Y \subseteq X$.*

(i) *If X is Hausdorff and Y is compact in X, then Y is closed in X.*

(ii) *If X is compact and Y is closed in X, then Y is compact in X.*

PROPOSITION 2.12. *Let X be a compact topological space and let $f : X \to \mathbb{R}$ be continuous. Then f attains both a minimum and a maximum on X.*

PROPOSITION 2.13. *Let X and Y be topological spaces such that X is compact and Y is Hausdorff, and let $f : X \to Y$ be a continuous bijection. Then $f : X \to Y$ is a homeomorphism.*

PROPOSITION 2.14. *Let X and Y be topological spaces such that Y is Hausdorff. Let A be a dense subset of X and let $f, g : X \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in A$. Then $f(x) = g(x)$ for all $x \in X$.*

CHAPTER 3

Topological vector spaces and separation theorems

In Section 3.1, we give basic definitions and results on topological vector spaces. In Section 3.2, we summarize some essential properties about quotient spaces and linear functionals of topological vector spaces. In Section 3.3, we prove a fundamental Hahn-Banach theorem for real topological vector spaces. Section 3.4 contains some consequences of the Hahn-Banach theorem, including several separation theorems and an extension theorem. The main references for this chapter are [**2**] and [**4**].

In this paper, we use the symbol $\mathbb{F}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

3.1. Topological vector spaces and seminorms

DEFINITION 3.1.1. A topological vector space (TVS) is a linear space X over $\mathbb{F}$ together with a topology such that with respect to this topology,

(i) the map $X \times X \to X$, $(x, y) \mapsto x + y$ is continuous;

(ii) the map $\mathbb{F} \times X \to X$, $(\alpha, y) \mapsto \alpha y$ is continuous.

In this case, it follows that for all x_1, $x_2 \in X$, if V is a neighborhood of $x_1 + x_2$, then there exists a neighborhood V_i of x_i $(i = 1, 2)$ such that $V_1 + V_2 \subseteq V$. Similarly, for all $x \in X$ and $\alpha \in \mathbb{F}$, if V is a neighborhood of αx, then for some $r > 0$ and some neighborhood W of x, we have $\beta W \subseteq V$ whenever $|\beta - \alpha| < r$.

DEFINITION 3.1.2. Let X be a linear space over $\mathbb{F}$. A function $\|\cdot\| : X \to [0, \infty)$ is called a norm on X if for all $x, y \in X$ and $\alpha \in \mathbb{F}$, we have

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \|x\|$;

(iii) $\|x + y\| \leqslant \|x\| + \|y\|$.

A function $p : X \to [0, \infty)$ satisfying (ii) and (iii) above is called a seminorm on X.

EXAMPLE 3.1.1. The function $\mathbb{R}^2 \to [0,\infty)$ defined by $x = (x_1, x_2) \mapsto |x_1|$ is a seminorm on $\mathbb{R}^2$, but not a norm since $x_1 = 0$ does not imply that $x = 0$.

DEFINITION 3.1.3. Let X be a linear space and let P be a family of seminorms on X. Let T be the topology on X that has a subbase consisting of the sets

$$\{x : p(x - x_0) < \varepsilon\},$$

where $p \in P$, $x_0 \in X$ and $\varepsilon > 0$. Thus a subset U of X is open if and only if for every x_0 in U, there are $p_1, \cdots, p_n$ in P and positive scalars $\varepsilon_1, \cdots, \varepsilon_n$ such that

$$\bigcap_{j=1}^{n} \{x \in X : p_j(x - x_0) < \varepsilon_j\} \subseteq U.$$

This topology T is called the topology defined by P.

LEMMA 3.1.2. *Let X be a linear space, let P be a family of seminorms on X, and let T be the topology on X defined by P. Let $x \in X$ and let $(x_i)_{i\in I}$ be a net in X. Then $p(x_i - x) \to 0$ for all $p \in P$ if and only if $x_i \to x$ in (X, T).*

PROOF. Suppose $p(x_i - x) \to 0$ for all $p \in P$. Let $N \in \mathcal{N}_x$. Then there exist $p, \cdots, p_n \in P$ and $\varepsilon_1, \cdots, \varepsilon_n > 0$ such that $\bigcap_{j=1}^{n} U_{p_j,\varepsilon_j} \subseteq N$, where

$$U_{p_j,\varepsilon_j} = \{y \in X : p_j(y - x) < \varepsilon_j\}.$$

Let $\delta = \min\{\varepsilon_1, \cdots, \varepsilon_n\}$. Then $\bigcap_{j=1}^{n} U_{p_j,\delta} \subseteq N$. Since $p_j(x_i - x) \to 0$ for $j = 1, \cdots, n$, there exists $i_0 \in I$ such that when $i \succcurlyeq i_0$, $p_j(x_i - x) < \delta$ for all $1 \leqslant j \leqslant n$. That is, for all $i \succcurlyeq i_0$, $x_i \in \bigcap_{j=1}^{n} U_{p_j,\delta} \subseteq N$. Therefore, $x_i \to x$ in (X, T).

Conversely, suppose $x_i \to x$ in (X, T). Let $p \in P$. Let $\varepsilon > 0$ and let

$$U_\varepsilon = \{y \in X : p(y - x) < \varepsilon\}.$$

Then U_ε is a neighborhood of x. Thus there exists $i_0 \in I$ such that $x_i \in U_\varepsilon$ for all $i \succcurlyeq i_0$. That is, $p(x_i - x) < \varepsilon$ for all $i \succcurlyeq i_0$. Therefore, $p(x_i - x) \to 0$. □

PROPOSITION 3.1.3. *Let X be a linear space, let P be a family of seminorms on X, and let X be equipped with the topology defined by P. Then*

(i) *X is a TVS;*

(ii) *each $p \in P$ is continuous on X.*

PROOF. (i) We just need show that the addition and scalar multiplication on X are continuous.

Let $x, y \in X$. Let (x_i) and (y_i) be two nets in X with $x_i \to x$ and $y_i \to y$. Let $p \in P$. Then

$$p((x_i + y_i) - (x + y)) = p((x_i - x) + (y_i - y)) \leqslant p(x_i - x) + p(y_i - y).$$

By Lemma 3.1.2, $p(x_i - x) \to 0$ and $p(y_i - y) \to 0$. So, $p((x_i + y_i) - (x + y)) \to 0$. By Lemma 3.1.2 again, $x_i + y_i \to x + y$. Therefore, the addition on X is continuous.

Let $\lambda \in \mathbb{F}$ and $x \in X$. Let (λ_i) be a net in $\mathbb{F}$ with $\lambda_i \to \lambda$, and let (x_i) be a net in X with $x_i \to x$. Then

$$p(\lambda_i x_i - \lambda x) = p((\lambda_i - \lambda)x_i + \lambda(x_i - x)) \leqslant |\lambda_i - \lambda| p(x_i) + |\lambda| p(x_i - x).$$

Since $p(x_i - x) \to 0$ (by Lemma 3.1.2), $p(x_i) \leqslant p(x_i - x) + p(x)$, and $\lambda_i \to \lambda$, we have

$$|\lambda_i - \lambda| p(x_i) \leqslant |\lambda_i - \lambda| p(x_i - x) + |\lambda_i - \lambda| p(x) \to 0.$$

Hence, $p(\lambda_i x_i - \lambda x) \to 0$. By Lemma 3.1.2, $\lambda_i x_i \to \lambda x$. Therefore, the scalar multiplication on X is continuous.

(ii) Let $p \in P$. Let (x_i) be a net in X with $x_i \to x \in X$. By Lemma 3.1.2, $p(x_i - x) \to 0$. Since $p(x_i - x) \geqslant p(x_i) - p(x)$ and $p(x - x_i) \geqslant p(x) - p(x_i)$, we have

$$|p(x_i) - p(x)| \leqslant p(x_i - x) \to 0.$$

Thus $p(x_i) \to p(x)$. Therefore, p is continuous on X. □

DEFINITION 3.1.4. A locally convex space (LCS) is a TVS whose topology is defined by a family P of seminorms such that $\bigcap_{p \in P} \ker(p) = \{0\}$.

THEOREM 3.1.4. *Let X be a TVS with the topology defined by a family P of seminorms on X. Then X is a LCS if and only if X is Hausdorff.*

PROOF. Suppose X is a LCS. Then $\Delta = \bigcap_{p\in P} \ker(p) = \{0\}$. Let $x, y \in X$ with $x \neq y$. Then $x - y \neq 0$, and thus $x - y \notin \Delta$. Hence, there exists $p \in P$ such that $p(x-y) > 0$. Let $\delta = p(x-y)$. Then $U_x = \{z \in X : p(z-x) < \frac{\delta}{3}\}$ is a neighborhood of x, $U_y = \{z \in X : p(z-y) < \frac{\delta}{3}\}$ is a neighborhood of y, and $U_x \cap U_y = \emptyset$. Therefore, X is Hausdorff.

Conversely, suppose X is Hausdorff. Let $x_0 \neq 0$. Then there exists a neighborhood N_0 of 0 such that $x_0 \notin N_0$. Thus there exist $p_i \in P$ and $\varepsilon_i > 0$ such that $\bigcap_{i=1}^{n} B_{p_i} \subseteq N_0$, where $B_{p_i} = \{y \in X : p_i(y) < \varepsilon_i\}$ $(i = 1, \cdots, n)$. So, $x_0 \notin \bigcap_{i=1}^{n} B_{p_i}$. It implies that $p_i(x) \neq 0$ for some i. Thus $x_0 \notin \bigcap_{p\in P} \ker(p)$. Therefore, X is a LCS. □

DEFINITION 3.1.5. Let X be a TVS and let E be a non-empty subset of X. The closed linear span of E, denoted by $\overline{\text{span}}(E)$, is the intersection of all closed linear subspaces of X which contain E.

LEMMA 3.1.5. *Let X be a TVS and let E be a non-empty subset of X. Then $\overline{\text{span}}(E) = \overline{\text{span}(E)}$.*

PROPOSITION 3.1.6. *Let X be a TVS and let A be a countable non-empty subset of X. Then $\overline{\text{span}}(A)$ is separable.*

3.2. Quotient spaces and linear functionals

DEFINITION 3.2.1. Let X be a linear space over $\mathbb{F}$ and let M be a linear subspace of X. For $x, y \in X$, define $x \sim y$ if $x - y \in M$. Then $\sim$ is an equivalence relation on X. For $x \in X$, let $[x] = \{y \in X : y \sim x\}$. Then $[x] = x + M = \{x + m : m \in M\}$. Let

$$X/M = \{[x] : x \in X\}.$$

Then X/M is a linear space over $\mathbb{F}$ with scalar multiplication and addition defined by $\alpha[x] = [\alpha x]$ and $[x] + [y] = [x + y]$ $(\alpha \in \mathbb{F},\ x, y \in X)$. The map $X \to X/M$, $x \mapsto [x]$ is linear and surjective, called the canonical quotient map.

PROPOSITION 3.2.1. *Let (X,T) be a topological space, let Y be a set, and let $q : X \to Y$ be a map. Then $T_Y = \{U : U \subseteq Y \text{ and } q^{-1}(U) \in T\}$ is the largest topology on Y that makes q continuous, called the quotient topology on Y induced by q.*

PROPOSITION 3.2.2. *Let (X,T) be a TVS and let M be a linear subspace of X. Let X/M be equipped with the quotient topology T_M induced by the canonical quotient map $Q : X \to X/M$. Then $Q : X \to X/M$ is open.*

PROOF. Let $U \in T$. Then we have $Q^{-1}(Q(U)) = U + M = \bigcup_{m\in M} (U + m)$. Since $U+m \in T$ for all $m \in M$, $Q^{-1}(Q(U)) \in T$. By the definition of the quotient topology, $Q(U) \in T_M$. Therefore, the map $Q : X \to X/M$ is open. □

PROPOSITION 3.2.3. *Let X be a linear space, let M be a linear subspace of X, and let p be a seminorm on X. Define $\overline{p} : X/M \to [0,\infty)$ by*

$$\overline{p}(x + M) = \inf\{p(x + y) : y \in M\}.$$

Then $\overline{p}$ is a seminorm on X/M.

If X is a LCS and P is the family of all continuous seminorms on X, then the family $\overline{P} = \{\overline{p} : p \in P\}$ defines the canonical quotient topology on X/M.

PROOF. Let $x \in X$. If $\alpha \in \mathbb{F} - \{0\}$, then

$$\begin{aligned}\overline{p}(\alpha x + M) &= \inf\{p(\alpha x + y) : y \in M\} = \inf\{|\alpha| p(x + \frac{y}{\alpha}) : y \in M\} \\ &= |\alpha| \inf\{p(x + z) : z \in M\} = |\alpha| \overline{p}(x + M).\end{aligned}$$

If $\alpha = 0$, then $\overline{p}(\alpha x + M) = \inf\{p(y) : y \in M\} = 0$. Hence,

$$\overline{p}(\alpha x + M) = |\alpha| \overline{p}(x + M) \text{ for all } x \in X \text{ and } \alpha \in \mathbb{F}.$$

Let $x_1, x_2 \in X$. Then

$$\begin{aligned}\overline{p}(x_1 + x_2 + M) &= \inf\{p(x_1 + x_2 + y) : y \in M\} \\ &= \inf\{p((x_1 + y_1) + (x_2 + y_2)) : y_1, y_2 \in M\}\end{aligned}$$

$$\begin{aligned}
&\leqslant \inf\{p(x_1+y_1)+p(x_2+y_2) : y_1, y_2 \in M\} \\
&= \inf\{p(x_1+y_1) : y_1 \in M\} + \inf\{p(x_2+y_2) : y_2 \in M\} \\
&= \overline{p}(x_1+M) + \overline{p}(x_2+M).
\end{aligned}$$

Therefore, $\overline{p}$ is a seminorm on X/M.

Suppose X is a LCS and P is the family of all continuous seminorms on X. Then $\overline{P} = \{\overline{p} : p \in P\}$ is a family of seminorms on X/M. Let T_P be the topology on X/M defined by $\overline{P}$, let $Q : X \to X/M$ be the canonical quotient map, and let T_M be the quotient topology on X/M induced by Q. We show below that $T_M = T_P$.

Note that $V \subseteq X/M$ is T_M-open if and only if $Q^{-1}(V)$ is open in X. So, we only have to prove that $V \subseteq X/M$ is T_P-open if and only if $Q^{-1}(V)$ is open in X.

Suppose V is T_P-open in X/M. Let $x_0 \in Q^{-1}(V)$. Then $x_0 + M \in V$. By the definition of T_p and Definition 3.1.3, there exist $p_i \in P$ and $\varepsilon_i > 0$ $(i = 1, \cdots, n)$ such that $\bigcap_{i=1}^{n} \{x + M \in X/M : \overline{p_i}(x + M - (x_0 + M)) < \varepsilon_i\} \subseteq V$. Let

$$U = \bigcap_{i=1}^{n} \{x + M \in X/M : \overline{p_i}(x + M - (x_0 + M)) < \varepsilon_i\}.$$

Then $x_0 + M \in U \subseteq V$. We have

$$\begin{aligned}
Q^{-1}(U) &= \bigcap_{i=1}^{n} Q^{-1}(\{x + M \in X/M : \overline{p_i}(x + M - (x_0 + M)) < \varepsilon_i\}) \\
&= \bigcap_{i=1}^{n} Q^{-1}(\{x + M \in X/M : \overline{p_i}(x - x_0 + M) < \varepsilon_i\}) \\
&= \bigcap_{i=1}^{n} \{x \in X : \inf\{p_i(x - x_0 + y) : y \in M\} < \varepsilon_i\}.
\end{aligned}$$

Now we show that for $1 \leqslant i \leqslant n$, we have

$$\{x \in X : \inf\{p_i(x - x_0 + y) : y \in M\} < \varepsilon_i\} = \bigcup_{y \in M} \big(y + \{x \in X : p_i(x - x_0) < \varepsilon_i\}\big).$$

Let $a \in X$ be such that $\inf_{y\in M} p_i(a - x_0 + y) < \varepsilon_i$. Let

$$\delta_i = \varepsilon_i - \inf_{y\in M} p_i(a - x_0 + y).$$

Then there exists $y_0 \in M$ such that

$$p_i(a - x_0 + y_0) < \inf_{y\in M} p_i(a - x_0 + y) + \delta_i = \varepsilon_i.$$

Let $z_0 = a + y_0$. Then $p_i(z_0 - x_0) < \varepsilon_i$. Since $a = (-y_0) + z_0$ and $-y_0 \in M$, $a \in \bigcup_{y\in M} \big(y + \{x \in X : p_i(x - x_0) < \varepsilon_i\}\big)$. Hence,

$$\{x \in X : \inf\{p_i(x - x_0 + y) : y \in M\} < \varepsilon_i\} \subseteq \bigcup_{y\in M} \big(y + \{x \in X : p_i(x - x_0) < \varepsilon_i\}\big).$$

Conversely, suppose $a = y_0 + z_0$ for some $y_0 \in M$ and $z_0 \in X$ with $p_i(z_0 - x_0) < \varepsilon_i$. Then $p_i(a - x_0 + (-y_0)) = p_i(z_0 - x_0) < \varepsilon_i$, and thus $\inf_{y\in M} p_i(a - x_0 + y) < \varepsilon_i$. So, $a \in \{x \in X : \inf\{p_i(x - x_0 + y) : y \in M\} < \varepsilon_i\}$. Therefore,

$$\{x \in X : \inf\{p_i(x - x_0 + y) : y \in M\} < \varepsilon_i\} = \bigcup_{y\in M} \big(y + \{x \in X : p_i(x - x_0) < \varepsilon_i\}\big).$$

Hence, $Q^{-1}(U) = \bigcap_{i=1}^{n} \bigcup_{y\in M} \big(y + \{x \in X : p_i(x - x_0) < \varepsilon_i\}\big)$, which is open in X. Since $x_0 + M \in U \subseteq V$, $x_0 \in Q^{-1}(U) \subseteq Q^{-1}(V)$. Therefore, $Q^{-1}(V)$ is open in X.

On the other hand, suppose $Q^{-1}(V)$ is open in X. Let $x_0 + M \in V$. Since X is a LCS, the topology on X is defined by a family P' of seminorms on X. By Proposition 3.1.3, $P' \subseteq P$. Since $x_0 \in Q^{-1}(V)$, by Definition 3.1.3, there exist $p_i \in P'$ and $\varepsilon_i > 0$ $(i = 1, \cdots, n)$ such that

$$\bigcap_{i=1}^{n} \{x \in X : p_i(x - x_0) < \varepsilon_i\} \subseteq Q^{-1}(V).$$

Let $p = \max\{p_1, \cdots, p_n\}$, $\varepsilon = \min\{\varepsilon_1, \cdots, \varepsilon_n\}$, and $B = \{x \in X : p(x - x_0) < \varepsilon\}$. Then $p \in P$, $\varepsilon > 0$, and $x_0 \in B \subseteq \bigcap_{i=1}^{n} \{x \in X : p_i(x - x_0) < \varepsilon_i\} \subseteq Q^{-1}(V)$. We have

$$Q(B) = \{x + M \in X/M : p(y - x_0) < \varepsilon \text{ for some } y \in x + M\}.$$

Now we show that

$$\begin{aligned}&\{x+M\in X/M : p(y-x_0)<\varepsilon \text{ for some } y\in x+M\}\\ =&\{x+M\in X/M : \inf_{m\in M} p(x+m-x_0)<\varepsilon\}.\end{aligned}$$

Let $a+M\in\{x+M\in X/M : p(y-x_0)<\varepsilon \text{ for some } y\in x+M\}$. Then there exists $b\in a+M$ such that $p(b-x_0)<\varepsilon$. Thus there exists $m_0\in M$ such that $b=a+m_0$ and $p(a+m_0-x_0)<\varepsilon$. Hence, $\inf_{m\in M} p(a+m-x_0)<\varepsilon$. Therefore,

$$a+M\in\{x+M\in X/M : \inf_{m\in M} p(x+m-x_0)<\varepsilon\}.$$

Conversely, let $a+M\in X/M$ be such that $\inf_{m\in M} p(a+m-x_0)<\varepsilon$. Let $\delta=\varepsilon-\inf_{m\in M} p(a+m-x_0)$. Then there exists $m_0\in M$ such that

$$p(a+m_0-x_0)<\inf_{m\in M} p(a+m-x_0)+\delta=\varepsilon.$$

Let $b=a+m_0$. Then $p(b-x_0)<\varepsilon$. Thus

$$a+M\in\{x+M\in X/M : p(y-x_0)<\varepsilon \text{ for some } y\in x+M\}.$$

So, we obtain that

$$\begin{aligned}&\{x+M\in X/M : p(y-x_0)<\varepsilon \text{ for some } y\in x+M\}\\ =&\{x+M\in X/M : \inf_{m\in M} p(x+m-x_0)<\varepsilon\}.\end{aligned}$$

Hence, we have

$$\begin{aligned}Q(B)&=\{x+M\in X/M : \inf_{m\in M} p(x+m-x_0)<\varepsilon\}\\ &=\{x+M\in X/M : \overline{p}((x+M)-(x_0+M))<\varepsilon\},\end{aligned}$$

which is T_p-open. Since $x_0\in B\subseteq Q^{-1}(V)$, $x_0+M\in Q(B)\subseteq V$. Therefore, V is T_p-open. □

If $(X, \|\cdot\|)$ is a normed space and M is a linear subspace of X, we define

$$\|x + M\| = \inf\{\|x + y\| : y \in M\} \ (x \in X).$$

PROPOSITION 3.2.4. *Let X be a normed space and let M be a closed linear subspace of X. Then the function $\|\cdot\|$ on X/M defined above is a norm on X/M.*

PROOF. By Proposition 3.2.3, the function $\|\cdot\|$ on X/M is a seminorm. Suppose that $x \in X$ and $\|x + M\| = 0$. Then there exists a sequence (y_n) in M such that $\|x + y_n\| \to 0$; that is, $(-y_n) \to x$. Since M is closed and the sequence $(-y_n)$ is in M, by Corollary 2.5, $x \in M$. Hence, $x + M = 0$. Therefore, $\|\cdot\|$ is a norm on X/M. □

PROPOSITION 3.2.5. *Let X be a normed space and let M be a closed linear subspace of X. Then the canonical quotient map $Q : X \to X/M$ is a bounded linear map and $\|Q\| \leqslant 1$.*

DEFINITION 3.2.2. Let X be a linear space and let M be a linear subspace of X. Then M is called a hyperplane in X if $dim(X/M) = 1$.

LEMMA 3.2.6. *Let M be a linear subspace of a linear space X over $\mathbb{F}$.*

(i) *M is a hyperplane in X if and only if $M = ker(f)$ for some non-zero linear functional f on X.*

(ii) *If f and g are linear functionals on X, then $ker(f) = ker(g)$ if and only if $g = \beta f$ for some $\beta \in \mathbb{F} - \{0\}$.*

PROPOSITION 3.2.7. *Let X be a TVS and let f be a linear functional on X. Then the following statements are equivalent.*

(a) *f is continuous.*

(b) *f is continuous at 0.*

(c) *f is continuous at some $x_0 \in X$.*

(d) *$ker(f)$ is closed in X.*

(e) *$x \mapsto |f(x)|$ is a continuous seminorm on X.*

(f) *f is bounded in some neighborhood of 0.*

If the topology on X is defined by a family P of seminorms on X, then each of (a)-(f) is equivalent to

(g) there exist $p_1, \cdots, p_n$ in P and positive scalars $\alpha_1, \cdots, \alpha_n$ such that

$$|f(x)| \leqslant \sum_{k=1}^{n} \alpha_k p_k(x) \text{ for all } x \in X.$$

PROOF. (a)⇒(c). This is obvious.

(c)⇒(b). Suppose (c) holds. Let (x_n) be a net in X with $x_n \to 0$ in X. Then $x_n + x_0 \to x_0$. Since f is linear and continuous at $x_0 \in X$, we have

$$f(x_n) + f(x_0) = f(x_n + x_0) \to f(x_0).$$

Hence, $f(x_n) \to 0$. Therefore, f is continuous at 0.

(b)⇒(d). Suppose (b) holds. Let $x \in X$ and let (x_n) be a net in $ker(f)$ with $x_n \to x$. Then $x_n - x \to 0$. Since f is continuous at 0, we have

$$f(x_n) - f(x) = f(x_n - x) \to f(0) = 0.$$

It follows that $f(x) = \lim_n f(x_n) = 0$; that is, $x \in ker(f)$. By Corollary 2.5, $ker(f)$ is closed.

(d)⇒(a). Suppose $ker(f)$ is closed in X. If $f = 0$, then f is continuous. Assume that $f \neq 0$. By Lemma 3.2.6, $ker(f)$ is a hyperplane in X. Then there exists a linear isomorphism $T : X/ker(f) \to \mathbb{F}$. Let $q : X \to X/ker(f)$ be the quotient map $x \mapsto x + ker(f)$, and let $g = T \circ q$. Then $g : X \to \mathbb{F}$ is linear and continuous. Now

$$x \in ker(f) \iff q(x) = 0 \iff T(q(x)) = 0 \iff g(x) = 0 \iff x \in ker(g).$$

Hence, $ker(f) = ker(g)$. By Lemma 3.2.6, $f = \beta g$ for some $\beta \in \mathbb{F} - \{0\}$. Therefore, f is continuous.

(a)⇒(e)⇒(b). This is obvious.

(b)⇔(f). Suppose $M > 0$ and $|f(x)| < M$ for all x in a neighborhood V of 0. Let $r > 0$ and let $W = (r/M)V$. Then W is a neighborhood of 0, and $|f(x)| < r$ for all $x \in W$. Hence, f is continuous at 0. Conversely, suppose f is continuous at

0. Let $A = \{\alpha \in \mathbb{F} : |\alpha| < 1\}$ and $B = f^{-1}(A)$. Then B is a neighborhood of 0 and $|f(x)| < 1$ for all $x \in B$.

In the rest of the proof, we assume that X is a LCS with the topology defined by a family P of seminorms on X.

(g)⇒(b). This follows immediately from Lemma 3.1.2.

(b)⇒(g). Suppose f is continuous at 0. Since $f(0) = 0$, for $\varepsilon = 1$, there exist $p_1, \cdots, p_n \in P$ and $\varepsilon_1, \cdots, \varepsilon_n > 0$ such that $|f(z)| < 1$ for all $z \in \bigcap_{i=1}^{n} u_{p_i,\varepsilon_i}$, where $u_{p_i,\varepsilon_i} = \{y \in X : p_i(y) < \varepsilon_i\}$ $(i = 1, \cdots, n)$. Let $\delta = \min\{\varepsilon_i : 1 \leqslant i \leqslant n\}$. Then $|f(z)| < 1$ for all $z \in \bigcap_{i=1}^{n} u_{p_i,\delta}$. Let $x \in X$. We consider the following two cases.

Case 1: $p_i(x) = 0$ for all i. Then for all $t > 0$, $p_i(tx) = tp_i(x) = 0$, and thus $tx \in \bigcap_{i=1}^{n} u_{p_i,\delta}$. So, $|f(tx)| < 1$ for all $t > 0$. It follows that $|f(x)| < \frac{1}{t}$ for all $t > 0$. Hence, $|f(x)| = 0$. Therefore, in this case, the inequality in (g) holds with any choice of positive scalars $\alpha_1, \cdots, \alpha_n$.

Case 2: $p_{i_0}(x) \neq 0$ for some i_0. Then

$$p_k\left(\frac{\delta}{2\sum_{i=1}^{n} p_i(x)}x\right) = \frac{\delta}{2}\frac{p_k(x)}{\sum_{i=1}^{n} p_i(x)} \leqslant \frac{\delta}{2} < \delta$$

for $k = 1, \cdots, n$. Hence, $\frac{\delta}{2\sum_{i=1}^{n} p_i(x)}x \in \bigcap_{i=1}^{n} u_{p_i,\delta}$. Therefore,

$$\left|f(\frac{\delta}{2\sum_{i=1}^{n} p_i(x)}x)\right| = \frac{\delta}{2}\frac{1}{\sum_{i=1}^{n} p_i(x)}|f(x)| < 1,$$

which implies that $|f(x)| \leqslant \sum_{i=1}^{n} \frac{2}{\delta} p_i(x)$.

Let $\alpha_i = \frac{2}{\delta}$ $(i = 1, \cdots, n)$. Then $|f(x)| \leqslant \sum_{i=1}^{n} \alpha_i p_i(x)$ for all $x \in X$. □

For the convenience of later use, we close this section with some results about linear functionals.

LEMMA 3.2.8. *Let X be a linear space over $\mathbb{C}$.*

(i) *Let $f : X \to \mathbb{R}$ be an $\mathbb{R}$-linear functional. Then $g(x) = f(x) - if(ix)$ $(x \in X)$ defines a $\mathbb{C}$-linear functional on X and $Re(g) = f$.*

(ii) *Let $g : X \to \mathbb{C}$ be a $\mathbb{C}$-linear functional and let $f = Re(g)$. Then $g(x) = f(x) - if(ix)$ $(x \in X)$ and f is an $\mathbb{R}$-linear functional on X.*

LEMMA 3.2.9. *Let X be a linear space over $\mathbb{F}$, let f be a linear functional on X, and let A be a convex subset of X. Then $f(A)$ is convex.*

LEMMA 3.2.10. *Let X be a linear space over $\mathbb{F}$ and let f, f_1, $\cdots$, f_n be linear functionals on X such that $\bigcap_{k=1}^{n} ker(f_k) \subseteq ker(f)$. Then there exist scalars $\alpha_1, \cdots, \alpha_n$ in $\mathbb{F}$ such that $f = \sum_{k=1}^{n} \alpha_k f_k$.*

PROOF. Let F be a subset of $\{1, \cdots, n\}$ such that $\bigcap_{i \in F} ker(f_i) \subseteq ker(f)$ but for all $k \in F$, $\bigcap_{i \in F - \{k\}} ker(f_i) \nsubseteq \bigcap_{i \in F} ker(f_i)$. Let us write $F = \{1, \cdots, n\}$.

Case 1: $n = 1$. In this case, $ker(f_1) \subseteq ker(f)$.

If $ker(f) = ker(f_1)$, then by Lemma 3.2.6, there exists $\alpha_1 \in \mathbb{F} - \{0\}$ such that $f = \alpha_1 f_1$.

If $ker(f_1) \subsetneqq ker(f)$, then there exists $y_0 \in ker(f) - ker(f_1)$. Let $x_0 = \frac{y_0}{f_1(y_0)}$. Then $f(x_0) = 0$ and $f_1(x_0) = 1$. Let $x \in X$ and let $y = x - f_1(x)x_0$. Then

$$f_1(y) = f_1(x) - f_1(x)f_1(x_0) = 0.$$

Since $ker(f_1) \subseteq ker(f)$, $f(y) = 0$. Hence, $f(y) = f(x) - f(x_0)f_1(x) = 0$; that is, $f(x) = f(x_0)f_1(x) = 0$ for all $x \in X$. Putting $\alpha_1 = 0$, we get $f = \alpha_1 f_1$.

Case 2: $n > 1$. Now $\bigcap_{j=1}^{n} ker(f_j) \subsetneqq \bigcap_{j \neq k} ker(f_j)$ for all $1 \leqslant k \leqslant n$.

Then for each $1 \leqslant k \leqslant n$, there exists $y_k \in \bigcap_{j \neq k} ker(f_j)$ such that $y_k \notin \bigcap_{j=1}^{n} ker(f_j)$. So, $f_k(y_k) \neq 0$ but $f_j(y_k) = 0$ for all $j \neq k$. Let $x_k = [f_k(y_k)]^{-1} y_k$. Then $f_k(x_k) = 1$ and $f_j(x_k) = 0$ if $j \neq k$. Let $x \in X$ and let $y = x - \sum_{k=1}^{n} f_k(x)x_k$. Then for each j,

$$f_j(y) = f_j(x) - \sum_{k=1}^{n} f_k(x)f_j(x_k) = f_j(x) - f_j(x)f_j(x_j) = f_j(x) - f_j(x) = 0.$$

Hence, $y \in \bigcap_{j=1}^{n} ker(f_j)$. Since $\bigcap_{j=1}^{n} ker(f_j) \subseteq ker(f)$,

$$f(y) = f(x) - \sum_{k=1}^{n} f_k(x) f(x_k) = 0;$$

that is, $f(x) = \sum_{k=1}^{n} f_k(x) f(x_k)$ for all $x \in X$. Taking $\alpha_k = f(x_k)$ $(k = 1, \cdots, n)$, we get $f = \sum_{k=1}^{n} \alpha_k f_k$. □

3.3. A Hahn-Banach theorem for real linear spaces

DEFINITION 3.3.1. Let X be a linear space. A function $q : X \to \mathbb{R}$ is called sublinear if

(i) for all $x, y \in X$, $q(x + y) \leqslant q(x) + q(y)$;

(ii) for $x \in X$ and $\alpha \geqslant 0$, $q(\alpha x) = \alpha q(x)$.

THEOREM 3.3.1. *Let X be a real linear space and let $q : X \to \mathbb{R}$ be a sublinear functional. Let M be a linear subspace of X and let $f : M \to \mathbb{R}$ be a linear functional such that $f(x) \leqslant q(x)$ for all $x \in M$. Then there exists a linear functional $F : X \to \mathbb{R}$ such that $F(x) \leqslant q(x)$ for all $x \in X$ and $F|_M = f$.*

PROOF. We can assume that M is a proper linear subspace of X.

First we assume that $\dim(X/M) = 1$. Let $x_0 \in X - M$. Then $\{x_0 + M\}$ is a basis of X/M, and $X = \{tx_0 + y : t \in \mathbb{R}, y \in M\}$. Now for all $x \in X$, there exist unique $t \in \mathbb{R}$ and $y \in M$ such that $x = tx_0 + y$.

Suppose that $F : X \to \mathbb{R}$ is such a linear extension of f. We want to see how F looks like in order to conclude that such F does exist. Write $\alpha = F(x_0)$. Then for all $t \in \mathbb{R}$ and $y \in M$,

$$F(tx_0 + y) = tF(x_0) + F(y) = tF(x_0) + f(y) = t\alpha + f(y).$$

Let $t > 0$. If $y_1 \in M$, then $F(tx_0 + y_1) = t\alpha + f(y_1) \leqslant q(tx_0 + y_1)$, and thus

$$\alpha \leqslant \frac{1}{t} q(tx_0 + y_1) - \frac{1}{t} f(y_1) = q(x_0 + \frac{1}{t} y_1) - f(\frac{1}{t} y_1).$$

Therefore,

$$\alpha \leqslant q(x_0 + y_1) - f(y_1) \text{ for all } y_1 \in M.$$

Similarly, if $y_2 \in M$, then $F(-tx_0 + y_2) = -t\alpha + f(y_2) \leqslant q(-tx_0 + y_2)$, and thus

$$\alpha \geqslant -\frac{1}{t}q(-tx_0 + y_2) + \frac{1}{t}f(y_2) = -q(-x_0 + \frac{1}{t}y_2) + f(\frac{1}{t}y_2).$$

Therefore,

$$\alpha \geqslant -q(-x_0 + y_2) + f(y_2) \text{ for all } y_2 \in M.$$

It follows that the number $\alpha = F(x_0)$ satisfies

$$-q(-x_0 + y_2) + f(y_2) \leqslant \alpha \leqslant q(x_0 + y_1) - f(y_1) \text{ for all } y_1, y_2 \in M.$$

Note that for all $y_1, y_2 \in M$,

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leqslant q(y_1 + y_2) \leqslant q(x_0 + y_1) + q(-x_0 + y_2);$$

that is, $-q(-x_0 + y_2) + f(y_2) \leqslant q(x_0 + y_1) - f(y_1)$ for all $y_1, y_2 \in M$. Hence,

$$\sup_{y \in M}(-q(-x_0 + y) + f(y)) \leqslant \inf_{y \in M}(q(x_0 + y) - f(y)).$$

Let $\alpha \in [\sup_{y \in M}(-q(-x_0 + y) + f(y)), \inf_{y \in M}(q(x_0 + y) - f(y))]$, and let

$$F(tx_0 + y) = t\alpha + f(y) \ (t \in \mathbb{R}, y \in M).$$

Since for all $x \in X$, there exist unique $t \in \mathbb{R}$ and $y \in M$ such that $x = tx_0 + y$, $F : X \to \mathbb{R}$ is well defined. For all $y_1, y_2 \in M$ and $t_1, t_2, \lambda \in \mathbb{R}$, we have

$$F(t_1x_0 + y_1 + \lambda(t_2x_0 + y_2)) = (t_1 + \lambda t_2)\alpha + f(y_1 + \lambda y_2)$$

$$= \ t_1\alpha + f(y_1) + \lambda(t_2\alpha + f(y_2)) = F(t_1x_0 + y_1) + \lambda F(t_2x_0 + y_2).$$

Hence, $F : X \to \mathbb{R}$ is linear. For all $y \in M$, we have $F(y) = 0 \cdot \alpha + f(y) = f(y)$. Thus $F|_M = f$. For all $t > 0$ and $y \in M$, we have

$$F(tx_0 + y) = t(\alpha + f(\frac{1}{t}y)) \leqslant t(q(x_0 + \frac{1}{t}y) - f(\frac{1}{t}y) + f(\frac{1}{t}y)) = q(tx_0 + y),$$

and

$$F(-tx_0+y) = t(-\alpha + f(\frac{1}{t}y)) \leqslant t(q(-x_0+\frac{1}{t}y) - f(\frac{1}{t}y) + f(\frac{1}{t}y)) = q(-tx_0+y).$$

Hence, $F(x) \leqslant q(x)$ for all $x \in X$.

For the general case, let S be the collection of all pairs (M_1, f_1), where M_1 is a linear subspace of X such that $M \subseteq M_1$ and f_1 is a linear extension of f with $f_1(x) \leqslant q(x)$ for all $x \in M_1$. Then $S \neq \emptyset$ since $(M, f) \in S$.

For $(M_1, f_1), (M_2, f_2) \in S$, define $(M_1, f_1) \preccurlyeq (M_2, f_2)$ if $M_1 \subseteq M_2$ and $f_2|_{M_1} = f_1$. Then $\preccurlyeq$ is a partial order on S. Let $C = \{(M_i, f_i) : i \in I\}$ be a chain in S, and let $\tilde{M} = \bigcup_{i\in I} M_i$. Then $\tilde{M}$ is a linear subspace of X since C is a chain in S. For $x \in \tilde{M}$, define $\tilde{f}(x) = f_i(x)$ if $x \in M_i$ for some $i \in I$. Suppose $x \in M_i \cap M_j$. Since C is a chain, we can assume that $(M_i, f_i) \preccurlyeq (M_j, f_j)$, and thus $f_i(x) = f_j(x)$. Therefore, $\tilde{f} : \tilde{M} \to \mathbb{R}$ is well defined. Let $x, y \in \tilde{M}$ and $\alpha \in \mathbb{R}$. Then $x, y \in M_j$ for some $j \in I$ and hence $x + \alpha y \in M_j$. Thus

$$\tilde{f}(x+\alpha y) = f_j(x+\alpha y) = f_j(x) + \alpha f_j(y) = \tilde{f}(x) + \alpha \tilde{f}(y).$$

Hence, $\tilde{f} : \tilde{M} \to \mathbb{R}$ is linear. Since $M \subseteq M_i \subseteq \tilde{M}$ and $\tilde{f}|_{M_i} = f_i$ for all $i \in I$, $\tilde{f}|_M = f$ and $\tilde{f} \leqslant q$ on $\tilde{M}$. Thus $(\tilde{M}, \tilde{f}) \in S$ and $(M_i, f_i) \preccurlyeq (\tilde{M}, \tilde{f})$ for all $i \in I$. Hence, $(\tilde{M}, \tilde{f})$ is an upper bound of C. By Zorn's lemma, $(S, \preccurlyeq)$ has a maximal element (Y, F).

Assume that $Y \neq X$. Let $x_0 \in X - Y$ and let $Z = \{tx_0 + y : t \in \mathbb{R}, y \in Y\}$. Then Z is a linear subspace of X containing M and $\dim(Z/Y) = 1$. By the proof above, there exists a linear functional $G : Z \to \mathbb{R}$ such that $G(x) \leqslant q(x)$ for all $x \in Z$ and $G|_Y = F$. Hence, $(Z, G) \in S$, $(Y, F) \preccurlyeq (Z, G)$ and $(Y, F) \neq (Z, G)$, which is contradicting to the fact that (Y, F) is a maximal element of $(S, \preccurlyeq)$. Therefore, $Y = X$; that is, F is a linear functional on X such that $F(x) \leqslant q(x)$ for all $x \in X$ and $F|_M = f$. □

3.4. Some consequences of the Hahn-Banach theorem

PROPOSITION 3.4.1. *Let X be a TVS, let G be an open convex subset of X that contains the origin, and let*

$$q(x) = \inf\{t : t > 0 \text{ and } x \in tG\} \ (x \in X).$$

Then q is a non-negative continuous sublinear functional on X and

$$G = \{x \in X : q(x) < 1\}.$$

PROOF. For all $x \in X$, we have $\frac{x}{n} \to 0$ when $n \to \infty$. Since G is a neighborhood of 0, there exists $n_0 > 0$ such that $\frac{x}{n_0} \in G$; that is, $x \in n_0 G$. Hence, $\{t > 0 : x \in tG\} \neq \emptyset$. Therefore, q is well defined and non-negative.

If $\alpha > 0$, then

$$\alpha q(x) = \alpha \inf\{t : x \in tG\} = \inf\{\alpha t : x \in tG\} = \inf\{\alpha t : \alpha x \in \alpha tG\} = q(\alpha x).$$

Note that $q(0) = \inf\{t : 0 \in tG\} = 0$. Hence, $q(\alpha x) = \alpha q(x)$ if $\alpha \geqslant 0$.

Let $x \in X$ and let $r = q(x)$. Then for every $\delta > 0$, there exists t_1 such that $0 < t_1 < r + \delta$ and $x \in t_1 G$. Now we claim that $aG \subseteq bG$ if $0 \leqslant a < b$. Let $g \in G$. Then $ag \in aG$. Since G is convex and contains 0, $(1 - \frac{a}{b})0 + \frac{a}{b} g \in G$, and thus $ag \in bG$. Hence, $aG \subseteq bG$. Since $x \in t_1 G$, by the claim above, $x \in t_1 G \subseteq (r + \delta)G$, and thus $x \in (r + \delta)G$ for all $\delta > 0$.

Now let $y \in X$ and $s = q(y)$. Then $y \in (s + \delta)G$ for all $\delta > 0$. Thus $\frac{x}{r+\delta} \in G$ and $\frac{y}{s+\delta} \in G$. Since G is convex, we have $\frac{r+\delta}{r+s+2\delta} \frac{x}{r+\delta} + \frac{s+\delta}{r+s+2\delta} \frac{y}{s+\delta} \in G$. It implies that $\frac{x+y}{r+s+2\delta} \in G$; that is, $x + y \in (r + s + 2\delta)G$. It follows that $q(x + y) \leqslant r + s + 2\delta$. Let $\delta \to 0$. Then $q(x + y) \leqslant r + s$; that is, $q(x + y) \leqslant q(x) + q(y)$. Therefore, q is a sublinear functional on X.

For any given $\varepsilon > 0$, we define $u = \varepsilon G \cap (-\varepsilon G)$. Since G contains 0 and εG and $-\varepsilon G$ are open, u is open and contains 0, and thus u is a neighborhood of 0. Let $x_0 \in X$ and let $(x_i)_{i \in I}$ be a net in X with $x_i \to x_0$. Then there exists $i_0 \in I$ such that $x_i - x_0 \in u$ for all $i \succcurlyeq i_0$. It follows that $x_i - x_0 \in \varepsilon G$ and $x_0 - x_i \in \varepsilon G$. Since q is

sublinear, we have

$$q(x_i) - q(x_0) \leqslant q(x_i - x_0) \leqslant \varepsilon \text{ and } q(x_0) - q(x_i) \leqslant q(x_0 - x_i) \leqslant \varepsilon;$$

that is, $|q(x_i) - q(x_0)| \leqslant \varepsilon$. Hence, $q(x_i) \to q(x_0)$. Therefore, q is a non-negative continuous sublinear functional on X.

Since X is a TVS, the map $\mathbb{F} \times X \to X$, $(\alpha, x) \mapsto \alpha x$ is continuous. Let $x_0 \in G$ and $\alpha = 1$. Then G is a neighborhood of x_0, and for some $r > 0$, there exists a neighborhood U of x_0 such that $\beta U \subseteq G$ whenever $|\beta - 1| < r$. Now let $\beta = \frac{r}{2} + 1$. Then $(1 + \frac{r}{2})U \subseteq G$. In particular, it follows that $x_0 \in (1 + \frac{r}{2})^{-1}G$. Thus

$$q(x_0) \leqslant (1 + \frac{r}{2})^{-1} < 1.$$

So, $G \subseteq \{x \in X : q(x) < 1\}$.

Conversely, suppose $q(x_0) < 1$ for some $x_0 \in X$. Let $\varepsilon = 1 - q(x_0)$ and let $a = 1 - \frac{\varepsilon}{2} = q(x_0) + \frac{\varepsilon}{2}$. By the proof in the 3rd paragraph, we have $x_0 \in aG$. Since G is convex and $a \in (0, 1)$, we have $x_0 = (1-a)0 + a\frac{x_0}{a} \in G$. Thus $\{x \in X : q(x) < 1\} \subseteq G$. Therefore, $G = \{x : q(x) < 1\}$. □

THEOREM 3.4.2. *Let X be a TVS and let G be an open convex non-empty subset of X that does not contain the origin. Then there exists a closed hyperplane M in X such that $M \cap G = \emptyset$.*

PROOF. Case 1. Suppose X is a TVS over $\mathbb{R}$. Let $x_0 \in G$ and let $H = x_0 - G$. Then $x_0 - x \in H$ for all $x \in G$. Now let $x_1, x_2 \in G$. Then for all $t \in [0, 1]$, we have

$$t(x_0 - x_1) + (1 - t)(x_0 - x_2) = x_0 - [tx_1 + (1 - t)x_2] \in x_0 - G = H,$$

since G is convex. Thus H is convex, and $0 = x_0 - x_0 \in H$. Therefore, H is an open convex set containing 0. By Proposition 3.4.1, there exists a non-negative continuous sublinear functional $q : X \to \mathbb{R}$ such that $H = \{x \in X : q(x) < 1\}$. Since G does not contain 0, we have $x_0 \notin H$, and thus $q(x_0) \geqslant 1$.

Let $Y = \{\alpha x_0 : \alpha \in \mathbb{R}\}$. Define $f_0 : Y \to \mathbb{R}$ by $f_0(\alpha x_0) = \alpha q(x_0)$. If $\alpha \geqslant 0$, then $f_0(\alpha x_0) = \alpha q(x_0) = q(\alpha x_0)$; if $\alpha < 0$, then $f_0(\alpha x_0) = \alpha q(x_0) \leqslant \alpha < 0 \leqslant q(\alpha x_0)$. Hence, $f_0 \leqslant q$ on Y. By Theorem 3.3.1, there exists a linear functional $f : X \to \mathbb{R}$ such that $f|_Y = f_0$ and $f \leqslant q$ on X. Put $M = ker(f)$. Then M is a hyperplane in X. Now if $x \in G$, then $x_0 - x \in H$, and thus

$$f(x_0) - f(x) = f(x_0 - x) \leqslant q(x_0 - x) < 1.$$

Hence, $f(x) > f(x_0) - 1 = q(x_0) - 1 \geqslant 0$ for all $x \in G$. Therefore, $M \cap G = \emptyset$.

Since $f \leqslant q$ on X and $q(x) < 1$ for all $x \in H$, $f(x) < 1$ for all $x \in H$. It follows that $f(-x) = -f(x) > -1$ for all $x \in H$. Thus $|f(x)| < 1$ for all $x \in H \cap (-H)$. Let $V = H \cap (-H)$. Then V is an open neighborhood of 0. By Proposition 3.2.7, $M = ker(f)$ is a closed hyperplane in X.

Case 2. Suppose X is a TVS over $\mathbb{C}$. Since X is also a TVS over $\mathbb{R}$, by Case 1 above, there exists a non-zero continuous $\mathbb{R}$-linear functional $f : X \to \mathbb{R}$ such that $(ker(f)) \cap G = \emptyset$. Let $F(x) = f(x) - if(ix)$ $(x \in X)$. Then by Lemma 3.2.8, F is a non-zero continuous $\mathbb{C}$-linear functional on X and $f = Re(F)$. Hence, $F(x) = 0$ if and only if $f(x) = f(ix) = 0$. Let $M' = ker(F)$. Then M' is a closed hyperplane in X and $M' \cap G \subseteq (ker(f)) \cap G = \emptyset$. □

Definition 3.4.1. Let X be a TVS. The dual space X^* of X is the linear space of all continuous linear functionals on X together with the pointwise linear operations.

Lemma 3.4.3. *Let X be a real TVS, let A be a non-empty open convex subset of X, and let $f \in X^*$ be non-zero. Then $f(A)$ is an open interval in $\mathbb{R}$.*

Proof. By Lemma 3.2.9, we only need show that $f(A)$ is open in $\mathbb{R}$. Let $x \in A$. Then $A - x$ is a neighborhood of 0 since A is open and $0 \in A - x$. Since $f \neq 0$, there exists $x_0 \in X$ such that $f(x_0) = 1$. Since the map $\mathbb{F} \to X, \alpha \mapsto \alpha x_0$ is continuous, there exists $\epsilon > 0$ such that $\alpha x_0 \in A - x$ whenever $|\alpha| < \epsilon$. Thus

$$f(x) + \alpha = f(x + \alpha x_0) \in f(A) \text{ if } |\alpha| < \epsilon.$$

Hence, $(f(x) - \epsilon, f(x) + \epsilon) \subseteq f(A)$. Therefore, $f(A)$ is open in $\mathbb{R}$. □

The next result is a Hahn-Banach separation theorem over real TVS. We will also prove some other separation theorems that follow from this one.

THEOREM 3.4.4. *Let X be a real TVS and let A and B be two disjoint non-empty convex sets in X with A open. Then there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(a) < \alpha \leqslant f(b) \text{ for all } a \in A \text{ and } b \in B.$$

If B is also open, then $f(b) > \alpha$ for all $b \in B$.

PROOF. Let $G = \{a - b : a \in A, b \in B\}$. Let $a_1, a_2 \in A$, let $b_1, b_2 \in B$, and let $t \in [0, 1]$. Then

$$t(a_1 - b_1) + (1 - t)(a_2 - b_2) = [ta_1 + (1 - t)a_2] - [tb_1 + (1 - t)b_2] \in G,$$

since A and B are convex. So, G is convex. Also, since $G = \bigcup_{b \in B} (A - b)$, G is open. Moreover, since $A \cap B = \emptyset$, $0 \notin G$. By Theorem 3.4.2, there exists a closed hyperplane M in X such that $M \cap G = \emptyset$. Let $f : X \to \mathbb{R}$ be a continuous linear functional such that $M = ker(f)$. Then $0 \notin f(G)$. Assume that there exist $x_1, x_2 \in G$ such that $f(x_1) > 0$ and $f(x_2) < 0$. Since G is convex, by Lemma 3.2.9, $f(G)$ is convex. Let $t_0 = \frac{f(x_2)}{f(x_2) - f(x_1)}$. Then $t_0 \in (0, 1)$ and

$$t_0 f(x_1) + (1 - t_0) f(x_2) = \frac{f(x_2)f(x_1)}{f(x_2) - f(x_1)} - \frac{f(x_1)f(x_2)}{f(x_2) - f(x_1)} = 0 \notin f(G),$$

contradicting that $f(G)$ is convex. Therefore, either $f(x) > 0$ for all $x \in G$ or $f(x) < 0$ for all $x \in G$.

Suppose $f(x) < 0$ for all $x \in G$. Then for all $a \in A$ and $b \in B$,

$$f(a - b) = f(a) - f(b) < 0;$$

that is, $f(a) < f(b)$. Let $\alpha = \sup\{f(a) : a \in A\}$. Then $\alpha \leqslant \inf\{f(b) : b \in B\}$. Since A is open, by Lemma 3.4.3, $f(a) < \alpha$ for all $a \in A$, and now $f(b) \geqslant \alpha$ for all $b \in B$. If B is also open, by Lemma 3.4.3 again, $f(b) > \alpha$ for all $b \in B$.

Suppose $f(x) > 0$ for all $x \in G$. Let $g = -f$. Then g is a continuous linear functional on X and $g(x) < 0$ for all $x \in G$. Thus the assertion follows by replacing f above by g. □

LEMMA 3.4.5. *Let X be a TVS, let K be a compact subset of X, and let V be an open subset of X such that $K \subseteq V$. Then there exists an open neighborhood U of 0 in X such that $K + U \subseteq V$.*

PROOF. Let $\mathcal{U}$ be the family of all open neighborhoods of 0. Assume that for each U in $\mathcal{U}$, $K + U$ is not contained in V. Then for each U in $\mathcal{U}$, there exist $x_U \in K$ and $y_U \in U$ such that $x_U + y_U \in X - V$. Order $\mathcal{U}$ by reverse inclusion. Then $\mathcal{U}$ is a directed set, and (x_U) and (y_U) are nets. Now $y_U \to 0$ in X. Since K is compact, by Theorem 2.10, (x_U) has a subnet (x_u) such that $x_u \to x$ for some $x \in K$. By Proposition 2.8, $y_u \to 0$. Hence, $x_u + y_u \to x$. Since $x_u + y_u \in X - V$, by Proposition 2.4, $x \in \overline{X - V} = X - V$, which is contradicting to the fact that $x \in K \subseteq V$. Therefore, there exists an open neighborhood U of 0 such that $K + U \subseteq V$. □

THEOREM 3.4.6. *Let X be a real LCS and let A and B be two disjoint non-empty closed convex subsets of X with B compact. Then there exist $f \in X^*$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that*

$$f(a) \leqslant \alpha < \alpha + \varepsilon \leqslant f(b) \text{ for all } a \in A \text{ and } b \in B.$$

PROOF. Since $B \subseteq X - A$, by Lemma 3.4.5, there exists an open neighborhood U_1 of 0 such that $B + U_1 \subseteq X - A$. Let P be the family of seminorms that defines the topology on X. Then by Definition 3.1.3, there exist $p_i \in P$ and $\varepsilon_i > 0$ such that $V_1 = \bigcap_{i=1}^{n} B_i \subseteq U_1$, where $B_i = \{x \in X : p_i(x) < \varepsilon_i\}$ $(i = 1, \cdots, n)$. Let $1 \leqslant i \leqslant n$ and $x_1, x_2 \in B_i$. Then for all $t \in [0, 1]$ and $x_3 = tx_1 + (1 - t)x_2$, we have

$$p_i(x_3) \leqslant p_i(tx_1) + p_i((1 - t)x_2) = tp_i(x_1) + (1 - t)p_i(x_2) < t\varepsilon_i + (1 - t)\varepsilon_i = \varepsilon_i.$$

Thus each B_i is convex. Hence, V_1 is an open convex subset of U_1. It follows that $B + V_1$ is an open convex set and $(B + V_1) \cap A = \emptyset$. By Theorem 3.4.4, there exist a

continuous linear functional f on X and $\alpha \in \mathbb{R}$ such that

$$f(a) \leqslant \alpha < f(c) \text{ for all } a \in A \text{ and } c \in B + V_1.$$

Since B is compact, by Proposition 2.12, $\inf\{f(b) : b \in B\} = f(b_0)$ for some $b_0 \in B$. Thus $f(b) \geqslant f(b_0) = \alpha + \varepsilon$ for all $b \in B$, where $\varepsilon = f(b_0) - \alpha > 0$ since $f(b_0) > \alpha$. Therefore, $f(a) \leqslant \alpha < \alpha + \varepsilon \leqslant f(b)$ for all $a \in A$ and $b \in B$. □

THEOREM 3.4.7. *Let X be a complex LCS and let A and B be two disjoint non-empty closed convex subsets of X with B compact. Then there exist $f \in X^*$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that*

$$Re(f(a)) \leqslant \alpha < \alpha + \varepsilon \leqslant Re(f(b)) \text{ for all } a \in A \text{ and } b \in B.$$

PROOF. Since a complex LCS is also a real LCS, by Theorem 3.4.6, there exist a continuous $\mathbb{R}$-linear functional f_1 on X, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$f_1(a) \leqslant \alpha < \alpha + \varepsilon \leqslant f_1(b) \text{ for all } a \in A \text{ and } b \in B.$$

By Lemma 3.2.8, $f(x) = f_1(x) - if_1(ix)$ is a $\mathbb{C}$-linear functional on X and $Re(f) = f_1$. Therefore,

$$Re(f(a)) \leqslant \alpha < \alpha + \varepsilon \leqslant Re(f(b)) \text{ for all } a \in A \text{ and } b \in B.$$

Clearly, $f : X \to \mathbb{C}$ is continuous since $f_1 : X \to \mathbb{R}$ is continuous. □

THEOREM 3.4.8. *Let X be a LCS, let M be a closed linear subspace of X, and let $x_0 \in X - M$. Then there exists $h \in X^*$ such that $h(x_0) = 1$ and $h|_M = 0$.*

PROOF. Since $\{x_0\}$ and M are disjoint closed convex sets in X with $\{x_0\}$ compact, by Theorems 3.4.6 and 3.4.7, there exist $g \in X^*$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$Re(g(m)) \leqslant \alpha < \alpha + \varepsilon \leqslant Re(g(x_0)) \text{ for all } m \in M.$$

Assume that $Re(g(m_0)) \neq 0$ for some $m_0 \in M$. Let $\lambda = \frac{\alpha+1}{Re(g(m_0))}$. Then $\lambda m_0 \in M$ and $Re(g(\lambda m_0)) = \lambda Re(g(m_0)) = \alpha + 1$, which is contradicting to $Re(g(\lambda m_0)) \leqslant \alpha$.

Hence, $Re(g(m)) = 0$ for all $m \in M$. By Lemma 3.2.8, $g(m) = 0$ for all $m \in M$. Since $Re(g(x_0)) > Re(g(0)) = 0$, we have $g(x_0) \neq 0$. Let $h = \frac{1}{g(x_0)} g \in X^*$. Then $h(x_0) = 1$ and $h(m) = 0$ for all $m \in M$. □

COROLLARY 3.4.9. *Let X be a LCS and let M be a linear subspace of X. Then*

$$\overline{M} = \bigcap \{\ker(f) : f \in X^* \text{ and } M \subseteq \ker(f)\}.$$

Therefore, M is dense in X if and only if 0 is the only element of X^* that vanishes on M.

THEOREM 3.4.10. *Let X be a LCS, let M be a linear subspace of X, and let $f \in M^*$. Then there exists $h \in X^*$ such that $h|_M = f$.*

PROOF. We can assume that $f \neq 0$. Let $M_0 = ker(f)$. Let $x_0 \in M$ be such that $f(x_0) = 1$. Then $x_0 \notin \overline{M_0}$ by the continuity of f. By Theorem 3.4.8, there exists $h \in X^*$ such that $h(x_0) = 1$ and $h(m) = 0$ for all $m \in \overline{M_0}$. Now for all $x \in M$, we have

$$f(x - f(x)x_0) = f(x) - f(x)f(x_0) = 0,$$

and thus $x - f(x)x_0 \in M_0 \subseteq \overline{M_0}$. Hence, $h(x) - f(x) = h(x - f(x)x_0) = 0$ for all $x \in M$. Therefore, $h|_M = f$. □

The proposition below holds by Theorem 3.3.1, Lemma 3.2.8 and Proposition 2.14.

PROPOSITION 3.4.11. *Let X be a normed space and let Y be a linear subspace of X. Let $f \in Y^*$. Then there exists $g \in X^*$ such that $g|_Y = f$ and $\|f\| = \|g\|$.*

Furthermore, if Y is dense in X, then the extension $g \in X^*$ of f is unique.

COROLLARY 3.4.12. *Let X be a normed space and let $x \in X$. Then*

$$\|x\| = sup\{|f(x)| : f \in X^* \text{ and } \|f\| \leqslant 1\}.$$

CHAPTER 4

Weak topology and weak-star topology

In Section 4.1, we consider the weak topology wk on a TVS X and the weak-star topology wk^* on its dual X^*, and give some of their basic properties. In Section 4.2, we study basic results on the dual of (X, wk) and (X^*, wk^*), and characterize the closed convex balanced hull and the closed linear span for sets in X and (X^*, wk^*) via bipolars and biannihilators, respectively. In Sections 4.3 and 4.4, we identify the dual of a quotient space with the annihilator of the subspace, and identify the dual of a subspace of a LCS with the quotient space of the annihilator of the subspace. The main references for this chapter are [**2**] and [**3**].

For $x \in X$ and $x^* \in X^*$, $\langle x, x^* \rangle$ and $\langle x^*, x \rangle$ both will stand for $x^*(x)$.

4.1. Definitions and basic properties

PROPOSITION 4.1.1. *Let X be a linear space and let f be a linear functional on X. Then the function $p_f : X \to [0, \infty), x \mapsto |f(x)|$ is a seminorm on X.*

Let T_1 and T_2 be two topologies on a set X. If $T_1 \subseteq T_2$, then we say that T_1 is weaker that T_2, or that T_2 is stronger than T_1.

DEFINITION 4.1.1. Let X be a TVS. The weak topology $\sigma(X, X^*)$ on X, also denoted by "wk", is the topology on X defined by the family $\{p_{x^*} : x^* \in X^*\}$ of seminorms on X.

PROPOSITION 4.1.2. *Let (X, T) be a TVS. Then (X, wk) is a TVS, and $\sigma(X, X^*)$ is weaker than T.*

PROOF. By Proposition 3.1.3, (X, wk) is a TVS. Let $x_0 \in X$ and let (x_i) be a net in X with $x_i \to x_0$ in (X, T). Then for all $x^* \in X^*$, $p_{x^*}(x_i - x_0) = |x^*(x_i) - x^*(x_0)| \to 0$. By Lemma 3.1.2, $x_i \to x_0$ in (X, wk). Therefore, $\sigma(X, X^*)$ is weaker than T. □

PROPOSITION 4.1.3. *Let X be a normed space. Then (X, wk) is a LCS.*

PROOF. Let $x_0 \in X - \{0\}$. Then by Theorem 3.4.8, there exists $x^* \in X^*$ such that $x^*(x_0) \neq 0$; that is, $p_{x^*}(x_0) \neq 0$. Hence, $x_0 \notin \bigcap_{x^* \in X^*} \{x : p_{x^*}(x) = 0\}$. Therefore, (X, wk) is a LCS. □

DEFINITION 4.1.2. Let X be a TVS and let $x \in X$. Then x defines a linear functional $\hat{x}$ on X^* via $\hat{x}(f) = f(x)$ $(f \in X^*)$.

DEFINITION 4.1.3. Let X be a TVS. The weak-star topology $\sigma(X^*, X)$ on X^*, also denoted by "wk^*", is the topology on X^* defined by the family $\{p_{\hat{x}} : x \in X\}$ of seminorms on X^*.

Since (X^*, wk^*) is a Hausdorff space, the proposition below follows immediately from Theorem 3.1.4.

PROPOSITION 4.1.4. *Let X be a TVS. Then (X^*, wk^*) is a LCS.*

By Proposition 3.1.3 and the definition of $\sigma(X^*, X)$, $\hat{x} \in (X^*, wk^*)^*$ for all $x \in X$. Clearly, the map $X \to (X^*, wk^*)^*$, $x \mapsto \hat{x}$ is linear. We will use $\widehat{X}$ to denote the linear subspace $\{\hat{x} : x \in X\}$ of $(X^*, wk^*)^*$.

PROPOSITION 4.1.5. *Let X be a TVS. Then the linear map $X \to \widehat{X}$, $x \mapsto \hat{x}$ is injective if and only if (X, wk) is a LCS.*

PROOF. Suppose the map $X \to \widehat{X}$, $x \mapsto \hat{x}$ is injective. Let $x \in \bigcap_{f \in X^*} ker(p_f)$. Then $\hat{x}(f) = f(x) = 0$ for all $f \in X^*$. Thus $\hat{x} = 0$ and hence $x = 0$. Therefore, (X, wk) is a LCS.

Conversely, suppose (X, wk) is a LCS. Let $x \in X$ be such that $\hat{x} = 0$. Then $f(x) = \hat{x}(f) = 0$ for all $f \in X^*$. Since (X, wk) is a LCS, $x = 0$. Therefore, the map $X \to \widehat{X}$, $x \mapsto \hat{x}$ is injective. □

We will see in the next section that for all TVS X, $\widehat{X} = (X^*, wk^*)^*$ and hence $\sigma(X^*, X) = \sigma(X^*, (X^*, wk^*)^*)$.

4.2. Duality

THEOREM 4.2.1. *Let X be a TVS. Then $(X, wk)^* = X^*$.*

PROOF. Let $g \in X^*$. Let $x \in X$ and let (x_i) be a net in X such that $x_i \xrightarrow{wk} x$. Then $\langle g, x_i\rangle \to \langle g, x\rangle$. So, g is weakly continuous on X. Therefore, $X^* \subseteq (X, wk)^*$. Conversely, let $f \in (X, wk)^*$. Then for all open sets G in $\mathbb{F}$, $f^{-1}(G)$ is open in (X, wk). Since $\sigma(X, X^*)$ is weaker than the original topology on X, $f^{-1}(G)$ is open in X. Hence, $f \in X^*$. Therefore, $(X, wk)^* \subseteq X^*$. □

Recall that for a TVS X, $\widehat{X}$ denotes the linear subspace $\{\hat{x} : x \in X\}$ of $(X^*, wk^*)^*$.

THEOREM 4.2.2. *Let X be a TVS. Then $(X^*, wk^*)^* = \widehat{X}$.*

PROOF. We only have to show that $(X^*, wk^*)^* \subseteq \widehat{X}$. Let $f \in (X^*, wk^*)^*$. By Proposition 3.2.7, there exist $x_1, \cdots, x_n \in X$ and positive scalars $c_1, \cdots, c_n$ such that

$$|f(x^*)| \leqslant \sum_{k=1}^{n} c_k |\langle x_k, x^*\rangle| \text{ for all } x^* \in X^*.$$

Thus $\bigcap_{k=1}^{n} ker(\widehat{x_k}) \subseteq ker(f)$. By Lemma 3.2.10, there exist $\alpha_1, \cdots, \alpha_n \in \mathbb{F}$ such that $f = \sum_{i=1}^{n} \alpha_i \widehat{x_i}$. Let $x = \sum_{i=1}^{n} \alpha_i x_i$. Then $f = \hat{x} \in \widehat{X}$. Hence, $(X^*, wk^*)^* \subseteq \widehat{X}$. □

THEOREM 4.2.3. *Let X be a LCS and let A be a convex subset of X. Then*

$$\overline{A} = \overline{A}^{wk} \text{ (the weak closure of } A \text{ in } X).$$

PROOF. Since $\sigma(X, X^*)$ is weaker than the original topology on X, every weakly closed set is closed in X. By the definition of a closure, we have $\overline{A} \subseteq \overline{A}^{wk}$.

Conversely, let $x \in X - \overline{A}$. Then by Theorems 3.4.6 and 3.4.7, there exist $x^* \in X^*$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that $Re\langle a, x^*\rangle \leqslant \alpha < \alpha + \varepsilon \leqslant Re\langle x, x^*\rangle$ for all $a \in \overline{A}$. Hence, $A \subseteq B = \{y \in X : Re\langle y, x^*\rangle \leqslant \alpha\}$. Since x^* is weakly continuous by Theorem 4.2.1, B is weakly closed in X, and thus $\overline{A}^{wk} \subseteq B$. Since $x \notin B$, we have $x \notin \overline{A}^{wk}$. Therefore, $\overline{A}^{wk} \subseteq \overline{A}$. □

The corollary below follows immediately from Proposition 2.3 and Theorem 4.2.3.

COROLLARY 4.2.4. *Let X be a LCS. Then a convex subset of X is closed if and only if it is weakly closed.*

DEFINITION 4.2.1. Let X be a TVS. Let $A \subseteq X$ and $B \subseteq X^*$. The polar of A, denoted by A°, is the subset of X^* defined by

$$A^\circ = \{x^* \in X^* : |\langle a, x^* \rangle| \leqslant 1 \text{ for all } a \in A\},$$

and the prepolar of B, denoted by ${}^\circ B$, is the subset of X defined by

$${}^\circ B = \{x \in X : |\langle x, b^* \rangle| \leqslant 1 \text{ for all } b^* \in B\}.$$

The bipolar of A is the set ${}^\circ(A^\circ)$ (also denoted by ${}^\circ A^\circ$), and the bipolar of B is the set $({}^\circ B)^\circ$ (also denoted by ${}^\circ B^\circ$).

DEFINITION 4.2.2. A subset S of a linear space is called a balanced set if $\alpha S \subseteq S$ for all scalars α in $\mathbb{F}$ with $|\alpha| \leqslant 1$.

PROPOSITION 4.2.5. *Let X be a TVS. Let $A \subseteq X$ and $B \subseteq X^*$.*

(a) *A° and ${}^\circ B$ are convex and balanced.*

(b) *If $A_1 \subseteq A$ and $B_1 \subseteq B$, then $A^\circ \subseteq A_1^\circ$ and ${}^\circ B \subseteq {}^\circ B_1$.*

(c) *If $\alpha \in \mathbb{F}$ and $\alpha \neq 0$, then $(\alpha A)^\circ = \alpha^{-1} A^\circ$ and ${}^\circ(\alpha B) = \alpha^{-1}({}^\circ B)$.*

(d) *$A \subseteq {}^\circ A^\circ$ and $B \subseteq {}^\circ B^\circ$.*

(e) *$A^\circ = ({}^\circ A^\circ)^\circ$ and ${}^\circ B = {}^\circ({}^\circ B^\circ)$.*

PROOF. It is trivial that (a) and (b) hold.

(c) Let $\alpha \in \mathbb{F} - \{0\}$. Let $x^* \in A^\circ$. Then for all $a \in A$, $|\langle \alpha a, \alpha^{-1} x^* \rangle| = |x^*(a)| \leqslant 1$. Thus $\alpha^{-1} x^* \in (\alpha A)^\circ$. Hence, $\alpha^{-1} A^\circ \subseteq (\alpha A)^\circ$; that is, $A^\circ \subseteq \alpha(\alpha A)^\circ$. Replacing α by α^{-1} and A by αA, we get $(\alpha A)^\circ \subseteq \alpha^{-1} A^\circ$. Therefore, $(\alpha A)^\circ = \alpha^{-1} A^\circ$. Similarly, we can prove that ${}^\circ(\alpha B) = \alpha^{-1}({}^\circ B)$.

(d) Let $a \in A$. Then $|\langle a, x^* \rangle| \leqslant 1$ for all $x^* \in A^\circ$. By the definition of a prepolar, $a \in {}^\circ A^\circ$. Therefore, $A \subseteq {}^\circ A^\circ$. Similarly, we have $B \subseteq {}^\circ B^\circ$.

(e) By (d), we have $A \subseteq {}^\circ A^\circ$. Then by (b), $({}^\circ A^\circ)^\circ \subseteq A^\circ$. Taking $B = A^\circ$ in (d), we have $A^\circ \subseteq {}^\circ(A^\circ)^\circ = ({}^\circ A^\circ)^\circ$. Therefore, $A^\circ = ({}^\circ A^\circ)^\circ$. The equality ${}^\circ B = {}^\circ({}^\circ B^\circ)$ can be obtained similarly. □

DEFINITION 4.2.3. Let X be a TVS. If $A \subseteq X$, then the closed convex hull (respectively, closed convex balanced hull) of A in X is the intersection of all closed convex (respectively, closed convex balanced) subsets of X that contain A.

THEOREM 4.2.6 (Bipolar theorem). *Let X be a LCS and let $A \subseteq X$. Then ${}^\circ A^\circ$ is the closed convex balanced hull of A in X. In particular, if A is convex and balanced, then ${}^\circ A^\circ = \overline{A}$.*

PROOF. Let A_1 be the closed convex balanced hull of A in X. Then $A_1 \subseteq {}^\circ A^\circ$, since ${}^\circ A^\circ$ is closed, convex and balanced, and $A \subseteq {}^\circ A^\circ$.

Let $x_0 \in X - A_1$. Since A_1 is a closed convex set, by Theorems 3.4.6 and 3.4.7, there exist $x^* \in X^*$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$Re\langle a_1, x^* \rangle \leqslant \alpha < \alpha + \varepsilon \leqslant Re\langle x_0, x^* \rangle \text{ for all } a_1 \in A_1.$$

Since A_1 is balanced, $0 \in A_1$, and hence $Re\langle 0, x^* \rangle = 0 < \alpha$. So, we can replace x^* with $\alpha^{-1}x^*$ in the above. It follows that there exists $\delta > 0$ such that

$$Re\langle a_1, x^* \rangle \leqslant 1 < 1 + \delta \leqslant Re\langle x_0, x^* \rangle \text{ for all } a_1 \in A_1.$$

If $a_1 \in A_1$ and $\langle a_1, x^* \rangle = |\langle a_1, x^* \rangle| e^{i\theta}$ for some $\theta \in \mathbb{R}$, since $|e^{-i\theta}| = 1$ and A_1 is balanced, we have $e^{-i\theta} a_1 \in A_1$, and thus

$$|\langle a_1, x^* \rangle| = Re\langle e^{-i\theta} a_1, x^* \rangle \leqslant 1 < Re\langle x_0, x^* \rangle.$$

Hence, $x^* \in A_1^\circ$. Since $A \subseteq A_1$, by Proposition 4.2.5, $A_1^\circ \subseteq A^\circ$. It follows that $x^* \in A^\circ$. On the other hand, since $|\langle x_0, x^* \rangle| \geqslant Re\langle x_0, x^* \rangle > 1$, we have $x_0 \notin {}^\circ A^\circ$. Therefore, ${}^\circ A^\circ \subseteq A_1$. □

THEOREM 4.2.7 (Bipolar theorem, dual version). *Let X be a TVS and let $B \subseteq X^*$. Then $^\circ B^\circ$ is the wk^* closed convex balanced hull of B in X^*. In particular, if B is convex and balanced, then $^\circ B^\circ = \overline{B}^{wk^*}$.*

PROOF. Let $Y = (X^*, wk^*)$. Then Y is a LCS and $Y^* = \widehat{X}$ by Proposition 4.1.4 and Theorem 4.2.2. By Theorem 4.2.6, the bipolar $^\circ(B^\circ)$ of B in Y is the wk^* closed convex balanced hull of B in X^*. On the other hand, due to Definition 4.2.1, we have $B^\circ = \{\hat{x} \in \widehat{X} : |\langle x, x^*\rangle| \leqslant 1 \text{ for all } x^* \in B\} = \widehat{^\circ B}$, and thus we get

$$\begin{aligned} ^\circ(B^\circ) &= \{y \in Y : |\langle y, \hat{x}\rangle| \leqslant 1 \text{ for all } x \in {}^\circ B\} \\ &= \{x^* \in X^* : |\langle x^*, x\rangle| \leqslant 1 \text{ for all } x \in {}^\circ B\} = {}^\circ B^\circ. \end{aligned}$$

Therefore, $^\circ B^\circ$ is the wk^* closed convex balanced hull of B in X^*. □

DEFINITION 4.2.4. Let X be a TVS. Let $A \subseteq X$ and $B \subseteq X^*$. The annihilator of A, denoted by $A^\perp$, is the subset of X^* defined by

$$A^\perp = \{x^* \in X^* : \langle a, x^*\rangle = 0 \text{ for all } a \in A\},$$

and the pre-annihilator of B, denoted by ${}^\perp B$, is the subset of X defined by

$${}^\perp B = \{x \in X : \langle x, b^*\rangle = 0 \text{ for all } b^* \in B\}.$$

The biannihilator of A is the set ${}^\perp(A^\perp)$, and the biannihilator of B is the set $({}^\perp B)^\perp$.

PROPOSITION 4.2.8. *Let X be a TVS. Let $A \subseteq X$ and $B \subseteq X^*$. Then $A^\perp$ is a weak-star closed linear subspace of X^*, and ${}^\perp B$ is a closed linear subspace of X.*

PROOF. Let $a \in A$. Then $\hat{a} \in \widehat{X} = (X^*, wk^*)^*$ by Theorem 4.2.2. It follows from Proposition 3.2.7 that $\ker(\hat{a})$ is a weak-star closed linear subspace of X^*. Therefore, $A^\perp = \bigcap_{a \in A} \ker(\hat{a})$ is a weak-star closed linear subspace of X^*.

For each $x^* \in B$, by Proposition 3.2.7, $\ker(x^*)$ is a closed linear subspace of X. Hence, ${}^\perp B = \bigcap_{x^* \in B} \ker(x^*)$ is a closed linear subspace of X. □

PROPOSITION 4.2.9. *Let X be a LCS and let $A \subseteq X$. Then ${}^\perp(A^\perp) = \overline{\text{span}}(A)$.*

PROOF. Since $A^\perp \subseteq X^*$, by Proposition 4.2.8, ${}^\perp(A^\perp)$ is a closed linear subspace of X. By the definition of $A^\perp$, $A \subseteq {}^\perp(A^\perp)$, and thus $\overline{\text{span}}(A) \subseteq {}^\perp(A^\perp)$. Assume that $x_0 \in {}^\perp(A^\perp) - \overline{\text{span}}(A)$. Since $\overline{\text{span}}(A)$ is a closed linear subspace of X, by Theorem 3.4.8, there exists $f \in X^*$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in \overline{\text{span}}(A)$. In particular, $f(x) = 0$ for all $x \in A$; that is, $f \in A^\perp$. Thus $f(x_0) = 0$, which is a contradiction. Therefore, ${}^\perp(A^\perp) = \overline{\text{span}}(A)$. □

PROPOSITION 4.2.10. *Let X be a TVS and let $B \subseteq X^*$. Then $({}^\perp B)^\perp = \overline{\text{span}}^{wk^*}(B)$.*

PROOF. Let $Y = (X^*, wk^*)$. Then Y is a LCS and $Y^* = \widehat{X}$ by Proposition 4.1.4 and Theorem 4.2.2. Let E denote the annihilator of B in Y^* and let F denote the biannihilator of B in Y. By Proposition 4.2.9, $F = \overline{\text{span}}^{wk^*}(B)$. By Definition 4.2.4, $E = \{\hat{x} \in \widehat{X} : \langle x, x^* \rangle = 0 \text{ for all } x^* \in B\} = \widehat{{}^\perp B}$, and thus we get

$$\begin{aligned} F &= \{y \in Y : \langle y, e \rangle = 0 \text{ for all } e \in E\} \\ &= \{x^* \in X^* : \langle x^*, x \rangle = 0 \text{ for all } x \in {}^\perp B\} = ({}^\perp B)^\perp. \end{aligned}$$

Therefore, $({}^\perp B)^\perp = \overline{\text{span}}^{wk^*}(B)$. □

4.3. The dual of a quotient space

THEOREM 4.3.1. *Let X be a TVS, let M be a closed linear subspace of X, and let X/M be equipped with the quotient topology induced by the canonical quotient map $Q : X \to X/M$. Then $\rho : (X/M)^* \to M^\perp, f \mapsto f \circ Q$ is a linear bijection.*

If $(X/M)^*$ has its weak-star topology $\sigma((X/M)^*, X/M)$ and $M^\perp$ has the relative $\sigma(X^*, X)$ topology, then $\rho : (X/M)^* \to M^\perp$ is a homeomorphism.

Furthermore, if X is a normed space, then $\rho : (X/M)^* \to M^\perp$ is an isometry.

PROOF. Let $f \in (X/M)^*$. Then for all $x \in M$,

$$\rho(f)(x) = (f \circ Q)(x) = f(x + M) = f(0) = 0;$$

that is, $\rho(f) \in M^\perp$. So, $f \mapsto f \circ Q$ defines a map from $(X/M)^*$ to $M^\perp$.

Clearly, ρ is linear. If $\rho(f) = f \circ Q = 0$, then $f = 0$ since Q is surjective. Hence, ρ is injective. Let $x^* \in M^\perp$. Define $f : X/M \to \mathbb{F}$ by $f(x + M) = \langle x, x^* \rangle$ $(x \in X)$. Then f is well defined, since if $x_1 + M = x_2 + M$, then $x_1 - x_2 \in M$ and hence

$$\langle x_1, x^* \rangle - \langle x_2, x^* \rangle = \langle x_1 - x_2, x^* \rangle = 0.$$

Obviously, $f : X/M \to \mathbb{F}$ is linear. Now $f \circ Q = x^* : X \to \mathbb{F}$ is continuous. By the definition of the quotient topology, $f : X/M \to \mathbb{F}$ is continuous. Thus $f \in (X/M)^*$ and $x^* = \rho(f)$. Hence, $\rho : (X/M)^* \to M^\perp$ is surjective. Therefore, ρ is a linear bijection between $(X/M)^*$ and $M^\perp$.

For a net (f_i) in $(X/M)^*$, we have $f_i \to 0$ in $\sigma((X/M)^*, X/M)$ if and only if $f_i(x + M) \to 0$ for all $x \in X$, and $\rho(f_i) \to 0$ in $\sigma(X^*, X)$ if and only if $\rho(f_i)(x) = f_i(x + M) \to 0$ for all $x \in X$. Therefore, $\rho : (X/M)^* \to M^\perp$ is a homeomorphism when $(X/M)^*$ has its weak*-topology and $M^\perp$ has its relative $\sigma(X^*, X)$ topology.

Suppose X is a normed space. Let $f \in (X/M)^*$. It follows from Proposition 3.2.5 that

$$\|\rho(f)\| = \|f \circ Q\| \leqslant \|f\| \cdot \|Q\| \leqslant \|f\|.$$

Let $(x_n + M)$ be a sequence in X/M such that $\|x_n + M\| < 1$ and $|f(x_n + M)| \to \|f\|$. Since for each n, there exists $y_n \in M$ such that $\|x_n + y_n\| < 1$, we have

$$\|\rho(f)\| \geqslant |\rho(f)(x_n + y_n)| = |f(x_n + M)| \to \|f\|.$$

Hence, $\|\rho(f)\| \geqslant \|f\|$. Therefore, $\rho : (X/M)^* \to M^\perp$ is an isometry. □

4.4. The dual of a subspace

THEOREM 4.4.1. *Let X be a LCS, let M be a closed linear subspace of X, and let $r : X^* \to M^*$ be the restriction map and $Q : X^* \to X^*/M^\perp$ be the canonical quotient map. Then r induces a linear bijection $\tilde{r} : X^*/M^\perp \to M^*$ given by $\tilde{r}(f + M^\perp) = f|_M$.*

If $X^*/M^\perp$ has the quotient topology when X^* is equipped with $\sigma(X^*, X)$ and M^* has its weak-star topology $\sigma(M^*, M)$, then $\tilde{r} : X^*/M^\perp \to M^*$ is a homeomorphism.

Furthermore, if X is a normed space, then $\tilde{r} : X^*/M^\perp \to M^*$ is an isometry.

PROOF. Clearly, $\tilde{r} : X^*/M^\perp \to M^*, f + M^\perp \mapsto f|_M$ is well defined and linear. If $\tilde{r}(f + M^\perp) = f|_M = 0$, then $f \in M^\perp$ and thus $f + M^\perp = 0$. Hence, $\tilde{r}$ is injective. By Theorem 3.4.10, for all $f \in M^*$, there exists $F \in X^*$ such that $F|_M = f$. Hence, $r : X^* \to M^*$ is surjective. Since $\tilde{r} \circ Q = r$, $\tilde{r} : X^*/M^\perp \to M^*$ is surjective. Therefore, $\tilde{r} : X^*/M^\perp \to M^*$ is a linear bijection.

Let $X^*/M^\perp$ be equipped with the quotient topology when X^* is equipped with $\sigma(X^*, X)$ and M^* is equipped with weak-star topology $\sigma(M^*, M)$. Let $g \in X^*$ and let (g_i) be a net in X^* with $g_i \to g$ in $\sigma(X^*, X)$. Then $g_i|_M \to g|_M$ in $\sigma(M^*, M)$. Hence, $r : X^* \to M^*$ is wk^*-wk^* continuous. Since $\tilde{r} \circ Q = r$, by the definition of the quotient topology, $\tilde{r} : X^*/M^\perp \to M^*$ is continuous.

Recall that for all $x \in X$, $p_{\hat{x}}(x^*) = |\langle x, x^* \rangle|$ $(x^* \in X^*)$ is a seminorm on X^*. By Proposition 3.2.3, the quotient topology on $X^*/M^\perp$ is defined by the seminorms $\{\overline{p_{\hat{x}}} : x \in X\}$, where $\overline{p_{\hat{x}}}(x^* + M^\perp) = \inf\{|\langle x, x^* + z^* \rangle| : z^* \in M^\perp\}$.

Let $x \in X - M$. We claim that $\overline{p_{\hat{x}}} = 0$. Let $x^* \in X^*$. By Theorem 3.4.8, there exists $h \in M^\perp$ such that $h(x) = -x^*(x)$. Thus $\overline{p_{\hat{x}}}(x^* + M) \leqslant |\langle x, x^* + h \rangle| = 0$. That is, $\overline{p_{\hat{x}}}(x^* + M) = 0$ for all $x^* \in X^*$. Hence, $\overline{p_{\hat{x}}} = 0$ for all $x \in X - M$.

Now let $(x_i^* + M^\perp)$ be a net in $X^*/M^\perp$ such that $\tilde{r}(x_i^* + M^\perp) = x_i^*|_M \xrightarrow{wk^*} 0$ in M^*. If $x \in X - M$, then by the claim above, $\overline{p_{\hat{x}}}(x_i^* + M^\perp) = 0$. If $x \in M$, then $\overline{p_{\hat{x}}}(x_i^* + M^\perp) \leqslant |\langle x, x_i^* \rangle| \to 0$. Thus $\overline{p_{\hat{x}}}(x_i^* + M^\perp) \to 0$ for all $x \in X$. Since the quotient topology on $X^*/M^\perp$ is defined by the seminorms $\{\overline{p_{\hat{x}}} : x \in X\}$, by Lemma 3.1.2, $x_i^* + M^\perp \to 0$ in $X^*/M^\perp$. Hence, $\tilde{r}^{-1} : M^* \to X^*/M^\perp$ is continuous. Therefore, $\tilde{r} : X^*/M^\perp \to M^*$ is a homeomorphism.

Suppose X is a normed space. Let $f \in X^*$. Then for all $g \in M^\perp$,

$$\|f|_M\| = \|(f + g)|_M\| \leqslant \|f + g\|.$$

Taking the infimum over all $g \in M^\perp$, we get $||f|_M|| \leqslant \|f + M^\perp\|$. Now let $\varphi \in M^*$. Then by Proposition 3.4.11, there exists $f \in X^*$ such that $f|_M = \varphi$ and $\|f\| = \|\varphi\|$. Thus $\|f|_M\| = \|\varphi\| = \|f\| \geqslant \|f + M^\perp\|$. Hence, $\|f + M^\perp\| = \|f|_M\|$. Therefore, $\tilde{r} : X^*/M^\perp \to M^*$ is an isometry. □

CHAPTER 5

Banach-Alaoglu theorem, Goldstine theorem, and reflexivity and separability of normed spaces

We begin this chapter with Banach-Alaoglu theorem and Goldstine theorem, which say that for a normed space X, the closed unit ball of X^* is weak-star compact and the closed unit ball of X^{**} is the weak-star closure in X^{**} of the canonical image of the closed unit ball of X. In Section 5.2, we give a number of characterizations of reflexive normed spaces. We prove that every reflexive space is weakly sequentially complete, and show that the converse is not true by checking the non-reflexivity of the classical weakly sequentially complete space ℓ_1. In Section 5.3, we prove that X is separable if and only if the closed unit ball of X^* is weak-star metrizable. The main references for this chapter are [**1**], [**2**], [**3**], and [**6**].

5.1. Banach-Alaoglu theorem and Goldstine theorem

For a normed space X, the closed unit ball of X is denoted by ball X.

THEOREM 5.1.1 (Banach-Alaoglu theorem). *Let X be a normed space. Then ball X^* is weak-star compact.*

PROOF. Let $D_x = \{\alpha \in \mathbb{F} : |\alpha| \leqslant 1\}$ for each $x \in \text{ball } X$. Let

$$D = \prod \{D_x : x \in \text{ball } X\}.$$

For each $x \in \text{ball } X$, since D_x is a bounded closed subset of $\mathbb{F}$, D_x is compact in $\mathbb{F}$. By Tychonoff's theorem, D is compact with the product topology. Also, we equip ball X^* with the relative weak-star topology of X^*. Define $\tau : \text{ball } X^* \to D$ by $\tau(x^*)(x) = \langle x, x^* \rangle$ $(x \in \text{ball } X)$. That is, $\tau(x^*)$ is the element of the product space D whose x coordinate is $\langle x, x^* \rangle$. Let $x^* \in \text{ball } X^*$ and let (x_i^*) be a net in ball X^* with

$x_i^* \xrightarrow{wk^*} x^*$. Then for each $x \in \text{ball } X$,

$$\tau(x_i^*)(x) = \langle x, x_i^* \rangle \to \langle x, x^* \rangle = \tau(x^*)(x).$$

By Proposition 2.2, $\tau(x_i^*) \to \tau(x^*)$ in D. Therefore, $\tau : \text{ball } X^* \to D$ is continuous.

Suppose $\tau(x_1^*) = \tau(x_2^*)$ for $x_1^*, x_2^* \in \text{ball } X^*$. Then we have $\langle x, x_1^* \rangle = \langle x, x_2^* \rangle$ for all $x \in \text{ball } X$ and hence for all $x \in X$, and thus $x_1^* = x_2^*$. Hence, $\tau : \text{ball } X^* \to D$ is injective. Let $x^* \in \text{ball } X^*$ and (x_i^*) be a net in ball X^* with $\tau(x_i^*) \to \tau(x^*)$ in D. Then for all $x \in \text{ball } X$, $\tau(x_i^*)(x) \to \tau(x^*)(x)$. Thus $\langle x, x_i^* \rangle \to \langle x, x^* \rangle$ for all $x \in \text{ball } X$ and hence for all $x \in X$. That is, $x_i^* \xrightarrow{wk^*} x^*$ in ball X^*. Therefore, $\tau^{-1} : \tau(\text{ball } X^*) \to \text{ball } X^*$ is continuous, and hence $\tau : \text{ball } X^* \to \tau(\text{ball } X^*)$ is a homeomorphism.

Let $f \in D$ and let (x_i^*) be a net in ball X^* with $\tau(x_i^*) \to f$ in D. Then for all x in ball X, $\lim_i \langle x, x_i^* \rangle = \lim_i \tau(x_i^*)(x) = f(x)$ exists. Let $x \in X$. Choose $\alpha \neq 0$ such that $||\alpha x|| \leqslant 1$. Then $\lim_i \langle x, x_i^* \rangle = \alpha^{-1} \lim_i \langle \alpha x, x_i^* \rangle$ exists. Now define $F : X \to \mathbb{F}$ by $F(x) = \lim_i \langle x, x_i^* \rangle$. Clearly, $F : X \to \mathbb{F}$ is linear, and $|F(x)| = |f(x)| \leqslant 1$ for all $x \in \text{ball } X$ since $f(x) \in D_x$. It implies that $F : X \to \mathbb{F}$ is a bounded linear functional with $||F|| \leqslant 1$; that is, $F \in \text{ball } X^*$. Note that $\tau(F)(x) = \langle x, F \rangle = f(x)$ for all $x \in \text{ball } X$. Thus $f = \tau(F) \in \tau(\text{ball } X^*)$. By Corollary 2.5, $\tau(\text{ball } X^*)$ is closed in D. It follows from Proposition 2.11 that $\tau(\text{ball } X^*)$ is compact. Therefore, ball X^* is weak-star compact since ball X^* is homeomorphic to $\tau(\text{ball } X^*)$. □

For a normed space X, we have $|\hat{x}(x^*)| = |x^*(x)| \leqslant \|x^*\| \cdot \|x\|$ for all $x \in X$ and $x^* \in X^*$, and thus $\widehat{X} \subseteq X^{**}$.

PROPOSITION 5.1.2. *Let X be a normed space. Then map $X \to X^{**}, x \mapsto \hat{x}$ is a linear isometry, called the canonical embedding of X to X^{**}.*

PROOF. Clearly, the map $X \to X^{**}, x \mapsto \hat{x}$ is linear. Let $x \in X$. Then

$$\|\hat{x}\| = sup\{|\hat{x}(x^*)| : x^* \in X^* \text{ and } \|x^*\| \leqslant 1\} = sup\{|x^*(x)| : x^* \in X^* \text{ and } \|x^*\| \leqslant 1\}.$$

It follows from Corollary 3.4.12 that $\|\hat{x}\| = \|x\|$. Therefore, the map $X \to X^{**}, x \mapsto \hat{x}$ is a linear isometry. □

The proof of the Goldstine theorem given below was suggested by Dr. Hu.

THEOREM 5.1.3 (Goldstine theorem). *Let X be a normed space. Then ball X^{**} is the weak-star closure of $\widehat{\text{ball } X}$ in X^{**}, where $\widehat{\text{ball } X} = \{\hat{x} : x \in \text{ball } X\}$.*

PROOF. Let $Y = (X^*, \|\cdot\|)$ and let $B = \widehat{\text{ball } X}$. Then Y is a LCS and B is a convex balanced subset of Y^*. Thus by Theorem 4.2.7, we only have to show that $^\circ B^\circ = \text{ball } X^{**}$. By definition, we have

$$\begin{aligned} {}^\circ B &= \{y \in Y : |\langle y, b\rangle| \leqslant 1 \text{ for all } b \in B\} \\ &= \{f \in X^* : |\langle f, x\rangle| \leqslant 1 \text{ for all } x \in \text{ball } X\} = \text{ball } X^*. \end{aligned}$$

Hence, $^\circ B^\circ = (^\circ B)^\circ = \{x^{**} \in X^{**} : |\langle x^{**}, f\rangle| \leqslant 1 \text{ for all } f \in \text{ball } X^*\} = \text{ball } X^{**}$. □

5.2. Reflexivity of normed spaces

Two linear spaces V and W over the same field are said to be isomorphic if there is a linear bijection $T : V \to W$. Such T is called a linear isomorphism from V to W. If two normed spaces X and Y are isometrically isomorphic, we write $X \cong Y$.

DEFINITION 5.2.1. A normed space X is reflexive if $X^{**} = \widehat{X}$.

When $\widehat{X}$ is equipped with the restriction of the norm on X^{**} to $\widehat{X}$, we have $X \cong \widehat{X}$ by Proposition 5.1.2. Since X^{**} is a Banach space, the corollary below is immediate.

COROLLARY 5.2.1. *Let X be a reflexive space. Then X is a Banach space.*

LEMMA 5.2.2. *Let X and Y be normed spaces with an isometric isomorphism $\phi : X \to Y$. Then the dual map $\phi^* : Y^* \to X^*, \lambda \mapsto \lambda \circ \phi$ is an isometric isomorphism.*

PROOF. Since $\phi : X \to Y$ an isometric isomorphism, $\phi^{-1} : Y \to X$ an isometric isomorphism. Let $f \in X^*$ and let $g = f \circ \phi^{-1}$. Then $g \in Y^*$ and $\phi^*(g) = g \circ \phi = f$.

Hence, $\phi^* : Y^* \to X^*$ is surjective. For all $\lambda \in Y^*$, we have

$$\|\lambda \circ \phi\| \leqslant \|\lambda\| \cdot \|\phi\| = \|\lambda\| = \|(\lambda \circ \phi) \circ \phi^{-1}\| \leqslant \|\lambda \circ \phi\| \cdot \|\phi^{-1}\| = \|\lambda \circ \phi\|.$$

That is, $\|\lambda \circ \phi\| = \|\lambda\|$ for all $\lambda \in Y^*$. Therefore, the dual map $\phi^* : Y^* \to X^*, \lambda \mapsto \lambda \circ \phi$ is an isometric isomorphism. □

PROPOSITION 5.2.3. *Let X and Y be normed spaces such that $X \cong Y$. Then X is reflexive if and only if Y is reflexive.*

PROOF. Since $X \cong Y$, there exists an isometric isomorphism $\phi : X \to Y$. By Lemma 5.2.2, $Y^* \cong X^*$ via ϕ^* and hence $X^{**} \cong Y^{**}$ via ϕ^{**}. Let $\pi_X : X \to X^{**}$ and $\pi_Y : Y \to Y^{**}$ be the canonical embeddings. Let $x \in X$ and let $f \in Y^*$. Then

$$\phi^{**}(\hat{x})(f) = \hat{x}(\phi^*(f)) = \hat{x}(f \circ \phi) = f(\phi(x)) = \widehat{\phi(x)}(f).$$

Thus $\phi^{**}(\hat{x}) = \widehat{\phi(x)}$ for all $x \in X$; that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X^{**} & \xrightarrow[\phi^{**}]{} & Y^{**} \end{array}$$

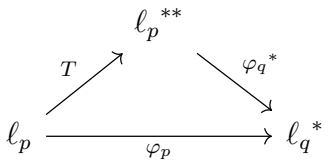
commutes. Since ϕ and ϕ^{**} are bijective, the commutativity of the diagram above implies that $\pi_X : X \to X^{**}$ is surjective if and only if $\pi_Y : Y \to Y^{**}$ is surjective; that is, X is reflexive if and only if Y is reflexive. □

For each $1 \leqslant p < \infty$, let ℓ_p be the linear space consisting of all sequences $x = (x_n)$ in $\mathbb{F}$ for which $\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty$. Then $\|\cdot\|_p$ is a norm on ℓ_p, and ℓ_p is a Banach space with respect to this norm.

Let ℓ_∞ be the linear space consisting of all bounded sequences $x = (x_n)$ in $\mathbb{F}$. Then $\|x\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$ defines a norm on ℓ_∞, and ℓ_∞ is a Banach space with respect to this norm. Let c_0 be the linear space of all sequences in $\mathbb{F}$ that converge to 0. Then c_0 is a closed linear subspace of ℓ_∞, and hence c_0 is a Banach space.

REMARK 5.2.4. Let $1 < q \leqslant \infty$ and let $p = \frac{q}{q-1}$. Then $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $x = (x_n) \in \ell_q$. Then $g_x(y) = \sum_{n=1}^{\infty} x_n y_n$ $(y = (y_n) \in \ell_p)$ defines a bounded linear functional on ℓ_p. In fact, $\varphi_q : \ell_q \to {\ell_p}^*, x \mapsto g_x$ is an isometric isomorphism. Also, for each $z = (z_n) \in \ell_1$, $f_z(y) = \sum_{n=1}^{\infty} z_n y_n$ $(y = (y_n) \in c_0)$ defines a bounded linear functional on c_0, and $\varphi_1 : \ell_1 \to {c_0}^*, z \mapsto f_z$ is an isometric isomorphism. See Theorem 6.13 and Corollary 6.14 in [**6**] for the proof of the isomorphisms $\ell_q \overset{\varphi_q}{\cong} {\ell_p}^*$ and $\ell_1 \overset{\varphi_1}{\cong} {c_0}^*$.

Now let $1 < p < \infty$ and let $q = \frac{p}{p-1}$. Then $\ell_p \cong {\ell_q}^*$ and $\ell_q \cong {\ell_p}^*$ via φ_p and φ_q, respectively. By Lemma 5.2.2, $\varphi_q{}^* : {\ell_p}^{**} \to {\ell_q}^*, \lambda \mapsto \lambda \circ \varphi_q$ is an isometric isomorphism. Let $T : \ell_p \to {\ell_p}^{**}$ be the canonical embedding. Then for $x = (x_n) \in \ell_p$ and $y = (y_n) \in \ell_q$, we have $\varphi_q{}^*(\hat{x})(y) = \hat{x}(\varphi_q(y)) = \varphi_q(y)(x) = \sum_{n=1}^{\infty} x_n y_n = \varphi_p(x)(y)$. Thus $\varphi_q{}^* \circ T = \varphi_p$; that is, the diagram



commutes. Since φ_p and $\varphi_q{}^*$ are bijective, the diagram implies that the embedding $T : \ell_p \to {\ell_p}^{**}$ is surjective. Therefore, we have the following proposition.

PROPOSITION 5.2.5. *Let $1 < p < \infty$. Then ℓ_p is a reflexive space.*

By Proposition 5.1.2, the map $X \to \widehat{X}, x \mapsto \hat{x}$ is a linear bijection. Then the proposition below holds obviously.

PROPOSITION 5.2.6. *Let X be a normed space. Equip X with the weak topology and $\widehat{X}$ with the relative weak-star topology of X^{**}. Then the map $X \to \widehat{X}, x \mapsto \hat{x}$ is a homeomorphism.*

PROPOSITION 5.2.7. *Let X be a normed space. Then $\widehat{X}$ is norm closed in X^{**} if and only if X is a Banach space.*

PROOF. Suppose X is a Banach space. Since $\widehat{X} \cong X$, $\widehat{X}$ is a Banach space. By Proposition 2.6, $\widehat{X}$ is norm closed in X^{**}. Conversely, suppose $\widehat{X}$ is norm closed in

X^{**}. Let (f_n) be a Cauchy sequence in $\widehat{X}$. Then (f_n) is a Cauchy sequence in X^{**}, and hence $f_n \to f$ for some $f \in X^{**}$. Since $\widehat{X}$ is norm closed in X^{**}, by Corollary 2.5, $f \in \widehat{X}$. Thus $\widehat{X}$ is complete. Since $X \cong \widehat{X}$, X is a Banach space. □

The corollary below follows immediately from Proposition 5.2.7 and Corollary 4.2.4.

COROLLARY 5.2.8. *Let X be a normed space. Then the following statements are equivalent.*

(a) *X is complete.*
(b) *$\widehat{ball\ X}$ is norm closed in X^{**}.*
(c) *$\widehat{ball\ X}$ is weakly closed in X^{**}.*

PROPOSITION 5.2.9. *Let X be a normed space and let Y be a dense linear subspace of X. Then $X^* \cong Y^*$ via $\tau^* : f \mapsto f|_Y$, where $\tau : Y \to X$ is the inclusion map.*

PROOF. By Proposition 3.4.11, $\tau^* : X^* \to Y^*$ is surjective, and

$$\|\tau^*(f)\| = \|f|_Y\| = \|f\| \text{ for all } f \in X^*.$$

Hence, $\tau^* : X^* \to Y^*$ is an isometric isomorphism. □

For any normed space X, since $\widehat{X} \subseteq X^{**}$, we have $\sigma(X^*, X) \subseteq \sigma(X^*, X^{**})$.

THEOREM 5.2.10. *Let X be a normed space. Then the following statements are equivalent.*

(a) *X is reflexive.*
(b) *$\sigma(X^*, X) = \sigma(X^*, X^{**})$.*
(c) *ball X is weakly compact in X.*

Furthermore, each of (a)-(c) implies the following

(d) *X^* is reflexive,*

and (a)-(d) are equivalent if X is a Banach space.

PROOF. (a)⇒(b). Suppose X is reflexive. We only need to show that $\sigma(X^*, X^{**}) \subseteq \sigma(X^*, X)$. Let $f \in X^*$ and let (f_i) be a net in X^* with $f_i \to f$ in $\sigma(X^*, X)$. Then for all $x \in X$,

$$\langle \hat{x}, f_i \rangle = \langle f_i, x \rangle \to \langle f, x \rangle = \langle \hat{x}, f \rangle.$$

Since $\widehat{X} = X^{**}$, $f_i \to f$ in $\sigma(X^*, X^{**})$. Therefore, $\sigma(X^*, X^{**}) \subseteq \sigma(X^*, X)$.

(b)⇒(a). Suppose $\sigma(X^*, X) = \sigma(X^*, X^{**})$. Then

$$X^{**} = (X^*, wk)^* = (X^*, wk^*)^* = \widehat{X}$$

by Theorems 4.2.1 and 4.2.2. Therefore, X is reflexive.

(a)⇒(c). Suppose X is reflexive. By Proposition 5.1.2,

$$\widehat{\text{ball } X} = \text{ball } \widehat{X} = \text{ball } X^{**}.$$

It follows from Banach-Alaoglu theorem that $\widehat{\text{ball } X}$ is weak-star compact in X^{**}. By Proposition 5.2.6, ball X is weakly compact in X.

(c)⇒(a). Suppose (c) holds. By Proposition 5.2.6, $\widehat{\text{ball } X}$ is weak-star compact in X^{**}. Since the weak-star topology on X^{**} is Hausdorff, by Proposition 2.11, $\widehat{\text{ball } X}$ is $\sigma(X^{**}, X^*)$ closed in X^{**}. By Theorem 5.1.3, $\widehat{\text{ball } X}$ = ball X^{**}. It follows from Proposition 5.1.2 that

$$\widehat{X} = \widehat{\text{span}(\text{ball } X)} = \text{span}(\widehat{\text{ball } X}) = \text{span}(\text{ball } X^{**}) = X^{**}.$$

Therefore, X is reflexive.

(b)⇒(d). Suppose $\sigma(X^*, X) = \sigma(X^*, X^{**})$. Then ball X^* is $\sigma(X^*, X^{**})$ compact in X^* by Banach-Alaoglu theorem. Since X^* is a normed space, by the equivalence of (a) and (c) proved already, we obtain that X^* is reflexive.

In the rest of the proof, we assume that X is a Banach space.

(d)⇒(a). Suppose (d) holds (that is, $\widehat{X^*} = X^{***}$). Since X is a Banach space, by Corollary 5.2.8, $\widehat{\text{ball } X}$ is $\sigma(X^{**}, X^{***})$ closed in X^{**}. Note that $\sigma(X^{**}, X^{***}) = \sigma(X^{**}, X^*)$ by the equivalence of (a) and (b) for X^*. Therefore, $\widehat{\text{ball } X}$ is $\sigma(X^{**}, X^*)$ closed in X^{**}, and hence X is reflexive as shown in the proof of (c)⇒(a). □

The corollary below was suggested by Dr. Hu. However, the author was unable to give a complete proof.

COROLLARY 5.2.11. *Let X be a normed space. Then the following statements are equivalent.*

(a) *X is reflexive.*
(b) *$\widehat{ball\ X}$ is weak-star compact in X^{**}.*
(c) *$\widehat{ball\ X}$ is weak-star closed in X^{**}.*
(d) *$\widehat{ball\ X}$ is weakly compact in X^{**}.*

As seen below, (a)-(c) and (d) in Theorem 5.2.10 are not equivalent if the normed space is not complete but its completion is reflexive. For this case, we will show the negation for each of (a)-(c) directly without using their equivalence.

The following example and its proof were suggested by Dr. Hu.

EXAMPLE 5.2.12. Let X be a reflexive space and let Y be a dense linear subspace of X with $Y \neq X$ (e.g., $X = \ell_p$ with $1 < p < \infty$ and $Y = (c_{00}, \|\cdot\|_p)$). Then Y^* is reflexive, but Y does not satisfy any of (a)-(c) in Theorem 5.2.10.

PROOF. Let $\tau : Y \to X$ be the inclusion map. Then $X^* \cong Y^*$ via τ^* (cf. Proposition 5.2.9). By Theorem 5.2.10 and Proposition 5.2.3, Y^* is reflexive.

Since $Y \neq X = \overline{Y}$, Y is not closed in X. By Proposition 2.6, Y is not complete. It follows from Corollary 5.2.1 that Y is not reflexive.

Now we show that ball Y is not weakly compact in Y. Choose $x_0 \in X$ with $\|x_0\| = 1$ and $x_0 \notin Y$. Then there exists a sequence (y_n) in Y such that $y_n \to x_0$ by Proposition 2.4. Since $\|y_n\| \to \|x_0\| = 1$, we can assume that $y_n \neq 0$ for all n; furthermore, replacing y_n by $\|y_n\|^{-1} y_n$, we can take the sequence (y_n) from ball Y. Assume that ball Y is weakly compact in Y. Then (y_n) has a subnet (y_{n_α}) such that $y_{n_\alpha} \to y$ weakly in Y for some $y \in$ ball Y. Hence, $y_{n_\alpha} \to y$ weakly in X. Note that we also have $y_{n_\alpha} \to x_0$ weakly in X, since $\|y_{n_\alpha} - x_0\| \to 0$. The uniqueness of the

limit implies that $x_0 = y$, and hence $x_0 \in Y$, contradicting that $x_0 \notin Y$. Therefore, ball Y is not weakly compact in Y.

Finally, we show $\sigma(Y^*, Y) \neq \sigma(Y^*, Y^{**})$ by finding a net in Y^* that is convergent in $\sigma(Y^*, Y)$ but not in $\sigma(Y^*, Y^{**})$. Let $\mathcal{M}$ be the family of all finite dimensional linear subspaces of Y. Then $\mathcal{M}$ is a directed set under the inclusion order. Now each $M \in \mathcal{M}$ is closed in X. By Theorem 3.4.8, for each $M \in \mathcal{M}$, there exists $f^M \in X^*$ such that $f^M(x_0) = 1$ and $f^M|_M = 0$. Let $g^M = \tau^*(f^M) = f^M|_Y$. Then $(g^M)_{M\in\mathcal{M}}$ is a net in Y^* and $g^M(y) \to 0$ for all $y \in Y$. Therefore, $g^M \to 0$ in $\sigma(Y^*, Y)$. Note that $\tau^{**} : Y^{**} \to X^{**}$ is an isometric isomorphism since $X^* \cong Y^*$ via τ^*. Let $y^{**} \in Y^{**}$ be such that $\tau^{**}(y^{**}) = \widehat{x_0}$. Then for all $M \in \mathcal{M}$, we have

$$y^{**}(g^M) = y^{**}(\tau^*(f^M)) = \tau^{**}(y^{**})(f^M) = \widehat{x_0}(f^M) = f^M(x_0) = 1.$$

Hence, (g^M) is not convergent (to 0) in $\sigma(Y^*, Y^{**})$. □

Remark 5.2.13. Note that the isometric isomorphism $\tau^* : X^* \to Y^*$ given in Example 5.2.12 is not a wk^*-wk^* homeomorphism though, as an adjoint map, it is automatically wk^*-wk^* continuous.

Corollary 5.2.14. *Let X be a reflexive space and let M be a norm closed linear subspace of X. Then M is a reflexive space.*

Proof. Since ball M is norm closed in M and M is a norm closed linear subspace of X, ball M is norm closed in X. By Corollary 4.2.4, ball M is weakly closed in X. Since X is reflexive, by Theorem 5.2.10, ball X is $\sigma(X, X^*)$ compact in X. Since ball $M \subseteq$ ball X, by Proposition 2.11, ball M is $\sigma(X, X^*)$ compact in X. It follows from Theorem 3.4.10 that $X^*|_M = M^*$ and hence $\sigma(X, X^*)|_M = \sigma(M, M^*)$. Thus ball M is $\sigma(M, M^*)$ compact in M. By Theorem 5.2.10, M is reflexive. □

Definition 5.2.2. Let X be a normed space. A sequence (x_n) in X is called weakly Cauchy if for all $x^* \in X^*$, $(\langle x_n, x^*\rangle)$ is a Cauchy sequence in $\mathbb{F}$. X is called weakly sequentially complete if every weakly Cauchy sequence in X converges weakly.

THEOREM 5.2.15. *Every reflexive space is weakly sequentially complete.*

PROOF. Let (x_n) be a weakly Cauchy sequence in a reflexive space X. Then for each x^* in X^*, $(\langle x_n, x^*\rangle)$ is a Cauchy sequence in $\mathbb{F}$ and hence $(\langle x_n, x^*\rangle)$ is bounded. Thus $\sup_n |x^*(x_n)| < \infty$ for all $x^* \in X^*$; that is, $\sup_n |\widehat{x_n}(x^*)| < \infty$ for all $x^* \in X^*$. Since X^* is a Banach space and $\mathbb{F}$ with the absolute value norm is a normed space, by the Principle of Uniform Boundedness, $\sup_n \|\widehat{x_n}\| < \infty$. It follows from Proposition 5.1.2 that $M = \sup_n \|x_n\| = \sup_n \|\widehat{x_n}\| < \infty$.

Let $y_n = \frac{x_n}{M}$ $(n \in \mathbb{N})$. Then (y_n) is a weakly Cauchy sequence in ball X. Since X is reflexive, by Theorem 5.2.10, ball X is weakly compact in X. By Theorem 2.10, (y_n) has a subnet $(y_{n'})$ such that $y_{n'} \xrightarrow{wk} y$ for some $y \in$ ball X. Thus $(x_{n'})$ is a subnet of (x_n) such that $x_{n'} \xrightarrow{wk} x = My \in X$. Let $x^* \in X^*$. Then $\langle x_n, x^*\rangle \to \alpha$ for some $\alpha \in \mathbb{F}$, since (x_n) is weakly Cauchy in X. By Proposition 2.8, $\langle x_{n'}, x^*\rangle \to \alpha$. Since $x_{n'} \xrightarrow{wk} x$ in X, by the uniqueness of limit in $\mathbb{F}$, $\langle x_n, x^*\rangle \to \alpha = \langle x, x^*\rangle$. Hence, $x_n \xrightarrow{wk} x \in X$. Therefore, X is weakly sequentially complete. □

Note that the converse of Theorem 5.2.15 is not true. In fact, as shown below, ℓ_1 is not reflexive, though it is weakly sequentially complete (see Proposition 2.3.12 in [**1**] for the proof of the fact that ℓ_1 is weakly sequentially complete).

EXAMPLE 5.2.16. Recall that $\ell_\infty \cong {\ell_1}^*$ and $\ell_1 \cong {c_0}^*$ via the maps φ_∞ and φ_1, respectively (cf. Remark 5.2.4). It follows from Lemma 5.2.2 that the dual map ${\varphi_1}^* : {c_0}^{**} \to \ell_1^*, \lambda \mapsto \lambda \circ \varphi_1$ is an isometric isomorphism.

Let $\tau : c_0 \to \ell_\infty$ be the inclusion map. Then $\tau : c_0 \to \ell_\infty$ is not surjective since $x_0 = (1, 1, \cdots) \in \ell_\infty - c_0$. Let $T : c_0 \to {c_0}^{**}$ be the canonical embedding. Similar arguments as given in Remark 5.2.4 shows that the diagram

$$\begin{array}{ccc} c_0 & \xrightarrow{\tau} & \ell_\infty \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \varphi_\infty} \\ {c_0}^{**} & \xrightarrow[{\varphi_1}^*]{} & {\ell_1}^* \end{array}$$

commutes. Since φ_∞ and ${\varphi_1}^*$ are bijective and $\tau : c_0 \to \ell_\infty$ is not surjective, the embedding $T : c_0 \to {c_0}^{**}$ is not surjective. Hence, c_0 is not reflexive. By Theorem 5.2.10, ${c_0}^*$ is not reflexive. Since $\ell_1 \cong {c_0}^*$, ℓ_1 is not reflexive by Proposition 5.2.3.

5.3. Separability of normed spaces

DEFINITION 5.3.1. A topological space X is said to be metrizable if the topology on X is induced by a metric on X.

THEOREM 5.3.1. *Let X be a normed space. Then ball X^* is weak-star metrizable if and only if X is separable.*

PROOF. We can assume that $X \neq \{0\}$. Suppose X is separable. By Proposition 2.7, ball X is separable. Let $C = \{x_1, x_2, x_3, \cdots\}$ be a countable dense subset of ball X. Define $d : \text{ball } X^* \times \text{ball } X^* \to [0, \infty)$ by $d(x^*, y^*) = \sum\limits_{n=1}^{\infty} \frac{|\langle x_n, x^* - y^* \rangle|}{2^n}$. For all $x^*, y^* \in \text{ball } X^*$ and $n \in \mathbb{N}$, we have

$$\frac{|\langle x_n, x^* - y^* \rangle|}{2^n} \leqslant \frac{|\langle x_n, x^* \rangle| + |\langle x_n, y^* \rangle|}{2^n} \leqslant \frac{2}{2^n} = 2^{1-n},$$

and thus $d(x^*, y^*) \leqslant \sum\limits_{n=1}^{\infty} 2^{1-n} = 2 < \infty$. Hence, the function d is well defined.

Let $x^*, y^* \in \text{ball } X^*$. Then $d(x^*, x^*) = 0$, $d(x^*, y^*) \geqslant 0$, and $d(x^*, y^*) = d(y^*, x^*)$. Suppose $d(x^*, y^*) = 0$. Then $|\langle x_n, x^* - y^* \rangle| = 0$ for all $n \in \mathbb{N}$. Since C is dense in ball X, by Proposition 2.14, $\langle x, x^* - y^* \rangle = 0$ for all $x \in \text{ball } X$ and hence for all $x \in X$. Thus $x^* = y^*$. Hence, $d(x^*, y^*) = 0$ if and only if $x^* = y^*$. Also, for all $x^*, y^*, z^* \in \text{ball } X^*$, we have

$$d(x^*, y^*) \leqslant \sum_{n=1}^{\infty} \frac{|\langle x_n, x^* - z^* \rangle|}{2^n} + \sum_{n=1}^{\infty} \frac{|\langle x_n, z^* - y^* \rangle|}{2^n} = d(x^*, z^*) + d(z^*, y^*).$$

Therefore, d is a metric on ball X^*.

Let T be the topology on ball X^* induced by d, let σ be the weak-star topology on ball X^*, and let I be the identity map from (ball X^*, σ) to (ball X^*, T). To show that (ball X^*, σ) is metrizable, we only have to show that I is a homeomorphism. By Banach-Alaoglu theorem, (ball X^*, σ) is compact. Since I is a bijection from a

compact space to a Hausdorff space, by Proposition 2.13, we only need to show that $I : (\text{ball } X^*, \sigma) \to (\text{ball } X^*, T)$ is continuous. Let $x^* \in \text{ball } X^*$ and let $(x^*_\alpha)_{\alpha\in A}$ be a net in ball X^* with $x^*_\alpha \xrightarrow{\sigma} x^*$. Then $\langle x_n, x^*_\alpha - x^*\rangle \to 0$ for all n. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $2^{1-N} < \frac{\epsilon}{2}$. For each $n \in \{1, \cdots, N\}$, there exists $\alpha_n \in A$ such that $|\langle x_n, x^*_\alpha - x^*\rangle| < \frac{2^n\epsilon}{2N}$ for all $\alpha \in A$ with $\alpha \succcurlyeq \alpha_n$. Since A is a directed set, there exists $\alpha_0 \in A$ such that $\alpha_0 \succcurlyeq \alpha_i$ for all $1 \leqslant i \leqslant N$. Hence, we have

$$\begin{aligned} d(x^*_\alpha, x^*) &= \sum_{n=1}^{N} \frac{|\langle x_n, x^*_\alpha - x^*\rangle|}{2^n} + \sum_{n=N+1}^{\infty} \frac{|\langle x_n, x^*_\alpha - x^*\rangle|}{2^n} \\ &< \sum_{n=1}^{N} \frac{\epsilon}{2N} + \sum_{n=N+1}^{\infty} 2^{1-n} = \frac{\epsilon}{2} + 2^{1-N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $\alpha \in A$ with $\alpha \succcurlyeq \alpha_0$. Thus $x^*_\alpha \xrightarrow{d} x^*$. Hence, $I : (\text{ball } X^*, \sigma) \to (\text{ball } X^*, T)$ is continuous. Therefore, ball X^* is weak-star metrizable.

Conversely, suppose ball X^* is weak-star metrizable. Then there exists a metric d on ball X^* such that the weak-star topology on ball X^* is induced by d. For each $n \in \mathbb{N}$, let $U_n = \{x^* \in \text{ball } X^* : d(x^*, 0) < \frac{1}{n}\}$. Then each U_n is open in (ball X^*, σ) with $0 \in U_n$, and $\bigcap_{n=1}^{\infty} U_n = \{0\}$. Let $n \in \mathbb{N}$. Since $0 \in U_n$, by the definition of the weak-star topology on ball X^* and Definition 3.1.3, there exist $x^n_i \in X$ and $\epsilon^n_i > 0$ $(i = 1, \cdots, m_n)$ such that

$$\bigcap_{i=1}^{m_n} \{x^* \in \text{ball } X^* : |\langle x^n_i, x^*\rangle| < \epsilon^n_i\} \subseteq U_n.$$

Let $F_n = \{x^n_1, \cdots, x^n_{m_n}\}$ and let $F = \bigcup_{n=1}^{\infty} F_n$. Then

$$\{x^* \in \text{ball } X^* : |\langle x, x^*\rangle| = 0 \text{ for all } x \in F_n\} \subseteq U_n$$

for all n, F is countable, and $F^\perp$ is a linear subspace of X^*. Let $x^* \in \text{ball } F^\perp$. Then $|\langle x, x^*\rangle| = 0$ for all $x \in F_n$ and $n \in \mathbb{N}$. Thus $x^* \in \bigcap_{n=1}^{\infty} U_n = \{0\}$. It follows that ball $F^\perp = \{0\}$. Hence, $F^\perp = \text{span}(\text{ball } F^\perp) = \{0\}$. By Propositions 3.1.6 and 4.2.9, ${}^\perp(F^\perp) = \overline{\text{span}}(F)$ is separable. Therefore, $X = {}^\perp\{0\} = {}^\perp(F^\perp)$ is separable. □

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