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Abstract: We consider the joint acquisition and pricing problem where the retailer sells multiple products with uncertain demands and the suppliers provide all unit quantity discounts. The problem is to determine the optimal acquisition quantities and selling prices so as to maximize the retailer's expected profit, subject to a budget constraint. This is the first extension to consider supplier discounts in the constrained multi-product newsvendor pricing problem. We establish a mixed integer nonlinear programming (MINLP) model to formulate the problem, and develop a Lagrangian based solution approach. Computational results for the test problems involving up to thousand products are reported, which show that the Lagrangian based approach can obtain high quality solutions in a very short time.

Keywords: Newsvendor, pricing, discount, acquisition, uncertain demand;

1. Introduction

Due to demand uncertainty, the matching of supply and demand is a constant challenge faced by a retailer. Product acquisition and pricing are used as two levers in the retailer's up-stream and down-stream to better match supply and demand. A retailer can use pricing to manage demand and increase the revenue, and optimize acquisition quantity or inventory level to reduce the mismatch and cost by exploiting economies of scale. How to integrate both pricing and acquisition decisions under uncertain demand is a challenging problem. The situation becomes more complicated when suppliers provide quantity discounts: the retailer can procure products at a lower unit price if the acquisition quantity is over a certain value - the threshold; however, since the demand is uncertain, the retailer's overstocking risk will increase. Through setting a suitable price, the retailer can reduce overstocking risk and increase revenue. Thus coordinating the acquisition decision and pricing with uncertain demands becomes more practical and challenging when suppliers offer quantity discounts. Motivated by the observation, this research investigates the joint acquisition and pricing problem with uncertain demand and supplier discounts. The problem is to determine the optimal ordering quantities and selling prices simultaneously so as to maximize the retailer's expected profit.

The problem is an extension of the newsvendor problem. The newsvendor problem is a classical model that is used to optimize the ordering quantity under uncertain demand. Due to its practical and theoretical importance, the newsvendor problem has been widely studied. Khouja (1999) presented a comprehensive review and classified the extensions of the newsvendor problem into eleven categories. Among those extensions are multi-product

acquisition, newsvendor pricing, and supplier discounts.

Extensions to multi-product involve two or more products, usually with resource constraints. The constrained multi-product newsvendor model was first proposed by Hadley and Whitin (1963). Since ordering multiple products under budget or other constraints is common, the constrained multi-product newsvendor problem is widely studied in the last two decades. Representative work in this area includes that by Lau and Lau (1995 and 1996), Erlebacher (2000), Abdel-Malek and Montanari (2005a, 2005b), and Niederhoff (2007). Incorporating pricing decision into the newsvendor problem was first presented by Whitin (1955), where selling price and stocking quantity are determined simultaneously. Then it was extensively studied by Petruzzi and Dada (1999), Webster and Weng (2008), and Chen and Bell (2009). Another important extension of the newsvendor problem is to take into account the supplier discount, which is a common policy for suppliers to promote their products. The notable work includes those of Pantumsinchai and Knowles (1991), Khouja (1996), Lin and Kroll (1997), and Zhang (2010).

So far, the three extensions to multi-product, pricing, and supplier discounts have been widely studied separately. To the best of our knowledge, it is the first investigation in the literature that studies these three issues in one integrated model. As discussed before, through the integrated model, the coordination of the up-stream and down-stream's decisions makes the problem more practical and challenging. Our objective is to develop the optimal acquisition and selling policy for the retailer, who faces uncertain demand and supplier discounts. Since suppliers provide quantity discounts, the product costs are piecewise linear. We develop a Mixed Integer Nonlinear Programming (MINLP) model to formulate the problem, and present a Lagrangian based solution approach, which is very efficient for large scale instances.

An outline of this paper is as follows. Section 2 provides a brief literature review of the

related research. Section 3 presents the MINLP model for the problem. A Lagrangian based solution approach is developed in Section 4, and numerical examples and computational results are presented in Section 5. We finally conclude the paper in Section 6.

2. Related Research

There are numerous works that address the newsvendor problem and various extensions. Here we mainly review the related studies on the extensions to multi-product, quantity discount, and newsvendor pricing. For a more comprehensive review, the reader is referred to Khouja (1999). A brief comparison of the features of the reviewed papers with the proposed model in this research is illustrated in Table 1.

[Insert Table 1 here]

The multi-product acquisition under uncertain demand is usually modeled as the multi-product newsvendor problem. A budget or other resource constraints are always associated with the problem otherwise it can be treated as a single product newsvendor problem. The Hadley and Whitin (1963) first presented a formulation for the constrained multi-product newsvendor problem and developed a solution method for the problem. Then Lau and Lau (1995, 1996) presented a formulation and a solution procedure for the multi-product constrained newsvendor problem, which can efficiently solve large scale problems involving 1000 products. Abdel-Malek and Montanari (2005a, 2005b) investigated the solution spaces for the multi-product newsvendor problem with one and two constraints respectively. Abdel-Malek and Areeratchakul (2007) developed a quadratic programming model for the multi-product newsvendor problem with side constraints, which can be solved by familiar linear programming software packages such as Excel Solver and Lingo. Niederhoff (2007) presented an approximation method for the multi-product multi-constraint

newsvendor problem by approximating the objective function with the piecewise linear interpolates. Zhang et al. (2009) presented a binary solution algorithm for the multi-product newsvendor problem with budget constraint. More articles on the multi-product newsvendor problem are included in Table 1.

Quantity discount is a common and effective policy for suppliers to promote their products. Quantity discount is based on the quantity of an item purchased - promoting the buyer to order large quantities of a given item. Pantumsinchai and Knowles (1991) formulated a single-period inventory problem with the consideration of standard container size discounts. Khouja (1995) formulated a newsvendor problem in which multiple discounts are used to sell excess inventory, while Khouja and Mehrez (1996) studied the multi-product constrained newsvendor problem under progressive multiple discounts. Khouja (1996) studied the newsvendor problem that considers both multiple discounts used by retailers to sell excess inventory and all-units quantity discounts offered by the suppliers. However, the model does not consider any resource constraint. Lin and Kroll (1997) investigated the single-item newsvendor problem with quantity discount and dual performance measure consideration. The solution approaches for the all unit quantity discount and incremental discount are developed. Zhang (2010) introduced supplier discounts to the constrained newsvendor problem, and presented a mixed integer nonlinear programming model. A Lagrangian heuristic is developed to solve the problem. However, the problem does not consider pricing decision.

By incorporating pricing into the newsvendor problem, Whitin (1955) first investigated the optimization problem of determining the stocking quantity and selling price simultaneously under uncertain demand environment. Petruzzi and Data (1999) presented a comprehensive review and some meaningful extensions for the newsvendor pricing problem. Parlar and Weng (2006) studied the effects of coordinating pricing and production decisions

on the improvement of a firm's position in a price-competitive environment and found that by coordinating their pricing and production decisions, the firm can increase their profitability, especially when conditions are unfavorable. Karakul (2008) studied the joint pricing and procurement of fashion products in the existence of clearance markets. Serel (2008) studied a single-period inventory and pricing problem where the risky supply is considered. Webster and Weng (2008) investigated a joint ordering and pricing problem for a manufacturing and distribution supply chain for fashion products. Chen (2009) addressed the simultaneous determination of price and inventory replenishment when customers return product to the firm. Pan et al. (2009) constructed a two-period model to determine the pricing and ordering problem for a dominant retailer with uncertain demand in a declining price environment.

As shown in Table 1, the three extensions to the newsvendor problem have been widely investigated separately. There are a couple of articles that combine two of those three extensions in the newsvendor problem but none on all those three. The model presented in this paper enriches the newsvendor problem by considering pricing, quantity discount, and multiple products simultaneously. The properties of the newsvendor pricing problem with supplier quantity discounts are studied and a solution approach is developed based on Lagrangian method.

3. Model formulation

In this section, the multi-product acquisition and pricing problem is formulated as a MINLP model, which is developed based on the following assumptions.

The retailer sells multiple products. It is assumed that the demand for each product is independent. We also assume that the demand is price-sensitive and stochastic: the relationship between demand and price is $\tilde{D}_i(p_i,u_i)=D_i(p_i)+u_i$, where $D_i(p_i)=a_i-b_ip_i$ (where $a_i\geq 0$ and $b_i>0$) is the expected demand and u_i is the stochastic term defined on the range $[A_i,B_i]$ with mean μ_i and standard deviation σ_i. In order to assure that the

demand is nonnegative for some range of p, A_i should not be less than $-a_i$. This assumption for the demand has been applied widely in revenue management and operations research literature (Petruzzi and Dada 1999; and Pan et al. 2009).

We assume that the retailer has a budget constraint and suppliers provide all-unit quantity discounts. For the details on all-unit quantity discounts, the reader is referred to Khouja (1996) and Zhang (2010).

3.1 Notation

The following notations are used to formulate the problem:

Indices:

i = 1,..., I: index of products

k_i = the number of price discounts for product i offered by suppliers

j = 1,..., k_i: index of price segments for product i offered by suppliers.

Parameters:

c_{ij} = the unit acquisition price of product i after discount on discount segment j

d_{ij}^{L} = the lower bound on the quantity of product i on discount segment j

d_{ij}^{U} = the upper bound on the quantity of product i on discount segment j

B_G = the available budget for the retailer

g_i = the estimated understocking cost (the loss of goodwill) of one unit of product i

s_i = the estimated overstocking cost of one unit of product i

$D_i(p_i) = a_i - b_i p_i$, the price-dependent expected demand function for product i

$\tilde{D}_i(p_i, u_i) = D_i(p_i) + u_i$, the price-dependent stochastic demand function for product i

u_i = stochastic term defined on the range $[A_i, B_i]$ with mean μ_i, in this paper $A_i = -a_i$, $B_i = +\infty$ and $\mu_i = 0$

$f_i(\cdot)$, $F_i(\cdot)$ = pdf and cdf of the distribution of u_i.

We define the following decision variables:

p_i = the retail price of product i

q_{ij} = the acquisition quantity of product i at discount segment j (explained in the model description)

y_{ij} = 1 if the retailer buys product i at discount segment j; otherwise 0.

Following Petruzzi and Dada (1999), we also define:

$$z_i = \sum_{j=1}^{k_i} q_{ij} - D_i(p_i) = \sum_{j=1}^{k_i} q_{ij} - (a_i - b_i p_i).$$

Where $\sum_{j=1}^{k_i} q_{ij}$ is the acquisition quantity of product i. The introduction of the decision variable z_i facilitates the modeling and analysis of the problem: there is overstocking cost if z_i is larger than u_i; otherwise understocking cost occurs. From the definition, we can see that the lower bound on z_i is $-a_i$ (where both acquisition quantity and the price are set to zero), which equals the lower bound A_i on u_i. The upper bound on z_i can be infinite. Thus the variable z_i has the range $[A_i, B_i]$. Actually, it is common to apply the range of u_i to variable z_i (Petruzzi and Dada 1999; Pan et al. 2009).

3.2 Model

The model for the joint acquisition and pricing problem can be formulated as:

Max

$$\sum_{i=1}^{I} \left\{ \int_{A_i}^{z_i} \left(p_i [D_i(p_i) + u_i] - s_i [z_i - u_i] \right) f_i(u_i) du_i + \int_{z_i}^{B_i} \left(p_i [D_i(p_i) + z_i] - g_i [u_i - z_i] \right) f_i(u_i) du_i \right\} - \sum_{i=1}^{I} \sum_{j=1}^{k_i} c_{ij} q_{ij} \quad (1)$$

subject to

$$\sum_{i=1}^{I} \sum_{j=1}^{k_i} c_{ij} q_{ij} \le B_G, \quad (2)$$

$$q_{ij} \geq d_{ij}^{L} y_{ij} \qquad \forall i, j, \tag{3}$$

$$q_{ij} \leq d_{ij}^{U} y_{ij} \qquad \forall i, j, \tag{4}$$

$$\sum_{j=1}^{k_i} y_{ij} = 1 \qquad \forall i, \tag{5}$$

$$D_i(p_i) + z_i = \sum_{j=1}^{k_i} q_{ij} \qquad \forall i, \tag{6}$$

$$A_i \leq z_i \leq B_i, \qquad \forall i, \tag{7}$$

$$p_i \geq 0, \;\; q_{ij} \geq 0, \;\; y_{ij} \in \{0,1\}, \qquad \forall i, j. \tag{8}$$

The objective function is to maximize the retailer's expected profit: the first term of represents the expected revenue minus the overstocking cost when the ordering quantities are above the actual demand levels; the second term represents the expected revenue minus the understocking costs when the ordering quantities are lower than the actual demands; the revenue is evaluated based on selling quantity, which is equal to $D_i(p_i) + u_i$ when overstocking, or $D_i(p_i) + z_i$ for understocking; the third term is the total acquisition cost. Constraint (2) is the budget limitation. Constraints (3)-(5) are the quantity discount constraints: (3) and (4) ensure the amount purchased from the supplier at the price level positions in the corresponding discount interval. Constraints (5) ensure only one discount level is eventually applied, which implies that for each product i only one of q_{ij}, $j = 1,..., k_i$, could be non-zero. Constraints (6) give the relationship between acquisition quantity and deterministic demand, as the definition on z_i. Constraints (7) give the bounds on z_i. Constraints (8) are nonnegative and integral constraints.

The formulae given by (1) to (8) is a MINLP model. It is hard to obtain the exact optimal solution to the problem, especially for large scale instances. In the next section we propose a Lagrangian based approach to solve the problem.

4. Solution Approach

The Lagrangian based approach consists of the following three phases: first, we construct the Lagrangian relaxation problem by relaxing the budget constraint (2); second, the Lagrangian relaxation problem is solved by bisection algorithm. The solution obtained may violate the budget constraint (2). Thus, in the last phase, a feasibility algorithm is developed to construct a feasible solution. Details of each phase are presented in the following.

4.1 Lagrangian Relaxation

By introducing a Lagrange multiplier λ, we relax the budget constraint (2) and construct the following Lagrangian relaxation problem:

Max

$$LR = \sum_{i=1}^{I}\left\{\int_{A_i}^{z_i}\left(p_i\left[D_i(p_i)+u_i\right]-s_i\left[z_i-u_i\right]\right)f_i(u_i)du_i+\int_{z_i}^{B_i}\left(p_i\left[D_i(p_i)+z_i\right]-g_i\left[u_i-z_i\right]\right)f_i(u_i)du_i\right\} - \sum_{i=1}^{I}\sum_{j=1}^{k_i}c_{ij}q_{ij}+\lambda\left(B_G-\sum_{i=1}^{I}\sum_{j=1}^{k_i}c_{ij}q_{ij}\right) \quad (9)$$

subject to (3)-(8).

Then the relaxed problem can be decomposed into *I* single-product subproblems:

Subproblem LRp_i**:**

Max

$$LRp_i = \int_{A_i}^{z_i}\left(p_i\left[D_i(p_i)+u_i\right]-s_i\left[z_i-u_i\right]\right)f_i(u_i)du_i+\int_{z_i}^{B_i}\left(p_i\left[D_i(p_i)+z_i\right]-g_i\left[u_i-z_i\right]\right)f_i(u_i)du_i - \sum_{j=1}^{k_i}(1+\lambda)c_{ij}q_{ij} \quad (10)$$

subject to (3)-(8).

Substituting (10) into (9), the relaxed problem can be written as

Max $LR = \sum_{i=1}^{I} LRp_i + \lambda B_G$

subject to (3)-(8).

4.1.1 Properties for single product newsvendor pricing problem

In subproblem LRp_i, constraint (5) ensures that only one discount segment can be selected. It implies that the solution to subproblem LRp_i must locate in one interval of k_i discount levels. Thus, we can solve subproblem LRp_i by solving the k_i subproblems and each of them is associated with one price level. Then the best solution of the k_i subproblems is the optimal solution of subproblem LRp_i. Actually, as Proposition 2 shows later on, it is unnecessary to solve all subproblems in many situations.

For price level j, we have the following **subproblem LRp_{ij}:**

Max

$$LRp_{ij} = \int_{A_i}^{z_i} \left(p_i\left[D_i(p_i)+u_i\right]-s_i\left[z_i-u_i\right]\right) f_i(u_i)du_i + \int_{z_i}^{B_i} \left(p_i\left[D_i(p_i)+z_i\right]-g_i\left[u_i-z_i\right]\right) f_i(u_i)du_i - c_{ij}(1+\lambda)\left[D_i(p_i)+z_i\right] \quad (11)$$

subject to

$$D_i(p_i)+z_i \geq d_{ij}^L \quad (12)$$

$$D_i(p_i)+z_i \leq d_{ij}^U \quad (13)$$

$$A_i \leq z_i \leq B_i, \quad (7)$$

$$p_i \geq 0. \quad (8)$$

We first introduce two Lemmas for the objective function of LRp_{ij}, which present the solution approach for the problem LRp_{ij} without considering the discount interval constraints (12) and (13).

Lemma 1. For a fixed z_i, the optimal selling price to maximize the objective function LRp_{ij} is determined uniquely as a function of z_i:

$$p_i^{j*} \equiv p_i^j(z_i) = p_i^{j0} - \frac{\Theta(z_i)}{2b_i},$$

where $\Theta(z_i) = \int_{z_i}^{B_i} (u - z_i) f_i(u) du$ and $p_i^{j0} = \frac{a_i + b_i c_{ij}(1+\lambda) + \mu_i}{2b_i}$.

Lemma 1 has been introduced by Petruzzi and Dada (1999).

Substituting $p_i^{j*} = p_i^j(z_i)$ into LRp_{ij}, the optimization becomes a maximization over the single variable z_i: $\underset{z_i}{Maximize}\, LRp_{ij}\left[z_i, p_i^j(z_i)\right]$. Petruzzi and Dada (1999) present the sufficient conditions for the unimodality of the objective function LRp_{ij}. For our problem, these conditions can be described as follows.

Lemma 2: If $a_i - b_i(c_{ij} + \lambda c_{ij} - 2g_i) + A_i > 0$ and the $F_i(z_i)$ is a distribution function satisfying the condition $2r_i(z_i)^2 + dr_i(z_i)/dz_i > 0$ for $A_i \le z_i \le B_i$, where $r_i(z_i) \equiv f_i(z_i) / [1 - F_i(z_i)]$ is the hazard rate, then function $LRp_{ij}\left[z_i, p_i^j(z_i)\right]$ is unimodal in z_i, and there is an unique z_i^{j*} in the region $[A_i, B_i]$ that satisfies $dLRp_{ij}\left[z_i, p_i^j(z_i)\right] / dz_i = 0$.

All the following propositions and algorithms are developed based on the assumption that the conditions in Lemma 2 are satisfied. Lemmas 1 and 2 provide a way to find the optimum in the region $[A_i, B_i]$ for function LRp_{ij}, but mostly not for the subproblem LRp_{ij} since the solution satisfies constraint (7) but may violate constraint (12) or (13). We call the solution realizable if it satisfies constraints (12) and (13). The following propositions are presented for the situation when the solution is unrealizable.

We proceed to analyze the situation when constraint (12) is violated. As indicated in Lemma 2, function LRp_{ij} is unimodal in z_i. Thus, if constraint (12) is violated, then the optimal solution to subproblem LRp_{ij} is obtained at the discount break point, that is, $D_i(p_i) + z_i = d_{ij}^L$. It follows that $p_i = \frac{a_i + z_i - d_{ij}^L}{b_i}$. We define $p_i^{jL}(z_i) = \frac{a_i + z_i - d_{ij}^L}{b_i}$ and

substitute it into function LRp_{ij}. Then the optimization of function LRp_{ij} in $D_i(p_i)+z_i=d_{ij}^L$ becomes a maximization over the single variable z_i: $\underset{z_i}{Maximize}\ LRp_{ij}\left[z_i,p_i^{jL}(z_i)\right]$.

Let p_i^{jL*} and z_i^{jL*} denote the optimal solution of function LRp_{ij} in $D_i(p_i)+z_i=d_{ij}^L$. We have the following proposition.

Proposition 1: When the ordering quantity is set to d_{ij}^L, the solution to maximize function LRp_{ij} is unique, and $p_i^{jL*}=\dfrac{a_i+z_i^{jL*}-d_{ij}^L}{b_i}$ where z_i^{jL*} is the unique solution in the region $[A_i,B_i]$ that satisfies $dLRp_{ij}\left[z_i,p_i^{jL}(z_i)\right]/dz_i=0$.

The proof of proposition 1 is provided in the Appendix. Proposition 1 provides the way to solve subproblem LRp_{ij} when the solution to function LRp_{ij} violates constraint (12). Similarly, we can solve subproblem LRp_{ij} for the situation that the solution to function LRp_{ij} violates constraint (13). We omit the parallel analysis since the situation does not happen in our algorithm, which is discussed in the next section.

The following propositions give the relationships among the solutions of the subproblems LRp_{ij}, $j=1,...,k_i$, i.e., with different price levels.

Proposition 2: Let LRp_{ij}^* denote the maximum value of function LRp_{ij}, for $j=1,...,k_i$. If the conditions for Lemma 2 are satisfied, then we have $LRp_{i,j-1}^*<LRp_{ij}^*$.

Proposition 3: The optimal solutions z_i^{j*} and p_i^{j*} to maximize function LRp_{ij}, for $j=1,...,k_i$, satisfy $z_i^{j*}<z_i^{j+1*}$, and $D_i\left(p_i^{j*}\right)+z_i^{j*}<D_i\left(p_i^{j+1*}\right)+z_i^{j+1*}$.

Proofs of Propositions 2 and 3 are provided in the Appendix. Proposition 2 implies that we do not need to solve the subproblems with higher price levels if the solution to function LRp_{ij} at a price level is realizable. Proposition 3 shows that the optimal order quantity for lower price level is larger than that for higher price level.

4.1.2 Solution algorithm for subproblem LRp_i

Based on Lemmas 1-2 and Propositions 1-3, an algorithm for subproblem LRp_i, called as Algorithm A, is developed. The main steps of Algorithm A are described below and a flow chart is presented in Figure 1.

Let z_i^* and p_i^* denote the optimal solutions for subproblem LRp_i.

Algorithm A:

Step 0: Initialization

Initialize and set $j = k_i$.

Step 1:

Calculate the optimal z_i^{j*} and p_i^{j*} for maximizing function LRp_{ij}:

z_i^{j*} is obtained by Lemma 2 and p_i^{j*} is obtained by Lemma 1.

Step 2:

If $d_{ij}^L \le D_i\left(p_i^{j*}\right) + z_i^{j*} \le d_{ij}^U$, $j^S = j$ and go to Step 4;

Otherwise go to Step 3.

Step 3:

Calculate the optimal p_i^{jL*} and z_i^{jL*} for maximizing function LRp_{ij} with $D_i(p_i) + z_i = d_{ij}^L$ according to Proposition 1.

Set $j = j - 1$ and go to Step 1.

Step 4:

$$\left\{z_i^*, p_i^*\right\} = \arg\max\left\{ LRp_{ij}\left(p_i^{j*}, z_i^{j*}\right) \middle| j = j^S, LRp_{ij}\left(p_i^{jL*}, z_i^{jL*}\right) \middle| j > j^S \right\}.$$

[Insert Figure 1 here].

The algorithm starts with the lowest price. Step 1 calculates the optimal solution for function LRp_{ij}. Step 2 checks if the optimal ordering quantity is realizable, i.e.,

$d_{ij}^L \le D_i\left(p_i^{j*}\right)+ z_i^{j*} \le d_{ij}^U$. If so, according to Propositions 2 and 3 we don't need to calculate the optimal solutions to LRp_{ij} for $j < j^S$, since their optimal values are less than that of LRp_{ij^S} . Otherwise, by Proposition 2 we know that the optimal ordering quantity at this discount segment is the discount break point. Thus Step 3 calculates the optimal p_i^{jL*} and z_i^{jL*} for maximizing function LRp_{ij} with $D_i(p_i)+ z_i = d_{ij}^L$. Then the algorithm goes back to Step 1 to calculate the optimal solution to the next discount segment. Step 4 evaluates LRp_{ij^S} and compares with all the optimal solutions of subproblem LRp_{ij}, for $j > j^S$. Then the best solution is the optimum of subproblem LRp_i . Algorithm A is similar to the procedure followed to solve the single item newsvendor problem with quantity discount (Lin and Kroll 1997), but we incorporate the pricing decision into the problem.

As indicated before, the algorithm does not need to consider the situation that the solution to function LRp_{ik_i} violates constraint (13): Usually, the upper bound on the discount segment with the lowest price is a large value up to infinite, thus the optimal ordering quantity for function LRp_{ik_i} is either within price range or less than $d_{ik_i}^L$. If the solution is in the discount range, the algorithm stops; otherwise go to the second lowest price. According to Proposition 3, we know the solution to function LRp_{ij} must satisfy constraint (13).

4.2 Solving the Lagrangian Dual Problem

For a given value of λ, the Lagrangian relaxation problem provides an upper bound to the original problem. Lagrangian Dual Problem is to find the optimal Lagrangian multiplier that minimizes the upper bound.

4.2.1 Bisection algorithm

We first set λ=0 and solve subproblems LRp_i , for i=1,…,I, by Algorithm A. If

$\sum_{i=1}^{I}\sum_{j=1}^{k_i} c_{ij}q_{ij} \le B_G$, it indicates that the capacity constraint (2) is non-operative, and the optimal solutions with λ=0 are optimal to the original problem. Otherwise, we need to solve the Lagrangian dual problem to find the optimal Lagrange multiplier to minimize the upper bound. To solve the dual problem, the bisection iteration algorithm is introduced as follows:

Step 0: Set $\lambda_1 = 0$ and $\lambda_2 = \lambda_{\max}$ ($\lambda_{\max}$ is explained in the next section).

Step 1: Let $\lambda = (\lambda_1 + \lambda_2)/2$, solve all the subproblems LRP_i for $i = 1,..., I$, by Algorithm A, and get their optimal solutions z_i^* and p_i^*.

Step 2: Calculate B_{error}.

Step 3: If $abs(B_{error}) \le \delta_1$ or $abs(\lambda_1 - \lambda_2) \le \delta_2$, then Stop.

Step 4: If $B_{error} > 0$, then set $\lambda_1 = \lambda$; else set $\lambda_2 = \lambda$. Go to Step 1.

Where, we define $B_{error} = \sum_{i=1}^{I}\sum_{j=1}^{k_i} c_{ij}q_{ij} - B_G$. δ_1 and δ_2 are parameters for stop criteria. In our case, $\delta_1 = 1$ and $\delta_2 = 0.001$.

Our computational experiments show that: for small scale problems such as involving less than 20 products, the algorithm stops with condition $abs(B_{error}) \le \delta_1$; for large scale problems such as involving hundreds of products, the algorithm stops with condition $abs(\lambda_1 - \lambda_2) \le \delta_2$. It implies that the bisection algorithm can obtain the optimal solutions for small scale problems, but for large scale problems, the dual solution from the bisection algorithm may violate budget constraint (2). A feasibility algorithm is needed to construct a feasible solution when the budget constraint is violated.

4.2.2 Observations

In each iteration of the bisection algorithm, Algorithm A is repeatedly employed to solve subproblems LRP_i for $i = 1,..., I$. From Algorithm A we can see that a number of the following nonlinear equations should be solved:

$$dLRp_{ij}\left[z_i, p_i^j(z_i)\right]/dz_i = 0, \quad j = j^S, ..., k_i, \tag{a}$$

$$dLRp_{ij}\left[z_i, p_i^{jL}(z_i)\right]/dz_i = 0, \quad j = j^S, ..., k_i. \tag{b}$$

Equation (a) is used to find the optimal solution to maximize function LRp_{ij}, while equation (b) is used to get the optimal solution to maximize function LRp_{ij} in $D_i(p_i) + z_i = d_{ij}^L$, that is, the optimal solution at discount break point *j*. It is time-consuming to solve these nonlinear equations. The following property of equations (b) can used to reduce their solving times.

Observation 1: The solutions of equations (b) are independent of the Lagrange multiplier λ.

Since $\dfrac{dLRp_{ij}\left[z_i, p_i^{d_{ij}^L}(z_i)\right]}{dz_i} = \left(\dfrac{d_{ij}^L - a_i - 2z_i}{b_i} - g_i - s_i\right)F_i(z_i) + \dfrac{1}{b_i}\int_{A_i}^{z_i} uf_i(u)du + \left(\dfrac{d_{ij}^L}{b_i} + g_i\right)$, the equations (b) are independent of the Lagrange multiplier λ. Observation 1 implies that the optimal selling price and the optimal value of z_i at the price break point are constant and they don't vary with the change of λ. We only need to solve equation (b) at most one time for each product at each discount break point.

Observation 2: There is an upper bound on the Lagrange multiplier λ.

In terms of conditions in Lemma 2, we should keep $a_i - b_i(c_{ij} + \lambda c_{ij} - 2g_i) + A_i > 0$ for all *i, j*, in order to make sure that every equation (a) has an unique root, that is,

$$\lambda < \frac{a_i + A_i + 2g_i b_i}{b_i c_{ij}} - 1, \quad \forall i, j.$$

Let $\lambda_{\max} = \min\left\{\dfrac{a_i + A_i + 2g_i b_i}{b_i c_{ij}} - 1 \middle| \forall i, j\right\}$, which gives the upper bound on λ.

As pointed by Lau and Lau (1995, 1996), when the budget capacity is too small, some ordering quantities during the process of the bisection algorithm may be negative, which are

infeasible. Thus, in each iteration, the ordering quantities for all products, $q_i = D_i(p_i) + z_i$ for i=1,…,I, should be checked and the negative ones are set to zero.

4.3 Feasibility Algorithm

The solution obtained by the bisection algorithm may violate the budget constraint. We have defined that $B_{error} = \sum_{i=1}^{I}\sum_{j=1}^{k_i} c_{ij}q_{ij} - B_G$. If $B_{error} > 0$, then the solution is infeasible. While $B_{error} < 0$ implies the solution is feasible but the budget is not sufficiently utilized. Hence, we develop a feasibility procedure to either adjust the dual solution to be feasible or improve the solution. The feasibility algorithm is described as follows.

Step 0: Sort the products in the descending order in terms of unit acquisition cost.

Step 1: If $B_{error} > 0$, decrease the acquisition quantities of the products in the order until the total budget reaches its balance.

Step 2: If $B_{error} < 0$, increase the acquisition quantities for the products in the reverse order until the budget is fully utilized.

Step 3: Recalculate the optimal selling prices for the adjusted products by Proposition 1.

The basic idea of the feasibility algorithm is straightforward. Each product has an upper bound on the acquisition quantity, which is the optimal order quantity without the budget limitation. The acquisition quantity for each product adjusted in Step 2 cannot be more than the upper bound.

For most of the cases, the balance is reached by adjusting only one product's acquisition quantity since the Lagrangian dual solutions are very near to the optimal solution.

Figure 2 shows the complete structure of the Lagrangian based approach, which mainly consists of bisection algorithm and feasibility algorithm.

[Insert Figure 2 here]

5. Numerical Example and Computational Results

The proposed approach is tested on the randomly produced examples. The algorithms are implemented with Matlab. The computational experiences for all the examples are conducted on the IBM T60 laptop with Windows XP (Intel® Core™2 Duo CPU, 1GB of RAM).

5.1 Numerical example

This section presents a numerical example to illustrate our procedures. We investigated the supply and random price-dependent demand of fashion clothes such as printable garment at a retail store in China. This example is designed based on the investigation data from the acquisition and pricing process of sweatshirts with five styles. The available budget (B_G) for the sweatshirts is $ 125,000 and the random part (u_i) of the price-dependent demand is assumed to follow the normal distribution with a mean of zero. The supplier offers three discount segments: less than 2000, from 2000 to 4000, and over 4000. Other parameters for the example are shown in Table 2.

[Insert Table 2 here]

We apply the proposed Lagrangian approach to the instance. We first solve the Lagrangian dual problem with the bisection algorithm. In the dual solution, the budget required 124999.22, which is slightly less than the available budget of 125,000. Thus the dual solution is feasible. The upper bound obtained by the dual solution is 184,725.75, while the profit of the feasible solution is $184,725.44. The relative gap between the feasible solution and the bound, defined as *(upper bound-lower bound)/lower bound,* is 1.67E-07, which is very small. Thus the feasible solution is very close to the optimal solution. Since the budget required almost reaches the limit and the gap is so small, it is unnecessary to employ the feasibility algorithm to adjust the dual solution for this instance. The optimal order quantities and selling prices for the example are presented in Table 3.

[Insert Table 3 here]

5.2 Managerial analysis and comparisons

We use the above example to investigate the relationships between the solutions and critical parameters to gain some insights to the joint ordering and pricing problem.

The relationship between the expected profit and the budget capacity

We observe the change of the expected profit by varying the budget capacity from 115,000 to 175,000 while the other parameters are fixed. The relationship between the profit and the budget capacity is shown in Figure 3. It can be seen that the total expected profit increases when the budget capacity increases, but it keeps unchanged beyond about 145,000. It indicates that the profit can be increased by adding the available budget, but does not increase any more after a certain point, since the additional budget is not fully utilized due to the limitation on demands.

[Insert Figure 3 here]

The acquisition policy versus the standard deviation of a product's demand

To observe impacts of demand uncertainty on the acquisition policy, Figure 4 illustrates how the acquisition quantities of the products change when the standard deviation of product 1's demand varies from 200 to 2,000. It can be seen that the ordering quantities for products 4 and 5 keep unchanged, while the ordering quantity for product 1 increases and the ordering quantities for products 2 and 3 decrease, when the standard deviation increases from 200 to 800 and from 1400 to 2000. It implies that the retailer would shift the budget from the products with lower risk to the products with higher risk. It coincides with the result of classical newsvendor problem when the order quantity is above the demand average. But when the standard deviation varies from 800 to 1,400, the ordering quantities for all the products keep unchanged. This is because the ordering quantity for product 1 is at the discount break point 4,000. It shows that the acquisition policy is less sensitive to the uncertainty of demand when the ordering quantity is at the discount break point.

[Insert Figure 4 here]

The pricing policy versus the standard deviation of a product's demand

We also observe how demand uncertainty affects the pricing policy by varying the standard deviation of product 1's demand from 200 to 2,000. From Figure 5, it is interesting to see that the price of product 1 decreases when the standard deviation varies form 200 to 800 and from 1,400 to 2,000 respectively, and increases when the standard deviation varies from 800 to 1,400. Note from Figure 4 that the order quantity of product 1 increases when the standard deviation varies form 200 to 800 and from 1,400 to 2,000. Thus retailer reduces the selling price to induce demand increase. When the standard deviation varies from 800 to 1,400 the product 1 keeps the ordering quantity unchanged at 4000, which is the discount break point. In the situation the retailer reduces the understocking risk through increasing the selling price.

[Insert Figure 5 here]

Comparison of the solutions of discount case with that of non-discount case

Since the problem without discount is a special case of the problem with discounts, the non-discount case can also be solved by the Lagrangian based solution approach, and the solutions are presented in Table 3. The optimal profit is $ 175,191.11, which is a little less than that of the discount case. Comparing the solutions for the two cases, we can see that the ordering quantities in the discount case are more than that of the non-discount case, while the selling prices in the discount case are less than that of the non-discount case. It indicates that the supplier quantity discounts can promote the retailer orders more products, and the retailer can increase the profit by offering lower selling prices to customers. Therefore both suppliers and retailers can benefit from the supplier discount policy.

5.3 Performances of the Solution Approach

In order to further test the performance of the solution approach, thirty test problems

with the sizes of 20, 200 and 1000 products are randomly produced based on our investigation data, and then solved by the proposed solution approach. Table 4 illustrates the running times and solution gaps for the problems. According to Table 4, the maximal relative gap for the small size examples is 9.59E-04 while the average gap is 2.21E-04, and the computational times for all the small examples are within 2 seconds. For the middle size examples, the maximal and average gaps are 2.00E-04 and 2.01E-05 respectively, and the computational times are less than 16 seconds. For the large size examples, the maximal and average gaps are 3.86E-05 and 6.20E-06 respectively, which are less than that of small and middle size problems. The running time for the problem with 1000 products is no more than 80 seconds. It is concluded that our solution approach can present very good solutions for all scale examples in a short computational time.

[Insert Table 4 here]

6 Conclusions

This paper investigates the multi-product acquisition and pricing problem when uncertain demands and supplier quantity discounts are present. We illustrate that the problem is an extension of the newsvendor pricing problem. It is the first work to combine supplier discounts with the constrained multi-product newsvendor pricing problem. The combination makes the problem more practical and challenging. Through the proposed MINLP model and solution approach, this research provides the retailer an effective way to use both acquisition and pricing decisions as levers to better match demand and supply, and increase the profit under the circumstance.

We analyze the properties of the newsvendor pricing problem with supplier quantity discounts. Based on the properties, we develop a Lagrangian based solution approach. The bisection algorithm is applied to solve the Lagrangian dual problem to obtain an upper bound. Thirty numerical examples are randomly produced for testing the approach. The

computational results show that the proposed solution approach can provide high quality solutions in terms of the relative gaps, in a very short time. Through a numerical example, we also compare the solutions of two cases with and without the discounts. It is found that the discount schemes have important impacts on the newsvendor's acquisition and pricing policies: order more from suppliers and offer lower prices to customers, which benefit suppliers and also increase newsvendor's profit.

In this paper, we only consider the retailer's budget constraint, while in practice the retailer may have multiple resource constraints. Thus, a natural extension of our study is to consider the problem with multiple constraints. In this scenario, bisection algorithm is unsuitable, and the subgradient algorithm can be employed to solve such more complicated problem. Another extension to our problem is to take into account other discount schemes, such as incremental quantity discount, volume discount, and bundle discount, which are also common in current business practice. Then new models and solution approaches need to be investigated.

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Appendix

Proof of Proposition 1.

$$LRp_{ij}\left[z_i, p_i^{jL}(z_i)\right] = \int_{A_i}^{z_i}\left[\frac{a_i + z_i - d_{ij}^L}{b_i}\left(d_{ij}^L - z_i + u_i\right) - s_i\left(z_i - u_i\right)\right] f_i(u_i)du_i$$
$$+ \int_{z_i}^{B_i}\left(\frac{a_i + z_i - d_{ij}^L}{b_i} d_{ij}^L - g_i\left[u_i - z_i\right]\right) f_i(u_i)du_i - c_{ij}(1+\lambda)d_{ij}^L.$$

The first and second derivatives of $LRp_i\left[z_i, p_i^{jL}(z_i)\right]$ are as follows:

$$\frac{dLRp_{ij}\left[z_i, p_i^{jL}(z_i)\right]}{dz_i} = \left(\frac{d_{ij}^L - a_i - 2z_i}{b_i} - g_i - s_i\right)F_i(z_i) + \frac{1}{b_i}\int_{A_i}^{z_i} uf_i(u)du + \left(\frac{d_{ij}^L}{b_i} + g_i\right), \text{ and}$$

$$\frac{d^2 LRp_i\left[z_i, p_i^{jL}(z_i)\right]}{dz_i^2} = -\frac{2}{b_i}F_i(z_i) - \left(\frac{a_i + z_i - d_{ij}^L}{b_i} + g_i + s_i\right)f_i(z_i)$$

$$= -\frac{2}{b_i}F_i(z_i) - \left(\frac{a_i + z_i - d_{ij}^L}{b_i} + g_i + s_i\right)f_i(z_i) < 0.$$

Thus the first derivative $\frac{dLRp_{ij}\left[z_i, p_i^{jL}(z_i)\right]}{dz_i}$ is a monotonically decreasing function of z_i.

Since $d_{ij}^L << +\infty = B_i$,

$$\frac{dLRp_{ij}\left[B_i, p_i^{jL}(B_i)\right]}{dz_i} = \frac{2d_{ij}^L - a_i - 2B_i + \mu_i}{b_i} - g_i - s_i < 0,$$

and $\frac{dLRp_{ij}\left[A_i, p_i^{jL}(A_i)\right]}{dz_i} = \frac{d_{ij}^L}{b_i} + s_i > 0.$

Therefore there is a unique z_i^{jL*} in the region $[A_i, B_i]$ that satisfies $dLRp_{ij}\left[z_i, p_i^{jL}(z_i)\right]/dz_i = 0$. □

Proof of proposition 2:

Let z_i^{j*} and p_i^{j*} denote the optimal solutions to maximize function LRp_{ij}, for $j = 1,...,k_i$. Then we have $LRp_{ij}^* = LRp_{ij}\left[z_i^{j*}, p_i^{j*}\right]$.

Since $LRp_{ij}\left[z_i^{j*}, p_i^{j*}\right] > LRp_{ij}\left[z_i^{j-1*}, p_i^{j-1*}\right]$ and

$$LRp_{ij}\left[z_i^{j-1*}, p_i^{j-1*}\right] - LRp_{i,j-1}\left[z_i^{j-1*}, p_i^{j-1*}\right] = \left(c_{i,j-1} - c_{ij}\right)\left[D\left(p_i^{j-1*}\right) + z_i^{j-1*}\right] > 0,$$

we have $LRp_{ij}\left[z_i^{j*}, p_i^{j*}\right] > LRp_{i,j-1}\left[z_i^{j-1*}, p_i^{j-1*}\right]$. □

Proof of proposition 3:

Let

$$R_{ij}(z_i) \equiv \frac{dLRp_{ij}(z_i, p_i^j(z_i))}{dz_i} = -\left[(1+\lambda)c_{ij} + g_i\right] + \left[\frac{a_i + b_i c_{ij}(1+\lambda) + \mu_i}{2b_i} + g_i + s_i - \frac{\Theta(z_i)}{2b_i}\right]\left[1 - F_i(z_i)\right]$$

$$= -c_{ij}\frac{1+\lambda+F_i(z_i)}{2} - g_i + \left[\frac{a_i + \mu_i}{2b_i} + g_i + s_i - \frac{\Theta(z_i)}{2b_i}\right]\left[1 - F_i(z_i)\right],$$

for $j = 1,...,k_i$.

As z_i^{j*} and p_i^{j*} are the optimal solutions to LRp_{ij}, for $j = 1,...,k_i$, from lemma 2 we know that $R_{ij}(z_i^{j*}) = 0$, for $j = 1,...,k_i$.

Since $R_{ij}(z_i^{j+1*}) - R_{i,j+1}(z_i^{j+1*}) = -\frac{1+\lambda+F_i(z_i^{j+1*})}{2}(c_{ij} - c_{i,j+1}) < 0$,

$R_{ij}(z_i^{j+1*}) < R_{i,j+1}(z_i^{j+1*}) = 0$.

Furthermore $R_{ij}(A) = \frac{a_i - b_i(c_{ij} + \lambda c_{ij} - 2g_i) + A_i}{2b_i} > 0$.

Therefore the unique root z_i^{j*} for $R_{ij}(z_i)$ belongs to range (A_i, z_i^{j+1*}), that is, $z_i^{j*} < z_i^{j+1*}$.

Let $q_i^{j*} = D_i(p_i^{j*}) + z_i^{j*}$, $j = 1,...,k_i$.

Substitute $p_i^{j*} = p_i^j(z_i^{j*})$ into q_i^{j*}, and we obtain

$$q_i^{j*} = \frac{a_i - b_i c_{ij}(1+\lambda) - \mu_i}{2} + \frac{\Theta(z_i^{j*})}{2} + z_i^{j*}.$$

Function $q_i^j(z_i) = \frac{a_i - b_i c_{ij}(1+\lambda) - \mu_i}{2} + \frac{\Theta(z_i)}{2} + z_i$ monotonously increases in z_i, for $j = 1,...,k_i$, as $\frac{dq_i^j(z_i)}{dz_i} = \frac{1+F_i(z_i)}{2} > 0$.

$z_i^{j*} < z_i^{j+1*}$, thus $q_i^j(z_i^{j*}) < q_i^j(z_i^{j+1*})$.

$q_i^{j+1}(z_i^{j+1*}) - q_i^j(z_i^{j+1*}) = \frac{b_i(c_{ij} - c_{i,j+1})(1+\lambda)}{2} > 0$, therefore $q_i^j(z_i^{j+1*}) < q_i^{j+1}(z_i^{j+1*})$.

We can get $q_i^j(z_i^{j*}) < q_i^{j+1}(z_i^{j+1*})$, that is, $D_i(p_i^{j*}) + z_i^{j*} < D_i(p_i^{j+1*}) + z_i^{j+1*}$.□

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