A LESSON PLAN WITH AN ARC MIDPOINT

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Abstract

The article introduces an explicit way of locating the arc midpoint in the Cartesian plane, which is consistent for both x- and y-coordinates and is technically accessible for students starting as young as fifteen. The authors give the proof of the statement using two trigonometry identities, and discuss some materials for innovative lessons on the arc midpoint computation that could enrich and enhance curriculum.

Introduction

Circumference is one of the most nearly perfect and most important lines in mathematics and science. Its segments, arcs, and their midpoints occur in thousands of theoretic and real-world problems. Along with the linear midpoint formula, the arc midpoint is beneficial for students' mathematics learning in general, and for their performance in coordinate geometry in particular. The logic conjunction "**iff,**" used in the arc midpoint statement, means "if and only if."

Arc Midpoint Computation

Let the origin-centered arc of radius r in the Cartesian plane (see Figure 1) have the endpoints A and B with x-coordinates a, b respectively, and midpoint M with x-coordinate μ. Then,

$$2\mu = \pm\sqrt{(r+a)(r+b)} \pm \sqrt{(r-a)(r-b)}\ , \qquad (1)$$

where the **first** radical has " $-$ " **iff** the arc makes a negative x-intercept, and the **second** radical has " $+$ " **iff** the arc makes a positive x-intercept.

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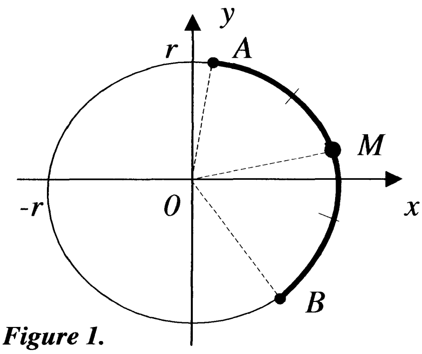

Figure 1.

The Same Relationship Holds for y-Values — Proof of this statement is shown below. Note that details of the proof are accessible only for strong mathematics students who are learning trigonometry at the advanced level. Consider additional propositions that are used in the main proof.

Two Identities

For any $p, q \in [-1, 1]$

$$\sin^{-1} p + \sin^{-1} q = 2\sin^{-1} A, \tag{2}$$

$$\cos^{-1} p + \cos^{-1} q = 2\cos^{-1} A, \tag{3}$$

where $2A = \sqrt{(1+p)(1+q)} - \sqrt{(1-p)(1-q)}$.

To prove identity (3), we denote $p = \cos\alpha$, $q = \cos\beta$, and $\gamma = \dfrac{\alpha+\beta}{2}$, where $\alpha, \beta, \gamma \in [0, \pi]$. Then, its left side of identity (3) is simplified to $\cos^{-1}(\cos\alpha) + \cos^{-1}(\cos\beta) = \alpha + \beta = 2\gamma$. Since $\alpha, \beta \in [0, \pi]$, then $\cos\dfrac{\alpha}{2} \geq 0$ and $\cos\dfrac{\beta}{2} \geq 0$. Using this, let us simplify its right side:

$$2\cos^{-1} A = 2\cos^{-1}\frac{1}{2}\left(\sqrt{(1+\cos\alpha)(1+\cos\beta)} - \sqrt{(1-\cos\alpha)(1-\cos\beta)}\right) =$$

$$2\cos^{-1}\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = 2\cos^{-1}\left(\cos\frac{\alpha+\beta}{2}\right) = 2\gamma,$$

and identity (3) is proved. Identity (2) is a simple corollary of identity (3), indeed:

$$\sin^{-1} p + \sin^{-1} q = \frac{\pi}{2} - \cos^{-1} p + \frac{\pi}{2} - \cos^{-1} q = \pi - \left(\cos^{-1} p + \cos^{-1} q\right) =$$

$$\pi - 2\cos^{-1} A = 2\left(\frac{\pi}{2} - \cos^{-1} A\right) = 2\sin^{-1} A,$$

and identity (2) is also proved.

Arc Midpoint Computation Proof

Denote $p = a/r$, $q = b/r$, and $m = \mu/r$. Consider angles $\alpha = \cos^{-1} p$, $\beta = \cos^{-1} q$, and $\gamma = \cos^{-1} m$ that radii *OA, OB,* and *OM* form with positive part of x-axes, respectively.

There are four cases.

Case 1. The arc does not have x-intercepts. Then, $\gamma = (\alpha + \beta)/2$, and therefore $m = \cos\gamma = \cos\frac{1}{2}\left(\cos^{-1} p + \cos^{-1} q\right)$. Using identity (3), we get $m = A = \frac{1}{2}\left(\sqrt{(1+p)(1+q)} - \sqrt{(1-p)(1-q)}\right)$. Hence, $2\mu = \sqrt{(r+a)(r+b)} - \sqrt{(r-a)(r-b)}$, and Case 1 is proved.

Case 2. The arc has a positive x-intercept, but does not have a negative one. Then, $\gamma = |\alpha - \beta|/2$, and therefore $m = \cos\frac{1}{2}\left(\cos^{-1} p - \cos^{-1} q\right)$. In addition, using identities (2) and (3), it is easy to see that $\left|\cos^{-1} p - \cos^{-1} q\right| = 2\cos^{-1}\frac{1}{2}\left(\sqrt{(1+p)(1+q)} + \sqrt{(1-p)(1-q)}\right)$ also holds for any $p, q \in [-1, 1]$. From here, we get $m = \frac{1}{2}\left(\sqrt{(1+p)(1+q)} + \sqrt{(1-p)(1-q)}\right)$. Hence, $2\mu = \sqrt{(r+a)(r+b)} + \sqrt{(r-a)(r-b)}$, and Case 2 is also proved.

Case 3. The arc has a negative x-intercept, but does not have a positive one. This part of the proof is similar to Case 2 with $\gamma = \pi - \frac{1}{2}|\alpha - \beta|$.

Case 4. The arc has two x-intercepts. This part of the proof is similar to Case 1 with $\gamma = \pi - \frac{1}{2}(\alpha + \beta)$.

Proof for y-values can be achieved similarly using identity (2) or otherwise. ***Q.E.D.***

Thoughts and Materials for Lessons

The first lesson on the new topic could begin from recalling the midpoint formula and illustrating with a quick example. Then, an analogy with the arc midpoint computation and its diagram can be made. The diagram is an important part of the computation. After the theory of computation is introduced, and before considering numerical examples, it is useful to have preliminary exercises to help students understand the logic of two "$\pm$ decisions." Through such exercises, the teacher ensures that students use the conjunction **iff** properly. Several diagrams, representing different locations of the arcs, may be shown on the board, and students could be asked to determine signs of both radicals in the formula (1) based on the particular location of the arc. For example, for the arc shown in Figure 1, the first radical has "$+$" because an arc does not have a negative x-intercept, and the second radical has "$+$" because an arc does have a positive x-intercept. Or, for the arc shown in Figure 2, the first radical has "$-$" because an arc does have a negative x-intercept, and the second radical has "$-$" because an arc does not have a positive x-intercept. When preliminary "$\pm$practice" is finished, numerical examples could be discussed. In the following examples, we provide a selection of sample problems where the *exact answer is to be found without using a calculator.*

<u>Example A:</u> An origin-centered arc of radius 50, located as shown in Figure 1, has the ends at $x = 14$ and $x = 25$. Find the x-coordinate of its midpoint.

<u>Example B:</u> An arc, with radius 40 and the center at origin, is located above the x-axis. If it begins and ends at $x = -24$ and $x = 9$, what is the x-value of its midpoint?

<u>Example C:</u> An arc has its center at $(0,0)$ and radius 82. It starts at $y = 18$ in quadrant II, passes through quadrant III and ends in quadrant IV at $y = -1$. What is the y-value of the arc's midpoint?

Solution A: We are given $r = 50$, $a = 14$, $b = 25$. As previously discussed, in this case both radicals in the arc midpoint formula (1) have "$+$" hence,

$$2\mu = \sqrt{(50+14)(50+25)} + \sqrt{(50-14)(50-25)} = 40\sqrt{3} + 30\ ,$$

and $20\sqrt{3} + 15$ is the answer.

Solution B: We are given $r = 40$, $a = -24$, $b = 9$. In this case, the first radical has "$+$," since the arc does not have a negative x-intercept, and the second radical has "$-$," since the arc does not have a positive x-intercept. Using formula (1), we get

$$2\mu = \sqrt{(40-24)(40+9)} - \sqrt{(40+24)(40-9)} = 28 - 8\sqrt{31}\ ,$$

and $14 - 4\sqrt{31}$ is the answer.

Solution C: $r = 82$, $a = 18$, $b = -1$ are given. In this case, the first radical is "$-$," since the arc does have a negative y-intercept, and the second radical has "$-$," since the arc does not have a positive y-intercept. Hence,

$$2\mu = -\sqrt{(82+18)(82-1)} - \sqrt{(82-18)(82+1)} = -90 - 8\sqrt{83}\ ,$$

and $-45 - 4\sqrt{83}$ is the answer.

Applications of the arc midpoint computation to the real-world problems could be planned for the next lesson. In such problems, both exact and rounded answers could be requested, and a calculator should be used for evaluating radicals.

Problem

A water tank (T), a grain bin (B), and a storage unit (S) are located on the circle (see Figure 2). T is 0.6 km away from the center C and equidistant from S and B. If S is located 0.4 km south of center C and B is located 0.2 km north of C, how far north of C is T located?

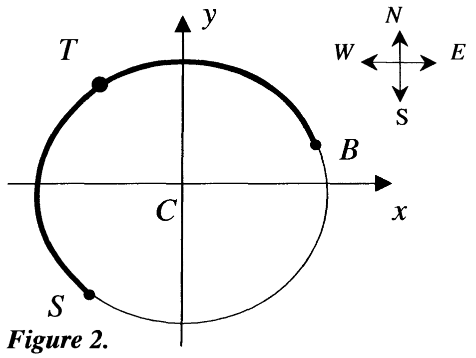

Figure 2.

Solution

Introduce the coordinate system with origin at C, x-axis pointed east and y-axis pointed north. Note that T is a midpoint of the arc STB. Using y-coordinates and 1 unit = 100 m, we have $a = -4$, $a = 2$, $r = 6$. For the first radical in (1) we chose "+," since the arc does not have a negative y-intercept. For the second radical in (1), we chose "+" since the arc does have a positive y-intercept. Then, formula (1) gives the y-value of

T: $\frac{1}{2}\left(\sqrt{(6-4)(6+2)}+\sqrt{(6+4)(6-2)}\right)=2+\sqrt{10}$. Hence, T is located $100\left(2+\sqrt{10}\right)$ m north of C (or 516 m north of C).

New problems for further practice could be prepared using various real-world situations that involve arcs and their midpoints.