BUILDING MIDDLE-LEVEL MATHEMATICS TEACHERS' CAPACITIES AS TEACHERS AND LEADERS: THE MATH IN THE MIDDLE INSTITUTE PARTNERSHIP

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Abstract

This article describes professional development for middle-level mathematics teachers offered through the Math in the Middle Institute Partnership, a National Science Foundation-funded project to build teachers' capacities to improve mathematics learning for all students. An overview of the project, including descriptions of its goals and curriculum are provided. Detailed descriptions of two mathematics courses and one pedagogy course are offered. The mathematics courses included here are the introductory course to the Math in the Middle Institute, as well as one of the final math courses of the Institute in which participants apply mathematical knowledge and processes to real-world problems. The pedagogy course features curriculum that enables teachers to acquire an understanding of the nature and purpose of action research, and launches teachers into planning and implementing systematic inquiry in their own mathematics classrooms around topics of their choosing. The varied abilities of teachers, as well as growth in teachers' mathematical and pedagogical capacities, are represented by several samples of student work provided within the article. In addition, mathematical and pedagogical products of student work are also provided through the project's URL links.

Improving teacher quality is identified as a national need in mathematics education and one many universities and schools across the country are working in partnership to try to address. This article describes a professional development project aimed at improving mathematics teaching and learning in the middle grades. An overview of the project, along with a close look at several of its course offerings, are presented highlighting mathematical and pedagogical goals, challenges, and accomplishments.

Introduction

The Math in the Middle Institute Partnership (M^2) is a partnership among mathematicians and mathematics educators at the University of Nebraska-Lincoln (UNL), and mathematics

teachers and administrators in the Lincoln Public Schools (LPS) and Nebraska's *Rural Educational Service Units* (ESU's). The aim of the Partnership is to develop intellectual leaders in middle-level mathematics (fifth through eighth grades) by investing in strengthening the capacities of teachers. This will, in turn, improve student achievement in mathematics and hopefully reduce achievement gaps in the mathematical performance of diverse student populations in Nebraska. The work of M^2 is informed by and provides evidence-based contributions to research on learning, teaching, and teacher professional development. The endeavor is funded by the National Science Foundation (NSF) and led by four co-principal investigators: W. James "Jim" Lewis, UNL Department of Mathematics; Ruth Heaton and Tom McGowan, UNL Department of Teaching, Learning, and Teacher Education (TLTE); and, Barbara Jacobsen, Curriculum Director for the Lincoln Public Schools.

The Math in the Middle Institute Partnership includes three major components. One is the M^2 Institute, a multi-year institute that offers participants a coherent program of study to deepen their mathematical knowledge for teaching and to develop their leadership skills. The second one is the use of mathematics learning teams, led by M^2 teacher participants and supported by school administrators and university faculty, which are intended to develop collegiality, help teachers align their teaching with state standards, and assist teachers in examining their instructional and assessment practices. The third and final component is a research initiative that transforms the M^2 Institute and the M^2 mathematics learning teams into laboratories for educational improvement and innovation.

Because more than half of Nebraska's population is located in rural areas and in towns of less than 25,000 people, Math in the Middle also focuses attention on the challenges and opportunities faced by mathematics teachers who teach in rural communities. We have established partnerships with sixty-seven school districts and fifteen of the seventeen ESU's across the State of Nebraska (the two ESU's not included in the Partnership represent urban school districts). The priority that Math in the Middle gives to concerns of rural education will permit it to make a unique contribution to the needs of students in rural schools and research in mathematical education [1].

The research agenda has two main foci: one is on understanding teachers' capacities to translate the mathematical knowledge and habits of mind acquired through professional development opportunities of M^2 into changes in classroom practice; the other is on understanding how changes in mathematics teaching practice translate into measurable improvement in student performance. We are particularly interested in how M^2 teachers support

one another, as well as other staff, in their individual schools in improving mathematics instruction. A description and preliminary findings from collaborative research with the Distributed Leadership Studies project are presented in an article also appearing in this *Journal* issue [2]. Although the learning teams and research initiative are significant features of the project, this article focuses on the M^2 Institute.

The Math in the Middle Institute

The M^2 Institute is designed to offer content rich courses intended to develop teachers' mathematical knowledge and knowledge of effective classroom pedagogy, and to conduct an action research project, thereby building their capacities as teachers and positioning them to be leaders among their peers. The Institute culminates in one of two degrees: a Master of Arts for Teachers (MAT) with a Specialization in the Teaching of Middle-Level Mathematics from the College of Arts and Science; or, a Master of Arts (MA) degree from the College of Education and Human Sciences. The participants go through the 25-month program in cohorts. To date, two cohorts of participants have completed the program, with the third and fourth cohorts scheduled to complete the program in Summer 2008 and Summer 2009, respectively. Across the four cohorts, 136 teachers were accepted into the program. The M^2 Institute has seen very few drop-outs as sixty teachers have already earned a master's Degree and seventy more remain active in the program.

The Curriculum

The *Principles and Standards*, *The Mathematical Education of Teachers*, and *Foundations for Success*, guide our goals for the pedagogical and mathematical content for teachers across the curriculum of the Math in the Middle Institute [3-5]. The Institute consists of twelve courses, including seven in the Department of Mathematics, one in the Department of Statistics, three in education offered by TLTE, and a capstone course that can be taken through either the Department of Mathematics or TLTE, depending on an individual teacher's master's program. Descriptions of each course can be found on the M^2 website [6]. The following is a list of these M^2 Institute courses:

     MATH 800T: Mathematics as a Second Language
     MATH 802T: Functions, Algebra, and Geometry for Middle-Level Teachers
     MATH 804T: Experimentation, Conjecture, and Reasoning
     MATH 805T: Discrete Mathematics for Middle-Level Teachers
     MATH 806T: Number Theory and Cryptology for Middle-Level Teachers
     MATH 807T: Using Mathematics to Understand Our World
     MATH 808T: Concepts of Calculus for Middle-Level Teachers
     STAT 892: Statistics for Middle-Level Teachers

TEAC 800: Inquiry into Teaching and Learning
TEAC 801: Curriculum Inquiry
TEAC 888: Teacher as Scholarly Practitioner
Capstone Course: Integrating the Learning and Teaching of Mathematics

In mathematics, we chose to create eight new mathematics courses designed to offer a challenging curriculum for middle-level teachers. The Department of Statistics developed *Stat 892: Statistics for Middle-Level Teachers*. In the Department of Teaching, Learning, and Teacher Education (TLTE), three courses are required of all students who earn a master of arts degree (*TEAC 800, 801*, and *889*). Faculty from TLTE approved a plan to offer special sections of each course (as well as *TEAC 888*, a course in action research) that meet the goals of these courses, but when possible, do so in the context of mathematics teaching and learning. The *Capstone Course* is an integrated mathematics and pedagogy experience that assists teachers in transferring the mathematics and pedagogy they have learned at the Institute to their classroom practices, and helps teachers plan for their emerging roles as leaders.

Across all of the mathematics courses is an overarching goal of helping middle-level mathematics teachers develop *mathematical habits of mind*. Mathematical habits of mind represent a deeper view of what it means to do mathematics, based on orientations mathematicians bring to their work, and the expectations for mathematical understandings for preK-12 students [7-9]. As a project, we continue to construct and reconstruct our own understanding of the phrase. Here is the project's current working definition, presented as a set of skills and dispositions of a mathematical thinker. A mathematical thinker with well-developed habits of mind:

- Understands which tools are appropriate when solving a problem;
- Is flexible in his/her thinking;
- Uses precise mathematical definitions;
- Understands that there exist multiple paths to a solution;
- Is able to make connections between what one knows and the problem;
- Knows what information in the problem is crucial to its being solved;
- Is able to develop strategies to solve a problem;
- Is able to explain solutions to others;
- Knows the effectiveness of algorithms within the context of the problem;
- Is persistent in the pursuit of a solution;
- Displays self-efficacy while doing problems; and

- Engages in meta-cognition by monitoring and reflecting on the processes of conjecturing, reasoning, proving, and problem solving.

We are also working to understand mathematical pedagogical habits of mind, an extension of the construct, as a means of understanding the dispositions teachers may bring to their development of these ways of thinking with their middle-level students [10].

There are essentially two types of courses taken by Math in the Middle participants: on-line courses (taken during the school year), and on-site courses (completed during the summer months). The distance courses are completed over the length of a standard semester while the on-site courses are completed in one to two weeks' time. Regardless of which type of course, they have several features in common.

In all M^2 courses, homework is assigned, collected, reviewed, and graded (in some fashion) on a regular basis. Homework assignments include a variety of problems, including ones that are computational in nature to "Habits of Mind" problems which require extensive problem solving, explanation, and mathematical justification. Participants are encouraged to collaborate on assignments in whatever groups are convenient, but to submit their work individually.

Most M^2 courses divide the class into subgroups, each assigned to a member of the instructional team. These groups convene daily (during on-site courses) in order to discuss homework and other course content. These small groups are an important feature for the courses, as participants who are hesitant to present their work or ask questions before the entire class are frequently more comfortable doing so in the smaller setting.

The M^2 courses typically culminate in a course portfolio containing the following: 1) a set of problems and solutions selected by the student to be representative of course accomplishments; 2) student written reflections about the nature of course learning; and, 3) solutions to what is referred to as an "End-of-Course Problem Set." Because our goal is to help teachers reach a point where they can successfully solve the problems we assign, we permit the teachers to submit solutions, receive feedback, and revise.

The one- or two-week Summer Institute courses are inspired by the system used by the Vermont Mathematics Initiative [11]. Courses meet eight hours each day for five days with homework assigned each evening. We believe this approach to instruction is respectful of the many demands on a teacher's time. The academic year courses are best described as "blended distance education courses." By this, we mean that there is an on-campus component and a

distance education component for each course. For the two on-campus days, the class meets eight hours each day with a homework assignment overnight. Ideally, this portion of the course will cover about 40% of the course, thus making the distance education portion of the course a reasonable "add-on" to the teachers' other duties.

For the distance education portion of academic year courses, we use *Blackboard*®, PC NoteTaker™, e-mail, and Macromedia Breeze communication network software in working with teachers. Use of technology is also embedded in many of the courses, whether they are on-line or face-to-face. Each participant receives a TI-84 Plus Silver Edition calculator and uses it for several purposes, one of which is to graph more complex functions (e.g., exponential functions, trig functions, higher degree polynomials) to promote the idea that a calculator can be a tool in exploring more complicated mathematics than they might otherwise be able to study.

An Expanded Examination of the Institute: A Look at Three Courses

In order to convey a range of ways we try to meet our goals—offering challenging mathematical and pedagogical content to teachers, supporting teachers to be successful, integrating mathematics and pedagogy, and making central the idea of developing habits of mind of a mathematical thinker)—we offer a closer look at three courses within the Institute. These courses are: *MATH 800T: Mathematics as a Second Language*; *MATH 807T: Using Mathematics to Understand Our World*; and, *TEAC 888: Teacher as Scholarly Practitioner*.

Mathematics as a Second Language

A primary focus of *Mathematics as a Second Language (MSL)*, the first course of the Institute, is on understanding mathematics as a language. This course lays the foundation for developing the "habits of mind of a mathematical thinker." Course goals include understanding numbers (arithmetic), developing number sense, and introducing algebra as a means of communicating mathematical ideas; that is, thinking about numbers as adjectives, and the nouns those adjectives modify. This course stresses a deep understanding of the basic operations of arithmetic, as well as the interconnected nature of arithmetic, algebra, and geometry. The following topics are included: a comparison of arithmetic and algebra; the process of solving equations; an understanding of place value and the history of counting; an understanding of inverse processes; an awareness of the geometry of multiplication; a recognition of the many meanings of division; a comparison of rational and irrational numbers, and an understanding of the 1-dimensional geometry of numbers. We borrowed this course and its content materials from

the Vermont Mathematics Initiative [13]. One "innovation" offered by our Institute is the introduction of what our teachers have come to call, "Habits of Mind" Problems.

As the first course of the Institute, we are challenged to begin to understand who these teachers are as learners of mathematics, what their mathematical strengths and needs are, and how best to meet their varied needs. Participants teach fifth through eighth grades, yet enter the Institute with differing mathematical backgrounds and teaching experience. While some participants enter having been a college math major and teach grades 7-12 (including some who teach calculus), the majority have degrees in elementary education and many may have only taken one or two college mathematics courses.

As the course progresses, participants are assigned problem sets that reinforce the course topics. In addition, participants work special "Habits of Mind" problems that challenge them to develop their problem solving and adaptive reasoning ability. "The Triangle Game" is one such problem [14]. Students were asked to respond to the following five parts of the problem: 1) Find a way to put the numbers 1-6 at each point on the triangle to create equal side sums; 2) Is there more than one way to get equal side sums? 3) Is it possible to have two different side sums? What are the smallest and largest possible sums and why? 4) What side sums are possible? 5) What is a possible generalization of The Triangle Game? In The Triangle Game, one must use the numbers one through six, placing one number at each vertex and edge midpoint in such a way that each side (two vertices plus one midpoint) has the same sum. Two of the possible solutions for part one are shown below in Figure 1.
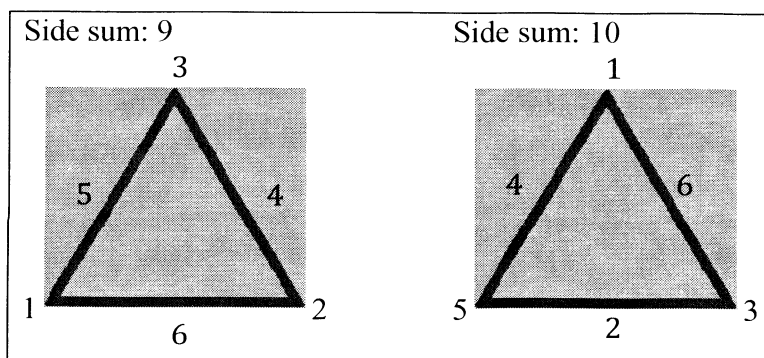

Figure 1. Two possible solutions for The Triangle Game.

Students' work across The Triangle Game problem varied tremendously, ranging from teachers who gave partial answers or grappled with what it means to justify and generalize solutions, to

those who already had great capacity to reason and communicate their ideas. Three variations in student work are shown in Figures 2-4. Figure 2 represents the only work Student A did on the five parts of the problem.
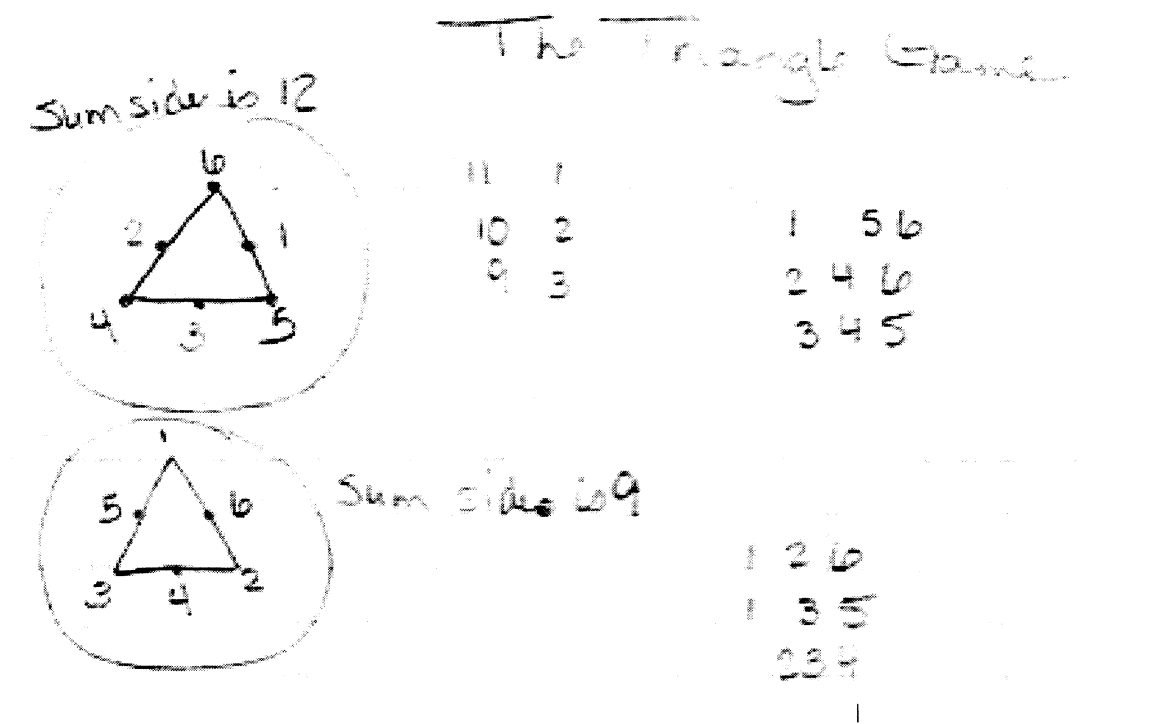

Figure 2. Student A's work on The Triangle Game.

She was elementary certified and entered the program with very few formal mathematics courses and low mathematical self-efficacy. Her solution shows efforts to explore numbers to find two possible solutions. Figure 3 represents the work of Student B, a middle-level certified teacher, who teaches fifth and sixth grade mathematics.
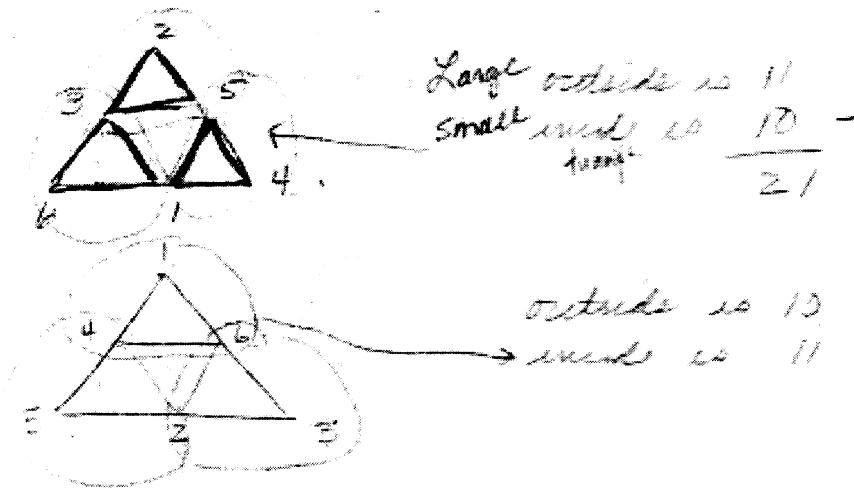
Figure 3. Student B's work on Part 3 of The Triangle Game.

Student B's work explores an interesting relationship among the arrangement of numbers in the solutions that she found. While this may be evidence that she came to our program with a stronger mathematics background than Student A, she still misuses the term "generalization" and she uses terms, such as "large outside," without defining them.

A third participant, an eighth grade teacher with a secondary certification offers evidence of even better mathematical sophistication at this early point in our program (see Figure 4). Her solution included the following justification that nine is the smallest possible side sum.

To get the "side sum" with the SMALLEST value for the sum, you would have to put the 3 smallest numbers at the vertices. The 3 larger numbers would then be put at the midpoints by placing the largest (6) between the smallest (1 and 2), the next largest (5) between the next smallest (1 and 3). That leaves only one place for the 4 to go (between the 2 and 3). This creates a side sum of 9.

5.) A possible generalization of The Triangle Game would be:

To get the the 'side sum' with the SMALLEST value for the sum, you would have to put the 3 smallest #s at the vertices. The 3 larger #s would then be put at the midpoint by placing the largest (6) between the smallest (1+2), the next largest (5) between the next smallest (1+3). That leaves only one place for the 4 to go (between the 2+3). This creates a side sum of 9.

To get a larger side sum, reverse that process. Place the 3 largest (4, 5+6) at the vertices. Then the midpoints would be placed with the smallest (1) between the largest pair (6+5), then (2) between (6+4) and finally the largest of the smaller #s (3) between the smallest sum of the pair of larger #s (5+4) to create sums of 12.

Figure 4. Student C's work on Part 5 of The Triangle Game.

The goals of this course and across the Institute as a whole are to meet these varied mathematical needs of the participants by making mathematical content accessible to all students, guiding the development of sound mathematical reasoning, and providing rigorous mathematical challenges. Generally, students are positive about the course and find that they are capable of doing challenging mathematics and experiencing success. When asked in a course evaluation what contributed most to their learning, participants offered a variety of responses, including group work, challenging yet feasible assignments, and looking at problems from multiple perspectives. One teacher wrote:

> It stretched my thinking so much that I was physically sore—I called it a mathematical hangover. However, it was welcomed. I felt like I knew many of the concepts (not all), but showing why was the key.

Using Mathematics to Understand Our World

Using Mathematics to Understand Our World (*UMW*) is one of the final mathematics courses offered within Math in the Middle. It is offered in the second spring semester as a distance learning class, designed around a series of projects in which participants examine the mathematics underlying several socially relevant questions which arise in a variety of academic disciplines (i.e., *real-world* problems). Participants learn to extract the mathematics out of the problem in order to construct models to describe them. The models are then analyzed using skills developed in this or previous mathematics courses. One key challenge for this class is learning to deal with the "messiness" inherent in using mathematics to model real-world problems. Such mathematical models frequently entail difficult mathematical ideas—ones frequently not encountered by elementary and middle-level teachers.

The primary goal of the course is to broaden students' mathematical perspectives by exposing them to a variety of interdisciplinary settings to which mathematical topics can be applied. Three additional course goals include the development of mathematical modeling and problem solving skills, an improved ability to read technical reports and research articles, and the refinement of written mathematical communication skills.

For each project assigned during the course, original documentation (such as government reports, data, and research articles) is provided whenever possible so that students develop an appreciation for the very real role mathematics plays in society. An overview of the six course projects can be found on the M^2 website [6]. Students then work in groups to complete the following basic pattern of activities:

- Study the problem and essential background information;
- Identify mathematical aspects of the problem to develop and analyze an appropriate mathematical model;
- Use the model and its analysis to understand more complex versions of the problem as described in research articles or other documentation; and,
- Submit written reports summarizing results.

Specific mathematical content includes exponential growth and decay, logarithmic functions, Newton's Law of Cooling, simulations, graphing data, making predictions, analysis of the effects of error, probability, and quality control. The disciplines to which the mathematics is applied include biology, medicine, natural science, forensics, finance, and industry.

Teachers strengthen their communication skills in mathematics by working collaboratively, sharing ideas on discussion boards, and submitting written descriptions and justifications of their mathematical models and solutions. Their written reports incorporate mathematics into language intended for non-mathematical audiences, thereby developing teachers' skills in articulating connections between a mathematical study and its concrete applications. The course affords teachers the opportunities to apply the mathematical knowledge they have learned in previous courses to new kinds of problems. While teachers find the course challenging, most appreciate the opportunity to do mathematics in the context of real-world applications. In a final course evaluation, one participant commented:

> This class stimulated my thinking and changed my views about how to incorporate real-world problems/projects in the mathematics classroom. I now see how using projects with the math embedded can provide enough student practice of procedures while giving students the experience of how mathematics is used out in the real world.

Teacher as Scholarly Practitioner

Teacher as Scholarly Practitioner introduces participants to the theory and practice of teacher-led inquiry into effective practice. The course prepares teachers to engage in a classroom-based action research project to be conducted during the second spring semester while simultaneously taking the *Using Mathematics to Understand Our World* course. Participants read and synthesize educational research related to their chosen action research topic, and also seek official university approval (Institutional Review Board [IRB]) for their planned projects.

The course provides opportunities to examine the theoretical underpinnings, issues, concerns, and methodologies of practitioner-based inquiry. Intended outcomes include an understanding of the following concepts: 1) teaching as not separate from research; 2) theory and practice as interdependent and constantly shifting in response to the educational environment; 3) inquiry as being central to the education process; and, 4) practitioner research as stemming from educators' questions of and reflections on their everyday practice and desire to improve teaching and learning. Teachers make plans for systematically examining some aspect of their own teaching based on a topic of their own choosing.

Teacher as Scholarly Practitioner builds on the academic reading and writing practiced in two previous M^2 pedagogy courses: *Inquiry Into Teaching and Learning*, and *Curriculum Inquiry*. *Inquiry Into Teaching and Learning* introduces educational research in a variety of forms. Participants build skills in locating, reading, analyzing, evaluating, and synthesizing educational research. Participants develop professional writing skills and work collaboratively to build knowledge in disciplined inquiry. As part of the ongoing evaluation of M^2 courses, the *Inquiry Into Teaching and Learning* course was moved from the summer to the spring semester in order to give more time for participants to be immersed in reading and writing. The *Curriculum Inquiry* course focuses on helping participants gain a deeper understanding of mathematics curriculum development, including historical and contemporary issues influencing curriculum planning and educational change. The course challenges participants to see curriculum extending beyond textbooks. Participants engage in detailed curricular analysis of their own mathematics curriculum as they deepen their understanding of curricular issues.

Teacher as Scholarly Practitioner offers participants opportunities to be deeply engaged in academic inquiry. One of the challenges for learners in this course includes learning how to write good research questions that are narrow, yet detailed enough to guide a disciplined inquiry. While each teacher participant chooses his or her own topic for the action research project, most *research questions* are related to making changes in current practices or trying something for the first time related to the following topics: problem solving, communication (oral or written), cooperative learning, assessment, homework, or vocabulary. Teachers must gather at least three sources of data for each of three research questions they are required to ask. The types of data used include, but are not limited to: pre-/post-surveys, student interviews, examples of student written work (e.g., in class, homework, tests) and teacher journal.

Students plan the course in the second fall semester and carry out classroom data collection in the spring, while also taking *UMW*. Participants are expected to write about their

research studies; for many, this is their first serious venture into scholarly writing [15]. Expectations for the depth of data analysis and length of the paper vary by degree, with TLTE graduates writing in-depth summative projects while graduates from the Department of Mathematics write much briefer reports and, instead, spend much of their time just prior to graduation on individual *Mathematics as a Second Language* (*MAT*) expository papers and a mathematics exam [16]. Having experienced cycles of inquiry first hand, we hope teachers will continue to try new things while teaching and study what happens based on their learning in the Institute.

Building Capacity

    We have observed M^2 teachers grow tremendously in their capacities to engage in the learning of challenging mathematics across their involvement at the Institute. For example, in one of the *MAT* expository papers, a student was asked to grapple with "The Polygon Game" [16]. Her explanation is outlined here:

    Take a regular, *n*-sided polygon (i.e., a regular *n*-gon) and the set of numbers, {1, 2, 3, …, (2*n*-2), (2*n*-1), 2*n*}. Place a dot at each vertex of the polygon and at the midpoint of each side of the polygon. Take the numbers and place one number beside each dot. A *side sum* is the sum of the number assigned to any midpoint plus the numbers assigned to the vertex on either side of the midpoint. A *solution* to the game is any polygon with numbers assigned to each dot for which all side sums are equal; i.e., for which you have *equal side sums*. The most general problem we might state is, "Find all solutions to The Polygon Game."

    In assigning this topic, we wanted her to analyze carefully a complete solution to The Triangle Game: reasoning carefully, offering a discussion about the importance of careful definition, and discussing opportunities to use algebra or geometry to solve problems. We hoped she would state and find solutions to "The Square Game" and explore comparable games for larger polygons (see Appendix A). Her work exceeded our expectations in several ways. For example, she argued that for any *n*-gon, each solution has a "dual solution," found by replacing the value *i* by (2*n* + 1) − *i* at each point. She not only found all solutions for The Square Game, but also for "The Pentagon Game" and "The Hexagon Game." These solutions offered new insights. For example, The Pentagon Game has solutions for 14 (and its dual, 19), but no solution for 15 or 18. Furthermore, both 16 and 17 have two uniquely different solutions that are not a transformation of each other. In perhaps the most interesting result in the paper, she uses modular

arithmetic to show that for any n-gon where n is odd, there is an Equal Side Sum solution S = $5(n+3)/2$.

Conclusion

Readers of this article will be pleasantly surprised to learn that this paper is the work of a fifth grade classroom teacher. The entire article is posted on our website [16]. We offer this as an example, coupled with teachers' earliest work in the Institute on The Triangle Game (see Figures 2, 3, and 4) to illustrate the sort of intellectual growth and mathematical capacity building we see in the participants as a result of the Institute. Understanding how this mathematical knowledge translates into more thoughtful teaching can be seen, to some degree, in the short term, by reading teachers' action research projects [16]. Long-term impact of teachers' new mathematical capacities on classroom practice is yet to be fully understood.

Acknowledgment

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Appendix A
M^2 Student Solution to The Polygon Game

Solution to All Polygons

Conjecture: One solution to every polygon will have a side of $n + 2n + 1$, where n = the number of vertices on the polygon, giving a side sum of $3n + 1$. Consider the following examples, all of which are a lower solution of the 2 center solutions in the range of possible solutions:

Triangle: <u>3, 6, 1</u>, 4, 5, 2   Side Sum = 10

Square: <u>4, 8, 1</u>, 7, 5, 2, 6, 3   Side Sum = 13

Pentagon: <u>5, 10, 1</u>, 8, 7, 6, 3, 4, 9, 2   Side Sum = 16

Hexagon: <u>6, 12, 1</u>, 10, 8, 4, 7, 9, 3, 5, 11, 2 Side Sum = 19

Notice that in each example the underlined numbers represent a side sum that is consistent with the expression $n + 2n + 1$. So, to see if this would be true for all polygons, I randomly chose an octagon, fixed the expression as a given side and checked for solutions.

Octagon: <u>8, 16, 1</u>, 13, 11, 12, 2, 9, 14, 5, 6, 4, 15, 3, 7, 10   Side Sum = 25

Decagon: <u>10, 20, 1</u>, 14, 16, 11, 4, 15, 12, 13, 6, 7, 18, 8, 5, 17, 9, 3, 19, 2 Side Sum = 31

The $n + 2n + 1$ still works!

Finally, with this last conjecture, my exploration of the polygon game comes to an end. I have been able to determine all solutions to the triangle game, the square game, the pentagon game and the hexagon game. I have then been able to use that information to find patterns that allowed me to explore n-gons in two different ways, from which I can determine two solutions to any odd sided polygon and one solution to any even sided polygon. Of course I can also use the concept of duality, which instantly doubles the number of solutions that I find!