THE KAPREKAR ROUTINE AND OTHER DIGIT GAMES FOR UNDERGRADUATE EXPLORATION

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Abstract

The Kaprekar Routine is a famous mathematical procedure involving the digits of a positive integer. This paper offers natural generalizations of the routine, states and proves related results, and presents many open problems that are suitable for mathematical research at the undergraduate level. In the process, we shed light on some interesting facts about *digit games*.

Introduction

Undergraduate senior research is a typical capstone experience. Additionally, it is usually an integral part of the assessment cycle of any undergraduate Mathematics curriculum. The Supporting Assessment of Undergraduate Mathematics (SAUM) initiative of the Mathematical Association of America (MAA) is an invaluable resource for mathematics departments looking for guidance in this regard. Of course, there is great variability in the degree to which departments make this requirement comprehensive. Furthermore, it is often difficult to find good and reasonable problems suitable for undergraduate research. These problems need to be clearly posed, and it must be easy to generate useful examples. Most of all, they must be solvable (in a semester or a year). There is a body of problems that satisfy the above criteria in an area of number theory that we will call "digit games." These are problems that involve the properties of some arithmetical function or manipulation of the digits of a whole number. In studying them, students are afforded the opportunity to experience many of the important aspects of mathematical research:

- Read and Understand the Problem — These problems are typically comprehensible with a knowledge of arithmetic and exponents;
- Pose Problems — Students are given a chance to pose their own problems either by extending known results to a more general setting or solving an open problem in a special case; and,

- Learn from Empirical Data — It is easy to generate many examples using a computer or calculator that students can subsequently use to formulate conjectures.

Elizabeth N. Chaille's senior thesis, *Kaprekar Type Routines For Arbitrary Bases* is a wonderful case study of just this type of exploration [1]. Many of these problems have well-known names with rich histories. The following is a short list of classes of integers defined by special properties involving their digits:

- Narcissistic Numbers — An n-digit number is said to be n-narcissistic if it is equal to the sum of the nth power of its digits (e.g., $153 = 1^3 + 5^3 + 3^3$).
- Niven (n-Harshad) Numbers — A positive integer is an n-Harshad (or Niven) number if it is divisible by the sum of its digits in base $n \geq 2$ (examples: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 20, 21, 24, \ldots$ are 10-Harshad numbers).
- Vampire Numbers — A number v is said to be a vampire number if v has an even number n of digits and $v = ab$ where a and b are both $\frac{n}{2}$-digit numbers and made up of the original digits of v in any order. Pairs of trailing zeros are not allowed (examples: $1260, 1395, 1435, \ldots$)

This paper is motivated by a digit game called the Kaprekar Routine. In 1949, the Indian mathematician D. R. Kaprekar discovered a procedure that, when applied to any positive 4-digit integer, not all of whose digits are the same, converges to the fixed point 6174 in at most seven iterations [2-3]. Now known as the "Kaprekar Routine," the procedure is described below.

1) Pick any 4-digit (base 10) number, not all of whose digits are the same.
2) Rearrange the digits in decreasing order to obtain a new 4-digit number B.
3) Rearrange the digits in increasing order to obtain a new 4-digit number A.
4) Find the difference $B - A$.
5) Repeat the process with $B - A$ with leading zeros added when necessary.

The first questions a student might ask are: 1) what is special about the three or four digits? and, 2) what is special about base 10? We can describe the Kaprekar Routine more generally as follows: for any positive m-digit integer N in base r, not all of whose digits are the same, arrange its digits in descending order yielding the integer N_d and ascending order yielding the integer N_a, treating the results as $m-$digit integers by adding leading zeros if necessary. Now continue to perform the above routine on the result of the difference $(N_d - N_a)$.

It is well known that when applied to decadic (base 10) 3-digit integers, the Kaprekar Routine converges to the unique fixed point 495 in at most six iterations. In fact, Klaus and Seok showed exactly what happens using the Kaprekar Routine on any 3-digit integer in an arbitrary base in their paper entitled, "The Determination of Kaprekar Convergence and Loop Convergence of All Three-Digit Numbers" [4-5]. While variations to the Kaprekar Routine have been studied in the past, we extend the results found in Klaus and Seok's paper to fourteen other natural variations of the Kaprekar Routine for 3-digit integers in an arbitrary base [4, 6]. We shall see that two such variations yield analogous results to the Kaprekar Routine and interestingly, all three routines share a nice property that the other twelve routines fail to possess. In a later section, we have provided different and less complicated proofs of the results on the Classical Kaprekar Routine found in Klaus and Seok's paper [4]. Not surprisingly, the Kaprekar Routine has long been a source of open problems. We continue this tradition by providing several problems, together with a few simple proofs for possible exploration by undergraduate students.

Terminology and Notation

Let abc be a 3-digit positive integer in base r and without loss of generality assume that $a \geq b \geq c$. Further assume that a,b,c are not all the same. Thus, $a > c$ for otherwise, $a = b = c$.

Let $S = (\alpha, \beta)$, where α and β are distinct permutations of the set $\{1,2,3\}$. A Kaprekar Routine of Type S is one in which at each stage the digits of a positive integer are rearranged in the order α and β, respectively, and the integer corresponding to the rearrangement β is subtracted from the integer corresponding to the rearrangement α, adding leading zeros when necessary. For some routines, it is possible that this subtraction will produce a negative result. In such cases, we will use the absolute value of the result when reordering the integer to use in the next iteration. For example, in the classical routine, each iterate is obtained by reordering the digits in descending order minus the integer obtained by reordering the digits in ascending order, so the classical routine is of Type $(123, 321)$. Note that our notation implies that 1 corresponds to the digit a, 2 corresponds to the digit b, and 3 corresponds to the digit c.

There are fifteen possible Kaprekar type routines for 3-digit integers. Table 1 shows the routines, as well as the result of the first iteration for each of them. It is easy to see that for $m-$digit integers, there are $[1+2+3+\cdots+(m!-1)]$ possible Kaprekar type routines. Observe that the routine of Type $(123,321)$ is the Classical Kaprekar Routine. For a given positive integer m, base r and subtraction routine A, a positive $m-$digit integer K in base r is called the *m-digit, r-adic Kaprekar constant of Type A* if all $m-$digit integers, not all of whose digits are the same, in base r converge to K after a finite number of iterations of routine A. Thus, 495 is the 3-digit, decadic Kaprekar constant of Type $(123,321)$ while 6174 is the 4-digit, decadic Kaprekar constant of Type $(1234,4321)$. In fact, it is known that in the decadic case, 495 and 6174 are the only Kaprekar constants [7-8]. When referring to the Classical Kaprekar Routine, we will omit mention of its type. Here, we will study the Kaprekar type routines on all positive, 3-digit integers, and will denote the integer abc by the ordered triple (a,b,c).

Table 1
All Possible Kaprekar Type Routines and the Result of the First Iteration

Type	Subtraction Order	Result After First Iteration
$(123,132)$	$abc-acb$	$(b-c)r-(b-c)$
$(123,213)$	$abc-bac$	$(a-b)r^2-(a-b)r$
$(123,231)$	$abc-bca$	$(a-b)r^2+(b-c)r-(a-c)$
$(123,312)$	$abc-cab$	$(a-c)r^2-(a-b)r-(b-c)$
$(123,321)$	$abc-cba$ (Classical)	$(a-c)r^2-(a-c)$
$(132,213)$	$acb-bac$	$(a-b)r^2-(a-c)r+(b-c)$
$(132,231)$	$acb-bca$	$(a-b)r^2-(a-b)$
$(132,312)$	$acb-cab$	$(a-c)r^2-(a-c)r$
$(132,321)$	$acb-cba$	$(a-c)r^2-(b-c)r-(a-b)$
$(213,231)$	$bac-bca$	$(a-c)r-(a-c)$
$(213,312)$	$bac-cab$	$(b-c)r^2-(b-c)$
$(213,321)$	$bac-cba$	$(b-c)r^2+(a-b)r-(a-c)$
$(231,312)$	$bca-cab$	$(b-c)r^2-(a-c)r+(a-b)$
$(231,321)$	$bca-cba$	$(b-c)r^2-(b-c)r$
$(312,321)$	$cab-cba$	$(a-b)r-(a-b)$

Routines With Kaprekar Constants

It is easy to see that when applying a Kaprekar type routine to a positive 3-digit integer in a fixed base r, there are only three possible outcomes: the routine converges to 0, a non-trivial fixed point, or a cycle. The routines that are especially interesting are those for which Kaprekar constants exist.

Type $(132,312)$ Routine

Theorem 1. If $r>1$ is an even integer, then all positive 3-digit integers in base r, not all of whose digits are equal, converge to the unique fixed point given by $(\frac{r-2}{2},\frac{r}{2},0)$ in at most $(\frac{r}{2}+1)$ iterations of the Kaprekar Routine of Type $(132,312)$.

Proof. Let abc be a positive 3-digit integer in base r. Without loss of generality, assume that $a\geq b\geq c$ with $a>c$. According to Table 1, the result of the first iteration of the Kaprekar Routine of Type $(132,312)$ is $(a-c)(r^2-r)$. We may also express the result of the first iteration as the triple $(a-c-1,r-a+c,0)$. Since $a>c$, the possible values of $a-c$ are $1,2,3,...,r-1$. Therefore, the possible values of the triples are $(0,r-1,0),(1,r-2,0)\cdots,(r-1,0,0)$. We will show that each of the first iterates converge to $(\frac{r-2}{2},\frac{r}{2},0)$ in at most $(\frac{r}{2}+1)$ iterations by calculating the sequence of iterates for each possibility. Table 2 shows these results.

For $0\leq n\leq\frac{r}{2}$, we have $r-n\geq n$. Hence, it is easy to check that, in this situation, if $a-c=n$, the first iteration of the Type $(132,312)$ routine yields $(n-1,r-n,0)$. The next iterate is always $(r-n-1,n,0)$. In general, for $k>1$ the $(k+1)^{th}$ iterate is found by reordering the triple $(r-[k+n-1],k+n-2,0)$ and subtracting to obtain

$(r-[k+n-1],0,k+n-2)-(0,r-[k+n-1],k+n-2)=(r-[k+n],k+n-1,0)$.

This continues until

$$r-[k+n-1]=\tfrac{r-2}{2}$$

and

$$k+n-2=\tfrac{r}{2}$$

which occurs, for the first time, when $k=\frac{r}{2}+2-n$. Note that this value of k is a solution to the previous system of equations.

Table 2
Sequences of Iterates after k Type (132, 312); Iterations for r Even

$n \backslash k$	1	2	3	$\cdots$	$\frac{r}{2}+1$
1	$(0,r-1,0)$	$(r-2,1,0)$	$(r-3,2,0)$	$\cdots$	$(\frac{r-2}{2},\frac{r}{2},0)$
2	$(1,r-2,0)$	$(r-3,2,0)$	$\cdots$	$(\frac{r-2}{2},\frac{r}{2},0)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{r}{2}$	$(\frac{r-2}{2},\frac{r}{2},0)$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r-2$	$(r-2,1,0)$	$(r-3,2,0)$	$\cdots$	$(\frac{r-2}{2},\frac{r}{2},0)$	
$r-1$	$(r-1,0,0)$	$(r-2,1,0)$	$(r-3,2,0)$	$\cdots$	$(\frac{r-2}{2},\frac{r}{2},0)$

Note the symmetry in the cases $(\frac{r}{2}+1),(\frac{r}{2}+2),\cdots,(r-1)$ and $1,2,\cdots(\frac{r}{2}-1)$. Let $\frac{r}{2}<m\leq r-1$; it is clear that in this case we have $m\geq r-m$. If we let $a-c=m$, then the second iteration of the Type $(132,312)$ routine yields $(m-1,r-m,0)$. Letting $n=r-m+1$, we have $0<n\leq\frac{r}{2}$. Solving for m, we find that the triple $(m-1,r-m,0)$ becomes $(r-n,n-1,0)$ one of the second iterates already listed. Hence, in every case, the iterates must converge to $(\frac{r-2}{2},\frac{r}{2},0)$. Thus, substituting $n=r-m+1$ in the expression $k=\frac{r}{2}+1-n$, we see that the number of iterations required in this case is given by $k=m+1-\frac{r}{2}$.

Finally, we show that $(\frac{r-2}{2},\frac{r}{2},0)$ is a fixed point under the Kaprekar Routine of Type $(132,312)$. Performing a Type $(132,312)$ iteration on $(\frac{r-2}{2},\frac{r}{2},0)$ gives

$$(\frac{r}{2},0,\frac{r-2}{2})-(0,\frac{r}{2},\frac{r-2}{2})=(\frac{r-2}{2},\frac{r}{2},0).$$

This completes the proof. △

Example 1. For $r=14$ (i.e., base 14), the integer $(3,4,7)$ converges to the Kaprekar type constant $(6,7,0)$ in at most $\frac{14}{2}+1=8$ iterations of Type $(132,312)$. In fact, we see that convergence occurs in exactly five iterations: $(3,4,7)\to(3,10,0)\to(9,4,0)\to(8,5,0)\to(7,6,0)\to(6,7,0)$.

Corollary 1. Let $c_{(132,312)}$ be the smallest number of iterations necessary for a three-digit integer in an even base r to converge to $(\frac{r}{2},0,\frac{r-2}{2})-(0,\frac{r}{2},\frac{r-2}{2})=(\frac{r-2}{2},\frac{r}{2},0)$, the Kaprekar constant of Type $(132,312)$, then

$$c_{(132,312)}=\begin{cases}\frac{r}{2}+2-(a-c) & \text{if } (a-c)<\frac{r}{2},\\ 1 & \text{if } (a-c)=\frac{r}{2},\\ (a-c)+1-\frac{r}{2} & \text{if } (a-c)>\frac{r}{2}.\end{cases}$$

Theorem 2. If $r>1$ is an odd integer, then all positive 3-digit integers in base r, not all of whose digits are equal, converge to an element of the 2-cycle given by $(\frac{r-1}{2},\frac{r-1}{2},0)\leftrightarrow(\frac{r-3}{2},\frac{r+1}{2},0)$ in at most $\frac{r+1}{2}$ iterations of the Kaprekar Routine of Type $(132,312)$.

Proof. The beginning of this proof is identical to the proof of Theorem 1. In fact, for $0\le n\le\frac{r}{2}$, we also have the k^{th} iteration, for $k>1$ to be $(r-[k+n-1],k+n-2,0)$ where $n=a-c$. As in Theorem 1, this continues until $r-[k+n-1]=\frac{r-1}{2}$ and $k+n-2=\frac{r-1}{2}$, which occurs at $k=\frac{r+3}{2}-n$. The symmetry found in Theorem 1, for the cases with $\frac{r}{2}<m\le r-1$, is also the same. Here, we find the number of iterations required in this case by replacing n with $r-m+1$ in the equation $k=\frac{r+3}{2}-n$ to obtain $k=m-\frac{r-1}{2}$.

Finally, we show that $(\frac{r-1}{2},\frac{r-1}{2},0)\leftrightarrow(\frac{r-3}{2},\frac{r+1}{2},0)$ is in fact a 2-cycle. Performing the Type $(132,312)$ routine on $(\frac{r-1}{2},\frac{r-1}{2},0)$ yields $(\frac{r-1}{2},0,\frac{r-1}{2})-(0,\frac{r-1}{2},\frac{r-1}{2})=(\frac{r-3}{2},\frac{r+1}{2},0)$ and now performing the Type $(132,312)$ routine on $(\frac{r-3}{2},\frac{r+1}{2},0)$ yields $(\frac{r-3}{2},0,\frac{r+1}{2})-(0,\frac{r-3}{2},\frac{r+1}{2})=(\frac{r-1}{2},\frac{r-1}{2},0)$. This completes the proof. Table 3 illustrates what we have just shown. $\vartriangle$

Table 3
Sequences of Iterates after k Type (132, 312); Iterations for r Odd

$n\backslash k$	1	2	3	$\cdots$	$\frac{r+1}{2}$
1	$(0,r-1,0)$	$(r-2,1,0)$	$(r-3,2,0)$	$\cdots$	$(\frac{r-1}{2},\frac{r-1}{2},0)$
2	$(1,r-2,0)$	$(r-3,2,0)$	$\cdots$	$(\frac{r-1}{2},\frac{r-1}{2},0)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{r-1}{2}$	$(\frac{r-1}{2},\frac{r-1}{2},0)$				
$\frac{r+1}{2}$	$(\frac{r-1}{2},\frac{r-1}{2},0)$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r-2$	$(r-2,1,0)$	$(r-3,2,0)$	$\cdots$	$(\frac{r-1}{2},\frac{r-1}{2},0)$	
$r-1$	$(r-1,0,0)$	$(r-2,1,0)$	$(r-3,2,0)$	$\cdots$	$(\frac{r-1}{2},\frac{r-1}{2},0)$

Example 2. For $r=13$, the integer $(3,4,7)$ converges to the 2-cycle $(6,6,0)\leftrightarrow(5,7,0)$ in at most $\frac{13+1}{2}=8$ iterations of Type $(132,312)$. In fact, we see that convergence occurs in exactly five iterations: $(3,4,7)\rightarrow(3,9,0)\rightarrow(8,4,0)\rightarrow(7,5,0)\rightarrow(6,6,0)\rightarrow(5,7,0)$.

Corollary 2. Let $l_{(132,312)}$ be the smallest number of iterations of Type $(132,312)$ necessary for a three-digit integer in an odd base r to converge to an element of the 2-cycle $(\frac{r-3}{2},\frac{r+1}{2},0)\leftrightarrow(\frac{r-1}{2},\frac{r-1}{2},0)$, then

$$l_{(132,312)}=\begin{cases}\dfrac{r+3}{2}-(a-c) & \text{if } (a-c)<\dfrac{r-1}{2},\\[2mm] 1 & \text{if } (a-c)=\dfrac{r-1}{2},\\[2mm] (a-c)-\dfrac{r-1}{2} & \text{if } (a-c)>\dfrac{r-1}{2}.\end{cases}$$

Type $(213,231)$ Routine

Theorem 3. If $r>1$ is an even integer, then all positive 3-digit integers in base r, not all of whose digits are equal, converge to the unique fixed point given by $(0,\frac{r-2}{2},\frac{r}{2})$ in at most $(\frac{r}{2}+1)$ iterations of the Kaprekar Routine of Type $(213,231)$.

Proof. As expected, the proof is identical in strategy to that of Theorem 1. Once the reader generates the table of first iterates, the remainder of the proof follows from an easy emulation of the arguments in Theorem 1.

Example 3. For $r = 14$, the integer $(2,11,7)$ converges to the constant $(0,6,7)$ in at most $\frac{14}{2}+1=8$ iterations of Type $(213,231)$. In fact, we see that convergence occurs in exactly three iterations: $(2,11,7) \to (0,8,5) \to (0,7,6) \to (0,6,7)$.

Corollary 3. Let $c_{(213,231)}$ be the smallest number of iterations of Type $(213,231)$, necessary for a three-digit integer in an even base r to converge to the Kaprekar constant $(0,\frac{r-2}{2},\frac{r}{2})$. Then,

$$c_{(213,231)} = \begin{cases} \dfrac{r}{2}+2-(a-c) & \text{if } (a-c) < \dfrac{r}{2}, \\ 1 & \text{if } (a-c) = \dfrac{r}{2}, \\ (a-c)+1-\dfrac{r}{2} & \text{if } (a-c) > \dfrac{r}{2}. \end{cases}$$

Theorem 4. If $r > 1$ is an odd integer, then all positive 3-digit integers in base r, not all of whose digits are equal, converge to an element of 2-cycle given by $(0,\frac{r-1}{2},\frac{r-1}{2}) \leftrightarrow (0,\frac{r-3}{2},\frac{r+1}{2})$ in at most $\frac{r+1}{2}$ iterations of the Kaprekar Routine of Type $(213,231)$.

Proof. This time, as expected, the proof is similar to that of Theorem 2.

Example 4. For $r = 13$, the integer $(2,11,7)$ converges to the 2-cycle $(0,6,6) \leftrightarrow (0,5,7)$ in at most $\frac{13+1}{2}=7$ iterations of Type $(213,231)$. In fact, we see that convergence occurs in exactly four iterations: $(2,11,7) \to (0,8,4) \to (0,7,5) \to (0,6,6) \to (0,5,7)$.

Corollary 4. Let $l_{213,231)}$ be the smallest number of iterations of Type $(213,231)$ necessary for a three-digit integer in an odd base r to converge to an element of the 2-cycle $(0,\frac{r-3}{2},\frac{r+1}{2}) \leftrightarrow (0,\frac{r-1}{2},\frac{r-1}{2})$, then

$$l_{(213,231)} = \begin{cases} \dfrac{r+3}{2} - (a-c) & \text{if } (a-c) < \dfrac{r-1}{2}, \\ 1 & \text{if } (a-c) = \dfrac{r-1}{2}, \\ (a-c) - \dfrac{r-1}{2} & \text{if } (a-c) > \dfrac{r-1}{2}. \end{cases}$$

Type $(123,321)$ **Routine—Classical Kaprekar**

In this section, we present proofs of the main results in Klaus and Seok's paper using the techniques already outlined [4].

Theorem 5. If $r > 1$ is an even integer, then all positive 3-digit integers in base r, not all of whose digits are equal, converge to the unique fixed point given by $(\frac{r-2}{2}, r-1, \frac{r}{2})$ in at most $(\frac{r}{2}+1)$ iterations of the Kaprekar Routine [4].

Proof. As before, the technique used to prove Theorem 1 works perfectly for the Classical Kaprekar Routine. From Table 1, in this case, the possibilities for the first iterates when expressed as a triple are $(a-c-1, r-1, r-a+c)$. Since $a > c$, the possible first iterates are $(0, r-1, r-1), (1, r-1, r-2), \cdots, (r-2, r-1, 1)$. Table 4 shows the iterates. Notice that we can obtain this table by simply making the second digit $r-1$ and removing the trailing zero digit from the table found in Theorem 1. △

Table 4
Sequences of Iterates after *k* Classical Kaprekar; Iterations for *r* Even

$n \backslash k$	1	2	3	$\cdots$	$\frac{r}{2}+1$
1	$(0, r-1, r-1)$	$(r-2, r-1, 1)$	$(r-3, r-1, 2)$	$\cdots$	$(\frac{r-2}{2}, r-1, \frac{r}{2})$
2	$(1, r-1, r-2)$	$(r-3, r-1, 2)$	$\cdots$	$(\frac{r-2}{2}, r-1, \frac{r}{2})$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{r}{2}$	$(\frac{r-2}{2}, r-1, \frac{r}{2})$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r-2$	$(r-2, r-1, 1)$	$(r-3, r-1, 2)$	$\cdots$	$(\frac{r-2}{2}, r-1, \frac{r}{2})$	
$r-1$	$(r-1, r-1, 0)$	$(r-2, r-1, 1)$	$(r-3, r-1, 2)$	$\cdots$	$(\frac{r-2}{2}, r-1, \frac{r}{2})$

Example 5. For $r = 14$, the integer $(1, 9, 11)$ converges to the constant $(6, 13, 7)$ in at most $\frac{14}{2}+1 = 8$ iterations of the Kaprekar Routine. In fact, we see that convergence occurs in

exactly four iterations: $(1,9,11) \to (9,13,4) \to (8,13,4) \to (7,13,6) \to (6,13,7)$.

Corollary 5. Let c be the smallest number of iterations of the Kaprekar Routine necessary for a three-digit integer in an even base r to converge to the Kaprekar constant $(\frac{r-2}{2}, r-1, \frac{r}{2})$, then

$$c = \begin{cases} \frac{r}{2} + 2 - (a-c) & \text{if } (a-c) < \frac{r}{2}, \\ 1 & \text{if } (a-c) = \frac{r}{2}, \\ (a-c) + 1 - \frac{r}{2} & \text{if } (a-c) > \frac{r}{2}. \end{cases}$$

See Klaus and Seok [4].

Theorem 6. If $r > 1$ is an odd integer, then all positive 3-digit integers in base r, not all of whose digits are equal, converge to an element of the 2-cycle given by $(\frac{r-1}{2}, r-1, \frac{r-1}{2}) \leftrightarrow (\frac{r-3}{2}, r-1, \frac{r+1}{2})$ in at most $\frac{r+1}{2}$ iterations of the Kaprekar Routine [4].

Proof. Finally, we exhibit the table of iterates for the Classical Kaprekar Routine for odd bases. Again notice the similarities to Table 4. In this case, the possibilities for the first iterates are $(0,r-1,r-1),(1,r-1,r-2),\cdots,(r-2,r-1,1)$. Table 5 shows the iterates. Again, it is easy to check that $(\frac{r-1}{2}, r-1, \frac{r-1}{2}) \leftrightarrow (\frac{r-3}{2}, r-1, \frac{r+1}{2})$ is, in fact, a 2-cycle. △

Table 5
Sequences of Iterates after *k* Classical Kaprekar; Iterations for *r* Odd

$n \backslash k$	1	2	3	$\cdots$	$\frac{r+1}{2}$
1	$(0,r-1,r-1)$	$(r-2,r-1,1)$	$(r-3,r-1,2)$	$\cdots$	$(\frac{r-1}{2}, r-1, \frac{r-1}{2})$
2	$(1,r-1,r-2)$	$(r-3,r-1,2)$	$\cdots$	$(\frac{r-1}{2}, r-1, \frac{r-1}{2})$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{r-1}{2}$	$(\frac{r-1}{2}, r-1, \frac{r-1}{2})$				
$\frac{r+1}{2}$	$(\frac{r-1}{2}, r-1, \frac{r-1}{2})$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r-2$	$(r-2,r-1,1)$	$(r-3,r-1,2)$	$\cdots$	$(\frac{r-1}{2}, r-1, \frac{r-1}{2})$	
$r-1$	$(r-1,r-1,0)$	$(r-2,r-1,1)$	$(r-1,r-1,2)$	$\cdots$	$(\frac{r-1}{2}, r-1, \frac{r}{2})$

Example 6. For $r = 13$, the integer $(1,9,11)$ converges to the 2-cycle $(6,12,6) \leftrightarrow (5,12,7)$ in at most $\frac{13+1}{2} = 7$ iterations of the Kaprekar Routine. In fact, we see that convergence occurs in exactly five iterations:
$(1,9,11) \to (9,12,3) \to (8,12,4) \to (7,12,5) \to (6,12,6) \to (5,12,7)$.

Corollary 6. Let l be the smallest number of iterations of the Kaprekar Routine necessary for a three-digit integer in an odd base r to converge to an element of the 2-cycle $(\frac{r-1}{2}, r-1, \frac{r-1}{2}) \leftrightarrow (\frac{r-3}{2}, r-1, \frac{r+1}{2})$, then

$$l = \begin{cases} \dfrac{r+3}{2} - (a-c) & \text{if } (a-c) < \dfrac{r-1}{2}, \\ 1 & \text{if } (a-c) = \dfrac{r-1}{2}, \\ (a-c) - \dfrac{r-1}{2} & \text{if } (a-c) > \dfrac{r-1}{2}. \end{cases}$$

See Klaus and Seok [4].

The Decadic Story

At this point the obvious question is, "do any of the other Kaprekar type routines yield similar results?" Table 6 lists all fixed points and cycles for each of the Kaprekar type routines in base 10. This table was generated using information from a *MATLAB®* program that simply checked every possibility. This is a great opportunity to use technology for generating a variety of examples that may help identify interesting patterns. We see that Types $(123,321)$, $(132,312)$, and $(213,231)$ are the most interesting routines and all demonstrate similar properties. That is to say, they exhibit Kaprekar constants. It is not known whether the other results in Table 6 may be generalized to other bases.

Table 6
All Possible Kaprekar Type Routines for Base 10 with Resulting Fixed Points and Cycles

Type	Subtraction Order	Fixed Points	Cycles
$(123,132)$	$abc-acb$	$(0,0,0)$	none
$(123,213)$	$abc-bac$	$(0,0,0)$	$(8,1,0)(6,3,0)(2,7,0)(4,5,0)(0,9,0)$
$(123,231)$	$abc-bca$	none	$(4,0,5)(1,3,5)(2,1,6)$
$(123,312)$	$abc-cab$	$(4,5,9)$	$(3,7,8)(4,8,6)$
$(123,321)$	$abc-cba$ (Classical)	$(4,9,5)$	none
$(132,213)$	$acb-bac$	$(0,5,4)$	$(1,6,2)(3,5,1)$
$(132,231)$	$acb-bca$	$(0,0,0)$	none
$(132,312)$	$acb-cab$	$(4,5,0)$	none
$(132,321)$	$acb-cba$	none	$(4,8,6)(3,7,8)(4,5,9)$
$(213,231)$	$bac-bca$	$(0,4,5)$	none
$(213,312)$	$bac-cab$	$(0,0,0)$	$(8,9,1)(6,9,3)(2,9,7)(4,9,5)(0,9,9)$
$(213,321)$	$bac-cba$	none	$(3,5,1)(1,6,2)(0,5,4)$
$(231,312)$	$bca-cab$	$(4,0,5)$	$(2,1,6)(1,3,5)$
$(231,321)$	$bca-cba$	$(0,0,0)$	none
$(312,321)$	$cab-cba$	$(0,0,0)$	$(0,8,1)(0,6,3)(0,2,7)(0,4,5)(0,0,9)$

Open Problems

Note that in Chaille's *Kaprekar Type Routines For Arbitrary Bases*, the student obtained many partial results toward a complete exploration of all Kaprekar type routines for 4-digit integers in an arbitrary base [1]. However, as is customary of any good research, Chaille's thesis also poses many new interesting open problems. As promised, this branch of recreational mathematics offers a plethora of accessible open problems for student exploration. We present six such problems.

1) Complete the analysis of the fifteen Kaprekar type routines for 3-digit integers in all bases.
2) Classify what happens in an arbitrary base for the Classical Kaprekar Routine (i.e., of Type $(1234,4321)$ for 4-digit integers.
3) Complete the analysis of the 275 other Kaprekar type routines for 4-digit integers in all bases.
4) If possible, generalize Problems 1 and 2 to arbitrary $m-$digit integers where $m \geq 5$.
5) Define and explore other reorder-subtract routines that are not of the

Kaprekar type.

6) Study the connections between the properties of the symmetric group S_n and Kaprekar type "digit games" on $n-$digit integers in an arbitrary base.

Concluding Remarks

In conclusion, we observe that Types $(123,321)$, $(132,312)$, and $(213,231)$ offer interesting results since they are the only three Kaprekar type routines for 3-digit integers for which the first iterates depend solely on the difference between the largest(a) and smallest(c) digits. Since we require that $a > c$, this difference is guaranteed to be positive. However, in the 4-digit case, the existence of 6174 as a Kaprekar constant is a seemingly perplexing phenomenon because the Type $(1423,4123)$ routine (the 4-digit version of the Type $[132,312]$ yields the 5-cycle $4527 \rightarrow 4509 \rightarrow 8109 \rightarrow 8163 \rightarrow 6327$. It would be nice to be able to discover the properties that guarantee Kaprekar constants for the other 275 Kaprekar type routines for 4-digit integers and to determine if there *are* some, other than the obvious ones like Type $(1234,3214)$. Digit games are a truly wonderful source of problems for undergraduate research that often lead to surprising results. ■

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