HIROSHI HARUKI'S LEMMA WITH *MATHEMATICA*

J. BOYD
St. Christopher's School,
Richmond, VA 23226
boydj@stcva.org

Introduction

My first mathematics course when I was a freshman at Hampden-Sydney College long, long ago was analytic geometry. I was absolutely amazed. I found that I did not have to be smart to prove things. Sometimes I did not even have to think! I just had to be willing to work hard and compute, compute, compute. Analytic geometry is the blue collar geometer's most powerful, most useful, and most versatile tool.

Now, forty-five years later, I have learned a little bit about *Mathematica* and can raise blue collar and completely non-elegant geometry to a new high, or sink it to a new low. High and low depend upon point of view.

Recently, I came across Hiroshi Haruki's lemma [1]. I had no idea how to prove its lovely result, but did not bother either to go to the library or to search the Internet. I simply began to compute.

Hiroshi Haruki's Lemma

Suppose that non-intersecting chords $\overline{AB}$ and $\overline{CD}$ are drawn in a circle and that point G belongs to arc $\overset{\frown}{AB}$ which stands opposite to chord $\overline{CD}$. Suppose further that $\overline{CG}$ intersects $\overline{AB}$ at point P and that $\overline{DG}$ intersects $\overline{AB}$ at point Q as shown in Figure 1. Let $AP = a$, $PQ = b$, and $QB = c$. Then, for each location of P on arc $\overset{\frown}{AB}$, the ratio ac/b is a constant.

Proof

Here is an analytic proof of the lemma implemented with *Mathematica*:

Let us draw our chords in the unit circle centered at $(0,0)$. No generality will be lost by letting the chord $\overline{CD}$ be drawn parallel to the x-axis. We take the coordinates of A, B, C, D, and G in the following way:

$$\begin{aligned}&A:(\cos\gamma,\sin\gamma),\\&B:(\cos\beta,\sin\beta),\\&C:(-\cos\theta,-\sin\theta),\\&D:(\cos\theta,-\sin\theta),\ \text{and}\\&G:(\cos\alpha,\sin\alpha).\end{aligned}$$

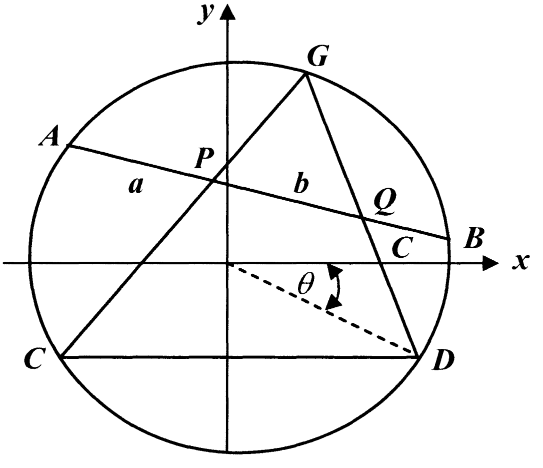

Figure 1. The unit circle for Hiroshi Haruki's lemma.

In our computations, we number chords $\overline{CG}$, $\overline{DG}$, and $\overline{AB}$ as 1, 2, and 3, respectively.

An equation for chord 1 is $y+\sin\theta=\left(\dfrac{\sin\alpha+\sin\theta}{\cos\alpha+\cos\theta}\right)(x+\cos\theta)$.

An equation for chord 2 is $y+\sin\theta=\left(\dfrac{\sin\alpha+\sin\theta}{\cos\alpha-\cos\theta}\right)(x+\cos\theta)$.

An equation for chord 3 is $y-\sin\beta=\left(\dfrac{\sin\beta-\sin\gamma}{\cos\beta-\cos\gamma}\right)(x-\cos\beta)$.

We denote the point at which chords 1 and 3 intersect as P and take the coordinates of the point to be (xP, yP). We denote the point at which chords 2 and 3 intersect as Q and take the coordinates of that point to be (xQ, yQ).

Now, we find the coordinates (xP, yP) of P by solving the equations of chords 1 and 3 simultaneously.

```
sol13 = Solve[{y + Sin[θ] == (Sin[α] + Sin[θ]) (x + Cos[θ]) / (Cos[α] + Cos[θ]),
   y - Sin[β] == (Sin[β] - Sin[γ]) (x - Cos[β]) / (Cos[β] - Cos[γ])}, {x, y}]
{{x → -(-Cos[β] Cos[θ] Sin[α] + Cos[γ] Cos[θ] Sin[α] - Cos[α] Cos[γ] Sin[β] - Cos[γ] Cos[θ] Sin[β] +
       Cos[α] Cos[β] Sin[γ] + Cos[β] Cos[θ] Sin[γ] + Cos[α] Cos[β] Sin[θ] - Cos[α] Cos[γ] Sin[θ]) /
     (-Cos[β] Sin[α] + Cos[γ] Sin[α] - Cos[α] Sin[β] + Cos[θ] Sin[β] -
       Cos[α] Sin[γ] - Cos[θ] Sin[γ] - Cos[β] Sin[θ] + Cos[γ] Sin[θ]),
  y → -(Cos[γ] Sin[α] Sin[β] + Cos[θ] Sin[α] Sin[β] - Cos[β] Sin[α] Sin[γ] - Cos[θ] Sin[α] Sin[γ] -
       Cos[α] Sin[β] Sin[θ] + Cos[γ] Sin[β] Sin[θ] + Cos[α] Sin[γ] Sin[θ] - Cos[β] Sin[γ] Sin[θ]) /
     (Cos[β] Sin[α] - Cos[γ] Sin[α] - Cos[α] Sin[β] - Cos[θ] Sin[β] + Cos[α] Sin[γ] +
       Cos[θ] Sin[γ] + Cos[β] Sin[θ] - Cos[γ] Sin[θ])}}
```

```
xP = FullSimplify[x /. sol13]
{((Cos[α] + Cos[θ]) Sin[β - γ] - (Cos[β] - Cos[γ]) Sin[α - θ]) / (-Sin[α - β] + Sin[α - γ] + Sin[β - θ] - Sin[γ - θ])}
```

```
yP = FullSimplify[y /. sol13]
{(Sin[α] ((Cos[γ] + Cos[θ]) Sin[β] - (Cos[β] + Cos[θ]) Sin[γ]) -
    (Sin[β - γ] - Cos[α] (-Sin[β] + Sin[γ])) Sin[θ]) /
  (-Sin[α - β] + Sin[α - γ] + Sin[β - θ] - Sin[γ - θ])}
```

Next, we find the coordinates (xQ, yQ) of Q by solving the equations of chords 2 and 3 simultaneously.

```
sol23 = Solve[{y + Sin[θ] == (Sin[α] + Sin[θ]) (x - Cos[θ]) / (Cos[α] - Cos[θ]),
   y - Sin[β] == (Sin[β] - Sin[γ]) (x - Cos[β]) / (Cos[β] - Cos[γ])}, {x, y}]
{{x → - (Cos[β] Cos[θ] Sin[α] - Cos[γ] Cos[θ] Sin[α] - Cos[α] Cos[γ] Sin[β] - Cos[γ] Cos[θ] Sin[β] -
      Cos[α] Cos[β] Sin[γ] - Cos[β] Cos[θ] Sin[γ] + Cos[α] Cos[β] Sin[θ] - Cos[α] Cos[γ] Sin[θ]) /
    (-Cos[β] Sin[α] + Cos[γ] Sin[α] + Cos[α] Sin[β] - Cos[θ] Sin[β] -
      Cos[α] Sin[γ] + Cos[θ] Sin[γ] - Cos[β] Sin[θ] + Cos[γ] Sin[θ]),
  y → - (Cos[γ] Sin[α] Sin[β] - Cos[θ] Sin[α] Sin[β] - Cos[β] Sin[α] Sin[γ] + Cos[θ] Sin[α] Sin[γ] -
      Cos[α] Sin[β] Sin[θ] + Cos[γ] Sin[β] Sin[θ] + Cos[α] Sin[γ] Sin[θ] - Cos[β] Sin[γ] Sin[θ]) /
    (Cos[β] Sin[α] - Cos[γ] Sin[α] - Cos[α] Sin[β] + Cos[θ] Sin[β] + Cos[α] Sin[γ] -
      Cos[θ] Sin[γ] + Cos[β] Sin[θ] - Cos[γ] Sin[θ])}}
```

```
xQ = FullSimplify[x /. sol23]
{((Cos[α] - Cos[θ]) Sin[β - γ] + (-Cos[β] + Cos[γ]) Sin[α - θ]) / (-Sin[α - β] + Sin[α - γ] - Sin[β - θ] + Sin[γ - θ])}
```

```
yQ = FullSimplify[y /. sol23]
{(Sin[α] ((Cos[γ] - Cos[θ]) Sin[β] + (-Cos[β] + Cos[θ]) Sin[γ]) +
    (Sin[β - γ] + Cos[α] (-Sin[β] + Sin[γ])) Sin[θ]) /
  (-Sin[α - β] + Sin[α - γ] - Sin[β + θ] + Sin[γ - θ])}
```

We compute a^2, b^2, and c^2 by using the distance formula.

```
asq = FullSimplify[(Cos[γ] - xP) ^2 + (Sin[γ] - yP) ^2]
{4 Cos[(γ-θ)/2]^2 Sec[1/2 (α - β - γ + θ)]^2 Sin[(α-γ)/2]^2}
```

```
csq = FullSimplify[(Cos[β] - xQ) ^2 + (Sin[β] - yQ) ^2]
{4 Csc[1/2 (α - β - γ - θ)]^2 Sin[(α-β)/2]^2 Sin[(β+θ)/2]^2}
```

```
bsq = FullSimplify[(xP - xQ) ^2 + (yP - yQ) ^2]
{4 Cos[θ]^2 Csc[1/2 (α - β - γ - θ)]^2 Sec[1/2 (α - β - γ + θ)]^2 Sin[(α-β)/2]^2 Sin[(α-γ)/2]^2}
```

Finally, we compute ac/b. The result is independent of angle α. Therefore, ac/b is independent of the location of point G on arc $\overset{\frown}{AB}$.

```
FullSimplify[asq * csq / bsq]
{4 Cos[(γ-θ)/2]^2 Sec[θ]^2 Sin[(β+θ)/2]^2}
```

```
ratio = √FullSimplify[asq * csq / bsq]
{2 √(Cos[(γ-θ)/2]^2 Sec[θ]^2 Sin[(β+θ)/2]^2)}
```

Thus, we have proven Hiroshi Haruki's lemma, a lovely result indeed. We own it and can use it as our own in exploring more Euclidean geometry, a country which seems to have no boundaries.

Reference

[1] R. Honsberger, "Mathematical Gems," *The Two-Year College Mathematics Journal*, **14**(1) (1983) 2.