A CASE FOR PROOF MAKING FOR PROSPECTIVE MIDDLE SCHOOL TEACHERS

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Abstract

In this article, we discuss how we, as mathematics teacher educators, might help our prospective middle school teachers develop a disposition toward mathematics that involves making sound arguments and, more generally, making proofs about mathematical ideas. First, we illustrate what we mean by making sound mathematical proofs. We then use this definition to characterize what ninety-two prospective middle school teachers consider to be proof making, based on a survey that we administered in their first mathematics course. Following our findings, we discuss how we, as teacher educators, might realign our instruction to provide opportunities for prospective teachers to develop new understandings about proof making.

A Case For Proof Making For Prospective Middle School Teachers

In this article, we discuss how we, as mathematics teacher educators, might help our middle school education students develop a disposition toward mathematics that involves making sound arguments and, more generally, making proofs about mathematical ideas. The reason that we address this issue is related to the results from a recent survey that we administered to our middle school education students enrolled in their first mathematics course to fulfill their general education requirements. The students' responses were surprising. On the one hand, they seemed to understand, at least intuitively, that they needed to prove that a statement was mathematically true. On the other hand, many students generated specific cases, but did not develop general arguments to justify the validity of a mathematical statement.

Here, we report some of the findings related to one of the questions from the survey we administered to our students. To develop our discussion, we begin by illustrating what we mean by making sound mathematical proofs. We then use this definition to characterize what our college students consider to be proof making. We then discuss how we, as teacher educators, might realign our instruction to provide opportunities for prospective teachers to develop new understandings about proof making.

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Proof Making

To accomplish our task, we reconstruct one of the classic proofs of the Pythagorean theorem: the sum of the squares of the legs, of lengths a and b, of the right triangle is equal to the square of the hypotenuse, of length c (i.e., $a^2 + b^2 = c^2$). As is well known, there are countless proofs of this theorem (over three hundred). The approach we illustrate relies on the principle that the area of a figure is equal to the sum of the areas of all the non-overlapping subsections of that figure. To begin, use four congruent right triangles with legs of lengths a and b and hypotenuse of length c (see Figure 1). Arrange the triangles to construct a square with sides of length $a + b$ that contains an inscribed square with sides of length c.

Figure 1. Right triangle and square.

Using the right triangle on the left, one can rotate copies of the triangle to make a square with sides of length $a + b$ that contains an inscribed square with sides of length c.
Once we have made the inscribed square, we need to be sure that we have in fact inscribed a square using the four right triangles. (We leave it to the reader to establish that we have in fact formed an inscribed square. Hint: Consider the straight angle formed by an angle of the square and the two non-right angles of the right triangle.)

We are now ready to prove the Pythagorean theorem. To do so, we use the given information to develop an equation for the area of the larger square with side $a + b$ in two different ways. Once we have done so, we can complete the proof by simplifying equal algebraic expressions.

> **Proof:** The area of the larger square of length $a + b$ is $(a + b)^2$ or $a^2 + 2ab + b^2$. The area of the larger square can also be represented by the sum of the areas of the four right triangles and the area of the square with side c. (This sum is $4 \times 1/2ab + c^2$ or $2ab + c^2$.) This second expression, which also represents the area of the larger square, is equal to the first expression, $a^2 + 2ab + b^2$. So we have the equation $a^2 + 2ab + b^2 = 2ab + c^2$. Since the two expressions are equal and $2ab = 2ab$, then $a^2 + b^2$ must be equal to c^2, our desired result.

To better understand what we mean by proof making, we suggest that making the above argument is only part of the proof. At this juncture, the instructor should encourage the student to decide if the result is true for ***all*** right triangles. That is, the student might next wish to explore if the result holds when one assumes one leg is greater than the other, or if the legs have the same length. The student might then develop three different cases, and thus handle all possible types of right triangles for which to "test" if the result is essentially the same. That is, the student must reason, that whether $a > b$, $a < b$ or even $a = b$, the sum of the squares of the lengths of the legs of the triangle will again be the equal to the square of the length of the hypotenuse. When the student understands a similar result is achieved for each of these cases, then the student has moved from the particular to the general, and thus has engaged in sound mathematical proof making.

We will return to this example when we discuss the college students' arguments for supporting a mathematical claim. As we do so, we juxtapose different views of what counts as generality and, ultimately, the extent to which one convinces others as well as oneself of a particular mathematical result.

We use two types of arguments, *inductive* and *deductive*, to characterize the types of arguments the students give. An inductive argument is given when the student uses examples to illustrate that a certain claim is true. Although the student might give numerous examples, all of which are mathematically correct, the student does not make a generalization as to why these examples hold true. By way of contrast, when the student develops a general argument, one that is true for all cases, such as the arguments we made above for the Pythagorean theorem, the student is said to make a deductive argument. The challenge for instructors is to support students as they develop arguments of both types. The ultimate instructional goal, however, is for students to move seamlessly from the particular to the general in order to convince themselves that a claim holds true. As we characterize the prospective middle school teachers' arguments, the distinction we make between these two types of arguments will be very useful in helping us understand the types of arguments they made. We now provide some background about the project.

Background

In our discussion, we highlight the results of ninety-two prospective teachers' responses to one of the items from the IDLS Mathematics Assessment. The survey consisted of seven questions that addressed algebra or geometry principles. These items were adapted from questions Knuth, Choppin, Slaughter, and Sutherland devised to characterize middle school students' understanding of proof [1].

For each question, the prospective teachers were prompted to make arguments on why certain mathematical ideas held true. For instance, one of the questions posed the following problem:

> Mei discovers a number trick. She takes a number and multiplies it by five and then adds twelve. She then subtracts the starting number and divides the result by four. She notices the answer she gets is three more than the number she started with. Mei uses the number seven, and arrives at an answer of ten, three more than the number she started with. Malaika (Mei's friend) doesn't think this will happen again, so she tries the trick with another number, ten. After Malaika arrives at an answer of thirteen, she and Mei decide that they always get a result that is three more than the start number. Do you think they are right? How would you convince a classmate that you would always get a result that is three more than the starting number?

Following Knuth and Waring, we characterized the types of arguments that the prospective teachers gave for each item on the survey [1,2]. Here, we specifically highlight the students' responses to the above survey item about Mei and Malaika. We share these particular results because the prospective students' responses were remarkably consistent, as well as instructive, as to how we might rethink our instructional practice as mathematics teacher educators.

Preliminary Findings

Students' responses fell along two general "categories." In their responses, students either tested a few cases or they offered extreme or random cases to show that the claim was true. By way of contrast, students who moved beyond giving specific examples to support the claim stated that they needed to make a general argument, but they were not always able to produce a mathematically sound argument.

Of the ninety-two prospective teachers, nine either did not respond or responded that they had no idea how to convince a classmate; e.g., the comment of Student 5 is similar to the types of comments these students made. Thirteen students gave general arguments. These students, like Student 1, did not give specific examples, but rather generated a formula. Here we suggest that Student 1's response is a more general, deductive argument of sorts. Interestingly, students who gave algebraic arguments of this type were not always successful in making a sound argument. In fact, only six of the thirteen students that gave general arguments did so successfully. In such cases, it was not uncommon for students to make algebraic errors when they developed their arguments. Note, for instance, that Student 2 attempted to generalize the conditions of the problem, but could not do so because she made several errors as she simplified the initial algebraic expression.

Student 1: Yes, come up with an equation: $\frac{5x+12-x}{4}=\frac{4x+12}{4}=x+3$

Student 2: I think they are right because you are doing the same process over and over, just changing the starting and ending numbers. Would show them examples of high and low numbers. Tried to do it algebraically but it wouldn't work out:
((5n) + 12) – 10)/4 = n + 3; 5n/4 + 3 – 10 = n + 3...
5n – 28 = 4n + 12; n = 4n

Student 3: No, they only tested it two times. It could be right, but they need to continue testing. By showing some sort of equation that shows why.

Student 4: I don't think that the girls are right.

I would probably do examples until they are convinced that answer always works out.

Student 5: It seems like they are. I have no idea!

Sixteen students suggested that one needed to develop a formula, although they did not provide a formula to justify why the claim held true. As the explanation of Student 3 illustrates, these students at least recognized that they needed to provide a more general explanation, or make a deductive argument, but for some reason did not choose to develop this formula. These students, too, moved beyond providing specific examples to support the claim.

Fifty-four students supported their claims by providing two or more examples. Like the response given by Student 4, they often suggested that one must give many more examples in order to convince Mei and Malaicha that the procedure always worked. As such, we characterize these types of arguments as inductive rather than deductive. Although these students did not develop general arguments, their responses were somewhat promising. The students did attempt to justify why the claim did or did not hold.

In light of these preliminary findings, we are again reminded of what constitutes proof making. When proof making, one often moves back and forth between generating examples and developing more general arguments. The process of moving between these two types of reasoning is imperative because one can gain insight as one develops a convincing argument. Thus, the students' responses are instructive. They suggest that we might wish to consider how we can support prospective teachers so that they might engage in this type of proof making. In the final section, we specifically address this issue in light of teacher preparation.

Final Remarks

In our discussion, we have illustrated what we mean by making sound mathematical proofs. Also, we have exemplified the important role of both inductive and deductive arguments in this process. Here, we return to these issues as they relate to teaching prospective teachers. To do so, we address three issues.

The first issue relates to what we, as teacher educators, might consider as we plan for instruction. When prospective teachers engage in mathematical tasks, what might we do to support them as they develop mathematical arguments? From a pedagogical standpoint, we must develop activities that make it possible for students to generate cases and/or examples that support mathematical claims. These activities can provide natural entry points for students to explore how their examples might be generalized. Of course, the instructor's role in the process is critical. As one who is privy to what sound proof making is, the instructor must capitalize on instances in which claims are made, challenged, refined, or even refuted. As part of this process, the instructor might ask questions, such as: Does this always work? Does it matter which numbers we use? At the very least, when prospective teachers make mathematical claims, we need to ask questions that make it possible for them to move between engaging in inductive and deductive arguments. If we wish our prospective teachers to embrace what it means to engage in doing mathematics with understanding, we must provide these types of experiences for them. If prospective teachers do not have these experiences, we could not expect them to recognize or capitalize on such instances when they are teaching middle school students.

Second, we must help students "see" the utility of making and supporting generalizations. When they generate a formula that works for all cases, such as Student 1 did in our example, they must recognize the power of this type of generalization. That is, they need to convince themselves and possibly others, that algebraically,
$[(5n + 12) - n]$ divided by $4 = (4n + 12)/4 = n + 3$. They must realize that their choice of n does not affect this generalization. We suspect that only then will they appreciate the mathematical utility of proof making.

And finally, we must help students understand that mathematics is not an exact science *per se*. If we draw on our understanding of the history of mathematics, we realize that some mathematical claims that appeared to hold true were later proven to be unfounded. So although we treat some mathematical claims as infallible, they are only such as long as they work for us. We rightly may need to adjust, refine, or refute a claim if it later is proven untrue under different conditions.

Why might we wish to realign our instructional practice as we work with prospective teachers? In helping our students experience mathematics by making sound mathematical proofs, we make it possible for them to engage in an activity that mirrors how mathematicians work. More importantly, they and perhaps the students they will eventually teach, will recognize that they are building sound mathematical ideas and in turn, are reconstructing mathematics—a mathematics that has its origins in their own mathematical activity. ■

References

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