STUDENT WORK SECTION

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The first paper in this section describes work that was undertaken by three students from Saint Catherine's School in Richmond, Virginia, in a class offered in the school's minimester. The name of the course was The Return of Hard Problems – The Sequel.

The second paper was written by Wendy Griffin who is a teacher and Chair of the Department of Mathematics at Liberty Middle School in Hanover County, Virginia. This paper was written while she was on a leave at Virginia Commonwealth University as a National Science Foundation GK-12 Fellow.

MATHEMATICS FROM CHINA TO VIRGINIA BY WAY OF SINGAPORE

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Abstract

Our article follows from an interesting concurrence of mathematical and educational lines. At least the concurrence seems so to us and we hope that those who read on will agree. The lines or streams are a joint minimester program at St. Catherine's and St. Christopher's Schools, an interest in problem solving, and a Singapore connection. We shall describe the lines first and then describe the mathematics that we found at their intersection.

Minimester at St. Catherine's and St. Christopher's Upper Schools

The academic session at our two schools is divided into three trimesters and a two-week minimester. The minimester takes place during the two weeks in late February and early March between the end of the second trimester and the start of spring break. All girls in the St. Catherine's Upper School and all seniors, as well as some juniors from St. Christopher's, participate in the minimester program. Regular academic courses (with the exception of certain Advanced Placement courses) for minimester students are suspended. Participants may choose

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two-week activities from a varied and impressive array of options which includes independent projects and study, academic and recreational courses, workshops in literature and the arts, travel, community service, and internships sponsored by local businesses and professional groups.

The yearly cycles of weather, psychology, and academic stamina seem in phase at their dismal low points as winter nears its end. Spring and the vacation which signals its advent are blocked by a succession of tests and due dates for term papers and projects; and, for upperclassmen, the stresses which attend the college applications process are heightened. Thus by its timing, minimester is intended to provide recreation and refreshment for the students of the two schools and for their teachers as well.

Every minimester activity must have an academic component, and credit for acceptable participation in activities and completion of courses is recorded on each student's transcript. Outstanding performances by students are honored and noted on their transcripts as well, while credit is withheld when performances are not satisfactory. In certain of the activities and courses, academic rigor may be properly relaxed a bit in light of the acknowledged intent that the program provide pleasure during a period of academic doldrums. However, each student must participate in at least one rigorous course or activity.

The major planning and direction for the minimester come from St. Catherine's. Courses, workshops, and other activities are led by St. Catherine's teachers, visiting faculty, and to a lesser extent, St. Christopher's teachers. Each year for the past three years (2000, 2001, 2002), the Department of Mathematics at St. Christopher's has offered a course focused on problem solving. In 2000, problem solving and learning to use *Mathematica* were strongly tied together, and subsequently *Mathematica* has been used when appropriate in a natural manner. Posing problems has always been a part of the courses; and clear, unambiguous statements of problems and expositions of their solutions have been emphasized. Hence, the writing of mathematics has been an important component of the work.

Throughout participation by the St. Christopher's mathematics faculty, writing has been an integral part of the mathematics course. In 1989, Virginia Mathematics Teacher published an article which grew out of that year's spring minimester study of several problems famous in the history of mathematics [1]. Also from time to time, that journal has published in its "Problem Corner" problems which were posed and solved in one or another minimester class.

Problem Solving

The title of the course for 2001 was *Hard Problems* and the title of the 2002 course was *The Return of Hard Problems—The Sequel*. Despite the forbidding titles, students did sign up for the classes, and the first three authors of this article comprise the class for the sequel of 2002; the fourth author was the teacher.

The opening paragraph of the published course description attempts to justify the course and to say something about what mathematicians and mathematics teachers mean when they talk about "problem solving," as opposed to working "the odd-numbered exercises at the end of the chapter."

> Every mathematician has his or her own personal mathematical frontier. No matter where that frontier is located, the mathematician will find challenging problems along that boundary which separates what is known from what is not yet understood. Such boundaries are, of course, somewhat fuzzy. By attacking such problems, the mathematician will advance his or her personal frontier. If the mathematician is working at the boundary of his or her discipline, the mathematician advances the frontier of mathematics itself.

The distinguished Canadian mathematician and problemist Murray Klamkin possesses far more clout than do the present four authors. Thus, it seems a good idea to cite his remarks to clinch the matter. He wrote:

> . . . problems and questions beget more problems and questions in an unending cycle. These problems and questions are the lifeblood of mathematics. Smaller problems lead to larger problems which in turn lead to substantial mathematical research [2].

Just as baseball is a game of failures, so is problem solving. A real problem solver knows that he will never solve all of the problems that he or she considers. If the sole criterion for success is a correct and complete answer at the bottom of a page, then good solution averages over many attempts would probably approximate the batting averages of good major league hitters who experience failure roughly two-thirds of their times at bat. But the real problemist does not keep score. Learning new mathematics, gaining better understanding of old mathematics by testing one's knowledge against new configurations of what is given and what is to be found, and the establishment of connections between seemingly disparate parts of the body of mathematical theory are the true rewards of problem solving.

Posing and writing good problems are at least as important as solving them. Mathematics is filled with conjectures and theorems named for the proposer rather than the solver. That proposers become eponyms recognizes that they were the ones who alerted the mathematical world to the significance of some idea. In a similar way, the problem editors of journals also recognize the importance of developing good problems. If one's problem is accepted for publication, the proposer's name appears when the problem is first published. His or her name is repeated with the restatement of the problem when the solution is published in a later issue of the journal. If the editors choose to reproduce the proposer's own solution, his or her name is given again. As a result, the proposer's name appears two or three times while the names of the other solvers appear only once.

"Problem solving" does have a particular meaning and can be taken as a specialized pursuit within mathematics. It must be acknowledged that the statement, "He is a problem solver" with an implied "only" between the "is" and the "a" should not be taken as a compliment. When a mathematician uses the description as a "put-down," it probably means that the mathematician feels that a colleague or competitor is missing "the big picture" and is wasting time and energy on relatively trivial matters. Problem solving is an important activity that produces fun as well as results, while it adds a game-like, competitive aspect to learning mathematics. However, it can be overdone at the expense of the systematic development of the various major branches of mathematics. Thus, a two-week minimester course seems an ideal setting for a fairly intense engagement in problem solving.

The Singapore Connection

Mr. Willie Yong of the SCT Publishing Company of Singapore produces a lovely journal, *Mathematics and Informatics Quarterly* (informatics is a synonym for information science used chiefly in Great Britain). This journal is largely devoted to problem solving at an advanced secondary level. It is published in English, but receives problems and manuscripts from all over the world. The names of some of the contributors would be familiar to those who read the problem sections in popular American journals. The last author of this article has the privilege of serving on the editorial board of Mr. Yong's journal, and one of his responsibilities is to provide smoothly flowing versions of often elegant and challenging problems, solutions, and articles submitted by non-native speakers of English. Some of the problems even arrive at St. Christopher's written in the contributors' own languages; but, the mathematical notation usually makes the content of the problems clear. Thus from far away places and via Singapore, there has come to St. Christopher's a large supply of problems to challenge the minimester mathematicians of recent years.

Mr. Yong often includes in his packets of material to be read and rewritten items that he thinks will be of interest to the students of St. Christopher's and St. Catherine's. One of these items caught the special attention of the minimester class. It was a single sheet of paper covered for the most part with diagrams and equations, but also containing a small amount of text written in Chinese characters. It is reproduced below as Figure 1. The hand written notations on the page were made by Mr. Yong. Mr. Yong's last suggestion—"Have fun"—might well serve as a motto for his journal.

The class agreed upon a project. The students would translate the text, and the students and teacher together would attempt to understand the mathematics. Then, if the results of the project seemed of sufficient interest, a manuscript describing the class and the mathematics from Singapore would be prepared and submitted for publication in an appropriate journal.

第三次活动

让每个小组派代表向全班公布他们的设计方案．学生们的创造力令人惊呀，提出了下面一些设计方案．

1 方案一

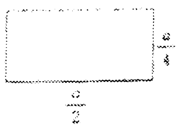

图1

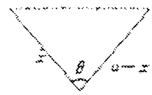

图2

$S_{max}=0.125a^2$

2 方案二 $S=\frac{1}{2}x\cdot(a-x)\cdot\sin\theta$

$$\leqslant\frac{1}{2}x(a-x)=\frac{a^2}{8}-\frac{1}{2}\left(x-\frac{a}{2}\right)^2$$

$$\leqslant\frac{a^2}{8}$$

当 $\theta=90°$，且 $x=\frac{a}{2}$ 时，$S_{max}=\frac{a^2}{8}=0.125a^2$

3 方案三

$$S=\frac{1}{2}\left(\frac{a}{2}+\frac{a}{2}+2\cdot\frac{a}{4}\cdot\sin\theta\right)\frac{a}{4}\cos\theta$$

$$=\frac{a^2}{16}(2+\sin\theta)\cos\theta$$

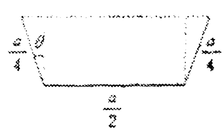

图3

学生通过取特殊角，查三角函数表，借助于计算器列出了下表，由此得 $S_{max}\approx0.138a^2$

θ	90°	60°	45°	30°	20°	10°	0
S	0	$0.09a^2$	$0.12a^2$	$0.135a^2$	$0.138a^2$	$0.134a^2$	$0.125a^2$

（此种情形下的答案是 $\theta=21°28'$ 时，S 最大）

4 方案四

$$S=\frac{1}{2}\left(\frac{2a}{3}+2\cdot\frac{a}{3}\sin\theta\right)\cdot\frac{a}{3}\cdot\cos\theta$$

$$=\frac{a^2}{9}(1+\sin\theta)\cos\theta$$

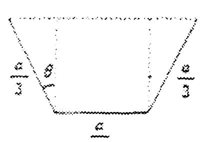

图4

通过试算，学生得出 $\theta=30°$ 时，$S_{max}=0.144a^2$

（学生没猜错）

5 方案五

$$S=\frac{1}{2}\left(2\cdot\frac{a}{4}+2\cdot\frac{3}{8}a\cdot\sin\theta\right)\cdot\frac{3a}{8}\cdot\cos\theta$$

$$=\frac{3}{64}a^2(2+3\sin\theta)\cos\theta$$

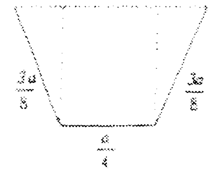

图5

学生试算出 $\theta=35°$ 时，$S_{max}=0.143a^2$，

（答案是 $\theta=34°02'$ 时，S 最大）

6 方案六

$$S=4S_\triangle=4\times\frac{1}{2}\cdot\frac{a}{4}\cdot\frac{a}{8}\cdot\mathrm{tg}67.5°$$

$$\approx0.151a^2$$

图6

$\theta=67.5°$

7 方案七

$$S=5S_\triangle$$

$$=5\cdot\frac{1}{2}\cdot\frac{a}{5}\cdot\frac{a}{10}\cdot\mathrm{tg}72°$$

$$\approx0.154a^2$$

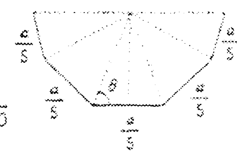

图7

$\theta=72°$

8 方案八

$\because \pi r=a$

$\therefore r=\frac{a}{\pi}$

$\therefore S=\frac{1}{2}\pi r^2$

$=\frac{a^2}{2\pi}$

$\approx\boxed{0.159a^2}$

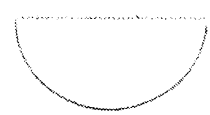
图8

LARGEST

Given a wire of length a construct the following cross-sections and determine the maximum area.

○ ≡Chinese equivalent of numerals

Figure 1

MATHEMATICS FROM SINGAPORE

Mathematics from China by Way of Singapore

Translation from the Chinese reveals that the page is a record of "Exercise III." It represents a series of student calculations most likely intended to support and motivate a well-known problem in the calculus of variations. A piece of wire of length a is bent to form the incomplete boundary of a plane, convex figure. That is, a segment would have to be added between the ends of the wire to close the figure. The problem is to discover how to bend the wire to define the plane figure of largest area.

American students and teachers sometimes refer to an equivalent problem as the "gutter problem," although a far more romantic name is "Dido's problem." Suppose that a gutter is to be created from a long strip of metal which has uniform width a. The top of the gutter is open, and the quantity of water that the gutter can carry away is proportional to its cross sectional area. In this version, the problem is to discover the shape of the cross section that will maximize the cross sectional area of the gutter, subject to the condition that the length of the cross sectional curve has the fixed value a.

It may be that the results obtained by the different pupils in the mathematics class in China are being presented to the class as a whole or, perhaps, to visitors to the classroom. Anyway, the text above the first set of diagrams represents what the teacher says to introduce the presentation: "Let the representatives of each group explain their plans to the entire class. The creativity of the students has been amazing. The students have suggested the following plans." Then Plans 1 through 8 are developed.

The sense of the calculations is obvious from the diagrams and students should observe an almost uniformly increasing area as the curve more nearly approximates a semicircle. A sample of several of the computations will be given in the next section. Students who have performed these calculations and been led by their teacher to ponder their meaning will most certainly have gained an understanding of a famous problem. A lovely proof by contradiction that the semicircle is the curve that yields the largest cross sectional area for a fixed length is given by Ivan Niven in *Maxima and Minima Without Calculus* [3]. The proof hinges upon the well-known theorem that an angle inscribed in a semicircle is a right angle. Niven also explains why the problem bears the name of Dido, Queen of Carthage.

Calculations — The calculations below are referenced to the diagrams as numbered in Figure 1. One must imagine the class discussion which accompanied these calculations and the conclusions to which the young students were led. Each diagram possesses the symmetry of a reflection across its vertical center line, and the fixed length of the wire in each instance is a.

Diagram 1 — The wire forms three sides of a rectangle with altitude and base of lengths $a/4$ and $a/2$ units, respectively. If the rectangle represents the cross section of a gutter, it presents a cross sectional area of $a^2/8 = 0.125a^2$ sq. units to water flow.

Diagram 2 — The wire forms two sides of a triangle. The lengths of the two sides are x and a - x, and the angle between the two sides has measure θ. Therefore, the area of the triangle is $x(a - x)\sin\theta/2$. The area is largest when $x = a/2$ and $\theta = 90^\circ$. It is interesting that this largest area ($0.125a^2$) has the same value as the area of the rectangle immediately above.

Diagram 3 — The wire forms the base and non-parallel sides of an isosceles trapezoid. Each of the non-parallel sides is $a/4$ units long and the base is $a/2$ units long. The angle between an altitude and each of the non-parallel sides is θ. The area of the trapezoid is $a(a + a(\sin\theta)/2)\cos\theta/8$. A series of computations indicates that the area will attain its maximum value of approximately $0.138a^2$ near $\theta = 20^\circ$. The methods of differential calculus yield a critical value of $21.4707^\circ = 21^\circ\ 28'$ for angle θ. In the next diagram, the wire is bent so that the three given sides of the trapezoid are all congruent.

Diagram 4 — In this figure, each segment of the wire is $a/3$ units long. The area of the trapezoid becomes $a^2(1 + \sin\theta)\cos\theta/9$. An angle $\theta = 30^\circ$ yields a maximum area of $0.144a^2$ as may be verified by differential calculus.

The trapezoid in this case is half of a regular hexagon, and in the case illustrated by the sixth diagram, the figure is half of a regular octagon. By now, students ought to anticipate that the area of the figure will increase as the shape of figure approaches that of a semicircular region. The final diagram to be considered in detail is that which appears next to last in Figure 1.

Diagram 7 — The figure is half of a regular decagon which may be partitioned into five isosceles triangles with base $a/5$ and altitude $a(\tan 72^\circ)/10$. Thus, the area of the figure is $(1/2)5(a/5)(a(\tan 72^\circ)/10) = a^2(\tan 72^\circ)/20$ which has the approximate value of $0.154a^2$ sq. units.

The exercise concludes with the computation of the area of a semicircle of length a. The radius of such a semicircle is a/π and its area is $(1/2)\pi(a/\pi)^2 = a^2/(2\pi)$ which is approximately $0.159\,a^2$.

It seems clear that students who participate with goodwill in these exercises and pay attention to the meaning of their results will learn a great deal of geometry.

Conclusion

One needs to ask whether or not the minimester excursions into problem solving have been successful. Total student enrollment has not been large—roughly fifteen students over the past three years. However, those who participated did so enthusiastically. Students and teachers not directly involved have derived benefit as well. There really is truth to the old adage about "casting one's bread upon the waters." By means of the minimester course, ideas were set afloat that do not find expression in the general mathematics curricula of the two schools. Ideas floated on a sea of young minds do have their eventual return.

If the question is asked, the answer is "Yes." If the question is not asked, the answer not given is still "Yes." At St. Catherine's and St. Christopher's, a convenient time in crowded schedules was available for the class, there was already in place an interest in problem solving, and there was a seemingly inexhaustible source of problems from Singapore to consider. All that was needed was the addition of enthusiasm and hard work.

References

[1] L. Crittenden, G. Case, and J.N. Boyd, "Professor Bearfon's Needle Problem," *Virginia Mathematics Teacher*, **16** (1989) 10-11.

[2] M.S. Klamkin, "Forward," in S. Raboniwitz (ed.), *Index to Mathematical Problems 1980-1984*, MathPro Press, Westford, MA, 1992.

[3] I. Niven, *Maxima and Minima Without Calculus*, The Mathematical Association of America, Washington, DC, 1981.