# REVISION OF A NON-EUCLIDEAN GEOMETRY COURSE BASED ON THE VAN HIELE MODEL OF THE DEVELOPMENT OF GEOMETRIC THOUGHT

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## Abstract

This paper describes the revision of a course in non-Euclidean geometry to incorporate active student learning. The design of the course and the sequence of lessons were based on the van Hiele model of the development of geometric thought.

## Introduction

The nature of high school geometry courses has changed over the years, with some high schools adopting a mixture of both formal and informal approaches to geometry where formal proof is also combined with visualization, problem solving, and applications [1]. Some high schools also offer courses integrating both algebra and geometry. Consequently, students enter college with a variety of geometric knowledge and prospective teachers must be prepared to teach a variety of geometry courses.

At Mary Washington College (MWC), the students who are certifying to teach mathematics in grades 6-12 must complete a major in mathematics. While they are very strong students, the prospective secondary teachers at MWC have indicated that they do not feel as well prepared to teach geometry as other topics in mathematics.

This article describes the revision of a course in non-Euclidean geometry at MWC based on the recommendations from the *Principles and Standards for School Mathematics* [2], the *Professional Standards for Teaching Mathematics* [3], *Moving Beyond Myths: Revitalizing Undergraduate Education* [4], and *Educating Teachers of Science, Mathematics and Technology: New Practices for the New Millennium* [5]. The design of the course was also based on the van Hiele model for the development of geometric thought.



The Journal of Mathematics and Science: Collaborative Explorations Volume 5 (2002) 25 - 36

## Need to Offer a Geometry Course

The non-Euclidean geometry course was designed in the mid-1980s as a 300-level mathematics course for mathematics majors, and it has always been recommended for those students who plan to teach mathematics in high school. I taught this course several times. However, due to a variety of circumstances, primarily difficulties with staffing, the course had not been taught for ten years. Both the mathematics and education departments saw the need to offer a geometry course again on a regular basis. I asked to teach the course and it was scheduled to be offered in the Spring 2002 semester.

## Design of the Course

During the summer of 2001, I received support from the Virginia Collaborative for Excellence in the Preparation of Teachers (VCEPT) to redesign the geometry course. The course would continue to be for mathematics majors and recommended for those who plan to teach. I had two basic considerations as I thought about revising the course: what content should I include, and what instructional strategies should I use? In considering the content, I questioned the topics that future teachers need to know in order to teach geometry in middle and high school, the students' prior knowledge of geometry, and how I should balance depth and breadth of coverage. Therefore, I first reviewed the Virginia Standards of Learning for geometry in grades 6-12 [6], the NCTM Standards [2], and various college geometry textbooks.

After reviewing these materials and seeing the scope of knowledge recommended for teachers, I questioned whether or not keeping the focus on non-Euclidean geometry was appropriate for the course, or whether the course should include a more substantial review of Euclidean geometry and more breadth of geometric topics. What was the best kind of course for these students to take to prepare them to teach geometry? It appeared that there would be several options for such a course. A college geometry course could focus more or less on an axiomatic development of geometry. It may or may not include topics, such as transformations, vectors, both 2-and 3-dimensional shapes, projective geometry, and non-Euclidean geometry.

Recently, there have been negative commentaries that the curriculum in the United States is a mile wide and an inch thick. Many reports and articles have called for teachers to develop a deep understanding of the subjects they will teach [1,5,7,8]. *The Mathematical Education of Teachers* states, "A major goal of a collegiate geometry course should be to deepen prospective teachers' understanding of standard Euclidean theorems and principles and their skill in use of axiom-based reasoning." [1] The report goes on to say, however, that prospective teachers should also be acquainted with other aspects of geometry and includes as examples, the geometry of the sphere, conic sections, artistic notions of perspective, Platonic solids, tilings, fractals, and applications such as computer graphics and robotics. However, the report cautions that, "Fitting all of those topics into one college geometry course that also gives an in-depth axiomatic development of Euclidean geometry runs a clear risk of covering ground without developing depth of understanding...it seems promising to survey some topics quickly and then treat a selected few in depth."

I finally decided to keep the focus of this course on non-Euclidean geometry and also try to weave in some of the other topics with which teachers must be familiar. There were several reasons why I decided to keep the focus on non-Euclidean geometry and, after having taught the course, I am very happy that I chose to do so. I believe that the design of a geometry course should be guided by the nature and level of the course and the backgrounds of students who will be taking the course. While it is essential that high school teachers have a thorough understanding of Euclidean geometry, since this course was designed for mathematics majors I wanted it to be more than simply a review of their high school geometry course. In developing the axiomatic systems for non-Euclidean geometry, we reviewed postulates and theorems from Euclidean geometry in more depth. Looking at alternate hypotheses and proving theorems that seem to contradict their common sense help students appreciate the importance of axioms and definitions and help them view Euclidean geometry from a different perspective. Another reason to keep the focus on the development of non-Euclidean geometry is that I have found that students enjoy learning about these different kinds of geometry. Many of our mathematics majors take a course in the history of mathematics, where non-Euclidean geometry is discussed briefly. This seems to whet their interest in the subject and they want to learn more. However, in addition to the focus on the axiomatic development of non-Euclidean geometry, I also wove in other topics the students need to be able to teach, such as rigid motions and three-dimensional solids. I felt secure that the students had studied other topics recommended for future teachers, such as coordinate geometry, matrices and graph theory, in other courses.

## Course Goals and Topics

After reviewing several textbooks, I decided to adopt the following text: *College Geometry: A Discovery Approach* [9]. We focused on the content of most sections in Chapters 2, 3, 4, and 6. However, I did supplement it with other materials and resources. The goals for the course were the following:

- students will review and extend the concepts and theorems of Euclidean geometry;
- students will develop their abilities to construct logical mathematical proofs in various axiomatic systems;
- students will learn about the historical developments of Euclidean and non-Euclidean geometries;
- students will learn the basic concepts and theorems of hyperbolic (Lobachevskian) and spherical (elliptic, Riemannian) geometries.

The course began with an introduction to axiomatics and proof, then examined the axioms and theorems of absolute geometry (geometry without a parallel postulate), and then focused on the results that follow from the Euclidean parallel postulate, the hyperbolic parallel postulate, and the elliptic parallel postulate.

## Pedagogical Considerations

For the past five years, I had been working with other faculty at MWC and other colleges and universities throughout Virginia as part of VCEPT. VCEPT's primary goal was to better prepare future teachers of students in grades K-8 to teach mathematics and science. I had concentrated on strengthening our program for those students who were enrolled in our elementary teacher preparation program and had designed two new courses. In so doing, I had relied upon educational research on learning and teaching, professional standards, and the VCEPT guidelines for course development. Since teachers teach they way they were taught, I tried to model the same recommended teaching strategies that I espoused for elementary mathematics teachers throughout the course. I wanted to apply some of these reform methods of teaching to this geometry course that would be taken by our future secondary mathematics teachers. In general, I wanted this course to be one in which there was a community of learners actively participating in class and working together to maximize their learning. To achieve this result, I relied upon the van Hiele levels of geometric learning and Phases of Learning as a guide in planning the course and the lessons.

## van Heile Levels and Phases

Two mathematics teachers from the Netherlands, who were also husband and wife, Pierre van Hiele and Dina van Hiele-Geldof, devised a model of the development of geometric thought in the 1950s. However, their works did not receive substantial interest in the United States until the 1980s when some of their major writings were translated into English. The van Hieles proposed that students progress sequentially through five levels of reasoning. At Level 0 (Visualization), a person recognizes shapes holistically without paying attention to relevant attributes and may actually focus on irrelevant attributes. A person claims a square is a square simply because it looks like a square. If a square is not oriented so that its sides are drawn vertically and horizontally but instead are on a slant, the person may not believe it is a square. At Level 1 (Analysis), the person can focus more analytically on the relevant attributes of a shape, such as the number and properties of sides and angles, and is not distracted by irrelevant attributes. For example, the person will say that a square has four equal sides, or four square corners, and knows that the orientation of the square on the page does not matter. At Level 2 (Informal Deduction), the person develops an understanding of relationships among shapes and can use informal deduction to justify observations and verify properties. For example, the person knows that a square is a kind of rectangle and a rectangle is a type of parallelogram. A person reasoning at Level 3 (Deduction) can write formal proofs of theorems. This is the level at which we hope students in a college preparatory, high school geometry course are functioning. However, many of these students are still at Level 2 or below. The highest level, Level 4 (Rigor), is highly abstract and reserved for serious students who are typically studying geometry at the college level where axioms themselves are studied and different geometric systems can be compared. A course in non-Euclidean geometry would fall, at least partially, in this last category [10,11].

The van Hieles asserted that students progress through these levels sequentially without skipping a level. A student's progress depends on the content and kind of instruction he or she receives rather than on age. If there is a mismatch between the level of instruction and the student's level of thought, learning may not occur. In order to facilitate a student's progress within a particular level, the van Hieles proposed that instruction be developed according to five sequential Phases of Learning. The initial phase, Phase 1, is Inquiry/Information where the teacher and students begin to discuss the topics so that the teacher can learn what prior knowledge the students have and the students learn what they will be studying. In Phase 2, Directed Orientation, the students explore the topics through the use of materials and structured activities. Phase 3 is Explication where students discuss what they have observed and exchange ideas. Phase 4 is Free Orientation where students work on more complex tasks. These tasks may be open-ended, involve multiple steps, and have a variety of solution methods. In working on these tasks, students may become aware of connections and relationships among the topics and objects they are studying. The final phase, Phase 5, is Integration where students review, summarize, and synthesize what they have learned. When the students have progressed through Phase 5, they should be ready to advance to the next level of geometric thought [10,11].

In designing the course in non-Euclidean geometry, I kept the van Heile levels in mind in two different ways. First, I wanted this course to have the students reason very abstractly, at Level 4. However, I also wanted to make sure that all of the students were ready for that level of abstract reasoning, so I knew that I might need to treat topics at a lower level first. I also wanted to give the students examples of learning at these lower levels in order to prepare them for teaching geometry in middle and high schools.

For each topic that we discussed, I tried to follow the van Hiele Phases of Learning in addition to considering the levels of geometric thought. When working in the lower levels, we progressed more rapidly through the phases; when we dealt with material at the higher levels, we progressed more slowly. In Phase 1, Inquiry/Information, I introduced the topic and helped the students recall their prior learning through questions, discussions, and occasional worksheets, and tried to motivate their interest. In addition to being a review for the students and orienting them to what we would be learning, this knowledge of their background helped me better plan future lessons. In Phase 2, Directed Orientation, I usually gave the students a problem to solve or a fairly structured activity to guide their learning of the content. I looked for worthwhile mathematical tasks that students could work on individually or together that would present them with the concepts we would be studying. Often, this involved drawings or manipulative materials. The textbook had special small units in many sections entitled, "Moments for Discovery" that were often appropriate for this Phase 2. In addition to these, I used problems from the text and other resources. The students also worked on constructions (such as orthogonal circles in a model for hyperbolic geometry) and guided "mini proofs" that could be combined later.

The students in this class were always anxious to move to Phase 3, Explication, so they could share what they had learned from their activities in Phase 2 or ask for clarification on problems they were having. When students asked questions, I tried very hard to turn the question to other members of the class, rather than answering it immediately myself. This promoted good discussion and after a while, the students naturally asked questions to one another and responded. It was a true pleasure to hear all these mathematical discussions taking place in class. For Phase 4, students worked on the more difficult problems or wrote proofs. While there was some collaboration at this phase, I encouraged students to first work individually, perhaps as part of their homework assignment, and then share their results and help one another with problems. Before moving on to the next chapter, or even the next section in the textbook, I conducted a review primarily by asking questions and sometimes making lists to help the students consolidate their learning, clarify any misconceptions, and fill in any gaps in their understanding. These reviews were extremely important, especially as the course material got more involved and abstract. On the last day of class, the students themselves organized and guided a review session to prepare for the final examination.

The following is an example of how we moved through the Phases of Learning in studying about parallel projections. In Phase 1, I questioned the students on what they remembered about similar triangles and what they had learned in their other courses about mappings. For Phase 2, I asked each student to take out a lined piece of notebook paper and I gave each student a blank, 4x6-inch index card. I challenged the students to divide the long side of the index card into five congruent sections using only the lines on the paper. After a few minutes, they excitedly discovered that if you slant the index card so that one corner touches a line on the paper and the corner on the other end of the long side touches the fifth line down from the first line, then the remaining lines divide the index card into five congruent sections. This activity and the discussion that followed in Phase 3 then led to proving the "Side-Splitting Theorem" and homework problems in Phase 4. Reviewing the concept in Phase 5 was done at the end of the chapter.

Using these Phases of Learning was an excellent way to help the students progress through their geometric learning at an appropriate pace in which they were challenged, but there were no gaps in learning. The students greatly enjoyed working with materials, solving problems, and participating in class discussions. While they sometimes struggled with writing the proofs individually, they helped one another and would often remind each other of the activities we had done that related to the theorems to get a better understanding of the concepts involved. They greatly appreciated taking a step to the side occasionally for the reviews, during which I could almost see the puzzle pieces fitting together in their brains. During one class when I was running short on time, I abandoned progressing through the phases and simply slapped a proof on the board. The students clearly did not like this approach. One of the best students, even though she understood each step in the proof, proclaimed, "I don't believe it." She and the others expected to understand what they were proving, to clearly see it, and have it make sense.

Using the van Hiele model, I found that when students worked at the higher more abstract levels of geometry, writing proofs at level 3, and especially in learning about the non-Euclidean geometries at level 4, they were well prepared to do so, having had a solid foundation at the more concrete lower levels. They were able to understand and write proofs in both Euclidean and non-Euclidean geometries.

## Examples of Course Activities

I have described below some particular lessons and activities in the course.

<u>History of Non-Euclidean Geometry</u> — Students find the history of how non-Euclidean geometry developed to be very interesting. Our textbook discussed some of the mathematicians involved and how several had tried to prove the Euclidean parallel postulate. However, I thought a more thorough treatment of the historical developments and a deeper look into some of the mathematicians involved would bring the subject to life. Nevertheless, I did not want to resort just to lecturing on the topic. So, several weeks before we started working with the non-Euclidean geometries in depth, I composed a list of ten mathematicians who had been instrumental in the development of non-Euclidean geometry. These were Euclid, Saccheri, Lambert, Lobachevsky, Wolfgang (Farkas) Bolyai, Janos Bolyai, Gauss, Legendre, Riemann, and Beltrami. There were ten students in the class and I allowed the students to pick which mathematician they would like to portray. They then were to research the lives and work of these mathematicians and present the information to the class as if they themselves were the actual people. The role-playing was presented in approximate chronological order, starting with Euclid. Most of the students assumed what they thought would be the personal demeanor of their character. Students made reference to each others' characters personally during their

presentations, such as telling Euclid he shouldn't have assumed his fifth postulate and finding flaws in each others' work. Gauss, in particular, took a lot of abuse. The students thoroughly enjoyed this drama, and did a wonderful job in portraying their geometers. Throughout the rest of the course, I would ask a person who played a particular geometer to contribute when introducing a topic or answering a question pertaining to his work. For example, when a question arose about a Saccheri quadrilateral, instead of answering it myself, I referred it to the student who portrayed Saccheri. Likewise, I heard students asking each other questions that their mathematician should be able to answer. The student who played Gauss was so impressed with his initial research that he continued to read more about Gauss, and stopped by my office several times to discuss what he had learned. On the final examination, there were twenty fill-in-the-blank questions on the development of non-Euclidean geometry and the roles of the various mathematicians. Although it had been over a month since we had had the dramatic portrayal of these geometers, the students all remembered the information very well, with most students answering all the questions correctly. In their comments on the course evaluations, students indicated that they thought doing this role-playing was an excellent way to learn about the mathematicians and the development of the field.

Tessellations — In designing the course, I wanted to include some alternate forms of assessment, in addition to the three tests, and show some connections between mathematics and other fields. I also wanted to quickly review the rigid motions of reflection, rotation, and translation in the course, since the students would have to teach these in middle or high school. Therefore, I assigned a project where the students were to design a tessellation through using rotations and translations, and then present it to the class. Going through the Phases of Learning, we first spent a few minutes reviewing rigid motions. Second, we experimented with pattern blocks and sets of plastic polygons to discover the types of polygons that would tile the plane. Then, we discussed regular polygons and angle sums and why certain polygons would tile and others would not. Next, we looked at various works of Escher and learned how to make a "unit cell" by starting with a polygon that would tile, and then cutting a piece from one side and rotating or translating the cut-out piece to produce a unit cell that would tile the plane. For Phase 4, the students created their own beautiful tessellations. For Phase 5, we reviewed the concepts when the students described how they created it to the class. Afterward, we posted the tessellations on the departmental bulletin board. Several students said this would be a project they would use when they teach.

Spherical Geometry — After discussing hyperbolic non-Euclidean geometry for several weeks, I wanted to spend some time on spherical geometry. Initially, we briefly discussed how in this geometry there were no parallel lines, and I drew spheres and circles on the board to try to represent this model. However, since it was in 3-dimensions, it was more difficult to visualize. I had ordered several sets of the Lenart Sphere. Each set consisted of a plastic sphere, three hemi-spherical acetate sheets, erasable pens for the acetate sheets, a spherical compass, and a spherical protractor. Each student had his/her own set. After identifying what all the components were, I led the class through a guided discovery lesson based on the materials that had come with the instructor's guide. For example, students drew and constructed great circle "lines" on the sphere to see that two of these lines could never be parallel. Students also drew spherical triangles and measured the angles. After comparing answers and some discussion, the students decided that the sum of the angles of these triangles would be between 180 and 540 degrees. The students thoroughly enjoyed working with these physical models and several stated that they would certainly not have been able to understand the concepts without them.

## Course Outcomes

Naturally, I will make some revisions when I teach the course again; however, all evidence indicates that the course was a success. While I admit it took work to develop each day's lesson, it was definitely worth the time I spent to try to involve the students actively in their learning. I thoroughly enjoyed teaching the course, much more so than I had when I taught it before, and it was obvious that the students also enjoyed the course very much. Most importantly, the students did very well learning the material in the course. We spent more time in class progressing through the Phases of Learning, using materials and in discussions, and less time on writing proofs than we had when I taught the course before. Nevertheless, the students were equally if not better able to write the proofs of propositions in non-Euclidean geometry that were on the final examination.

On the final course evaluation, the overall rating for this class was 4.78 on a five-point scale. This rating and the ratings in all of the six subsections of the evaluation instrument (course organization and planning; communication; faculty/student interaction; assignments, tests, and grading; and, student effort and involvement) were higher than the corresponding mean ratings for the MWC upper level mathematics courses, the MWC upper level courses, and the mean for four-year institutions provided by the evaluation company. In fact, the ratings on four of the six subsections ranked above the $90^{th}$ percentile of all the scores for the four-year institutions.

Every student wrote positive comments about the course. One student commented, "I thought this course was fantastic and so much fun. I personally loved it!!!" Another student wrote, "I truly enjoyed this class. My interest in geometry grew tremendously. I enjoyed discovering, proving, and constructing things on my own." Another asked, "When will we be able to take the second half of *Non-Euclidean Geometry*?" Several of the students remarked that not only had they learned geometry, but also how to teach it. One student told me that she hopes she enjoys teaching as much as I do.

On one of the last days of the class, one student asked the others if they remembered a skit at their freshman orientation where students were depicted signing up for a course in non-Euclidean geometry, and it was portrayed as being the most intimidating and incomprehensible course that was offered at MWC. Several chuckled and said that they did indeed remember the skit. Then they commented, much relieved, "And it was not like that at all." ■

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