STUDENT WORK SECTION

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PATTERNS IN THE SAND: A MATHEMATICAL EXPLORATION OF CHLADNI PATTERNS

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Chladni Patterns are formed when sand settles at the nodes of two dimensional standing waves, excited on a metallic plate which is driven at a resonant frequency. By considering a two-dimensional rectangular membrane with fixed boundary and constant density as an idealized model of the metal plate, a formula for predicting the Chladni Patterns that will form at certain frequencies can be found. In addition to mathematically exploring these mysterious patterns, I have created my own "Chladni Patterns" in the lab.

The Genesis

> "Seeing M. Chladni's experiments during his stay in Paris excited my interest anew. I began studying...desiring to come to appreciate those difficulties that [were] brought to mind." -Sophie Germain

Why is it that when a metallic plate, covered with a fine powder, and is vibrated at certain frequencies, beautiful patterns form in the powder? What can be learned from these wonderful "Chladni Patterns"? These questions have intrigued some of the greatest mathematicians, physicists, and even national leaders: Galileo, Laplace, Legendre, Poisson, Gauss, Chladni, Germain, and Napoleon.

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The phenomenon of these patterns was first reported by Galileo Galilei almost 400 years ago. Galileo reports in his book, *Dialogues Concerning Two New Sciences:*

> "As I was scraping a brass plate with a sharp iron chisel in order to remove some spots from it and was running the chisel rather rapidly over it, I once or twice, during many strikes, heard the plate emit a rather strong and clear whistling sound: on looking at the plate more carefully, I noticed a long row of fine streaks parallel and equidistant from one another. Scraping with the chisel over and over again, I noticed that it was only when the plate emitted this hissing noise that any marks were left upon it; when the scraping was not accompanied by this sibilant note there was not the least trace of such marks." [1]

Galileo reported this phenomenon, but not a lot was learned about it until the early 1800's when a man named Ernst Chladni became intrigued by it. Ernst Florens Chladni was born in Wittenberg, Saxony in 1756. Raised an only child, Chladni was educated for a career in law. At age nineteen, he took up music and it was through this interest in music that he became fascinated with the patterns which eventually were named for him [2].

Chladni would produce these beautiful patterns by taking differently shaped pieces of metal or glass and sprinkling powder on them. He would then bow along the edge of the metal and the patterns would mysteriously appear.

It was during such an exhibition that Napoleon was first introduced to these patterns and became intrigued. He was "struck by the impact which the discovery of a rigorous theory capable of explaining all the phenomena revealed by these experiments would have on the advancement of physics and analysis." [3] Because of this interest, Napoleon urged the First Class of the Institute of France to create an incentive for solving the mystery which surrounded these patterns.

The First Class was the section of the Science Academy devoted to mathematics and physics. At the time, there were a number of well-known mathematicians in the First Class: Lagrange, Biot, Laplace, Legendre. The First Class offered a *prix extraordinaire* of 3000 francs in April 1809 [3] to be awarded to the one who could provide an explanation for Chladni Patterns.

This prize was not won easily or quickly. Because of the rules of the prize, none of the men who comprised the First Class were allowed to submit explanations. The prize was actually reset twice before it was eventually awarded.

In January of 1816, Sophie Germain was awarded the *prix extraordinaire*.

In addition to the work for which Sophie won the *prix extraordinaire*, she made great contributions to the fields of number theory and analysis. Sophie remained secluded from society for her entire life. Maybe this seclusion was the result of the fact that she was a woman in a male-dominated field, or that being an educated woman was out of the ordinary in her day and age. Although she was not shy about her mathematical work, she kept out of the public sector as much as possible.

In 1829, Sophie was stricken with breast cancer. During her two year battle with the disease, she continued to work on mathematics. She succumbed to the fight and died on June 27, 1831 at the age of 55. Unfortunately, Sophie died just a short while before she was to receive an honorary degree from the University of Göttingen, for which Gauss had requested she be considered [4].

Vibrating Membrane of Constant Density

Solution to the Wave Equation in 2-D

What Galileo was witnessing in his laboratory and what Chladni was producing were actually two-dimensional standing waves. The men had stumbled upon the natural frequencies of the plates and by driving them at these frequencies, they were able to produce standing waves. In one dimension, nodes of standing waves are single points, but in two dimensions the nodes are lines. By understanding these nodal lines, we can solve the mystery of Chladni Patterns.

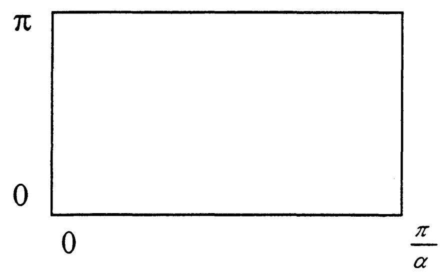


Figure 2.1: Membrane with Dimensions $\frac{\pi}{\alpha}$ by π

Consider a constant density, rectangular membrane. This membrane is completely elastic and has dimensions $\frac{\pi}{\alpha}$ by π. The shape constant, α, denotes the ratio of the height of the membrane to the width. We give the membrane an initial displacement, no initial velocity and denote the vertical displacement of the membrane at position (x, y) at time t by $u(x, y, t)$. The wave equation in two dimensions governs the motion of the membrane if air-resistance is disregarded. (For a dimensional analysis of this equation [in one-dimension] see appendix.)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where v^2 is the square of the velocity of the wave. For simplicity, we set $v^2 = 1$.

The initial conditions are $u(x, y, 0) = f(x, y)$ where $f(x, y)$ is the initial displacement at (x, y) and $u_t(x, y, 0) = 0$.

The boundary conditions are

$$u(0, y, t) = 0 = u\left(\frac{\pi}{a}, y, t\right) \quad \text{and} \quad u(x, 0, t) = 0 = u(x, \pi, t).$$

(The membrane is fixed on all four sides.)

Using the technique of separation of variables, write $u(x,y,t) = X(x)Y(y)T(t)$. Substitution into the wave equation gives

$$Y(y)T(t)X''(x) + X(x)T(t)Y''(y) = X(x)T(t)T''(t)$$

$$\text{or} \qquad \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} - \frac{Y''(y)}{Y(y)}. \tag{2}$$

Since the left hand side is a function solely of x, while the right hand side is a function only of y and t and since the equation must be true for all x, y and t, each side must be equal to the same constant.

$$\frac{X''}{X} = -\alpha \qquad \text{and} \qquad \frac{T''}{T} - \frac{Y''}{Y} = -\alpha .$$

$$u(0,y,t) = 0 = u\left(\frac{\pi}{a}, y, t\right)$$

$$\text{and} \qquad u(x,0,t) = 0 = u(x,\pi,t).$$

By rearranging the equation dealing with Y and T, three ordinary differential equations are obtained:

$$\frac{X''}{X} = -\alpha \tag{3}$$

$$\frac{Y''}{Y} = -\beta \tag{4}$$

$$\frac{T''}{T} = -\alpha - \beta \tag{5}$$

where α and β are constants.

Look at (3). The general solution to this ODE is

$$X(x) = c_1 \sin\left(\sqrt{\alpha}\, x\right) + c_2 \cos\left(\sqrt{\alpha}\, x\right) \tag{6}$$

where c_1 and c_2 are constants.

Using the boundary condition $u(0, y, t) = X(0)Y(y)T(t) = 0,$ we see that $X(0) = 0$. Thus, c_2 must be 0.

Therefore, $X(x) = c_1 sin\left(\sqrt{a}\, x\right)$. It is also known that $u\left(\frac{\pi}{a}, y, t\right) = X\left(\frac{\pi}{a}\right)Y(y)T(t) = 0.$ Using this gives

$$X\left(\frac{\pi}{a}\right) = c_1 \sin\left(\sqrt{\alpha}\,\frac{\pi}{a}\right) = 0.$$

This only happens when

$$\sqrt{\alpha}\,\frac{\pi}{\alpha} = n\pi \quad \text{where} \quad n \in Z$$

$$\text{or} \qquad \alpha = \alpha^2 n^2 \quad \text{where} \quad n \in Z.$$

So finally,

$$X(x) = c_1 \sin(anx) \quad \text{where} \quad n \in Z. \tag{7}$$

Take a look at (4). This is essentially the same equation as (3) and it is not surprising that

$$Y(y) = d_1 \sin(my) \quad \text{where} \quad m \in Z. \tag{8}$$

and d_1 is a constant. Equation (5) is similar to (3) and (4), except that α and β have already been determined, $\alpha = a^2n^2$ and $\beta = m^2$. So the general solution to (5) is

$$T(t) = a_1 \sin\left(\sqrt{a^2n^2 + m^2}\, t\right) + a_2 \cos\left(\sqrt{a^2n^2 + m^2}\, t\right)$$

where a_1 and a_2 are constants.

Use the initial condition which dictates there is no initial velocity, $u_t(x, y, 0) = 0$, to find that $a_1 = 0$. Therefore,

$$T(t) = a_2 \cos\left(\sqrt{a^2n^2 + m^2}\, t\right). \tag{9}$$

Thus, the general solution for $u(x,y,t)$ is

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin(anx)\sin(my)\cos\left(\sqrt{a^2n^2 + m^2}\, t\right). \tag{10}$$

The coefficients, b_{mn}, can be found by using the initial condition that gives the membrane an initial displacement, $u(x,y,0)=f(x,y)$.

$$u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin(anx)\sin(my) = f(x,y).$$

This is a 2-dimensional Fourier sine series and, in fact, b_{mn} can be found by solving for the Fourier Coefficients of this series. So,

$$b_{nm} = \frac{4a}{\pi^2}\int_0^{\pi}\int_0^{\frac{\pi}{n}} f(x,y)\sin(anx)\sin(my)\,dx\,dy.$$

So the general solution to the two dimensional wave equation when applied to the membrane is

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}b_{nm}\sin(anx)\sin(my)\cos\left(\sqrt{a^2n^2+m^2}\,t\right) \tag{11}$$

where

$$b_{nm}=\frac{4a}{\pi^2}\int_0^{\pi}\int_0^{\frac{\pi}{n}}f(x,y)\sin(anx)\sin(my)dxdy.$$

It can be assumed that *f(x,y)* is continuous since this fits with the physical situation.

Nodal Lines

Now that an equation for the vertical displacement of a constant density membrane has been derived, what does this say about the mysterious Chladni Patterns? Well, Chladni Patterns are formed when powder or sand settles at the nodal lines of a plate which is being driven at one of its natural (or resonant) frequencies. Using this equation for the vertical displacement, let's try to identify these natural frequencies and predict the nodal lines.

In general, an infinite sum of periodic functions is not periodic. To have a periodic function, and therefore find a common period, the frequencies of each term must be rational multiples of one another.

In this case, for an arbitrary initial displacement, *f(x,y)*, the general solution will not be a periodic function with respect to *t*. The angular frequency for the n, m^{th} term is $\sqrt{a^2n^2+m^2}$, which is in general irrational. As n and m step through integer values, the frequencies are not rational multiples of one another. Therefore, the overall function is not periodic and it does not even make sense to talk about a common period. Only periodic functions admit of resonance (the whistling sound noticed by Galileo in conjunction with the patterns) and combinations with a common period are necessary to form standing waves.

Certain initial displacements do lead to solutions which are periodic functions. For example, consider an initial displacement *f(x,y) = sin(ax)sin(y)*. With this displacement,

$$b_{nm} = 4\begin{cases} \frac{1}{4}, & \text{if } n = m = 1 \\ 0, & \text{if } n \neq m \neq 1 \end{cases}.$$

Therefore, $b_{11} = 1$ and $b_{nm} = 0$ for all $n \neq m \neq 1$. This produces a very nice expression for the vertical displacement of the membrane.

$$u(x,y,t) = \sin(ax)\sin(y)\cos\left(\sqrt{a^2+1}t\right) \qquad (12)$$

This function is periodic in t with angular frequency $\omega_{11} = \left(\sqrt{a^2+1}\right)$.

Now $u(x,y,t)$ is a function which is periodic in t and governs the vertical displacement of the membrane. But what about nodal lines?

Nodes occur where the membrane is not moving, i.e, where the amplitude of the vertical displacement is zero for all time t. From (12), it can be seen that the amplitude is $sin(ax)sin(y)$. (Note that $n = m = 1$.) Where is this zero? The amplitude is zero when $sin(ax)=0$ *or* $sin(y) = 0$. This only happens when $ax = c\pi$ or $y = d\pi$. So

$$x = \frac{c\pi}{a} \quad \text{or} \quad y = d\pi \quad \text{where} \quad c,d \in Z.$$

Since the membrane has dimensions $\frac{\pi}{a}$ by π, consider c and d to be only 1 or 0. Therefore, the nodal lines are $x = 0$, $x = \frac{\pi}{a}$, $y = 0$ and $y = \pi$ and the membrane will look like a one by one grid.

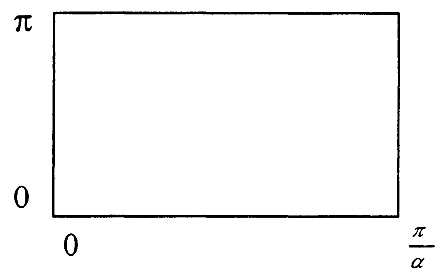


Figure 2.2: A 1 x 1 grid.

Since the boundary is fixed, it is not surprising that there is no movement along the edge.

Now consider a solution in which $n = 2$ and $m = 1$. This gives a vertical displacement of $u(x,y,t) = \sin(2ax)\sin(y)\cos\left(\sqrt{4a^2+1}t\right)$. This is a periodic function with angular frequency $\omega_{21} = \left(\sqrt{4a^2+1}\right)$.

Here the amplitude is zero when $x = \frac{c\pi}{2a}$ or $y = d\pi$ for $c, d \in Z$.

In this case, consider $c = 0,1,2$ and $d = 0,1$ because of the physical features of the membrane. The nodal lines on this membrane form a two by one grid, as in Figure 2.3.

For one final example, look at the case where $n = 4$ and $m = 5$. The displacement is $u(x,y,t) = \sin(4ax)\sin(5y)\cos\left(\sqrt{16a^2+25}t\right)$. The nodal lines appear at $x = \frac{c\pi}{4a}$ and $y = \frac{d\pi}{5}$. Here c is an integer from 0 to 4 and d is an integer from 0 to 5. The nodal lines form a four by five grid as in Figure 2.4.

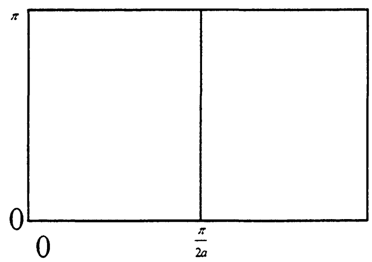


Figure 2.3: A 2 x 1 grid

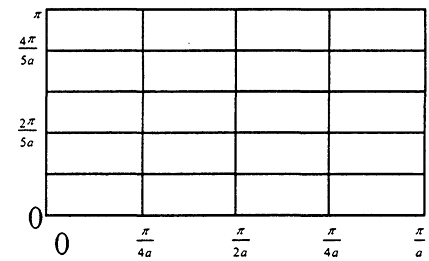


Figure 2.4: A 4 x 5 grid

In general, it is then easy to see that, if the vertical displacement of the membrane is $u(x,y,t) = \sin(anx)\sin(my)\cos\left(\sqrt{a^2n^2 + m^2}\,t\right)$, then the amplitude is zero when

$$x = \frac{c\pi}{an} \quad \text{and} \quad y = \frac{d\pi}{m} \quad \text{where} \quad c,d \in Z$$

with $0 \leq c \leq n$ and $0 \leq d \leq m$.

Using this information, it is clear that the displacement pattern (the pattern formed by the nodal lines) is in general an n by m grid.

Now that a way to predict mathematically what the nodal lines will look like has been determined, what do all of these assumptions mean in the real world? What is being examined is the case where the membrane is given an initial displacement. This particular initial displacement is chosen so that it will produce a standing wave. From this initial displacement and the standing wave which it produces, the frequency at which the membrane will vibrate is determined.

For example, in the case where the vertical displacement is $u(x,y,t) = \sin(ax)\sin(y)\cos\left(\sqrt{a^2 + 1}t\right)$, the nodal lines can be predicted.

In an ideal world, this standing wave would never dissipate and the membrane would continue to vibrate until someone stopped it. The frequency of vibration is determined by the time component of $u(x,y,t)$. Here the angular frequency is $\sqrt{a^2+1}$. This means that, disregarding air-resistance, if a membrane is given an initial displacement of $f(x,y) = sin(ax)\ sin(y)$, it will continuously vibrate at an angular frequency of $\sqrt{a^2+1}$ and produce a standing wave and nodal lines.

In general, if the membrane is given an initial displacement which produces a standing wave of the form

$$u(x,y,t) = \sin(anx)\sin(my)\cos\left(\sqrt{a^2n^2+m^2}\,t\right),$$

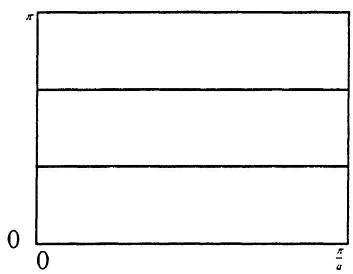
the membrane will vibrate at an angular frequency of $\sqrt{a^2n^2+m^2}$. The frequencies that you get by cycling through integer values of m and n are called the natural frequencies.

If the membrane is covered in a fine powder or sand, and then given an initial displacement which produces vibrations at one of these natural frequencies, the powder will settle into the nodal lines and patterns, of which Chladni was so fond. The displacement patterns which are depicted in Chladni patterns are called the natural modes of vibration of the membrane. Three of these modes are shown in Figures 2.2, 2.3, 2.4.

These natural modes of vibration are what Galileo noticed over 400 years ago. By scraping a metal plate with a metal chisel, he was able to vibrate the plate at one of its resonant frequencies, completely by chance. In his case, instead of sand, it was the metal filings that had been generated by the scraping, that fell into the nodal lines and formed the patterns.

Producing Chladni Patterns

In addition to working with the mathematics presented above, I also produced my own "Chladni Patterns" in the lab. (For methods, please contact me.) The physical set-up had several differences from the mathematical model. The patterns produced were on a free edge plate. The most obvious difference between this set-up and the mathematical model is that the boundary conditions are different. Another difference is that the idealized model assumes a membrane which by definition has no depth, stiffness, or volume. In the actual set-up, the plate has stiffness and an appreciable depth. The free boundary changes the problem significantly. Rayleigh describes the problem of calculating the modes of vibration for a plate with free boundary as "extremely difficult."[5] Rayleigh did find though, that the (0,*m*) mode is just like the fixed boundary case. The nodal lines are parallel to the y-axis. Likewise, the (*n*,0) mode has nodal lines running parallel to the x-axis.



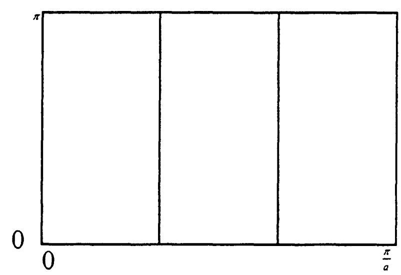


Figure 3.1: (*0, m*) Mode of Vibration Figure 3.2: (*n, 0*) Mode of Vibration

The difference from the fixed boundary case comes when the modes are combined to get the (*n, m*) mode of vibration. When this is done, the lines tend to bend toward each other and are no longer independent of one another (as in the fixed boundary membrane) (5).

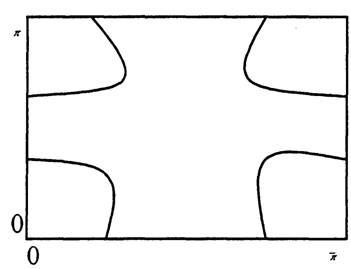


Figure 3.3: (n, *m*) Mode of Vibration

These shapes are evident in the patterns produced in the lab. (See following pictures.)

Another difference between the work done in the lab and the problem solved mathematically is that in the real world, there is air-resistance. Because of that, a plate or membrane can be given an initial condition, but it can not be expected that the plate will vibrate forever. In the ideal world of mathematics, this is exactly what is claimed. If a membrane is given a certain initial condition, it will vibrate at a natural frequency and produce a standing wave which will never dissipate. To deal with air-resistance in the lab, the plate can be *driven* at a natural frequency. When the driving force is cut off, the sound which is produced by the vibration of the plate can actually be heard. This sound does dissipate because of air-resistance and other damping forces, but it does ring for a few seconds. This sound is the same sound Galileo heard over 400 years ago when he was scraping his plate.

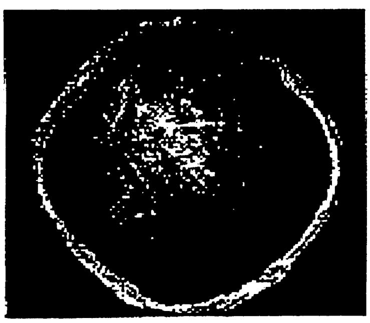


Figure 3.4: Frequency 408.1 Hz

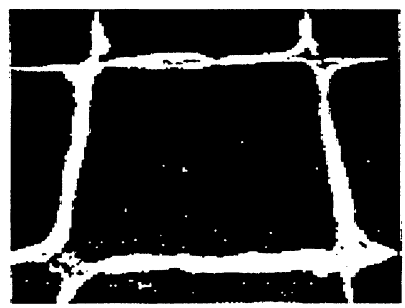


Figure 3.5: Frequency 1057.0 Hz

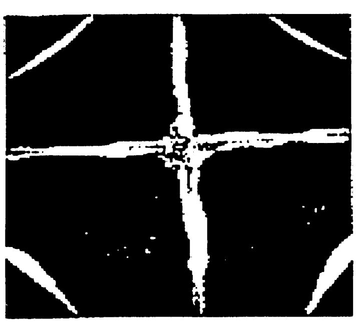


Figure 3.6: Frequency 1281.0 Hz

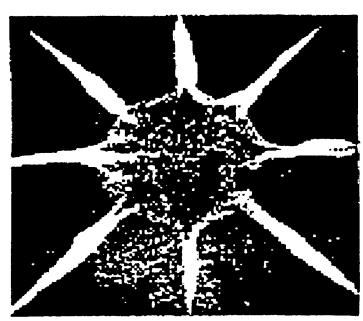


Figure 3.7: Frequency 1147.7 Hz

A Dimensional Analysis of Wave Equation

In order to verify quickly that the wave equation in one dimension makes sense, let's perform a dimensional analysis.

The variables in the wave equation and their units are as follows:

$$[u(x, y, t)] = \text{length} = l$$

$$[x] = \text{length} = l$$

$$[\,t] = \text{time} = t$$

$$\{\upsilon^2] = \left[\frac{\text{tension}}{\text{density}}\right] = \frac{\text{force}}{\text{density}} = \frac{\text{mass} * \dfrac{\text{length}}{\text{time}^2}}{\dfrac{\text{mass}}{\text{length}}} = \frac{l^2}{t^2}.$$

Let's look at the units on the left hand side of the wave equation:

$$\left[\frac{\partial^2 u}{\partial x^2}\right] = \left[\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\right] = \left[\frac{\partial}{\partial x}\right]\left[\frac{\partial u}{\partial x}\right] = \frac{1}{l} * \frac{l}{l} = \frac{1}{l}.$$

Now let's examine the units on the right hand side of the wave equation and hope that they come out to be the same as those on the left hand side:

$$\left[\frac{1}{\upsilon^2}\frac{\partial^2 u}{\partial x^2}\right] = \left[\frac{1}{\upsilon^2}\frac{\partial}{\partial t}\frac{\partial u}{\partial t}\right] = \left[\frac{1}{\upsilon^2}\right]\left[\frac{\partial}{\partial t}\right]\left[\frac{\partial u}{\partial t}\right] = \frac{t^2}{l^2} * \frac{1}{t} * \frac{l}{t} = \frac{1}{l}.$$

The units on each side of the wave equation match, so the wave equation makes sense. ■

References

[1] G. Galilei, *Dialogues Concerning Two New Sciences*, The MacMillan Co., Chicago, 1939, quoted in J. McLaughlin, "Good Vibrations," American Scientist, 86, 342.

[2] M. Waller, *Chladni Figures: A Study in Symmetry*, G.Bell and Sons, Ltd., London, 1961.

[3] L. Bucciarelli and N. Dworsky, *Sophie Germain: An Essay in the History of the Theory of Elasticity*, D.Reidel Publishing Co., Boston, 1980.

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[5] N. Fletcher and T. Rossing, *The Physics of Musical Instruments*, Springer-Verlag, New York, 1991.