REPORT ON A COURSE FOR PROSPECTIVE HIGH SCHOOL MATHEMATICS TEACHERS

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The author describes his design for a course entitled *Secondary School Mathematics from an Advanced Viewpoint*. He adds subjective comments on how his design has worked in practice.

Introduction

The School of Education and the Department of Mathematics at Brooklyn College jointly offer a program, leading to B.A. and B.S. degrees, for students who intend to teach mathematics in secondary school. These students are required to take, as mathematics courses, three semesters of calculus and one each of linear algebra, abstract algebra, advanced calculus, foundations of geometry, an introduction to probability and statistics, a one-credit course in problem-solving, and other electives. They are also required to take *Mathematics 46*, a four-credit course entitled, *Secondary School Mathematics from an Advanced Viewpoint*.

This article concerns my experience with designing *Maths 46*, which I have now taught twice. I am grateful to members of NYCETP, especially Rosamond Welchman, for many helpful discussions.

Most of the students in *Maths 46* are seniors; they are already student-teaching in Brooklyn high schools. However, when, at the beginning of the semester, I ask my students what electives they have taken or are taking, their responses show that typically they have not studied differential equations and their use in mathematical modelling, far less partial differential equations; nor, the algebra, geometry and calculus of the complex number system; nor, the theory of eigenvalues and eigenvectors of linear transformations, and their significance in applications — to name only a few of the major areas of mathematics. In the absence of these studies, my students cannot possibly have an adequate conception of what mathematics is about or how it is used in the quantitative sciences. Moreover, if they themselves do not have an appropriately

mature appreciation of mathematics, how can they inspire their students with a genuine idea of its importance and beauty?

These observations have led me to question the appropriateness, for Brooklyn College, of the prevailing view that the principal objective of a course on secondary school mathematics from an advanced perspective should be to examine the content of high school courses from the perspective of the college curriculum. An equally important objective of such a course should be to teach some parts of the college curriculum which are particularly suited to shedding light on high school mathematics, and to which the students have not yet been exposed. The main purpose of this article is to report on what I did to balance these and other objectives.

Objectives of the Course
This section consists of quotations (in edited form) from a handout distributed to the class at its first meeting.

> In this course, I want to show you some approaches to the understanding and teaching of mathematics which I hope you will be able to use by having, in my own teaching of you, an example from which you can choose to follow or diverge; and, by using your own experience as a student to give you insight into how to help the students whom you will teach.

> Ideas about curriculum, the structure of classroom space and time, and methods of assessing children's progress are changing rapidly in official circles; some of these ideas will be tried in this course. You may be expected to learn, and then teach, in ways quite different from those you are used to. Because of these innovations, as well as the usual stress and excitement that go hand-in-hand with learning, you may find yourself from time to time feeling: on the one hand, anxious, bored, confused, totally lost, inadequate, despairing, angry at me or angry at yourself; and, on the other hand, curious, enthusiastic or proud of something you've accomplished. **One objective of this course** is to give you opportunities to recognise these feelings in yourself, discuss them openly and

constructively, and then, prepare to put what you learn about yourself to the best use when you become a teacher.

THE STRUCTURE OF MATHEMATICS

I think that there is a rough hierarchy in levels of understanding of mathematics and corresponding modes of learning, just as in the case of English and other subjects. The following table illustrates this point.

	English	**Mathematics**	**Learning Mode**
Power of the discipline	Prayer, poetry, legal brief, political speech	Unity of all maths, aesthetic beauty, power of the quantitative world view	Inspiration from a teacher (or book)
Pleasure from the discipline	Reading for fun, creative writing, and speaking	Experiments in science and probability based on maths, maths-based games, puzzles	Cooperative learning, discovery by students
Competence in the discipline	Understanding a lease or instruction manual, writing a résumé, correct spelling, and grammar	Arithmetic and algebraic skills, ability to convert verbal situations into maths problems	Constant practice/ drill

I hope that in this course you will have experience of all three levels of learning/understanding maths, and of going back and forth between them. Drill is necessary to become competent enough to enjoy the pleasurable part of maths and to appreciate its power. Having an overview of maths gives you guidance in selecting drill and experiments/games so as best to direct children towards future goals.

The Design of the Course
(1) Mathematical Content

I divided the course into four modules: (a) differential equations; (b) the algebra and geometry of the complex number system; (c) the classification of congruences of the Euclidean plane; and, (d) an introduction to *Mathematica*. In choosing these topics, and in selecting the material to be presented in each, I was guided by the pedagogical

objectives mentioned above. I wanted to be able to show my students that more advanced mathematics evolves from high school maths; or, to put it another way, that high school mathematics can be presented so as to foreshadow more advanced maths. I also wanted to show them what too often fails to become apparent from an undergraduate education: that all branches of mathematics are interconnected.

For example, we can use *Mathematica* to illustrate the multiplication of complex numbers. This prepares the way for showing that complex multiplication can be performed geometrically using similar triangles. At a more advanced level, a complex-linear mapping $f(z)=az+b$, where a, b, and z are complex numbers and $a=1$, is a Euclidean congruence, consisting of a rotation about 0 through angle $arg(a)$ followed by a translation by b; we can use either algebra or geometry to show that f is a pure rotation through $arg(a)$ about some other point.

As another example, in the module on differential equations, I treated the equations $x''=0$, $x'=kx$ and $x'=kx(a-x)$. I also covered the corresponding difference equations. I like to do this because difference equations can be treated at several levels of school mathematics. They can be solved numerically. The calculation can be automated on a programmable calculator — or, in *Maths 46*, using *Mathematica*. Difference equations also lead to solutions using arithmetic and geometric sequences and their sums. *Mathematica* can also solve the logistic difference equation, leaving us the simpler task of verifying that its solution is correct.

In general, the students used *Mathematica* to enhance their solutions of problems in the differential equations and complex number modules; for example, by plotting together the vector field and a few solutions of a first-order differential equation.

(2) Teaching style

I had the students (I had fourteen the first time I taught the course, seven the second) form themselves into groups of three or four for the duration of the semester. Some of the material was presented in lecture format, some as projects to be carried out by the students in groups, and some in a mixture of lecture and group work.

The module on the arithmetic and geometry of complex numbers was presented in the form of problem-sets for the students to work on in groups. The problem-sets included a small amount of instructional material. The problems ranged from straight-forward calculations to propositions to prove based on the results of those calculations. The intention was for the students to learn the mathematics involved by a process of guided discovery.

In the module on the classification of Euclidean congruences, I gave some introductory lectures on what it means to have a classification system for congruences, and on the proof that every congruence can be expressed as the composition of at most three reflections. Then, the students worked in groups on problem-sets intended to guide them to the discovery of: (a) how to perform translations, half-turns, rotations, reflections, and glide reflections; (b) how to diagnose a given congruence as one of these types; and, (c) an introduction to the group structure of the set of all congruences.

The material on complex numbers and most of the material on congruences could be presented as a process of guided discovery because the students had sufficient background in algebra and geometry to understand on their own the concepts introduced in the problem-sets. I did not use this approach to teach differential equations to these students, who had never seen them before. They all had had experience with anti-differentiation, of course; but, to go from there to the concept of what differential equations are and how they are applied is a very big leap, greater than most students can manage without active assistance from a teacher. So, I lectured on differential and difference equations. For the same reason, I also lectured on the classification of congruences and the geometry of complex-linear functions.

When *Mathematica* was to be used, the class met in the Mathematics Department's computer laboratory, where every student had a terminal with access to *Mathematica*, version 3.0. The students varied in their familiarity with working in the *Windows* environment; and so I encouraged them to ask for and receive help from each other. Of course, I also did what I could to help them straighten out their *Mathematica* notebooks.

(3) Assessments

I assigned either homework or problem-sets with each module except the one on *Mathematica*. For each module, I chose a subset of the assignment, and required each group to submit to me their collaborative work on that subset. I told the class that each group member was responsible, if not for solving all the problems himself or herself, at least for understanding all the solutions that were submitted. All the students who worked on a paper received the same grade for that paper. My intention was to foster an uncompetitive, cooperative spirit. The students who were able to solve the problems would benefit from explaining their solutions to the other members of their groups who, in turn, would benefit from having the problems explained to them in a context where (I hoped) they would feel very free to ask questions. I also allowed groups to resubmit a paper after it had been marked, and to receive the improved grade in place of the previous one. I wanted thereby to encourage the ideas that: (a) what matters is learning the material as best possible, rather than learning what can be managed in a specified time; and, (b) that a grade is not a permanent attribute of a person.

Some of the homework required students to use *Mathematica*; sometimes, they were only encouraged to use it. I did not assign homework specifically within the module on *Mathematica*. This is because my students varied so much in their familiarity with programming, and even with the *Windows* environment (as I mentioned before) what would be a simple task for one student might be an extremely difficult one for another.

I gave a take-home midterm examination on the module on differential and difference equations. It contained the instruction that *Mathematica* must be used in some way, but I left it up to the individual student whether to do no more than graph the solution function, or to do much more. The degree to which *Mathematica* was used did not affect the grade.

The final examination was a two-hour, closed-book affair. I told the class it would consist of problems taken from a specified subset of the problems they had been assigned to submit. My idea was to reward those students who had fully participated in their groups' efforts rather than signing their names to papers without having completely understood them. *Mathematica*-related questions did not appear on the final exam.

Evaluation/Critique of the Course

My main goals for the students in the course were achieved, in that:

- They experienced working cooperatively in groups.
- They had some experience with the process of making guided discoveries.
- They were openly and overtly invited to consider my own performance as their teacher, as a model to follow in some respects and to differ from in others.
- They experienced assessment methods other than closed-book, limited-time examinations.
- They understood that differential equations (and difference equations) are useful in mathematical modelling.
- They made acquaintance with the complex number system, and reviewed some high school algebra and geometry, and polar coordinates in that context.
- They reviewed some high school geometry in the broader context of studying the group of congruences of the Euclidean plane.

Several students have told me that they had never before been required to work in groups; that, after some initial repugnance, they had discovered the benefit and pleasure of doing so; and, that they were ready to use the same idea in their own teaching. They were animated and engaged in their group work during class time, and it was sometimes hard to get them to leave at the end of the period.

My impression is that the students learned more thoroughly from group work on projects than from the lecture-homework format. This is not surprising. In the former case, they work at a speed comfortable to themselves; in the latter, at the much faster speed comfortable to the lecturer. Moreover, the kind of material which I presented as group projects was less abstract, and therefore more accessible, than what I presented in lectures.

If I teach the course again, I shall try to find ways to reduce the amount of lecturing, without sacrificing mathematical honesty. For example, I shall not teach how to solve differential equations (separation of variables is the only method I used). Instead, I shall use *Mathematica* to discover a solution, which leaves us with the much simpler task of verifying that the solution is correct. This plan will also reinforce the idea that *Mathematica* is an ally, not just another body of material to be mastered.

I regret that the process of learning by making discoveries had to be abbreviated. The students should have worked more examples before the conjecture, to which they were supposed to be led, was introduced. But a true process of guided discovery is very time-consuming, and I felt pressure to cover more mathematical content than that method would have allowed. At least my students had some experience of the situation that they themselves will no doubt encounter as teachers, struggling against pressure of time and syllabus.

Assessment Methods

(a) Homework

There was a pervasive problem with assignments being turned in late — often very late. This problem is especially pernicious in the lecture-homework format, where a timely discussion of difficulties with homework serves to clarify and reinforce the lecture material. These valuable opportunities are lost if the homework is postponed.

There were several causes for the lateness of the students' papers:

(i) My attitude toward my students was one cause. I enjoyed the collegial atmosphere of a small course, in which we were all both teachers and learners, and the excitement of trying out new teaching methods. My interest in discussing, as the occasion arose, my own teaching style as exemplified in the course, and its good and bad aspects, was, I think, novel and useful to them, if only as a demonstration that these matters may be discussed openly. But it did have the drawback of tending to blur the distinction between them as students and me as teacher when it came to enforcing deadlines on assignments. My mistake was that I did not want to spoil the good mood by playing the role of the Old-Fashioned, Authoritarian Bad Guy who insisted on firm deadlines.

(ii) For the most part, I composed the assignments one whole module at a time. This led to relatively large assignments which were due relatively infrequently, say every three or four weeks. In the future, I shall break the assignments into smaller units, with one falling due every week. Also, in the lecture-homework format, I shall specify one problem near the beginning of the assignment, which I shall require the students to prepare for oral discussion by the very next class. I shall make it clear that these discussions are not counted toward the course grade, and that a student's contribution to

the discussion need consist of no more than saying where he or she got stuck on the problem.

(iii) In most of the groups, each student would take responsibility for one aspect of the work: solving the first third of the problems, or writing up the final paper from notes and drafts. If any one student delayed performing his or her part, the whole paper was delayed. In future, I shall require each group to plan to have *something* ready to turn in on the due date, which can later be updated and improved.

Conclusion

In my view, the course *Mathematics 46* is obliged to attend to several goals, which are not entirely mutually compatible. For example, I want my students to have some personal experience of working in small groups, and of learning by guided discovery; but these modes of learning take more time than the traditional lecture format, and therefore operate counter to the objective of covering essential mathematical content. For another example, there is a tension between teaching that is directed to short-term needs of student teachers in their classrooms, and teaching that is directed toward providing a long-range perspective on mathematics.

It seems to me that in any undergraduate program directed toward training future high school mathematics teachers, there should be a course that fills a niche similar to that of *Mathematics 46*; and, that any such course will have to be designed as a compromise between various conflicting objectives similar to those I have mentioned. One major difficulty in designing such a course would be avoided if the mathematics electives offered to or required of these students gave them an adequate overview of what the mathematical enterprise is about. Then the necessity — which I, at any rate, felt —of incorporating a relatively broad advanced mathematical content into this course would be obviated, and one could focus more on pedagogical matters and an overview of high school mathematics.

My purpose in writing this paper has been to raise some issues which, I think, must be taken into account when designing a course along the lines of *Mathematics 46*. I also hope that reporting on my own experience with designing and teaching the course will be of use to anyone preparing to teach a similar course.   ∎

BIO

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