ALGEBRA AND CALCULUS FOR ALL?

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Motivation for the paper

The motivation for this paper, "Algebra and Calculus for All?", comes from three different sources. The first is a longstanding belief that I have had, that not only do algebra and calculus play analogous roles in high school and college, but almost all the issues that apply to one also apply to the other.

The second source of the title came in June 1994, when I was asked to give a talk at a calculus conference in Ann Arbor, Michigan, sponsored by the Calculus Consortium centered at Harvard University. The rhetoric being used to discuss calculus reform at that conference was virtually identical to the rhetoric that we have been using in the University of Chicago School Mathematics Project (UCSMP) since its inception. I was struck by the similarity.

The third spur for this paper came in January of 1995, when Texas Instruments announced that, by the end of 1995, they would be selling a calculator that could do algebraic manipulation – dare I say "abstract" manipulation. The TI-92 does not only the manipulations one normally finds in algebra but also the manipulations found in calculus. There had been Hewlett-Packard calculators that did this. But the TI-92 is easier to use, and cheaper, though still not inexpensive.

This was not the first technology to make us wonder about the relationships between algebra and calculus. From the first function-graphing program there have been obvious questions of how much manipulation is necessary to graph a function, or to find its maxima or minima on an interval, two tasks that have been among the major reasons for studying calculus.

The Parallel Roles of Algebra and Calculus Courses

"Algebra" and "calculus" each have two meanings. Each is part of a large area of

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mathematical thought - calculus is in the area of mathematics known to mathematicians as analysis, and algebra in schools is part of the field of mathematics that studies algebraic structures.

But these names also are strongly associated with specific courses, and I will first discuss the courses. These courses play parallel roles in high school and college in a large number of ways.

a. They are fixtures of the curriculum; the first course; a sign of arrival.
Algebra is the first course in high school mathematics. If you take algebra before 9th grade, you are considered to be taking it early. If you do not take algebra in 9th grade, you are taking remedial mathematics. Calculus is the first course in college mathematics. If you take calculus before your freshman year of college, you are considered to be taking it early. If you do not take calculus as a freshman, you are taking remedial mathematics. Thus algebra and calculus are fixtures of the curriculum, fixed in time and place as the first year usually in a new school. They are a sign that you have arrived at a new level of schooling.

b. They are prerequisites to a great deal of future work.
They are important fixtures. Algebra is the necessary precursor to the rest of high school mathematics. Whether or not you use much algebra in your geometry course, you must pass algebra to get into geometry. Calculus plays the same role for college level mathematics. You may not use much calculus in your abstract algebra course, but you won't be allowed in unless you have passed calculus.

c. They are normally offered at a variety of levels of difficulty.
Not everyone goes on to take more mathematics beyond these courses. And so there have developed various levels. Honors level. Regular level. Basic level. Courses taught with rigor for the better students. Courses taught as drill and practice for the lower-performing students.

d. They are filters.
If you are a student in the lowest of the levels, you are greatly handicapped if you wish to go on to take more mathematics. You do not have the prerequisites. If you do not perform well at the level you take, whether it be a high level course or a low level course, you will be

dropped a level. In these ways, both algebra and calculus act as filters. But they are different filters. No one wants algebra in high school to be a filter. On the other hand, calculus in college is used as a prerequisite to some majors simply to reduce the number of people who will take the major. Business majors in many places must take calculus, but many never encounter its uses while they are undergraduates even though calculus was required. Calculus is an overt filter; algebra is more covert.

e. It is normal for the courses to be taken one year earlier by the best students.
In 1955, the Advanced Placement program of the College Board began, and calculus began to be offered to the best students in 12th grade. At the very beginning, most schools tried to cram four years of high school mathematics into three so that they would have students ready for the AP exam, but after a while it became more popular to offer an 8th grade algebra course. Now the 8th grade algebra course for the best students is a fixture in most school systems. The latest evidence is that about 6% of students take a full calculus course in high school; about 20% of students are taking a year of algebra in 8th grade.

f. When given earlier, the course tends to be harder.
The AP Calculus course is not an easy calculus course. In many schools, the 8th grade algebra course is harder than the corresponding 9th grade course. There is a justifiable reason for this – the best students can handle such a course. But that means that many more students could take the course earlier if it were not made so difficult. We have ample evidence that average 8th graders can take algebra, and the best can do it at 7th grade. In some other countries, 7th graders normally study some algebra and 11th graders begin calculus. Although algebra in the 8th grade is being taken by an increasing number of students, it is still not standard practice. But it needs to become standard practice, because this is the only way that students have time to learn the statistics and discrete mathematics and other topics that they now should have before entering college.

g. Some people think that the ideas can be learned much earlier than this.
And there are those who believe even 7th grade is far too late. Henry Borenson touts his "Hands-On Equations" for 3rd, 4th, and 5th graders, the same grades in which Project SEED has been teaching algebra ideas for a couple of decades. And for many years I have had materials called "Calculus for Seven-Year-Olds", written by Don Cohen of Champaign,

Illinois. Thus, although the courses have fixed places in the traditional curriculum, there are many who believe that the ideas could be taught much earlier.

h. Hurdles are in place to discourage early work with the subject.
However, there are hurdles in place to discourage early work with the subject. When the student comes to a high school having had an algebra course, the student is examined closely. He or she is very likely to be given a difficult test with a good probability that the score will not be high enough to pass out of the algebra course. The school does not trust others to teach algebra. The school does not want to consider the fact that when you take a course from one teacher and then have to take a final exam months later written by someone else, you are unlikely to score as high as if you had taken the course from the exam-writer. The same situation applies in colleges. The colleges do not trust the high schools to teach calculus. A placement test is given that is not necessarily over the content of the courses students have had, and the results are used to justify not allowing the student to place out of a semester or two of calculus - and often used to put the student back a year or two.

The similarities I have mentioned between algebra and calculus have almost nothing to do with the content of the courses. Now let me move to the content, to the mathematics itself.

Parallel Views of the Content of Algebra and Calculus

We have been taught for all our lives, and our parents, and their parents were taught as well, two statements that we were to take as axioms about mathematics. The first was: *Mathematics is abstract.* And we have been taught that there are levels of abstraction. Algebra is more abstract than arithmetic. The concept of "variable" is viewed as a major step up from arithmetic, which is viewed as concrete. The concept of "limit" is viewed as a step up from algebra. Calculus is more abstract than algebra.

a. Algebra is more abstract than arithmetic; calculus is more abstract than algebra.
This view of mathematics gives unsuccessful students an out. They can say that they learn concretely, that they are visual learners, that they need practical examples, and that mathematics just doesn't fit the bill. Because mathematics is abstract, we were also led to believe: *Mathematics is difficult.* And there are levels of difficulty.

b. Algebra is more difficult than arithmetic; calculus is more difficult than algebra.

The ideas about abstraction and difficulty are so ingrained in the psyche of many mathematicians and mathematics educators, that often when mathematics is not taught abstractly, or if it is taught so that students do not have difficulty, then it is not considered mathematics, or as good mathematics.

c. If algebra is not taught abstractly, then it is not good algebra; if calculus is not taught abstractly, then it is not good calculus.

d. If algebra is not difficult, then it is not good algebra; if calculus is not difficult, then it is not good calculus.

Those who are unsuccessful in mathematics can take refuge in their ignorance because mathematics is not supposed to be able to be learned by all. Those who are successful in mathematics bask in the glory of having succeeded where only a minority achieve success, wallow in the pride of having knowledge that only a few are granted the opportunity to acquire. Thus, as a consequence of this, society concludes:

e. Algebra is not for all students; calculus is for even fewer.

Because algebra is more abstract than arithmetic, many elementary school teachers avoid algebra entirely, and many high school teachers want it that way. They would prefer that a student come in without having had any algebra so that they can teach the student from the beginning. And so it is with calculus. The concept of infinite processes leading to limits is viewed by many calculus teachers as a very difficult idea. Calculus teachers are notorious for wanting high schools not to teach any calculus – not even introductions. And some high school teachers go along with this and do not introduce calculus ideas for fear of doing something wrong. A very common belief thus arises.

f. Introductory work in these areas without a full course is not wise except for the small percentage of very bright students.

An Alternate View of Algebra and Calculus

In 1623, Galileo wrote: "Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth." (Galileo, *The Assayer, 1623*)

At the time of this quote, algebra was in its infancy, and analytic geometry had yet to be invented. There was no probability theory. The basic vocabulary and symbols of calculus would be first introduced about sixty years later by Newton and Leibniz. The f(x) function notation, the symbol for π, and the abbreviations sin and cos would not come into the language until Euler over 100 years later. The invention of statistical displays – bar graphs, circle graphs, and the like – were even further in the future, and statistical theory was nonexistent.

The language Galileo wrote about was only a small piece of the edifice that today we call mathematics. The language available to Galileo described only part of the physical world. Today's mathematical language underlies financial dealings worldwide, describes a wealth of characteristics of all sorts of phenomena, is integral in high-speed communication through words and pictures, and models far more of the physical world than was available in Galileo's time.

All of my curriculum work, in which I first examined courses built around concepts of pure mathematics and mathematical systems, and later with applications of mathematics, has convinced me that mathematics is not inherently the most abstract of subjects, but made so; that it is no more abstract than English or any other language. It has also convinced me that mathematics is not inherently the most difficult of subjects, but made so; it is no more difficult than learning a new language. Furthermore, just as it is more difficult to learn a language when you are older, the very delays that we think make it more likely that children will succeed in learning algebra and later calculus make the subjects more difficult. But notice, and it is very significant, that the discussion now has turned from the *courses* of algebra and calculus to the *areas of mathematics* in which algebra and calculus are customarily the first courses.

What Have We Learned about Algebra?

Why do I feel as I do?

a. Algebra starts earlier than its formal study, whether we want it to or not. The equation $3 + \square = 7$ is algebra; the use of a square is no different than the use of a letter. So 1st or 2nd grade students do algebra problems, we just don't tell them–perhaps because we don't want to scare the teachers. For many decades, formulas like $I = prt$ or $A = \pi r^2$ have been studied prior to algebra. And now almost everyone introduces students to graphing, to equation-solving, to properties like the distributive property, before the formal course called algebra.

b. The use of applications concretizes algebra, motivates it; makes it easier. We know that algebra can be approached theoretically, such as through field properties, but this approach does not work with many students. On the other hand, we also know that we can approach algebra through formulas, and through generalizations of patterns, and that this approach does work. It isn't automatic, it doesn't come in one day. But situating the algebra in contexts which give reasons for studying the subject at the same time that they illustrate the concepts changes one's view of algebra forever. I know there are many here who could not return to the way they used to teach algebra.

c. We have found no age cut-off with respect to the learning of variable. We should not have been surprised. Variable is supposed to be an abstract concept, but variables are introduced quite early in some countries' curricula, such as Russia, and seem to be easier to learn early. Is the use of a letter such as A for area or x for an unknown any more abstract than the use of the symbol p for the sound "puh"? Surely there is an age cut-off; babies will not learn variables. But at the secondary level, from grade 6 or 7 up, it seems that the earlier the better.

d. There *are* prerequisites to learning algebra. I do not wish to be interpreted as believing that you can just go in and learn algebra. If a symbol is to stand for a number, as variables usually do, you have to know something about numbers. You have to know that a number can be represented in many ways, that 6 can be written as $\sqrt{36}$ and as $\frac{12}{2}$ and as 6.000, and that if $x = 6$, then x can be written in any of those ways, too. You need to know what it means for one number to be close to another in value. If you have the expression x + y, you need to know what the + sign means independently of what the numbers x and y are. We should work on these meanings before the concentrated study of algebra because such work is necessary for success in algebra.

e. The distinction between arithmetic and algebra is often hard to make.

Suppose you are working with a spreadsheet. You instruct the computer to take the numbers in cells A2 and B2 and put the sum in cell C2. So you write =A2+B2 in the cell C2. Are you doing arithmetic or algebra? Technology has blurred the distinction between the two. While working with spreadsheets and graphics technology, students use algebra without realizing it.

The distinction was blurred, however, well before there were computers. Consider any ABC. Is A a variable? It is in the sense that it may stand for any point. But we think of it as a specific point because it is such a familiar idea.

When I studied algebra, plotting points on the coordinate plane was considered algebra. So was any work with negative numbers. We never did one day of that work before algebra. Yet we have found graphing and negative numbers to be easy when it is contextualized, and so we are comfortable today not calling it algebra. Thus virtually all middle school or junior high school students are learning algebra today, whether or not they take a formal course with that name. Thus we must conclude:

f. Virtually all students can learn algebra.

Does What We Know about Algebra Apply to Calculus?

Do these statements and does this conclusion apply to calculus? Could we replace "algebra" with "calculus" in the statements above? True or false?

a. Calculus starts earlier than its formal study, whether we want it to or not.
b. The use of applications concretizes calculus, motivates it; makes it easier.
c. We have found no age cut-off with respect to the learning of limits.
d. There are prerequisites to learning calculus.
e. The distinction between algebra and calculus is often hard to make.
f. Virtually all students can learn calculus.

Let us take the statements one at a time.

The first introduction to limits that students see is certainly not in calculus. Infinite decimals involve limits. In geometry, students see limits of inscribed and circumscribed polygons for approximating π. For volumes, there is Cavalieri's principle which suggests the idea of finding volume by summing very thin slices. The volume of a sphere is the sum of the volumes of infinitely many pyramids. There are the successive approximations of solutions to equations, and of course there is rate of change. So the ideas of calculus start earlier than its formal study, whether or not we want that to be the case.

Most newer calculus projects make strong use of applications for exactly the same reasons that we use applications in all of our secondary school mathematics courses. The authors of these materials know that calculus can be approached theoretically such as through ϵ-δ definitions, but they also know that such an approach does not work with many if not most students. The use of applications does concretize calculus, motivate it, and make it easier to learn.

There are prerequisites to learning calculus. Students must be familiar with slope and rate of change from algebra; with area and volume formulas from geometry; with summation notation; with function notation. They must be able to chunk algebraic expressions so that they think of $f(x+h) - f(x)$ as a single number, and not as some undecipherable expression. And very importantly, they must learn to think of a function as a single object that can be represented in various ways, for instance, by a graph, by an equation, by a rule. They must be able to distinguish the function f from its values f(x).

Technology has blurred the distinction between algebra and calculus. Suppose you are asked to determine the maximum point of the function V, where

$$V(x) = x(20 - 2x)(24 - 2x).$$

Some of you may find this function V to be familiar. It is the volume of the rectangular surface formed by cutting out squares of size x from a 20 X 24 rectangle, and then folding up the sides. If you use the trace function on an automatic grapher to estimate this maximum, are you doing arithmetic, algebra or calculus? There is an algebraic expression which is being evaluated, but the grapher is doing the calculation automatically, and all you see are the arithmetic values. Yet you are working on a calculus problem.

All algebra teachers in all algebra courses have students who solve algebra problems without doing formal algebra. They use arithmetic. So it should not surprise you that calculus problems can be solved without formal calculus.

In the algebra of even a decade ago, we might have multiplied the three factors of V(x) to represent V(x) as a polynomial so that we could differentiate V with respect to x more easily. But if we can find the answers to max-min problems without formal differentiation, we don't need that manipulation. Even before the existence of a symbol manipulator, the technology took away our motivation to do some of the manipulation in algebra.

But you do not need to be using computers to not know whether you are doing calculus or some other subject. When calculus ideas are used in geometry, or infinite sums are shown for an infinite decimal, the ideas of calculus lurk but few students realize it. And we have found no age cutoff with respect to those learnings.

Can all students learn calculus? It is hard to believe they can't, if the approach is reasonable. Should all students take the course called calculus? That is a different question, one beyond the scope of this talk.

Algebra/Calculus K-12 and the Question of Integrated Curricula

Today, as soon as one mentions the algebra *course* to some people, they cringe. Mathematics knows no strict boundaries, they say. There should not be a full year devoted to algebra. People have said this about calculus, too.

As many of you know, I have either authored, co-authored, or actively edited virtually every page of the UCSMP texts and teacher editions. Before UCSMP began, I had authored or co-authored four texts for each of the years of high school. All these courses tended to integrate areas of mathematics that were previously often separated: the integration of transformations and matrices into algebra and geometry; the integration of applications into all courses; the consideration of both algebra and geometry in the first four UCSMP courses; the combining of functions and statistics in *Functions, Statistics and Trigonometry*; the interplay of the discrete and the continuous in *Precalculus and Discrete Mathematics*.

Yet I have been uneasy with those who believe that our curriculum should be decompartmentalized. One reason I have been uneasy is that I believe the courses I have worked on to be more integrated than many of the curricula that other people call integrated, including the curricula of Japan or Russia or the curricula written to follow the New York State Regents exams. To me, an integrated curriculum must have connections not only within lessons and chapters and units, but also between and among the units. One of the major developments in mathematics early in this century was the demonstration that all mathematics, from arithmetic to geometry, from sets to calculus, could be considered as part of one logical system. Consequently, covering a bunch of unrelated mathematics topics in a given year without relating them is not integration, but disintegration. It destroys one of the most basic principles underlying the field of mathematics.

Still, there is an important reason for not having the traditional one-year algebra course. The most powerful argument against the *traditional* year course in algebra is that only the best students really understood what they were doing when they finished. Almost no country in the world tried to do what we tried with first-year algebra; that is, to take students from a position of almost no familiarity with the subject to become familiar with some of the most complicated aspects of manipulation of the symbols – all in a single year. This practice does result in failure for a significant number of students. But we have learned how to avoid most of the failures:

(1) Introduce students to the main ideas of algebra well before making them central in a course.

(2) Develop algebraic ideas in context, not as symbols without meaning.

(3) While concentrating on algebra, involve all other mathematics both as motivation for doing the algebra and as avenues for application.

(4) Do not expect the most difficult ideas to be learned in one year but return to them again and again as often as needed.

The first three of these were not characteristics of the traditional way in which we approached algebra. And, in many schools, these principles are still not followed. If you have that traditional view of how algebra is taught, you are correct to want to change it. In our work with UCSMP we have tried to follow all four principles.

And, in the UCSMP curriculum, we have applied these principles to calculus. The main ideas of calculus are introduced over six years. Some of the other new curricula are beginning to do the same. The following concepts are those which need to be done over a minimum of three years:

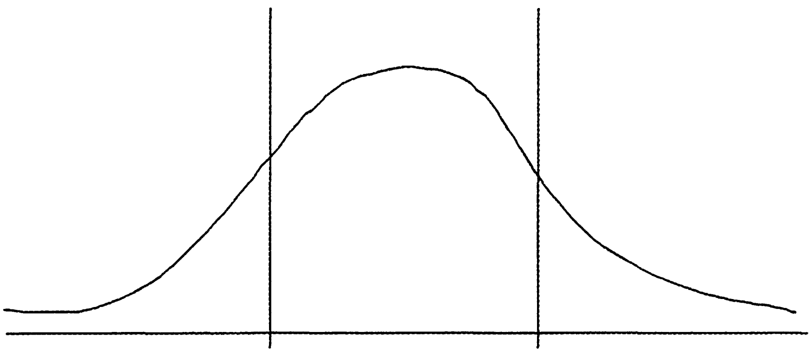
Idea
Inequality
Distance with coordinates
Area
Rate
Infinity
Rate of change
Sequence
Function
Limit
Areas on coordinate systems
max-min
Σ

And these two topics should be introduced at least once before the study of a formal calculus. They are too important to delay:

derivatives
integrals

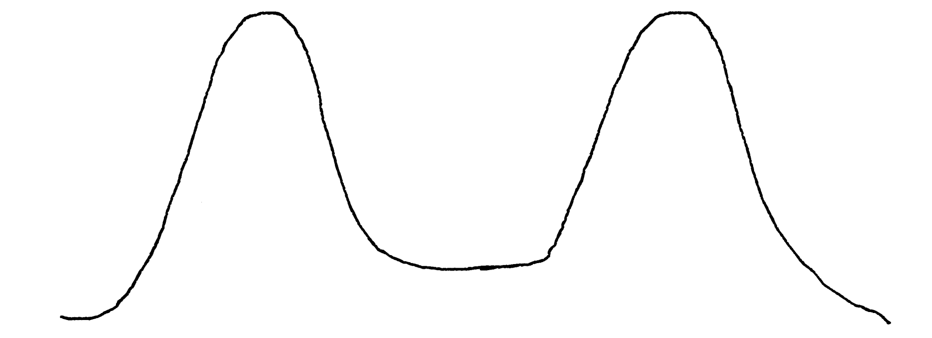
Virtually all of these ideas should be developed in context, and except for the derivative and integral they should be developed more than once before the student studies a formal calculus course. I think this is an optimal way to approach to prepare students for calculus.

In general, one might say that the normal approach is the best approach for studying an area of mathematics. The area should have a time when it is the center of attention, but both before and after that it should be examined, too.



center of
attention

There is a moral to this for calculus. The present-day year-long or 1.5-year-long calculus course is unwise if students have none of the course before. But, if attention is given to the major concepts in many of the preceding years, then it is a good idea to put all those ideas together, synthesize them, show how they are all related.



concentrating on a topic twice

Algebra is so important that it is customary to concentrate on it twice, in a first year course with major thought to variables in expressions, and in a second year course with more attention to expressions in functions. A bimodal normal curve represents this approach. In this approach, we don't give up algebra in between the modes, or before or after the years in which there is concentration. I think that some of the newest integrated curricula agree with this idea, but want many more modes - perhaps one mode for algebra each year. Maybe calculus should be done that way, too. The jury is out on this.

Regardless of your view about integrated curricula, the best approach is to develop an area over many years. Then, when the time is ready, the area should be studied in some detail so that all of those ideas that were done separately over the years can be seen as related to each other, so that logical connections can be made, so that students can see what is known and what is not known about the area, what problems are simple, and what problems are quite difficult. While this is being done, other areas should not be neglected. Calculus and geometry and probability and statistics are passengers on the algebra train, but when the calculus train comes, it needs its own passengers - some algebra, geometry, probability, and statistics from earlier years, but also differential equations, complex variables, algebraic structures that students might encounter in later years. Students' lack of early exposure to these later topics in early undergraduate mathematics is surely one of the reasons for student difficulty with them in later undergraduate mathematics.

As an aside, all of this applies to geometry. To introduce geometry with a full-year course is silly. It takes a while to learn concepts like measurement, similarity, congruence, and transformations. That is why a good curriculum pays strong attention to geometry both before and after the one year in which geometry is the centerpiece. That attention is absolutely necessary if you wish students to perform well.

The Question of Manipulative Skill

So far, this analysis of algebra and calculus has dealt primarily with the properties, uses, and representation dimensions of understanding mathematics, what some people have called the conceptual understandings of mathematics. Now let us turn to the question of manipulative skill.

We cannot ignore the availability of symbol manipulators. To do so is like ignoring available paper-and-pencil algorithms. By a symbol manipulator, what I mean is technology in which you can input ax = b without specific values for a and b, and get x = b/a as output; in which you can put two rational expressions to be added, subtracted, multiplied, or divided, and get the sum, difference, product, or quotient in lowest terms as output; in which you can input a polynomial of virtually any degree and get its factored form either over the rationals, reals, or complex numbers as output; in which you can input an equation for a function and get its derivative as output; in which you can input a definite integral and obtain its value or input an indefinite integral and obtain an expression for it. Symbol manipulators were available on large mainframes even 20 years ago; they became available for smaller machines in the early 1980s; they became available in user-unfriendly form on calculators at least a half-dozen years ago; and now they have become available in user friendly form.

Many of the skills in algebra are important regardless of the later mathematics one wishes to study. Squares and square roots are found in geometry and statistics and physics. Linear and exponential equations and functions are found here, there, and everywhere, as the song goes. Applications can be found for polynomials, too, and we have them in our books. And there are applications of rational expressions and rational functions, but not as many of these, and they are not so elementary, so we should not teach them in first or second year algebra but delay them until the year before calculus. But much of the manipulation that has been in traditional algebra courses has nothing to do with these kinds of situations; it is manipulation contrived so that the student can do the further contrived manipulations that will be encountered in calculus. As an example, consider this question from a traditional algebra course:

$$\text{Simplify} \qquad \frac{x^2-5x+4}{3x^2-13x+4}$$

Why was it in traditional courses? Mostly because years later students might encounter the following question in a calculus course:

$$\text{Find} \qquad \lim_{(x\to 4)} \frac{x^2-5x+4}{3x^2-13x+4}$$

At the time of calculus we might argue that a student ought to be able to factor a quadratic by hand, but that does not give much reason for doing it in first-year algebra. When we built our curriculum we could not justify this kind of manipulation until the pre-calculus course.

When studying rational functions, there is for the first time more than a contrived reason for having such manipulation.

But it's not clear to me that students of the future need to be able even to do this kind of manipulation by hand. The purpose of the calculus exercise is not to factor the numerator and denominator; it is to notice that there are ways to analyze limits even when both numerator and denominator of a fraction are 0 at the limit. Once we know that, we can ask technology to do the factoring. The advantage of asking technology to do the factoring is that we concentrate on the idea rather than on the technique, on the end rather than the means. Without the technology, we may feel limited in the types of expressions that can be dealt with by this technique. The technology shows us the power of the technique.

One of the interesting things that technology does is make us aware of techniques that we may have not have ever been taught, of if taught, we may not have mastered. We can obtain partial fractions with ease whenever we wish and now ask the important question: why do we want partial fractions? Many students learn to separate out a rational expression into partial fractions, but have no idea why they are doing that. It is no different than a student who struggles with long division and then, after mastering the algorithm, is not able to recognize the situations in which division is needed. No student should leave an algebra course without being able to recognize situations in which algebra is appropriate. An adult of the future might ask: Why did *I* take algebra? But no adult should ever ask what algebra is good for.

The appearance of inexpensive technology that does algebra – and we do expect that the technology will become cheaper – will enable us to concentrate even more on the whys of mathematics. We will be able to teach students why algebra is important and why calculus is important without being burdened by the need to teach them how to deal with every expression or function. And we will no longer have to say that you have to learn this or that paper-and-pencil skill because it is needed for calculus.

We must not forget that to the general populace today, complicated algebra and even easy calculus are viewed as the province of an elite minority of society. Will you tell your friends that you went to a conference to learn about the teaching of algebra, geometry, statistics, and calculus? You know what many of your friends think. Either they view you as a genius or they view you as weird. Take heart in that this is the way that people who knew arithmetic were viewed 500 years ago. The technology algorithms make algebra and calculus automatic just as our long multiplication and long division algorithms have made arithmetic automatic, and they will in the long run cause algebra and calculus to be available to all, and much less threatening.

Messages

What are the messages for the algebra student and the algebra teacher of today? The first is that algebra is a language with a set of skills, properties, uses, and representations that is not learned in a single year or two. Like other languages, it takes many years to become fluent in it, and even students who have studied it for four or five years may not achieve that fluency. Learning the language should start in elementary school, even by 6th grade you are still at the early stages, in *Algebra* you first visit the country in which this language is native, in *Geometry* you apply that language, in a second year of algebra you revisit the country, and then in pre-calculus and statistics and discrete mathematics and throughout your study of mathematics and its applications you use this language, continually learning more about it through your use.

The second message is that when you concretize algebra, when you and your students discuss its applications, when you discuss it at all, you are transporting your students into the land of algebra in order to make them more fluent in the language. But if you just treat the subject abstractly, without context, then you are acting as if algebra is a dead language, and then there is no land to which you can now transport your students.

The third message is that students must learn how to use today's technology, just as they learned how to use yesterday's. Yesterday's technology included the use of tables for squares and square roots, logarithms, and trigonometric functions, and all sorts of algorithms for rewriting expressions and solving equations and inequalities. Today's technology incorporates the tables and has new algorithms. Yesterday's technology is much more difficult to use than today's, but today's is not trivial. It too has to be taught.

Today's symbol manipulator technology acts like a dictionary for the language. It tells which expressions mean the same thing. It indicates synonyms for complicated expressions, and because we are dealing with mathematics, you have the opportunity to show how one creates synonyms. Learning a foreign language is very difficult if you have no dictionary because people keep using words you do not understand and because you are always unsure of how certain words are spelled. In an analogous way, the technology will help students learn algebra. The interesting outcome from having so much technology around is that our courses become simultaneously both more conceptual *and* more applied.

There will be teachers who don't want their students to have the technology. They will say that the student no longer has to work, that the student no longer has to think. They do not understand that, like any language, algebra exists for many reasons, and the purpose of teaching algebra is so that students can communicate, reason, and problem-solve in the language, not so that they can find synonyms or learn to spell. Obviously students should not have to look up every word in the dictionary, but they must learn how to use the dictionary.

The fourth message is that the previous messages also apply to the learning of calculus. When we teach inequalities, distance, area, rate, infinite decimals, slope, sequences, functions, limits, summation notation, maximum or minimum problems, we are engaged in the first stages of the learning of calculus. We should not shy away from making the connections, because the learning of calculus also requires many years for its fluency. And we must use the technology.

The fifth and last message is that, with these new developments, for the first time it is reasonable to believe that the ideas of algebra and calculus *can* be learned by virtually all students. But no person will believe that these ideas *should* be learned by all students unless he or she is convinced that the subject is important enough to warrant such a hallowed position in the schooling of students. Both the natural universe and that part that has been made by humankind may be written in the language of mathematics, but if we mathematics teachers do not teach the connections to that universe, we cannot expect others to do it for us. The developments of the last generation are a challenge to all of us, but they make the teaching of mathematics in this generation as exciting as it ever has been, and for the first time they have made the important ideas of algebra and calculus accessible to all. ■

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