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DELAY DIFFERENTIAL EQUATIONS AND THEIR APPLICATION TO MICRO ELECTRO MECHANICAL SYSTEMS

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

Delay differential equations have a wide range of applications in engineering. This work is devoted to the analysis of delay Duffing equation, which plays a crucial role in modeling performance on demand Micro Electro Mechanical Systems (MEMS). We start with the stability analysis of a linear delay model. We also show that in certain cases the delay model can be efficiently approximated with a much simpler model without delay. We proceed with the analysis of a non-linear Duffing equation. This model is a significantly more complex mathematical model. For instance, the existence of a periodic solution for this equation is a highly nontrivial question, which was established by Struwe [3]. The main result of this work is to establish the existence of a periodic solution to delay Duffing equation. The paper [7] claimed to establish the existence of such solutions, however their argument is wrong. In this work we establish the existence of a periodic solution under the assumption that the delay is sufficiently small.

Chapter 1

Introduction to modeling MEMS via delay equations

Micro Electro Mechanical Systems (MEMS) are nano devices which transfer electromagnetic signals into mechanical vibrations. MEMS are widely used in various sensors, including smartphones, inkjet printers, accelerometers in cars (airbag deployment), game controllers, projectors etc. Typically, every MEMS responds to the electromagnetic signals in its own way. Therefore, since different applications can require different design parameters, the utility of a particular MEMS is often limited. Furthermore, the MEMS performance is usually not deterministic, but is subject to various discrepancies caused by process variations, packaging stresses, thermal drift, energy losses, and various sources of noise. Identically-fabricated MEMS do not perform identically, and their performance can vary by tens of percent. Prior efforts to compensate for such process variation include post-fabrication mechanical or electrical tuning. Nevertheless, dynamic and accurate means of tuning effective mass, damping, and stiffness are yet to be proposed. The motivation behind the current project is modeling and studying

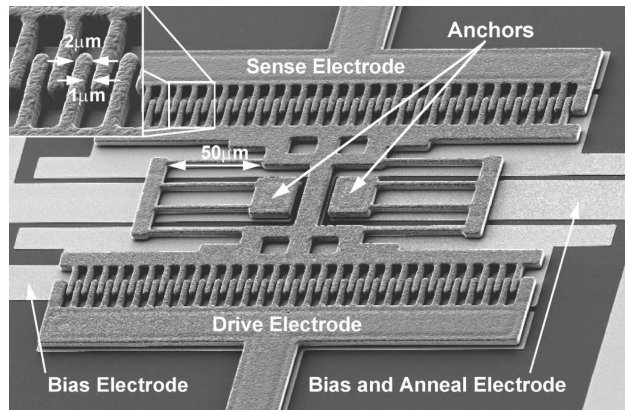


Figure 1.1: MEMS (figure from Wikipedia).

the properties of POD (Performance on Demand) MEMS, whose response to the electromagnetic signals can be controlled by the user. By being able to easily adjust the properties of these MEMS, we aim at creating a universal MEMS device with a larger

range of applications as compared to the conventional MEMS. Furthermore, improving the performance of MEMS often requires pushing the limits of lithography, materials, and structural mechanics. The control technology proposed in this project greatly improves the dynamics of MEMS. This performance will expand the utility of MEMS far beyond what is available to date.

We start with second order linear ordinary differential equations (ODE) with time delay. Delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Delay comes from the fact that it takes some time for the signal to arrive from the controller to the device. Let a sinusoidal electrostatic driving force $F_{dr} = F_{dr,0}e^{i\omega t}$ be applied to the prove mass through the comb drives, where ω is the drive frequency and $F_{dr,0}$ is the amplitude of the drive force. Electrical tuning of stiffness, damping, and mass is applied by electrostatic feedback forces that are proportional to displacement, velocity, and acceleration. Thus the equation of motion is

$$M\ddot{x} + D\dot{x} + Kx = F_0 \cdot \cos(\omega_0 t) - D_e \dot{x}(t - \tau) - K_e x(t - \tau) \quad (1.1)$$

which is an example of a second order linear delay differential equation with constant coefficients.

This thesis is organized as follows. In Section 2 we will start with understanding the stability properties of the linear model with delay, which may be found, e.g. in [1]. We will then focus our attention on nonlinear models, such as Duffing equation. Surprisingly enough, even the existence of a periodic solution in this case is a highly nontrivial problem, which was done, e.g. by Struwe [3] using a variant of a fixed point argument. We will summarize this idea here in Section 3. The presence of delay in the nonlinear model makes it even more challenging. We were surprised to find out how little is known about the classic Duffing equation with delay, despite its importance to applications, such as modeling MEMS devices. One result which we found [5], decribed in Section 4 below, deals with the stability of the periodic solution for delay Duffing equation with small nonlinearities. However, to the best of our knowledge the comprehensive analysis of the delay Duffing equation (with not necessarily small nonlinearity) is missing. We were able to find one result [7], where the authors claimed to had established the existence of periodic solution to such equation. However, we found a mistake in their argument. Therefore the main result of this work, presented in Section 5, is Lemma 6, which “saves” the argument of [7] and thus leads to the existence of a periodic solution for delay Duffing equation.

Chapter 2

Stability Analysis for Linear Delay Differential Equations

Our goal in this section is to understand for which parameter, the equation (1.1) is stable or unstable. This is done in the professor's Misiats paper [1]. To start with, let us recall the stability with no delay.

2.1 Stability with no delay

EXAMPLE 1. Consider

$$\ddot{u} + D\dot{u} + Ku = 0 \tag{2.1}$$

The characteristic equation is

$$\begin{aligned} \lambda^2 + \lambda D + K &= 0; \\ \lambda_{1,2} &= \frac{-D \pm \sqrt{D^2 - 4K}}{2}; \end{aligned}$$

If $D^2 - 4K \geq 0$, the equation is stable if $-D + \sqrt{D^2 - 4K} \leq 0$, or $K \geq 0$

If $D^2 - 4K \leq 0$, the equation is stable if $D \geq 0$. Hence, we conclude that the equation is stable if and only if $D \geq 0$ and $K \geq 0$.

2.2 D-subdivision method

We now consider the stability of Delay Differential Equations of type (1.1). We will analyze their stability by using so called D-subdivision method.

Zeros of the characteristic equation for (1.1) with a fixed delay τ , obtained from substituting $x = e^{\lambda t}$ into the homogeneous version of equation (1.1), are continuous functions of the coefficients. Let us bisect the space of coefficients into regions by hypersurfaces points of which correspond to quasipolynomial having at least one zero on the imaginary axis (the case of $z=0$ is not excluded). Such a partition is called D-subdivision. The

points in each region for such a D-subdivision correspond to quasipolynomials with the same number of zeros with a positive real part (counting multiplicity). This is true since a change in the number of zeros with a positive real part can occur only when the zero passes through the imaginary axis, that is, when a point at space of coefficients passes through a boundary of a domain of a D-subdivision.

Thus, for each domain U_k of D-subdivision, there is a number k which is the number of zeros with positive real parts of the quasipolynomial that is determined by the points of this domain. Among the domains of this D-subdivision are regions u_0 (if they exists) that correspond to quasipolynomials that have no roots with a positive real parts. These regions are regions of asymptotic stability.

EXAMPLE 2

$$\ddot{x} + a \cdot \dot{x}(t-1) + b \cdot x(t-1) = 0 \quad (2.2)$$

We find the domain of stability in the space of coefficients a and b . The characteristic equation is

$$\lambda^2 + (a\lambda + b) \cdot e^{-\lambda} = 0 \quad (2.3)$$

Note that this transcendental equation has infinitely many solutions, which makes the space of solutions of the delay equation infinite dimensional. This is a drastic difference to the case with no delay, where the space is only two dimensional.

If we start with looking for real solutions for the equation (2.3), we may notice that if $b < 0$ the equation $\lambda^2 = -(a\lambda + b) \cdot e^{-\lambda}$ always has a positive root $\lambda_0 > 0$, which automatically makes the equation unstable whenever $b < 0$.

We now look for pure imaginary solutions of (2.3) i.e. $\lambda = iy$ since these solutions form a borderline between stability and instability. Since, $e^{iy} = \cos y + i \sin y$, we have:

$$-y^2 + (aiy + b)(\cos y - i \sin y) = 0.$$

Separating real and imaginary parts, we have

$$\begin{cases} -y^2 + a y \sin y + b \cos y = 0; \\ a y \cos y - b \sin y = 0, y \neq 0; \end{cases}$$

or

$$\begin{cases} a = y \sin y, \\ b = y^2 \cos y, \text{ for } y \in (0; \infty) \end{cases}$$

This is a system of parametric equations in y . By plotting the curves for a and b , along with the necessary condition $b \geq 0$ for stability, we are able to distinguish the region U_0 , U_1 , U_2 etc. in the (a, b) plane, with 0, 1, 2 etc. roots with positive real part. In particular, U_0 is the stability region.

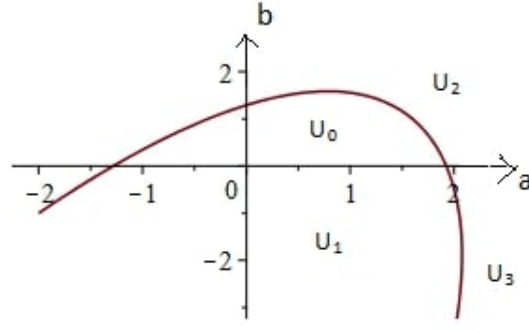


Figure 2.1: Stability domain.

2.3 Application to Linear Models of POD MEMS

The oscillations of POD MEMS can be accurately modeled by

$$F_{dr} = M\ddot{x}(t) + D\dot{x}(t) + Kx(t) + M_e\ddot{x}(t - \tau) + D_e\dot{x}(t - \tau) + K_ex(t - \tau). \quad (2.4)$$

Here M , D and K are positive constant mass, damping and stiffness of the device (standard MEMS model), M_e , D_e and K_e are the input from the user (i.e. the parameters which we can control), and τ is the delay. Note that the presence of M_e , D_e and K_e is what distinguishes a usual MEMS model from POD MEMS model. We also assume that the driving force is oscillatory, i.e. $F_{dr} = F_0 \cdot e^{i\omega t}$.

In this section we will investigate the stability of (2.4) as well as derive the existence of so called effective equation. We will follow closely the ideas from [1]. By effective equation we will mean the equation without delay, which yet shares the same features as the equation with delay.

We can write the general solution of (2.4) as

$$x(t) = X^{TR}(t) + X^{SS}(t)$$

where $X^{TR}(t)$ is the general transient solution, which solves the homogeneous equation

$$0 = M\ddot{x}(t) + D\dot{x}(t) + Kx(t) + M_e\ddot{x}(t - \tau) + D_e\dot{x}(t - \tau) + K_ex(t - \tau) \quad (2.5)$$

and $X^{SS}(t)$ is the steady - state (particular) solution of (2.4).

Definition 1. *The equation*

$$F_{dr} = M_{eff}\ddot{x} + D_{eff}\dot{x} + K_{eff}x \quad (2.6)$$

is called an effective equation, if:

(I) Both (2.4) and (2.6) have the steady - state solution X^{SS} ;

(II) Both (2.4) and (2.6) are stable (that is transients decay with time)

Let start with checking the condition (I). Assuming the absence of resonance (i.e. $e^{i\omega t}$ is not a solution of (2.5)), the steady - state solution of (2.4) can be found in the form:

$$X^{SS}(t) = x_0(\omega) \cdot e^{i(\omega t - \phi)} \quad (2.7)$$

Substituting (2.7) into (2.4), we have:

$$\frac{F_{dr,0}}{x_0} \cdot e^{i\phi} = -M\omega^2 + i\omega D + K + e^{-i\omega} \cdot [-M_e\omega^2 e^\tau + i\omega D_e e^\tau + K_e e^\tau] \quad (2.8)$$

and substitute (2.7) into (2.6), we will have:

$$\frac{F_{dr,0}}{x_0} \cdot e^{i\phi} = -M_{eff}\omega^2 + i\omega D_{eff} + K_{eff} \quad (2.9)$$

By comparing the real and imaginary parts of (2.8) and (2.9), we will have:

$$-M_{eff} \cdot \omega^2 + K_{eff} = -M\omega^2 + K - M_e\omega^2 \cdot \cos(\omega\tau) + \omega D_e \sin(\omega\tau) + K_e \cos(\omega\tau) \quad (2.10)$$

and

$$\omega D_{eff} = \omega D + \omega D_e \cos(\omega\tau) - K_e \sin(\omega\tau) + M_e \omega^2 \sin(\omega\tau) \quad (2.11)$$

The effective damping is uniquely defined by (2.11) as

$$D_{eff} = D + D_e \cos(\omega\tau) - K_e \omega^{-1} \sin(\omega\tau) + M_e \omega \sin(\omega\tau) \quad (2.12)$$

On the other hand, there is some freedom in formula (2.10), in choosing the effective stiffness and mass. Another words, we can choose any real number to be the effective mass, $M_{eff} = \alpha \in \mathbb{R}$, in which case (2.10) becomes

$$\begin{cases} M_{eff} = \alpha; \\ K_{eff} = \alpha\omega^2 - M\omega^2 + K - M_e\omega^2 \cos(\omega\tau) + \omega D_e \sin(\omega\tau) + K_e \cos(\omega\tau) \end{cases} \quad (2.13)$$

Condition (II) is related to stability. Using D-section method, we start with the analysis of the characteristic equation for (2.4):

$$Mz^2 + Dz + K + e^{-\tau z} \cdot (M_e z^2 + D_e z + K_e) = 0 \quad (2.14)$$

Substituting $z = iy$ into the homogeneous equation (2.5) and solving for D_e and K_e , will give us:

$$\begin{cases} D_e(y) = (My^2 - K)\sin((\tau y)/y) - D\cos(\tau y); \\ K_e(y) = (My^2 - K)\cos(\tau y) - Dy\sin(\tau y) + M_e y^2. \end{cases} \quad (2.15)$$

where $y \geq 0$ is a parameter, by the equations (2.15) there is defined a spiral in (D_e, K_e) - plane. Together with the line $K_e = -K$ along which $z = 0$ solves (2.14), for fixed M_e this spiral defines a partition of the (D_e, K_e) - plane. By varying M_e , we get a corresponding partition of (M_e, D_e, K_e) - space. Furthermore, since M , D and K are positive, due to the result in Example 1 it remains to choose the subset from this partition that contains $(M_e, D_e, K_e) = (0, 0, 0)$. Equation (9) is stable for any triple (M_e, D_e, K_e) chosen from this subset. Families of stability domains are plotted in Figs. 2 a and 2 b, and parameterized for several M_e and τ values, as a function of D_e/D versus K_e/K .

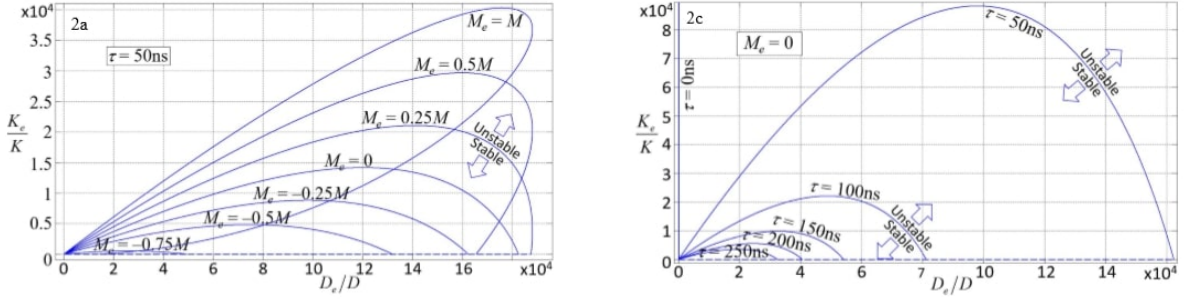


Figure 2.2: Domains of PODMEMS stability (figures from [1])

Note that the effective coefficients, given by (2.13), are determined up to a constant α . The freedom of choosing α is very important since this way we can always ensure that the effective equation is stable whenever the delay equation is. In particular, if D_{eff} (which is independent of α) is positive, we choose $\alpha > 0$ large enough so that both M_{eff} and K_{eff} are positive. Vice versa, if $D_{eff} < 0$, we choose $\alpha < 0$ small enough to ensure M_{eff} and K_{eff} are also negative. In either of the cases, the corresponding effective equation will be stable due to the main result in the Example 1.

Chapter 3

Nonlinear Duffing Equation.

We now consider a more accurate, as well as more mathematically rich, nonlinear model of MEMS. At the heart of this model lies the Duffing equation

$$\ddot{u} + D\dot{u} + Ku + \alpha u^3 = F\cos\omega_0 t \quad (3.1)$$

which is nonlinear due to the presence of the cubic term αu^3 . It is worth noting that this cubic term completely changes the behavior of this system, as compared to the linear case. In particular, the equation (3.1) with $\alpha = 0$ can have only ω_0 -periodic steady state solutions. Yet the numerical calculations for the equation

$$\ddot{u} + 0.3\dot{u} - u + u^3 = F\cos\omega_0 t \quad (3.2)$$

with $u(0) = 1$ and $\dot{u}(0) = 0$ indicate a wide range of behaviors of the solutions [6]. Namely, this initial value problem has:

- ω_0 -periodic solution for $F = 0.2$;
- $2\omega_0$ -periodic solution for $F = 0.28$;
- $4\omega_0$ -periodic solution for $F = 0.29$;
- $5\omega_0$ -periodic solution for $F = 0.37$;
- chaotic behavior for $F = 0.5$ and
- $2\omega_0$ -periodic solution for $F = 0.65$.

Thus even establishing the existence of a periodic solution to a nonlinear Duffing equation is a highly nontrivial question.

3.1 Analytic results with small nonlinearities

The natural simplification is the assumption that the parameter $|\alpha|$ is small. This scenario was considered, e.g. in the paper [2]. In this case, ignoring the high order terms, the first two terms in the expansion of the solution in the powers of α are periodic:

$$u_0(t) = a \cdot \cos(\omega_0 t - \gamma) + \frac{\alpha a^3}{32K} \cdot \cos(3\omega_0 t - 3\gamma) + o(\alpha)$$

The amplitude a in then satisfies

$$\left[(\omega_0 - \sqrt{K} - \frac{3\alpha a^2}{8\sqrt{K}})^2 + \frac{D^2}{4} \right] \cdot a^2 = \frac{F_0^2}{4K}.$$

The authors investigated the stability of the periodic solution of (3.1) with small α . In particular, they showed that if

$$a_* < a < a_{**} \quad (3.3)$$

where a_* and a_{**} are the two roots of the equation

$$\frac{F_0^4}{9\alpha^2 K \cdot a^8} = \frac{F_0^2}{4K \cdot a^2} - \frac{D^2}{4} \quad (3.4)$$

then the equation is unstable.

3.2 Periodic Solutions of Nonlinear Duffing Equation

In this subsection we will study the existence of periodic solutions to the Duffing equation in the case when α is not necessarily small. More generally, we will outline the main ideas how to establish the existence of periodic solution for

$$\ddot{x} + f(t, x, \dot{x}) = 0 \quad (3.5)$$

using the method from the paper of Struwe [3]. Assume f satisfies the following conditions:

- (I) For fixed $T > 0$, the function $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the Caratheodory condition;
- (II) \exists a continuous $g : \mathbb{R} \rightarrow \mathbb{R}$, a function $h \in L^2([0, T])$, and a constant $K > 0$ such that

$$\begin{aligned} |f(t, x, y) - g(x)| &\leq K(|x| + |y|) + h(t), \\ g(x)/x &\rightarrow \infty, \text{ as } |x| \rightarrow \infty. \end{aligned} \quad (3.6)$$

The typical choice of g is $g(x) = \alpha x^3(t)$.

We will state some lemmas which we will use to prove the main result of this subsection.

In what follows, denote $\zeta = (\xi, \eta)$, and let $x = x(\zeta, \cdot)$ be the solution of (3.5) with $x(\tau) = \xi$ and $x'(\tau) = \eta$. We will also use the notation $z = (x, x') = z(\zeta, \cdot)$

Lemma 1. *For any constant $C > 0$ there exists a constant $C_1 > 0$ such that for any $t \in I$ we have $|z(\zeta; t)| > C$ whenever $|\zeta| > C_1$*

Lemma 2. *For any $\delta > 0$ there exists a constant $C = C(\delta)$ such that any solution of equation (32) has a zero in any interval $J \in I$ of width $|J| \geq \delta$, whenever $\|x\|_{W^{1,\infty}(J)} := \|x\|_{L^\infty(J)} + \|x'\|_{L^\infty(J)} \geq C$.*

Theorem 1. [3] *Assume conditions (I)-(II) are satisfied by f . Then there exists a periodic solution of (3.5).*

Proof. Firstly, let us assume that f is Lipschitz with respect to the phase space variables and then we will be approximating f by Lipschitzian function in order to obtain the general case. The mapping $F : \zeta \rightarrow z(\zeta; T)$ is a continuous mapping of \mathbb{R}^2 into \mathbb{R}^2 . According to Lemmas 1 and 2 there exist constant $s_1, s_2 > s_0$ such that the mapping $\Psi : \{\zeta : |\zeta| > s_0\} \rightarrow \mathbb{R}$, defined by $\Psi(\zeta) = |\Theta(\zeta; T) - \Theta(\zeta; 0)|$ is continuous and such that

$$\inf\{\Psi(\zeta) : |\zeta| \geq s_2\} - \sup\{\Psi(\zeta) : |\zeta| \leq s_1\} > 2\pi$$

Let M be the compact subset of \mathbb{R}^2 , defined by

$$M = \{\zeta(s) : s_1 \leq |\zeta(s)| \leq s_2, F(\zeta(s)) = \lambda \zeta(s) \text{ for some } \lambda \in \mathbb{R}\}$$

The existence of a periodic solution for Lipschitzian f now is a consequence of the following fixed point theorem

Theorem 2. [4]. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping, M is a compact subset of \mathbb{R}^n such that 0 is contained in bounded component of the complement of M and such that $F(m) = \lambda m$ for any $m \in M$ with some $\lambda = \lambda(m) \in \mathbb{R}$. Then F has a fixed point.*

□

Remark. We may assert the existence of a fixed point of F in the convex hull of M ; hence there exists a periodic solution with norm $\leq \sup\{b(\zeta) : |\zeta| \leq s_2\}$

To sum up, if a function measurable in the first and continuous with respect to the remaining arguments and it satisfies the condition (3.6) then we have a periodic solution of

$$\ddot{x} + f(t, x, \dot{x}) = 0.$$

Chapter 4

Dynamics of a Time Delayed Duffing Oscillator with small nonlinearities

In this section we will be focusing on the results of the article [5], devoted to the primary resonance of the time delayed Duffing oscillator solved by means of the multiple scales method. The classic Duffing oscillator with delayed displacement is governed by a second order non-linear differential equation with delay:

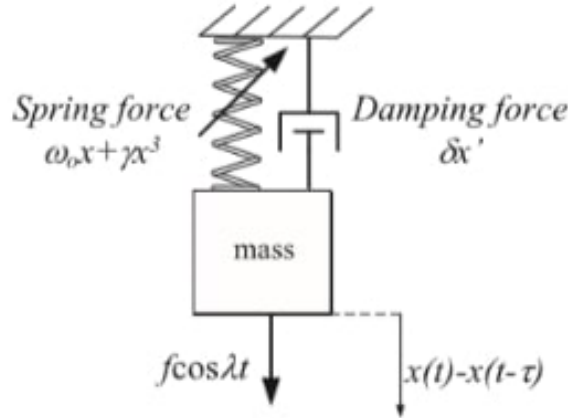


Figure 4.1: Duffing system with a delay (figure from [5])

$$\ddot{x}(t) + \delta \dot{x}(t) + \omega^2 x(t) + \gamma x(t)^3 = \alpha(-\mu x(t) + x(t - \tau)) + f \cdot \cos(\lambda t) \quad (4.1)$$

where as before, D is damping, ω is natural frequency of a linear system, γ is a small coefficient representing non-linear stiffness, α is an amplitude of delay, f is an amplitude of external force, λ is a frequency of external excitation, τ is a time delay and μ is a switching parameter: if $\mu = 1$, then term $\mu x(t)$ produces only an increase of linear stiffness of the system. Therefore (4.1) can be transformed using the substitution $\omega_0^2 = (\omega^2 + \alpha)$:

$$\ddot{x}(t) + \delta \dot{x}(t) + \omega_0^2 x(t) + \gamma x(t)^3 = \alpha x(t - \tau) + f \cdot \cos(\lambda t) \quad (4.2)$$

4.1 Existence of an approximate periodic solution.

Assuming that α is small, the authors in [5] found the approximate periodic solution using the methods similar to [2] with no delay. By using of the multiple scale time method the equation (4.2) is solved analytically. Let us introduce the fast scale T_0 and the two slow scales T_1 and T_2 of time:

$$T_0 = t, T_1 = \epsilon t, T_2 = \epsilon^2 t \quad (4.3)$$

Then a solution in the second order approximation is in the form:

$$x(t) = x_0(T_0, T_1, T_2) + \epsilon x_1(T_0, T_1, T_2) + \epsilon^2 x_2(T_0, T_1, T_2) \quad (4.4)$$

$$x(t - \tau) = x_\tau = x_{0\tau}(T_0, T_1, T_2) + \epsilon x_{1\tau}(T_0, T_1, T_2) + \epsilon^2 x_{2\tau}(T_0, T_1, T_2) \quad (4.5)$$

Using the chain rule, the time derivative is transformed in accordance with the following expressions:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} \quad (4.6)$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \epsilon^2 \left(2 \frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2} \right) \quad (4.7)$$

Comparing the power of ϵ , we will have a set of equations in a successive perturbation order:

$$\epsilon^0 : \frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \quad (4.8)$$

$$\epsilon^1 : \frac{\partial^2 x_1}{\partial T_0^2} + 2 \frac{\partial^2 x_0}{\partial T_0 \partial T_1} + D \frac{\partial x_0}{\partial T_0} + x_1 + K x_0^3 - \alpha x_{0\tau} - F_0 \cdot \cos(T_0 + \sigma T_2) = 0 \quad (4.9)$$

$$\epsilon^2 : \frac{\partial^2 x_2}{\partial T_0^2} + 2 \frac{\partial^2 x_1}{\partial T_0 \partial T_1} + 2 \frac{\partial^2 x_0}{\partial T_0 \partial T_2} + \frac{\partial^2 x_0}{\partial T_1^2} + D \frac{\partial x_1}{\partial T_0} + 3K x_0^2 x_1 - \alpha x_{1\tau} + x_2 = 0 \quad (4.10)$$

The solution of (4.9) can be expressed in the complex form :

$$x_0(T_0, T_1, T_2) = A(T_1, T_2) e^{iT_0} + \bar{A}(T_1, T_2) e^{-iT_0} \quad (4.11)$$

$$x_{0\tau}(T_0, T_1, T_2) = A(T_1, T_2) e^{i(T_0 - \tau)} + \bar{A}(T_1, T_2) e^{-i(T_0 - \tau)} \quad (4.12)$$

We may now find χ_1 :

$$\chi_1(T_0, T_1, T_2) = \frac{1}{8}K \cdot A(T_1, T_2)e^{3iT_0} + \frac{1}{8}K \cdot \bar{A}(T_1, T_2)e^{-3iT_0}, \quad (4.13)$$

$$\chi_{1\tau}(T_0, T_1, T_2) = \frac{1}{8}K \cdot A(T_1, T_2)e^{3i(T_0-\tau)} + \frac{1}{8}K \cdot \bar{A}(T_1, T_2)e^{-3i(T_0-\tau)} \quad (4.14)$$

Omitting the higher order terms, we may get the leading order expression for A . To this end, write A in polar form:

$$A(T_1) := \frac{1}{2}a(T_1)e^{i\beta(T_1)}. \quad (4.15)$$

We have

$$\begin{aligned} & -\frac{1}{4}\epsilon i f e^{i\sigma T_2 - i\beta(t)} - \frac{1}{16}\epsilon^2 i \alpha f e^{-i\tau + i\sigma T_2 - i\beta(t)} + \frac{1}{16}\epsilon^2 D f e^{i\sigma T_2 - i\beta(t)} - \frac{1}{4}\epsilon i \alpha a(t) e^{-i\tau} - \\ & -\frac{1}{4}\epsilon D a(t) - \frac{1}{16}\epsilon^2 i \alpha^2 a(t) e^{-2i\tau} - \frac{1}{16}\epsilon^2 i D^2 a(t) + \frac{3}{32}\epsilon^2 i K f a(t)^2 e^{i\sigma T_2 - i\beta(t)} - \\ & -\frac{3}{64}\epsilon^2 i K f a(t)^2 e^{-i\sigma T_2 + i\beta(t)} + \frac{3}{16}\epsilon i K a(t)^3 + \frac{9}{64}\epsilon^2 i \alpha K a(t)^3 e^{-i\tau} - \frac{3}{64}\epsilon^2 i \alpha K a(t)^3 e^{i\tau} + \\ & + \frac{3}{32}\epsilon^2 D K a(t)^3 - \frac{5}{512}\epsilon^2 i K^2 a(t)^5 - \frac{1}{2}a'(t) - \frac{1}{2}i a(t) \beta'(t) = 0 \end{aligned} \quad (4.16)$$

Now we will separate real and imaginary parts and get a system of ODEs for a and ϕ :

$$\begin{aligned} \dot{a} = & -\frac{1}{2}\epsilon D a + \frac{3}{16}\epsilon^2 D K a^3 + \frac{1}{8}\epsilon^2 D f \cos(\phi) - \frac{1}{2}\epsilon \alpha a \cdot \sin(\tau) + \\ & + \frac{3}{8}\epsilon^2 \alpha K a^3 \sin(\tau) - \frac{1}{4}\epsilon^2 \alpha^2 a \cos(\tau) \sin(\tau) - \frac{1}{8}\epsilon^2 \alpha f \cos(\phi) \sin(\tau) + \frac{1}{2}\epsilon f \sin(\phi) - \\ & - \frac{9}{32}\epsilon^2 K f a^2 \sin(\phi) + \frac{1}{8}\epsilon^2 \alpha f \cos(\tau) \sin(\phi) \end{aligned} \quad (4.17)$$

$$\begin{aligned} \dot{\phi} = & -\frac{3}{8}\epsilon K a^2 + \frac{15}{256}\epsilon^2 K^2 a^4 + \frac{1}{8}\epsilon^2 D^2 + \sigma + \frac{1}{2}\epsilon \alpha \cos(\tau) - \frac{3}{16}\epsilon^2 \alpha K a^2 \cos(\tau) + \\ & + \frac{1}{8}\epsilon^2 \alpha^2 \cos^2(\tau) + \frac{1}{2} \frac{\epsilon f \cos(\phi)}{a} - \frac{3}{32}\epsilon^2 K f \cdot a \cos(\phi) + \frac{1}{8} \frac{\epsilon^2 \alpha f \cos(\tau) \cos(\phi)}{a} - \\ & - \frac{1}{8}\epsilon^2 \alpha^2 \sin^2(\tau) - \frac{1}{8} \frac{\epsilon^2 D f \sin(\phi)}{a} + \frac{1}{8} \frac{\epsilon^2 \alpha f \sin(\tau) \sin(\phi)}{a} \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} a(t) &= a, \\ \dot{a}(t) &= \dot{a}, \\ \dot{\beta}(t) &= \sigma - \dot{\phi} \end{aligned} \quad (4.19)$$

The amplitude and phase of the steady state solution may now be implicitly expressed via $f_1 = 0$ and $f_2 = 0$, where

$$f_1 := f_{11} + \frac{3}{16}DKa^3 + \frac{1}{8}Df\cos(\phi) + \frac{3}{8}\alpha Ka^3\sin(\tau) - \frac{1}{4}\alpha^2 a\cos(\tau)\sin(\tau) - \frac{1}{8}\alpha f\cos(\phi)\sin(\tau) - \frac{9}{32}Kf \cdot a^2\sin(\phi) + \frac{1}{8}\alpha f\cos(\tau)\sin(\phi) \quad (4.20)$$

$$f_2 := f_{12} + \frac{15}{256}K^2a^4 + \frac{1}{8}D^2 - \frac{3}{16}\alpha Ka^2\cos(\tau) + \frac{1}{8}\alpha^2\cos^2(\tau) - \frac{3}{32}Kf \cdot a\cos(\phi) + \frac{1}{8}\frac{\alpha f\cos(\tau)\cos(\phi)}{a} - \frac{1}{8}\alpha^2\sin^2(\tau) - \frac{1}{8}\frac{Df\sin(\phi)}{a} + \frac{1}{8}\frac{\alpha f\sin(\tau)\sin(\phi)}{a} \quad (4.21)$$

Here f_{11} and f_{12} are obtained from the first approximation of the multiple scales method, which are defined as follows:

$$f_{11} = -\frac{1}{2}Da - \frac{1}{2}\alpha a\sin(\tau) + \frac{1}{2}f\sin(\phi) \quad (4.22)$$

$$f_{21} = -\frac{3}{8}Ka^2 + (\lambda - \omega_0) + \frac{1}{2}\alpha\cos(\tau) + \frac{1}{2}\frac{f\cos(\phi)}{a} \quad (4.23)$$

4.2 Stability of the approximate solutions

In order to analyse the stability of steady-state solutions, equations (4.20) and (4.21) are linearized with respect to a and ϕ . The Jacobian matrix is defined as

$$J = \begin{pmatrix} \frac{df_1}{da} & \frac{df_1}{d\phi} \\ \frac{df_2}{da} & \frac{df_2}{d\phi} \end{pmatrix} \quad (4.24)$$

where f_1 and f_2 are defined in (4.20) and (4.21). We can rewrite the characteristic equation as follow

$$s^2 + \text{Tr}(J)s + \text{Det}(J) = 0 \quad (4.25)$$

where s is an eigenvalue of the Jacobian matrix. The trace (Tr) and the determinant (Det) are defined as

$$\begin{aligned} \text{Tr}(J) = & -\frac{1}{2}D + \frac{9}{16}a^2KD - \frac{fD\cos(\phi)}{8a} - \frac{1}{2}\alpha\sin(\tau) + \frac{9}{8}a^2\alpha K\sin(\tau) - \\ & -\frac{1}{4}\alpha^2\cos(\tau)\sin(\tau) + \frac{f\alpha\cos(\phi)\sin(\tau)}{8a} - \frac{f\sin(\phi)}{2a} - \\ & -\frac{15}{32}afK\sin(\phi) - \frac{f\alpha\cos(\tau)\sin(\phi)}{8a} \end{aligned} \quad (4.26)$$

$$\text{Det}(J) = A \cdot \cos\phi + B \cdot \cos^2\phi + C \cdot \sin\phi + D \cdot \sin^2\phi + E \cdot \cos\phi\sin\phi \quad (4.27)$$

According the Routh-Hurwitz criterion, the system has:

- 1) stable solutions corresponding to a stable focus when $\text{Tr}(J) < 0$ and $\text{Det}(J) > 0$,
- 2) unstable solutions corresponding to unstable focus when $\text{Tr}(J) > 0$ and $\text{Det}(J) > 0$,
- 3) unstable solutions corresponding to saddle point at $\text{Det}(J) > 0$

Using the explicit expressions for f_{11} and f_{12} , the trace (Tr_1) and the determinant (Det_1) may be simplified as

$$\text{Tr}_1(J) = -D - \alpha\sin(\tau), \quad (4.28)$$

$$\begin{aligned} \text{Det}_1(J) = \frac{1}{64} (6a^2K(3a^2 - 8(\omega_0 - \lambda_0) - 4\alpha\cos\tau) + \\ + (3a^2K - 8(\omega_0 - \lambda_0) - 4\alpha\cos\tau)^2 + 16(D + \alpha\sin\tau)(D + \alpha\sin\tau)) \end{aligned} \quad (4.29)$$

Equation (4.28) is independent of non-linearity, the resonance amplitude, and hence the excitation. The critical value of the delay amplitude (α_{cr}) can be easily found from equation (4.28)

$$\alpha_{cr} = D \quad (4.30)$$

If $\alpha < \alpha_{cr}$ the system has only stable solutions. On the other hand, when $\alpha > \alpha_{cr}$ the stability of a solution is influenced by the time delay parameter $\Omega = \frac{2\pi}{\tau}$.

Chapter 5

Main Results: Existence of Periodic Solution for General Delay Duffing Equation

We are now ready to formulate the main result of this work. Consider

$$\ddot{x}(t - \tau) + g(x(t - \tau)) = p(t) \quad (5.1)$$

where g is a locally Lipschitz continuous function, τ is a positive constant, $p : \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $T > 0$ the minimal period.

Assume that g satisfies the superlinear growth condition

$$(S_p) \text{ (superlinear) } g(x)/x \rightarrow +\infty \text{ as } |x| \rightarrow +\infty$$

We then have the following Theorem:

Theorem 3. *Assume that condition S_p holds. Then, for τ small enough, there is at least one periodic solution for (5.1).*

The problem was originally posed in [7]. However, after the careful analysis, we discovered a mistake in their argument (namely in Lemmas 2.6 and 2.7 [7], or Lemmas 7 and 8 presented below). We contacted the authors and they confirmed that their argument is incomplete. In this section we suggest a way how to fix the problem in their argument. Namely, we claim that the results of Lemmas 7 and 8 hold for τ small enough.

Let us rewrite (5.1) as a system

$$\dot{x} = y, \dot{y} = -g(x(t - \tau)) + p(t) \quad (5.2)$$

Consider the solution $(x(t), y(t)) = (x(t; x_0; y_0), y(t; x_0; y_0))$ of (5.2) satisfying the initial value condition

$$x(0; x_0; y_0) = x_0; \quad y(0; x_0; y_0) = y_0$$

We assume that g satisfies the following condition:

$$(g_0) \quad \lim_{|x(t-\tau)| \rightarrow +\infty} g(x(t-\tau)) = +\infty$$

It is easy to show that, superlinear condition (S_p) implies condition (g_0) , on the other hand the inverse does not hold.

Lemma 3. *Assume that condition (g_0) holds. Then every solution $(x(t), y(t))$ of (5.2) is defined on the whole t -axis.*

Proof. Set

$$H(t) = \int_0^t g(x(s-\tau))y(s)ds, \quad P(t) = \int_0^t p(s)y(s)ds$$

Now we define the potential function

$$V(t) = V(x(t), y(t)) = \frac{1}{2}y^2(t) + H(t) - P(t).$$

Then according to the fundamental theorem of calculus we have:

$$V'(t) = y(t)y'(t) + g(x(t-\tau))y(t) - p(t)y(t) = y(t)(y'(t) + g(x(t-\tau)) - p(t)) = 0 \quad (5.3)$$

Therefore, $V(T) = C$.

Thus

$$|V'(t)| \leq V(t) + M_2 \quad (5.4)$$

where $|C| \leq M_2$. From (5.4) we have that, for $t \in [t_0, t_0 + \tau]$ with $t_0 \in \mathbb{R}$, $\tau > 0$,

$$V(t) \leq V(t_0)e^\tau + M_2e^\tau \quad (5.5)$$

which implies that there is no blow-up for solution $(x(t), y(t))$ in any finite interval $[t_0, t_0 + \tau]$. \square

According to our conditions, the function g is locally Lipschitz continuous, from previous lemma, we know that $(x(t), y(t))$ exists on the entire t -axis. Let us define a function $\mathcal{R} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$

$$\mathcal{R}(x, y) = x^2 + y^2. \quad (5.6)$$

Lemma 4. *Assume that condition g_0 holds. Then there exists a positive constant $m \in \mathbb{N}^+$ such that $\mathcal{R}(x(t), y(t)) \rightarrow +\infty$ for $t \in [0, mT]$ as $\mathcal{R}(x_0, y_0) \rightarrow +\infty$.*

Proof. From (5.5), it is easy to see that

$$(V(0) + A_1)e^{-T} \leq V(t) \leq (V(0) + A_1)e^T, \quad (5.7)$$

for $t \in [0, mT]$, which shows that $\mathcal{R}(x(t), y(t)) \rightarrow +\infty$ uniformly for $t \in [0, T]$ as $\mathcal{R}(x_0, y_0) \rightarrow +\infty$ \square

Remark 1. For a fixed constant $R_1 > 0$ there is a constant $R_2 \geq R_1$ such that

$$\mathcal{R}(x_0, y_0) \leq R_1 \Rightarrow \mathcal{R}(x(t), y(t)) \leq R_2, \quad \text{for } t \in [0, mT];$$

$$\mathcal{R}(x_0, y_0) \geq R_2 \Rightarrow \mathcal{R}(x(t), y(t)) \geq R_1, \quad \text{for } t \in [0, mT]$$

From previous lemma we know that if $\mathcal{R}(x_0, y_0)$ is a sufficiently large number, then $x^2(t) + y^2(t) \neq 0$, $t \in [0, mT]$. Therefore, we can make a polar coordinate transformation:

$$\begin{cases} x(t) = r(t) \cdot \cos \theta(t); \\ y(t) = r(t) \cdot \sin \theta(t). \end{cases} \quad (5.8)$$

(when $\mathcal{R}(x_0, y_0)$ is a sufficiently large number). Under this transformation, (5.2) becomes

$$\begin{cases} \dot{r} = [r \cos \theta - g(r(t - \tau)) \cdot \cos \theta(t - \tau) - p(t)] \cdot \sin \theta, \\ \dot{\theta} = -\left[\sin^2 \theta + \frac{1}{r} (g(r(t - \tau)) \cdot \cos \theta(t - \tau) - p(t)) \cdot \cos \theta \right]. \end{cases} \quad (5.9)$$

The main idea of establishing the periodic solution of (5.9) is based on a fixed point argument. We will summarize this idea below.

For $R_2 > R_1$ let A be an annular region $B(0, R_2) \setminus B(0, R_1)$.

Definition 2. $F := (F_1, F_2) : A \rightarrow \mathbb{R}^2 \setminus \{0\}$ is called a *twist* if

$$\begin{pmatrix} F_1(r, \theta) \\ F_2(r, \theta) \end{pmatrix} = \begin{pmatrix} f(r, \theta) \\ \theta + g(r, \theta) \end{pmatrix}$$

where f and g are continuous and periodic in θ , and g satisfies the twist condition

$$g(R_1, \theta)g(R_2, \theta) < 0$$

The following Lemma was proved in [8]

Lemma 5. If F is a twist in A , then F has at least one fixed point in $B(0, R_2)$.

Denote by $(r(t), \theta(t)) = (r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ the solution of (5.9) satisfying $r(0) = r_0$ and $\theta(0) = \theta_0$. Denote $\mathcal{D} = \{(r, \theta) : r > 0, \theta \in \mathbb{R}\}$. Then the map $\mathcal{P} : \mathcal{D} \rightarrow \mathcal{D}$ defined by

$$\mathcal{P} : (r_0, \theta_0) \rightarrow (r_1, \theta_1) = (r(T, r_0, \theta_0), \theta(T, r_0, \theta_0))$$

is a continuous homeomorphism from \mathcal{D} to itself. Furthermore, fixed point of \mathcal{P} correspond to periodic solutions of (5.1). In order to proceed, we will need the following Lemma, which is the main result of this work:

Lemma 6. (Misiats, Ospanov) Let $x(t)$ be a solution of (5.1) with $g(y) = y^3$, satisfying

$$x'(T) < 0, \quad x(T) \geq N \quad (5.10)$$

for some $N > 1$. Then there is $\tau = \tau(N) > 0$ such that $x(t) > \frac{1}{2}$ for $t \in [T, T + \tau]$.

Proof. Let x_0 be the periodic solution of the Duffing equation with no delay

$$\ddot{x}_0(t) + x_0^3(t) = p(t)$$

such that $x_0(T) > N$. This solution exists by the main result of Struwe. What do we know about $x_0(t)$?

- $x_0(t)$ is periodic (continuously differentiable);
- $\max |x'_0(t)| \leq c$;
- So, if $x_0(T) > N \Rightarrow x_0(T + \tau) > 1$ for τ small enough.

Also $x_0(t)$ satisfies $\ddot{x}_0(t) + x_0^3(t - \tau) = \ddot{x}_0(t) + x_0^3(t) + \tau a(t) = p(t)$, where $a(t)$ is some periodic function (depending only on x_0) satisfying $|a(t)| \leq c$, $t \in [T, T + \tau]$. So,

$$\ddot{x}_0(t) + x_0^3(t - \tau) = p(t) + \tau a(t) \quad (5.11)$$

Now let x_1 be the solution of (5.1) with the same boundary conditions as x_0 on $[T - \tau, T]$, i.e.

$$\ddot{x}_1(t) + x_1^3(t - \tau) = p(t). \quad (5.12)$$

Subtracting (5.11) and (5.12) we have

$$\ddot{x}_1(t) - \ddot{x}_0(t) + x_1^3(t - \tau) - x_0^3(t - \tau) = \tau a(t) \quad (5.13)$$

or

$$\ddot{x}_1(t) - \ddot{x}_0(t) + (x_1(t - \tau) - x_0(t - \tau)) \cdot (x_1^2(t - \tau) + x_1(t - \tau) \cdot x_0(t - \tau) + x_0^2(t - \tau)) = \tau a(t) \quad (5.14)$$

Using the elementary inequality

$$|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$$

we have

$$x_1^2(t - \tau) + x_1(t - \tau) \cdot x_0(t - \tau) + x_0^2(t - \tau) \geq \frac{x_1^2(t - \tau)}{2} + \frac{x_0^2(t - \tau)}{2} \geq \frac{N^2}{2} \geq \frac{1}{2} \quad (5.15)$$

since $x_0(t - \tau) > \frac{1}{2}$. On the other hand, in view of the condition (5.10),

$$x_1^2(t - \tau) + x_1(t - \tau) \cdot x_0(t - \tau) + x_0^2(t - \tau) \leq C_0 \quad (5.16)$$

for some $C > 0$. Denote $y(t) := x_1(t) - x_0(t)$. We claim that this difference is small for small τ . More precisely, $y(t)$ solves

$$\ddot{y}(t) + y(t - \tau) \cdot b(t) = \tau a(t). \quad (5.17)$$

Generally speaking, (5.17) is a nonlinear equation since both a and b depend on x_0 and x_1 , and therefore on y , i.e. $a = a(t, y)$ and $b = b(t, y)$. However, a priori estimates on a and b , namely

$$|a(t)| \leq c, \quad \frac{1}{2} \leq b(t) \leq 3N^2 \quad (5.18)$$

(which follows from (5.15)) allow us to treat (5.17) as semilinear. Namely, for a fixed solution y consider the linear equation

$$\ddot{v}(t) + \tilde{v}(t - \tau) \cdot b(t, y) = \tau a(t, y). \quad (5.19)$$

The change of variables $\tau v := \tilde{v}$ yields

$$\ddot{v}(t) + v(t - \tau) \cdot b(t, y) = a(t, y). \quad (5.20)$$

The general solution of the linear equation (5.20) now has the form:

$$v(t) := v_{SS}(t) + v_{TR}(t)$$

where $v_{SS}(t)$ is a fixed steady state (particular) solution of the nonhomogeneous equation (5.20), which is periodic given when a is periodic, and the transient solution $v_{TR}(t)$ solves the homogeneous equation

$$\ddot{v}_{TR}(t) + v_{TR}(t - \tau) \cdot b(t, y) = 0. \quad (5.21)$$

Equation (5.21) is a linear delay equation with non-constant coefficients. However, the a priori estimates (5.18) enable us to apply the delay analog of Sturm-Picon comparison principle [9] which states that all the solutions of (5.21) are oscillatory (and hence bounded on finite time intervals). Furthermore, the zeros of the solutions of (5.21) are confined between the zeros of the solutions of $\ddot{w}(t) + \frac{1}{2}w(t - \tau) = 0$ and $\ddot{w}(t) + 3N^2w(t - \tau) = 0$, which may be found explicitly. Thus we may conclude that there is a constant $C > 0$ such that for $|v(t)| \leq C$, $t \in [T, T + \tau]$, hence

$$|\tilde{v}(t)| \leq C\tau, \quad t \in [T, T + \tau].$$

But since y is also a solution of (5.19), due to uniqueness of solution y must coincide with \tilde{v} , hence

$$|y(t)| \leq C\tau, \quad t \in [T, T + \tau].$$

Therefore, we have $|y(t)| = |x_1(t) - x_0(t)| \leq C\tau$. Since $x_0(t) > 1$, we have

$$x_1(t) \geq x_0(t) - C\tau \geq 1 - C\tau \geq \frac{1}{2}, \quad t \in [T, T + \tau],$$

which is what we wanted to prove. \square

We may now resume the arguments from [7].

Lemma 7. *Assume that condition g_0 holds. Then there exists a positive constant ρ_0 such that, for $\mathcal{R}(x_0, y_0) \geq \rho_0$, $\theta'(t) < 0$, $t \in [0, mT]$.*

Proof. From the condition g_0 we can say that there exists a positive, large enough, constant N such that

$$g(x(t - \tau)) - p(t) > 0, \quad |x(t - \tau)| \geq N, \quad t \in [0, mT]$$

Step 1. If $x(t - \tau) \geq N$, we know that $\cos\theta(t - \tau) > 0$, and $y(t) \geq 0$ or $y(t) \leq 0$

Case 1. If $y(t) \geq 0$, from first equation of (5.2) and Taylor expansion, namely: $x(t - \tau) = x(t) - x'(t) \cdot \tau$ we know that $x(t) \geq x(t - \tau) \geq N$ since $\tau > 0$. So, $\cos\theta(t) > 0$, then

$$\frac{d\theta}{dt} = - \left[\sin^2\theta + \frac{g(x(t - \tau)) - p(t)}{r(t)} \cdot \cos\theta \right] < 0$$

At this point we noticed the mistake in their argument. The authors claimed, in case 2, that if $y(t) \leq 0$ then $\theta(t) \in [-\frac{\pi}{2}, 0]$. This is simply incorrect, since when inequality $y(t) \leq 0$ is equivalent to $\theta(t) \in [-\pi, 0]$. Thus the case $\theta(t) \in [-\pi, -\frac{\pi}{2}]$ has to be ruled out, which is the case for τ small enough due to our Lemma 6.

Case 2. If $y(t) \leq 0$, we know that $\theta(t) \in [-\pi, 0]$. So, $\cos\theta(t) > 0$, then

$$\frac{d\theta}{dt} = - \left[\sin^2\theta + \frac{g(x(t-\tau)) - p(t)}{r(t)} \cdot \cos\theta \right] < 0$$

Step 2. Similarly, if $x(t-\tau) \leq -N$, we have

$$\frac{d\theta}{dt} = - \left[\sin^2\theta + \frac{g(x(t-\tau)) - p(t)}{r(t)} \cdot \cos\theta \right] < 0$$

Step 3. If $|x(t-\tau)| \leq N$, for a sufficiently large (then r is sufficiently large), we have $\sin^2\theta > 1/2$

$$\sin^2\theta > 1/2 \quad \frac{|g(x(t-\tau)) - p(t)|}{r(t)} \leq \frac{1}{4}, \quad t \in [0, mT]$$

So we get,

$$\frac{d\theta}{dt} = - \left[\sin^2\theta + \frac{g(x(t-\tau)) - p(t)}{r(t)} \cdot \cos\theta \right] < 0$$

Now we conclude, that we have $\frac{d\theta}{dt} < 0$ whenever $r \gg 1$ □

The lemma implies that $\theta(t)$ decreases strictly when r is large enough. Let's denote by $\tau(r_0, \theta_0)$ the time its take for the solution $(r(t), \theta(t))$ to makes one turn around the origin.

Lemma 8. Assume that condition (S_p) holds, let $m \in \mathbb{N}^+$. Then for an arbitrary large integer $N \in \mathbb{N}^+$, there exists a large enough constant $\Lambda_0 > 0$ such that

$$\theta(mT; r_0, \theta_0) - \theta_0 < -2N\pi$$

for $(x_0, y_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \{(x, y) \mid \mathcal{R}(x, y) = \Lambda, \Lambda \geq \Lambda_0\}$

Proof. Let $z = (x, y)$ and set Γ_{z_0} be such that $z = z(t) = (x(t), y(t))$, $t \in \mathbb{R}$, is a solution of (72) satisfying the initial value condition $z(0) = z_0 = (x_0, y_0)$.

Without loss of generality, we can say that there exist a large enough constant c_0 and a positive constant $\epsilon \ll 1$ such that $R_1 := R_1(\epsilon) > ((2c_0)^2 + \epsilon^2 c_0^2)/\epsilon^2$. Furthermore, let $R_2 := R_2(\epsilon) > R_1$. From remark 1, we know that $\mathcal{R}(x, y) \geq R_1$ if $\mathcal{R}(x_0, y_0) \geq R_2$.

When $0 \leq t \leq mT$, we choose $0 = t_0 < t_1 < \dots < t_6$ so that

- $D_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x(t-\tau)| \leq c_0, y(t) \geq 0, t \in [t_0, t_1]\};$
- $D_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x(t-\tau) \geq c_0, 0 \leq y(t) < +\infty, t \in [t_1, t_2]\};$

- $D_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x(t - \tau) \geq c_0, -\infty < y(t) \leq 0, t \in [t_2, t_3]\};$
- $D_4 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x(t - \tau)| \leq c_0, y(t) \leq 0, t \in [t_3, t_4]\};$
- $D_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x(t - \tau) \leq -c_0, -\infty < y(t) \leq 0, t \in [t_4, t_5]\};$
- $D_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x(t - \tau) \leq -c_0, 0 \leq y(t) < +\infty, t \in [t_5, t_6]\}$

From previous lemma, we know that there exist t_i , $i = 0, \dots, 6$, such that t_i satisfy the above properties for large enough $\mathcal{R}(x_0, y_0)$. Next we shall estimate $t_1 - t_0$, $t_2 - t_1$, $t_3 - t_2$, $t_4 - t_3$, $t_5 - t_4$, $t_6 - t_5$, respectively.

Step 1. Estimate of $t_1 - t_0$, $t_4 - t_3$. From the first equation in 5.2 and the choice of R_1 , we have

$$x'(t - \tau) = y(t - \tau) \geq (R_1 - c_0^2)^{\frac{1}{2}} \geq \frac{2c_0}{\epsilon},$$

and thus we obtain

$$2c_0 \geq x(t_1 - \tau) - x(t_0 - \tau) = \int_{t_0}^{t_1} x'(t - \tau) dt \geq \frac{2c_0(t_1 - t_0)}{\epsilon},$$

hence $t_1 - t_0 \leq \epsilon$. With obvious changes in the proof, we can obtain the estimate $t_4 - t_3 \leq \epsilon$.

Step 2. Estimate of $t_2 - t_1$, $t_6 - t_5$. For $x^2 + y^2 = r^2$, $0 \leq y < +\infty$, we know that there exist large enough B such that $0 \leq y(t) \leq B < +\infty$. **Once again, the statement $y(t) \geq 0$ is justified by our Lemma 6. Originally it was not justified in the paper [7].** From (S_p) we know that there exist large enough K such that

$$\frac{g(x(t - \tau))}{x(t - \tau)} \geq \frac{K(c_0 + B\tau)}{c_0} := K_1, |x(t - \tau)| \geq c_0.$$

Let $t \in [t_2, t_1]$.

$$t_2 - t_1 = \int_{\theta(t_1)}^{\theta(t_0)} dt = \int_{\theta(t_1)}^{\theta(t_0)} \frac{d\theta}{\frac{d\theta}{dt}} = \int_{\theta(t_1)}^{\theta(t_0)} \frac{d\theta}{\sin^2\theta + (g(x(t - \tau)) - p(t)/r(t))\cos\theta}.$$

As $x(t)$ is increasing and $0 \leq y(t) \leq B < +\infty$, we have $x(t) \geq x(t - \tau) \geq c_0$ and

$$x(t) - x(t - \tau) = \int_{t-\tau}^t x'(s) ds \leq B\tau,$$

i.e.

$$\frac{x(t)}{x(t - \tau)} - 1 \leq \frac{B\tau}{x(t - \tau)} \leq \frac{B\tau}{c_0}.$$

So,

$$\frac{x(t - \tau)}{x(t)} \geq \frac{c_0}{c_0 + B\tau} \tag{5.22}$$

From (76), for $r(t)$ large enough we have

$$\begin{aligned} \sin^2\theta + \frac{g(r(t-\tau)\cos\theta(t-\tau)) - p(t)}{r(t)} \cdot \cos\theta &= \sin^2\theta + \frac{g(r(t-\tau)\cos\theta(t-\tau))}{r(t-\tau)\cos\theta(t-\tau)} \times \\ \frac{r(t-\tau)\cos\theta(t-\tau)}{r(t)\cos\theta(t)} \cos^2\theta - \frac{p(t)}{r(t)} \cos\theta &\geq \sin^2\theta + K_1 \cdot \frac{c_0}{c_0 + B\tau} \cos^2\theta - \frac{p(t)}{r(t)} \cos\theta = \\ &= \sin^2\theta + K \cos^2\theta - \frac{p(t)}{r(t)} \cos\theta \geq \frac{1}{2}(\sin^2\theta + K \cos^2\theta) \end{aligned} \quad (5.23)$$

For the initial data large enough, namely as $\mathcal{R}(x_0, y_0) \rightarrow \infty$.
So,

$$t_2 - t_1 \leq \int_0^{\pi/2} \frac{2d\theta}{\sin^2\theta + K \cos^2\theta} = \frac{2}{\sqrt{K} \arctg(tg\theta/\sqrt{K})} \Big|_0^{\pi/2} = 2 \frac{\pi}{2\sqrt{K}} = \frac{\pi}{\sqrt{K}} \ll 1$$

Similarly, we can estimate:

$$t_6 - t_5 \leq \frac{\pi}{\sqrt{K}} \ll 1$$

Step 3. Estimate of $t_3 - t_2, t_5 - t_4$.

$$t_3 - t_2 = \int_{\theta(t_3)}^{\theta(t_2)} dt = \int_{\theta(t_3)}^{\theta(t_2)} \frac{d\theta}{\frac{d\theta}{dt}} = \int_{\theta(t_3)}^{\theta(t_2)} \frac{d\theta}{\sin^2\theta + (g(x(t-\tau)) - p(t)/r(t)) \cos\theta}.$$

Since $y(t) \leq 0$, we have $0 < x(t) \leq x(t-\tau)$, so there exists a constant $\sigma > 1$ such that So,

$$\frac{x(t-\tau)}{x(t)} \geq \sigma \quad (5.24)$$

From (78), for $r(t)$ large enough we have

$$\begin{aligned} \sin^2\theta + \frac{g(r(t-\tau)\cos\theta(t-\tau)) - p(t)}{r(t)} \cdot \cos\theta &= \sin^2\theta + \frac{g(r(t-\tau)\cos\theta(t-\tau))}{r(t-\tau)\cos\theta(t-\tau)} \times \\ \frac{r(t-\tau)\cos\theta(t-\tau)}{r(t)\cos\theta(t)} \cos^2\theta - \frac{p(t)}{r(t)} \cos\theta &\geq \sin^2\theta + K_1 \cdot \sigma \cos^2\theta - \frac{p(t)}{r(t)} \cos\theta = \\ &= \sin^2\theta + K_1 \sigma \cos^2\theta - \frac{p(t)}{r(t)} \cos\theta \geq \frac{1}{2}(\sin^2\theta + K_1 \cdot \sigma \cos^2\theta) \end{aligned} \quad (5.25)$$

For the initial data large enough, namely as $\mathcal{R}(x_0, y_0) \rightarrow \infty$.
So,

$$t_3 - t_2 \leq \int_0^{\pi/2} \frac{2d\theta}{\sin^2\theta + K_1 \sigma \cos^2\theta} = \frac{2}{\sqrt{K_1 \sigma} \arctg(tg\theta/\sqrt{K_1 \sigma})} \Big|_0^{\pi/2} = 2 \frac{\pi}{2\sqrt{K_1 \sigma}} = \frac{\pi}{\sqrt{K_1 \sigma}} \ll 1$$

Similarly, we can estimate:

$$t_5 - t_4 \leq \frac{\pi}{\sqrt{K_1 \sigma}} \ll 1$$

□

We are now ready to establish the existence of a periodic solution.

Let $z(t) = (x(t), y(t))$ satisfies (5.2) with the initial conditions $z_0 = (x_0, y_0) = (r_0 \cos \theta, r_0 \sin \theta) = (r_0, \theta_0)$. Then $\theta(T; r_0, \theta_0) = \theta_0(T, z_0)$. Consider

$$\Delta_1(z_0) = \theta(T, z_0) - \theta(0, z_0)$$

It is continuous at z_0 . Firstly, we take an appropriately large constant α_1 . Then there exists a positive integer K_1 such that $\inf \Delta_1(z_0) > -2K_1\pi$, $|z_0| = \alpha_1$, So

$$\theta(T, z_0) - \theta(0, z_0) > -2K_1\pi, |z_0| = \alpha_1 \quad (5.26)$$

On the other hand, from Lemma 8, there exists a constant $b_1 > \alpha_1$, such that

$$\theta(T, z_0) - \theta(0, z_0) < -2K_1\pi, |z_0| = b_1 \quad (5.27)$$

Consider the Poincare map associated to (5.2):

$$\mathcal{P} : \langle r_0, \theta \rangle \rightarrow \langle r(T, z_0), \theta(T, z_0) \rangle,$$

in the annular region \mathcal{A}_∞ , such that $\alpha_1^* \leq |z| \leq b_1^*$. From (5.26) and (5.27), it is twist in \mathcal{A}_∞ . Therefore, by Lemma 5, there exists at least one fixed point for the map \mathcal{P} , that is to say $\zeta = \langle \phi, \psi \rangle \in B(0, b_1^*)$, with

$$\theta(T, \zeta) - \theta(0, \zeta) = -2K_1\pi. \quad (5.28)$$

Thus $z = (t, \zeta)$ is a T-periodic solution of (5.1).

Bibliography

- [1] J. Clark, O. Misiats and S. Sayed, "Electrical Control of Effective Mass, Damping, and Stiffness of MEMS Devices", in Proc. IEEE Browse Journals and Magazines, 2016
- [2] A. Seiranian and Y. Wang, "On stability of Periodic Solutions for Duffing's Equation", Reports of the National Academy of Sciences of Armenia, 111, no.4, 2011.
- [3] M. Struwe, "Multiple Solution of Anticoercive Boundary Value Problems for a Class of Ordinary Differential Equations of Second Order", Journal of differential equations 37, pp. 285-295, 1979
- [4] K.Fucik and J.Lovicar, "Periodic solution of the equation $x'' + g(x) = p$ ", Casopis Pest. Mat. 100 (1975), pp. 160-175
- [5] R.Rusinek, A.Weremczuk, K.Kecik and J. Warminski, "Dynamics of a Time Delayed Duffing Oscillator", International Journal of Non-Linear Mechanics 65, pp. 98-106, 2014
- [6] I. Kovacic, M. Brennan, "The Duffing Equation, Wiley, Chichester, West Sussex", UK; Hoboken, NJ, 2011
- [7] Z.Cheng and J. Ren, "Harmonic and Subharmonic Solutions for Suplinear Duffing Equation with delay", Topological Method in Nonlinear Analysis 47, No 2, 2016
- [8] W.Ding, "Fixed points of twist mappings and periodic solutions of ordinary differential equations" Acta Math. Sinica 25, 1982.
- [9] M. Du and M.Kwong, "Sturm Comparison Theorems for Second-Order Delay Equations", Journal of Mathematical Analysis and Applications 152, 1990