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Abstract

Wallpaper patterns have been shown to be decomposable into standing waves of plane vibrations [6]. Previously unexplored are the sounds that arise from these vibrations. The main result of this paper is that each wallpaper type (square, hexagonal, rectangular, generic) has its own distinctive family of pitches relative to a fundamental. We review the method to make wallpaper with wave functions and describe new musical scales for each type, including initial attempts to use the scales: a movie showing vibrations of wallpaper patterns with 3- and 6-fold symmetry inspired a new piece by American composer William Susman, commissioned by the San Jose Chamber Orchestra, Barbara Day Turner, conductor. The piece, "In a State of Patterns," was premiered on March 25, 2018.

Introduction

Numbers are connected to music through physics: the standing waves of a violin string fixed at both ends correspond to sounds that are 1, 2, 3, and so on times a fundamental musical pitch. When the object making the sounds is not as simple as a string, the analysis is more complicated, but well understood. For instance, a circular drumhead produces a spectrum of sounds where the simple sequence 1, 2, 3, . . . is replaced by the zeroes of the not-so-well-known Bessel functions [1].

Even in the simple case of vibrations of a string, the application of standing waves to the practice of music quickly becomes complicated. Efforts to base musical scales on whole number ratios inevitably run into difficulties, which many past Bridges papers have addressed [8, 9, 10, 11]. One celebrated solution that connects musical practicality to the reality of physics is the *well-tempered* scale, which replaces whole number ratios by nearest powers of $2^{1/12}$. This is the scale we hear from modern pianos and most of us are accustomed to hearing notes produced by this reasonably satisfying approximation.

The same mathematics that models the violin string was behind my first article about wallpaper [7]. I took the strange leap of modeling a wallpaper pattern by a periodic deformation of an infinite flexible planar membrane, as if it were a drumhead. For instance, it is not too difficult to imagine all the red squares of an infinite checkerboard bulging up while the black ones bulge down. When such a membrane is released from rest, there is a simple model to predict how it progresses in time. My original intent was to construct patterns by determining the standing waves of these infinite wallpaper drums. An unplanned consequence is that every pattern I construct by this method has a natural way to vibrate. And since vibrations have frequencies, there is a natural connection to music, the exploration of which is the topic of this paper.

We first review some basics about conventional musical scales: how notes arise from whole number ratios and why approximations are commonly used. To make the article self-contained, we review how to make wallpaper patterns by superimposing standing waves, with formulas to show how patterns evolve naturally in time [7]. Then we describe the new scales that arise from the frequencies of these wallpaper waves—music from vibrating wallpaper.

Since wallpaper falls naturally into categories that depend on the underlying translation lattice—the square, hexagonal, rectangular, rhombic, and generic lattices—the musical scales are similarly organized. After we see precise formulas for the new notes sounded by hypothetical wallpaper drums, I speculate about practical ways to realize them in music: some pitches fall close enough to notes of a well tempered scale to suggest music that can be played on traditional instruments; in other cases, the new frequencies fall exactly between the cracks of the piano. We must use tone generators to hear those pitches exactly.

The new ideas are not just speculation. In the summer of 2017, I showed movies of vibrating wallpaper to Barbara Day Turner, founding conductor of the San Jose Chamber Orchestra. She planned a concert called "A Touch of Tech" for March, 2018 and suggested that the right composer might be able to write a piece using these ideas. Turner commissioned American composer William Susman [13] for the job. The new piece, "In a State of Patterns," accompanies wallpaper movies colored by a sequence of California photographs. It premiered on March 25, 2018 and a synthesized version is available as a supplement to this paper. The audience and at least one critic [3] deemed our collaboration a success, though it is surely not the end of the story of music from vibrating wallpaper.

Sounds and Whole Number Ratios

We start with the idea that the collection of pure sounds produced by a violin string correspond to motions of special, simple shapes of the string. If we assume the string has unit length, these are modeled by functions of a length variable, x:

$$\sin(n\pi x) \text{ for } n = 1, 2, 3, \ldots. \tag{1}$$

Each of the shapes has its own way to vibrate, with higher vibrations corresponding to higher values of n. You can hear the nth one by touching the string lightly at a point $1/n$th of the way along the string and bowing. I will refer to numbers that play a role like n plays here as *frequency numbers*. The sound we hear when $n = 1$ is called the *fundamental*. Throughout our discussion, all pitches are described in relation to a fundamental, which we take to be C. What do the rest sound like? It turns out that these are the fundamental sounds of Western music.

One basic understanding is that people seem perfectly able to identify one sound with another that vibrates at twice the speed. The sounds corresponding in our model to $n = 1$ and $n = 2$ are said to be "one octave apart." They sound like the same pitch even though the second sound is higher. This is the first step toward what I will call a *Pythagorean* sensibility: whole number ratios in frequencies give the most pleasing sounds.

Since musicians tend to lump together two sounds whose frequencies differ by a multiple of two, then the sound corresponding to $n = 3$ should be identified with a sound whose frequency is $3/2$ times the fundamental. If the fundamental is middle C (525.6 Hz), then this new note is G (788.4 Hz). We continue with successive integers, rejecting $n = 4$ as a new note, because we identify it with the fundamental, arriving at $5/4$ (which we know as E), rejecting $n = 6$ as $3/2$, and reducing $n = 7$ to $7/4$, which we hear as B ♭.

Our first four notes spell out a pleasing chord sung by many a barbershop quartet, C E G B♭. This chord and those derived from it form the basis for much of Western music.

Push comes to shove when we name D as $9/8$ from $n = 9$ and expect the distance from C to D to match that from D to E. The fact is:

$$1 \cdot \frac{9}{8} \cdot \frac{9}{8} = \frac{81}{64} = 1.265625 \neq \frac{5}{4} = 1.25.$$

So much for scales based purely on whole number ratios.

The *well tempered* solution is to define 12 notes by $N_j = 2^{j/12}$, so that $N_{12} = 2$, indicating a perfect octave. The ratio of successive pitches is a constant $2^{1/12}$, which is known as a *half step*. This is such a perfect mathematical solution, one might think it marks the end of the discussion of scales. No. For instance, instead of the sweet sound of C and E together, the *major third*, with a euphonious ratio of $5/4$, the piano gives us sounds in a ratio of 1.259921050, just a little too high. Enough human ears are sufficiently sensitive to make this a problem, one that we will not pursue much further.

We will, however, address one other interesting feature of the well tempered scale. At its exact center, the note $N_6 = \sqrt{2}$ is the *tritone*, considered grating to some ears (but famously featured by Saint-Saëns in *Danse Macabre*). From a Pythagorean point of view, it is way out there: the first note anywhere near to the tritone in our sequence of Western tones uses $n = 23$. We have to lower it four octaves before we can hear a ratio of $23/16 = 1.4375$, which is not so close to the tempered tritone. As our wallpaper scales develop, we will find much use for this shunned interval.

Wallpaper Drums

Given my background in differential geometry and analysis, it might not be too surprising that I rebelled against the idea that wallpaper patterns consist of copies of a discrete motif, stamped out across the plane. I saw a wallpaper pattern as a function of two variables, one that is invariant under two independent translations.

To make a long story short, suppose we want to make a pattern invariant under specific translations in two independent directions and possibly some other symmetries. In other words, we have in mind a *wallpaper group*, which is a group, G, of Euclidean isometries whose translations are generated by two smallest independent translations. A *wallpaper function* with group G is any $f(x, y)$ that satisfies

$$f(\gamma(x, y)) = f(x, y) \text{ for } \gamma \in G,\ \ x, y \in \mathbb{R}.$$

To construct wallpaper functions, Fourier analysis comes to the rescue. It turns out that every such function can be decomposed (in a suitable technical sense) into a sum of multiples of *standing waves*, by which I mean solutions to the partial differential equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\lambda f.$$

Functions like f are called *eigenfunctions* of the *Laplacian* Δ and the numbers λ are called *eigenvalues*.

We are compressing an important part of the story: if f is an eigenfunction of the Laplacian with eigenvalue λ, then multiplying f by $\cos(\sqrt{\lambda}t)$ produces a solution to the wave equation $\Delta f = \partial^2 f/\partial t^2$. Solutions to this linear equation correctly model the motion of vibrating membranes, as long as the vertical displacement is not too high, so the sounds we describe really could be heard.

Backing up for a moment to the case of the vibrating string, the relevant Laplacian is just $\partial^2 f/\partial x^2$. The boundary conditions $f(0) = f(1) = 0$, meaning that the (conveniently unit-length) string is fixed at both ends, restrict the solutions to be exactly the sine functions in (1). Inspection shows the eigenvalues to be n^2 for $n = 1, 2, 3, \ldots$. If a string were shaped into one of these waves and released from rest, its future progress in time would be $\sin(n\pi x)\cos(n\pi t)$, provided we measure time in suitable units. The temporal vibration from the cosine factor makes this one repeat n times as often as the base wave does, justifying the name *frequency*. In general, frequencies are square roots of eigenvalues. Note that space and time are defined only up to scale. The important thing is the sequence of eigenvalues in relation to the first one.

In the case of wallpaper drums, the wallpaper invariance plays the role of the boundary condition. The family of standing waves for a given wallpaper group depends only on the translations that belong to the group. We will show the details for wallpaper drums where the lattice unit is the unit square and summarize results for other cases when we apply the results to music.

The Square Symmetry The simplest examples to understand involve wallpaper groups whose translations are generated by $\tau_1(x, y) = (x + 1, y)$ and $\tau_2(x, y) = (x, y + 1)$. Evidently, every function of the form

$$E_{n,m}(x, y) = e^{2\pi i(nx+my)} \text{ where } n, m \in \mathbb{Z} \tag{2}$$

satisfies the given partial different equation and is invariant under all the given translations. Perhaps surprisingly, there is a sense in which *every* sufficiently nice wallpaper function for this group can be written as a sum of these fundamental waves.

For artistic purposes, we probably just want to superimpose a few of these building blocks. The set of functions

$$\left\{ \sum_{\text{finite}} a_{n,m} e^{2\pi i(nx+my)} \mid n, m \in \mathbb{Z} \right\}$$

gives plenty of room to play with wallpaper functions.

The complex-valued wave $E_{n,m}$ can be checked to be an eigenfunction of the Laplacian with eigenvalue $4\pi^2(n^2 + m^2)$. Therefore, the correct way to evolve it in time is to multiply it by $\cos\left(2\pi\sqrt{n^2 + m^2}t\right)$.

We started talking about vertical displacements of a flexible infinite membrane and are now looking at a set of complex-valued functions. If we really wanted to deform physical wallpaper drums, we could just superimpose the real and imaginary parts of these fundamental waves. For artistic purposes, we stick with complex-valued functions and use the *domain coloring algorithm* [6], which we quickly describe.

Suppose we want to find the color to use for a pixel in an image, which we have associated with a point in the plane, (x, y). A wallpaper function determines a complex number at each point of the plane, so we can use the real and imaginary parts of $f(x, y)$ to locate the horizontal and vertical location of a pixel in a photograph, which we then use as the color for our pixel. (If the complex number is too large, we need to specify a default color for overruns, usually black.)

Once we have wave functions with the correct translational symmetry, there is just a little more work to achieve additional symmetries. If we want a pattern with 4-fold rotational symmetry, we lock four waves together in packets, defining

$$S_{n,m}(x, y) = \left(e^{2\pi i(nx+my)} + e^{2\pi i(-ny+mx)} + e^{2\pi i(-nx-my)} + e^{2\pi i(ny-mx)}\right)/4.$$

Each wave is a 90° rotation of the next, so the sum of the four is rotationally invariant.

My favorite wallpaper group is called p4g by the International Union of Crystallographers and known as 4*2 in Conway's orbifold system [5, 6]. The glide symmetry present in p4g patterns arises from pairing certain waves together; computation shows that every function in the linear space

$$\mathcal{F}_{\text{p4g}} = \left\{ \sum a_{n,m} S_{n,m} \mid a_{n,m} \in \mathbb{C},\ a_{n,m} = (-1)^{n+m} a_{m,n} \right\}$$

has p4g symmetry. One example appears on the bottom left in Figure 1, where I chose a function of the form

$$a_{1,1}S_{1,1} + a_{1,2}(S_{1,2} - S_{2,1}) + a_{2,3}(S_{3,2} - S_{3,2}).$$

It was colored with the pixels from a photograph of peppers and a pumpkin. In the summer of 2016, students at Bowdoin College wrote *SymmetryWorks*, a public domain graphical interface that facilitates choices of frequency pairs and complex coefficients. It is freely available on GitHub [12].

The rest of the bottom row of the figure shows a primitive animation, where each wave $S_{n,m}(x, y)$ moves forward in time according to the formula

$$S_{n,m}(x, y)\cos\left(2\pi\sqrt{n^2 + m^2}t\right).$$

Wallpaper Scales

We have shown that if we beat an infinite wallpaper drum with square symmetry, the pure sounds we hear will correspond to frequency multiples

$$\sqrt{n^2 + m^2} \text{ for } n, m, \in \mathbb{Z}.$$

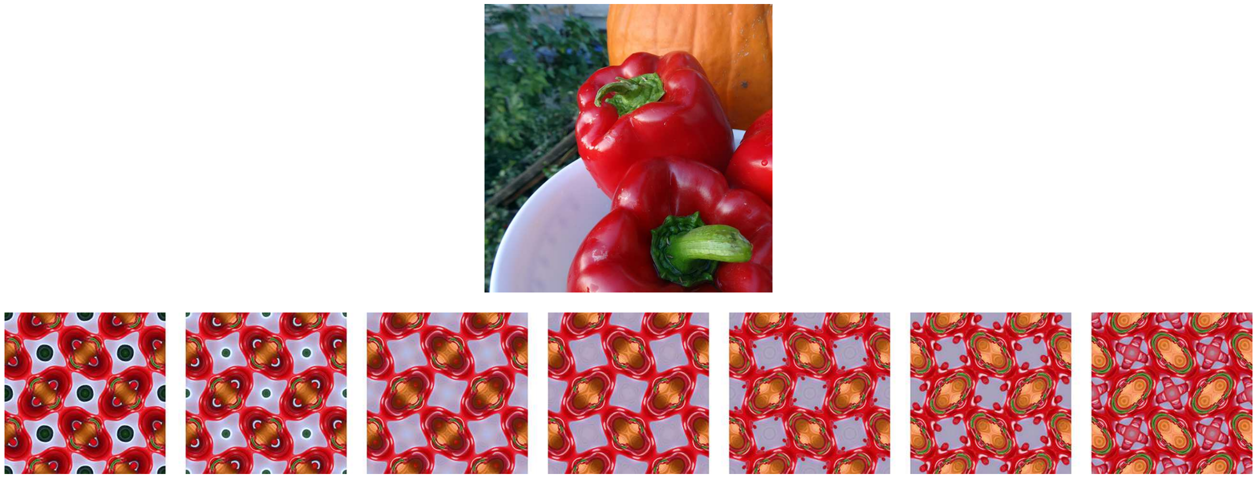

Figure 1: *A primitive animation of vibrating wallpaper: a p4g function, colored using a vegetable still life, evolves in time.*

To see how these can be realized as musical pitches, we first need to know which numbers appear in this form. The example of n^2+m^2 is ground zero in the theory of *quadratic forms* [4]. In this case, a classical result tells us that numbers delivered by the form as outputs have a special property in their prime factorizations: every prime of the form $4k+3$ must appear an even number of times. So $\sqrt{5}$ and $\sqrt{45}$ are legitimate frequencies, but $\sqrt{7}$ and $\sqrt{21}$ are not.

The succession of frequency numbers for the square lattice can be written, in order, as $1, \sqrt{2}, 2, \sqrt{5}, 3, \sqrt{8}, \sqrt{10}, \sqrt{13}, 4, \sqrt{17}, \sqrt{18}, \sqrt{20}, 5, \ldots$. When we realize these as frequencies, starting with a base tone that we may as well call C, the very first overtone is $\sqrt{2}$, the tritone that was abhorrent to certain classical ears! After a discardable octave for 2, the next note is $\sqrt{5}$. By the principle of octave reduction, this becomes $\sqrt{5}/2 \approx 1.118$, which can pass for a well-tempered 1.122, or D though it might sound a bit flat. Moving up, to identify just the novel tones, we pass over the G we knew we would find as 3/2, reject $\sqrt{8}$ as twice $\sqrt{2}$, and find $\sqrt{10} = \sqrt{2}\sqrt{5}$. This is a tritone up from the D we found from $\sqrt{5}$, so if we take D as a new note in our tonality, this next discovery would sound like A♭.

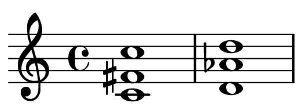

Figure 2: *A chord progression with frequencies based on the lowest eigenvalues in the spectrum of vibrating square wallpaper.*

Before we have finished with the first two octaves of our investigation (frequency numbers between 1 and 4), we have already found an interesting chord progression to replace the C E G B♭ that we plausibly associate with the violin string. Figure 2 shows the notes separated into two chords (doubling the octaves). I cannot find a direct quote, but this is certainly reminiscent of parallel sequences from Debussy's preludes.

Table 1 shows the first few *novel* frequencies for the vibrating square wallpaper drum. The violin frequencies (3/2, 5/4, and so on) still are present in the wallpaper drum, but we omit them for brevity. For each tone, we give the reduced frequency—diving by 2 to find a number in the interval $[1,2]$—the well-tempered note below it, the well-tempered note above it, and an indication of how the pitch would be perceived. A flattish note is indicated by ↓ and a sharpish one by ↑. If a note falls exactly between the cracks, as with $\sqrt{17}$, we list both notes, C/C♯ in this case.

Table 1: *Frequencies and associated well-tempered pitches from the spectrum of a vibrating square wallpaper drum. The top row lists square roots of values of the quadratic form $n^2 + m^2$, excluding whole numbers.*

Eigenvalue	1	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{10}$	$\sqrt{13}$	$\sqrt{17}$	$\sqrt{18}$	$\sqrt{20}$	$\sqrt{29}$
Red. Frequency	1	1.414	1.118	1.581	1.803	1.031	1.061	1.118	1.346
Lower WT	1.0	1.414	1.059	1.498	1.782	1.	1.059	1.059	1.335
Upper WT	1.0	1.414	1.122	1.587	1.888	1.0598	1.122	1.122	1.414
Heard as	C	F♯	D↓	A♭ ↓	B♭ ↑	C/C♯	C♯	D↓	F↑

If we really want to avoid finding notes we've seen before, we might have noticed that $\sqrt{10}$ gives a note that is as high above $\sqrt{2}$ as $\sqrt{5}$ is above the fundamental, after reducing by one octave. Once we have identified the tritone interval, we can naturally include a tritone up from every new note in our scales. The same might be said of $\sqrt{18}$, which should seem like a perfect fifth above $\sqrt{2}$. Indeed, it worked out to be a rather accurate C♯. And $\sqrt{20}$ is just an octave up from $\sqrt{5}$.

The Hexagonal Lattice Pattern types that use the hexagonal lattice require their own special waves. Rather than write the formulas, we note that computations produce waves similar to those in (2) [6]. Frames from a movie of vibrating wallpaper of type p31m appear in Figure 3.

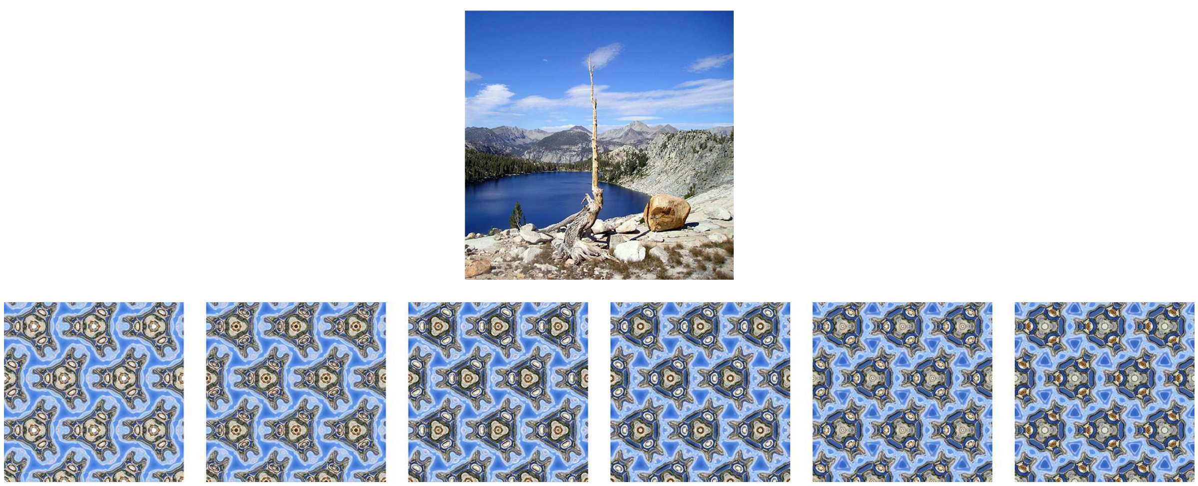

Figure 3: *Frames from the movie to accompany "In a State of Patterns," by William Susman. A p31m function, colored with a Sierra Nevada scene, evolves in time.*

The important thing for musical purposes is that the eigenvalues of waves periodic with respect to the hexagonal lattice are (proportional to)

$$n^2 - nm + m^2 \text{ where } n, m \in \mathbb{Z}.$$

This is another celebrated quadratic form about which much is known. For instance, although taking $m = 0$ shows that the form takes on every square-integer value, the form never delivers a number with an odd power of 5 or 13 in its factorization; this time, odd powers of primes congruent to 2 modulo 3 are the ones off-limits, creating a distinctively different spectrum of sounds.

Table 2: *Frequencies and associated well-tempered pitches from the spectrum of a vibrating hexagonal wallpaper drum. The top row lists square roots of prime values of the quadratic form* $n^2 - nm + m^2$.

Eigenvalue	1	$\sqrt{3}$	$\sqrt{7}$	$\sqrt{13}$	$\sqrt{19}$	$\sqrt{31}$	$\sqrt{43}$
Red. Frequency	1	1.732	1.323	1.803	1.090	1.392	1.639
Lower WT	1.0	1.682	1.260	1.782	1.0595	1.335	1.587
Upper WT	1.0	1.782	1.335	1.888	1.122	1.414	1.682
Heard as	C	A/B♭	F↓	B♭ ↑	C♯/D	F♯ ↓	A♭/A

Table 2 includes only the pitches that give us notes absent from the vibrating string spectrum. We also leave out numbers like $\sqrt{21}$ and $\sqrt{39}$, which give already-discovered intervals built on novel base tones. This one is sort of a mess. Everyone loves wallpaper patterns with 3- and 6-fold symmetry, but that first interval is rather strange, falling right between a major sixth and minor seventh. Nevertheless, when I realize it with an online tone generator, it sounds pleasant, though unfamiliar. Figure 4 shows a fairly pleasing chord progression, constructed from well tempered equivalents of the first few overtones, including one violin overtone (1.5 or G): 1-$\sqrt{3}$, $\sqrt{7}$-$\sqrt{19}$, and 1.5-$\sqrt{13}$. Note pairs indicate pitches that fall in the cracks of the piano. In a supplementary video file, you can watch the wallpaper vibrate and hear these sounds.

Figure 4: *Left: A chord progression built from the lowest vibrations of a vibrating hexagonal wallpaper drum. To indicate pitches that do not easily round to well tempered ones, two notes are shown half a step apart. Right: A similar sequence for a* $\phi \times 1$ *rectangular drum.*

Golden Wallpaper Other wallpaper types generically give rise to non-integer quadratic forms, so the connection to number theory breaks down. Nonetheless, I found a pleasant scale associated with rectangular wallpaper in a $\phi \times 1$ grid, where ϕ is the golden ratio. Now the quadratic form is $\phi^2 n^2 + m^2$. Table 3 shows some of the pitches and Figure 4 has the scale. Why did I leave out a note arising from $\phi^2 + 3^2$? When we perform octave reduction on this note, it works out to be almost the same pitch as $4\phi^2 + 1$!

Table 3: *Frequencies and associated well-tempered pitches from the spectrum of a vibrating* $\phi \times 1$ *wallpaper drum. The top row lists square roots of values of the quadratic form* $\phi^2 n^2 + m^2$, *excluding whole numbers.*

Eigenvalue	1	ϕ	$\sqrt{\phi^2+1}$	$\sqrt{\phi^2+4}$	3	$\sqrt{4\phi^2+1}$	$\sqrt{\phi^2+4^2}$	3ϕ
Red. Frequency	1	1.618	1.902	1.286	1.5	1.694	1.079	1.214
Lower WT	1.0	1.587	1.888	1.260	1.498	1.682	1.059	1.189
Upper WT	1.0	1.682	2	1.888	1.335	1.587	1.122	1.260
Heard as	C	A♭ ↑	B	E	G	A	C♯	E

Conclusion

For wallpaper patterns of various types, we have identified pitch collections that arise naturally from the vibrations of that pattern type. Making movies of vibrating wallpaper is now fairly easy, using an adaptation of the open-source *SymmetryWorks* software [12]; writing music based on these scales remains an open-ended experiment. In the end, Susman's composition, "In a State of Patterns," did not use the special scales based on wallpaper patterns. Instead, Susman took a post-minimalist approach stimulated by the shapes and rhythms he saw in the films (3- and 6-fold symmetries). The audience loved it and reviews were positive [3]. Video clips are available: one short passage of vibrating wallpaper with notes from the hexagonal scale and a five-minute version of Susman's piece with my animations. Between the visual potential of vibrating wallpaper and the new scales to accompany it, there is rich territory to be explored.

Acknowledgements

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