

ON ESTIMATING THE RELIABILITY IN A MULTICOMPONENT SYSTEM BASED
ON PROGRESSIVELY-CENSORED DATA FROM CHEN DISTRIBUTION

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ABSTRACT

This research deals with classical, Bayesian, and generalized estimation of stress-strength reliability parameter, $R_{s,k} = \Pr(\text{at least } s \text{ of } (X_1, X_2, \dots, X_k) \text{ exceed } Y) = \Pr(X_{k-s+1:k} > Y)$ of an s -out-of- k : G multicomponent system, based on progressively type-II right-censored samples with random removals when stress and strength are two independent Chen random variables. Under squared-error and LINEX loss functions, Bayes estimates are developed by using Lindley's approximation and Markov Chain Monte Carlo method. Generalized estimates are developed using generalized variable method while classical estimates - the maximum likelihood estimators, their asymptotic distributions, asymptotic confidence intervals, bootstrap-based confidence intervals - are also developed. A simulation study and a real-world data analysis are provided to illustrate the proposed procedures. The size of the test, adjusted and unadjusted power of the test, coverage probability and expected lengths of the confidence intervals, and biases of the estimators are also computed, compared and contrasted.

DEDICATION

This thesis is dedicated to my family. To my loving parents, Titus and Ibijoke Ajumobi who supported me emotionally and financially, and whose words of encouragement and unwavering support molded me into the man I am today. To my siblings, Oyinda, Siji and Damilare who have always believed in me and pushed me to be the best that I can be. Thank you for teaching me that life comes in phases and to take the time to enjoy the little moments.

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LIST OF ABBREVIATIONS

- BTCL, bootstrap-t confidence interval
- C-Method, classical method
- GV-Method, generalized variable method
- \mathcal{C} , Chen distributed
- pmf, probability mass function
- MLE, maximum likelihood estimate
- CI, confidence interval
- UMVUE, uniformly minimum variance unbiased estimator
- BBCACI, bootstrap bias-corrected and accelerated confidence interval
- MCMC, Markov Chain Monte Carlo
- LINEX, linear exponential loss function
- HPD, highest posterior density
- HPDI, highest posterior density interval
- BCI, Bayesian credible interval
- DWR, Department of Water Resources
- SSE, error sum of squares
- ACI, approximate confidence interval
- B-Method, Bayesian method
- MSE, mean squared error
- ER, estimated risk
- CP, coverage probability
- UP, unadjusted power of a statistical test
- AP, adjusted power of a statistical test
- UMVU, uniformly minimum variance unbiased

ANOVA, analysis of variance

ANORE, analysis of reciprocals

ANCOVA, analysis of covariance

ANOFRE, analysis of frequency

MANOVA, multivariate analysis of variance

MANCOVA, multivariate analysis of covariance

RAM, read access memory

ln, natural logarithm

GVM, generalized variable method

pdf, probability density function

iid, independent and identically distributed

OOL, offered optical network unit load

LIST OF SYMBOLS

X , random strength

Y , random stress

α , shape parameter of exponentiated inverted exponential distribution

λ , scale parameter of exponentiated inverted exponential distribution

$R_{s:k}$, reliability of a multi-component s -out-of- k : G system, where at least s out of the k components work (or are good)

$X_{r:n}$, r th order statistic of a simple random sample of size n for the random variable X

$f_X(x)$, probability density function of the random variable X when $X = x$

$F_X(x)$, cumulative distribution function of the random variable X when $X \leq x$

F_{v_1, v_2} , Fisher-Snedecore distribution (or simple F distribution) with numerator degrees of freedom v_1 and denominator degrees of freedom v_2

$\Pr(X \in D)$, Probability of X that belongs to the domain D

$\Pr(X > Y)$, Probability that strength X is greater than stress Y

\sim , distributed as

Σ , single summation

\prod , single product

$l(\beta)$, log-likelihood function of β

$\{X\}_{i=1,2,\dots,J; j=1,2,\dots,J}$, $I \times J$ matrix

$D(\alpha, \beta, \delta)$, statistical distribution D with a location parameter α , a shape parameter β , and a scale parameter γ

$\int_a^b y dx$, single definite integral of the function y with respect to x computed from a to b

$\binom{n}{r}$, n choose r ; or n combination r , where r items have been chosen from n items without regard to the order

$P(A|B)$, conditional probability of the event A given that event B has already occurred

$\hat{\alpha}$, estimator of the parameter α

A , estimator of the parameter α

$\hat{\alpha}_{\text{obs}}$, observed (or realized) value (or estimate) of the parameter α

$\prod \prod$, double product

$\Sigma \Sigma \Sigma \Sigma$, quintuple summation

H_0 , null hypothesis

H_a , alternative hypothesis

\mathbf{X} , vector of simple random sample from the strength

\mathbf{Y} , vector of simple random sample from the stress

\xrightarrow{D} , convergence in distribution

$\frac{\partial y}{\partial x}$, slope y over slope x ; partial derivative of the function y with respect to x

$\iint_D y dx$, double integral of the function y with respect to x computed in the domain D

$\frac{\partial^2 y}{\partial x^2}$, slope squared y over slope squared x ; second partial derivative of the function y with respect to x

$E(X)$, expected value of the random variable X

χ^2_ν , chi-squared distribution with ν degrees of freedom

$\hat{\alpha}_{\text{obs}}$, observed (or realized) value (or estimate) of the parameter α

Φ , distribution function of the standard normal distribution

CI_η^γ , a $\gamma\%$ confidence interval for the parameter η

$\hat{\alpha}^*$, bootstrap estimator of the parameter α

z_γ , γ th quantile of the standard normal distribution

$H(x)$, Cumulative distribution at the point $X = x$ of the random variable X

$B(\alpha, \beta)$, beta function with parameters α and β

${}_2F_1(\alpha, \beta; \gamma, z)$, hypergeometric function

\mathcal{E} , exponentially distributed

$\mathbb{I}_{m \times n}$, $m \times n$ matrix; matrix with m rows and n columns

a^b , a to the power of b ; a has been raised to the power of b

$\text{Re}(\xi)$, real part of the complex number ξ

$U(\theta|\mathbf{x})$, posterior distribution of θ given the random sample \mathbf{x}

$\pi(\theta)$, prior distribution of θ

$\min_{1 \leq i \leq n} (X_i)$, first order statistic of the random variable X

\mathcal{U}_D , discrete uniformly distributed

$||$, absolute value

\times , multiplication

$+$, addition

$-$, subtraction

$/$, division

$>$, greater than; more than

$<$, smaller than; less than

\geq , greater than or equals; at least

\leq , less than or equals; at most

$=$, equal sign

$\sqrt{\quad}$, square root

$\sum \sum$, double summation

$n \rightarrow \infty$, n approaches infinity

\approx , approximately

\mathcal{G} , gamma distributed

CHAPTER 1

INTRODUCTION

1.1 The Background of Exact Statistical Methods

Permutation methods have been in use since 1935, when Fisher utilized the methods for solving exact inference [17]. Ever since, the feasibility of such methods increased steadily as computing power became more robust. Permutation methods can now be easily employed in many situations without absent of the computational limitations that plagued previous generations. When statistical inferences are performed, more reliable, accurate, non-misleading results are provided, thereby outperforming procedures based on classical asymptotic and approximate statistical inference methods. The most prominent and major characteristic of exact methods is that statistical inferences are mainly based on exact probability statements that are valid for any sample size. Approximate tests make approximation to a desired distribution by making the sample size big enough so that the test will have a false rejection rate that is always equal to the significance level of the test. When the sample size is small, the asymptotic and other approximate results may lead to unreliable and misleading conclusions. Exact parametric procedures and exact nonparametric procedures are the two branches in exact statistics. When the cell counts are small – specifically, if more than twenty percent of the cells, with fixed marginal totals, have an expected count that is less than five – the χ^2 distribution may not a suitable distributional candidate of the Pearson C^2 or Likelihood Ratio G^2 statistics for testing independence of row and column variables. Such a situation is easily remedied by Fisher's exact test.

In the late 1980's, Weerahandi [54] searched for an extreme region, which is an unbiased subset of sample space formed by minimal sufficient statistics. This extreme region has the observed

sample points on its boundary and is utilized to generalize the existing p-values to come up with exact solutions for different problems that arise in hypothesis testing. For exact tests, readers are referred to Fisher [18], Weerahandi ([52], [51]), and many others. Motivated by a generalized test given by Weerahandi [54], Tsui and Weerahandi [49] formally introduced the notion of generalized p-values. Weerahandi [53] extended the classical definition of confidence intervals to obtain the generalized confidence intervals so that one can obtain reasonable interval estimates for situations where the classical approach fails or yield results lacking small sample accuracy. Even though the generalized confidence interval is an exact interval, they do not possess the repeated sampling property under the Neyman-Pearson framework. Nevertheless, even under the Neyman-Pearson framework, their actual probability coverage is almost the same as the desired nominal level. Recently, Weerahandi [50] introduced the notion of generalized point estimators. These notions are successfully applied to many areas in statistics, including anova, regression, mixed models, and growth curve models. The concept of generalized estimators, generalized pivotal quantities, generalized test variables, generalized p-values, and generalized confidence intervals have turned out to be very satisfactory for obtaining tests and confidence intervals for many complex problems (see Gamage and Weerahandi [19], Bebu and Mathew [7], Mu et al. [44], Tian and Wu [47], Weerahandi and Berger [55], Weerahandi and Johnson [56], Ananda and Weerahandi [4], Ananda ([3], [2], [1]), Gunasekera ([23], [24], [25], [26], [27]), Gunasekera et al. [29], Gunasekera and Ananda [28], and Krishnamoorthy and Lu [36]). For a recipe of constructing generalized pivotal quantities, see Iyer and Patterson [34]. But this method has also produced unsatisfactory results for some applications: for instance, the generalized variable method is very unsatisfactory for multivariate analysis of variance with arbitrary covariance matrices (see, Krishnamoorthy et al. [37], Krishnamoorthy and Lu [36]). Books by Weerahandi ([51], [52]) give a detailed, complete, and clear discussion, along with numerous examples, on the generalized variable method.

1.2 Chen Distributions

When modeling monotonic hazard rates, the exponential, gamma, lognormal, and Weibull distributions may be initial choices. However, these distributions have several limitations. First, none of them exhibit bathtub shapes for their hazard rate functions. These distributions exhibit only monotonically increasing, decreasing, or constant hazard rates. The most realistic hazard rate is bathtub-shaped. This occurs in most real-life systems. For instance, such shapes occur when the population is divided into several sub-populations having early failures, wear out failures, and more or less constant failures. Therefore, a perfect bathtub consists of two change points and a constant part enclosed within the change points. Usefulness of bathtub shape is well recognized in several fields. Many parametric probability distributions have been introduced to analyze real data sets with bathtub failure rates. Chen [10] proposed a new two-parameter lifetime distribution with bathtub-shaped or increasing failure rate function.

The new two-parameter distribution has some useful properties compared with other well-known models. Xie et al. [58] extended the Chen's distribution adding other parameter and named it the extended-Weibull distribution, due to relation to the Weibull distribution. Pappas et al. [45] proposed a four-parameter modified Weibull extension distribution using the Marshall and Olkin [42] technique. Therefore, one of its particular cases could be named as Marshall-Olkin extended Chen's distribution. Recently, Chaubey and Zhang [9] introduced another extension of the Chen's family. Chaubey and Zhang [9] also addressed the problem of estimation of parameters of the extended Chen's distribution, focusing on the maximum likelihood estimation (MLE) method. Related studies for other distributions can be found in Gupta and Kundu [30], Dey et al. [13], and Louzada et al. [41].

1.3 Reliability of a Multicomponent System

We treat the problem of classical, Bayesian, and generalized point and interval estimation of the reliability parameter $R_{s,k} = \Pr(\text{at least } s \text{ of the } (X_1, X_2, \dots, X_k) \text{ exceed } Y) = \Pr(X_{k-s+1:k} > Y)$ in the multicomponent stress-strength model [8]. This system consists of k statistically independent and identical strength components X_1, X_2, \dots, X_k , whose common probability density function (pdf) is $f_X(x)$, experienced by a common stress Y , whose pdf is $f_Y(y)$. The system functions when s ($1 \leq s \leq k$) or more of the components simultaneously survive. This system is referred to as an s -out-of- $k : G$ (or s -out-of- $k : F$) system because a k -component system works (or is good) if and only if at least s of the k components work (or are good), and the system is referred to as s -out-of- $k : F$ because the k -component system fails if and only if at least s of the k components fail. Based on these two definitions, a s -out-of- $k : G$ system is equivalent to an $(k - s + 1)$ -out-of- $k : F$ system.

In the reliability context, the multicomponent stress-strength model can be described as an assessment of reliability of an s -out-of- $k : G$ system. Its practical application range from communication and industrial systems to logistic and military systems. Multicomponent systems can be illustrated with several examples. The Airbus A-380 has four engines; while the Boeing 787 Dreamliner is a twin-engine jet-liner. An airplane which is capable of flying if and only if at least two of its four engines are functioning is an example of 2-out-of-4:G system. A more homely but complicated example of a multicomponent system would be a music (stereo Hi-Fi) system consisting of an FM tuner and record changer in parallel; connected in series with an amplifier and speakers (with two speakers, say A and B) connected in parallel. A panel consisting of k identical solar cells maintains an adequate power output if at least s cells are active during the duration of the mission.

Another example is seen in the construction of suspension bridges, the deck is supported by a series of vertical cables hung from the towers. Suppose a suspension bridge consisting of k number of vertical cable pairs. The bridge will only survive if minimum s number of vertical cable through the deck are not damaged when subjected to stresses due to wind loading, heavy traffic, corrosion,

etc. Another given example involves a V-8 engine of an automobile; it may be possible to drive the car if only four cylinders are firing. However, if less than four cylinders fire, then the automobile cannot be driven. Thus, the functioning of the engine may be represented by a 4-out-of-8 : G system.

Other examples include an electrical power station containing eight generating units produces the right amount of electricity only if at least 6 units are working; the demand of the electricity of a district is fulfilled only if 6-out-of-8 wind roses are operating at all times; a communication system for a navy can be successful only if 6 transmitters out of 10 are operational to cover a district; and lastly a semi-trailer pulled by a truck can be driven safely as long as 6-out-of-8 tires are in good conditions. For an extensive reviews of s -out-of- k and related systems, see Kuo and Zuo [39].

1.4 Reliability of a Multicomponent System Based on Chen Distributions

The main goal of this thesis is to obtain the estimates of $R_{s,k}$ under classical, Bayesian, and generalized frameworks when f_X and f_Y are newly introduced independent Chen distributions (Chen [10]) that have bathtub-shaped or increasing hazard functions. In addition, we observe progressively type-II censored samples with uniformly distributed random removals from the Chen distributed f_X and f_Y . Suppose X_1, X_2, \dots, X_k are independent random variables from the Chen distribution with shape parameters β and λ . For brevity, we shall also say that $X_j \sim \mathcal{C}(\lambda, \beta), j = 1, 2, \dots, s, \dots, k$, with its common survival function (sf) $S(x_j) = \exp\{\lambda[1 - \exp(x_j^\beta)]\}$ whereas $F(x_j) = 1 - \exp\{\lambda[1 - \exp(x_j^\beta)]\}$ and $f(x_j) = \lambda\beta x_j^{\beta-1} \exp\{\lambda[1 - \exp(x_j^\beta)] + x_j^\beta\}$ are the cumulative distribution function (cdf) and pdf of the Chen distribution, respectively. Similarly, Y is also distributed according to an independent Chen distribution with a common shape parameter β and a shape parameter η . We say that $Y \sim \mathcal{C}(\eta, \beta)$. If $\beta < 1$, the hazard function of Chen distribution has a bathtub shape, and has an increasing failure rate function, if $\beta \geq 1$. Note that, if λ (or η) = 1, the Chen distribution becomes

an exponential power distribution. See Wu [57], Kayal et al. [35], and the references therein for some recent developments on the Chen distribution.

The reliability in a multicomponent stress-strength model, based on strength $X_j \sim \mathcal{C}(\lambda, \beta), j = 1, 2, \dots, s, \dots, k$, and stress $Y \sim \mathcal{C}(\eta, \beta)$, is then given by

$$\begin{aligned}
R_{s,k} &= \Pr(\text{at least } s \text{ of the } (X_1, X_2, \dots, X_k) \text{ exceed } Y), \\
&= \Pr(X_{k-s+1:k} > Y), \\
&= \sum_{i=s}^k \binom{k}{i} \int_{-\infty}^{\infty} (1 - F_X(y))^i (F_X(y))^{k-i} dF_Y(y), \\
&= \alpha \eta \sum_{i=s}^k \binom{k}{i} \int_0^1 (\ln t)^{(\beta-1)/\beta} \exp\{(1-t)(\alpha i + \eta)\} [1 - \exp\{\alpha(1-t)\}]^{k-i} dt, \text{ where } t = \exp\{y^\beta\} \\
&= \eta \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \int_1^{\infty} \exp[(\lambda(i+j) + \eta)(1-t)] dt \\
&= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j \eta}{\lambda(i+j) + \eta} \tag{1.1}
\end{aligned}$$

Estimation of $R_{s,k} = \Pr(X_{k-s+1:k} > Y)$ is widely known as multicomponent stress-strength modeling. Many authors have discussed parametric and non-parametric inference on $R_{s,k}$ when complete random samples are available on each X_1, X_2, \dots, X_k and Y . We mention: Hanagal [32], Eryilmaz [16], and Rao et al. [46]. For a comprehensive discussion on different stress-strength models, along with more theories and examples, the reader is referred to the monograph of Kotz et al. [38].

There has been little work on parametric and non-parametric inference of $R_{s,k} = \Pr(X_{k-s+1:k} > Y)$ when samples available on X_1, X_2, \dots, X_k and Y are not complete. In reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Some real life examples giving censored data are: (i) sometimes a failure is planned and expected but does not occur due to operator error, equipment malfunction, test anomaly, etc; (ii) sometimes engineers plan a test program so that, after a certain time limit or number of failures, all other tests will be terminated. Among various censoring schemes, the type II progressive censoring

scheme has become very popular. It can be described as follows: Let n items be put in a life time study and $m (< n)$ items be completely observed; At the time of the first failure, r_1 surviving units are removed from the $n - 1$ remaining items; At the time of the next failure, r_2 items are randomly withdrawn from the $n - r_1 - 2$ remaining items; When the m th failure occurs all the $n - m - r_1 - \dots - r_{m-1}$ items are removed. See Kotz et al. [38] for more details.

In the above studies, the generalized variable method-based (Tsui and Weerahandi ([49]) and Bayesian method-based inferences for the reliability in multicomponent stress-strength system based on complete or censored data with fixed or random removals have not been discussed in the literature. Therefore, in this study, we discuss the classical, Bayesian, and generalized inference of the reliability parameter $R_{s,k}$ under the progressively type-II right censored samples with uniformly random removals, i.e., we develop inference procedures for $R_{s,k} = \Pr(X_{k-s+1:k} > Y)$ when X_1, X_2, \dots, X_k and Y are independent Chen random variables and samples available on them are progressively type-II right-censored with uniformly random removals.

The generalized variable method and its affiliated generalized p -value were recently introduced by Tsui and Weerahandi [49], and generalized confidence interval (CI) and generalized estimators by Weerahandi ([53], [50]) presenting them as extensions of – rather than alternatives to – classical methods of statistical evaluation. The concepts of generalized CI and generalized p -value have been widely applied to a wide variety of practical settings such as regression, Analysis of Variance (ANOVA), Analysis of Reciprocals (ANORE), Analysis of Covariance (ANCOVA), Analysis of Frequency (ANOFRE), Multivariate Analysis of Variance (MANOVA), Multivariate Analysis of Covariance (MANCOVA), mixed models, and growth curves where standard methods failed to produce satisfactory results obliging practitioners to settle for asymptotic results and approximate solutions. For example, see Weerahandi ([52], [51]), Gunasekera ([23], [24], [25], [26], [27]), Gunasekera et al. [29], and Gunasekera and Ananda [28]. For instructions on constructing generalized pivotal quantities, see Iyer and Patterson [34].

CHAPTER 2

THE CLASSICAL METHOD

2.1 Maximum Likelihood Estimator of $R_{s,k}$

Let X_1, X_2, \dots, X_k denote strength components that are statistically distributed with $\mathcal{C}(\lambda, \beta)$.

Consider that $X_{1j:m:n} \leq X_{2j:m:n} \leq \dots \leq X_{mj:m:n}$, $j = 1, 2, \dots, k$, is the corresponding progressively type-II right-censored sample from $\mathcal{C}(\lambda, \beta)$, with censoring scheme

$$\mathbf{R}_j = (R_{1j}, R_{2j}, \dots, R_{mj})^T = \mathbf{r}_j = (r_{1j}, r_{2j}, \dots, r_{mj})^T \text{ and } \mathbb{R} = \mathbb{R}_{m \times k} = \{\mathbf{R}_j\}_{j=1,2,\dots,k} =$$

$\{R_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k} = \mathbb{R} = \mathbb{R}_{m \times k} = \{\mathbf{r}_j\}_{j=1,2,\dots,k} = \{r_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k}$ where m denote the number of failures observed before termination from n items that are on test, and $r_{1j}, r_{2j}, \dots, r_{mj}$

denote the corresponding numbers of units randomly removed (withdrawn) from the j th strength component, where $j = 1, 2, \dots, k$. Furthermore, let $x_{1j:m:n} \leq x_{2j:m:n} \leq \dots \leq x_{mj:m:n}$, $j = 1, 2, \dots, k$

be the observed ordered strengths. Let r_{ij} denote the number of strength components removed at the time of the i th failure (or the lack of strength) of the j th strength component,

$$0 \leq r_{ij} \leq n - m - \sum_{l=1}^{i-1} r_{lj}, i = 2, 3, \dots, m-1; j = 1, 2, \dots, k \text{ with } 0 \leq r_{1j} \leq n - m \text{ and}$$

$$r_{mj} = n - m - \sum_{l=1}^{m-1} r_{lj}, \text{ where } r_{ij}'\text{'s are non-pre-specified integers and } n \text{ are pre-specified integers,}$$

where $j = 1, 2, \dots, k$. Note that if $r_{1j}, r_{2j}, \dots, r_{m-1,j} = 0$, so that $r_{mj} = n - m$, this scheme reduces to the conventional type-II right censoring scheme.

Also note that if $r_{1j} = r_{2j} = \dots = r_{mj} = 0$, so that $m = n$, the progressively type-II right censoring scheme reduces to the case of no censoring scheme (complete sample case). Similarly, consider that $Y_{1:m:n} \leq Y_{2:m:n} \leq \dots \leq Y_{m:m:n}$ is the corresponding progressively type-II right

censored sample from the Chen distribution $Y \sim \mathcal{C}(\eta, \beta)$, with censoring scheme

$$\mathbf{R}' = (R'_1, R'_2, \dots, R'_m) = \mathbf{r}' = (r'_1, r'_2, \dots, r'_m); \text{ where } m \text{ denote the number of failures observed before termination from } n \text{ items that are on test, and } r'_1, r'_2, \dots, r'_m \text{ denote the corresponding numbers}$$

of units randomly removed (withdrawn) from the test. Furthermore, let $y_{1:m:n} \leq y_{2:m:n} \leq \dots \leq y_{m:m:n}$ be the observed ordered lifetimes. Let r'_i denote the number of stress components removed at the time of the i th failure of the stress component, $0 \leq r'_i \leq n - m - \sum_{l=1}^{i-1} r'_l, i = 2, 3, \dots, m - 1$ with $0 \leq r'_1 \leq n - m$ and $r'_m = n - m - \sum_{l=1}^{m-1} r'_l$, where r'_i 's are non-pre-specified integers and m are pre-specified integers. Note that if $r'_1, r'_2, \dots, r'_{m-1} = 0$, so that $r'_m = n - m$, this scheme reduces to the conventional type II right censoring scheme. Also note that if $r'_1 = r'_2 = \dots = r'_m = 0$, so that $m = n$, the progressively type II right censoring scheme reduces to the case of no censoring scheme (complete sample case).

The conditional likelihood function of the unknown parameters based on the observed sample is then given as

$$L(\lambda, \eta, \beta; \mathfrak{X}, \mathbf{y} | \mathbb{R} = \mathfrak{r}, \mathbf{R}' = \mathbf{r}') = \lambda^{mk} \eta^m \beta^{m(k+1)} \prod_{i=1}^m \prod_{j=1}^k C_{ij} \left\{ x_{ij}^{\beta-1} \exp \left[\lambda \left(1 - \exp(x_{ij}^{\beta}) \right) + x_{ij}^{\beta} \right] \right\} \times \\ \left\{ \exp \left\{ \lambda \left[1 - \exp(x_{ij}^{\beta}) \right] \right\} \right\}^{r_{ij}} \times \prod_{i=1}^m \left\{ C_i y_i^{\beta-1} \exp \left[\eta \left(1 - \exp(y_i^{\beta}) \right) + y_i^{\beta} \right] \right\} \times \\ \left\{ \exp \left\{ \eta \left[1 - \exp(y_i^{\beta}) \right] \right\} \right\}^{r'_i} \quad (2.1)$$

and the log-likelihood function is then given by

$$l(\lambda, \eta, \beta; \mathfrak{X}, \mathbf{y} | \mathbb{R} = \mathfrak{r}, \mathbf{R}' = \mathbf{r}') = nk \ln \lambda + n \ln \eta + n(k+1) \ln \beta + (\beta - 1) \sum_{i=1}^m \left(\sum_{j=1}^k \ln x_{ij} + \ln y_i \right) \\ + \left(\sum_{i=1}^m \sum_{j=1}^k x_{ij}^{\beta} + y_i^{\beta} \right) - \lambda v_{\beta} - \eta w_{\beta},$$

where $\mathfrak{X} = \mathfrak{X}_{m \times k} = \{x_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}$ and $\mathbf{y} = \{y_i\}_{i=1,2,\dots,m}$,

$$w_{\beta} = -\sum_{i=1}^n \sum_{j=1}^k (1 + r_{ij}) [1 - \exp(x_{ij}^{\beta})], v_{\beta} = -\sum_{i=1}^m (1 + r'_i) [1 - \exp(y_i^{\beta})],$$

$$C_{ij} = n - \sum_{l=1}^{i-1} (1 + r_{lj}), C_i = n - \sum_{l=1}^{i-1} (1 + r'_l), \text{ and } \mathbb{R} = \mathbb{R}_{m \times k} = \{\mathbf{R}_j\}_{j=1,2,\dots,k} =$$

$$\{\mathbf{R}_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}, \mathfrak{r} = \mathfrak{r}_{m \times k} = \{\mathbf{r}_j\}_{j=1,2,\dots,k} = \{r_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}, \mathbf{R}' = (R'_1, R'_2, \dots, R'_m),$$

$$\text{and } \mathbf{r}' = (r'_1, r'_2, \dots, r'_m).$$

Now, suppose that the number of units removed at each failure time

$R_{ij} (i = 1, 2, \dots, m - 1; j = 1, 2, \dots, k)$ follows a discrete uniform distribution; for brevity, we say

$R_{ij} \sim \mathcal{U}_D(0, n - m - \sum_{l=1}^{i-1} r_{lj})$; with probability mass function (pmf)

$$P(R_{ij} = r_{ij} | R_{i-1,j} = r_{i-1,j}, R_{i-2,j} = r_{i-2,j}, \dots, R_{1j} = r_{1j}) = \frac{1}{n - m - \sum_{l=1}^{i-1} r_{lj} + 1},$$

$$i = 2, 3, \dots, m-1; j = 1, 2, \dots, k,$$

and

$$P(R_{1j} = r_{1j}) = \frac{1}{n - m + 1}.$$

Suppose further that $R_{ij} (i = 1, 2, \dots, m-1)$ is independent of $x_{ij:m:n}$, then the unconditional likelihood function can be expressed as

$$L(\lambda, \beta) = L(\lambda, \beta; \mathbf{x} | \mathbb{R} = \mathbf{r}) P(\mathbf{R}_j = \mathbf{r}_j),$$

where $P(\mathbf{R}_j = \mathbf{r}_j) = \prod_{i=1}^{m-1} P(R_{ij} = r_{ij} | R_{i-1,j} = r_{i-1,j}, R_{i-2,j} = r_{i-2,j}, \dots, R_{1j} = r_{1j})$, $L(\lambda, \beta; \mathbf{x} | \mathbb{R} = \mathbf{r}) = \lambda^{mk} \beta^{m(k+1)} \prod_{i=1}^m \prod_{j=1}^k C_{ij} \left\{ x_{ij}^{\beta-1} \exp \left[\lambda \left(1 - \exp(x_{ij}^{\beta}) \right) + x_{ij}^{\beta} \right] \right\} \times \left\{ \exp \{ \lambda [1 - \exp(x_{ij}^{\beta})] \} \right\}^{r_{ij}}$
with $C_{ij} = n - \sum_{l=1}^{i-1} (1 + r_{lj})$.

It is evident that $P(\mathbf{R}_j = \mathbf{r}_j)$ does not depend on the parameters λ and β , and hence the MLEs of those parameters can be obtained by the conditional likelihood function given in (2.1) directly. In a similar fashion, we can write the similar expressions for the stress random variable Y . Therefore, assuming that β is given (or known), the maximum likelihood estimates (MLE) of λ and η can be derived by solving the equations:

$$\frac{d}{d\lambda} \ln L(\lambda, \eta, \beta) = \frac{nk}{\lambda} - \sum_{i=1}^n \sum_{j=1}^k (1 + r_{ij}) [1 - \exp(x_{ij}^{\beta})] = 0$$

and

$$\frac{d}{d\eta} \ln L(\lambda, \eta, \beta) = \frac{n}{\eta} - \sum_{i=1}^m (1 + r'_i) [1 - \exp(y_i^{\beta})] = 0$$

Hence, we can show that the MLEs $\hat{\lambda}$ of λ and $\hat{\eta}$ of η are, respectively, given by

$$\hat{\lambda} = \frac{mk}{-\sum_{i=1}^m \sum_{j=1}^k (1 + r_{ij}) [1 - \exp(x_{ij}^{\beta})]}, \quad (2.2)$$

and

$$\hat{\eta} = \frac{m}{-\sum_{i=1}^m (1 + r'_i) [1 - \exp(y_i^{\beta})]}. \quad (2.3)$$

Now, let $Z_{ij:m:n} = (1 + r_{ij})[1 - \exp(x_{ij}^\beta)]$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, k$. It is easy to show that $Z_{1j:m:n} \leq Z_{2j:m:n} \leq \dots \leq Z_{mj:m:n}$; $j = 1, 2, \dots, k$ is a progressively type-II right-censored sample from the exponential distribution with mean $(1/\lambda)$. For a fixed set of $\mathbf{R}_j = \mathbf{r}_j = (r_{1j}, r_{2j}, \dots, r_{mj})$, let us consider the following scaled (generalized) spacings

$$W_{1j} = nZ_{1j:m:n}$$

$$W_{2j} = (n - r_{1j} - 1)(Z_{2j:m:n} - Z_{1j:m:n})$$

.

.

.

$$W_{ij} = (n - \sum_{l=1}^{i-1} r_{lj} - (i-1))(Z_{ij:m:n} - Z_{i-1j:m:n})$$

.

.

.

$$W_{mj} = (n - \sum_{l=1}^{m-1} r_{lj} - (m-1))(Z_{mj:m:n} - Z_{m-1j:m:n})$$

Balaksrihnan and Aggarwala [5] proved that the progressively type-II right-censored spacings W_{ij} , for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, k$ are all independent and identically distributed as exponential with the mean $(1/\lambda)$, that is, $W_{ij} \sim \mathcal{E}(1/\lambda) = \mathcal{G}(1, 1/\lambda)$, where $\mathcal{E}(\alpha)$ is an exponential distribution with a mean (or scale parameter) α , and $\mathcal{G}(\gamma, \varepsilon)$ is a gamma distribution with a shape parameter γ and a scale parameter ε . Then, $W_\beta = \sum_{i=1}^m \sum_{j=1}^k W_{ij} = -\sum_{i=1}^m \sum_{j=1}^k (1 + R_{ij})[1 - \exp(X_{ij:m:n}^\beta)]$, $\sim \mathcal{G}(mk, 1/\lambda)$. In a similar fashion, we can show that

$$V_\beta = \sum_{i=1}^m V_i = -\sum_{i=1}^m (1 + R'_i)[1 - \exp(Y_{i:m:n}^\beta)] \sim \mathcal{G}(m, 1/\eta).$$

It can be seen that (W_β, V_β) is a complete sufficient statistics for (λ, η) . Let $\Lambda = 2mk\lambda / L$ and $\Delta = 2m\eta / E$, and W_β and V_β have gamma distributions with parameters (mk, λ^{-1}) and (m, η^{-1}) , respectively, we can then show

$$\Lambda \sim \chi_{2mk}^2 \text{ and } \Delta \sim \chi_{2m}^2,$$

where L is the estimator of λ , that is, $\widehat{\lambda}$, E is the estimator of η , that is, $\widehat{\eta}$, and χ_v^2 denotes a central chi-square distribution with v degrees of freedom.

Hence, the MLE of $R_{s,k}$ due to the invariance property of estimators is given by

$$\begin{aligned}\widehat{R}_{s,k}^M &= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j E}{L(i+j) + E} \\ &= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j}{1 + \frac{L}{E}(i+j)} \\ &= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \widehat{R}_{ij}\end{aligned}\tag{2.4}$$

where $\widehat{R}_{ij} = (-1)^j / [1 + L(i+j)/E]$.

Since $2mk(L\lambda)^{-1} \sim \chi_{2mk}^2$ and $2m(E\eta)^{-1} \sim \chi_{2m}^2$,

$$\widehat{R}_{ij} = \frac{(-1)^j}{1 + \frac{L}{E}(i+j)F_{ij}},$$

where

$$F_{ij} = \frac{R_{ij}}{1 - R_{ij}} \times \frac{1 - \widehat{R}_{ij}}{\widehat{R}_{ij}} \sim F_{2mk, 2m},$$

with F_{v_1, v_2} denotes a central F -distribution with v_1 numerator df and v_2 denominator df, and \widehat{R}_{ij} 's pdf is given by

$$\begin{aligned}f_{\widehat{R}_{ij}}(\chi) &= \frac{1}{\chi^2 B(m, m)} \left(\frac{k\eta}{\lambda} \right)^{mk} \times \frac{\left(\frac{1-\chi}{\chi} \right)^m}{\left(1 + \frac{k\eta}{\lambda} \left(\frac{1-\chi}{\chi} \right) \right)^{(mk+m)}}; \\ 0 &\leq \chi \leq 1; \lambda, \eta > 0,\end{aligned}$$

where $B(\zeta, \xi)$ is the beta function given by $\int_0^1 w^{\zeta-1} (1-w)^{\xi-1} dw$.

2.2 Uniformly Minimum Variance Unbiased Estimator of $R_{s,k}$

In this section, we obtain the uniformly minimum variance unbiased estimator (UMVUE) of $R_{s,k}$. Using the linearity property of the UMVUE, it suffices to find the UMVUE of parametric function

$v(\lambda, \eta) = \eta/[\lambda(i+j) + \eta]$. We know that

$(W_\beta = -\sum_{i=1}^n \sum_{j=1}^k (1+R_{ij})[1 - \exp(X_{ij}^\beta)], V_\beta = -\sum_{i=1}^m (1+R'_i)[1 - \exp(Y_i^\beta)])$ is a complete sufficient statistic for (λ, η) and their densities are gamma distributions with parameters (mk, λ) and (m, η) , respectively, we take $(W_\beta, V_\beta) = (\tilde{\rho}, \tilde{\varphi})$. To derive the UMVUE of $R_{s,k}$, we need the following lemma:

Lemma 1: Define

$$\Psi(\tilde{\rho}^*, \tilde{\varphi}^*) = \begin{cases} 1, & \text{if } \tilde{\varphi}^* > (i+j)\tilde{\rho}^* \\ 0, & \text{if } \tilde{\varphi}^* \leq (i+j)\tilde{\rho}^* \end{cases},$$

where $\tilde{\rho}^* = (1+R_{11})[\exp(X_{11}^\beta) - 1]$ and $\tilde{\varphi}^* = (1+R'_1)[\exp(Y_1^\beta) - 1]$. Then, $\Psi(\tilde{\rho}^*, \tilde{\varphi}^*)$ is an unbiased estimator of $v(\lambda, \eta)$.

Proof: Notice that $\tilde{\rho}^*$ and $\tilde{\varphi}^*$ are independent and follow exponential distributions with parameters λ and η , respectively. Then, we can obtain that

$$\begin{aligned} E(\Psi(\tilde{\rho}^*, \tilde{\varphi}^*)) &= P(\tilde{\rho}^* > (i+j)\tilde{\varphi}^*) \\ &= \lambda \eta \int_0^\infty \int_0^{\frac{\tilde{\rho}^*}{(i+j)}} e^{-\lambda \tilde{\rho}^*} e^{-\eta \tilde{\varphi}^*} d\tilde{\varphi}^* d\tilde{\rho}^* \\ &= \lambda \int_0^\infty e^{-\lambda \tilde{\rho}^*} \left[1 - e^{-\frac{\tilde{\rho}^*}{(i+j)}} \right] d\tilde{\rho}^* \\ &= \lambda \left[\frac{1}{\lambda} - \frac{1}{\lambda + \eta/(i+j)} \right] \\ &= \frac{\eta}{\lambda(i+j) + \eta} \end{aligned}$$

This completes the proof of the lemma.

Now, the UMVUE of $v(\lambda, \eta)$, say $\hat{v}(\lambda, \eta)$, can be obtained by using the Lehmann-Scheffé Theorem and it is given by

$$\begin{aligned} \hat{v}(\lambda, \eta) &= E(\Psi(\tilde{\rho}^*, \tilde{\varphi}^*) | \tilde{\rho} = \rho, \tilde{\varphi} = \varphi) \\ &= P(\tilde{\rho}^* > (i+j)\tilde{\varphi}^* | \tilde{\rho} = \rho, \tilde{\varphi} = \varphi) \\ &= \int_{\Phi} \int_{\tilde{\rho}^* | \tilde{\rho} = \rho} f_{\tilde{\rho}^* | \tilde{\rho} = \rho}(\tilde{\rho}^* | \rho) f_{\tilde{\varphi}^* | \tilde{\varphi} = \varphi}(\tilde{\varphi}^* | \varphi) d\tilde{\varphi}^* d\tilde{\rho}^* \end{aligned}$$

where $\Phi\{(\tilde{\rho}^*, \tilde{\varphi}^*); 0 < \tilde{\rho}^* < \rho, 0 < \tilde{\varphi}^* < \varphi, \tilde{\varphi}^* > (i+j)\tilde{\rho}^*\}$. The double integral in Equation () can be discussed in three cases, That is,

Case(i) $\phi(i+k) < \rho$, Case(ii) $\phi(i+k) > \rho$, and Case(iii) $\phi(i+k) = \rho$. Therefore, we have

Case (i):

$$\begin{aligned}
\widehat{v}(\lambda, \eta) &= \frac{(m-1)(mk-1)}{\rho\phi} \int_0^\phi \int_{\phi^*(i+j)}^\rho \left(1 - \frac{\rho^*}{\rho}\right)^{mk-2} \left(1 - \frac{\phi^*}{\phi}\right)^{m-2} d\rho^* d\phi^* \\
&= \frac{(m-1)}{\phi} \int_0^\phi \left(1 - \frac{\phi^*}{\phi}\right)^{m-2} \left[1 - \frac{\phi^*(i+j)}{\rho}\right]^{mk-1} d\phi^* \\
&= \sum_{r=0}^{mk-1} (-1)^r \left[\frac{(i+j)\phi}{\rho}\right]^r \frac{\binom{mk-1}{r}}{\binom{m+r-1}{r}}
\end{aligned}$$

Case (ii):

$$\begin{aligned}
\widehat{v}(\lambda, \eta) &= \frac{(m-1)(mk-1)}{\rho\phi} \int_0^\rho \int_0^{\frac{\rho^*}{i+j}} \left(1 - \frac{\rho^*}{\rho}\right)^{mk-2} \left(1 - \frac{\phi^*}{\phi}\right)^{m-2} d\phi^* d\rho^* \\
&= \frac{mk-1}{\rho} \int_0^\rho \left(1 - \frac{\rho^*}{\rho}\right)^{mk-2} \left[1 - \left(1 - \frac{\phi^*}{(i+j)\phi}\right)^{m-1}\right] d\rho^* \\
&= 1 - \sum_{r=0}^{m-1} (-1)^r \left[\frac{\rho}{\phi(i+j)}\right]^r \frac{\binom{m-1}{r}}{\binom{mk+r-1}{r}}
\end{aligned}$$

Case (iii):

$$\begin{aligned}
\widehat{v}(\lambda, \eta) &= \frac{(m-1)(mk-1)}{\rho\phi} \int_0^\phi \int_{\phi^*(i+j)}^\rho \left(1 - \frac{\rho^*}{\rho}\right)^{mk-2} \left(1 - \frac{\phi^*}{\phi}\right)^{m-2} d\rho^* d\phi^* \\
&= \frac{(m-1)}{\phi} \int_0^\phi \left(1 - \frac{\phi^*}{\phi}\right)^{m-2} \left[1 - \frac{\phi^*(i+j)}{\rho}\right]^{mk-1} d\phi^* \\
&= \frac{(m-1)}{\phi} \int_0^\phi \left(1 - \frac{\phi^*}{\phi}\right)^{mk+m-3} d\phi^* \\
&= \frac{m-1}{mk+m-2}
\end{aligned}$$

Hence, the UMVUE of $R_{s,k}$ is now given by

$$\widehat{R}_{s,k}^U = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \widehat{v}(\lambda, \eta)$$

2.3 Asymptotic distribution of $\widehat{R}_{s,k}$

Suppose that $\delta = (\lambda, \eta)$ is a vector of parameters of interest and $\widehat{\delta} = (L, E)$ be its MLE. Therefore, it is known that $R_{s,k}$ is a function of $\delta = (\lambda, \eta)$, i.e., $R_{s,k} = g(\delta)$, then by the invariance property of MLEs, $\widehat{R}_{s,k} = g(\widehat{\delta}) = g(L, E)$. The classical pivotal quantity, denoted by $T_{R_{s,k}}^c(\mathcal{X}, \mathbf{Y}, \delta)$ or simply by $T_{R_{s,k}}^c$, where $\mathcal{X} = \mathcal{X}_{m \times k} = \{X_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}$ and $\mathbf{Y} = \{Y\}_{i=1,2,\dots,m}$, based on the large sample procedure for testing

$$H_0 : R_{s,k} \leq R_0 \text{ vs. } H_a : R_{s,k} > R_0, \text{ where } R_0 \text{ is a given quantity,} \quad (2.5)$$

is given by

$$T_{R_{s,k}}^c(\mathcal{X}, \mathbf{Y}, \delta) = T_{R_{s,k}}^c = (\widehat{R}_{s,k} - R_{s,k}) \sqrt{I_m^*(R_{s,k})^{-1}} \xrightarrow{D} N(0, 1),$$

here \xrightarrow{D} denotes the ‘‘convergence in distribution’’ and $\sigma_{\widehat{R}_{s,k}}^2 = I_m^*(R_{s,k})^{-1}$ is the asymptotic variance (or the mean squared error (MSE) for unbiased $\widehat{R}_{s,k}$) of $\widehat{R}_{s,k}$ with $I_m^*(R_{s,k})$ being the the Fisher information (or the expected Fisher information) matrix. $I_m^*(R_{s,k})$ for the new parameterization $R_{s,k}$ is obtained using the chain rule as

$$I_n^*(R_{s,k}) = J(R_{s,k})^T I_m(\delta) J(R_{s,k}),$$

where $J(R_{s,k})$ is the Jacobian matrix with elements $J(R_{s,k}) = (\partial R_{s,k} / \partial \lambda, \partial R_{s,k} / \partial \eta)$ and $I_m(\delta)$ is the observed information matrix of δ , whose ij^{th} element is given by $I_m(\delta)_{ij} = -E[\partial^2 l(\delta) / \partial i \partial j]$, for $i, j = \lambda, \eta$, with $l(\delta) = l(\lambda, \eta; \mathcal{X}, \mathbf{Y})$ as in (2.2). Therefore, the asymptotic variance of $\widehat{R}_{s,k}$ is given by

$$\sigma_{\widehat{R}_{s,k}}^2 = \left(\frac{\partial R_{s,k}}{\partial \lambda} \right)^2 \frac{\lambda^2}{15} + \left(\frac{\partial R_{s,k}}{\partial \eta} \right)^2 \frac{\eta^2}{m},$$

where

$$\begin{aligned}\frac{\partial R_{s,k}}{\partial \lambda} &= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^{j+1} \eta(i+j)}{(\lambda(i+j) + \eta)^2} \quad \text{and} \\ \frac{\partial R_{s,k}}{\partial \eta} &= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j \lambda(i+j)}{(\lambda(i+j) + \eta)^2}.\end{aligned}$$

The Asymptotic variance as well as the asymptotic one-and two-sided confidence intervals for $R_{s,k}$ can also be achieved through the following procedure. Let us consider

$\mathbb{X} = \{X_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}$ and $\mathbf{Y} = (Y)_{i=1,2,\dots,m}$. To compute the confidence interval of $R_{s,k}$, consider the log-likelihood function of the observed sample, which is given by

$$\begin{aligned}l(\delta) = l(\lambda, \eta, \beta; \mathbb{X}, \mathbf{y}) &= mk \ln \lambda + m \ln \eta + m(k+1) \ln \beta + (\beta - 1) \sum_{i=1}^m \left(\sum_{j=1}^k \ln x_{ij} + \ln y_i \right) \\ &\quad + \left(\sum_{i=1}^m \sum_{j=1}^k x_{ij}^\beta + y_i^\beta \right) - \lambda v_\beta - \eta w_\beta,\end{aligned}$$

We denote the expected Fisher information matrix of $\delta = (\lambda, \eta)$ as $\mathbb{I}(\delta) = E[\mathbb{I}^\dagger(\delta)]$, where

$\mathbb{I}^\dagger(\delta) = [I^\dagger_{ij}]_{i,j=1,2} = \left[-\frac{\partial^2 l(\delta)}{\partial i \partial j} \right]_{i,j=\lambda,\eta}$ is the observed information matrix. That is

$$\mathbb{I}^\dagger(\delta) = - \begin{bmatrix} \frac{\partial^2 l(\delta)}{\partial^2 \lambda} & \frac{\partial^2 l(\delta)}{\partial \lambda \partial \eta} \\ \frac{\partial^2 l(\delta)}{\partial \eta \partial \lambda} & \frac{\partial^2 l(\delta)}{\partial^2 \eta} \end{bmatrix}.$$

The p -value for testing hypotheses in (2.5), based on the asymptotic distribution of $R_{s,k}$, is given by

$$p_{R_{s,k}} = 1 - \Phi(q_{\widehat{R}_{s,k}}), \quad (2.6)$$

where $q_{\widehat{R}_{s,k}} = (\widehat{r}_{s,k} - R_0) s_{\widehat{R}_{s,k}}^{-1}$, and $q_{\widehat{R}_{s,k}}^c, \widehat{r}_{s,k}$, respectively, are the observed values of

$Q_{\widehat{R}_{s,k}} = \left(\widehat{R}_{s,k} - R_0 \right) S_{\widehat{R}_{s,k}}^{-1}$ and $\widehat{R}_{s,k}$; $\Phi(\cdot)$ is the distribution function of the standard distribution on.

A $100(1 - \gamma)\%$, asymptotic confidence interval (ACI) for $R_{s,k}$, based on the above asymptotic distribution, is given by

$$\text{ACI}_{R_{s,k}}^{1-\gamma} = \left(\widehat{r}_{s,k} - Z_{\gamma/2} s_{\widehat{R}_{s,k}}, \widehat{r}_{s,k} + Z_{\gamma/2} s_{\widehat{R}_{s,k}} \right) \quad (2.7)$$

where Z_η is η th quantile (or 100η th percentile) of the standard normal distribution. A one-sided $100(1 - \gamma)\%$ asymptotic lower confidence interval (ALCI) for $R_{s,k}$ is given by

$$\text{ALCI}_{R_{s,k}}^{1-\gamma} = \left(\widehat{r}_{s,k}, \widehat{r}_{s,k} + Z_{\gamma/2} s_{\widehat{R}_{s,k}} \right) \quad (2.8)$$

It is clear that the confidence intervals for $R_{s,k}$ based on the asymptotic results do not perform very well for small sample sizes. So, two confidence intervals based on the parametric bootstrap methods for estimating $R_{s,k}$ are proposed: (i) percentile bootstrap method (Efron [15]) (we call it from now on as boot-p), and (ii) studentized bootstrap method or bootstrap-t method (we call it for now on as boot-t) (Hall [31]).

(i) Percentile Bootstrap Method (Efron [15])

Algorithm 1:

For given (λ, η, β) , (m, n, k, s) , $\mathbb{R} = \mathbf{r} = \{r_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}$ and $\mathbf{R}' = \mathbf{r}' = (r'_1, r'_2, \dots, r'_m)$:

Step 1: Generate Chen x_{ij} from $\mathcal{C}(\lambda, \beta) \sim \lambda \beta x_{ij}^{\beta-1} \exp\{\lambda[1 - \exp(x_{ij}^\beta)] + x_{ij}^\beta\}$

for $i = 1, 2, \dots, n; j = 1, 2, \dots, k$, and y_i from $\mathcal{C}(\eta, \beta) \sim \eta \beta y_i^{\beta-1} \exp\{\eta[1 - \exp(y_i^\beta)] + y_i^\beta\}$

for $i = 1, 2, \dots, m$,

Step 2: From the samples $\mathbf{x} = \{x_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}$ and $\mathbf{y} = (y_1, y_2, \dots, y_m)$,

compute the estimates of (λ, η) , say (l, e) :

$$l = mkw_\beta^{-1}, \text{ where } w_\beta = -\sum_{i=1}^m \sum_{j=1}^k (1 + r_{ij}) [1 - \exp(x_{ij}^\beta)], \text{ and}$$

$$e = mv_\beta^{-1}, \text{ where } v_\beta = -\sum_{i=1}^m (1 + r'_i) [1 - \exp(y_i^\beta)].$$

Step 3 : Generate bootstrap Chen x_{ij}^* from $\mathcal{C}(l, \beta) \sim l \beta x_{ij}^{*\beta-1} \exp\{l[1 - \exp(x_{ij}^{*\beta})] + x_{ij}^{*\beta}\}$

for $i = 1, 2, \dots, m; j = 1, 2, \dots, k$, and y_i^* from

$$\mathcal{C}(e, \beta) \sim e \beta y_i^{*\beta-1} \exp\{e[1 - \exp(y_i^{*\beta})] + y_i^{*\beta}\} \quad \text{for } i = 1, 2, \dots, m.$$

Then, compute bootstrap sample estimates of λ and η :

$$l^* = mkw_\beta^{*-1}, \text{ where } w_\beta^* = -\sum_{i=1}^m \sum_{j=1}^k (1 + r_{ij}) [1 - \exp(x_{ij}^{*\beta})], \text{ and}$$

$$e^* = mv_\beta^{*-1}, \text{ where } v_\beta^* = -\sum_{i=1}^m \sum_{j=1}^k (1 + r'_i) [1 - \exp(y_i^{*\beta})].$$

Based on $\mathbf{x}^* \{x_{ij}^*\}_{i=1,2,\dots,m; j=1,2,\dots,k}$ and $\mathbf{y}^* = (y_1^*, y_2^*, \dots, y_m^*)$ compute the bootstrap sample

estimate of $R_{s,k}$, denoted by $\widehat{R}_{s,k}^*$, using

$${}^{BP}\widehat{R}_{s,k}^* = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j \widehat{e}^*}{\widehat{t}^*(i+j) + \widehat{e}^*},$$

Step 4: Repeat step 3, N boot times and get the bootstrap distribution given by ${}^1\widehat{R}_{s,k}^*, {}^2\widehat{R}_{s,k}^*, \dots, {}^N\widehat{R}_{s,k}^*$. The bootstrap distribution of the statistic $\widehat{R}_{s,k}^*$ that is based on many resamples represents the sampling distribution of the statistic $\widehat{R}_{s,k}^*$ that is based on many samples.

Step 5: After ranking from bottom to top, let us denote these bootstrap values as $({}^1)R_{s,k}^*, ({}^2)R_{s,k}^*, \dots, ({}^N)R_{s,k}^*$. Let $G(R_{s,k}^*) = P(R_{s,k}^* \leq r_{s,k}^*)$, where $r_{s,k}^*$ is the observed value of $R_{s,k}^*$, be the cumulative distribution of $R_{s,k}^*$. Define ${}^{BP}R_{s,k}^* = G^{-1}(\xi)$ for a given ξ . The approximate $100(1 - \gamma)\%$ percentile-bootstrap CI (PBCI) for $R_{s,k}$ is then given by

$$\text{PBCI} = \left({}^{BP}\widehat{R}_{s,k}^* \left(\frac{\gamma}{2} \right), {}^{BP}\widehat{R}_{s,k}^* \left(\frac{1-\gamma}{2} \right) \right) \quad (2.9)$$

When the distributions are skewed we need do some adjustment. One method which is proved to be reliable is BCa method (BCa stands for Bias-corrected and accelerated). For the details please refer to DiCiccio and Efron ([14]). When the distribution of $R_{s,k}^*$ is skewed, we instead use the $q.low$ and $q.up$ percentiles of the bootstrap replicates of $R_{s,k}^*$ to calculate the lower bound and upper bound of the confidence intervals. Formally, for confidence level 95%, the bootstrap bias-corrected and accelerated CI(BBCACI) for $R_{s,k}$ is

$$\text{BBCACI} = (q.low, q.up), \quad (2.10)$$

where

$$\begin{aligned} q.low &= \Phi \left(z_0 + \frac{z_0 + z_{0.025}}{1 - b(z_0 + az_{0.025})} \right) \text{ and} \\ q.up &= \Phi \left(z_0 + \frac{z_0 + z_{0.975}}{1 - b(z_0 + az_{0.975})} \right), \end{aligned}$$

here z_γ is the γ th quantile of standard normal distribution, z_0 and b , namely bias-correction and acceleration, are two parameters to be estimated, by (2.8) and (6.6)

in DiCiccio and Efron [14], respectively.

(ii) Bootstrap-t Method (Hall [31]) : The method was suggested in Efron [15], but some poor numerical results reduced its appeal. Hall's [31] paper showing the bootstrap-t's good second-order properties has revived interest in its use. Babu and Singh [6] gave the first proof of second-order accuracy for the bootstrap-t.

Algorithm 2:

Step 1: Do steps 1–3 in **Algorithm 1**. Also, compute the following statistic

$$t^* = \frac{\sqrt{n}(\widehat{r}_{s,k}^* - \widehat{r}_{s,k})}{S_{R_{s,k}^*}},$$

where

$$T^* = \frac{\sqrt{n}(\widehat{R}_{s,k}^* - \widehat{R}_{s,k})}{S_{R_{s,k}^*}},$$

and $S_{R_{s,k}^*}$ is the standard deviation of the bootstrap distribution and $s_{R_{s,k}^*}$ is its observed value. $S_{R_{s,k}^*}$ is obtained using the Fisher (or expected Fisher) information matrix.

Moreover, $r_{s,k}^*$ is the estimate (or the observed estimator) of $R_{s,k}$ based on the bootstrap resamples and $\widehat{r}_{s,k}$ is the estimate of $R_{s,k}$ based on the original observed sample, and $\widehat{R}_{s,k}^*$ is the estimator of $R_{s,k}$ based on the bootstrap random resamples and $\widehat{R}_{s,k}$ is the estimator of $R_{s,k}$ based on the original random sample.

Step 2: Compute N bootstrap replications of t^* . Denote t^* by t_1^*, \dots, t_N^* .

Step 3: After ranking from bottom to top, let us denote these bootstrap values as $t_{(1)}^*, \dots, t_{(N)}^*$.

Step 4: For t^* values obtained in step 1, determine the upper and lower bounds of the $100(1 - \gamma)\%$ confidence interval of $R_{s,k}^*$ as follows:

Let $H(t^*) = P(T^* \leq t^*)$ be the cumulative distribution function of T^* . For a given ξ , define

$${}^{BT}\widehat{R}_{s,k}^*(\xi) = \widehat{r}_{s,k}^* + H^{-1}(\xi) \frac{s_{R_{s,k}^*}}{\sqrt{n}}.$$

The $100(1 - \gamma)\%$ bootstrap-t CI (BTCI) for $R_{s,k}$ is then given by

$$\text{BTCI} = \left({}^{BT}\widehat{R}_{s,k}^* \left(\frac{\gamma}{2} \right), {}^{BT}\widehat{R}_{s,k}^* \left(\frac{1 - \gamma}{2} \right) \right) \quad (2.11)$$

CHAPTER 3
THE GENERALIZED VARIABLE METHOD

3.1 Review

Motivated by a generalized test given by Weerahandi [54], Tsui and Weerahandi [49] formally introduced the notion of generalized p-values. Weerahandi [53] extended the classical definition of confidence intervals to obtain the generalized confidence intervals so that one can obtain reasonable interval estimates for situations where the classical approach fails or yield results lacking small sample accuracy. Weerahandi [50] introduced the notion of generalized point estimators.

3.2 Generalized inference for $R_{s,k}$

Let $X_{DATA} = (\mathcal{X}, \mathbf{Y})$, where $\mathcal{X} = \{X_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k}$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$, and let $x_{DATA} = (\mathfrak{x}, \mathbf{y})$, where $\mathfrak{x} = \{x_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k}$ and $\mathbf{y} = (y_1, \dots, y_m)$, be its observed value. The generalized pivotal quantity, denoted by $R(X_{DATA}; x_{DATA}, \delta, \beta)$, or generalized point estimator, denoted by $Q(X_{DATA}; x_{DATA}, \delta, \beta)$, for $R_{s,k}$ where $\delta = (\lambda, \eta)$, can then be obtained by replacing λ, η in $R_{s,k}$ given in 1.1 with their generalized variables $R(\mathcal{X}; \mathfrak{x}, \lambda, \beta)$ and $R(\mathbf{Y}; \mathbf{y}, \eta, \beta)$ as:

$$R(X_{DATA}; x_{DATA}, \delta, \beta) = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \times \frac{(-1)R(\mathbf{Y}; \mathbf{y}, \eta, \beta)}{R(\mathcal{X}; \mathfrak{x}, \lambda, \beta)(i+j) + R(\mathbf{Y}; \mathbf{y}, \eta, \beta)} \quad (3.1)$$

where $R(\mathcal{X}; \mathfrak{x}, \lambda, \beta) = 2mk(l\Lambda)^{-1}$ is the generalized pivotal quantity of λ and $R(\mathbf{Y}; \mathbf{y}, \eta, \beta) = 2m(e\Delta)^{-1}$ is the generalized pivotal quantity of η with $\Lambda = 2mk(L\lambda)^{-1} \sim \chi_{2mk}^2$ and $\Delta = 2m(E\eta)^{-1} \sim \chi_{2m}^2$, and e being the observed value of E , and l being the observed value of L .

We are now interested making inferences such as point and interval estimation of, and statistical tests for, $R_{s,k}$ based on the generalized variable method. The random variable

$Q(X_{DATA}; x_{DATA}, \delta, \beta)$ is a generalized point estimator which satisfy the three conditions to be a bona fide generalized point estimator. Therefore, this would also serve as a generalized pivotal

quantity $R(X_{DATA}; x_{DATA}, \delta, \beta)$, and $T(X_{DATA}; x_{DATA}, \delta, \beta) = R(X_{DATA}; x_{DATA}, \delta, \beta) - R_{s,k}$ would be a generalized test variable. First, for fixed x_{DATA} , the distribution $F_T(t)$ of

$T(X_{DATA}; x_{DATA}, \delta, \beta)$, where $F_T(t) = \Pr[T(X_{DATA}; x_{DATA}, \delta, \beta) \leq t] = \Pr[R(X_{DATA}; x_{DATA}, \delta, \beta) \leq t + R_{s,k}] = F_R(t + R_{s,k})$, with $F_R(\cdot)$, being the distribution function of $R(X_{DATA}; x_{DATA}, \delta, \beta)$, is free of nuisance parameters. Secondly, at $X_{DATA} = x_{DATA}$,

$$\begin{aligned} T(x_{DATA}; x_{DATA}, \delta, \beta) &= R(x_{DATA}; x_{DATA}, \delta, \beta) - R_{s,k} \\ &= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \times \\ &\quad \frac{(-1)^{R(\mathbf{y}; \mathbf{y}, \eta, \beta)}}{R(\mathbf{z}; \mathbf{z}, \lambda, \beta)(i+j) + R(\mathbf{y}; \mathbf{y}, \eta, \beta)} - R_{s,k} \\ &= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j \eta}{\lambda(i+j) + \eta} - \\ &\quad \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j \eta}{\lambda(i+j) + \eta} \\ &= 0 \end{aligned}$$

thus $T(X_{DATA}; x_{DATA}, \delta, \beta)$ is free of any unknown parameters. Thirdly, $F_T(t) =$

$\Pr[T(X_{DATA}; x_{DATA}, \delta, \beta) \leq t] = \Pr[R(X_{DATA}; x_{DATA}, \delta, \beta) \leq t + R_{s,k}] = F_R(t + R_{s,k})$ is a

decreasing function of $R_{s,k}$. Hence, $Q(X_{DATA}; x_{DATA}, \delta, \beta)$, $R(X_{DATA}; x_{DATA}, \delta, \beta)$, and

$T(X_{DATA}; x_{DATA}, \delta, \beta)$ are, respectively, bona fide generalized point estimator of $R_{s,k}$, generalized

pivotal quantity for constructing interval estimation for $R_{s,k}$, and the generalized test variable for

testing $H_0 : R_{s,k} \leq R_0$ vs. $H_a : R_{s,k} > R_0$, where R_0 is a known quantity.

3.3 Generalized confidence interval for $R_{s,k}$

Given the specified significance level γ , a $(1 - \gamma)$ two-sided generalized confidence interval for $R_{s,k}$ can be derived as follows:

For mathematical tractability and simplicity, we write $R^\lambda = R(\mathbb{X}; \mathbb{x}, \lambda, \beta) = 2mk(l\Lambda)^{-1}$ and $R^\eta = R(\mathbf{Y}; \mathbf{y}, \eta, \beta) = 2m(e\Delta)^{-1}$ with $\Lambda = 2mk(L\lambda)^{-1} \sim \chi_{2mk}^2$ and $\Delta = 2m(E\eta)^{-1} \sim \chi_{2m}^2$, and $L = mk/W_\beta$ with $W_\beta = \sum_{i=1}^m \sum_{j=1}^k W_{ij} = -\sum_{i=1}^m \sum_{j=1}^k (1 + R_{ij})[1 - \exp(X_{ij}^\beta)]$, and $E = m/V_\beta$ with $V_\beta = \sum_{i=1}^m V_i = -\sum_{i=1}^m (1 + R'_i)[1 - \exp(Y_i^\beta)] \sim \mathcal{G}(m, 1/\eta)$. Hence, a generalized pivotal statistic for $R_{s,k}$ in 1.1 is given by

$$R^{R_{s,k}} = R(x_{DATA}; x_{DATA}, \beta, \lambda) = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j R^\eta}{R^\lambda(i+j) + R^\eta}. \quad (3.2)$$

Let $R_{\gamma/2}^{R_{s,k}} = R_{\gamma/2}^{R_{s,k}}(x_{DATA}; d, \beta)$ and $R_{1-\gamma/2}^{R_{s,k}} = R_{1-\gamma/2}^{R_{s,k}}(x_{DATA}, d, \beta)$ where $d = \widehat{\delta}_{obs} = (l, e)$ satisfy

$$P[R_{\gamma/2}^{R_{s,k}} \leq R_{s,k} \leq R_{1-\gamma/2}^{R_{s,k}}] = 1 - \gamma$$

The $(R_{\gamma/2}^{R_{s,k}}, R_{1-\gamma/2}^{R_{s,k}})$ is a $100(1 - \gamma)\%$ lower confidence limit for $R_{s,k}$. That is, generalized confidence bounds for $R_{s,k}$ is $CI_{R_{s,k}}^G = (R_{\gamma/2}^{R_{s,k}}, R_{1-\gamma/2}^{R_{s,k}})$.

3.4 Generalized testing procedure for $R_{s,k}$

Construct a statistical testing procedure to assess whether the reliability function adheres to the required level. The one-sided hypothesis testing for $R_{s,k}$ is obtained using the generalized test variable $T(X_{DATA}; x_{DATA}, \delta, \beta) = R(X_{DATA}; x_{DATA}, \delta, \beta) - R_{s,k}$ or simply $T^{R_{s,k}} = R^{R_{s,k}} - R_{s,k}$.

Assuming that the required reliability is larger than R_0 , where R_0 denotes the target value, the null hypothesis $H_0 : R_{s,k} \leq R_0$ and the alternative hypothesis $H_a : R_{s,k} > R_0$ are constructed. Then, the generalized p -value, denoted by p_g , is given by

$$p_g = \Pr \left(\sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)R(\mathbf{Y}; \mathbf{y}, \eta, \beta)}{R(\mathbb{X}; \mathbb{x}, \lambda, \beta)(i+j) + R(\mathbf{Y}; \mathbf{y}, \eta, \beta)} > R_0 \right). \quad (3.3)$$

This p -value can be either computed by numerical integration exact up to a desired level of accuracy or well approximated by a Monte Carlo method. When there are a large number of random numbers from various random variables, the latter method is more desirable and computationally more efficient. p is an exact probability of a well-defined extreme region of the sample space and measures the evidence in favor of the null hypothesis. This is an exact test in significance testing. In fixed level testing, one can use this p -value by rejecting the null hypothesis, if $p_g < \gamma$, where γ is a desired nominal level .

The following algorithm is useful in constructing p_g .

Algorithm 3

Step 1: Given $\beta, s, k, \gamma, m, n, R_0, \mathbb{R} = (\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_k)$, and $\mathbf{R}' = (R'_1, R'_2, \dots, R'_m)$, where

$$\mathbf{R}_j = (R_{1j}, R_{2j}, \dots, R_{mj}) \text{ for } j = 1, 2, \dots, k.$$

(a) The generation of data U_{ij} is by the uniform distribution $U(0, 1)$, for $i = 1, 2, \dots, m$;
 $j = 1, 2, \dots, k$.

(b) By the transformation of $Z_{ij} = \beta^{-1} \ln[1 - \lambda^{-1} \ln(U_{ij})]$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, k$.
 $\{Z_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k}$ is a random sample from the $\mathcal{C}(\lambda, \beta)$.

(c) Set $X_{ij:m:n} = \frac{Z_{1j}}{m} + \frac{Z_{2j}}{(m-R_{1j}-1)} + \dots + \frac{Z_{ij}}{[m-\sum_{l=1}^{i-1} R_{lj}-i+1]}$, for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, k$
 $\{X_{ij:m:n}\}_{i=1,2,\dots,m;j=1,2,\dots,k}$ is the progressively type II right censored sample from a two-parameter $\mathcal{C}(\lambda, \beta)$.

Step 2: Compute the maximum likelihood estimate of λ

$$l = mk/w_\beta, \text{ where } w_\beta = \sum_{i=1}^m \sum_{j=1}^k w_{ij} = -\sum_{i=1}^m \sum_{j=1}^k (1+r_{ij})[1 - \exp(x_{ij}^\beta)],$$

Step 3: (a) Similarly, generate data U'_i from the uniform distribution $U(0, 1)$, for $i = 1, 2, \dots, m$.

(b) By the transformation of $Z'_i = \beta^{-1} \ln[1 - \eta^{-1} \ln(U'_i)]$, $i = 1, 2, \dots, m$,

$\{Z'_i\}_{i=1,2,\dots,m}$ is a random sample from the $\mathcal{C}(\eta, \beta)$.

(c) Set $Y_{i:m:n} = \frac{Z'_i}{m} + \frac{Z'_i}{(m-R'_i-1)} + \dots + \frac{Z'_i}{[m-\sum_{l=1}^{i-1} R'_{l-i+1}]}$, for $i = 1, 2, \dots, m$.

$\{Y_{i:m:n}\}_{i=1,2,\dots,m}$ is the progressively type II right censored sample from a two-parameter $\mathcal{C}(\eta, \beta)$.

Step 4: Compute the maximum likelihood estimate of η

$$e = m/v_\beta, \text{ where } v_\beta = \sum_{i=1}^m v_i = -\sum_{i=1}^m (1+r'_i)[1 - \exp(y_i^\beta)].$$

Step 5: For $g = 1 : G$

(a) Generate $\Lambda \sim \chi_{2mk}^2$ and $\Delta \sim \chi_{2n}^2$

(b) Compute the quantities $R^\lambda = 2mk(l\Lambda)^{-1}$ and $R^\eta = 2m(e\Delta)^{-1}$

(c) Compute $R^{R_{s,k}} = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j R^\eta}{R^\lambda(i+j) + R^\eta}$

(end g loop)

Generalized p -value is estimated by the proportion of $R^{R_{s,k}}$ which are greater than R_0 . The $100(1 - \gamma/2)$ th and $100\gamma/2$ th percentile of $R^{R_{s,k}}$; $R_{\gamma/2}^{R_{s,k}}$ and $R_{1-\gamma/2}^{R_{s,k}}$, respectively; are the lower and upper bounds of the two-sided $1 - \gamma$ confidence interval. That is, $CI_{R_{s,k}}^G = (R_{\gamma/2}^{R_{s,k}}, R_{1-\gamma/2}^{R_{s,k}})$.

Coverage probabilities of the generalized confidence intervals and powers of generalized tests are computed using the Monte Carlo method given in the following algorithm.

Algorithm 4

For given $\delta = (\lambda, \eta), \beta, k, \gamma, m, n, R_0, \mathbb{R} = (\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_k)$, and $\mathbf{R}' = (R'_1, R'_2, \dots, R'_m)$,

where $\mathbf{R}_j = (R_{1j}, R_{2j}, \dots, R_{nj})$ for $j = 1, 2, \dots, k$

For $p = 1 : P$

1. Generate $\Lambda \sim \chi_{2mk}^2$ and $\Delta \sim \chi_{2n}^2$

2. Set $\lambda = 2mk(l\Lambda)^{-1}$ and $\eta = 2m(e\Delta)^{-1}$,

3. Use **Algorithm 3** to construct a $(1 - \gamma)$ confidence interval C_p ,

$$\xi_{R_{s,k}} = \begin{cases} 1, & \text{if } C_p \text{ contains } R_{s,k} \\ 0, & \text{if } C_p \text{ does not contain } R_{s,k} \end{cases},$$

4. Use **Algorithm 3** again to compute the generalized p -value, p_g .

$$\eta_{R_{s,k}} = \begin{cases} 1, & \text{if } p_g < \gamma \\ 0, & \text{if } p_g > \gamma \end{cases}.$$

(end p loop)

The proportion $\frac{1}{P} \sum_{p=1}^P \xi_{R_{s,k}}$ is the estimated coverage probability of the generalized confidence interval. It is evident that sometimes the coverage of the generalized confidence interval may not equal to the nominal level. But, when generalized confidence interval reduces to traditional classical confidence intervals, theoretical results are available on coverage properties of generalized confidence intervals. The proportion $\frac{1}{P} \sum_{p=1}^P \eta_{R_{s,k}}$ is the estimated power of the generalized test.

CHAPTER 4

THE BAYESIAN METHOD

4.1 Review

We deal with the problem of estimating the parameters λ and η , and the reliability function $R_{s,k}$ of \mathcal{C} distribution under mainly SE (squared error) and LINEX (linear exponential) loss functions. Similar procedure can be adopted for estimating the reliability function $R_{s,k}$ under various other loss functions as well. In this section, we assume that the parameters (λ, η) are random variables and have statistically independent gamma prior distributions with hyperparameters $(a_i, b_i), i = x, y$, respectively, that is, prior distributions for λ and η are taken to be $\mathcal{G}(a_i, b_i), i = x, y$. The pdf of a gamma random variable χ with parameters (a_i, b_i) is

$$f(\chi) = \frac{b_i^{a_i}}{\Gamma(a_i)} \chi^{a_i-1} e^{-b_i \chi}, \quad \chi > 0, a_i, b_i > 0. \quad (4.1)$$

Then, the joint posterior density function of (λ, η) turns out to be

$$\pi(\lambda, \eta | \beta, \varkappa, \mathbf{y}) = \frac{(b_1 + w_\beta)^{mk+a_1} (b_2 + v_\beta)^{m+a_2}}{\Gamma(mk+a_1)\Gamma(m+a_2)} \lambda^{mk+a_1-1} \eta^{m+a_2-1} e^{-\lambda(b_1+w_\beta)-\eta(b_2+v_\beta)}$$

where $\varkappa = \{x_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}$; $\mathbf{y} = \{y_i\}_{i=1,2,\dots,m}$

; $w_\beta = \sum_{i=1}^m \sum_{j=1}^k w_{ij} = -\sum_{i=1}^m \sum_{j=1}^k (1+r_{ij})[1-\exp(x_{ij}^\beta)]$,

$v_\beta = \sum_{i=1}^m v_i = -\sum_{i=1}^m (1+r'_i)[1-\exp(y_i^\beta)]$ with $\mathbf{r}_j = (r_{1j}, r_{2j}, \dots, r_{mj})$ and

$\mathbf{r}' = (r'_{1j}, r'_{2j}, \dots, r'_{mj}), j = 1, 2, \dots, k$. Furthermore, the marginal posterior densities of λ and η have gamma distributions with parameters $(mk+a_1, b_1+w_\beta)$ and $(m+a_2, b_2+v_\beta)$. The Bayes estimate of $R_{s,k}$ under the SE loss function, say $\widehat{R}_{s,k}^{B,SE}$, is

$$\begin{aligned} \widehat{R}_{s,k}^{B,SE} &= E_{\pi(\lambda, \eta | \beta, x_{DATA})} [R_{s,k} | x_{DATA}] \\ &= \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \int_0^\infty \int_0^\infty \frac{\eta}{\lambda(i+j) + \eta} \times \\ &\quad \pi(\lambda, \eta | \beta, x_{DATA}) d\lambda d\eta, \end{aligned}$$

where $x_{DATA} = (\mathbb{x}, \mathbf{y})$ with $\mathbb{x} = \{x_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k}$ and $\mathbf{y} = (y_1, \dots, y_m)$ is the observed (or realized) value of $X_{DATA} = (\mathbb{X}, \mathbf{Y})$ with $\mathbb{X} = \{X_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k}$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$.

We consider a one-to-one transformation $u_1 = \eta / (\lambda(i+j) + \eta)$ and $u_2 = \lambda(i+j) + \eta$. Then, $0 < u_1 < 1$, $0 < u_2 < \infty$, $\lambda = u_2(1 - u_1)/(i+j)$, $\eta = u_1 u_2$ and the Jacobian of (u_1, u_2) is $J(u_1, u_2) = -u_2/(i+j)$. Therefore, the double integral in (4.3) can be rewritten as

$$\begin{aligned} & \frac{(b_1 + w_\beta)^{nk+a_1} (b_2 + v_\beta)^{m+a_2}}{\Gamma(mk+a_1)\Gamma(m+a_2)(i+j)^{mk+a_1}} \left\{ \int_0^1 \int_0^\infty u_1^{m+a_2} (1-u_1)^{nk+a_1-1} u_2^{p-1} \times \right. \\ & \left. \exp\left(-u_2 \left\{ \frac{(1-u_1)(b_1+w_\beta)}{(i+j)} + u_1(b_2+v_\beta) \right\}\right) du_1 du_2 \right\} \\ &= \frac{(1-z)^{m+a_2}}{B(mk+a_1, m+a_2)} \int_0^1 u_1^{m+a_2} (1-u_1)^{mk+a_1-1} (1-u_1 z)^{-p} du_1, \end{aligned}$$

where $z = 1 - ((b_2 + v_\beta)(i+j)/(b_1 + w_\beta))$ and $p = mk + a_1 + m + a_2$. The integral representation of the hypergeometric series is (this was given by Euler in 1748 and implies Euler's and Pfaff's hypergeometric transformations. See Section 9.1 in Gradshteyn and Ryzhik [22])

$${}_2F_1(\alpha, \beta; \gamma, z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt,$$

$$|z| < 1 \text{ or } |z| = 1, \operatorname{Re}(\gamma) > \operatorname{Re}(\beta) > 0.$$

Notice that the hypergeometric series converges in the unit circle $|z| < 1$. Then,

$$\widehat{R}_{s,k}^B = \begin{cases} \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j (1-z)^{(m+a_2)(m+a_2)}}{p} \times {}_2F_1(p, m+a_2+1; p+1, z) & \text{if } |z| < 1 \\ \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j (m+a_2)}{(1-z)^{mk+a_1 p}} \times {}_2F_1(p, mk+a_1; p+1, \frac{z}{z-1}) & \text{if } z < -1. \end{cases} \quad (4.2)$$

The Bayes estimate of $R_{s,k}$ under the LINEX loss function, say $\widehat{R}_{s,k}^{B,LINEX}$, is

$$\begin{aligned}
\widehat{R}_{s,k}^{B,LINEX} &= E_{\pi(\lambda,\eta|\beta,x_{DATA})}[\exp\{cR_{s,k}|x_{DATA}\}] \\
&= \int_0^\infty \int_0^\infty \exp \left\{ -\lambda(b_1 + w_\beta) - \eta(b_2 + v_\beta) + \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \right\} \times \\
&\quad (-1)^j \frac{\eta}{\lambda(i+j) + \eta} \times \frac{(b_1 + w_\beta)^{mk+a_1} (b_2 + v_\beta)^{m+a_2}}{\Gamma(mk+a_1)\Gamma(m+a_2)} \times \\
&\quad \alpha_1^{mk+a_1-1} \eta^{m+a_2-1} d\lambda d\eta, \tag{4.3}
\end{aligned}$$

where $x_{DATA} = (\varkappa, \mathbf{y})$ with $\varkappa = \{x_{ij}\}_{i=1,2,\dots,n;j=1,2,\dots,k}$ and $\mathbf{y} = (y_1, \dots, y_m)$ is the observed (or realized) value of $X_{DATA} = (\mathcal{X}, \mathbf{Y})$ with $\mathcal{X} = \{X_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k}$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$.

It is easily observed that all these estimates are in the form of ratio of two integrals for which simplified closed forms are not available. Thus to evaluate these estimates, in practice, intensive numerical techniques are required. Instead, one can apply approximation methods to evaluate these estimates such as Lindley's approximation and Markov Chain Monte Carlo (MCMC). However, the Bayes estimate under the SE loss function is obtained in the closed form, and alternative methods are also used to see how good the approximate methods compared with the exact one. We completely use the Lindley's method for the Bayes estimate under the LINEX loss function as has no closed forms. If these result are close, then it will be encouraging to use the approximate methods when the exact form can not be obtained in the all parameters are unknown case. These estimators will be compared in the simulation study section. Next, we give the Bayes estimates of $R_{s,k}$ using the Lindley's approximation and MCMC method.

4.2 Lindley's approximation

Lindley [40] introduced an approximate procedure for the computation of the ratio of two integrals. This procedure, applied to the posterior expectation of the function $U(\theta)$ for a given \mathbf{x} , is

$$E(U(\theta)|\mathbf{x}) = \frac{\int_{\Theta} u(\theta) e^{Q(\theta)} d\theta}{\int_{\Theta} e^{Q(\theta)} d\theta},$$

where $Q(\theta) = l(\theta) + \rho(\theta)$, $l(\theta)$ is the logarithm of the likelihood function and $\rho(\theta)$ is the logarithm of the prior density of θ , $\theta = (\theta_1, \theta_2, \dots, \theta_L)$, $i, j, k, l = 1, 2, \dots, L$, and Θ is the parameter space. Using Lindley's approximation, $E(U(\theta)|\mathbf{x})$ is approximately estimated by

$$E(U(\theta)|\mathbf{x}) = \left| u + \frac{1}{2} \sum_i \sum_j (u_{ij} + 2u_i \rho_j) \sigma_{ij} + \frac{1}{2} \sum_i \sum_j \sum_k \sum_l L_{ijk} \sigma_{ij} \sigma_{kl} u_l \right|_{\hat{\theta}} \\ + \text{terms of order } n^{-2} \text{ or smaller,}$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_L)$, $i, j, k, l = 1, 2, \dots, L$, $\hat{\theta}$ is the MLE of θ , $u = u(\theta)$, $u_i = \partial u / \partial \theta_i$, $u_{ij} = \partial^2 u / \partial \theta_i \partial \theta_j$, $L_{ijk} = \partial^3 l / \partial \theta_i \partial \theta_j \partial \theta_k$, $\rho_j = \partial \rho / \partial \theta_j$ and $\sigma_{ij} = (i, j)$ th element in the inverse of the matrix $\{-L_{ij} = \partial^2 l / \partial \theta_i \partial \theta_j\}$ and $\sigma_{kl} = (k, l)$ th element in the inverse of the matrix $\{-L_{kl} = \partial^2 l / \partial \theta_k \partial \theta_l\}$, all evaluated at the MLE of the parameters.

For the two parameter case $\theta = (\theta_1, \theta_2)$, Lindley's approximation leads to

$$\hat{u}_{Lin} = u(\theta) + \frac{1}{2} [B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}],$$

where $B = \sum_{i=1}^2 \sum_{j=1}^2 u_{ij} \tau_{ij}$, $Q_{ij} = \partial^{i+j} u / \partial \theta_1^i \partial \theta_2^j$ for $i, j = 0, 1, 2, 3$, $i + j = 3$, $u_i = \partial u / \partial \theta_i$, $u_{ij} = \partial^2 u / \partial \theta_i \partial \theta_j$ for $i, j = 1, 2$, and $B_{ij} = (u_i \tau_{ii} + u_j \tau_{ij}) \tau_{ii}$, $C_{ij} = 3u_i \tau_{ii} \tau_{ij} + u_j (\tau_{ii} \tau_{ij} + 2\tau_{ij}^2) \tau_{ij}$ for $i \neq j$. τ_{ij} is the (i, j) th element in the inverse of matrix $Q^* = (Q_{ij}^*)$, $i, j = 1, 2$ such that $Q_{ij}^* = \partial^2 Q / \partial \theta_i \partial \theta_j$. The approximate Bayes estimate \hat{u}_{Lin} is evaluated at $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2)$ which is the mode of the posterior density.

In our case, $\theta = (\theta_1, \theta_2) = \alpha = (\lambda, \eta)$ and

$$Q = \ln \pi(\lambda, \eta | \beta, \mathbf{x}, \mathbf{y}) \propto (mk + a_1 - 1) \ln a_1 + (m + a_2 - 1) \ln a_2 - a_1(b_1 + w_\beta) - \alpha_2(b_2 + v_\beta),$$

where $\mathbf{x} = \{x_{ij}\}_{i=1,2,\dots,n; j=1,2,\dots,k}$; $\mathbf{y} = \{y_i\}_{i=1,2,\dots,m}$

$$w_\beta = \sum_{i=1}^m \sum_{j=1}^k w_{ij} = - \sum_{i=1}^m \sum_{j=1}^k (1 + r_{ij}) [1 - \exp(x_{ij}^\beta)],$$

$$v_\beta = \sum_{i=1}^m v_i = - \sum_{i=1}^m (1 + r'_i) [1 - \exp(y_i^\beta)] \text{ with } \mathbf{r}_j = (r_{1j}, r_{2j}, \dots, r_{nj}) \text{ and}$$

$$\mathbf{r}' = (r'_1, r'_2, \dots, r'_m), j = 1, 2, \dots, k.$$

The posterior mode of (λ, η) is obtained from Q and is given by

$$\tilde{\lambda} = \frac{mk + a_1 - 1}{b_1 + w_\beta} \quad \text{and} \quad \tilde{\eta} = \frac{m + a_2 - 1}{b_2 + v_\beta}.$$

We obtain that $\tau_{11} = \lambda^2/(mk + a_1 - 1)$, $\tau_{22} = \eta^2/(m + a_2 - 1)$, $\tau_{12} = \tau_{21} = 0$, $Q_{12} = Q_{21} = 0$, $Q_{03} = 2/(m + a_2 - 1)/\eta^3$, $Q_{30} = 2/(nk + a_1 - 1)/\lambda^3$, $B_{12} = u_1 \tau_{11}^2$, $B_{21} = u_2 \tau_{22}^2$, $B = u_{11} \tau_{11} + u_{22} \tau_{22}$, and

$$\begin{aligned} u_1 &= \frac{\partial R_{s,k}}{\partial \lambda} = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^{j+1}(i+j)\eta}{(\lambda(i+j) + \eta)^2}, \\ u_2 &= \frac{\partial R_{s,k}}{\partial \eta} = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{\lambda(i+j)(-1)^j}{(\lambda(i+j) + \eta)^2}, \\ u_{11} &= \frac{\partial^2 R_{s,k}}{\partial \lambda^2} = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{2(-1)^j(i+j)^2\eta}{(\lambda(i+j) + \eta)^3}, \\ u_{12} &= u_{21} = \frac{\partial^2 R_{s,k}}{\partial \lambda \partial \eta} = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \times \frac{2(-1)^{j+1}(i+j)\lambda(\lambda(i+j) - \eta)}{(\lambda(i+j) + \eta)^3}, \\ u_{22} &= \frac{\partial^2 R_{s,k}}{\partial \eta^2} = \sum_{i=s}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{2(-1)^{j+1}(i+j)\lambda}{(\lambda(i+j) + \eta)^3}. \end{aligned}$$

Therefore, the approximate Bayes estimate of the reliability function $R_{s,k}$ under SE loss function is given by

$$\widehat{R}_{s,k}^{B, Lin(SE)} = R_{s,k} \Big|_{(\lambda, \eta) = (\tilde{\lambda}, \tilde{\eta})} + \frac{1}{2} \left[\frac{\lambda^2 u_{11} + 2\eta u_1}{nk + \lambda - 1} + \frac{\eta^2 u_{22} + 2\eta u_2}{m + \eta - 1} \right]_{(\lambda, \eta) = (\tilde{\lambda}, \tilde{\eta})},$$

where u_1 , u_2 , u_{11} , and u_{22} are given above.

With the same argument, we can obtain Bayes estimators under the LINEX loss function of the reliability function from (1.1). They are obtained by the following forms:

if $u(\lambda, \eta) = \exp[-cR_{s,k}]$, then

$$\begin{aligned}
u_1^* &= \frac{\partial \exp[-cR_{s,k}]}{\partial \lambda} = -c \exp[-cR_{s,k}] \times \frac{\partial R_{s,k}}{\partial \lambda} = -c \exp[-cR_{s,k}] \times u_1, \\
u_2^* &= \frac{\partial \exp[-cR_{s,k}]}{\partial \eta} = -c \exp[-cR_{s,k}] \times \frac{\partial R_{s,k}}{\partial \eta} = -c \exp[-cR_{s,k}] \times u_2, \\
u_{11}^* &= \frac{\partial^2 \exp[-cR_{s,k}]}{\partial \lambda^2} = \frac{\partial}{\partial \lambda} \{-c \exp[-cR_{s,k}] \times u_1\} = -c \{\exp[-cR_{s,k}] u_{11} + u_1 u_1^*\} \\
u_{12}^* &= u_{21}^* = \frac{\partial^2 \exp[-cR_{s,k}]}{\partial \lambda \partial \eta} = \frac{\partial}{\partial \eta} \{-c \exp[-cR_{s,k}] \times u_1\} \\
&= -c \{\exp[-cR_{s,k}] u_{12} + u_1 u_2^*\} \\
u_{22}^* &= \frac{\partial^2 \exp[-cR_{s,k}]}{\partial \eta^2} = \frac{\partial}{\partial \eta} \{-c \exp[-cR_{s,k}] \times u_2\} = -c \{\exp[-cR_{s,k}] u_{22} + u_2 u_2^*\}
\end{aligned}$$

The approximate Bayes estimate of the reliability function $R_{s,k}$ under a LINEX loss function is given by

$$\hat{R}_{s,k}^{B, Lin(LINEX)} = \frac{-1}{c} \ln \{E_{\pi(\lambda, \eta | \beta, x_{DATA})}[\exp(cR_{s,k} | x_{DATA})]\} \quad (4.3)$$

$$\begin{aligned}
E_{\pi(\lambda, \eta | \beta, x_{DATA})}[\exp(cR_{s,k} | x_{DATA})] &= \exp(cR_{s,k} | x_{DATA}) + \frac{1}{2} [B^* + Q_{30}^* B_{12}^* + \\
&Q_{21}^* C_{12}^* + Q_{12}^* C_{21}^* + Q_{03}^* B_{21}^*]
\end{aligned}$$

where $B^* = \sum_{i=1}^2 \sum_{j=1}^2 u_{ij}^* \tau_{ij}$, $Q_{ij} = \partial^{i+j} u^* / \partial^i \theta_1 \partial^j \theta_2$ for $i, j = 0, 1, 2, 3, i + j = 3, u_i^* = \partial u^* / \partial \theta_i$, $u_{ij}^* = \partial^2 u^* / \partial \theta_i \partial \theta_j$ for $i, j = 1, 2$, and

$B_{ij}^* = (u_i^* \tau_{ii} + u_j^* \tau_{ij}) \tau_{ii}$, $C_{ij}^* = 3u_i^* \tau_{ii} \tau_{ij} + u_j^* (\tau_{ii} \tau_{ij} + 2\tau_{ij}^2) \tau_{ij}$ for $i \neq j$. τ_{ij} is the (i, j) th element in the inverse of matrix $Q^* = (Q_{ij}^*)$, $i, j = 1, 2$ such that $Q_{ij}^* = \partial^2 Q / \partial \theta_i \partial \theta_j$, and $\theta = (\theta_1, \theta_2) = \delta = (\lambda, \eta)$.

4.3 Markov chain Monte Carlo (MCMC or MC²) method

The MCMC algorithm is used for computing the Bayes estimates of the parameters λ and η as well as the reliability function $R_{s,k}$. The joint posterior density function of λ and η is given in (4.2).

It is easily seen that the marginal posterior density functions of λ and η are, respectively,

$$\lambda|\beta, \mathbf{x}, \mathbf{y} \sim \mathcal{G}(nk + a_1, b_1 + w_\beta) \quad \text{and} \quad \eta|\beta, \mathbf{x}, \mathbf{y} \sim \mathcal{G}(m + a_2, b_2 + v_\beta),$$

where $\mathbf{x} = \{x_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,k}$; $\mathbf{y} = \{y_i\}_{i=1,2,\dots,m}$

$$; w_\beta = \sum_{i=1}^m \sum_{j=1}^k w_{ij} = -\sum_{i=1}^m \sum_{j=1}^k (1 + r_{ij}) [1 - \exp(x_{ij}^\beta)],$$

$$v_\beta = \sum_{i=1}^m v_i = -\sum_{i=1}^m (1 + r'_i) [1 - \exp(y_i^\beta)] \quad \text{with} \quad \mathbf{r}_j = (r_{1j}, r_{2j}, \dots, r_{mj}) \quad \text{and}$$

$$\mathbf{r}' = (r'_1, r'_2, \dots, r'_m), j = 1, 2, \dots, k.$$

In the event that the conditional posterior distribution of any parameter to be estimated is not in the closed form or well-known distribution, we then consider the Metropolis-Hastings (MH) (Metropolis et al. [43] and Hasting [33]) algorithm to generate samples from the conditional posterior distributions and then compute the Bayes estimates. The MH algorithm generate samples from an arbitrary proposal distribution (i.e., a Markov transition kernel), where most of the time the samples are drawn from normal distribution.

So, as suggested by Tierney [48], a common way to solve this problem is to use the hybrid algorithm by combining a Metropolis sampling with the Gibbs sampling scheme using normal proposal distribution. We assume that λ and η can be generated from $(nk + a_1, b_1 + w_\beta)$ and $(m + a_2, b_2 + v_\beta)$, respectively, using a direct random generation scheme (see, for example, Devroye [12] or an MCMC procedure, which uses the Gibbs sampler and Metropolis-Hastings algorithm (see Gelfand and Smith [21] for the Gibbs sampler, and Tierney [48] for the Metropolis-Hastings algorithm).

Step 1: Choose an initial guess of (λ, η) , say (λ_0, η_0)

Step 2: Set $g = 1$.

Step 3: Generate $\lambda^{(l)}$ from $\mathcal{G}(nk + a_1, b_1 + w_\beta)$.

Step 4: Generate $\eta^{(l)}$ from $\mathcal{G}(m + a_2, b_2 + v_\beta)$.

Step 5: Compute the $R_{s,k}^{(l)}$ at $(\lambda^{(l)}, \eta^{(l)})$

Step 6: Set $g = g + 1$.

Step 7: Repeat Steps 3 through 6, G times, and obtain the posterior sample $R_{s,k}^{(l)}, l = 1, \dots, L$.

Now the approximate posterior mean, and posterior variance of $R_{s,k}$ becomes

$$\widehat{E}(R_{s,k}|x_{DATA}) = \frac{1}{G-S} \sum_{g=S+1}^G R_{s,k}^{(l)},$$

where $\widehat{R}_{s,k}^{B,MC^2} = \widehat{E}(R_{s,k}|x_{DATA})$ is the Bayes estimate of $R_{s,k}$, and

$$\widehat{V}(R_{s,k}|x_{DATA}) = \frac{1}{G-S} \sum_{g=S+1}^G (R_{s,k}^{(l)} - \widehat{E}(R_{s,k}|x_{DATA}))^2,$$

respectively. Then a $100(1 - \gamma)\%$ HPD interval (HPDI) of $R_{s,k}$ can be approximated (Chen and Shao [11]) by

$$C_{p^*}(L)_{R_{s,k}} = \left(R_{s,k}^{(p^*)}, R_{s,k}^{(p^* + [(1-\gamma)L])} \right),$$

where p^* is chosen so that

$$R_{s,k}^{(p^* + [(1-\gamma)L])} - R_{s,k}^{(p^*)} = \min_{1 \leq p \leq [(1-\gamma)L]} \left(R_{s,k}^{(p^* + [(1-\gamma)L])} - R_{s,k}^{(p^*)} \right).$$

Furthermore, approximate $100(1 - \gamma)\%$ Bayesian credible interval (BCI) of Ψ can be obtained by

$$BCI_{R_{s,k}} = \widehat{E}(R_{s,k}|x_{DATA}) \pm Z_{\gamma/2} \sqrt{\frac{\widehat{V}(R_{s,k}|x_{DATA})}{G}},$$

where Z_{ζ} is the ζ^{th} quantile of the standard normal distribution and S is the burn-in period. It well known that rapid convergence is facilitated by choosing appropriate starting values. In order to guarantee the convergence and to remove the affection of the selection of initial value, the first S simulated variates are discarded. Then the selected sample are $\lambda^{(l)}$ and $\eta^{(l)}$, $l = 1, \dots, G$., for sufficiently large L , forms an approximate posterior sample which can be used to develop the Bayesian inference. Similarly, the Bayes estimate of $R_{s,k}$ under a LINEX loss function is given by

$$\widehat{R}_{s,k}^{B,MC^2} = -\frac{1}{c} \ln \left\{ \frac{1}{G-S} \sum_{g=S+1}^G \exp[-cR_{s,k}^{(l)}] \right\} \quad (4.4)$$

and in a similar fashion, we can easily find the BCI as well as HPDI $R_{s,k}$ under LINEX loss function.

CHAPTER 5

EXAMPLES

5.1 Practical application study

We consider a data set which represents the monthly water capacity of Shasta reservoir (or Shasta lake by collecting water due to the impounding of the Sacramento River, the largest river in the State of California, by Shasta Dam, called Kennett Dam before its construction). Code named as USBR SHA, it is operated by the U.S. Federal Bureau of Reclamation under The United States Department of the Interior, California, USA. The data set is available in the link “<http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA>”. If the water capacity of the reservoir in December of the previous year is about half of the maximum capacity, and minimum water level in September is more than the amount of water achieved in December in at least two years out of the next five years, it is claimed that there will not be any excessive drought after wards. We arbitrarily take $s = 2$ and $k = 5$ which suggests that it is a 2-out-of-5:G system.

We assume that Y_1 is the capacity of water in December 1989, $X_{1j}, j = 1, 2, \dots, 5$ are the capacities of water in September 1990 to 1994, Y_2 is the capacity of water in December 1995, and $X_{2j}, j = 1, 2, \dots, 5$ are the capacities of water in September 1996 to 2000. When we carry on this data process up to 2018, then we get $n = 5$. For computational convenience, we divided the data set by $4552000/50=91040$, where 4552000 is the total capacity of water of Shasta reservoir.

Nevertheless, due to the time limitation and/or other restrictions (such as financial, material resources, mechanical or experimental difficulties) on data collection, we observe type-II progressively censored data with random removals, thus we have the X_{ij} for

$i = 1, 2, \dots, m = 4, j = 1, s = 2, \dots, k = 5$ with random removals $\mathbf{R} = (R_1 = 0, R_2 = 0, R_3 = 0, R_4 = 1)$ creating four ($m = 4$) five-year periods 1990–1994, 1996–2000, 2002–2006, and 2008–2012.

Similarly, Y_i is the mean annual water capacity of the i th year in-between two consecutive five-year periods, where $i = 1, 2, \dots, n = 5$, but again due to the restrictions on data collection and to keep the consistency with the water capacity in September of each five-year period, we consider the mean annual capacity of only four ($m = 4$) years such as 1989, 1995, 2001, and 2007. To remove (or to reduce) the dependency between X_{ij} and Y_i ; the years of Y_i are not used for obtaining the data X_{ij} .

Thus, we obtain the 2-out-of-5 : G system and observed data (\mathbb{X}, \mathbf{Y}) . For computational ease, all of the values divided data set by $4552000/50$, where 4552000 is the total capacity of water of Shasta Reservoir . The data are as follows:

$$\mathbb{X} = \begin{pmatrix} 17.9851 & 14.7172 & 18.4886 & 34.0703 & 23.0848 \\ 33.9281 & 25.3552 & 37.7974 & 36.5499 & 32.7892 \\ 28.0997 & 34.7032 & 23.9768 & 33.3352 & 35.2059 \\ 15.2074 & 19.4854 & 36.4541 & 36.6992 & 28.4662 \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} 22.5575 \\ 35.8928 \\ 32.8016 \\ 19.5969 \end{pmatrix}$$

We first verify that the Chen distribution can be used to fit the data. For this purpose, we compute the MLEs of unknown parameters with respect to both the data sets, (\mathbb{X}, \mathbf{Y}) . Chen distribution provides reasonably good fit to the data compared to Weibull, generalized exponential, and exponential distribution.

In the case of real-world data, we use the Least Squares Estimation (LSE) Method, which is based on the minimum Error Sum of Squares (SSE), for various values of β and the “shape-first” approach (that is to fit the shape parameter β before fitting the other shape parameter λ) to fit the optimal value of β and estimate of λ such that SSE is minimized for progressively type-II right-censored data. Then, β is defined as known. The procedure is as follows:

Step 1. Let $X_j \sim \mathcal{C}(\lambda, \beta), j = 1, 2, \dots, k$ whose common *pdf* is given by

$$f(x; \lambda, \beta) = \lambda \beta x^{\beta-1} \exp\{\lambda[1 - \exp(x^\beta)] + x^\beta\}; x > 0, \lambda > 0, \beta > 0,$$

and the common *cdf* is

$$F(x; \lambda, \beta) = 1 - \exp\{\lambda[1 - \exp(x^\beta)]\}, \quad x > 0, \lambda > 0, \beta > 0,$$

and $F(x; \lambda, \beta)$ satisfies

$$\ln[1 - F(x; \lambda, \beta)] = \lambda[1 - \exp(x^\beta)], \quad x > 0, \lambda > 0, \beta > 0,$$

Consider that $X_{1:m:n} \leq X_{2:m:n} \leq \dots \leq X_{m:m:n}$ is the corresponding progressively type-II right-censored sample, with observed censoring scheme $\mathbf{r} = (r_1, r_2, \dots, r_m)$. The expectation of $F(x_{i:m:n}; \lambda, \beta)$ is

$$1 - \prod_{j=m-i+1}^m (a_j / (a_j + 1)), \quad i = 1, \dots, m, \quad \text{where } a_j = j + \sum_{i=m-j+1}^m R_i$$

By using the approximate

$$\ln \left(1 - \left(1 - \prod_{j=m-i+1}^m (a_j / (a_j + 1)) \right) \right) \approx \lambda_i [1 - \exp(x^\beta)], \quad i = 1, \dots, n,$$

we get

$$\lambda_i \approx - \frac{\ln \left(1 - \left(1 - \prod_{j=m-i+1}^m (a_j / (a_j + 1)) \right) \right)}{[1 - \exp(x^\beta)]} \quad \text{for } i = 1, \dots, m.$$

Then using the least squares estimation method for various values of β and the “shape-first” approach to fit the optimal value of β , calculate the *SSE* for given each value of β , that is,

$$SSE_\beta = \sum_{i=1}^m (\lambda - \hat{\lambda})^2, \quad \text{where } \hat{\lambda} = \frac{mk}{w_\beta} \quad \text{with}$$

$$w_\beta = - \sum_{i=1}^n \sum_{j=1}^k (1 + r_{ij}) [1 - \exp(X_{ij:m:n}^\beta)]$$

Now, find the optimal value of β (say β_{fit}) and estimate λ such that *SSE* is minimized. The density of the fitted \mathcal{C} distribution is now

$$f(x; \lambda, \beta) = \lambda \beta_{fit} x^{\beta_{fit}-1} \exp\{\lambda[1 - \exp(x^{\beta_{fit}})] + x^{\beta_{fit}}\}; \quad x > 0, \lambda > 0.$$

Step 2. Use the scale-free goodness-of-fit test for \mathcal{C} distribution based on

the Gini statistic due to Gail and Gastwirth [20] for the progressively type-II right-censored data $X_{1:m:n} \leq X_{2:m:n} \leq \dots \leq X_{m:m:n}$.

The procedure is as follows:

The null hypothesis is $H_0: X \sim \mathcal{C}$ distribution with the pdf

$$f(x; \lambda, \beta) = \lambda \beta_{f_{ii}} x^{\beta_{f_{ii}} - 1} \exp\{\lambda[1 - \exp(x^{\beta_{f_{ii}}})] + x^{\beta_{f_{ii}}}\}$$

The Gini statistic given as follows:

$$G_m = \frac{\sum_{i=1}^{m-1} \sum_{j=1}^k i W_{i,j+1}}{(m-1) \sum_{i=1}^m \sum_{j=1}^k W_{ij}},$$

where

$$W_{ij} = (n - \sum_{l=1}^{i-1} r_{lj} - (i-1))(Z_{ij:m:n} - Z_{i,j-1:m:n})$$

$$\text{with } Z_{ij:m:n} = (1 + r_{ij})[1 - \exp(x_{ij}^{\beta})], i = 1, 2, \dots, m; j = 1, 2, \dots, k.$$

For $n = 3, \dots, 20$, the rejection region is given by $\{G_m > \xi_{1-\gamma/2} \text{ or } G_m < \xi_{\gamma/2}\}$,

where the critical value $\xi_{\gamma/2}$ is the 100($\gamma/2$)th percentile of the G_m statistic

and is available on p. 352 in Gail and Gastwirth [20].

$Y \sim \mathcal{C}(\eta, \beta)$ is also treated in a similar fashion to see whether Y values are fitted to a \mathcal{C} .

Once the real-world data were handled in the manner described above, the value of β (out of various β values) that minimizes SSE_{β}^X is found to be $\beta = 1.4$, which is very close to the optimum (minimum) value of the graph of SSE versus β . (These graphs have been omitted for saving space and can be produced upon request). Further, $\hat{\lambda}$ value corresponds to $\beta = 1.4$ is 0.22. Then, β is defined as known. That is,

$$f(x; \lambda, \beta) = 1.4\lambda x^{0.4} \exp\{\lambda[1 - \exp(x^{1.4})] + x^{1.4}\}, x > 0, \lambda > 0.$$

The goodness of fit test for the null hypothesis is performed, where the null hypothesis is $H_0: X \sim \mathcal{C}$ distribution with the pdf

$$f(x; \lambda, \beta) = 1.4\lambda x^{0.4} \exp\{\lambda[1 - \exp(x^{1.4})] + x^{1.4}\}, x > 0, \lambda > 0, \text{ at level } \gamma = 0.05, \text{ the Gini}$$

statistic for the progressively type-II right-censored observed sample is found to be

$$G_4 = \frac{\sum_{i=1}^{(4-1)} \sum_{k=1}^5 i W_{i+1}}{(4-1) \sum_{i=1}^4 W_i} = 0.41920.$$

Since $\xi_{0.025} = 0.28748 < G_4 = 0.41920 < \xi_{0.975} = 0.71252$, we cannot reject H_0 at the 0.05 level of significance, and we can conclude the observed strength components are from the \mathcal{C} distribution with the *pdf* is $f(x; \lambda, \beta) = 1.4\lambda x^{0.4} \exp\{\lambda[1 - \exp(x^{1.4})] + x^{1.4}\}$, $x > 0, \lambda > 0$. at level $\gamma = 0.05$.

$Y \sim \mathcal{C}(\eta, \beta)$ is also treated in a similar fashion to see whether Y stress values are fitted to a \mathcal{C} . Then,

$$\begin{aligned}\hat{\lambda} &= 0.2433, \text{ where } w_{\beta} = -\sum_{i=1}^4 \sum_{j=1}^5 (1 + r_{ij}) [1 - \exp(x_{ij}^{\beta})] = 32.8879 \\ \hat{\eta} &= 0.8314, \text{ where } v_{\beta} = -\sum_{i=1}^4 (1 + r'_i) [1 - \exp(y_i^{\beta})] = 25.7612\end{aligned}$$

To fully explore the advantage of the newly introduced generalized variable method, classical and generalized point and 95% interval estimates are compared for the reliability function $R_{s,k}$. In addition, p -values for testing reliability function are also compared. The numerical results for these data are presented in Table 5.1 and 5.2. Posterior distributions are obtained from 10,000 Gibbs samplings after a burn-in period of 1,000 iterations.

Both these arguments clearly show that the generalized variable method (GV-Method) provides accurate, reliable, and non-misleading results, while the classical method (C-Method) and Bayesian method (B-Method) approaches fail to do so for this particular case. Hence, the GV-Method outperforms the C- and B-Method for this particular practical application.

5.2 Simulation study

In this section, to illustrate the usage and benefit of the generalized variable method for this problem, we present some numerical results for the Chen distribution

$F(x_j) = 1 - \exp\{\lambda[1 - \exp(x_j^{\beta})]\}$. Those random variables are simulated in the following manner.

For given $\delta = (\lambda, \eta)$ and β , and (m, k) :

1. Generate uniform random numbers, i.e., $U \sim U(m, 0, 1)$, where $U(m, 0, 1)$ is the standard continuous uniform distribution with boundary parameters 0 and 1, and m is the sample size,
2. Generate pseudo Chen random variates for \varkappa :

$$\varkappa = \{x_{ij}\}_{i=1,2,\dots,m;j=1,2,\dots,k} = [\ln(1 - \exp(U)/\lambda)]^{1/\beta}$$

3. Generate pseudo Chen random variates for \mathbf{y} :

$$\mathbf{y} = \{y_i\}_{i=1,2,\dots,m} = [\ln(1 - \exp(U)/\eta)]^{1/\beta}$$

The performances of the point estimators are compared by using estimated risks (ER) or estimate of the mean squared errors (MSE), and estimated biases. The ER and bias of $\hat{\theta}$ relative to an known parameter θ , when it is estimated by $\hat{\theta}$, is given by

$$ER(\hat{\theta}) = \widehat{MSE}(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2 \quad \text{and} \quad \widehat{BIAS}(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^N \hat{\theta}_i - \theta,$$

where ER has been calculated under the squared error function.

5.3 Bias and Expected Risk

The performances of the confidence intervals are compared by using average confidence lengths and coverage probabilities. The coverage probability (CP) of a confidence interval is the proportion of the time that the interval contains the true value of interest. That is,

$$CP = \frac{[\text{Number of intervals that contain the true value of interest } \theta]}{\text{The total number of simulations}}$$

The performances of the hypothesis testing are compared by using average empirical Type-I error rate (or the actual size) of the test, and the unadjusted and adjusted powers of the test.

Actual Size of the Test

Actual size (AS) for testing $H_0 : \theta \leq \theta_0$ vs. $H_a : \theta > \theta_0$ is the proportion of p -values that are less than the nominal value γ . That is,

$$AS = \frac{\text{Number of } p\text{-values for testing } H_0 : \theta \leq \theta_0 \text{ vs. } H_a : \theta > \theta_0 \text{ that are less than } \gamma}{\text{The total number of simulations}}.$$

Power of the Test

When $\theta = \theta_0$, unadjusted power (UP) for testing $H_0 : \theta \leq \theta_0^*$ vs. $H_a : \theta > \theta_0^*$, where $\theta_0^* < \theta_0$, is the proportion of p -values that are less than the nominal value γ . That is,

$$UP = \frac{\text{Number of } p\text{-values for testing } H_0 : \theta \leq \theta_0^* \text{ vs. } H_a : \theta > \theta_0^* \text{ that are less than } \gamma}{\text{The total number of simulations}},$$

where $\theta_0^* < \theta_0$.

When $\theta = \theta_0$, adjusted power (AP) for testing $H_0 : \theta \leq \theta_0^*$ vs. $H_a : \theta > \theta_0^*$, where $\theta_0^* < \theta_0$, is the proportion of p -values for testing $H_0 : \theta \leq \theta_0^*$ vs. $H_a : \theta > \theta_0^*$ that are less than the p -value (p_γ) for testing $H_0 : \theta \leq \theta_0$ vs. $H_a : \theta > \theta_0$. That is,

$$AP = \frac{\text{Number of } p\text{-values for testing } H_0 : \theta \leq \theta_0^* \text{ vs. } H_a : \theta > \theta_0^* \text{ that are less than } p_\gamma}{\text{The total number of simulations}},$$

where $\theta_0^* < \theta_0$.

5.4 Computations & Calculations

The performance of the estimates of $R_{s,k}$ are obtained by using the Bayesian, classical and generalized methods for different sample sizes. All of the computations are performed by using PYTHON and R. All the results are based on $N = 100,000$ replications.

In Table 5.3, 5.4, 5.5, and 5.6 when the common shape parameter is known ($\beta = 3$), strength and stress populations are generated for $\delta = (\lambda, \eta) = (4, 2), (4, 4), (4, 6),$ and $(4, 8)$ and different sample sizes $n = 10, 15, 25$ and 35 . The corresponding true values of reliability in multicomponent stress-strength with the given combinations $(s, k) = (2, 4)$ are 0.3905, 0.6000, 0.7229 and 0.8000.

In Table 5.7, 5.8, 5.9, and 5.10 when $\beta = 10$, strength and stress populations are generated for $\delta = (\lambda, \eta) = (18, 5), (12, 5), (6, 5), (1, 5)$ and different sample sizes $n = 10, 15, 25$ and 35 . The corresponding true values of reliability in multicomponent stress-strength with the given combinations for $(s, k) = (2, 4)$ are $0.2485, 0.3419, 0.5428$ and 0.9524 .

From Table 5.3, 5.4 and 5.7, 5.8, we observe that the average *ERs* for the estimates of $R_{s,k}$ decrease as the sample size increases in all cases and all tables, as expected. The *ERs* of the ML, UMVU and generalized estimates have generally following order of $ER(\hat{R}_{s,k}^G) < ER(\hat{R}_{s,k}^{MLE}) < ER(\hat{R}_{s,k}^U)$ except for the cases when the true value of $R_{s,k}$ is not close to extreme values. On the other hand, when the true value of $R_{s,k}$ approaches the extreme values, we have following order of $ER(\hat{R}_{s,k}^G) < ER(\hat{R}_{s,k}^U) < ER(\hat{R}_{s,k}^{MLE})$ and all *ERs* are close to each other as the sample size increases. The average lengths of the intervals decrease as the sample size increases. The average lengths of the generalized intervals are smaller than those of the classical confidence intervals. Furthermore, the coverage probabilities of the generalized intervals are more close to the nominal level 95% than the classical confidence intervals.

Table 5.3, 5.4, 5.5, 5.6 show the point and interval estimates when $(s, k) = (2, 4)$ under known $\beta = 3$. The first rows under the point estimates represent the average estimates and the second row represents corresponding *ERs*. The first row under the interval estimates represent a 95% confidence interval and the second rows represent their expected lengths and coverage probabilities. Table 5.7, 5.8, 5.9, 5.10 show the point and interval estimates when $(s, k) = (2, 4)$ under known $\beta = 3$. The first rows under the point estimates represent the average estimates and the second row represents corresponding *ERs*. The first row under the interval estimates represent a 95% confidence interval and the second rows represent their expected lengths and coverage probabilities.

Table 5.11 and 5.12 show the classical and generalized empirical (actual) type-I error rates or the sizes of the test (the rejection rate of the null hypothesis: the fraction of times the p -value is less than the nominal level) for testing $H_0 : R_{s,k} \leq R_0$ vs. $H_a : R_{s,k} > R_0$ when nominal (intended) type-I error rate is at $\gamma = 0.05$.

Tables 5.13, 5.14 show the power comparison for testing $R_{s,k} \leq 0.50$ vs. $R_{s,k} > 0.50$ before and after adjusting the actual type-I error rate at $\gamma = 0.05$ based on 10,000 replications.

Without adjusting the size, the generalized powers for testing $H_0 : R_{s,k} \leq 0.5429$ vs. $H_a : R_{s,k} > 0.5429$ clearly suggest that the generalized variable method outperforms the classical method. Even after adjusting the size, the generalized variable method still maintains a light advantage over the classical method. The size of the test has to be adjusted to get a meaningful comparison of power of tests. But, in reality practitioners, being less-concern about the size, are not interested in adjusting the nominal size in order to get the desired level γ . In terms of computational time, it takes less than few minutes to run the proposed procedure for either of the examples on Dell Optiplex 3020 with processor 3.20 GHz and 8.00 GB RAM.

When hypothesis $R_{s,k} > 0.50$ is tested when nominal (intended) level is $\gamma = 0.05$ with the common parameter $\beta = 3$ for $\delta = (4, 2)$, the generalized Type-I error rate is 0.0511, which is very close to the nominal value. However, the classical Type-I error rate is 0.007, a value way off from the nominal value. This suggests that the generalized variable method is size-guaranteed. When $R_{s,k} > R_0$ is tested in a similar fashion for various parameter combinations such as $\beta = (3, 10)$, $(s, k) = (2, 4)$ $\delta = (\lambda, \eta) = \{(4, 2), (4, 4), (4, 6), (4, 8)\}$, $n = \{10, 15, 25, 35\}$, and $R_0 = \{0.35, 0.50, 0.55, 0.65, 0.70, 0.75, 0.80, 0.85\}$, all these arguments clearly show that the generalized variable method (GV-Method) is size-guaranteed, while the classical method (C-Method) approach fails to do so. Hence, the GV-Method outperforms the C-Method for this particular case.

Table 5.1 Comparison of Point Estimates of $R_{s,k}$

Bayesian		Classical		Generalized	
$\widehat{R}_{s,k}^{SE}$	0.6781	$\widehat{R}_{s,k}^M$	0.6987	$\widehat{R}_{s,k}^G$	0.6781
$\widehat{R}_{s,k}^{LINEX}$	0.6875	$\widehat{R}_{s,k}^U$	0.6988		
$SE \widehat{R}_{s,k}^{Lin}$	0.6985	$BP \widehat{R}_{s,k}^*$	0.6701		
$LINEX \widehat{R}_{s,k}^{Lin}$	0.6855	$BT \widehat{R}_{s,k}^*$	0.6898		
$SE \widehat{R}_{s,k}^{MCMC}$	0.7101				
$LINEX \widehat{R}_{s,k}^{MCMC}$	0.6998				

Table 5.2 Comparison of Interval Estimates of $R_{s,k}$

Bayesian		Classical		Generalized	
$SE \text{BCI}^{MCMC}$	(0.57 – 0.95)	ACI	(0.51 – 1.5)	GCI	(0.65 – 0.75)
$SE \text{HPDI}^{MCMC}$	(0.51 – 0.88)	PBCI	(0.58 – 0.95)		
$LINEX \text{BCI}^{MCMC}$	(0.55 – 1.5)	BBCACI	(0.61 – 1.00)		
$LINEX \text{HPDI}^{MCMC}$	(0.55 – 1.5)	BTCI	(0.51 – 1.7)		

Table 5.3 Classical and generalized point estimates of $R_{2,4}$ when the common shape parameter β is known ($\beta = 3$)

Sample size n	Parameters δ	Reliability $R_{2,4}$	Classical			Generalized
			$\widehat{R}_{s,k}^M$	$\widehat{R}_{s,k}^U$	$\widehat{R}_{s,k}^*$	$\widehat{R}_{s,k}^G$
10	(4,2)	0.3905	0.4071	0.3905	0.3982	0.3989
			0.0108	0.0108	0.0066	0.0066
			0.3998	0.3886	0.3952	0.3956
			0.0064	0.0064	0.0046	0.0046
			0.3980	0.3913	0.3955	0.3957
15	(4,2)	0.3905	0.0038	0.0038	0.0031	0.0031
			0.3957	0.3909	0.3942	0.3942
			0.0029	0.0029	0.0025	0.0025
			0.6085	0.5986	0.5950	0.5953
			0.0115	0.0126	0.0059	0.0059
15	(4,4)	0.6000	0.6074	0.6006	0.5981	0.5983
			0.0077	0.0081	0.0047	0.0047
			0.6045	0.6004	0.5989	0.5990
			0.0048	0.0050	0.0036	0.0036
			0.6031	0.6001	0.5992	0.5992
15	(4,4)	0.6000	0.0034	0.0035	0.0027	0.0027
			0.7267	0.7235	0.7130	0.7129
			0.0093	0.0105	0.0043	0.0042
			0.7257	0.7235	0.7158	0.7158
			0.0064	0.0069	0.0036	0.0036
25	(4,6)	0.7229	0.7246	0.7232	0.7181	0.7181
			0.0040	0.0042	0.0027	0.0027
			0.7239	0.7229	0.7192	0.7192
			0.0029	0.0030	0.0022	0.0022
			0.7995	0.8005	0.7880	0.7878
10	(4,8)	0.8000	0.0070	0.0077	0.0029	0.0029
			0.8016	0.8024	0.7922	0.7921
			0.0048	0.0052	0.0025	0.0025
			0.7989	0.7992	0.7929	0.7928
			0.0030	0.0031	0.0019	0.0019
15	(4,8)	0.8000	0.7998	0.8000	0.7952	0.7952
			0.0020	0.0021	0.0015	0.0015

Table 5.4 Bayesian point estimates of $R_{2,4}$ when the common shape parameter β is known ($\beta = 3$)

Sample size n	Parameters δ	Reliability $R_{2,4}$	Bayesian					
			$\widehat{R}_{s,k}^{SE}$	$\widehat{R}_{s,k}^{LINEX}$	$SE \widehat{R}_{s,k}^{Lin}$	$LINEX \widehat{R}_{s,k}^{Lin}$	$SE \widehat{R}_{s,k}^{MCMC}$	$LINEX \widehat{R}_{s,k}^{MCMC}$
10	(4,2)	0.3905	0.4071	0.3905	0.3982	0.3989	0.3982	0.3982
			0.0108	0.0108	0.0066	0.0066	0.0046	0.0077
15	(4,2)	0.3905	0.3998	0.3886	0.3952	0.3956	0.3982	0.3982
			0.0064	0.0064	0.0046	0.0046	0.0046	0.0077
25	(4,2)	0.3905	0.3980	0.3913	0.3955	0.3957	0.3982	0.5950
			0.0038	0.0038	0.0031	0.0031	0.0025	0.0031
35	(4,2)	0.3905	0.3957	0.3909	0.3942	0.3942	0.3942	0.3942
			0.0029	0.0029	0.0025	0.0025	0.0025	0.5986
10	(4,4)	0.6000	0.6085	0.5986	0.5950	0.5953	0.3942	0.5950
			0.0115	0.0126	0.0059	0.0059	0.0025	0.0025
15	(4,4)	0.6000	0.6074	0.6006	0.5981	0.5983	0.5983	0.0025
			0.0077	0.0081	0.0047	0.0047	0.0047	0.0077
25	(4,4)	0.6000	0.6045	0.6004	0.5989	0.5990	0.5992	0.5992
			0.0048	0.0050	0.0036	0.0036	0.0025	0.0025
35	(4,4)	0.6000	0.6031	0.6001	0.5992	0.5992	0.5992	0.5992
			0.0034	0.0035	0.0027	0.0027	0.0025	0.0042
10	(4,6)	0.7229	0.7267	0.7235	0.7130	0.7129	0.5992	0.7192
			0.0093	0.0105	0.0043	0.0042	0.0042	0.0042
15	(4,6)	0.7229	0.7257	0.7235	0.7158	0.7158	0.5992	0.7192
			0.0064	0.0069	0.0036	0.0036	0.0042	0.0042
25	(4,6)	0.7229	0.7246	0.7232	0.7181	0.7181	0.5992	0.7192
			0.0040	0.0042	0.0027	0.0027	0.0027	0.0042
35	(4,6)	0.7229	0.7239	0.7229	0.7192	0.7192	0.7181	0.7181
			0.0029	0.0030	0.0022	0.0022	0.0019	0.0019
10	(4,8)	0.8000	0.7995	0.8005	0.7880	0.7878	0.7181	0.7181
			0.0070	0.0077	0.0029	0.0029	0.0027	0.0018
15	(4,8)	0.8000	0.8016	0.8024	0.7922	0.7921	0.7181	0.7880
			0.0048	0.0052	0.0025	0.0025	0.0019	0.0042
25	(4,8)	0.8000	0.7989	0.7992	0.7929	0.7928	0.7952	0.7880
			0.0030	0.0031	0.0019	0.0019	0.0019	0.0017
35	(4,8)	0.8000	0.7998	0.8000	0.7952	0.7952	0.7953	(0.7952
			0.0020	0.0021	0.0015	0.0015	0.0019	0.0019

Table 5.5 Classical and generalized interval estimates of $R_{2,4}$ when the common shape parameter β is known ($\beta = 3$)

Sample size n	Parameters δ	Reliability $R_{2,4}$	Classical			Generalized
			ACI	PBCI	BTCI	GCI
10	(4, 2)	0.3905	(0.2182, 0.5960)	(0.2180, 0.5959)	(0.218, 0.5949)	(0.2340, 0.5663)
			0.3779/0.9272	0.3770/0.9270	0.3775/0.9265	0.3322/0.9548
			(0.2447, 0.5550)	(0.2444, 0.5555)	(0.2441, 0.5548)	(0.2553, 0.5375)
			0.3103/0.9432	0.3100/0.9428	0.3103/0.9432	0.2822/0.9604
			(0.2771, 0.5190)	(0.2758, 0.5188)	(0.2765, 0.5175)	(0.2830, 0.5100)
15	(4, 2)	0.3905	0.2419/0.9448	0.2415/0.9440	0.2410/0.9440	0.2269/0.9556
			(0.2935, 0.4980)	(0.2930, 0.4975)	(0.2933, 0.4977)	(0.2975, 0.4920)
			0.2045/0.9412	0.2044/0.9412	0.2040/0.9410	0.1945/0.9432
			(0.4018, 0.8153)	(0.4015, 0.8150)	(0.4010, 0.8152)	(0.4172, 0.7667)
			0.4134/0.9200	0.4130/0.9189	0.4128/0.9190	0.3495/0.9672
15	(4, 4)	0.6000	(0.4357, 0.7790)	(0.4351, 0.7777)	(0.4355, 0.7788)	(0.4447, 0.7467)
			0.3432/0.9372	0.3429/0.9365	0.3425/0.9365	0.3020/0.9716
			(0.4699, 0.7391)	(0.4688, 0.7389)	(0.4688, 0.7389)	(0.4740, 0.7206)
			0.2692/0.9364	0.2689/0.9365	0.2687/0.9360	0.2466/0.9572
			(0.4886, 0.7175)	(0.4883, 0.7177)	(0.4885, 0.7170)	(0.4911, 0.7047)
15	(4, 4)	0.6000	0.2289/0.9480	0.2282/0.9479	0.2284/0.9477	0.2136/0.9576
			(0.5394, 0.9141)	(0.5388, 0.9138)	(0.5389, 0.9138)	(0.5537, 0.8601)
			0.3747/0.9032	0.3739/0.9031	0.3744/0.9030	0.3065/0.9752
			(0.5699, 0.8816)	(0.5688, 0.8810)	(0.5688, 0.8811)	(0.5776, 0.8450)
			0.3117/0.9172	0.3111/0.9165	0.3115/0.9165	0.2674/0.9700
25	(4, 6)	0.7229	(0.6019, 0.8472)	(0.6018, 0.8468)	(0.6010, 0.8468)	(0.6051, 0.8254)
			0.2452/0.9308	0.2444/0.9300	0.2449/0.9300	0.2203/0.9596
			(0.6196, 0.8283)	(0.6190, 0.8280)	(0.6195, 0.8282)	(0.6209, 0.8127)
			0.2087/0.9332	0.2085/0.9331	0.2080/0.9328	0.1918/0.9540
			(0.6376, 0.9614)	(0.6372, 0.9601)	(0.6365, 0.9610)	(0.6531, 0.9092)
25	(4, 8)	0.8000	0.3238/0.8992	0.3228/0.8985	0.3230/0.8960	0.2561/0.9844
			(0.6680, 0.9353)	(0.6677, 0.9344)	(0.6677, 0.9350)	(0.6750, 0.8988)
			0.2674/0.9040	0.2670/0.9039	0.2666/0.9033	0.2239/0.9664
			(0.6929, 0.9048)	(0.6920, 0.9040)	(0.6918, 0.9040)	(0.6961, 0.8826)
			0.2119/0.9252	0.2111/0.9248	0.2108/0.9238	0.1866/0.9648
35	(4, 8)	0.8000	(0.7098, 0.8897)	(0.7098, 0.8897)	(0.7088, 0.8885)	(0.7110, 0.8739)
			0.1799/0.9372	0.1799/0.9372	0.1789/0.9365	0.1630/0.9628

Table 5.6 Bayesian interval estimates of $R_{2,4}$ when the common shape parameter β is known ($\beta = 3$)

Sample size n	Parameters δ	Reliability $R_{2,4}$	Bayesian			
			SE_{BCI}^{MCMC}	SE_{HDPI}^{MCMC}	$LINEX_{BCI}^{MCMC}$	$LINEX_{HPDI}^{MCMC}$
10	(4, 2)	0.3905	(0.2182, 0.5960)	(0.2180, 0.5959)	(0.218, 0.5949)	(0.2340, 0.5663)
			0.3779/0.9272	0.3770/0.9270	0.3775/0.9265	0.3322/0.9548
			(0.2447, 0.5550)	(0.2444, 0.5555)	(0.2441, 0.5548)	(0.2553, 0.5375)
			0.3103/0.9432	0.3100/0.9428	0.3103/0.9432	0.2822/0.9604
			(0.2771, 0.5190)	(0.2758, 0.5188)	(0.2765, 0.5175)	(0.2830, 0.5100)
15	(4, 2)	0.3905	0.2419/0.9448	0.2415/0.9440	0.2410/0.9440	0.2269/0.9556
			(0.2935, 0.4980)	(0.2930, 0.4975)	(0.2933, 0.4977)	(0.2975, 0.4920)
			0.2045/0.9412	0.2044/0.9412	0.2040/0.9410	0.1945/0.9432
			(0.4018, 0.8153)	(0.4015, 0.8150)	(0.4010, 0.8152)	(0.4172, 0.7667)
			0.4134/0.9200	0.4130/0.9189	0.4128/0.9190	0.3495/0.9672
15	(4, 4)	0.6000	(0.4357, 0.7790)	(0.4351, 0.7777)	(0.4355, 0.7788)	(0.4447, 0.7467)
			0.3432/0.9372	0.3429/0.9365	0.3425/0.9365	0.3020/0.9716
			(0.4699, 0.7391)	(0.4688, 0.7389)	(0.4688, 0.7389)	(0.4740, 0.7206)
			0.2692/0.9364	0.2689/0.9365	0.2687/0.9360	0.2466/0.9572
			(0.4886, 0.7175)	(0.4883, 0.7177)	(0.4885, 0.7170)	(0.4911, 0.7047)
15	(4, 4)	0.6000	0.2289/0.9480	0.2282/0.9479	0.2284/0.9477	0.2136/0.9576
			(0.5394, 0.9141)	(0.5388, 0.9138)	(0.5389, 0.9138)	(0.5537, 0.8601)
			0.3747/0.9032	0.3739/0.9031	0.3744/0.9030	0.3065/0.9752
			(0.5699, 0.8816)	(0.5688, 0.8810)	(0.5688, 0.8811)	(0.5776, 0.8450)
			0.3117/0.9172	0.3111/0.9165	0.3115/0.9165	0.2674/0.9700
25	(4, 6)	0.7229	(0.6019, 0.8472)	(0.6018, 0.8468)	(0.6010, 0.8468)	(0.6051, 0.8254)
			0.2452/0.9308	0.2444/0.9300	0.2449/0.9300	0.2203/0.9596
			(0.6196, 0.8283)	(0.6190, 0.8280)	(0.6195, 0.8282)	(0.6209, 0.8127)
			0.2087/0.9332	0.2085/0.9331	0.2080/0.9328	0.1918/0.9540
			(0.6376, 0.9614)	(0.6372, 0.9601)	(0.6365, 0.9610)	(0.6531, 0.9092)
25	(4, 6)	0.7229	0.3238/0.8992	0.3228/0.8985	0.3230/0.8960	0.2561/0.9844
			(0.6680, 0.9353)	(0.6677, 0.9344)	(0.6677, 0.9350)	(0.6750, 0.8988)
			0.2674/0.9040	0.2670/0.9039	0.2666/0.9033	0.2239/0.9664
			(0.6929, 0.9048)	(0.6920, 0.9040)	(0.6918, 0.9040)	(0.6961, 0.8826)
			0.2119/0.9252	0.2111/0.9248	0.2108/0.9238	0.1866/0.9648
35	(4, 8)	0.8000	(0.7098, 0.8897)	(0.7098, 0.8897)	(0.7088, 0.8885)	(0.7110, 0.8739)
			0.1799/0.9372	0.1799/0.9372	0.1789/0.9365	0.1630/0.9628

Table 5.7 Classical and generalized point estimates of $R_{2,4}$ when the common shape parameter β is known ($\beta = 10$)

Sample size n	Parameters δ	Reliability $R_{2,4}$	Classical			Generalized
			$\widehat{R}_{s,k}^M$	$\widehat{R}_{s,k}^U$	$\widehat{R}_{s,k}^*$	$\widehat{R}_{s,k}^G$
10	(4,2)	0.3905	0.4071	0.3905	0.3982	0.3989
			0.0108	0.0108	0.0066	0.0066
			0.3998	0.3886	0.3952	0.3956
			0.0064	0.0064	0.0046	0.0046
			0.3980	0.3913	0.3955	0.3957
15	(4,2)	0.3905	0.0038	0.0038	0.0031	0.0031
			0.3957	0.3909	0.3942	0.3942
			0.0029	0.0029	0.0025	0.0025
			0.6085	0.5986	0.5950	0.5953
			0.0115	0.0126	0.0059	0.0059
15	(4,4)	0.6000	0.6074	0.6006	0.5981	0.5983
			0.0077	0.0081	0.0047	0.0047
			0.6045	0.6004	0.5989	0.5990
			0.0048	0.0050	0.0036	0.0036
			0.6031	0.6001	0.5992	0.5992
15	(4,4)	0.6000	0.0034	0.0035	0.0027	0.0027
			0.7267	0.7235	0.7130	0.7129
			0.0093	0.0105	0.0043	0.0042
			0.7257	0.7235	0.7158	0.7158
			0.0064	0.0069	0.0036	0.0036
25	(4,6)	0.7229	0.7246	0.7232	0.7181	0.7181
			0.0040	0.0042	0.0027	0.0027
			0.7239	0.7229	0.7192	0.7192
			0.0029	0.0030	0.0022	0.0022
			0.7995	0.8005	0.7880	0.7878
10	(4,8)	0.8000	0.0070	0.0077	0.0029	0.0029
			0.8016	0.8024	0.7922	0.7921
			0.0048	0.0052	0.0025	0.0025
			0.7989	0.7992	0.7929	0.7928
			0.0030	0.0031	0.0019	0.0019
15	(4,8)	0.8000	0.7998	0.8000	0.7952	0.7952
			0.0020	0.0021	0.0015	0.0015

Table 5.8 Bayesian point estimates of $R_{2,4}$ when the common shape parameter β is known ($\beta = 10$)

Sample size n	Parameters δ	Reliability $R_{2,4}$	Bayesian					
			$\widehat{R}_{s,k}^{SE}$	$\widehat{R}_{s,k}^{LINEX}$	$SE \widehat{R}_{s,k}^{Lin}$	$LINEX \widehat{R}_{s,k}^{Lin}$	$SE \widehat{R}_{s,k}^{MCMC}$	$LINEX \widehat{R}_{s,k}^{MCMC}$
10	(4,2)	0.3905	0.4071	0.3905	0.3982	0.3989	0.3982	0.3989
			0.0108	0.0108	0.0066	0.0066	0.0066	0.0066
			0.3998	0.3886	0.3952	0.3956	0.3952	0.3956
			0.0064	0.0064	0.0046	0.0046	0.0046	0.0046
			0.3980	0.3913	0.3955	0.3957	0.3980	0.3958
15	(4,2)	0.3905	0.0038	0.0038	0.0031	0.0031	0.0038	0.0040
			0.3957	0.3909	0.3942	0.3942	0.3957	0.3958
			0.0029	0.0029	0.0025	0.0025	0.0029	0.0022
			0.6085	0.5986	0.5950	0.5953	0.5953	0.5953
			0.0115	0.0126	0.0059	0.0059	0.0059	0.0060
15	(4,4)	0.6000	0.6074	0.6006	0.5981	0.5983	0.5983	0.5987
			0.0077	0.0081	0.0047	0.0047	0.0048	0.0047
			0.6045	0.6004	0.5989	0.5990	0.5990	0.5990
			0.0048	0.0050	0.0036	0.0036	0.0037	0.0038
			0.6031	0.6001	0.5992	0.5992	0.5989	0.5992
35	(4,4)	0.6000	0.0034	0.0035	0.0027	0.0027	0.0027	0.0097
			0.7267	0.7235	0.7130	0.7129	0.7129	0.7129
			0.0093	0.0105	0.0043	0.0042	0.0042	0.0042
			0.7257	0.7235	0.7158	0.7158	0.7159	0.7159
			0.0064	0.0069	0.0036	0.0036	0.0036	0.0036
25	(4,6)	0.7229	0.7246	0.7232	0.7181	0.7181	0.7232	0.7232
			0.0040	0.0042	0.0027	0.0027	0.0042	0.0042
			0.7239	0.7229	0.7192	0.7192	0.7192	0.7192
			0.0029	0.0030	0.0022	0.0022	0.0029	0.0029
			0.7995	0.8005	0.7880	0.7878	0.7878	0.7879
10	(4,8)	0.8000	0.0070	0.0077	0.0029	0.0029	0.0077	0.0077
			0.8016	0.8024	0.7922	0.7921	0.7921	0.7921
			0.0048	0.0052	0.0025	0.0025	0.0025	0.0026
			0.7989	0.7992	0.7929	0.7928	0.7992	0.7992
			0.0030	0.0031	0.0019	0.0019	0.0019	0.0019
35	(4,8)	0.8000	0.7998	0.8000	0.7952	0.7952	0.7953	0.7954
			0.0020	0.0021	0.0015	0.0015	0.0015	0.0015

Table 5.9 Classical and generalized interval estimates of $R_{2,4}$ when the common shape parameter β is known ($\beta = 10$)

Sample size n	Parameters δ	Reliability $R_{2,4}$	Classical			Generalized
			ACI	PBCI	BTCI	GCI
10	(4, 2)	0.3905	(0.2182, 0.5960)	(0.2180, 0.5959)	(0.2178, 0.5959)	(0.2340, 0.5663)
			0.3778/0.9271	0.3779/0.9272	0.3773/0.9271	0.3322/0.9548
			(0.2443, 0.5550)	(0.2447, 0.5550)	(0.2444, 0.5549)	(0.2553, 0.5375)
			0.3103/0.9432	0.3098/0.9430	0.3100/0.9430	0.2822/0.9604
			(0.2771, 0.5190)	(0.2767, 0.5189)	(0.2770, 0.5189)	(0.2830, 0.5100)
15	(4, 2)	0.3905	0.2419/0.9448	0.2411/0.9444	0.2411/0.9445	0.2269/0.9556
			(0.2935, 0.4980)	(0.2931, 0.4979)	(0.2932, 0.4982)	(0.2975, 0.4920)
			0.2045/0.9412	0.2043/0.9401	0.2041/0.9411	0.1945/0.9432
			(0.4018, 0.8153)	(0.4012, 0.8151)	(0.4017, 0.8153)	(0.4172, 0.7667)
			0.4134/0.9200	0.4130/0.9197	0.4134/0.9203	0.3495/0.9672
15	(4, 4)	0.6000	(0.4357, 0.7790)	(0.4355, 0.7789)	(0.4356, 0.7791)	(0.4447, 0.7467)
			0.3432/0.9372	0.3430/0.9370	0.3431/0.9372	0.3020/0.9716
			(0.4699, 0.7391)	(0.4691, 0.7388)	(0.4693, 0.7390)	(0.4740, 0.7206)
			0.2692/0.9364	0.2688/0.9361	0.2691/0.9361	0.2466/0.9572
			(0.4886, 0.7175)	(0.4885, 0.7175)	(0.4885, 0.7175)	(0.4911, 0.7047)
15	(4, 4)	0.6000	0.2289/0.9480	0.2283/0.9478	0.2284/0.9475	0.2136/0.9576
			(0.5394, 0.9141)	(0.5391, 0.9140)	(0.5391, 0.9140)	(0.5537, 0.8601)
			0.3747/0.9032	0.3742/0.9031	0.3744/0.9030	0.3065/0.9752
			(0.5699, 0.8816)	(0.5695, 0.8811)	(0.5697, 0.8815)	(0.5776, 0.8450)
			0.3117/0.9172	0.3112/0.9170	0.3116/0.9170	0.2674/0.9700
25	(4, 6)	0.7229	(0.6019, 0.8472)	(0.6009, 0.8467)	(0.6011, 0.8470)	(0.6051, 0.8254)
			0.2452/0.9308	0.2451/0.9303	0.2450/0.9300	0.2203/0.9596
			(0.6196, 0.8283)	(0.6191, 0.8282)	(0.6195, 0.8280)	(0.6209, 0.8127)
			0.2087/0.9332	0.2088/0.9333	0.2080/0.9330	0.1918/0.9540
			(0.6376, 0.9614)	(0.6371, 0.9611)	(0.6371, 0.9614)	(0.6531, 0.9092)
25	(4, 8)	0.8000	0.3238/0.8992	0.3233/0.8989	0.3233/0.8992	0.2561/0.9844
			(0.6680, 0.9353)	(0.6678, 0.9350)	(0.6680, 0.9352)	(0.6750, 0.8988)
			0.2674/0.9040	0.2671/0.9037	0.2673/0.9040	0.2239/0.9664
			(0.6929, 0.9048)	(0.6928, 0.9041)	(0.6929, 0.9045)	(0.6961, 0.8826)
			0.2119/0.9252	0.2111/0.9251	0.2111/0.9250	0.1866/0.9648
35	(4, 8)	0.8000	(0.7098, 0.8897)	(0.7092, 0.8892)	(0.7099, 0.8897)	(0.7110, 0.8739)
			0.1799/0.9372	0.1793/0.9371	0.1799/0.9371	0.1630/0.9628

Table 5.10 Bayesian interval estimates of $R_{2,4}$ when the common shape parameter β is known ($\beta = 10$)

Sample size n	Parameters δ	Reliability $R_{2,4}$	Bayesian			
			SE_{BCI}^{MCMC}	SE_{HPDI}^{MCMC}	$LINEX_{BCI}^{MCMC}$	$LINEX_{HPDI}^{MCMC}$
10	(4, 2)	0.3905	(0.2182, 0.5960)	(0.2180, 0.5959)	(0.2178, 0.5959)	(0.2340, 0.5663)
			0.3778/0.9271	0.3779/0.9272	0.3773/0.9271	0.3322/0.9548
			(0.2443, 0.5550)	(0.2447, 0.5550)	(0.2444, 0.5549)	(0.2553, 0.5375)
			0.3103/0.9432	0.3098/0.9430	0.3100/0.9430	0.2822/0.9604
15	(4, 2)	0.3905	(0.2771, 0.5190)	(0.2767, 0.5189)	(0.2770, 0.5189)	(0.2830, 0.5100)
			0.2419/0.9448	0.2411/0.9444	0.2411/0.9445	0.2269/0.9556
			(0.2935, 0.4980)	(0.2931, 0.4979)	(0.2932, 0.4982)	(0.2975, 0.4920)
			0.2045/0.9412	0.2043/0.9401	0.2041/0.9411	0.1945/0.9432
10	(4, 4)	0.6000	(0.4018, 0.8153)	(0.4012, 0.8151)	(0.4017, 0.8153)	(0.4172, 0.7667)
			0.4134/0.9200	0.4130/0.9197	0.4134/0.9203	0.3495/0.9672
			(0.4357, 0.7790)	(0.4355, 0.7789)	(0.4356, 0.7791)	(0.4447, 0.7467)
			0.3432/0.9372	0.3430/0.9370	0.3431/0.9372	0.3020/0.9716
15	(4, 4)	0.6000	(0.4699, 0.7391)	(0.4691, 0.7388)	(0.4693, 0.7390)	(0.4740, 0.7206)
			0.2692/0.9364	0.2688/0.9361	0.2691/0.9361	0.2466/0.9572
			(0.4886, 0.7175)	(0.4885, 0.7175)	(0.4885, 0.7175)	(0.4911, 0.7047)
			0.2289/0.9480	0.2283/0.9478	0.2284/0.9475	0.2136/0.9576
10	(4, 6)	0.7229	(0.5394, 0.9141)	(0.5391, 0.9140)	(0.5391, 0.9140)	(0.5537, 0.8601)
			0.3747/0.9032	0.3742/0.9031	0.3744/0.9030	0.3065/0.9752
			(0.5699, 0.8816)	(0.5695, 0.8811)	(0.5697, 0.8815)	(0.5776, 0.8450)
			0.3117/0.9172	0.3112/0.9170	0.3116/0.9170	0.2674/0.9700
15	(4, 6)	0.7229	(0.6019, 0.8472)	(0.6009, 0.8467)	(0.6011, 0.8470)	(0.6051, 0.8254)
			0.2452/0.9308	0.2451/0.9303	0.2450/0.9300	0.2203/0.9596
			(0.6196, 0.8283)	(0.6191, 0.8282)	(0.6195, 0.8280)	(0.6209, 0.8127)
			0.2087/0.9332	0.2088/0.9333	0.2080/0.9330	0.1918/0.9540
10	(4, 8)	0.8000	(0.6376, 0.9614)	(0.6371, 0.9611)	(0.6371, 0.9614)	(0.6531, 0.9092)
			0.3238/0.8992	0.3233/0.8989	0.3233/0.8992	0.2561/0.9844
			(0.6680, 0.9353)	(0.6678, 0.9350)	(0.6680, 0.9352)	(0.6750, 0.8988)
			0.2674/0.9040	0.2671/0.9037	0.2673/0.9040	0.2239/0.9664
15	(4, 8)	0.8000	(0.6929, 0.9048)	(0.6928, 0.9041)	(0.6929, 0.9045)	(0.6961, 0.8826)
			0.2119/0.9252	0.2111/0.9251	0.2111/0.9250	0.1866/0.9648
			(0.7098, 0.8897)	(0.7092, 0.8892)	(0.7099, 0.8897)	(0.7110, 0.8739)
			0.1799/0.9372	0.1793/0.9371	0.1799/0.9371	0.1630/0.9628

Table 5.11 Empirical (true) Type-I error rates for testing $H_0 : R_{s,k} \leq R_0$ vs. $H_a : R_{s,k} > R_0$ when nominal (intended) level is $\gamma = 0.05$ with the known common shape parameter ($\beta = 3$)

n	δ	$R_{1,3}$	R_0	Generalized	Bayesian	Classical	$R_{2,4}$	R_0	Generalized	Bayesian	Classical
10	(4, 2)	0.5429	0.50	0.0490	0.0059	0.0070	0.3905	0.35	0.0510	0.0510	0.0145
15				0.0450	0.0050	0.0058			0.0489	0.0541	0.0125
25				0.0510	0.0480	0.0060			0.0485	0.0478	0.0128
35				0.0491	0.0480	0.0063			0.0510	0.0478	0.088
10	(4, 4)	0.7500	0.70	0.0481	0.0030	0.0031	0.6000	0.55	0.0510	0.0510	0.0412
15				0.0510	0.0500	0.0281			0.0478	0.0510	0.0415
25				0.0503	0.0050	0.0017			0.0512	0.0478	0.0325
35				0.0540	0.0570	0.0125			0.0499	0.0510	0.0324
10	(4, 6)	0.8476	0.80	0.0479	0.0590	0.0254	0.7229	0.65	0.0510	0.0541	0.0254
15				0.0486	0.0480	0.0123			0.0502	0.0499	0.0213
25				0.0512	0.0480	0.0325			0.0513	0.0499	0.0215
35				0.01487	0.0059	0.0327			0.0499	0.0510	0.0113
10	(4, 8)	0.9000	0.85	0.0489	0.0590	0.0400	0.8000	0.75	0.0501	0.0541	0.0413
15				0.0466	0.0570	0.0328			0.4888	0.0510	0.0077
25				0.0485	0.0059	0.0214			0.4789	0.0541	0.0012
35				0.0512	0.0059	0.0415			0.0541	0.0510	0.0045

Table 5.12 Empirical (true) Type-I error rates for testing $H_0 : R_{s,k} \leq R_0$ vs. $H_a : R_{s,k} > R_0$ when nominal (intended) level is $\gamma = 0.05$ with the known common shape parameter ($\beta = 10$)

n	δ	$R_{1,3}$	R_0	Generalized	Bayesian	Classical	$R_{2,4}$	R_0	Generalized	Bayesian	Classical
10	(4, 2)	0.5429	0.50	0.0511	0.0478	0.0012	0.3905	0.35	0.0512	0.0498	0.0124
15				0.0513	0.0231	0.0045			0.0548	0.0088	0.0128
25				0.0489	0.0478	0.0078			0.0510	0.0145	0.0088
35				0.0485	0.0511	0.0099			0.0555	0.0498	0.0099
10	(4, 4)	0.7500	0.70	0.0478	0.0231	0.0012	0.6000	0.55	0.0478	0.0128	0.0100
15				0.0498	0.0222	0.0100			0.0498	0.0145	0.0099
25				0.0478	0.0125	0.0125			0.0457	0.0145	0.0145
35				0.0456	0.0231	0.0123			0.0498	0.0145	0.0179
10	(4, 6)	0.8476	0.80	0.0511	0.0511	0.0236	0.7229	0.65	0.0478	0.0400	0.0258
15				0.0509	0.0222	0.0223			0.0513	0.0088	0.0248
25				0.0478	0.0222	0.0145			0.0511	0.0145	0.0325
35				0.0499	0.0222	0.0128			0.0547	0.0547	0.0125
10	(4, 8)	0.9000	0.85	0.0456	0.0511	0.0222	0.8000	0.75	0.0555	0.0498	0.0410
15				0.0477	0.0478	0.0114			0.0547	0.0400	0.0124
25				0.0518	0.0231	0.0231			0.0512	0.0400	0.0400
35				0.0498	0.0231	0.0224			0.0478	0.0478	0.0128

Table 5.13 Comparison of powers for testing $H_0 : R_{2,4} \leq 0.5429$ vs $H_a : R_{2,4} > 0.5429$ without and after adjusting the size at $\gamma = 0.05$ when the common shape parameter is known ($\beta = 3$)

Parameters			Without adjusting the size			After adjusting the size		
n	δ	$R_{2,4}$	Generalized	Bayesian	Classical	Generalized	Bayesian	Classical
10	(4,2)	0.5429	0.1180	0.1180	0.0660	0.0500	0.0553	0.0553
15			0.1010	0.1180	0.0712	0.0998	0.0553	0.0621
25			0.1021	0.1180	0.0722	0.1000	0.0553	0.0702
35			0.1225	0.0712	0.0741	0.1198	0.0553	0.0715
10	(4,4)	0.7500	0.2222	0.0712	0.1215	0.1998	0.2000	0.1125
15			0.2112	0.0712	0.1015	0.2000	0.0715	0.1001
25			0.3125	0.0712	0.2451	0.2145	0.2000	0.1198
35			0.3546	0.2451	0.2415	0.3212	0.0715	0.2356
10	(4,6)	0.8476	0.4115	0.3999	0.3874	0.3998	0.2000	0.3789
15			0.4899	0.2451	0.3899	0.3454	0.0715	0.3877
25			0.5551	0.3999	0.3999	0.4597	0.2000	0.3845
35			0.5789	0.3999	0.4521	0.5412	0.4852	0.4511
10	(4,8)	0.9000	0.6889	0.3999	0.4887	0.5778	0.4852	0.4852
15			0.7888	0.8888	0.6552	0.6589	0.4852	0.5879
25			0.8888	0.8888	0.7858	0.7777	0.6666	0.6666
35			0.8994	0.8888	0.7889	0.8412	0.6666	0.6894

Table 5.14 Comparison of powers for testing $H_0 : R_{2,4} \leq 0.5429$ vs $H_a : R_{2,4} > 0.5429$ without and after adjusting the size at $\gamma = 0.05$ when the common shape parameter is known ($\beta = 10$)

Parameters			Without adjusting the size			After adjusting the size		
n	δ	$R_{2,4}$	Generalized	Bayesian	Classical	Generalized	Bayesian	Classical
10	(4,2)	0.5429	0.1001	0.1001	0.0125	0.0500	0.0500	0.0500
15			0.1254	0.1001	0.0245	0.1356	0.0500	0.0235
25			0.1540	0.1001	0.325	0.1389	0.0500	0.0245
35			0.1656	0.0245	0.0458	0.1478	0.0500	0.0889
10	(4,4)	0.7500	0.2789	0.0245	0.1225	0.1899	0.0500	0.0999
15			0.2889	0.0245	0.1458	0.2458	0.0500	0.1225
25			0.3211	0.0245	0.2451	0.2589	0.1899	0.1889
35			0.3489	0.3211	0.2898	0.3458	0.1899	0.1997
10	(4,6)	0.8476	0.4569	0.3211	0.3254	0.3888	0.1899	0.2458
15			0.4689	0.3211	0.3589	0.4125	0.1899	0.2789
25			0.4889	0.3211	0.3789	0.5478	0.2589	0.3458
35			0.5879	0.3211	0.4558	0.6521	0.2589	0.3333
10	(4,8)	0.9000	0.6655	0.3254	0.4789	0.7415	0.2589	0.3889
15			0.7889	0.3254	0.5511	0.8888	0.4215	0.4215
25			0.8994	0.6789	0.6654	0.8995	0.4215	0.4887
35			0.9994	0.6789	0.6789	0.9885	0.4215	0.6987

CHAPTER 6

OVERVIEW, SUMMARY, AND FUTURE RESEARCH

Overview

Classical inferences for the reliability of multicomponent stress-strength system using various underlying distributions have been discussed intensively and extensively in literature. To name few seminal papers: Hanagal [32], Eryilmaz [16], Rao et al. [46], and seminal works of many others. For a comprehensive discussion on different stress-strength models, along with more theories, applications, and examples, the interested parties are referred to the monograph of Kotz et al. [38]. In these studies, maximum likelihood estimator (MLE) and asymptotic confidence interval were obtained. The size of the test, adjusted and unadjusted power of the test, coverage probability and expected confidence lengths of the confidence interval, and biases of the estimator are also discussed. But mainly, the recently introduced generalized variable method (GVM) due to Tusi and Weerahandi [49] was not taken into consideration.

Therefore, in this research work, we mainly discuss the generalized inferences. To do that: firstly, for the classical inferences of $R_{s,k}$, MLE- and UMVUE-based, pivotal quantity for the hypothesis testing and interval estimation, where MLE is the maximum likelihood estimator and UMVUE is the minimum variance unbiased estimator, are developed. Secondly, for the Bayesian inference of $R_{s,k}$, exact and approximate point estimators are developed with the aid of Markov Chain Monte Carlo (MCMC or MC²) procedure using the Gibbs sampler and Metropolis-Hasting sampler, and of Lindley's approximation [40] procedure. Bayesian confidence intervals (BCI) as well as highest posterior density intervals (HPDI) are also computed. Finally, for the generalized inference of $R_{s,k}$, estimators, interval estimators, and hypothesis testing of $R_{s,k}$ are developed with the aid of the generalized variable method. The diagnostic testing procedures found in reliability analyses have a

wide variety of applications in economics, engineering, biostatistics, biomedical, and various other related-fields.

It is the opinion and strong belief of the author of this research thesis that the intensive and extensive research in this nature must be carried out to broaden the scope of, and to open new avenues for, the critical and rational thinking needed to produce new statistical methodologies and procedures to tackle the complex and complicated statistical problems found in artificial intelligence, machine learning, deep learning, and data analytics in this era of very advanced high technology and sciences. Furthermore, independent and collaborative research based on these new procedures done with the other interested parties will contribute in a great deal to the success and advancement of the statistical research and to fill vacuum in the statistical arena. A statistics major with a strong and robust background in this type of research will be a very competitive and beneficial advantage when they plan to enter into the workforce in future.

Over the years we have seen an increase in the number of students pursuing advanced degrees in statistics after graduated with Bachelor's and Master's degrees. This research will broaden the statistical knowledge of those students who are pursuing Ph.D. and are interested in doing research to contribute to the statistical arena, and also those who seek employment or internships in various institutions.

Summary

In Chapter II, we examine and review the classical inferences for $R_{s,k}$ with the aid of MLE and UMVUE.

In Chapter III, we provide generalized inference for $R_{s,k}$. This discussion has been built up for the generalized point and interval estimation as well as for the testing hypothesis in the face of nuisance parameters from different populations by using generalized p -value approach introduced

by Tsui and Weerahandi [54]. This new development, which has a promising approach for data modeling in reliability and survivability, has revolutionized the advanced-science- and hi-tech-based modern society. This technique is very useful for practitioners who have been performing inferences using the normality-assumption-based inferences even if they deal with small samples for the sake of the mathematical tractability and mere simplicity.

Reliability experts, who encounter several various system models with longer heavy-tailed distributions, can now easily remedy the difficulties using this newly introduced generalized variable method. In addition, this methodology is heavily used in agriculture, mechanical engineering, econometrics fields, etc. Practitioners in biostatistical and biomedical research, where each sample point is vital and expensive, can now comfortably use this generalized variable method to provide a significant test with power of testing procedures. This generalized p -value approach can easily be used to overcome the drawbacks of F-test's failure to detect significant experimental results.

In Chapter IV, we review and suggest the Bayesian inference for $R_{s,k}$ with the aid of MCMC, Gibbs sampler, Metropolis-Hasting sampler, and Lindley's approximation.

In Chapter V, simulation results for biases of the point estimators, coverage probability and mean confidence lengths of the interval estimators, and true type-I error rate control, unadjusted and adjusted power of the test are extensively and intensively discussed. In addition, extensive and intensive data analysis was performed for a real-world data set, which represents the monthly water capacity of Shasta reservoir with code name USBR SHA that is operated and maintained by the U.S. Federal Bureau of Reclamation under The United States Department of the Interior, California, USA. The data set is available in the link

<http://cdec.water.ca.gov/cgi-progs/queryMonthly?SHA>.

Complicated functions of parameters are not easily inferred exactly using classical approaches; in that sense, we here emphasize the importance of using the generalized variable method, which outperforms other available and exiting inferential methodologies in the face of nuisance parameters

Future Research

One of the major weaknesses and the drawbacks of the generalized variable method is that its non-applicability when the pivotal quantities are not distributed with standard distributions. But such situations are also tackled by using intensive and tedious numerical approaches, which are to be explored as future works. Moreover, the power guarantee has not been mathematically proved and is a major hot topic in the statistical arena. Furthermore, the advantages and drawbacks are summarized as follows;

Advantages of the proposed method:

1. Can handle complicated functions of parameters.
2. Various distribution-driven tests.
3. Valid for smaller samples as well as for the larger samples.
4. Can easily avoid the unnecessary large sample assumption.
5. Can avoid the unnecessary large sample assumption.
6. Can find exact solutions in the face of nuisance parameters.

Drawbacks of the proposed procedure:

1. p -values are not uniformly distributed.
2. If the estimators are not distributed with distributions with closed forms intensive numerical analysis has to be carried out.
3. Can not solve all situations unless the test variable satisfy the properties of Generalized Test Variable.

A compact and comprehensive final version of the thesis will be submitted to the Graduate Coordinating Committee of the Department of Mathematics and to the university's Graduate School. Collaborating with my advisor Dr. Gunasekera, several high quality advanced papers stemming from this research will be submitted to top peer-reviewed statistical/mathematical journals. In addition, papers will be submitted to the 2019 Joint Statistical Meetings (JSM) and 2019 8th International Conference on Biostatistics & Bioinformatics (CB&B) for the oral presentation. JSM is the largest gathering of statisticians in North America, attended by more than

6000 visitors across the globe, sponsored jointly with the American Statistical Association (ASA), Institute of Mathematical Statistics (IMS), International Biometric Society (IBS) (Eastern North American Region - ENAR and Western North American Region –WNAR), Statistical Society of Canada (SSC), International Chinese Statistical Association (ICSA), International Indian Statistical Association (IISA), International Society for Bayesian Analysis (ISBA), and Korean International Statistical Association (KISA). It will be held at the Baltimore Convention Center, Baltimore, Maryland from July 27 to August 01, 2019, and CB&B is sponsored by the Conference Series from September 16 to 17, 2019 in San Francisco, CA

Furthermore, building up from analyzing a two-component system, future research will focus on analyzing three-component or many-component systems. Another development in analysis of reliability is taking different type of censored, truncated, grouped, or merged data under Type-I or -II left-and right-censored data rather than taking type-II progressively right censored data uniformly removals thus paving the way for different aspects to be discussed.

Applicability, accessibility, and usability of exact non parametric procedures in reliability are also in serious consideration and hope to explore nonparametric new approaches coupled with the old ones to come up with methodology to tackle drastic, vague situations without taking the underlying distributions into account. In the future, we seek to expand the applications of this generalized p -value methodology expanding from reliability into other areas and fields such as data networking, econometrics, agriculture, actuarial field, insurance, etc.

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