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by

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Trading Motives in Asset Markets*

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Abstract

I study how trading motives in asset markets affect equilibrium outcomes and welfare. I focus on two types of trading motives – informational and allocational. I show that while a fully separating equilibrium is the unique equilibrium when trading motives are known, multiple equilibria exist when trading motives are unknown. Moreover, forcing traders to reveal their trading motives may harm welfare. I also use this model to study how an asset market may exit a fire sale equilibrium and how government programs may eliminate private information and improve agents' welfare.

JEL Codes: G12, D82, D83, G18

Keywords: asset markets, private information, competitive search, government intervention

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1 Introduction

Traders in asset markets trade for various reasons. In this paper I consider two types of motives, "allocational" and "informational". The allocational motive refers to trading for liquidity needs, and the informational motive refers to trading on private information.[1] For example, in the used car market a car owner may wish to sell because she needs money to purchase other goods and services, or because the car is a lemon. In the first case the motive is allocational, in that the trade is mutually beneficial. In the second case the motive is informational, in that the total surplus from the trade is zero and the owner only sells to take advantage of her private information on quality.

In the above example, two trading motives arise from two dimensions of private information – agents' liquidity needs and car quality. In this paper, I propose a model where there are two assets – a liquid and safe asset (fruit) and an illiquid and risky asset (tree). There are two sources of private information: liquidity needs and tree quality. I will use this model to answer the following questions: How do trading motives determine the market equilibrium? Are both trading motives always present when both dimensions of private information exist? Would forcing traders to reveal their motives improve welfare? Are there government programs that eliminate information frictions and improve trades welfare? How can this framework be applied to some well-known phenomena of the asset markets, for example the fire sales?

To understand the role played by trading motives, I first study a benchmark model where only the tree quality is private information and therefore only the allocational motive exists in equilibrium. I show that there exists a unique separating equilibrium in which sellers with different quality trees offer different prices and quantities for sale. Next, I assume that liquidity needs are also private information. I show that, depending on the distribution of tree quality, multiple equilibria exist, and in some equilibria some sellers pool to offer the same price and quantity. There are three main findings.

First, sellers with different quality trees and motives pool only when both liquidity needs and tree quality are private information. For sellers with liquidity needs, not selling trees means less liquidity available for consumption. Hence, there is an opportunity cost to holding trees, and this holding cost lowers the private value of the trees. As a result, sellers with trees of different common value may share similar private value. Because of the similar preferences, sellers with low quality trees can always successfully mimic the behavior of sellers with high quality trees. Consequently, different types of sellers pool in equilibrium.

Second, it is possible that only the allocational motive exists even when both dimensions of private information are present. The reason is that, since information-motivated sellers share the same private value of the assets with the buyers, the price must be higher than the common

[1] These two concepts are first coined by Vayanos (2001). There are other potentially interesting trading motives. Duffie (2011) shows that some banks in federal funds market trade to take advantage of their higher contact rates compared to other less active banks. In Yoon (2017) agents trade in OTC market to learn the price information on their assets.

value for the sale to be profitable to the sellers. Liquidity-motivated sellers, on the other hand, are willing to sell at prices lower than the common value because they gain from satisfying their liquidity needs. If buyers' profit (which is determined by the relative number of sellers) is high enough, tree prices in the market will be too low to attract information-motivated sellers. As a result, information-motivated sellers do not participate in the asset market because they cannot profit from their private information.

Third, more information needs not be better. If traders were forced to reveal their trading motives, this may harm welfare. If motives are known, information-motivated sellers will not be able to sell since other agents know they only trade to pass on lemons, and the separating equilibrium will be the only equilibrium. But separation is costly since sellers ration to signal their quality. If motives remain hidden, there may exist pooling of sellers, which increases trading volume. The downside of pooling is that now sellers with high quality trees have to subsidize the other sellers. I show that the benefit from selling more can outweigh the cost of subsidizing, and therefore welfare can be improved in the pooling equilibrium. Similar conclusions about the ambiguous effect of more information on welfare can also be found in models with financial assets serving as media of exchange (Andolfatto, 2010; Andolfatto et al., 2014; Dang et al., 2017). In those models, the disclosure of asset quality generates excessive volatility in prices and limits the assets' roles in transactions, whereas in my model the disclosure of motives increases the signaling cost and lowers the trading volume.

This paper belongs to the literature that studies competitive search equilibria in environments with private information.[2] It is closely related to the recent literature on two-dimensional private information in asset markets that consists of asset quality and sellers' preferences (Williams, 2016; Chang, 2018; Guerrieri and Shimer, 2018). Similar pooling of sellers can be found in Chang (2018), and like this paper, the main results build on the existence of such pooling. However, Williams (2016) shows that there always exists a fully-separating equilibrium if one drops the assumption of indivisible assets in Chang (2018) to allow partial retention of assets. Then it is unclear why pooling of sellers would necessarily emerge in equilibrium given the empirical evidence on partial retention.[3]

A major difference between this paper and Williams (2016) and Chang (2018) is that the holding cost is endogenous and the roles of buyers and sellers arise endogenously. I endogenize the holding cost by adding a goods market that opens after the asset market.[4] Agents who want to consume in the goods market then try to sell their illiquid assets in the asset market. Moreover, I allow agents to decide whether to be a buyer and/or a seller. In equilibrium, some sellers are also

[2]Guerrieri et al. (2010) are the first to propose a model where agents with private information search for matches with principles who use contracts to screen hidden types. Building on Guerrieri et al. (2010), Guerrieri and Shimer (2014) assume there is one-dimensional private information in asset quality and show that prices and trading probabilities are important signaling devices in equilibrium.

[3]Partial retention of assets can be found in the divestitures, venture capital, IPOs, SEOs, and mortgage-backed security pools (Williams, 2016).

[4]Geromichalos and Herrenbrueck (2016) and Jacquet (2018) have similar environments.

buyers, and some assets are not on the market. Under this setup, I show that even though assets are perfectly divisible and partial retention is allowed, an equilibrium with partial pooling is the unique equilibrium when some sellers' private value of assets is similar enough.

There are several other important differences between this paper and the existing literature. First, Guerrieri and Shimer (2018) impose relatively weak restrictions on off-equilibrium beliefs and focus on the welfare comparisons of the resulting multiple equilibria. In this paper, I instead employ two widely-used equilibrium refinement methods, the Intuitive Criterion (Cho and Kreps, 1987) and the Undefeated Equilibrium (Mailath et al., 1993), to test that the main welfare result is robust to different restrictions on off-equilibrium beliefs.

Second, Chang (2018) focuses on a type of semi-pooling equilibrium that exhibits some empirical features of a "fire sale". That is, in the semi-pooling equilibrium, some distressed sellers sell quickly at a highly depressed price. However, in Chang (2018), buyers enter the asset market after paying an entry cost. The zero profit condition dictates that buyers' search value is always constant and equal to the entry cost. I instead assume a fixed supply of buyers. When sellers' desire to sell increases as they become more distressed, a fixed supply allows buyers' share of the surplus to increase. The increase in buyers' surplus drives down the asset price and prompts low quality sellers with no liquidity needs to stop selling, breaking the fire sale equilibrium. I use the model to show how an asset market can "exit" a fire sale equilibrium as economic fundamentals change.

Third, compared to those three papers, my setup makes it easy to incorporate a government to the environment. In particular, I consider two government programs, an asset purchase program and a collateralized lending program. I show that it is possible for both programs to eliminate private information and improve agents' welfare. However, if the economy is hit with an unforeseen aggregate shock to asset quality or if there is not enough collateral, the programs may either become infeasible or fail to reduce information frictions.

The rest of the paper is organized as follows. Section 2 describes the physical environment. Section 3 solves a model with known trading motives. Section 4 returns to the full model with unknown trading motives and conducts welfare analysis. Section 5 studies the fire sale equilibria and how the asset market may exit a fire sale equilibrium. Section 6 discusses government interventions that may improve agents' welfare. Section 7 concludes the paper.

2 Model Environment

The economy has two assets, lasts for three periods, and has three types of agents. The two assets are fruit and trees.[5] The three periods are the "AM", the "GM" and the "FM", which are short

[5]The naming is meant to convey that the fruit is portable and safe (no uncertainty in quality) compared to trees. Therefore, the fruit is assumed to be the medium of exchange. Neither fruit nor trees are perishable because they both last three periods.

for asset market, goods market and fruit market. There is a continuum of agents who participate in the AM and FM. I refer to them as the "consumers". The consumers who also participate in the GM are the "shoppers", otherwise they are the "non-shoppers". There is also a continuum of "producers" who participate in the GM and FM. There is no discounting between periods.

At the beginning of the AM, all consumers are endowed with b units of fruit and a units of trees. Both fruit and trees are perfectly divisible. Next, an idiosyncratic consumption shock is realized. The shock happens with probability α. Consumers who receive the shock become "shoppers" and meet producers bilaterally with probability one in the GM. Upon meeting a producer, a shopper makes a take-or-leave it offer to purchase the GM good from the producer. The rest of the consumers become "non-shoppers" and do not participate in the GM. Shoppers possess a utility function $u(c)$ where c is the amount of GM good consumed. I assume $u'(.) > 0$, $u''(.) < 0$ and $u'(b) > 1$. The disutility incurred from producing c units of GM good is c. Because of anonymity and lack of record keeping, credit arrangements are not possible and producers only accept fruit as payment. In the FM, trees produce fruit, and consumers and producers consume fruit that is either brought from previous periods or from the trees they own.[6] All agents receive f units of utility from f units of fruit consumed. To summarize, a shopper's utility is given by $u(c) + f$. A non-shopper's utility is given by f. And a producer's utility is given by $-c + f$.

After the consumption shock but before the GM opens, an idiosyncratic shock to tree quality is also realized. The two shocks are independent of each other. A tree of quality δ generates δ units of fruit in the FM. I assume there are J different types of quality and $\delta_1 < \delta_2 < ... < \delta_J$. Define $\mathcal{J} = \{\delta_j\}_{j=1}^{J}$. The probability of $\delta = \delta_j$ is Δ_j. Also I assume both the consumption shock and the quality shock are private information in the AM. That is, consumers do not know whether others are shoppers or non-shoppers, and they do not know the quality of others' trees.

After the shocks are realized, an asset market opens and consumers trade fruit and trees. A consumer who wants to sell trees (the "sellers") posts a price-quantity pair (ψ, s) where ψ is the price of the tree in terms of fruit, and s is the quantity of trees they want to sell. I call each price-quantity pair (ψ, s) a "location". Consumers who want to buy trees (the "buyers") observe all the price-quantity pairs and also the buyer-seller ratio (θ) of each location before they decide where to buy. The meetings in the AM are bilateral. The probability of meeting a seller in a location with tightness θ is $q(\theta) = \min\{\theta^{-1}, 1\}$ and the probability of meeting a buyer is $p(\theta) = \min\{\theta, 1\}$. Consumers can also choose not to sell or buy any trees. I assume sellers can commit to the price-quantity pair (ψ, s) they post.

I summarize the sequence of events in the following figure.

[6] Since there is no discounting between periods and any consumers may become shoppers, the fruit is not consumed before the GM. Hence, it is without loss of generality to assume that fruit is only consumed in the FM.

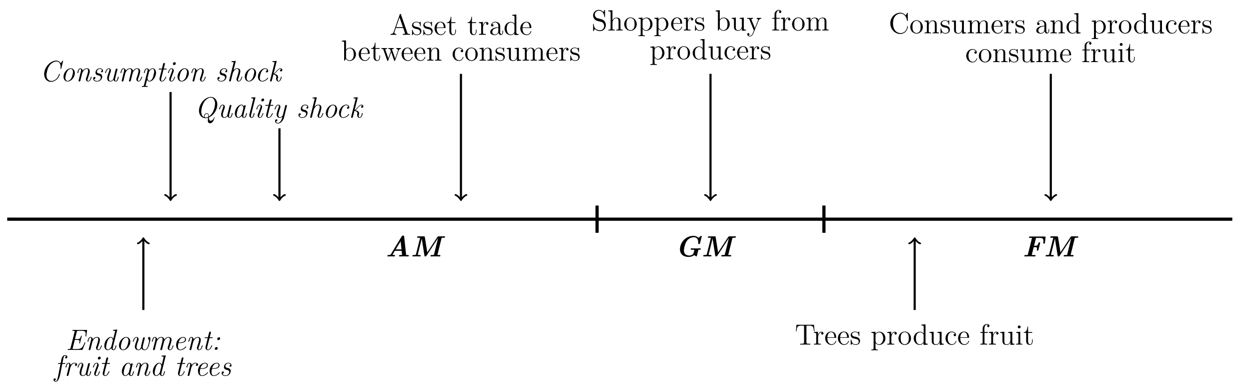

Figure 1: Model environment – sequence of events

3 A Benchmark Model: Trading Motives Known

In this benchmark model I assume the consumption shock is public information. There is only one dimension of private information – the quality of trees. To characterize the equilibrium, it is convenient to discuss first the GM and the FM. Suppose shoppers enter GM with the $\tilde{m}$ units of fruit. Then a shopper solves the following problem

$$\max_{c} u(c) - \ell \tag{3.1}$$

$$\text{s.t.} - c + \ell \geq 0 \tag{3.2}$$

$$\ell \leq \tilde{m} \tag{3.3}$$

where c is the amount of GM good purchased and ℓ is the amount of fruit transferred to the producers. The first constraint is the producer's participation constraint. Recall that ℓ units of fruit generate ℓ units of utility in the following FM. The second constraint is shopper's resource constraint. The shopper cannot spend more than what she brings to the GM. Let $c(\tilde{m})$ be the solution to the above problem conditional on $\tilde{m}$. Then it is easy to see that $c(\tilde{m}) = \min\{\tilde{m}, c^*\}$ where $u'(c^*) = 1$.

Recall that I assume $u'(b) > 1$. That is, the fruit endowment is not large enough for shoppers to achieve the efficient level of consumption. Then, since all consumers have utility linear in fruit in the FM but shoppers have higher value for fruit in the GM, shoppers will want to sell trees in exchange for fruit. In addition, since shoppers value fruit more than non-shoppers, sellers will always be shoppers and buyers will be non-shoppers.[7]

Now turn to consumer's problem in the AM. Let Ψ be the set of price-quantity pairs (ψ, s) posted in the equilibrium. A seller with quality δ trees solves the following problem

$$\max_{(\psi,s)\in\Psi\cup\emptyset} p(\theta(\psi,s))[u(c(b+\psi s)) - c(b+\psi s) - \delta s] + (1 - p(\theta(\psi,s)))[u(b) - b] \tag{3.4}$$

[7]This is no longer true if consumers do not know whether others are shoppers or non-shoppers. See Section 4.

where $\theta(\psi, s)$ is the market tightness at location (ψ, s). In words, (3.5) says that if a seller meets a buyer at location (ψ, s), which happens with probability $p(\theta(\psi, s))$, the seller obtains ψs units of fruit from the buyer, and together with the endowment b they allow $c(b+\psi s)$ amount of consumption in the following GM. In exchange, the seller transfers s units of trees to the buyer, which costs the seller δs amount of consumption in the FM. With probability $1 - p(\theta(\psi, s))$, the seller does not meet a buyer and consumes $c(b) = b$ amount of consumption in the following GM. A seller can always choose "$\emptyset$", which means they do not participate in the AM. I will show later that it is never optimal for sellers to acquire more fruit than what they need to achieve c^*. Hence, in equilibrium $c(b + \psi s) = b + \psi s$. Use this result to simplify the above equation and obtain

$$v^*_{1,j} = \max_{(\psi,s)\in\Psi\cup\emptyset} p(\theta(\psi, s))[u(b + \psi s) - u(b) - \delta_j s]. \tag{3.5}$$

where $v^*_{1,j}$ is the search value of sellers with quality δ_j trees. Next, buyers in the AM solve the following problem

$$v^*_0 = \max_{(\psi,s)\in\Psi\cup\emptyset} q(\theta(\psi, s))s \sum_{j'=1}^{J} \gamma(\psi, s; \delta_{j'})(\delta_{j'} - \psi) \tag{3.6}$$

where $\gamma(\psi, s; \delta_{j'})$ is the buyer's belief on the probability of tree quality in location (ψ, s) being $\delta_{j'}$ and $\sum_{j'=1}^{J} \gamma(\psi, s; \delta_{j'}) = 1$. In words, (3.6) says that if a buyer meets a seller at location (ψ, s), with probability $\gamma(\psi, s; \delta_{j'})$ the trees have quality $\delta_{j'} s$. Then the buyer receives s units of trees, which will give her $\delta_{j'} s$ units of fruit in the FM. In return, the buyer gives the seller ψs units of fruit. v^*_0 is the buyer's search value in the AM. Now we are ready to define the equilibrium in the AM.

Definition 3.1 *A competitive equilibrium in the AM is a set Ψ of price-quantity pairs (ψ, s), a vector $\{v^*_{1,j}\}_{j=1}^{J}$, a scalar v^*_0, functions $\theta : \Psi \to [0, \infty]$ and $\gamma : \Psi \times \mathcal{J} \to [0, 1]$, and an accumulative distribution function $F : \Psi \to [0, 1]$ that satisfy*
(1) Seller's and buyer's optimal behavior: for all $j \in \mathcal{J}$, $v^*_{1,j}$ satisfies equation (3.5); v^*_0 satisfies equation (3.6);
(2) Equilibrium beliefs: for all $j \in \mathcal{J}$ and for all $(\psi, s) \in \Psi$, $\gamma(\psi, s; \delta)$ satisfies Bayes Rule: $\gamma(\psi, s; \delta) = \mathbb{E}[\delta|(\psi, s)]$;
(3) Active markets: for all (ψ, s), $(\psi, s) \in \Psi$ only if it solves the maximization problem in (3.6) and it is feasible: $\psi s \le b$ and $s \le a$;
(4) Aggregate consistency: for all $j \in \mathcal{J}$ and for all $(\psi, s) \in \Psi$,

$$\Delta_j = \int_\Psi \gamma(\psi, s; \delta_j) dF(\psi, s) \text{ and } \frac{\alpha}{1 - \alpha} = \frac{1}{\int_\Psi \theta(\psi, s) dF(\psi, s)};$$

and
(5) No profitable deviations: there does not exist $S \subset \mathcal{J}$ and $(\psi, s, \theta) \in \mathbb{R}^3_+$ such that (ψ, s) is

feasible and

$$p(\theta)[u(b+\psi s) - u(b) - \delta_j s] > v_{1,j}^* \text{ for some } j \in S, \tag{3.7}$$

$$p(\theta)[u(b+\psi s) - u(b) - \delta_{j'} s] < v_{1,j'}^* \text{ for all } j' \in \mathcal{J} \backslash S, \tag{3.8}$$

$$q(\theta)s \sum_{j' \in S'} \gamma(\psi, s; \delta_{j'})(\delta_{j'} - \psi) \geq v_0^* \text{ for all } S' \subset S \tag{3.9}$$

where $\gamma(\psi, s; \delta)$ satisfies the Bayes' Rule.

Condition (1) says that sellers and buyers must choose optimally which location to sell or buy trees. Condition (2) puts restrictions on buyer's beliefs – they have to satisfy the Bayes' Rule given sellers' strategies. Condition (3) ensures that Ψ represents the set of "active" markets – it rules out the price-quantity pairs that are optimal for sellers under some function θ but not optimal for buyers and therefore are not posted in equilibrium. Condition (4) guarantees that the beliefs are consistent with the actual supply of trees, and that market tightness is consistent with actual supply of sellers and buyers.

Condition (5) is the Intuitive Criterion (Cho and Kreps, 1987; Rocheteau, 2008). It says that for any sequential equilibrium to exist, there must not exist an off-equilibrium offer (ψ, s) that makes at least one seller in set S strictly better off but makes everyone in $\mathcal{J} \backslash S$ strictly worse off, and is accepted by buyers given any belief system that puts no weights on sellers in $\mathcal{J} \backslash S$.

I solve the equilibrium in detail in Appendix A. I summarize the results in the following Proposition.

Proposition 3.1 *There exists a unique competitive equilibrium in the AM. In the equilibrium sellers with δ_j post (ψ_j, s_j) where $s_j \leq s_{j-1}$, $\psi_j > \psi_{j-1}$ and $\theta_j \leq \theta_{j-1}$ for all j.*

In Appendix A, I show that in general, shoppers do not sell all the trees they have. This marks a stark difference from asset market models with constant holding costs: sellers here care about both the prices and how much they have to sell, since sellers' liquidity needs decrease with the amount of liquidity they receive. If sellers' desire to sell does not vary with the trading volume, they will always sell all the trees.[8] This finding also implies that sellers actually sell *less* when the price is *high*, because a high price decreases the amount of trees that needs to be sold to satisfy certain liquidity needs.

Another noteworthy finding is that a fully separating equilibrium is the only equilibrium. This is not surprising as the Intuitive Criterion often rules out pooling equilibria. The reason is that the sellers with high quality trees have higher marginal cost of selling since they are giving up more FM consumption. Suppose, in any pooling equilibria, a high quality seller deviates by reducing the quantity they sell by a small amount, and the buyers can be kept at least as well off as before

[8] Section 3 of Williams (2016) describes exactly this situation.

as long as the decrease in quantity is offset by the increase in quality. Then, suppose some seller with lower quality trees tries to mimic this behavior. While the loss from selling less is the same for both sellers – they receive less fruit and consume less in the GM – the benefit from selling less is strictly larger for the high quality seller. Then, it is always possible for the high quality seller to find an offer that only benefits her and is acceptable by the buyers. This violates condition (5) of Definition 3.1.

While the Intuitive Criterion has been widely used in the signaling literature,[9] some have criticized its logical foundation (Mailath et al., 1993). The essence of the criticism is that if some sellers can deviate to become better off while convincing the buyers that they are the only ones to benefit, these sellers will always deviate in equilibrium. Then, the original equilibrium should not serve as the basis for comparison. In practice, the Intuitive Criterion tends to rule out pooling equilibria even when such equilibria are arguably more "realistic". For example, if there was only small amount of low quality trees in the economy, the Intuitive Criterion would still rule out any pooling equilibria. Since in this case the price distortion in a pooling equilibrium is likely to be smaller than the signaling cost in a fully separating equilibrium, all agents may be made better off by switching to a pooling equilibrium.

One solution to this problem is to use the Undefeated Equilibrium proposed by Mailath et al. (1993). A sequential equilibrium is undefeated if there does not exist another sequential equilibrium where at least some agents are strictly better off. Some recent papers (Bajaj, 2018; Madison, 2018) use this concept to endogenously select pooling and separating equilibria.

I choose the Intuitive Criterion instead of the Undefeated Equilibrium for two reasons. First, equilibrium characterization under the Undefeated Equilibrium depends heavily on the quality distribution (see Section 4.3). Without making strong assumptions about the distribution, the equilibrium characterization is not tractable. Second, the purpose of this and the following section is to highlight the role played by trading motives in shaping the equilibrium outcome, and it is achieved by comparing the separating equilibrium that exists when motives are known, with the pooling equilibrium that exists when motives are unknown. That is, I do not seek to find all "reasonable" equilibria in either case but the mechanism that allows the pooling equilibrium to exist.[10]

[9] See Riley (2001) for a review. In context similar to this paper where assets serve as direct or indirect median of exchange, it has been used by Nosal and Wallace (2007), Rocheteau (2008, 2011) and Madison (2018).

[10] However, when conducting welfare analysis, it is important to consider other reasonable equilibria, and therefore I also use the Undefeated Equilibrium to refine the equilibrium (see Section 4.3).

4 The Full Model: Trading Motives Unknown

4.1 One Extension to the Model Environment

In this section I assume agents do not know whether other agents are consuming or not in the GM. When agents' identities are known, there is no incentive to trade if a shoppers meets another shopper or if a non-shopper meets another non-shopper. However, if agents' identities are unknown, it is possible for some agents to take advantage of this private information. For example, a non-shopper with low quality trees can go to the locations with high prices to sell, even though she has no liquidity needs. The trading motives are therefore no longer apparent. It could be either "allocational" if shoppers are selling or "informational" if the non-shoppers are selling.

To accommodate agents' "informational" motive, I modify the model to allow agents to sell and buy trees at the same time. However, agents can only trade with fruit and trees they bring to the AM – they cannot use the fruit/trees they acquire from the market to trade again. None of the conclusions in Section 3 are affected by this modification because no agent buys and sells at the same time when trading motives are known.

I now rewrite consumer's problem in the AM to reflect the changes in environment. Shoppers' search value is given by

$$v_{1,j}^{**} = v_{1,j}^{*} + \tilde{v}_1^{*}. \tag{4.1}$$

where

$$v_{1,j}^{*} = \max_{(\psi,s)\in\Psi\cup\emptyset} p(\theta(\psi,s))[u(b+\psi s) - u(b) - \delta_j s] \tag{4.2}$$

and

$$\tilde{v}_1^{*} = \max_{(\psi,s)\in\Psi\cup\emptyset} q(\theta(\psi,s)) \left[s\sum_{j'=1}^{J} \gamma(\psi,s;\delta_{j'})\delta_{j'} + u(b-\psi s) - u(b) \right]. \tag{4.3}$$

$v_{1,j}^{*}$ is shoppers' search value in the AM when they sell optimally and $\tilde{v}_1^{*}$ is their search value when they buy optimally. The interpretation of (4.2) is the same as (3.5). (4.3) says that if a shopper meets a seller at location (ψ, s), with probability $\gamma(\psi, s; \delta_{j'})$ the trees have quality $\delta_{j'}s$. Then the shopper receive s units of trees, which will give her $\delta_{j'}s$ units of fruit in the FM. Note that the purchase will reduce her DM consumption by ψs.

Non-shoppers' search value is given by

$$v_{0,j}^{**} = v_0^{*} + \tilde{v}_{0,j}^{*}. \tag{4.4}$$

where

$$v_0^* = \max_{(\psi,s)\in\Psi\cup\emptyset} q(\theta(\psi,s))s\sum_{j'=1}^{J}\gamma(\psi,s;\delta_{j'})(\delta_{j'}-\psi) \tag{4.5}$$

and

$$\tilde{v}_{0,j}^* = \max_{(\psi,s)\in\Psi\cup\emptyset} p(\theta(\psi,s))s[\psi-\delta_j]. \tag{4.6}$$

Then v_0^* is non-shoppers' search value when they buy optimally and $\tilde{v}_{0,j}^*$ is their search value when they sell optimally. The interpretations are similar to (4.2) and (4.3). Now I define the competitive equilibrium in the AM.

Definition 4.1 *A competitive equilibrium in the AM is a set Ψ of price-quantity pairs (ψ,s), vectors $\{v_{1,j}^{**}\}_{j=1}^{J}$ and $\{v_{0,j}^{**}\}_{j=1}^{J}$, functions $\theta:\Psi\to[0,\infty]$ and $\gamma:\Psi\times\mathcal{J}\to[0,1]$, and an accumulative distribution function $F:\Psi\to[0,1]$ that satisfy*
(1) Shopper's and non-shopper's optimal behavior: for all $j\in\mathcal{J}$, $v_{1,j}^{**}$ satisfies equation (4.1) and $v_{0,j}^{**}$ satisfies equation (4.4);
(2) Equilibrium beliefs: for all $j\in\mathcal{J}$ and for all $(\psi,s)\in\Psi$, $\gamma(\psi,s;\delta)$ satisfies Bayes Rule: $\gamma(\psi,s;\delta)=\mathbb{E}[\delta|(\psi,s)]$;
(3) Active markets: for all (ψ,s), $(\psi,s)\in\Psi$ only if it solves the maximization problem in (4.4) and it is feasible: $\psi s\le b$ and $s\le a$;
(4) Aggregate consistency: for all $j\in\mathcal{J}$ and for all $(\psi,s)\in\Psi$,

$$\Delta_j=\int_\Psi\gamma(\psi,s;\delta_j)dF(\psi,s) \text{ and } \frac{\alpha}{1-\alpha}=\frac{1}{\int_\Psi\theta(\psi,s)dF(\psi,s)};$$

and
(5) No profitable deviations: there does not exist $S\subset\mathcal{J}$ and $(\psi,s,\theta)\in\mathbb{R}_+^3$ such that (ψ,s) is feasible and

$$v_{1,j}(\psi,s,\theta)>v_{1,j}^* \text{ for some } j\in S, \tag{4.7}$$

$$v_{1,j}(\psi,s,\theta)<v_{1,j}^* \text{ and } \tilde{v}_{0,j}(\psi,s,\theta)<\tilde{v}_{0,j}^* \text{ for all } j\in\mathcal{J}\backslash S, \tag{4.8}$$

$$q(\theta)s\sum_{j'\in S'}\gamma(\psi,s;\delta_{j'})(\delta_{j'}-\psi)\ge v_0^* \text{ for all } S'\subset S \tag{4.9}$$

where $\gamma(\psi,s;\delta_{j'})$ satisfies Bayes Rule. $v_{1,j}(\psi,s,\theta)$ and $\tilde{v}_{0,j}(\psi,s,\theta)$ are given by

$$v_{1,j}(\psi,s,\theta)=p(\theta)[u(b+\psi s)-u(b)-\delta_j s]$$

and

$$\tilde{v}_{0,j}(\psi,s,\theta)=p(\theta)s[\psi-\delta_j].$$

Condition (5) is similar to the condition (5) in Definition 3.1. It is also the Intuitive Criterion. Notice that in (4.7) I require any profitable deviation to at least benefit some shoppers. This is because deviations that only benefit non-shoppers will not be exploited in the equilibrium. If they are exploited, non-shoppers will expose their identity and will not be able to trade.

4.2 The Competitive Equilibria in the AM

To characterize the competitive equilibria, I first state some properties that any equilibrium must satisfy.

Lemma 4.1 *Shoppers do not purchase trees in the AM. That is,* $v_{1,j}^{**} = v_{1,j}^{*}$.

A shopper can purchase trees from either another shopper or a non-shopper. If the trees were worth buying, it would mean that the increase in FM consumption offsets the decrease in GM consumption. If the asset seller were another shopper, she would keep the trees themselves. If the seller were a non-shopper, since a shopper values fruit more than a non-shopper, the non-shopper would never sell the trees at a price low enough for the shopper, which means the trees must be of low quality. Hence, shoppers would never buy trees.

Lemma 4.2 *There does not exist a fully-separating equilibrium where non-shoppers sell trees.*

If it is profitable for non-shoppers to sell trees, the price must be higher than the trees' common value, δ. But then the price will not be acceptable to either shoppers or other non-shoppers. Hence, no one will knowingly buy trees from non-shoppers, and the non-shoppers will always try to pool with shoppers.

Next, consider the possibility of sellers pooling in equilibrium. Denote the set of sellers that pool in equilibrium as S^p.

Lemma 4.3 *(1)* S^p *may not be empty; and*
(2) let $\bar{\delta}^p = \sum_{j \in S^p} \delta_j \Delta_j$ *be average tree quality of set* S^p*. Then all sellers in* S^p *with* $\delta_j < \bar{\delta}^p$ *are non-shoppers.*

I explain the second point first. Since non-shoppers only sell their trees at prices higher than their common value, the sales have to be subsidized by the shoppers. Hence, non-shoppers never sell in a pool where the average quality is lower than theirs. The following corollary is derived directly from Lemma 4.3. It says that there are always non-shoppers with high quality trees who do not sell in the equilibrium because all prices posted by shoppers are too low.

Corollary 4.1 *There exists a* $\delta^n < \delta_J$ *such that non-shoppers with* $\delta > \delta^n$ *do not sell in the equilibrium.*

Now back to the first point of Lemma 4.3. It says the Intuitive Criterion does not rule out pooling equilibria. Recall that when motives are known, high quality sellers can always separate themselves from low quality sellers because the marginal (opportunity) cost of selling trees is strictly higher for sellers with higher quality trees. This is still true since utility is linear in fruit consumption for all consumers. However, when motives are unknown, some of the low quality sellers are non-shoppers and they have lower marginal benefits of selling. Consider the following preferences of a high quality shopper and a low quality non-shopper.

$$\text{Shopper's preference: } U_s(\psi, s, \delta_j) = \underbrace{u(b + \psi s) - u(b)}_{\text{marginal benefit: } \psi u'(b+\psi s) > \psi} - \underbrace{\delta_j s}_{\text{marginal cost: } \delta_j > \bar{\delta}^p}$$

$$\text{Non-shopper's preference: } U_n(\psi, s, \delta_{j'}) = \underbrace{\psi s}_{\text{marginal benefit: } \psi} - \underbrace{\delta_{j'} s}_{\text{marginal cost: } \delta_{j'} < \bar{\delta}^p}$$

First note that in equilibrium shoppers generally acquire less than what is needed to achieve the efficient level of consumption.[11] That is, $b + \psi s < c^*$ where $u'(c^*) = 1$. Then, the marginal benefit of selling is strictly larger for the shoppers since $u'(b + \psi s) > 1$. As a result, as long as $\delta_{j'}$ is not too small compared to δ_j, shoppers' preferences for fruit vs trees ($U_s(\psi, s, \delta_j)$) can be very similar to non-shoppers' preferences for fruit vs trees ($U_n(\psi, s, \delta_{j'})$). In other words, although the the common value of the trees of is different, the shoppers and non-shoppers have similar *private* value. Consequently, it is possible that any offers that make the shoppers strictly better off will also make the non-shoppers better off. As a result, the difference in marginal cost is no longer sufficient for separation and the non-shoppers and shoppers pool in equilibrium.

To further characterize the competitive equilibria, one needs to make assumptions concerning the distributions of tree quality and consumer types. In what follows, I first describe an equilibrium with full separation and only shoppers as sellers. Only one trading motive arises from two dimensions of private information because non-shoppers cannot profit from the private information on tree quality.

Proposition 4.1 *For all $\delta_1 > 0$, there exists an $\alpha(\delta_1)$ such that as long as $\alpha > \alpha(\delta_1)$, the unique competitive equilibrium in the AM is a fully separating equilibrium where only shoppers sell trees.*

To understand the proposition, first note that in equilibrium buyers' search value, v_0^*, increases with α.[12] A high v_0^* then decreases tree prices, making selling trees less attractive to the non-shoppers. If α is high enough, all the prices posted by the shoppers will be too low to be profitable for δ_1 non-shoppers and hence all non-shoppers. Note that non-shoppers cannot post a different price without being identified as non-shoppers since no other sellers have the incentive to deviate.

[11] This is because sellers either face the signaling cost in a separating equilibrium or the price distortion in a pooling equilibrium (except for those with the lowest quality trees). Hence they never choose to acquire enough fruit for efficient consumption.

[12] See Proposition A.3 in Appendix A for more details.

To characterize the types of equilibria where sellers pool, more assumptions concerning quality distribution and consumer type distribution are needed. To keep the analysis tractable, for the rest of this section I restrict my attention to the case where there are only two types of quality ($J = 2$). For simplicity, I also assume that trade in the asset market is not constrained by the quantity of fruit and trees possessed by the consumers.[13]

4.3 A Special Case: $J = 2$

Let the two types of quality be δ_1 (low type) and δ_2 (high type). Let $\Delta_1 = \Delta$ and $\Delta_2 = 1 - \Delta$ be the proportions of δ_1 (low quality) and δ_2 (high quality) trees. Define $\bar{\delta} = \frac{\Delta(1-\alpha)\delta_1 + (1-\Delta)\alpha\delta_2}{\Delta(1-\alpha)+(1-\Delta)\alpha}$ to be the average quality if δ_1 non-shopper and δ_2 shoppers pool. Also I assume $u(c) = \log(c)$.

To characterize the equilibrium, I look for collections of a *4-tuple* $(\delta_1, \delta_2, \Delta, \alpha)$ that allow seller pooling to exist. Denote these sets as $\mathcal{C}$'s, differentiated by their superscripts.[14] The next proposition shows the existence of a "semi-pooling" equilibrium in which all non-shoppers buy trees and all the shoppers sell trees. δ_1 non-shoppers both sell and buy trees, and they pool with δ_2 shoppers. The equilibrium is illustrated in Figure 2.

Proposition 4.2 *There exists $\mathcal{C}^p \subset \mathbb{R}^2_{++} \times (0,1)^2$ such that for all $(\delta_1, \delta_2, \Delta, \alpha) \in \mathcal{C}^p$, there exists a unique competitive equilibrium where δ_1 shoppers offer (ψ_1, s_1), and δ_2 shoppers and δ_1 non-shoppers offer (ψ_2, s_2). $s_1 = \frac{m^*}{\delta_1}$ and $\psi_1 = \delta_1$ where $u'(b + m^*) = 1$. $s_2 = \frac{m'}{\delta_2}$ and $\psi_2 = \bar{\delta}$ where $u'(b + m') = \frac{\delta_2}{\bar{\delta}}$. Furthermore, $\theta_1 = \theta_2 = 1$ and $v_0^* = 0$.*[15]

In the proof (Appendix B) I show that for the semi-pooling equilibrium to exist, we need the difference between δ_1 and δ_2 to be small. Recall that for the pooling equilibrium to exist, the overall preferences of δ_1 non-shopper and δ_2 shoppers have to be similar. If δ_1 is too small, the difference in marginal cost of selling will be too large to be offset by the difference in marginal benefit of selling. Then there may exist offers that only benefit δ_2 shoppers, and the semi-pooling equilibrium cannot exist because condition (5) of Definition 4.1 is violated.

[13] In Appendix A I solve a more general case without this assumption.

[14] Detailed descriptions of these sets can be found in Appendix B.

[15] I look for equilibria with $v_0^* = 0$ because they can be solved analytically (see the proof in Appendix B). A semi-pooling equilibrium with $v_0^* > 0$ exists but has to be solved numerically. See Section 5.

	Shoppers	Non-shoppers
High type (δ_2)	*Sell at* (ψ_2, s_2)	*Buy*
Low type (δ_1)	*Sell at* (ψ_1, s_1)	*Buy; and sell at* (ψ_2, s_2)

Figure 2: Illustration of the semi-pooling equilibrium

It is then interesting to know how consumers' welfare in the semi-pooling equilibrium compares with the fully separating equilibrium in Section 3. If the fully separating equilibrium implies higher welfare, forcing consumers to reveal their identities as shoppers and non-shoppers may be beneficial. The following proposition states that if we maintain the assumptions in Proposition 4.2, the high type (δ_2) shoppers are always worse off when the trading motives are unknown, while the low type (δ_1) shoppers are unaffected.

Proposition 4.3 *Suppose the assumptions in Proposition 4.2 holds so a semi-pooling equilibrium exists. Then compared to the fully separating equilibrium where trading motives are known, high type shopper's search value ($v^*_{1,2}$) is always smaller and low type shopper's search value ($v^*_{1,1}$) is unchanged.*

While the result is not surprising, it is also not completely obvious. Low type shoppers are unaffected because no other sellers compete with them in either case. For high type shoppers, because pooling with low type non-shoppers lowers average tree quality, they have to post a lower price. However, the lower price may also allow the high type shoppers to sell more because they now do not need to ration as much to prevent low type shoppers from deviating. That is, trading volume ($\psi_2 s_2$) may be higher when trading motives are unknown. The proposition simply says that the first force is stronger, and therefore high type shopper's search value is smaller.

However, this is not the whole picture. While high type shoppers are made worse off, low type non-shoppers are made better off because their share of the surplus is zero in the fully separating equilibrium. Now, since the consumption shock is random, all consumers can be shoppers or non-shoppers. Hence, we should measure welfare with consumers' expected search value in the AM before the two shocks are realized. Define

$$v_p^* = \alpha\Delta v_{1,1}^*(p) + \alpha(1-\Delta)v_{1,2}^*(p) + (1-\alpha)\Delta\tilde{v}_{0,1}^*(p)$$

and

$$v_s^* = \alpha\Delta v_{1,1}^*(s) + \alpha(1-\Delta)v_{1,2}^*(s)$$

where the letters in the brackets mean "semi-pooling equilibrium" (p) and "separating equilibrium" (s). Recall that $v_{1,j}^*$ is δ_j shopper's search value and $\tilde{v}_{0,1}^*$ is low type non-shopper's search value. Then, v_p^* is consumers' expected search value in a semi-pooling equilibrium and v_s^* is consumers' expected search value in a separating equilibrium. The next proposition says it is possible that v_p^* is larger than v_s^*.

Proposition 4.4 *There exists $\mathcal{C}^w \subset \mathcal{C}^p$ such that for all $(\delta_1, \delta_2, \Delta, \alpha) \in \mathcal{C}^w$, there exist a semi-pooling equilibria and a corresponding separating equilibrium where $v_p^* > v_s^*$. That is, hidden trading motives may improve welfare.*

In the proof (Appendix B) I show that, for the proposition to hold, δ_1 (low type) must not be too large relative to δ_2 (high type). This is because in the fully separating equilibrium, the smaller δ_1 is, the more high type shoppers have to ration to prevent low type shoppers from deviating. Therefore, when δ_1 is not too large, the semi-pooling equilibrium permits higher trading volume for high type shoppers. Now since $v_0^* = 0$ (see Proposition 4.2), buyers' share of the surplus is zero. Then, the sum of δ_2 shoppers' and δ_1 non-shoppers' search value must equal to the total surplus, which increases with the trading volume. That is, it must be that

$$\alpha(1-\Delta)v_{1,2}^*(p) + (1-\alpha)\Delta\tilde{v}_{0,1}^*(p) > \alpha(1-\Delta)v_{1,2}^*(s).$$

Since $v_{1,1}^*(p) = v_{1,1}^*(s)$ (see Proposition 4.3), we have $v_p^* > v_s^*$. Proposition 4.3 and 4.4 show that while *ex post* high type shoppers prefer no private information on trading motives, *ex ante* it can be welfare-improving to have unknown trading motives.

So far I have only used the Intuitive Criterion to refine the equilibrium. The Intuitive Criterion keeps the analysis tractable, making it easy to demonstrate the mechanism that induces seller to pool. However, it is inadequate for welfare analysis because the Intuitive Criterion may have ruled out reasonable equilibria that offer higher welfare to consumers. In the next proposition, I refine the equilibrium using the Undefeated Equilibrium (Mailath et al., 1993) and show that the above conclusion still holds. Recall that a sequential equilibrium is undefeated if there does not exist another sequential equilibrium where at least some agents are strictly better off.

Proposition 4.5 *There exists $\mathcal{C}^u \subset \mathcal{C}^w$ such that for all $(\delta_1, \delta_2, \Delta, \alpha) \in \mathcal{C}^u$, there exist a semi-pooling equilibrium and a corresponding separating equilibrium that are both undefeated and have the property that $v_p^* > v_s^*$.*

That is, for some $(\delta_1, \delta_2, \Delta, \alpha)$, both the separating equilibrium and the semi-pooling equilibrium are undefeated. This is equivalent to stating that both equilibria are Pareto dominant in their respective environments. Hence, the conclusion that the semi-pooling equilibrium may improve welfare is robust to alternative refinements.

Before I end this sub-section, there are a few things that need to be clarified. First, the conditions found in this sub-section (Proposition 4.2-4.5) are sufficient but may not be necessary. Second, depending on the parameter values, the following scenarios are also possible: (1) a semi-pooling equilibrium is not undefeated; (2) a semi-pooling is undefeated but not welfare-improving; and (3) a semi-pooling is undefeated and is welfare-improving compared to the separating equilibrium, but the separating equilibrium is not undefeated. In these cases, the conclusions in this section do not apply, and consumers may receive higher welfare if motives are known.[16]

5 Exiting Fire Sale Equilibria

In this section, I show how the asset market can "exit" the semi-pooling equilibrium discussed in Section 4.3. Chang (2018) shows that such semi-pooling equilibrium is useful in explaining the "fire sale" phenomenon. In particular, the semi-pooling equilibrium is able to generate the following empirical features of a fire sale: (1) distressed sellers sell quickly at a highly discounted price; and (2) an increase in the distress level leads to a larger price discount. One of the differences between Chang (2018) and this paper is that Chang (2018) assumes free entry of buyers, while in this paper the supply of buyers is fixed. With free entry, buyers' search value in the asset market are always constant and equal to the entry cost. But, if the supply of buyers is fixed, buyers' search value will respond to sellers' desire to sell. In what follows, I show how buyers' search value affects the market equilibrium, and how the asset market may transition from a semi-pooling equilibrium to a fully-separating equilibrium as economic fundamentals change.

I consider two variables: the relative quality of the low type sellers (δ_1) and shoppers' fruit endowment (b), because they determine how shoppers' private value of the assets compares to non-shoppers'. I focus on two equilibrium outcomes – the price and the trading volume faced by high type shoppers. Recall that the trading volume is the product of the price (ψ_2) and quantity sold (s_2). δ_1 is measured as a percentage of δ_2. b is measured as a percentage of the quantity needed for efficient consumption (c^*). The price ψ_2 measured as a percentage of δ_2. That is, $100\% - \psi_2$ equals to the price discount on δ_2 trees. The trading volume is measured as a percentage of the efficient volume, which is equal to $c^* - b$.

First, I fix b and vary δ_1. Figure 3 shows that, as δ_1 increases, the price in the semi-pooling equilibrium increases because the quality in the pool improves. However, the increase in price is

[16]In the first and the second case, consumer do receive higher welfare when motives are known. To see this, note that if the semi-pooling equilibrium is not undefeated, the only possible undefeated equilibrium is a pooling equilibrium (a one-price equilibrium), and consumers are always better off if shoppers do not have to pool with δ_1 non-shoppers. The second case is obvious.

slower than the improvement in average quality because buyers' search value (v_0) also increases with δ_1 (Figure 4). That is, buyers are taking a bigger and bigger "cut" of the surplus, preventing the improvement in price from keeping pace with the improvement in δ_1. The result is a kink where low type non-shoppers begin to leave the pooling market due to the low pooling price. As they leave the pooling market, the price starts to increase faster and trading volume starts to decrease to dissuade low type shoppers from deviating. Eventually, all low type non-shoppers will leave the pooling market and only shoppers sell trees, an equilibrium described in Proposition 4.1. I refer to this as the "exit" from the fire sale equilibria.

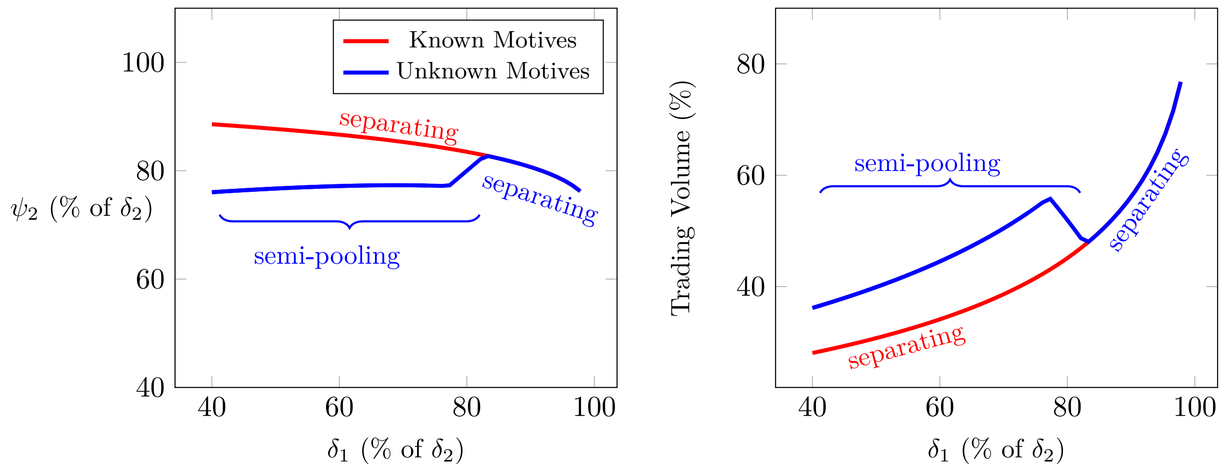

Figure 3: Equilibrium as a Function of Low Type's Quality (δ_1)

Why does buyers' search value, v_0^*, increase with δ_1? As δ_1 increases, the improvement in the pooling price makes high type sellers want to sell more and hence demand a higher buyer-seller ratio, which must be met with a higher v_0^* to clear the market. Intuitively, a fixed supply of buyers allows buyers' "bargaining power" to increase as sellers' desire to sell increases. Similarly, when motives are known, the signaling cost decreases with δ_1. Then the increase in high type sellers' desire to sell allows buyers to demand more in trading.

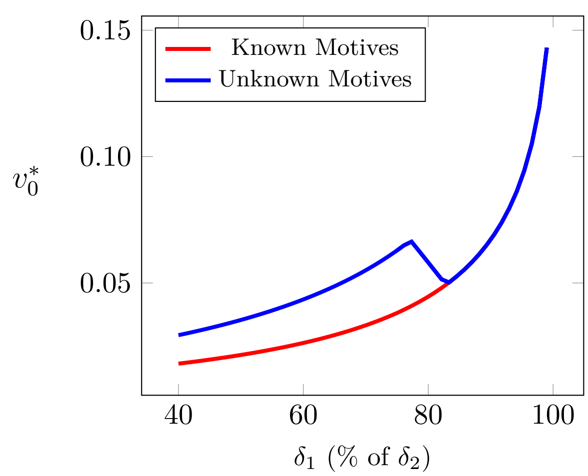

Figure 4: Buyers' Search Value (v_0^*) as a Function of Low Type's Quality (δ_1)

Next, I fix δ_1 and vary b. As b decreases, shoppers' holding cost increases and so is their desire to sell. Consequently, v_0^* increases (Figure 6), which drives down the price to a point where low type non-shoppers begin to leave the pooling market (the kink in the figures). The improvement in asset quality due to the exit of low type non-shoppers is offset by the increase in v_0^*, which explains the flat part in the left panel of Figure 5. Once the low type non-shoppers finish exiting, the decrease in price resumes. The trading volume increases, however, because the strong desire to sell outweighs the losses from selling at low prices. At this point, the asset market has "exited" the fire sale equilibria.

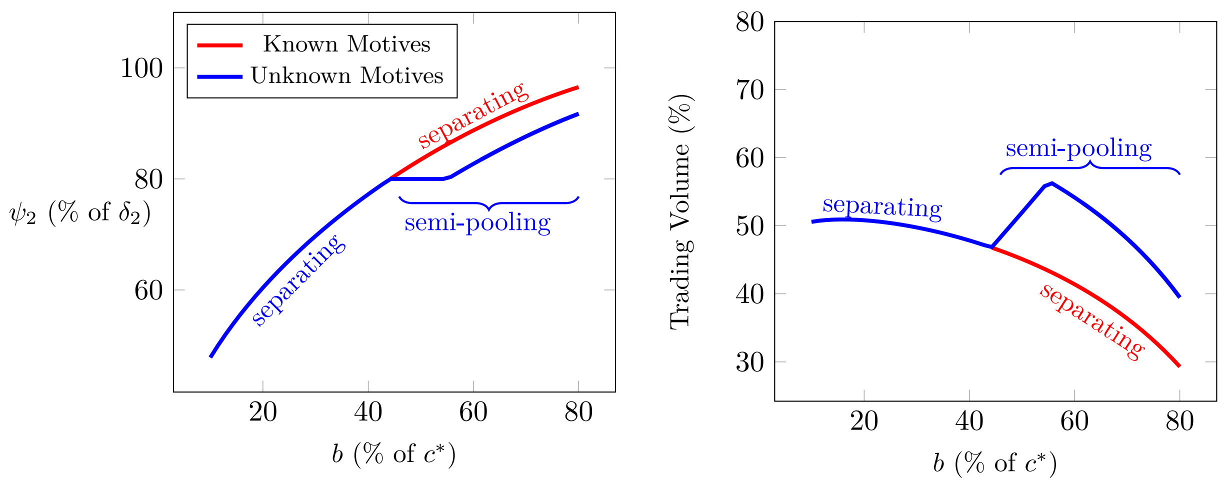

Figure 5: Equilibrium as a Function of Fruit Endowment (b)

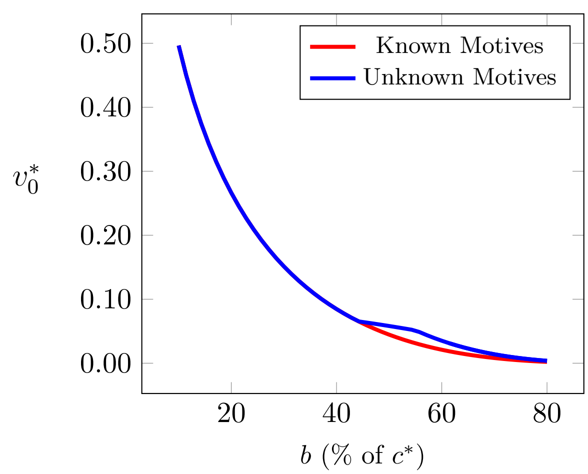

Figure 6: Buyers' Search Value (v_0^*) as a Function of Fruit Endowment (b)

To summarize, there are two forces that can lead the asset market out of a fire sale equilibrium. First, if the quality difference is big or if the holding cost is small, high type shoppers' private value of assets will be sufficiently different from low type non-shoppers'. Then, the high type shoppers will deviate. This is the force discussed in Section 4.3.[17] Second, if the quality difference between

[17] This force does not show up here because in the first scenario δ_1 is larger than 40% of δ_2, and in the second

assets is small or if the holding cost is big, high type shoppers' desire to sell will be high, which increases buyers' search value and drives down the price to a point where it is no longer profitable for low type non-shoppers to sell. Clearly, the second force does not exist if buyers' search value is fixed by a free entry condition.

6 Government Interventions

6.1 Before-shock Purchase

First I consider a government asset purchase program. Before the shocks are realized, the government issues bonds to purchase all trees from consumers. Denote the bonds as D. The bonds can be exchanged for fruit in the FM. The exchange rate is not state-contingent – one unit of bonds is exchangeable for one unit of fruit. The bonds are perfectly divisible and recognizable, and they cannot be counterfeited by agents. Hence, if the government can commit, the producers will treat the bonds as regular fruit.

At what price will consumers be willing to sell their trees to the government? Let $\bar{\delta} = \sum_{j=1}^{J} \Delta_j \delta_j$ be the post-shock average tree quality. Denote the price of trees in terms of bonds as P. Then for the government to break even, we need $P \leq \bar{\delta}$. Suppose $P = \bar{\delta}$ so the government does not earn any profit. Every agent now holds $D = \bar{\delta} \cdot a$ units of government bonds and b units of fruit. For the simplicity of exposition, let us consider the case where $b + D > c^*$ where $u'(c^*) = 1$ so there are enough bonds and fruit for efficient consumption. Then the value of selling the trees to the government compared to holding the original endowment (b, a) is

$$V^c = u(b + m^*) - u(b) - m^* \tag{6.1}$$

where m^* is given by $u'(b + m^*) = 1$. Consumers' search value in the asset market is given by

$$V^d = \alpha \sum_{j=1}^{J} \Delta_j v_{1,j}^{**} + (1 - \alpha) \sum_{j=1}^{J} \Delta_j v_{0,j}^{**} \tag{6.2}$$

where $v_{1,j}^{**}$ and $v_{0,j}^{**}$ are defined by (4.1) and (4.4), respectively. With some algebra, we can rewrite V^d as

$$V^d = \sum_{j=1}^{J} \Delta_j p(\theta_j) \left[u(b + m_j) - u(b) - m_j \right] \tag{6.3}$$

where $m_j = \psi_j s_j$ is the amount of fruit acquired by δ_j sellers, and $p(\theta_j)$ is the trading probability. Then, unless $m_j = m^*$ and $p(\theta_j) = 1$ for all j, $V^d < V^c$. In general, $m_j < m^*$, because sellers face

scenario b is smaller than 80% of c^* (see the figures). Given other parameter values, if δ_1 is smaller or b is larger, consumers' private value of assets will be different enough that δ_2 shoppers will deviate.

either the signaling cost or the price distortion in the asset market. Since the benefit of consuming close to $b + m^*$ is second-order but the signaling cost or the price distortion is first-order, agents consume less than $b + m^*$. Hence, we have $V^d < V^c$ and the program is strictly welfare-improving for consumers if $P = \bar{\delta}$. There must then exist a $\underline{P} < \bar{\delta}$ such that as long as $P \geq \underline{P}$, the consumers will be willing to sell their trees.

Such program can also be offered by a private agency and financed by the agency's debt, as long as the agency can commit to repayment. The following punishment can be introduced to guarantee repayment. Should the agency default, it is required to return all of the trees to their owners. That is, the trees serve as collateral. Note that the agency has to return *all* trees because otherwise it may default strategically on those with low quality trees.

Is it always a good idea to let a public or private agency purchase all the trees? For the asset purchase program to work, the agency needs to know the average quality of the trees. However, for buyers to participate in the asset market, this information is generally not required. For example, in the semi-pooling equilibrium discussed in Section 4.3, buyers need only know the density of the sellers that pool in equilibrium. In the fully-separating equilibrium discussed in Proposition 4.1, no knowledge about the density of each type is needed. Hence, whether the program works or not depends on what information is available in the market. Also, suppose that, on top of the idiosyncratic shocks, there is an unforeseen aggregate shock that worsens the average tree quality. Then the agency may not be able to commit to its debt. The asset market, however, can still operate under such shocks. Hence, while it is possible for the asset purchase program to improve welfare, it does not replace the asset market.

6.2 After-shock Lending

Now suppose the government can only intervene after the shocks are revealed. Hence, the government is subjected to the same information frictions – it cannot observe asset quality or trading motives. I assume that the government can always borrow fruit from an outside economy at no cost and then lend it to the consumers.[18] The loans are offered before the asset market opens. To obtain a loan from the government, the consumers have to post their trees as collateral. The loans are paid back in the FM after the dividends from the trees are realized. If the consumers default, the government can seize the dividends from the trees that are posted as collateral. Let the interest rate be r so if a consumer borrows e units of fruit, she needs to payback $(1 + r)e$ units in the FM. For every unit of fruit borrowed the government requires d units of trees posted as collateral. The government must break even after the FM.

Now consider a program featuring $r = 0$ and $d = 1/\delta_1$. Recall that I define δ_1 to be the lowest quality type. First, note that if a borrower with loan size e defaults in the FM, the government

[18] In a model with fiat money, a more natural assumption is that the central bank lends to liquidity-constrained agents. To avoid adding another asset to the environment, I assume there is a market where the government can borrow fruit but the consumers cannot. The conclusion is of course not affected by this detail.

can seize $ed\delta$ amount of fruit. Then as long as $d \geq 1/\delta_1$, no borrowers will default. Second, if $r \geq 0$ and there is no incentive to default, the non-shoppers will not borrow from the government. Third, if $r = 0$ then it is costless to borrow. Hence, the shoppers will borrow till they have enough fruit for efficient consumption. That is, they will borrow $c^* - b$ where $u'(c^*) = 1$. This will require $(c^* - b)/\delta_1$ units of trees as collateral.

Then, a collateralized lending program can effectively eliminate all private information in the market and help shoppers achieve the first best. One caveat is that the shoppers need to possess at least $(c^* - b)/\delta_1$ units of trees. When they do, private information concerning asset quality and trading motives has no effect no welfare. But when they do not, it reduces the pledgeability of high quality trees. Choosing r and d in this scenario is more tricky as the government now faces a trade-off. It can lower d to increase the pledgeability of the trees, but it will also attract the non-shoppers with low quality trees. More importantly, borrowers with low quality trees will always default in the FM. To break even, the government will have to raise r. But then this will prevent shoppers with high quality trees from consuming efficiently. The takeaway is that when the economy has enough assets to serve as collateral, such a lending program can avoid information frictions that exist in the decentralized market. But if the economy does not have enough assets, the lending program will face the same information frictions as the decentralized market.

7 Conclusion

In this paper I study how trading motives affect equilibrium outcomes and welfare. I focus on two types of trading motives – informational and allocational. I show that while a fully separating equilibrium is the unique equilibrium when trading motives are known, multiple equilibria exist when trading motives are unknown. Moreover, with unknown trading motives, there exists a semi-pooling equilibrium that may improve welfare relative to the separating equilibrium with known motives. I also use this model to study how the asset market can exit a fire sale equilibrium. Lastly, I discuss two government programs, an asset purchase program and a collateralized lending program, that may eliminate private information and improve agents' welfare.

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Appendix A Proofs: Section 3

In this section I show how to solve the competitive equilibrium in the AM when motives are known. Following Faig and Jerez (2006) and Guerrieri and Shimer (2014), I first take buyer's search value v_0^* as given. I then endogenize v_0^* to solve for the full equilibrium.

To solve the partial equilibrium where v_0^* is given, I follow Guerrieri et al. (2010) and solve a set of problems $\{P(\delta_j)\}_{j=1}^J$ first. And then I prove any solution to $\{P(\delta_j)\}_{j=1}^J$ is a partial equilibrium and any partial equilibrium is a solution to $\{P(\delta_j)\}_{j=1}^J$.

Let

$$\bar{v}_0 = u(\hat{c}) - u(b) - (\hat{c} - b)$$

be the total trade surplus if the trade in asset market helps the seller consumes efficient amount of consumption in the following GM. For any $j \in \mathcal{J}$ and $v_0 \in [0, \bar{v}_0]$, problem $P(\delta_j)$ is given by

$$v_{1,j} = \max_{\theta,\psi,s}\{\min\{\theta, 1\}[u(b + \psi s) - u(b) - \delta_j s]\} \tag{A.1}$$

$$\text{s.t. } v_0 \leq \min\{\theta^{-1}, 1\}s(\delta_j - \psi) \tag{A.2}$$

$$v_{1,j'} \geq \min\{\theta, 1\}[u(b + \psi s) - u(b) - \delta_{j'} s] \text{ for all } j' < j \tag{A.3}$$

$$s \leq a \tag{A.4}$$

$$\psi s \leq b. \tag{A.5}$$

The first constraint says the price-quantity pair, (ψ, s), must satisfy buyer's participation constraint. The second constraint requires that no sellers with quality worse than j have the incentive to deviate. And the last two constraints are agents' resource constraints.

The next proposition characterizes the solution to this set of problems. I find it convenient to redefine the choice variables as (θ, m) rather than (θ, ψ, s) where $m = \psi s$ is the units of fruit transferred in trade. I refer to m as the trade volume. I then calculate ψ and s from m. Before stating the proposition, I define some variables that will be useful in describing the solution.

Let $\{(\theta_j, \psi_j, s_j)\}_{j=1}^J$ be the solution to $\{P(\delta_j)\}_{j=1}^J$. Define $m_j \equiv \psi_j s_j$. Define m^* be such that

$$u'(b + m^*) = 1 \tag{A.6}$$

and $\bar{m}^*$ be such that

$$u'(b + \bar{m}^*)(v_0 + \bar{m}^*) = u(b + \bar{m}^*) - u(b). \tag{A.7}$$

Next, given $v_{1,1}$, define $m_j^\dagger$ for all $j > 1$ to be such that

$$v_{1,j-1} = u(b + m_j^\dagger) - (v_0^* + m_j^\dagger)\delta_{j-1}/\delta_j - u(b) \tag{A.8}$$

and θ_j^* to be such that

$$v_{1,j-1} = \theta_j^*[u(b+m_j) - (v_0^* + m_j)\delta_{j-1}/\delta_j - u(b)]. \tag{A.9}$$

Lastly, let

$$\underline{m}_j = \min\{\delta_j a - v_0, b\}, \tag{A.10}$$

the highest amount of fruit sellers can acquire.

Proposition A.1 *Assume $v_0 < \bar{v}_0$. The unique solution $\{(\theta_j, \psi_j, s_j)\}_{j=1}^J$ to $\{P(\delta_j)\}_{j=1}^J$ can be divided into the following cases:*
(1) $\underline{m}_1 \geq m^*$. In this case we have $\theta_1 = 1$, $m_1 = m^*$, and $v_{1,1} = u(b+m^*) - (v_0 + m^*) - u(b)$. For all $j > 1$ such that $m_j^\dagger > \bar{m}^*$, $\theta_j = 1$ and $m_j = m_j^\dagger$. For all $j > 1$ such that $m_j^\dagger \leq \bar{m}^*$, $m_j = \bar{m}^*$ and $\theta_j = \theta_j^*$;
(2) $\bar{m}^* \leq \underline{m}_1 < m^*$. In this case we have $\theta_1 = 1$, $m_1 = \underline{m}_1$, and $v_{1,1} = u(b+\underline{m}_1) - (v_0 + \underline{m}_1) - u(b)$. Again, for all $j > 1$ such that $m_j^\dagger > \bar{m}^*$, $\theta_j = 1$ and $m_j = m_j^\dagger$. For all $j > 1$ such that $m_j^\dagger \leq \bar{m}^*$, $m_j = \bar{m}^*$ and $\theta_j = \theta_j^*$;
(3) $\bar{m}^* > \underline{m}_1$. In this case we have $\theta_1 = 1$, $m_1 = \underline{m}_1$, and $v_{1,1} = u(b+\underline{m}_1) - (v_0 + \underline{m}_1) - u(b)$. For all $j > 1$, $m_j = \min\{\bar{m}^*, \underline{m}_j\}$ and $\theta_j = \theta_j^*$.
In all cases, $s_j = \frac{v_0 + m_j}{\delta_j}$ and $\psi_j = \frac{m_j}{s_j} = \frac{m_j}{v_0 + m_j}\delta_j$. $s_j \leq s_{j-1}$ for all $j > 1$. $\psi_j > \psi_{j-1}$ for all $j > 1$. $\theta_1 = 1$ and $\theta_j \leq \theta_{j-1}$ for all $j > 1$. Lastly, $v_{1,j-1} > v_{1,j}$ for all $j > 1$.

Proof of Proposition A.1: Before solving the problem let me first prove a claim that will simplify the problem. In what follows I refer to the constraint that involves j' in problem $P(\delta_j)$ as "constraint j'".

Claim A.1: Constraint (A.2) binds for all j and that $\theta_j \leq 1$ for all $j \geq 2$.

Proof. First it is easy to see that constraint (A.2) binds for $j = 1$ where constraint (A.3) disappears. Next, suppose constraint (A.2) binds for all $j' = 2, ..., j-1$ and suppose the inequality is strict for $P(\delta_j)$ with the solution being $\{\psi_j, s_j, \theta_j\}$. Now pick $\{\psi', s', \theta'\}$ such that

$$\begin{aligned}
&p(\theta_j)(u(b + \psi_j s_j) - u(b) - \delta_{j-1} s_j) = p(\theta')(u(b + \psi' s') - u(b) - \delta_{j-1} s') \\
&v_0 \leq q(\theta') s'(\delta_j - \psi') \\
&\psi' s' = \psi_j s_j \\
&s' < s_j \\
&\theta' < \theta_j.
\end{aligned}$$

Now let us check constraint (A.3). For any $\delta_{j'} < \delta_{j-1}$, we have

$$\begin{aligned} &p(\theta_j)(u(b+\psi_j s_j) - u(b) - \delta_{j'} s_j) + p(\theta_j) s_j(\delta_{j'} - \delta_{j-1}) \\ =&p(\theta')(u(b+\psi' s') - u(b) - \delta_{j'} s') + p(\theta') s'(\delta_{j'} - \delta_{j-1}). \end{aligned}$$

Since $s' < s_j$ and $\theta' < \theta_j$, we have

$$p(\theta_j)(u(b+\psi_j s_j) - u(b) - \delta_{j'} s_j) > p(\theta')(u(b+\psi' s') - u(b) - \delta_{j'} s')$$

so constraint (A.3) is still satisfied. But similarly we have

$$p(\theta')(u(b+\psi' s') - u(b) - \delta_j s') > p(\theta_j)(u(b+\psi_j s_j) - u(b) - \delta_j s_j),$$

a contradiction.

Next, I show that $\theta_j \leq 1$ for all $j \geq 2$. Now suppose for some j we have $\theta_j > 1$. Since (A.2) binds for j, then let $\theta' = 1$ and we must have

$$q(\theta') s_j(\delta_j - \psi_j) = s_j(\delta_j - \psi_j) > v_0 \tag{A.11}$$

while all other constraints are satisfied and seller's utility is unchanged. Hence (θ', ψ_j, s_j) is also optimal. But this is a contradiction since constraint (A.2) does not bind. *Q.E.D.*

Now let us consider $j = 1$. First, define $\bar{m} \equiv \psi s$. Then $\bar{m}$ denotes the fruit transferred from the buyer to the seller. Problem $P(\delta_1)$ becomes

$$v_{1,1} = \max_{\theta, \bar{m}} \{\theta(u(b+\bar{m}) - (v_0 + \bar{m}) - u(b))\} \tag{A.12}$$

$$\text{s.t. } \bar{m} \leq \min\{\delta_1 a - v_0, b\} \tag{A.13}$$

It is straightforward to formulate the solution: $\theta_1 = 1$ and $m_1 = \min\{m^*, \min\{\delta_1 b - v_0, b\}\}$ where $u'(b + m^*) = 1$ and hence $b + m^*$ is the amount of real balances that supports efficient level of consumption.

Next let us look at $j = 2$. It must be that constraint (A.3) binds. Suppose not, then problem $P(\delta_2)$ is the same as $P(\delta_1)$ except for the difference in tree quality. Then $j = 2$ sellers will choose $m_2 = m_1$ and $\theta_2 = 1$. But this is strictly better than what $j = 1$ sellers have: if they deviate they receive $u(b + m_1) - (v_0 + m_1)\delta_1/\delta_2 - u(b)$ So it is a contradiction. Using the binding constraint

(A.2) we can rewrite the problem as

$$
\begin{aligned}
& v_{1,2} = \max_{\theta,\bar{m}} \{\theta(u(b+\bar{m}) - (v_0+\bar{m}) - u(b))\} && \text{(A.14)}\\
\text{s.t. } & v_{1,1} = \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_1/\delta_2 - u(b)) && \text{(A.15)}\\
& v_{1,1} \le u(b+\bar{m}) - (v_0+\bar{m})\delta_1/\delta_2 - u(b) && \text{(A.16)}\\
& \bar{m} \le \min\{\delta_2 a - v_0, b\} && \text{(A.17)}
\end{aligned}
$$

where I call constraint (A.16) the "feasibility" constraint. It is derived from the result that $\theta \le 1$.

To solve this problem, for the moment let me ignore constraint (A.16) and the resource constraint. The problem becomes

$$
\max_{\bar{m}} \left\{ \frac{u(b+\bar{m}) - (v_0+\bar{m}) - u(b)}{u(b+\bar{m}) - d_1(v_0+\bar{m}) - u(b)} \right\} \tag{A.18}
$$

where $d_1 = \delta_2/\delta_1$. First order condition is given by

$$
\frac{(1-d_1)[u'(b+\bar{m})(v_0+\bar{m}) - u(b+\bar{m}) + u(b)]}{[u(b+\bar{m}) - d_1(v_0+\bar{m}) - u(b)]^2} = 0. \tag{A.19}
$$

It is easy to check that the second derivative is always negative for $\bar{m} \ge 0$. Also $u'(b+\bar{m})(v_0 + \bar{m}) - u(b+\bar{m}) + u(b) > 0$ when $\bar{m} = 0$ and $u'(b+\bar{m})(v_0+\bar{m}) - u(b+\bar{m}) + u(b) < 0$ when $\bar{m} \to \infty$. Then there is a unique solution to equation (A.19), which is given by

$$
u'(b+\bar{m}^*)(v_0+\bar{m}^*) = u(b+\bar{m}^*) - u(b). \tag{A.20}
$$

Note $\bar{m}^* \le c^* - b$ with equality only when $v_0 = \bar{v}_0$. That is, sellers in this case do not acquire enough real balances to support efficient consumption in the GM.

Now let $\underline{m}_2$, $\bar{m}^*$ and $m_2^\dagger$ be given by

$$
\begin{aligned}
& \underline{m}_2 = \min\{\delta_2 a - v_0, b\} && \text{(A.21)}\\
& u'(b+\bar{m}^*)(v_0+\bar{m}^*) = u(b+\bar{m}^*) - u(b) && \text{(A.22)}\\
& v_{1,1} = u(b+m_2^\dagger) - (v_0+m_2^\dagger)\delta_1/\delta_2 - u(b). && \text{(A.23)}
\end{aligned}
$$

It is easy to see that constraint (A.16) puts a upper bound on m_2. Constraint (A.17), on the other hand, puts a lower bound on m_2 because if $\bar{m}^* < m_2^\dagger$, constraint (A.17) will be violated. Then the optimal amount of real balances acquired by the asset seller is given by

$$
m_2 = \min\{\underline{m}_2, \max\{\bar{m}^*, m_2^\dagger\}\}, \tag{A.24}
$$

which concludes the solution to $P(\delta_2)$.

For cases where $j > 2$, the following claim gives the solution.

Claim A.2: Assume $v_0 < \bar{v}_0$. For all $j > 2$, constraint (A.3) binds for $j' = j - 1$ and is slack for all the other j''s.

Proof. The proof is long and I divide it into several cases.

Case 1: $\underline{m}_1 \geq m^*$.

Subcase 1.1: $\max\{\bar{m}^*, m_2^\dagger\} = \bar{m}^*$. In this case $m_2 = \bar{m}^*$ Let us start with $j = 3$ and proceed by induction.

$$
\begin{aligned}
& v_{1,3} = \max_{\theta,\bar{m}}\{\theta(u(b+\bar{m}) - (v_0+\bar{m}) - u(b))\} && \text{(A.25)} \\
\text{s.t. } & v_{1,1} \geq \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_1/\delta_3 - u(b)) && \text{(A.26)} \\
& v_{1,2} \geq \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_2/\delta_3 - u(b)) && \text{(A.27)} \\
& \theta \leq 1 && \text{(A.28)} \\
& \bar{m} \leq \min\{\delta_3 a - v_0, b\}. && \text{(A.29)}
\end{aligned}
$$

First, by the same reasoning in $P(\delta_2)$, at least one of the two resource constraints must bind. Now suppose constraint (A.26) binds but constraint (A.27) is slack. We can then substitute the constraint into the objective function just like in $P(\delta_2)$. It is easy to see that $\bar{m}^*$ is still the optimal because the solution to (A.20) is independent of δ_1/δ_3. It is feasible because by assumption sellers and buyers can still afford it. In addition, θ_3 given by binding constraint (A.26) is strictly less than one since

$$
u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_1/\delta_2 - u(b) < u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_1/\delta_3 - u(b)
$$

so $\theta_2 > \theta_3$. However, constraint (A.27) is violated by $(\theta_3, \bar{m}^*)$ because

$$
v_{1,1} = \theta_2(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_1/\delta_2 - u(b)) = \theta_3(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_1/\delta_3 - u(b))
$$

and $\theta_2 > \theta_3$ together imply

$$
\begin{aligned}
v_{1,2} &= \theta_2(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_2/\delta_2 - u(b)) \\
&= \theta_2(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_1/\delta_2 - u(b)) + \theta_2(v_0+\bar{m}^*)(\delta_1-\delta_2)/\delta_2 \\
&= \theta_3(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_1/\delta_3 - u(b)) + \theta_2(v_0+\bar{m}^*)(\delta_1-\delta_2)/\delta_2 \\
&= \theta_3(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_2/\delta_3 - u(b)) + \theta_2(v_0+\bar{m}^*)(\delta_1-\delta_2)/\delta_2 + \theta_3(v_0+\bar{m}^*)(\delta_2-\delta_1)/\delta_3 \\
&< \theta_3(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_2/\delta_3 - u(b))
\end{aligned}
$$

where the third equality is because by assumption constraint (A.26) binds. Now suppose constraint (A.27) binds but constraint (A.26) is slack, by similar arguments it is easy to see that all the constraints are satisfied and θ_3 is now given by binding constraint (A.27). Lastly, consider the case where both constraints bind. It is easy to see that this is not possible – at least one of them must be slack. Hence we have established that only $j' = 2$ constraint binds and the solution is given by $(\theta_3, \bar{m}^*)$ where θ_3 solves

$$v_{1,2} = \theta_3(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_2/\delta_3 - u(b)).$$

Now suppose that the claim holds for all $j' < j$. For $P(\delta_j)$ we can rewrite constraint (A.3) as

$$v_{1,j-1} = \theta_{j-1}(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j-1}/\delta_{j-1} - u(b)) \geq \theta(u(b+\bar{m}) - (v_0 + \bar{m})\delta_{j-1}/\delta_j - u(b))$$
$$v_{1,j-2} = \theta_{j-1}(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j-2}/\delta_{j-1} - u(b)) \geq \theta(u(b+\bar{m}) - (v_0 + \bar{m})\delta_{j-2}/\delta_j - u(b))$$
$$v_{1,j-3} = \theta_{j-2}(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j-3}/\delta_{j-2} - u(b)) \geq \theta(u(b+\bar{m}) - (v_0 + \bar{m})\delta_{j-3}/\delta_j - u(b))$$
...
$$v_{1,j-k} = \theta_{j-k+1}(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j-k}/\delta_{j-k+1} - u(b)) \geq \theta(u(b+\bar{m}) - (v_0 + \bar{m})\delta_{j-k}/\delta_j - u(b))$$
...
$$v_{1,1} = \theta_2(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_1/\delta_2 - u(b)) \geq \theta(u(b+\bar{m}) - (v_0 + \bar{m})\delta_1/\delta_j - u(b))$$

Again, at least one of the $j-1$ constraints must be binding. Let one of the binding constraints be $j' \in \{1, ..., j-1\}$. Now suppose $j' < j-1$. We can substitute the constraint into the objective function and $\bar{m}^*$ will be the optimal solution. θ_j is given by

$$v_{1,j'} = \theta_{j'+1}(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j'}/\delta_{j'+1} - u(b)) = \theta_j(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j'}/\delta_j - u(b))$$

and again $\theta_j < \theta_{j'+1}$. However, this implies that

$$\begin{aligned}
v_{1,j'+1} &= \theta_{j'+1}(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j'+1}/\delta_{j'+1} - u(b)) \\
&= \theta_{j'+1}(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j'}/\delta_{j'+1} - u(b)) + \theta_{j'+1}(v_0 + \bar{m}^*)(\delta_{j'} - \delta_{j'+1})/\delta_{j'+1} \\
&= \theta_j(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j'}/\delta_j - u(b)) + \theta_{j'+1}(v_0 + \bar{m}^*)(\delta_{j'} - \delta_{j'+1})/\delta_{j'+1} \\
&= \theta_j(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j'+1}/\delta_j - u(b)) + \theta_{j'+1}(v_0 + \bar{m}^*)(\delta_{j'} - \delta_{j'+1})/\delta_{j'+1} \\
&\quad\quad + \theta_j(v_0 + \bar{m}^*)(\delta_{j'+1} - \delta_{j'})/\delta_j \\
&< \theta_j(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j'+1}/\delta_j - u(b)).
\end{aligned}$$

where again the third equality is because by assumption constraint j' binds. That is, constraint $j'+1$ is violated. Hence $j' = j-1$. This also implies that $\theta_j < \theta_{j-1}$. Lastly, if one reverses the above arguments, together with the fact that $\theta_j < \theta_{j-1}$, it is easy to see that if the first constraint is held at equality, the rest of the inequalities are strict. Note that θ_j is given by

$$v_{1,j-1} = \theta_j(u(b+\bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j-1}/\delta_j - u(b)).$$

Subcase 1.2: $\max\{\bar{m}^*, m_2^\dagger\} = m_2^\dagger$. In this case, m_2 is given by

$$v_{1,1} = u(b + m_2) - (v_0 + m_2)\delta_1/\delta_2 - u(b)$$

and $\theta_2 = 1$. Now let us check $j = 3$. The feasibility constraints are

$$u(b + m_2) - (v_0 + m_2)\delta_1/\delta_2 - u(b) \leq u(b + \bar{m}) - (v_0 + \bar{m})\delta_1/\delta_3 - u(b) \tag{A.30}$$
$$u(b + m_2) - (v_0 + m_2)\delta_2/\delta_2 - u(b) \leq u(b + \bar{m}) - (v_0 + \bar{m})\delta_2/\delta_3 - u(b). \tag{A.31}$$

Note that there can only be one constraint that binds and if one of constraints binds, the other one must be slack and switch the direction of the inequality. Also, it must be constraint (A.31). If constraint (A.30) binds, it will imply that

$$v_{1,1} = u(b + m_2) - (v_0 + m_2)\delta_1/\delta_2 - u(b) < u(b + m_3) - (v_0 + m_3)\delta_1/\delta_3 - u(b)$$

because $m_3 < m_2$. Then $j = 1$ sellers will deviate. Hence, we have $\theta_3 = 1$ and $m_3 = m_3^\dagger$. For cases where $j > 3$, suppose that for all $j' < j$, the feasibility constraints satisfy that at least one of them binds and it is constraint $j - 1$. Then we can write

$$\begin{aligned}
&u(b + m_{j-1}) - (v_0 + m_{j-1})\delta_{j-1}/\delta_{j-1} - u(b) \leq u(b + m_j) - (v_0 + m_j)\delta_{j-1}/\delta_j - u(b)\\
&u(b + m_{j-1}) - (v_0 + m_{j-1})\delta_{j-2}/\delta_{j-1} - u(b) \leq u(b + m_j) - (v_0 + m_j)\delta_{j-2}/\delta_j - u(b)\\
&u(b + m_{j-2}) - (v_0 + m_{j-2})\delta_{j-3}/\delta_{j-2} - u(b) \leq u(b + m_j) - (v_0 + m_j)\delta_{j-3}/\delta_j - u(b)\\
&\ldots\\
&u(b + m_{j-k+1}) - (v_0 + m_{j-k+1})\delta_{j-k}/\delta_{j-k+1} - u(b) \leq u(b + m_j) - (v_0 + m_j)\delta_{j-k}/\delta_j - u(b)\\
&\ldots\\
&u(b + m_2) - (v_0 + m_2)\delta_1/\delta_2 - u(b) \leq u(b + m_j) - (v_0 + m_j)\delta_1/\delta_j - u(b).
\end{aligned}$$

Again if constraint $j' < j - 1$ binds, the constraint $j' + 1$ will be violated. Hence if there are constraints that bind, it must be the first constraint. Note in this case the rest of the constraints are satisfied automatically. Then $\theta_j = 1$ and $m_j = m_j^\dagger$.

It is easy to see that when the feasibility constraint binds, $m_j^\dagger < m_{j-1}^\dagger$, and hence it is possible that for some $j \geq 3$, $m_j^\dagger < \bar{m}^*$ and the feasibility constraint is slack at $\bar{m} = \bar{m}^*$. Suppose $j = 3$. Since $\bar{m}^* \leq m_2$, it is easy to show (by the same logic) that only constraint $j = 2$ binds. Next, suppose Claim A.2 is true for all $j' < j$ and that the feasibility constraint binds for all $P(\delta_{j'})$ with

$1 < j' < j-k$ where $0 \le k < j-3$ but is slack for $j-k \le j' \le j$. The incentive constraints are

$$
\begin{aligned}
&v_{1,j-1} = \theta_{j-1}(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_{j-1}/\delta_{j-1} - u(b)) \ge \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_{j-1}/\delta_j - u(b))\\
&v_{1,j-2} = \theta_{j-1}(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_{j-2}/\delta_{j-1} - u(b)) \ge \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_{j-2}/\delta_j - u(b))\\
&v_{1,j-3} = \theta_{j-2}(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_{j-3}/\delta_{j-2} - u(b)) \ge \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_{j-3}/\delta_j - u(b))\\
&\ldots\\
&v_{1,j-k-1} = \theta_{j-k}(u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_{j-k-1}/\delta_{j-k} - u(b)) \ge \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_{j-k-1}/\delta_j - u(b))\\
&v_{1,j-k-2} = u(b+m^\dagger_{j-k-1}) - (v_0+m^\dagger_{j-k-1})\delta_{j-k-2}/\delta_{j-k-1} - u(b) \ge \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_{j-k-2}/\delta_j - u(b))\\
&\ldots\\
&v_{1,1} = u(b+m^\dagger_2) - (v_0+m^\dagger_2)\delta_1/\delta_2 - u(b) \ge \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_1/\delta_j - u(b)).
\end{aligned}
$$

It can be shown that, provided that $\bar{m} = \bar{m}^*$, if any constraint $j' < j-1$ binds constraint $j'+1$ will be violated, and that if the first constraint binds the rest are satisfied automatically. To see the first point, the only noteworthy constraint is constraint $j-k-2$. Suppose it binds, first note that $\bar{m}^* < m^\dagger_{j-k-1}$. Second, because $\bar{m}^* \ge m^\dagger_j$, $\theta_j \le 1$ if $\bar{m} = \bar{m}^*$. Third, constraint $j-k-1$ can be written as

$$
u(b+m^\dagger_{j-k-1}) - (v_0+m^\dagger_{j-k-1})\delta_{j-k-1}/\delta_{j-k-1} - u(b) \ge \theta(u(b+\bar{m}) - (v_0+\bar{m})\delta_{j-k-1}/\delta_j - u(b)),
$$

which is violated if constraint $j-k-2$ binds. The second point follows from previous arguments in Subcase 1.1 and in this case.

Case 2: $\bar{m}^* \le \underline{m}_1 < m^*$.

Subcase 2.1: $\max\{\bar{m}^*, m^\dagger_2\} = \bar{m}^*$. Since $\underline{m}_j \ge m_1$ for all $j > 1$, no sellers or buyers are resource constrained. The solution then follows from Subcase 1.1.

Subcase 2.2: $\max\{\bar{m}^*, m^\dagger_2\} = m^\dagger_2$. Again no sellers or buyers will be resource constrained because $\bar{m}^* \le m^\dagger_2 < m_1$. This case then follows from Subcase 1.2.

Case 3: $\bar{m}^* > \underline{m}_1$.

Subcase 3.1: $\underline{m}_1 = b$. That is, sellers are constrained by the amount of cash carried by the buyers. Because every buyer carries the same amount of cash, all transactions will be constrained. δ_2 sellers will choose $m_2 = b$. For the rest of the problems $\{P(\delta_j)\}_{j=3}^J$, at least one incentive constraint must bind. It then follows from similar arguments that for all $j > 2$, only constraint $j' = j-1$ binds and all the others are slack, and that all sellers choose $m_j = b$. θ_j is given by

$$
v_{1,j-1} = \theta_j(u(b+b) - (v_0+b)\delta_{j-1}/\delta_j - u(b)).
$$

In what follows, denote $\delta_j a - v_0$ as δ^*_j.

Subcase 3.2: $\underline{m}_1 = \delta_1 a - v_0$ and $\min\{b, \delta^*_J\} = \delta^*_J < \bar{m}^*$. In this case, all sellers are asset constrained because buyers bring enough fruit and sellers would want to choose $\bar{m}^*$ but are limited by δ_j^*. We need to check if constraint (A.3) is satisfied when sellers choose δ_j^*. Consider $j = 3$ first. The incentive constraints are

$$\theta_2(u(b+\delta_2^*) - \delta_1 a - u(b)) \geq \theta_3(u(b+\delta_3^*) - \delta_1 a - u(b)) \tag{A.32}$$
$$\theta_2(u(b+\delta_2^*) - \delta_2 a - u(b)) \geq \theta_3(u(b+\delta_3^*) - \delta_2 a - u(b)). \tag{A.33}$$

Following the reasoning in previous proof, it is easy to see that the second constraint must bind and the first is slack. We can then prove by induction that for all $j > 3$, only constraint $j' = j-1$ binds and all the others are slack. Sellers then choose $m_j = \delta_j^*$. θ_j is given by

$$v_{1,j-1} = \theta_j(u(b+\delta_j^*) - \delta_j a - u(b)).$$

Subcase 3.3: $\underline{m}_1 = \delta_1 a - v_0$ and there exists $j > 1$ such that $\delta_j^* \geq \bar{m}^*$ but $b < \bar{m}^*$. Again for all $j' \geq j$, the sellers are cash constrained. It remains to show that incentive constraints hold when sellers with δ_j choose b. The constraints are

$$\begin{aligned}
&v_{1,j-1} = \theta_{j-1}(u(b+\delta_{j-1}^*) - \delta_{j-1} a - u(b)) \geq \theta_j(u(b+b) - (v_0+b)\delta_{j-1}/\delta_j - u(b))\\
&v_{1,j-2} = \theta_{j-1}(u(b+\delta_{j-1}^*) - \delta_{j-2} a - u(b)) \geq \theta_j(u(b+b) - (v_0+b)\delta_{j-2}/\delta_j - u(b))\\
&v_{1,j-3} = \theta_{j-2}(u(b+\delta_{j-2}^*) - \delta_{j-3} a - u(b)) \geq \theta_j(u(b+b) - (v_0+b)\delta_{j-3}/\delta_j - u(b))\\
&...\\
&v_{1,j-k} = \theta_{j-k+1}(u(b+\delta_{j-k+1}^*) - \delta_{j-k} a - u(b)) \geq \theta_j(u(b+b) - (v_0+b)\delta_{j-k}/\delta_j - u(b))\\
&...\\
&v_{1,1} = \theta_2(u(b+\delta_2^*) - \delta_1 a - u(b)) \geq \theta_j(u(b+b) - (v_0+b)\delta_1/\delta_j - u(b)).
\end{aligned}$$

Suppose that constraint $j' < j-1$ is binding. First note that $b + b < c^*$. Next, we know that $v_0 + b \leq \delta_j a$ so $(v_0+b)\delta_{j'}/\delta_j \leq \delta_{j'} a$. Since $b + b > b + \delta_{j'}^*$, it must be that $\theta_{j'+1} > \theta_j$. Then using the same method as before we can show that constraint $j'+1$ is violated. Then only constraint $j-1$ binds. In the cases where the sellers are asset constrained for all $P(\delta_{j'})$ with $1 \leq j' < j-k$ where $0 \leq k < j-2$ but is cash constrained for $j-k \leq j' \leq j$, the proof is similar to Subcase 3.1 and 3.2. To summarize, for all $j' < j-k$, we have $m_{j'} = \delta_{j'}^*$ and $\theta_{j'}$ is given by

$$v_{1,j'-1} = \theta_{j'}(u(b+\delta_{j'}^*) - \delta_{j'} a - u(b)).$$

For all $j' \geq j-k$, we have $m_{j'} = b$ and $\theta_{j'}$ is given by

$$v_{1,j'-1} = \theta_{j'}(u(b+b) - (v_0+b)\delta_{j'-1}/\delta_{j'} - u(b)).$$

Subcase 3.4: $\underline{m}_1 = \delta_1 a - v_0$ and there exists $j > 1$ such that $\delta_j^* \geq \bar{m}^*$ and $b > \bar{m}^*$. We can just replace b with $\bar{m}^*$ in the above case. Hence, for all $j' < j$, we have $m_{j'} = \delta^*_{j'}$ or $s = a$ and $\theta_{j'}$ is given by

$$v_{1,j'-1} = \theta_{j'}(u(b + \delta^*_{j'}) - \delta_{j'}a - u(b)).$$

For all $j' \geq j$, we have $m_{j'} = \bar{m}^*$ and $\theta_{j'}$ is given by

$$v_{1,j'-1} = \theta_{j'}(u(b + \bar{m}^*) - (v_0 + \bar{m}^*)\delta_{j'-1}/\delta_{j'} - u(b)).$$

Q.E.D.

Lastly I prove the final claim in Proposition A.1. In all cases, $s_j = \frac{v_0+m_j}{\delta_j}$ and $\psi_j = \frac{m_j}{s_j} = \delta_j \frac{m_j}{v_0+m_j}$. Notice that because m_j is weakly decreasing, s_j is strictly decreasing except in Case 3 where $s_j = a$ because agents are asset constrained. For ψ_j, it is easy to see that if m_j is constant (either $m_j = \bar{m}^*$ or agents are cash constrained), ψ_j will be strictly increasing. When agents are asset constrained in Case 3, $s_j = a$ and hence ψ_j must be strictly increasing as well. Also in Case 3 when agents go from asset constrained to cash constrained, because m_j will be bigger and s_j will be smaller, ψ_j will also be bigger. We are then left with cases in Case 1 and 2 where m_j decreases in j as a result of binding feasibility constraint. First, in Subcase 1.2, by Claim A.2 we have

$$u(b + \psi_1 s_1) - u(b) - \delta_1 s_1 = u(b + \psi_2 s_2) - u(b) - \delta_1 s_2.$$

Since we have shown that $s_2 < s_1$, then it must be that $\psi_2 > \psi_1$, otherwise the right hand side would be smaller since the right hand side increases in s_2 and ψ_2. Similar for other cases where the feasibility constraint binds. Now suppose the feasibility constraint binds for $j-1$ but not for j. Again by Claim A.2 we have

$$u(b + \psi_{j-1}s_{j-1}) - u(b) - \delta_{j-1}s_{j-1} = \theta_j[u(m + \psi_j s_j) - u(b) - \delta_{j-1}s_j].$$

Because $\theta_j \leq 1$, we must have

$$u(b + \psi_{j-1}s_{j-1}) - u(b) - \delta_{j-1}s_{j-1} \leq u(b + \psi_j s_j) - u(b) - \delta_{j-1}s_j$$

and hence $\psi_j > \psi_{j-1}$. Lastly, since

$$\begin{aligned} v_{1,j-1} &= \theta_{j-1}[u(\underline{+}\psi_{j-1}s_{j-1}) - u(b) - \delta_{j-1}s_{j-1}] \\ &= \theta_j[u(b + \psi_j s_j) - u(b) - \delta_{j-1}s_j] > v_{1,j}, \end{aligned}$$

we have $v_{1,j-1} > v_{1,j}$ for all $j > 1$. $\square$

The next proposition shows the connection between $\{P(\delta_j)\}_{j=1}^J$ and the partial equilibrium defined in Definition 3.1. In short, any solution to $\{P(\delta_j)\}_{j=1}^J$ is a partial equilibrium and any partial equilibrium is a solution to $\{P(\delta_j)\}_{j=1}^J$.

Proposition A.2 *(1) For any solution to $\{P(\delta_j)\}_{j=1}^J$, for all j, let $\Psi = \{(\psi_j, s_j)\}_{j=1}^J$ and $\theta(\psi_j, s_j) = \theta_j$; let $\gamma(\psi_j, s_j; \delta) = 1$ if and only if $\delta = \delta_j$; let $v_{1,j}^* = v_{1,j}$; and let $F(\psi_j, s_j) = \Delta_j$. Then $\{\Psi, \{v_{1,j}^*\}_{j=1}^J, \theta(.), \gamma(.), F(.)\}$ is a partial equilibrium;*
(2) For any partial equilibrium, let (ψ_j, s_j) be such that $\gamma(\psi, s; \delta_j) > 0$ and let $\theta_j = \theta(\psi_j, s_j)$. Then $\{(\psi_j, s_j, \theta_j)\}_{j=1}^J$ solves $\{P(\delta_j)\}_{j=1}^J$.

Proof of Proposition A.2: Now I prove any solution to $\{P(\delta_j)\}_{j=1}^J$ is a partial equilibrium and any partial equilibrium is a solution to $\{P(\delta_j)\}_{j=1}^J$. Note that because the above solution to $\{P(\delta_j)\}_{j=1}^J$ is unique, this proof implies that the partial equilibrium is unique.

Part 1: To show that the solution to $\{P(\delta_j)\}_{j=1}^J$ is a partial equilibrium, I look for for a partial equilibrium characterized by (θ_j, ψ_j, s_j). For all j, let $\Psi = \{(\psi_j, s_j)\}_{j=1}^J$ and $\theta(\psi_j, s_j) = \theta_j$; let $\gamma((\psi_j, s_j); \delta) = 1$ if and only if $\delta = \delta_j$; let $v_{1,j}^* = v_{1,j}$; and let $F(\psi_j, s_j) = \Delta_j$.

Let us check Condition (1) to (5). Condition (2) to (4) hold by construction. Buyer's optimal behavior also holds because for all j, $(\psi_j, s_j, \theta(\psi_j, s_j))$ offers utility v_0 to buyers. We need to show that Seller's optimal behavior is satisfied and Condition (5) satisfied.

By construction, for all j and for all $j' < j$, j' will not deviate to j. We only need to prove $j' > j$ will not deviate to j either. Let us proceed by induction. First note that from $\{P(\delta)\}$ we have

$$v_{1,j} = \theta_j[u(b + \psi_j s_j) - u(b) - \delta_j s_j] = \theta_{j+1}[u(b + \psi_{j+1} s_{j+1}) - u(b) - \delta_j s_{j+1}]. \tag{A.34}$$

Since $\theta_j \geq \theta_{j+1}$ and $s_j \geq s_{j+1}$ imply that $\theta_j s_j \geq \theta_{j+1} s_{j+1}$, we have

$$\begin{aligned} \theta_j[u(b + \psi_j s_j) - u(b) - \delta_{j+1} s_j] &< \theta_{j+1}[u(b + \psi_{j+1} s_{j+1}) - u(b) - \delta_{j+1} s_{j+1}] \\ &= v_{1,j+1}. \end{aligned} \tag{A.35}$$

Now suppose for some $j' > j+1$, the above equation is also true for $j' - 1$:

$$\theta_j[u(b + \psi_j s_j) - u(b) - \delta_{j'-1} s_j] < \theta_{j'-1}[u(b + \psi_{j'-1} s_{j'-1}) - u(b - \delta_{j'-1} s_{j'-1}]. \tag{A.36}$$

Again because $\theta_j s_j \geq \theta_{j'-1} s_{j'-1}$ and $\delta_{j'} > \delta_{j'-1}$, the following is also true

$$\theta_j[u(b + \psi_j s_j) - u(b) - \delta_{j'} s_j] < \theta_{j'-1}[u(b + \psi_{j'-1} s_{j'-1}) - u(b) - \delta_{j'} s_{j'-1}]. \tag{A.37}$$

Now apply (A.35) again to get

$$\theta_{j'-1}[u(b + \psi_{j'-1} s_{j'-1}) - u(b) - \delta_{j'} s_{j'-1}] < \theta_{j'}[u(b + \psi_{j'} s_{j'}) - u(b) - \delta_{j'} s_{j'}]. \tag{A.38}$$

Combine the last two inequalities to get

$$v_{1,j'} = \theta_{j'}[u(b + \psi_{j'} s_{j'}) - u(b) - \delta_{j'} s_{j'}] > \theta_j[u(b + \psi_j s_j) - u(b) - \delta_{j'} s_j] \tag{A.39}$$

which is what we want.

Now suppose there exists a set S that satisfies condition (3.7) to (3.9). Define

$$S^* = \{j \in S | \text{condition (3.7) is true}\}$$

and let $j^* = \min S^*$. Now suppose there exists $j \in S$ such that $j < j^*$, then set $S \backslash j$ also satisfies condition (3.7) to (3.9). Therefore, without loss of generality I assume $j^* = \min S$. Now consider the belief system that assigns all the weights to δ_{j^*}: $\gamma^*(\psi, s; \delta_{j^*}) = 1$. By assumption, (ψ, s) must be accepted under γ^*. Then we have a contradiction: δ_{j^*} seller is strictly better off while not violating any constraints in $P(\delta_j)$.

Part 2: In this part I show that any equilibrium defined by Definition 3.1 is a solution to $\{P(\delta_j)\}_{j=1}^J$. Aggregate consistency implies that for all j there exists (ψ, s) such that $\gamma(\psi, s; \delta_j) > 0$. Denote such (ψ, s) as (ψ_j, s_j). Let $\theta_j = \theta(\psi_j, s_j)$. I first show that (ψ_j, s_j, θ_j) satisfies constraints (A.2) to (A.5) for all j. Then I show that $\{(\psi_j, s_j, \theta_j)\}_{j=1}^J$ solves $\{P(\delta)\}$.

First, note that Buyer's optimal behavior and Active markets together imply that

$$v_0 = q(\theta_j) s_j (\delta_j - \psi_j)$$

for all j and that resource constraints are satisfied. That is, constraint (A.2), (A.4) and (A.5) are satisfied.

Next, Equilibrium beliefs and Seller's optimal behavior imply that

$$v_{1,j}^* = \min\{\theta_j, 1\}[u(b + \psi_j s_j) - u(b) - \delta_j s_j]$$

and that

$$v_{1,j}^* \geq \min\{\theta_{j'}, 1\}[u(b + \psi_{j'} s_{j'}) - u(b) - \delta_j s_{j'}] \text{ for all } j'.$$

Hence, constraint (A.3) is satisfied as long as $v_{1,j} = v_{1,j}^*$.

Lastly, I prove that $v_{1,j} = v_{1,j}^*$ for all j. That is, $\{(\psi_j, s_j, \theta_j)\}_{j=1}^J$ is a solution to $\{P(\delta)\}$. Let us proceed by induction. First, it is easy to see that $v_{1,1} = v_{1,1}^*$. Next, suppose $v_{1,j'} = v_{1,j'}^*$ for all $j' < j$. Now suppose $v_{1,j} > v_{1,j}^*$. That is, some (ψ, s, θ) satisfies the constraints of $P(\delta_j)$ and delivers higher utility to sellers. That is,

$$v_{1,j}^* < \min\{\theta, 1\}[u(b + \psi s) - u(b) - \delta_j s] \tag{A.40}$$

$$v_0 \leq \min\{\theta^{-1}, 1\} s(\delta - \psi) \tag{A.41}$$

$$v_{1,j'}^* \geq \min\{\theta, 1\}[u(b + \psi s) - u(b) - \delta_{j'} s] \text{ for all } j' < j \tag{A.42}$$

$$s \leq a \tag{A.43}$$

$$\psi s \leq m. \tag{A.44}$$

where (A.42) uses the induction assumption. Then there must exist (ψ', s', θ) such that

$$\psi' < \psi, \ \ s' < s \tag{A.45}$$

$$v^*_{1,j} < \min\{\theta, 1\}[u(b + \psi' s') - u(b) - \delta_j s'] \tag{A.46}$$

$$v_0 \leq \min\{\theta^{-1}, 1\} s'(\delta - \psi') \tag{A.47}$$

$$v^*_{1,j'} > \min\{\theta, 1\}[u(b + \psi s) - u(b) - \delta_{j'} s] \text{ for all } j' < j \tag{A.48}$$

$$s' \leq a \tag{A.49}$$

$$\psi' s' \leq m. \tag{A.50}$$

Now let the off-equilibrium offer be (ψ', s', θ) and $S = \{j'' \in \mathcal{J} | j'' \geq j\}$. It is easy to see that condition (3.7) to (3.9) are satisfied, violating condition (5). Hence, it must be that $v_{1,j} = v^*_{1,j}$. □

Now I am ready to solve the full equilibrium in the AM where v_0 is endogenized.

Proposition A.3 *There exists a unique competitive equilibrium* $\{\Psi, \{v^*_{1,j}\}_{j=1}^J, \theta(.), \gamma(.), F(.), v^*_0\}$ *for all* $\alpha \in (0, 1)$.

Proof of Proposition A.3: Proposition A.1 and A.2 show that the partial equilibrium can be divided into three cases: (1) $\underline{m}_1 \geq m^*$, (2) $\bar{m}^* \leq \underline{m}_1 < m^*$, and (3) $\bar{m}^* > \underline{m}_1$ where m^*, $\bar{m}_1$ and $\bar{m}^*$ are defined by (A.6), (A.10) and (A.7), respectively.

Case (1): let me repeat the equilibrium solution in this case. In this case we have $\theta_1 = 1$, $m_1 = m^*$, and $v^*_{1,1} = u(b + m^*) - (v_0 + m^*) - u(b)$. For all $j > 1$ such that $m^\dagger_j > \bar{m}^*$, $\theta_j = 1$ and $m_j = m^\dagger_j$. For all $j > 1$ such that $m^\dagger_j \leq \bar{m}^*$, $m_j = \bar{m}^*$ and $\theta_j = \theta^*_j$. $m^\dagger_j$ is defined by (A.8) and θ^*_j is defined by (A.9).

Let us first check $m^\dagger_2$. It is given by

$$u(b + m^*) - (v_0 + m^*) - u(b) = u(b + m^\dagger_2) - (v_0 + m^\dagger_2)\delta_1/\delta_2 - u(b). \tag{A.51}$$

Taking full derivative to get (note m^* should be regarded as a constant)

$$\frac{dm^\dagger_2}{dv_0} = \frac{-1 + \delta_1/\delta_2}{u'(b + m^\dagger_2) - \delta_1/\delta_2} < 0 \tag{A.52}$$

since $u'(b + m^\dagger_2) \geq 1$ by Proposition A.1. Similarly, for $m^\dagger_3$ we have

$$\frac{dm^\dagger_3}{dv_0} = \frac{(u'(b + m^\dagger_2) - 1)\frac{dm^\dagger_2}{dv} - 1 + \delta_1/\delta_2}{u'(b + m^\dagger_3) - \delta_2/\delta_3} < 0. \tag{A.53}$$

It is easy to see that $\frac{dm_j^\dagger}{dv_0} < 0$ for all j such that $m_j^\dagger > \bar{m}^*$. Now let us check $\bar{m}^*$. We have

$$\frac{d\bar{m}^*}{dv_0} = \frac{-u'(b+\bar{m}^*)}{u''(b+\bar{m}^*)(v_0+\bar{m}^*)} > 0. \tag{A.54}$$

Next I turn to the biggest j such that $m_j^\dagger \le \bar{m}^*$. If $j > 1$, we have

$$u(b+m_{j-1}^\dagger) - (v_0+m_{j-1}^\dagger) - u(b) = \theta_j^*[u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_{j-1}/\delta_j - u(b)]. \tag{A.55}$$

Define $\theta_j^*(v_0)$ to be

$$\theta_j^*(v_0) = \frac{u(b+m_{j-1}^\dagger(v_0)) - (v_0+m_{j-1}^\dagger(v_0)) - u(b)}{u(b+\bar{m}^*(v_0)) - (v_0+\bar{m}^*(v_0))\delta_{j-1}/\delta_j - u(b)}. \tag{A.56}$$

Take derivative and get

$$\begin{aligned}\frac{d\theta_j^*(v_0)}{dv_0} =& \left[(u'(b+m_{j-1}^\dagger)-1)\frac{dm_{j-1}^\dagger}{dv_0} - 1\right][u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_{j-1}/\delta_j - u(b)] \\ &- \left[(u'(b+\bar{m}^*)-\delta_{j-1}/\delta_j)\frac{d\bar{m}^*}{dv_0} - 1\right][u(b+m_{j-1}^\dagger) - (v_0+m_{j-1}^\dagger) - u(b)] \\ =& A+B\end{aligned}$$

where

$$\begin{aligned}A =& \left[(u'(b+m_{j-1}^\dagger)-1)\frac{dm_{j-1}^\dagger}{dv_0}\right][u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_{j-1}/\delta_j - u(b)] \\ &- \left[(u'(b+\bar{m}^*)-\delta_{j-1}/\delta_j)\frac{d\bar{m}^*}{dv_0}\right][u(b+m_{j-1}^\dagger) - (v_0+m_{j-1}^\dagger) - u(b)] < 0\end{aligned}$$

and

$$\begin{aligned}B =& -[u(b+\bar{m}^*) - (v_0+\bar{m}^*)\delta_{j-1}/\delta_j - u(b)] \\ &+ [u(b+m_{j-1}^\dagger) - (v_0+m_{j-1}^\dagger) - u(b)] \le 0.\end{aligned}$$

$B \le 0$ because $\theta_j^* \le 1$. Hence $\frac{d\theta_j^*(v_0)}{dv_0} < 0$. If $j = 1$ then it is still true that $A < 0$. Now apply some minor modifications to (A.55) and we have

$$\begin{aligned}\frac{d\theta_{j+1}^*(v_0)}{dv_0} =& \left[(u'(b+\bar{m}^*)-1)\frac{d\bar{m}^*}{dv_0}-1\right]\theta_j^*[u(b+\bar{m}^*)-(v_0+\bar{m}^*)\delta_j/\delta_{j+1}-u(b)]\\ &+\frac{d\theta_j^*(v_0)}{dv_0}[u(b+\bar{m}^*)-(v_0+\bar{m}^*)-u(b)][u(b+\bar{m}^*)-(v_0+\bar{m}^*)\delta_j/\delta_{j+1}-u(b)]\\ &-\left[(u'(b+\bar{m}^*)-\delta_j/\delta_{j+1})\frac{d\bar{m}^*}{dv_0}-1\right]\theta_j^*[u(b+\bar{m}^*)-(v_0+\bar{m}^*)-u(b)]\\ =&C+D\end{aligned}$$

where

$$\begin{aligned}D=&-\theta_j^*[u(b+\bar{m}^*)-(v_0+\bar{m}^*)\delta_j/\delta_{j+1}-u(b)]\\ &+\theta_j^*[u(b+\bar{m}^*)-(v_0+\bar{m}^*)-u(b)]<0.\end{aligned}$$

and

$$\begin{aligned}C=&\left[(u'(b+\bar{m}^*)-1)\frac{d\bar{m}^*}{dv_0}\right]\theta_j^*[u(b+\bar{m}^*)-(v_0+\bar{m}^*)\delta_j/\delta_{j+1}-u(b)]\\ &-\left[(u'(b+\bar{m}^*)-\delta_j/\delta_{j+1})\frac{d\bar{m}^*}{dv_0}\right]\theta_j^*[u(b+\bar{m}^*)-(v_0+\bar{m}^*)-u(b)]\\ &+\frac{d\theta_j^*(v_0)}{dv_0}[u(b+\bar{m}^*)-(v_0+\bar{m}^*)-u(b)][u(b+\bar{m}^*)-(v_0+\bar{m}^*)\delta_j/\delta_{j+1}-u(b)]\\ =&\frac{d\bar{m}^*}{dv_0}\theta_j^*(1-\delta_j/\delta_{j+1})[u'(b+\bar{m}^*)(v_0+\bar{m}^*)-u(b+\bar{m}^*)+u(b)]\\ &+\frac{d\theta_j^*(v_0)}{dv_0}[u(b+\bar{m}^*)-(v_0+\bar{m}^*)-u(b)][u(b+\bar{m}^*)-(v_0+\bar{m}^*)\delta_j/\delta_{j+1}-u(b)]\\ =&\frac{d\theta_j^*(v_0)}{dv_0}[u(b+\bar{m}^*)-(v_0+\bar{m}^*)-u(b)][u(b+\bar{m}^*)-(v_0+\bar{m}^*)\delta_j/\delta_{j+1}-u(b)]<0\end{aligned}$$

where the last equality is because of (A.7). We can then prove by induction that $\frac{d\theta_{j'}^*(v_0)}{dv_0}<0$ for all $j'>j$.

Define

$$g(v_0)\equiv\sum_{j=1}^{J}\theta_j(v_0)\Delta_j.$$

Let $j(v_0)$ be the biggest j such that $m_j^\dagger\le\bar{m}^*$. Then $j(v_0)$ decreases with v_0. Because for all $j'<j$ we have $\theta_{j'}=1$, and $\frac{d\theta_{j'}^*(v_0)}{dv_0}<0$ for all $j'>j$. Hence $g(v_0)$ is strictly decreasing in v_0. When $v_0=0$, $\bar{m}^*=0$ and hence $\theta_j=1$ for all j. When $v_0=\bar{v}_0$ we have $\theta_1\le 1$ and $\theta_j=0$ for all $j>1$.

Now define $\underline{\alpha}$ and $\bar{\alpha}$ be such that

$$\frac{\underline{\alpha}}{1-\underline{\alpha}} = \frac{1}{g(0)} \tag{A.57}$$

and

$$\frac{\bar{\alpha}}{1-\bar{\alpha}} = \frac{1}{g(\bar{v}_0)}. \tag{A.58}$$

Let $v_0(\alpha)$ be such that

$$\frac{\alpha}{1-\alpha} = \frac{1}{g(v_0(\alpha))}.$$

Then there exist a unique v_0 that the market clearing condition is satisfied. It is given by

$$v_0 = \begin{cases} 0, \text{ if } \alpha < \underline{\alpha}; \\ v_0(\alpha), \text{ if } \alpha \in [\underline{\alpha}, \bar{\alpha}]; \\ \bar{v}_0, \text{ if } \alpha > \bar{\alpha}. \end{cases} \tag{A.59}$$

When $v_0 = 0$, buyers are indifferent between participating or not and hence the buyer-seller ratio in each market when $\alpha < \underline{\alpha}$ is the sames when $\alpha = \underline{\alpha}$. Some buyers simply do not participate in the search market. Similarly, $v_0 = \bar{v}_0$, sellers are indifferent between participating or not and hence the buyer-seller ratio in each market when $\alpha > \bar{\alpha}$ is the sames when $\alpha = \bar{\alpha}$. Some sellers do not participate in the search market.

Case (2): Case (2) is similar to Case (1) except that, due to resource constraints, $\bar{v}_0$ cannot be achieved. Furthermore, it matters whether δ_1 is asset constrained or cash constrained. If δ_1 is cash constrained, define $\hat{v}_0 = u(b + \underline{m}_1) - \underline{m}_1 - u(b)$. We then have

$$v_0 = \begin{cases} 0, \text{ if } \alpha < \underline{\alpha}; \\ v_0(\alpha), \text{ if } \alpha \in [\underline{\alpha}, \hat{\alpha}]; \\ \hat{v}_0, \text{ if } \alpha > \hat{\alpha}. \end{cases} \tag{A.60}$$

where $\hat{\alpha}$ is defined by

$$\frac{\hat{\alpha}}{1-\hat{\alpha}} = \frac{1}{g(\hat{v}_0)}. \tag{A.61}$$

If δ_1 is asset constrained, we have to write $\underline{m}_1$ as $\underline{m}_1(v_0)$ and it decreases in v_0. It is easy to check that it is still true that $\frac{dm_2^\dagger}{dv_0} < 0$. Hence the proof in Case (1) still holds. Define $\hat{v}_0$ such that

$$u(b + \delta_1 a - \hat{v}_0) - \delta_1 a - u(b) = 0.$$

v_0 is given by (A.60).

Case (3): We need to check each subcases (see the proof of Proposition A.1). In Subcase 3.1, we have $m_j = b$ for all j. θ_j is given by

$$v_{1,j-1} = \theta_j(u(b+b) - (v_0+b)\delta_{j-1}/\delta_j - u(b)).$$

It is easy to check that $\frac{d\theta_2}{dv_0} < 0$. From there it is easy to show that $\frac{d\theta_j}{dv_0} < 0$ for all $j > 2$. Then in this case, define $\hat{v}_0 = u(b+b) - b - u(b)$ and v_0 is given by (A.60).

In Subcase 3.2 all sellers are asset constrained. θ_2 is given by

$$\theta_2 = \frac{u(b+\delta_1 a - v_0) - \delta_1 a - u(b)}{u(b+\delta_2 a - v_0) - \delta_1 a - u(b)} \tag{A.62}$$

and

$$\begin{aligned}
\frac{d\theta_2}{dv_0} =& - u'(b+\delta_1 a - v_0)[u(b+\delta_2 a - v_0) - \delta_1 a - u(b)] \\
&+ u'(b+\delta_2 a - v_0)[u(b+\delta_1 a - v_0) - \delta_1 a - u(b)] \\
\leq& - u'(b+\delta_1 a - v_0)[u(b+\delta_2 a - v_0) - \delta_1 a - u(b)] \\
&+ u'(b+\delta_1 a - v_0)[u(b+\delta_1 a - v_0) - \delta_1 a - u(b)] \\
=& - u'(b+\delta_1 a - v_0)[u(b+\delta_2 a - v_0) - u(b+\delta_1 a - v_0)] < 0
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{d\theta_3}{dv_0} =& \frac{d\theta_2}{dv_0}[u(b+\delta_2 a - v_0) - \delta_2 a - u(b)][u(b+\delta_3 a - v_0) - \delta_2 a - u(b)] \\
&- u'(b+\delta_2 a - v_0)\theta_2[u(b+\delta_3 a - v_0) - \delta_2 a - u(b)] \\
&+ u'(b+\delta_3 a - v_0)\theta_2[u(b+\delta_2 a - v_0) - \delta_2 a - u(b)] \\
\leq& \frac{d\theta_2}{dv_0}[u(b+\delta_2 a - v_0) - \delta_2 a - u(b)][u(b+\delta_3 a - v_0) - \delta_2 a - u(b)] \\
&- u'(b+\delta_2 a - v_0)\theta_2[u(b+\delta_3 a - v_0) - \delta_2 a - u(b)] \\
&+ u'(b+\delta_2 a - v_0)\theta_2[u(b+\delta_2 a - v_0) - \delta_2 a - u(b)] \\
=& \frac{d\theta_2}{dv_0}[u(b+\delta_2 a - v_0) - \delta_2 a - u(b)][u(b+\delta_3 a - v_0) - \delta_2 a - u(b)] \\
&- u'(b+\delta_2 a - v_0)[u(b+\delta_3 a - v_0) - u(b+\delta_2 a - v_0)] < 0.
\end{aligned}$$

Then we can prove by induction that $\frac{d\theta_j}{dv_0} < 0$ for all j. Now define $\hat{v}_0$ such that

$$u(b+\delta_1 a - \hat{v}_0) - \delta_1 a - u(b) = 0.$$

v_0 is given by (A.60).

In Subcase 3.3, for all $j' < j$, $\frac{d\theta_{j'}}{dv_0} < 0$ by the above arguments. For θ_j, we have

$$\theta_j = \frac{\theta_{j-1}[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)]}{u(b+b) - (v_0+b) - u(b)} \tag{A.63}$$

and

$$\begin{aligned}
\frac{d\theta_j}{dv_0} =& \frac{d\theta_{j-1}}{dv_0}[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)][u(b+b) - (v_0+b) - u(b)] \\
& - u'(b+\delta_2 a - v_0)\theta_2[u(b+b) - (v_0+b) - u(b)] \\
& + \theta_2[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)] \\
\leq & \frac{d\theta_{j-1}}{dv_0}[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)][u(b+b) - (v_0+b) - u(b)] \\
& - u'(b+\delta_2 a - v_0)\theta_2[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)] \\
& + u'(b+\delta_2 a - v_0)\theta_2[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)] < 0
\end{aligned}$$

where the second inequality is because $u'(b+\delta_2 a - v_0) > 1$ and $\theta_j < \theta_{j-1}$. Then define $\hat{v}_0$ such that

$$u(b+\delta_1 a - \hat{v}_0) - \delta_1 a - u(b) = 0.$$

v_0 is given by (A.60).

Subcase 3.4 is similar to Subcase 3.3. Only difference is that for all $j' \geq j$, we have $m_j = \bar{m}^*$. Then

$$\begin{aligned}
\frac{d\theta_j}{dv_0} =& \frac{d\theta_{j-1}}{dv_0}[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)][u(b+\bar{m}^*) - (v_0+\bar{m}^*) - u(b)] \\
& - u'(b+\delta_2 a - v_0)\theta_2[u(b+\bar{m}^*) - (v_0+\bar{m}^*) - u(b)] \\
& + \theta_2[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)] \\
& - \frac{d\bar{m}^*}{dv_0}[u'(b+\bar{m}^*) - 1]\theta_2[u(b+\delta_{j-1}a - v_0) - \delta_{j-1}a - u(b)]
\end{aligned}$$

which is the same as in Subcase 3.3 except for the last term. The last term is negative because $\frac{d\bar{m}^*}{dv_0} > 0$. Then define $\hat{v}_0$ such that

$$u(b+\delta_1 a - \hat{v}_0) - \delta_1 a - u(b) = 0.$$

v_0 is given by (A.60). $\square$

Appendix B Proofs: Section 4

Proof of Proposition 4.1: from the proof of Proposition A.1 and Proposition A.3 we know that the if $\alpha \leq 0.5$, $v_0^* = 0$ and the highest price in the fully separating equilibrium is given by δ_J, in which case non-shoppers will always try to sell trees. When $\alpha > 0.5$, v_0^* increases with α. While v_0^* increases, the prices in asset market decrease. If α is big enough so that v_0^* is high enough, markets shutdown one by one starting from market δ_J. For any δ_1, for the fully separating equilibrium to exist, we need $\frac{m_J}{v_0^*+m_J}\delta_j \leq \delta_1$ where δ_j is the highest quality asset market that is open. Such α always exists since if $v_0^* = \bar{v}_0 = u(b+m^*) - u(b) - m^*$ where $u'(b+m^*) = 1$,[19] only δ_1 market is open, in which case δ_1 non-shopper will not attempt to sell. Hence, for all δ_1 there exists an α such that the fully separating equilibrium is the unique equilibrium. □

Proof of Proposition 4.2: In this proof I restrict my attention to equilibria with $v_0^* = 0$ since it can be solved analytically. I first assume $v_0^* = 0$. Later I show there exist conditions under which it is true.

Since δ_1 non-shoppers do not sell with δ_1 shoppers, δ_1 seller's problem is the same as the complete information case where the amount they sell either allow them to reach efficient consumption or is constrained by resources.

To look for the semi-pooling equilibrium, I assume that δ_2 shoppers cannot separate themselves from δ_1 non-shoppers. The δ_2 shopper's problem can be written as

$$v_{1,2} = \max_{\theta,\psi,s} \{\min\{\theta, 1\}[u(b+\psi s) - u(b) - \delta_j s]\} \tag{B.1}$$

$$\text{s.t. } 0 \leq \min\{\theta^{-1}, 1\} s(\bar{\delta} - \psi) \tag{B.2}$$

$$v_{1,1} \geq \min\{\theta, 1\}[u(b+\psi s) - u(b) - \delta_1 s] \tag{B.3}$$

where $\bar{\delta} = \frac{\alpha\Delta_2\delta_2 + (1-\alpha)\Delta_1\delta_1}{\alpha\Delta_2 + (1-\alpha)\Delta_1}$. It is easy to check that Claim A.1 from the proof of Proposition A.1 still applies here. Then, we can rewrite the problem as

$$v_{1,2} = \max_{\theta,\bar{m}} \theta \left[u(b+\bar{m}) - u(b) - \bar{m}\frac{\delta_2}{\bar{\delta}}\right] \tag{B.4}$$

$$\text{s.t. } v_{1,1} \geq \theta \left[u(b+\bar{m}) - u(b) - \bar{m}\frac{\delta_1}{\bar{\delta}}\right] \tag{B.5}$$

$$\theta \leq 1. \tag{B.6}$$

[19] This is the case where trade in asset market is not resource constrained. $\bar{v}_0 < u(b+m^*) - u(b) - m^*$ if trade in asset market is resource constrained.

Let the $\bar{m}^\dagger$ and $\bar{m}^\ddagger$ be defined by

$$v_{1,1} = u(b + \bar{m}^\dagger) - u(b) - \bar{m}^\dagger \frac{\delta_1}{\bar{\delta}} \tag{B.7}$$

$$u'(b + \bar{m}^\ddagger) = \frac{\delta_2}{\bar{\delta}}. \tag{B.8}$$

Let m_2 be the solution to the above problem. Note that $\bar{m}^\ddagger$ is the unconstrained maximizer. $m_2 = \bar{m}^\ddagger$ if and only if constraint (B.5) is slack. Substitute in $v_0^* = 0$ and we have

$$m_2 = \min\{\bar{m}^\dagger, \bar{m}^\ddagger\}$$

and $\theta_2 = 1$.

Next, I look for conditions under which δ_2 shoppers cannot achieve separation. Consider (θ', m') such that

$$\theta' \Big[u(b + m') - u(b) - m'\Big] > v_{1,2} = u(b + m_2) - u(b) - m_2 \frac{\delta_2}{\bar{\delta}}. \tag{B.9}$$

That is, (θ', m') makes δ_2 shoppers strictly better off. Next, consider

$$\theta' m' - \theta' m' \frac{\delta_1}{\delta_2} > \tilde{v}_{0,1} = m_2 - m_2 \frac{\delta_1}{\bar{\delta}}. \tag{B.10}$$

That is, (θ', m') also makes δ_1 non-shoppers strictly better off. For the semi-pooling equilibrium to exist, we need (B.10) to be true for all (θ', m') that satisfy (B.9). Now consider the set $\mathcal{M}$ defined by

$$\mathcal{M} = \{(\theta, m) | \theta \left[u(b + m) - u(b) - (v_0 + m)\right] = v_{1,2}\}.$$

That is, for any $(\theta, m) \in \mathcal{M}$, the δ_2 shoppers are indifferent between it and (θ_2, m_2). Notice that the left hand sides of (B.9) and (B.10) both strictly increase in θ' and m'. Then, if for all $(\theta, m) \in \mathcal{M}$ we have $\theta m - \theta m \frac{\delta_1}{\delta_2} \geq \tilde{v}_{0,1}$, it must be true that any (θ, m) that makes δ_2 shoppers better off will make δ_1 non-shopper better off. Note that for all $(\theta, m) \in \mathcal{M}$ we have

$$\theta = \frac{v_{1,2}}{u(b + m) - u(b) - m}. \tag{B.11}$$

Substitute it into (B.10) to get

$$\frac{v_{1,2} \left[m - m\frac{\delta_1}{\delta_2}\right]}{u(b + m) - u(b) - m} \geq m_2 - m_2 \frac{\delta_1}{\bar{\delta}}. \tag{B.12}$$

Now define $g(m) = \frac{v_{1,2}\left[m - m\frac{\delta_1}{\delta_2}\right]}{u(b+m) - u(b) - m}$, the left hand side of expression (B.12). Next, take the derivative of $g(m)$ to get

$$g'(m) \propto \left(1 - \frac{\delta_1}{\delta_2}\right) \Big[-u'(b+m)m + u(b+m) - u(b)\Big] > 0. \tag{B.13}$$

To see why $g'(m) > 0$, note that the expression in the square bracket is strictly increasing in m and it is equal to 0 when $m = 0$. Hence, $g'(m) > 0$ for all $m > 0$. This means that when checking if expression (B.12) is true, we need only consider $(1, m^\diamond) \in \mathcal{M}$ where $m^\diamond$ is given by

$$u(b + m^\diamond) - u(b) - m^\diamond = v_{1,2} \tag{B.14}$$

because $\forall m$ such that $(\theta, m) \in \mathcal{M}$, $m^\diamond < m$. Now define $\delta_1^\diamond$ to be such that

$$m^\diamond \left(1 - \frac{\delta_1^\diamond}{\delta_2}\right) = m_2 \left(1 - \frac{\delta_1^\diamond}{\bar{\delta}}\right). \tag{B.15}$$

Then for all $\delta_1 > \delta_1^\diamond$, we have

$$m^\diamond \left(1 - \frac{\delta_1}{\delta_2}\right) > m_2 \left(1 - \frac{\delta_1}{\bar{\delta}}\right). \tag{B.16}$$

Next, define $\delta_1^\ddagger$ to be such that

$$v_{1,1} = u(b + m^\ddagger) - u(b) - m^\ddagger \frac{\delta_1^\ddagger}{\bar{\delta}}. \tag{B.17}$$

Note that $\delta_1^\ddagger < \bar{\delta}$. Then for all $\delta_1 > \delta^\ddagger$, constraint (B.5) is slack. This is important because now $m_2 = \delta^\ddagger$ and it does not depend on δ_1.

Let me summarize the above findings. Fix b, δ_2, $\bar{\delta}$. Given b, $v_{1,1}$ is pinned down. Now suppose $\delta_1 \geq \delta^\ddagger$, we have m_2 always given by the unconstrained optimization in (B.4), which pins down m_2 and $v_{1,2}$ given δ_2, $\bar{\delta}$ and b. Next, suppose $\delta_1 \geq \delta_1^\diamond$, and then (B.10) is always true whenever (B.9) is true. To put it in words, any deviation that makes δ_2 shoppers strictly better off is going to make δ_1 non-shoppers strictly better off too. That is, the pooling of δ_2 shoppers and δ_1 non-shoppers does not fail the test of the Intuitive Criterion.

The last thing left to be shown is that given δ_1, δ_2 and $\bar{\delta}$, there exists (α, Δ) such that $\alpha + (1-\alpha)\Delta \leq 1 - \alpha$, where the left hand side is the measure of sellers and the right hand side is the measure of sellers. The condition is sufficient for $v_0^* = 0$ (see proof of Proposition A.3). First rewrite it and get

$$\alpha \leq \frac{1 - \Delta}{2 - \Delta}. \tag{B.18}$$

Next, note that $\bar{\delta} = \frac{\alpha(1-\Delta)\delta_2+(1-\alpha)\Delta\delta_1}{\alpha(1-\Delta)+(1-\alpha)\Delta}$. Rewrite it to get

$$\alpha = \frac{\Delta(\bar{\delta}-\delta_1)}{\Delta(\bar{\delta}-\delta_1)+(1-\Delta)(\delta_2-\bar{\delta})}. \tag{B.19}$$

Combine (B.18) and (B.19) to get

$$\frac{\Delta(\bar{\delta}-\delta_1)}{\Delta(\bar{\delta}-\delta_1)+(1-\Delta)(\delta_2-\bar{\delta})} \leq \frac{1-\Delta}{2-\Delta}. \tag{B.20}$$

Rearrange to get

$$-(\delta_2-\bar{\delta})\Delta^2 + (2\delta_2-\bar{\delta}-\delta_1)\Delta - \delta_2 + \bar{\delta} \leq 0. \tag{B.21}$$

Now consider the function $h(\Delta) = -(\delta_2-\bar{\delta})\Delta^2 + (2\delta_2-\bar{\delta}-\delta_1)\Delta - \delta_2 + \bar{\delta}$. It reaches maximum $\frac{(4\delta_2-3\bar{\delta}-\delta_1)(\bar{\delta}-\delta_1)}{4(\delta_2-\bar{\delta})} > 0$ at $0 < \frac{2\delta_2-\bar{\delta}-\delta_1}{2\delta_2-2\bar{\delta}} < 1$. Then we only need the smaller root of $h(\Delta) = 0$ to be bigger than 0. That is

$$2\delta_2 - \bar{\delta} - \delta_1 - \sqrt{4\delta_2(\bar{\delta}-\delta_1) - 3\bar{\delta}^2 + 2\bar{\delta}\delta_1 + \delta_1^2} > 0. \tag{B.22}$$

Simplify it to get

$$(\delta_2 - \bar{\delta})^2 > 0, \tag{B.23}$$

which is always true. Now we can define the condition $\mathcal{C}^p$ on δ_1, δ_2, Δ and α that allows the semi-pooling equilibrium to exist.

$$\mathcal{C}^p = \{(\delta_1,\delta_2,\Delta,\alpha) \in \mathbb{R}^2_{++} \times (0,1)^2 | \delta_1 \geq \max\{\delta_1^{\ddagger}, \delta_1^{\diamond}\} \text{ and } \alpha + (1-\alpha)\Delta \leq 1-\alpha\}. \tag{B.24}$$

And the above reasoning shows that it is not empty. □

Proof of Proposition 4.3: Consider the following problem

$$v_{1,2}^* = \max_{\theta,\bar{m}} \theta \left[u(b+\bar{m}) - u(b) - \bar{m}\frac{\delta_2}{\delta} \right] \tag{B.25}$$

$$\text{s.t. } v_{1,1}^* \geq \theta \left[u(b+\bar{m}) - u(b) - \bar{m}\frac{\delta_1}{\delta} \right] \tag{B.26}$$

$$\theta \leq 1. \tag{B.27}$$

If we let $\delta = \delta_2$ then this problem is the one when trading motives are known. If we let $\delta = \bar{\delta}$ then this problem is the one when trading motives are unknown. If δ is close enough to δ_2, we know the solution to the above problem is

$$v_{1,1}^* = u(b+m_2) - u(b) - m_2\frac{\delta_1}{\delta} \tag{B.28}$$

and $\theta = 1$. Now take implicit derivatives and get

$$\frac{\partial m_2}{\partial \delta} = -\frac{m_2\frac{\delta_1}{\delta^2}}{u'(b+m_2) - \frac{\delta_1}{\delta}}. \tag{B.29}$$

Then

$$\begin{aligned}\frac{\partial v_{1,2}^*}{\partial \delta} &= \left[u'(b+m_2) - \frac{\delta_2}{\delta}\right]\frac{\partial m_2}{\partial \delta} + m_2\frac{\delta_2}{\delta^2} \\ &= \frac{\frac{u'(b+m_2)m_2}{\delta^2}(\delta_2 - \delta_1)}{u'(b+m_2) - \frac{\delta_1}{\delta}} > 0. \end{aligned} \tag{B.30}$$

This says that when we decreases δ from δ_2 to $\bar{\delta}$, first (B.26) binds, and the above equation says that $v_{1,2}$ decreases as δ decreases. Next, suppose that (B.26) does not bind, which is exactly the scenario Proposition 4.2 refers to. Then we have $v_{1,2}^* = u(b+m_2) - u(b) - m_2\frac{\delta_2}{\delta}$ and $u'(b+m_2) = \frac{\delta_2}{\delta}$. As a result

$$\frac{\partial v_{1,2}^*}{\partial \delta} = m_2\frac{\delta_2}{\delta^2} > 0. \tag{B.31}$$

That is, $v_{1,2}$ always decreases as δ decreases. □

Proof of Proposition 4.4: Let m_2^p denote the trading volume of δ_2 location in the semi-pooling equilibrium. Let m_2^s denote the trading volume of δ_2 location in the separating equilibrium. To prove the proposition, we want to show that $m_2^s < m_2^p$ because the total surplus in the semi-pooling equilibrium is shared between δ_1 non-shopper and δ_2 shopper, and it increases with m_2. Since m_1 is unchanged, a higher m_2 implies higher expected search value for agents.

Now define $\delta_1^\dagger$ to be such that

$$v_{1,1}^* = u(b+m_2^p) - u(b) - m_2^p\frac{\delta_1^\dagger}{\delta_2}.$$

Then fix δ_2, a necessary and sufficient condition for $m_2^s \le m_2^p$ to be true is that $\delta_1 \le \delta_1^\dagger$. That is, we then need $\delta_1^\dagger \ge \delta_1 \ge \max\{\delta^\diamond, \delta^\ddagger\}$. Now we can define the condition $\mathcal{C}^w$ on δ_1, δ_2, Δ and α that allows the semi-pooling equilibrium to improve welfare.

$$\mathcal{C}^w = \{(\delta_1, \delta_2, \Delta, \alpha) \in \mathcal{C}^p | \delta_1^\dagger \ge \max\{\delta^\ddagger, \delta^\diamond\} \text{ and } \delta_1 \in [\max\{\delta^\ddagger, \delta^\diamond\}, \delta_1^\dagger]\}. \tag{B.32}$$

We need to show that the set $\mathcal{C}^w$ is not empty. Suppose $\max\{\delta^\diamond, \delta^\ddagger\} = \delta^\ddagger$. Recall that $\delta^\ddagger$ is defined by (see the proof of Proposition 4.2)

$$v^*_{1,1} = u(b + m^\diamond) - u(b) - m^\diamond \frac{\delta_1^\ddagger}{\bar{\delta}}. \tag{B.33}$$

Since $\bar{\delta} < \delta_2$, it must be that $\delta_1^\dagger > \delta^\ddagger$. Hence, if $\max\{\delta^\diamond, \delta^\ddagger\} = \delta^\ddagger$, $\mathcal{C}^w$ is not empty.

Now suppose $\max\{\delta^\diamond, \delta^\ddagger\} = \delta^\diamond$. Recall $\delta^\diamond$ is defined by the following two expressions

$$u(b + m^\diamond) - u(b) - m^\diamond = u(b + m^\ddagger) - u(b) - m^\ddagger \frac{\delta_2}{\bar{\delta}} \tag{B.34}$$

$$m^\diamond \left(1 - \frac{\delta_1^\diamond}{\delta_2}\right) = m_2 \left(1 - \frac{\delta_1^\diamond}{\bar{\delta}}\right). \tag{B.35}$$

This case is trickier since even with explicit functional forms of $u(.)$ (which I assume to be *log*), it is no possible to derive explicit expressions. To see the set $\mathcal{C}^w$ is not empty, consider the following example: $b = 0.50$, $\delta_2 = 1.45$ and $\bar{\delta} = 1.00$, which gives rise to $\delta^\diamond = 0.90$, $\delta^\ddagger = 0.67$ and $\delta_1^\dagger = 0.97$. Then any $\delta_1 \in (0.90, 0.97)$ supports a semi-pooling equilibrium that improves welfare. $\square$

Proof of Proposition 4.5: First, I show the semi-pooling equilibrium proposed in Proposition 4.4 is always undefeated.[20] Note first that there does not exist another semi-pooling equilibrium where δ_2 shoppers are better off, because otherwise the maximization problem (B.4) is violated. The only other possible equilibrium is a pooling equilibrium. In the only undefeated pooling equilibrium, the δ_2 shoppers solves

$$\max_{\theta, \bar{m}} \theta \left[u(b + \bar{m}) - u(b) - \bar{m}\frac{\delta_2}{\bar{\delta}^p}\right] \tag{B.36}$$

$$\text{s.t. } v_{1,1} \le \theta \left[u(b + \bar{m}) - u(b) - \bar{m}\frac{\delta_1}{\bar{\delta}^p}\right]. \tag{B.37}$$

where $\bar{\delta}^p = \frac{\alpha\Delta_2\delta_2 + \Delta_1\delta_1}{\alpha\Delta_2 + \Delta_1}$ and $v_{1,1} = u(b + m^*) - u(b) - m^*$. That is, δ_2 shoppers takes δ_1 shoppers' full information search value as given and maximizes their search value. Now since in the semi-pooling equilibrium constraint (B.5) does not and $\bar{\delta}^p < \bar{\delta}$, the above constraint must bind. Then δ_2 shoppers' search value is strictly less than in the semi-pooling equilibrium while δ_1 shoppers' search value stays the same. Hence this equilibrium is defeated.

Next, for the fully separating equilibrium I derive conditions under which it is not defeated. Bajaj (2018) shows that a sufficient condition to determine if the separating equilibrium is defeated is whether it maximizes δ_2 shoppers' utility. That is, the separating equilibrium is undefeated if

[20] Note that this is not saying that any semi-pooling equilibrium will be undefeated.

and only if $v^s_{1,2} > v^p_{1,2}$ where

$$v^s_{1,2} = u(b + m^s_2) - u(b) - m^s_2 \tag{B.38}$$

$$v^*_{1,1} = u(b + m^s_2) - u(b) - m^s_2 \frac{\delta_1}{\delta_2} \tag{B.39}$$

and

$$v^p_{1,2} = \max_{\bar{m}} u(b + \bar{m}) - u(b) - \bar{m} \frac{\delta_2}{\bar{\delta}^{pk}} \tag{B.40}$$

$$\text{s.t. } v_{1,1} \le u(b + \bar{m}) - u(b) - \bar{m} \frac{\delta_1}{\bar{\delta}^{pk}}, \tag{B.41}$$

$$\bar{\delta}^{pk} = \Delta\delta_1 + (1 - \Delta)\delta_2. \tag{B.42}$$

Recall that for the semi-pooling equilibrium to exist, given δ_2, $\bar{\delta}$ and b, we put restrictions on the lower bound of δ_1. For the semi-pooling equilibrium to improve welfare, we put restrictions on the upper bound of δ_1. The goal now is that given δ_1, δ_2, $\bar{\delta}$ and b, we find the Δ that makes the separating equilibrium undefeated. Recall that we also have restrictions on Δ Proposition 4.1 – we need Δ to be not too big so that $v^*_0 = 0$. Recall

$$\alpha = \frac{\Delta(\bar{\delta} - \delta_1)}{\Delta(\bar{\delta} - \delta_1) + (1 - \Delta)(\delta_2 - \bar{\delta})}. \tag{B.43}$$

and we need $\alpha \le (1 - \alpha)(1 - \Delta)$. Now we can define the condition $\mathcal{C}^u$ on δ_1, δ_2, Δ and α that allows the semi-pooling equilibrium to improve welfare.

$$\mathcal{C}^u = \left\{ (\delta_1, \delta_2, \Delta, \alpha) \in \mathcal{C}^w \,\middle|\, \alpha = \frac{\Delta(\bar{\delta} - \delta_1)}{\Delta(\bar{\delta} - \delta_1) + (1 - \Delta)(\delta_2 - \bar{\delta})},\ \alpha \le (1 - \alpha)(1 - \Delta),\ \text{and } v^s_{1,2} \ge v^p_{1,2} \right\}. \tag{B.44}$$

To show that the set $\mathcal{C}^u$ is not empty, consider the following example: $b = 0.50$, $\delta_2 = 1.45$, $\bar{\delta} = 1.00$, and $\delta_1 = 0.95$. Obviously the example is in $\mathcal{C}^w$ (see the proof of Proposition 4.4). Now let $\underline{\Delta} = 0.40$ and $\bar{\Delta} = 0.71$. If $\Delta < \bar{\Delta}$, then $\alpha \le (1 - \alpha)(1 - \Delta)$. If $\Delta \ge \underline{\Delta}$, then the separating equilibrium is undefeated. Hence any $\Delta \in [\underline{\Delta}, \bar{\Delta}]$ is in $\mathcal{C}^u$ with α given by equation (B.43). □