# Generalized Scattering-Based Stabilization of Nonlinear Interconnected Systems 

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#### Abstract

The research presented in this thesis is aimed at development of new methods and techniques for stability analysis and stabilization of interconnections of nonlinear systems, in particular, in the presence of communication delays. Based on the conic systems' formalism, we extend the notion of conicity for the non-planar case where the dimension of the cone's central subspace may be greater than one. One of the advantages of the notion of non-planar conicity is that any dissipative system with a quadratic supply rate can be represented as a non-planar conic system; specifically, its central subspace and radius can be calculated using an algorithm developed in this thesis. For a feedback interconnection of two non-planar conic systems, a graph separation condition for finite-gain $\mathcal{L}_{2}$-stability is established in terms of central subspaces and radii of the subsystems' non-planar cones. Subsequently, a generalized version of the scattering transformation is developed which is applicable to non-planar conic systems. The transformation allows for rendering the dynamics of a non-planar conic system into a prescribed cone with compatible dimensions; the corresponding design algorithm is presented. The ability of the generalized scattering transformation to change the parameters of a system's cone can be used for stabilization of interconnections of non-planar conic systems. For interconnections without communication delays, stabilization is achieved through the design of a scattering transformation that guarantees the fulfilment of the graph separation stability condition. For interconnected systems with communication delays, scattering transformations are designed on both sides of communication channel in a way that guarantees the overall stability through fulfilment of the small gain stability condition. Application to stabilization of bilateral teleoperators with multiple heterogeneous communication delays is briefly discussed.

Next, the coupled stability problem is addressed based on the proposed scattering based stabilization techniques. The coupled stability problem is one of the most fundamental problems in robotics. It requires to guarantee stability of a controlled manipulator in contact with an environment whose dynamics are unknown, or at least not known precisely. We present a scattering-based design procedure that guarantees coupled stability while at the same time does not affect the robot's trajectory tracking performance in free space. A detailed design example is presented that demonstrates the capabilities of the scattering-based design approach, as well as its advantages in comparison with more conventional passivity-based approaches.

Finally, the generalized scattering-based technique is applied to the problem of stabilization of complex interconnections of dissipative systems with quadratic supply rates in the presence of multiple heterogeneous constant time delays. Our approach is to design local scattering transformations that guarantee the fulfilment of a multi-dimensional small-gain stability condition for the interconnected system. A numerical example is presented that illustrates the capabilities of the proposed design method.


Keywords: Stability, (Q, S, R)-dissipativity, non-planar conicity, scattering transformation, finite gain $\mathcal{L}_{2}$-stability, small-gain theorem, coupled stability

## Co-Authorship Statement

The work presented in Chapters 2-4 involves collaboration with my supervisor Dr. Ilia G. Polushin who contributed to theoretical developments, writing, and reviewing drafts of the manuscript. The work presented in Chapters 2-4 involves collaboration with my co-supervisor Dr. Rajni V. Patel who reviewed several drafts of the manuscript.

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## List of Symbols

- $\mathbb{R}, \mathbb{R}_{+}$denote sets of real numbers and nonnegative real numbers, respectively.
- $\mathbb{C}$ is the set of complex numbers.
- $\mathbb{R}^{n}$ denotes the $n$-dimensional Euclidean space over $\mathbb{R}$ with norm

$$
\|x\|=\sqrt{\sum_{i=1}^{n} x_{1}^{2}}, \quad x=\left[\begin{array}{c}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right] .
$$

- $\mathbb{R}^{n \times m}\left(\mathbb{C}^{n \times m}\right)$ is the set of real-valued (complex-valued) $n \times m$ matrices.
- $\mathbb{I}_{n}$ is identity $n \times n$ matrix.
- $\mathbb{O}_{n \times m}$ is zero $n \times m$ matrix.
- Given a vector $v, v \in \mathbb{C}^{n}$ (or a subspace $\Omega \subset \mathbb{R}^{n}$ ), $\operatorname{dim} v(\operatorname{dim} \Omega)$ is a dimension of the vector $v$ (subspace $\Omega$ ).
- Given square matrix $A \in \mathbb{R}^{n \times n}, \lambda(A)$ denotes the set of eigenvalues of $A$.
- Given a complex matrix $A \in \mathbb{C}^{m \times n}$,
- $A^{T}$ is a transposed matrix $A$,
- $A^{*}$ denotes the complex conjugate matrix to $A$,
$-\sigma_{\max }(A):=\max \left\{\sqrt{\lambda}: \lambda \in \lambda\left(A^{*} A\right)\right\}$ denotes the maximal singular value of $A$.
- Given a symmetric matrix $M \in \mathbb{R}^{n \times n}$, i.e. $M^{T}=M, \lambda^{+}(M)\left(\lambda^{-}(M)\right)$ denotes the set of its nonnegative (negative) eigenvalues,
- Given vectors $h_{1}, h_{2}, \ldots, h_{m} \in \mathbb{R}^{n}(m \leqslant n)$, $\operatorname{span}\left\{h_{1}, h_{2}, \ldots, h_{m}\right\}$ denotes a subspace spanned by the vectors $h_{1}, h_{2}, \ldots, h_{m}$.
- Given a linear subspace $\Omega \subset \mathbb{R}^{n}, \Omega^{\perp}$ determines its orthogonal complement.
- $\Pi_{\Omega}$ is a projector onto the subspace $\Omega \subset \mathbb{R}^{m}$,

$$
\forall h \in \mathbb{R}^{n} \quad \Pi_{\Omega} h \in \Omega, \quad(m \leqslant n)
$$

- Given $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}, A \otimes B \in \mathbb{R}^{m p \times n q}$ denotes the Kronecker product of $A$ and $B$,

$$
A \otimes B:=\left[\begin{array}{ccc}
a_{11} B & \cdots & a_{1 n} B \\
\vdots & \ddots & \vdots \\
a_{m 1} B & \cdots & a_{m n} B
\end{array}\right]
$$

- $\tanh (\cdot), \operatorname{sech}(\cdot)$ are the hyperbolic functions defined as follows:

$$
\tanh (s):=\frac{e^{s}-e^{-s}}{e^{s}+e^{-s}}, \quad \operatorname{sech}(s):=\frac{2}{e^{s}+e^{-s}}
$$

- For a set $\mathbb{K}, \operatorname{card}\{\mathbb{K}\}$ denotes the number of elements in $\mathbb{K}$.
- $\mathcal{L}_{2}^{n}\left(\mathbb{R}_{+}\right)$denotes the Hilbert space of Lebesque measurable functions $f: \mathbb{R}_{+} \mapsto \mathbb{R}^{n}$, which are square integrable, i.e.

$$
\|f\|_{2} \stackrel{\text { def }}{=}\left(\int_{0}^{+\infty}\|f(t)\|^{2} d t\right)^{1 / 2}<\infty
$$

- $[\mathrm{QSR}]$ is a real symmetric matrix $\left([\mathrm{QSR}] \in \mathbb{R}^{(m+p) \times(m+p)}\right)$ of the form

$$
[\mathrm{QSR}]=\left[\begin{array}{cc}
R & S \\
S^{T} & Q
\end{array}\right], \quad R=R^{T} \in \mathbb{R}^{m \times m}, Q=Q^{T} \in \mathbb{R}^{p \times p}, S \in \mathbb{R}^{m \times p}
$$

## Chapter 1

## Introduction

Numerous engineering applications deal with controlled interconnections of nonlinear systems. Examples of such applications can be found in many industries, including space and terrestrial exploration, mining, factory automation, power systems, medical robotics, and others. Development of control algorithms that guarantee stable behavior of systems' interconnections in the presence of various external inputs such as noise or disturbances and possibly time delays in the communication channels between subsystems is an important engineering problem.

One of the approaches commonly applicable to stability of nonlinear multi-input-multioutput (MIMO) systems is based on the notion of input-output stability [39, 44, 45], which describes system's behavior as a mapping of admissible inputs into appropriately chosen outputs. The input-output stability approach is especially useful in the case where the system's dynamics are uncertain. For example, in the problem of robot-environment interaction, the model of environment is usually not precisely known (or not known at all). The input-output stability theory is applicable to many intensively developing areas including robotics, teleoperation theory, and process control. It allows for solving the problems in the fields of robust and optimal control. The input-output stability theory was largely introduced by George Zames. In particular, in 1966, G. Zames [45] published theorems establishing input-output stability for interconnections of passive and (planar) conic systems, as well as a small-gain stability condition. An extension of this approach for (Q, S, R)-dissipative systems (i.e., dissipative systems with quadratic supply rates) has been developed by D.J. Hill and P.J. Moylan [14]. Afterwards, graph separation stability condition for general interconnected dynamical systems has been derived in [38]. However, among all these results, only the passivity and small-gain theorems are widely used.

It is worth to mention that the passivity-based approach cannot be applied directly in the case of time-delayed interconnections, since the communication block producing delays is not passive [2]. To address this issue, the scattering-based (or wave variable) stabilization tech-
nique has been developed within the teleoperator systems theory [2,31]. The basic idea behind the scattering-based approach is to render the communication channel passive by emulating the behavior of a lossless electrical transmission line. An analogy between the communication channel producing time-delay and a lossless transmission line was originally revealed by R.J. Anderson and M.W. Spong [2]. The fundamental result related to this analogy is formulated in terms of the scattering operator $\mathbb{S}$ determined as the map $\left(\mathbb{S}: \mathcal{L}_{2}^{n}\left(\mathbb{R}_{+}\right) \mapsto \mathcal{L}_{2}^{n}\left(\mathbb{R}_{+}\right)\right)$of effort $F$ plus flow $v$ into effort $F$ minus flow $v$, i.e. $F-v=\mathbb{S}(s) \cdot(F+v)$, where the flow $v$ is entering the system's ports, and the effort $F$ is measured across the system's ports. It was demonstrated in [2] that a system is passive if and only if the spectral norm of its scattering operator is not greater than one, i.e. $\|\mathbb{S}\| \leqslant 1$. Using this criterion, a scattering transformation can be designed that eliminates the delay-induced non-passivity of the communication channel and, consequently, stabilizes a teleoperation system. The scattering-based technique provides robust stabilizing control laws with respect to a wide variety of perturbations due to preserving passivity of the subsystems (e.g. master and slave) included in the interconnection. Moreover, this method can be partially extended to the case of time-varying delays [27]. In addition, it allows for improving performance of the control algorithm by tuning stiffness and damping gains involved into the control law. For example, an appropriate choice of control gains (impedance matching) leads to avoiding wave-reflection phenomena [31].

There exists, however, a number of issues associated with the passivity-based design of interconnected systems. Specifically, the passivity condition is often conservative and may be violated for many reasons, for instance, due to existence of time delays, actuator and/or sensor noise, or in the case where one of the subsystem behaves in a non-passive way. In particular, there is substantial evidence in the teleoperation literature indicating that the assumption of passivity imposed on the behavior of the human operator(s) and the environment can be violated. Examples include experimental evidence of non-passive behavior of the human operators when performing certain tasks [13], non-passivity arising in teleoperation of wheeled robots as a result of the wheel slippage [26], intrinsically non-passive behavior of the therapist in tele-rehabilitation systems during assistive therapy [3, 4], etc. Moreover, the assumption of passivity imposed on the behavior of the human operator(s) and the environment may be overly conservative, which results in unnecessary restrictions on the design of the local masters' and slaves' controllers. In particular, the requirement of passivity of the closed-loop master and slave subsystems is apparently conflicting with the trajectory tracking performance, and the rigorously proven results regarding the tracking properties of the passivity-based teleoperators are relatively weak (see for example [33]).

In recent works [15, 35], generalizations of the scattering transformation have been reported which are applicable to classes of systems more general than passive. These generalizations
are based on a representation of the scattering transformation as a rotation in the space of input-output variables. In particular, the study [35] elaborates on the most general version of the scattering transformation for stabilizing interconnections of arbitrary planar conic systems, particularly in the presence of communication delays. Planar conic systems are systems whose input-output characteristics belong to a dynamic conic sector on a plane [38, 44, 45]. Examples include passive systems, input-feedforward-output-feedback-passive (IF-OFP) nonlinear systems studied in [15], as well as systems with finite $\mathcal{L}_{2}$-gain, which all are special cases of the planar conic systems. However, the class of planar conic systems addressed in [35] is still limited in some important aspects. The most essential limitations are the assumption of equal number of input and outputs of a system, and the requirement that each input-output pair satisfies the uniform constraints imposed by the supply rate. Another limitation is that a feedback interconnection of two planar conic systems is, generally speaking, not a planar conic system. This circumstance complicates the analysis of complex interconnections within the framework of planar conic systems. This thesis is aimed to overcome these limitations by extending the notion of conicity to the non-planar case and generalize scattering-based stabilization approaches for the interconnections of non-planar conic and ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems.

In comparison with earlier results developed for the class of planar conic systems, the research presented in this thesis provides a natural way of expanding this notion to a more general class of non-planar conic systems. We demonstrate that the notion of non-planar conicity eliminates the limitations inherent to the planar case. In regard to the stabilization methods, this thesis develops the generalized version of the scattering transformation which is applicable to non-planar conic systems. Combination of the above described developments allows for an extension of the existing scattering-based stabilization methods to the case of interconnections of nonlinear non-planar conic systems, in particular, in the presence of heterogeneous constant delays in the communication channels between the subsystems. These new methods are subsequently applied to the coupled stability problem and to the stabilization problem of the complex interconnection of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ ) - dissipative systems, where they bring new results or improve the existing results, especially in those cases where the currently existing passivity-based approaches either are not applicable or result in overly conservative design methods.

### 1.1 Literature review

Stabilization technique developed in this thesis is based on the scattering transformation that has been extensively used in the teleoperator theory. Therefore, we start the literature review from a survey of stabilization methods that have been conventionally applied to the teleoperator systems, but extended to more general classes of nonlinear dynamic systems in the present study.

### 1.1.1 Scattering transformation

In the context of nonlinear systems, the scattering transformation first appeared in 1972 [1], where it was used to establish relationship between the passivity and the small gain theorems. Specifically, it was shown in (see [2, 11]) that application of the scattering transformation to inputs and outputs of a passive system turns it into a system with gain less than or equal to one.

Subsequently, the scattering-based approach was applied to solve the problem of stabilization of bilateral teleoperators in the presence of communication delays under the assumption of passivity of both the master and the slave subsystems (see [2]). To illustrate the approach,


Figure 1.1: Network representation of teleoperator
consider a teleoperator system shown in Figure 1.1, where the master manipulator, the communication block, and the slave manipulator are represented by two-port networks, while the human operator and the environment are represented by one-port networks [41]. In Figure 1.1, symbols $F_{e}, F_{s}, F_{m d}$, and $F_{h}$ denote the environmental force applied to the slave, the slave force signal transmitted over the communication channel, the desired master force, and the interaction force between the human operator and the master device, respectively. The symbols $\dot{x}_{m}$, $\dot{x}_{s d}$, and $\dot{x}_{s}$ denote the master velocity, the desired slave velocity, and the actual slave velocity, respectively.

An $n$-port network is characterized by a relationship between $n$ effort variables $F_{1}, F_{2}, \ldots$, and $F_{n}\left(F=\left[F_{1}, \ldots, F_{n}\right]^{T}\right)$ (force, voltage), and $n$ flow variables $\dot{x}=\left[\dot{x}_{1}, \ldots, \dot{x}_{n}\right]$ (velocity, current). For a linear time invariant (LTI) one-port network, this relationship can be described in the Laplace domain by the network's impedance $Z(s)$, according to the formula

$$
F(s)=Z(s) \dot{x}(s),
$$

where $F(s), \dot{x}(s)$ are the Laplace transforms of the effort $F(t)$ and flow $\dot{x}(t)$ respectively. In the case of a two-port LTI network, different representations are possible; one particular representation describes relationship between the effort and flow variables in terms of hybrid matrix $H(s)$, according to the equation

$$
\left[\begin{array}{c}
F_{1}(s)  \tag{1.1}\\
-\dot{x}_{2}(s)
\end{array}\right]=\left[\begin{array}{ll}
h_{11}(s) & h_{21}(s) \\
h_{12}(s) & h_{22}(s)
\end{array}\right]\left[\begin{array}{l}
\dot{x}_{1}(s) \\
F_{2}(s)
\end{array}\right]=H(s)\left[\begin{array}{l}
\dot{x}_{1}(s) \\
F_{2}(s)
\end{array}\right] .
$$

The sign of $\dot{x}_{2}$ is reversed here since $\dot{x}_{2}$ is assumed to exit the two-port network.
The definition of the scattering operator given in [2] goes as follows. The scattering operator $S: \mathcal{L}_{2}^{n} \mapsto \mathcal{L}_{2}^{n}$ is defined by the formula

$$
F(t)-\dot{x}(t)=S(F(t)+\dot{x}(t))
$$

where $\dot{x}$ is a flow entering the system's ports, and $F$ is the effort across the system's ports. For LTI systems, the scattering operator $S$ can be expressed in the Laplace domain as a scattering matrix $S(s)$ such that

$$
F(s)-\dot{x}(s)=S(s)(F(s)+\dot{x}(s)) .
$$

Consequently, for a two-port network, the scattering matrix $S(s)$ is related to the hybrid matrix $H(s)$ (1.1), according to the formula

$$
S(s)=\left[\begin{array}{cc}
1 & 0  \tag{1.2}\\
0 & -1
\end{array}\right](H(s)-I)(H(s)+I)^{-1}
$$

The following is a well-known definition of a passive $n$-port network. An $n$-port network is passive if for any independent set of $n$-port flows $\dot{x}=\left[\dot{x}_{1}, \ldots, \dot{x}_{n}\right]^{T} \in \mathcal{L}_{2}^{n}$ injected into the system and efforts $F=\left[F_{1}, \ldots, F_{n}\right]^{T} \in \mathcal{L}_{2}^{n}$ across the system, the following inequality holds

$$
\int_{0}^{\infty} F^{T}(t) \dot{x}(t) d t \geqslant-\beta_{0}
$$

where $\beta_{0} \geq 0$ represents the initial energy stored inside the system. In terms of the scattering operator, the following criteria for system passivity can be established.

Theorem 1.1.1. ( [11] Section VI.10], [2] Section III, Therem 3.1.]) A system is passive if and only if the spectral norm ${ }^{1}$ of its scattering operator is less than or equal to one.

Regarding the bilateral teleoperator system schematically shown in Figure 1.1, the conventional approach to its design is based on passivity considerations. Specifically, the local

[^0]controllers can be designed to make both the master and the slave manipulators passive [2, 33]. There is also a widely adopted assumption of passivity of the human operator which is based on the experimental results published in [20]. In addition, the environment can also frequently be considered as a passive subsystem. Since the interconnection of any number of passive networks is passive, one concludes that passivity of the communication block will result in passivity (and, therefore, stability) of the overall teleoperator system.

An interesting and somewhat counter-intuitive fact is that the constant communication delay block is not a passive system; in fact, it can generate energy and, as a result, destabilize the overall teleoperator system. To show this, consider an idealized communication channel characterized by a constant communication delay $T>0$, described by the formulas

$$
\begin{equation*}
F_{m d}(t)=F_{s}(t-T), \quad \dot{x}_{s d}(t)=\dot{x}_{m}(t-T) . \tag{1.3}
\end{equation*}
$$

The hybrid matrix that describe the communication channel (1.3) has a form

$$
H(s)=\left[\begin{array}{cc}
0 & e^{-s T}  \tag{1.4}\\
-e^{-s T} & 0
\end{array}\right]
$$

that provides the following scattering operator (1.2)

$$
S(s)=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{cc}
-1 & e^{-s T} \\
-e^{-s T} & -1
\end{array}\right]\left[\begin{array}{cc}
1 & e^{-s T} \\
-e^{-s T} & 1
\end{array}\right]^{-1}=\left[\begin{array}{cc}
-\tanh (s T) & \operatorname{sech}(s T) \\
\operatorname{sech}(s T) & \tanh (s T)
\end{array}\right]
$$

which is unbounded $(\|S\|=+\infty)$. Therefore, the pure delay block is not a passive system, and the delayed communication channel may generate energy, which in turn may potentially destabilize the teleoperator. A solution of this problem was proposed in [2]. It relies on application of the following scattering-based stabilization algorithm

$$
\begin{align*}
F_{m d}(t) & =F_{s}(t-T)+\dot{x}_{m}(t)-\dot{x}_{s d}(t-T),  \tag{1.5}\\
\dot{x}_{s d}(t) & =\dot{x}_{m}(t-T)+F_{m d}(t-T)-F_{s}(t)
\end{align*}
$$

that provides hybrid matrix $H_{s}(s)$ corresponding to the scattering-based communication channel (1.5)

$$
\left[\begin{array}{cc}
1 & -e^{-s T} \\
e^{-s T} & 1
\end{array}\right]\left[\begin{array}{c}
F_{m d}(s) \\
-\dot{x}_{s d}(s)
\end{array}\right]=\left[\begin{array}{cc}
1 & e^{-s T} \\
-e^{-s T} & 1
\end{array}\right]\left[\begin{array}{l}
\dot{x}_{m}(s) \\
F_{s}(s)
\end{array}\right] \Rightarrow H_{s}(s)=\frac{1}{1+e^{-2 s T}}\left[\begin{array}{cc}
1-e^{-2 s T} & 2 e^{-s T} \\
-2 e^{-s T} & 1-e^{-2 s T}
\end{array}\right]
$$

Passivity of the communication block then follows from Theorem1.1.1 since the scattering operator $S(s)$ computed by (1.2) has the spectral norm equal to one, namely

$$
S=\left[\begin{array}{cc}
0 & e^{-s T} \\
e^{-s T} & 0
\end{array}\right], \quad\|S\|=\sup _{\omega} \sqrt{\lambda\left(S^{*}(j \omega) S(j \omega)\right)}=\sup _{\omega} \sqrt{\lambda\left(\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right)}=1 .
$$

Thus, passivity of the scattering-based communication channel results in passivity of the overall teleoperation system, and guarantees its stability for all passive environments and passive human operator behavior.

### 1.1.2 Wave variables

The idea behind the algorithm (1.5) is clarified by addressing a conceptually similar but slightly more elaborated approach originally proposed in [31, 32]. Works [12, Section 2.8], [10, 31, [32,34] show that the communication block in a teleoperator system can be considered as a virtual lossless transmission line. As is well-known, a lossless transmission line has a passive behavior [12,31]. Consequently, within this approach, the communication channel inheriting properties of the transmission line is passive.

A lossless transmission line consists of a series of an infinitesimally small components comprised by inductances and capacitances between the two conductors (see Figure 1.2). Any


Figure 1.2: An element of a lossless transmission line.
element represents an infinitesimally short segment $\Delta x$ of the transmission line of the length $l$. Applying Kirchhoff's current and voltage laws at the each segment of the line (see Figure 1.2), the following equalities take place

$$
i(t, x)-i(t, x+\Delta x)=C \Delta x \frac{\partial v(t, x)}{\partial t}, \quad v(t, x)-v(t, x+\Delta x)=L \Delta x \frac{\partial i(t, x)}{\partial t}
$$

where $v(t, x)$ and $i(t, x)$ are the voltage and the current associated to the spatial variable $x \in[0, l]$. The passage to the limit as $\Delta x \rightarrow+0$ in the last expressions leads to the Telegrapher's equations that describe the behavior of the transmission line

$$
\begin{equation*}
\frac{\partial i(t, x)}{\partial x}=-C \frac{\partial v(t, x)}{\partial t} ; \quad \frac{\partial v(t, x)}{\partial x}=-L \frac{\partial i(t, x)}{\partial t} . \tag{1.6}
\end{equation*}
$$

Equalities (1.6) are equivalent to the system of the partial differential equations

$$
\begin{equation*}
\frac{\partial^{2} i(t, x)}{\partial x^{2}}=C L \frac{\partial^{2} i(t, x)}{\partial t^{2}}, \quad \frac{\partial^{2} v(t, x)}{\partial x^{2}}=L C \frac{\partial^{2} v(t, x)}{\partial t^{2}} . \tag{1.7}
\end{equation*}
$$

Solution of the system (1.7) has the structure (see [12])

$$
\begin{equation*}
i(t, x)=F_{2}(x+v t)-F_{1}(x-v t), \quad v(t, x)=b\left(F_{2}(x+v t)+F_{1}(x-v t)\right), \tag{1.8}
\end{equation*}
$$

where $F_{1}$ and $F_{2}$ are waveforms spreading in positive and negative directions respectively. Parameter $v=1 / \sqrt{L C}$ is a wave propagation velocity, and $b=\sqrt{L / C}$ is the impedance of the transmission line. The derived representation (1.8) of the solution of the Telegrapher's equations allows for transferring to the new so-called wave or scattering variables

$$
\begin{equation*}
s^{+}(t, x)=\sqrt{2 b} F_{1}(x-v t), \quad s^{-}(t, x)=\sqrt{2 b} F_{2}(x+v t) \tag{1.9}
\end{equation*}
$$

that are, in fact, scaled waves $F_{1}(\cdot)$ and $F_{2}(\cdot)$ traveling in positive and negative directions respectively. The scattering variables have a useful property related to their delayed values. Propagation delay $T_{p}$ is the time for which a wave passes through the transmission line of the length $l$ with propagation speed $v=1 / \sqrt{L C}$, i.e. $T_{p}=l / v=l \sqrt{L C}$. Substitution of $x=l=v T_{p}$ to the positive directed wave $s^{+}(t, x)$, and $x=0$ to the opposite wave $s^{-}(t, x)(1.9)$ provides the equalities

$$
\begin{gather*}
s^{+}(t, l)=\sqrt{2 b} F_{1}(l-v t)=\sqrt{2 b} F_{1}\left(v T_{p}-v t\right)=\sqrt{2 b} F_{1}\left(-v\left(t-T_{p}\right)\right)=s^{+}\left(t-T_{p}, 0\right) \\
s^{-}(t, 0)=\sqrt{2 b} F_{2}(v t)=\sqrt{2 b} F_{2}\left(l-v T_{p}+v t\right)=\sqrt{2 b} F_{2}\left(l+v\left(t-T_{p}\right)\right)=s^{-}\left(t-T_{p}, l\right) \\
s^{+}(t, l)=s^{+}\left(t-T_{p}, 0\right), \quad s^{-}(t, 0)=s^{-}\left(t-T_{p}, l\right) \tag{1.10}
\end{gather*}
$$

The relationship between current $i(t, x)$, voltage $v(t, x)$ and scattering variables is determined by the scattering transformation (or map) $\mathbb{S}$ (see [12])

$$
\left[\begin{array}{l}
s^{+}(t, x)  \tag{1.11}\\
s^{-}(t, x)
\end{array}\right]=\mathbb{S}\left[\begin{array}{l}
i(t, x) \\
v(t, x)
\end{array}\right], \quad \mathbb{S}=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
b^{1 / 2} & b^{-1 / 2} \\
-b^{1 / 2} & b^{-1 / 2}
\end{array}\right]
$$

Adopting this technique for the bilateral teleoperation, the standard analogy [33] is used, i.e. current $i(t, \cdot)$ is replaced with velocity $\dot{x}(t)$, and voltage $v(t, \cdot)$ is considered as force $f(t)$. Interconnections of the port variables now play role of the conroller and the terminations with the master and slave subsystems. The scattering scheme in the teleoperator model assumes transmition of the wave variables $s^{+}$and $s^{-}$through the delayed communication channel rather than original power variables (velocity $\dot{x}$ and force $f$ ). The corresponding scheme is depicted at Figure 1.3. Furthermore, according to the property (1.10) of wave variables, at the opposite site of the communication block, scattered signals $s^{+}(t, 0)$ and $s^{-}(t, l)$ take values $s^{+}(t-T, 0)=$ $s^{+}(t, l)$ and $s^{-}(t-T, l)=s^{-}(t, 0)$ respectively. Wave variables are standardly denoted as $\mathbf{u}_{j}, \mathbf{v}_{j}$ (index $j \in\{m, s\}$ indicates a side (master or slave) of the teleoperator transmitting/receiving the


Figure 1.3: Application of the scattering approach for the teleoperator model [32]
scattering signals), i.e.

$$
\left[\begin{array}{c}
\mathbf{u}_{m} \\
\mathbf{v}_{m}
\end{array}\right]=\left[\begin{array}{c}
s^{-}(t, 0) \\
s^{+}(t, 0)
\end{array}\right], \quad\left[\begin{array}{l}
\mathbf{u}_{s} \\
\mathbf{v}_{s}
\end{array}\right]=\left[\begin{array}{c}
s^{+}(t, l) \\
s^{-}(t, l)
\end{array}\right] .
$$

The power variables $\dot{x}, f$ are related to the wave variables $\mathbf{u}, \mathbf{v}$ through the scattering (wave) transformation $\mathbb{S}(1.11$, according to the formulas

$$
\left[\begin{array}{c}
\mathbf{v}_{m}  \tag{1.12}\\
\mathbf{u}_{m}
\end{array}\right]=\mathbb{S}\left[\begin{array}{c}
\dot{x}_{m} \\
f_{m}
\end{array}\right], \quad\left[\begin{array}{c}
\mathbf{u}_{s} \\
\mathbf{v}_{s}
\end{array}\right]=\mathbb{S}\left[\begin{array}{c}
\dot{x}_{s} \\
f_{s}
\end{array}\right] .
$$

In the new variables, according to the formulae (1.10), master-slave interconnection obeys the rule

$$
\begin{equation*}
\mathbf{u}_{s}(t)=\mathbf{u}_{m}(t-T), \quad \mathbf{v}_{m}(t)=\mathbf{v}_{s}(t-T) . \tag{1.13}
\end{equation*}
$$

The wave-based communication channel (1.13), (1.12) is passive (see [22, 31]). Indeed, assuming $f_{j}(t)=0$ and $\dot{x}_{j}(t)=0$ for all $t \leqslant 0(j \in\{m, s\})$, we have

$$
\begin{aligned}
& E(t)=\int_{0}^{t}\left(\dot{x}_{m}^{T}(\tau) f_{m}(\tau)-\dot{x}_{s}^{T}(\tau) f_{s}(\tau)\right) d \tau=\frac{1}{2} \int_{0}^{t}\left(\left|\mathbf{v}_{m}(\tau)\right|^{2}-\left|\mathbf{u}_{m}^{T}(\tau)\right|-\left|\mathbf{u}_{s}(\tau)\right|^{2}+\left|\mathbf{v}_{s}^{T}(\tau)\right|\right) d \tau= \\
& \frac{1}{2} \int_{0}^{t}\left(\left|\mathbf{v}_{m}(\tau)\right|^{2}-\left|\mathbf{v}_{s}^{T}(\tau-T)\right|-\left|\mathbf{v}_{m}(\tau-T)\right|^{2}+\left|\mathbf{v}_{s}^{T}(\tau)\right|\right) d \tau= \\
& \frac{1}{2}\left(\int_{0}^{t}\left(\left|\mathbf{v}_{m}(\tau)\right|^{2}+\left|\mathbf{v}_{s}^{T}(\tau)\right|\right) d \tau-\int_{-T}^{t-T}\left(\left|\mathbf{v}_{s}^{T}(\tau)\right|+\left|\mathbf{v}_{m}(\tau)\right|^{2}\right) d \tau\right)=\frac{1}{2} \int_{t-T}^{t}\left(\left|\mathbf{v}_{m}(\tau)\right|^{2}+\left|\mathbf{v}_{s}(\tau)\right|^{2}\right) d \tau \geqslant 0 .
\end{aligned}
$$

This reveals passivity of the communication channel, that ensures passivity of the overall teleoperator system as an interconnection of passive subsystems.

To establish connection of the wave-variable approach with the scattering-based scheme (1.5) proposed in [2], the scattering operator $\mathbb{S}(1.11)$ of the form $\mathbb{S}=\frac{1}{\sqrt{2}}\left[\begin{array}{rr}1 & 1 \\ 1 & -1\end{array}\right]$ provides an equivalent control law (for $b=1$ ) in terms of the wave-variables (1.12), (1.13).

Regardless the advantages of the scattering (or wave) variables approach that ensures robust and delay independent stabilization of the interconnected systems, the aforesaid method is
limited to the passive systems only. As a consequence, it imposes strict or sometimes excessive restrictions on the design of subsystems involved into the interconnection. This especially evident in the stabilization problem of robot-environment interactions where the environment may behave in a non-passive way. Similar problems occur in the teleoperator systems. In this concern, the next subsection surveys some examples of passivity violation that are reflected in the literature.

### 1.1.3 Examples of non-passive behavior

The assumption of passivity of the human operator can be traced back to the work of N. Hogan [20], where it was experimentally demonstrated that the behavior of a human operator's hand is passive when interacting with a passive manipulandum. More precisely, it was established in [20, 30] that, if a human operator holds a passive manipulandum at a fixed position in a workspace, then the natural reaction force of the human hand to small perturbations applied to manipulandum appears as if generated by a spatial spring with symmetrical apparent stiffness. Such a response can be considered as generated by a potential force field and, therefore, is passive. However, there are some evidence that the conventional assumption of passivity of the human operator behavior may not hold in all situations. For example, it was shown in [13] that, under some conditions, the behavior of the human operator can be non-passive; more precisely, the human behavior may exhibit passive or non-passive characteristics depending on the specific task performed by the human arm. In particular, positive energy can be produced when the human operator blocks the disturbance forces, or returns the hand back to the initial position.

On the other hand, numerous examples of non-passive environments can be found in the literature. For example, in [3, 4], a telerehabilitation problem is addressed where the patient plays the role of the human operator, while the therapist does the one of the environment. In this case, the passivity/nonpassivity of the environment depends on the task performed by the therapist. In the case of assistive therapy, the therapist (environment) applies assistive forces to help patient perform a task, thus necessarily producing energy, which makes this subsystem non-passive. Another example of non-passive environments can be found in the paper [26], which deals with analysis of the wheel slippage problem in mobile robotics. In this work, the slippage is modeled as the environment termination for the slave wheeled mobile robot. Authors demonstrated that fluctuations of the slippage may cause the environment termination to exhibit non-passive behavior; specifically, the slave can be non-passive when the rate of change of slippage is negative.

In addition, the requirement of passivity imposes restrictions on the behavior of the slave
subsystem which, in many cases, do not allow for achieving position tracking. This is due to the fact that the classical passivity property imposes integral constraints on the relationship between velocities and forces, which may be in contradiction with the requirement for the slave device to follow the position and velocity of the master. For example, a substantial number of known passivity-based results are summarized in the survey paper [33]; however, in all these results, only boundedness of trajectories are guaranteed during free motion, while the convergence of velocities to zero and the master/slave positions to each other is guaranteed only when the operator releases the master device (more precisely, under the assumption that the human force is equal to zero). Finally, in multi-master-multi-slave (MMMS) teleoperator systems, where the slave manipulators differ in size, the scaled passivity can be violated by direct physical contact between slaves' arms.

Aforementioned difficulties can be partially overcome by revealing the nature of the scattering transformation. The next subsection goes over the results devoted to the generalized scattering-based methods that are applicable for nonlinear dynamic systems satisfying weaker conditions compared to passivity constraints.

### 1.1.4 Generalization of the scattering-based approach and planar conic systems

In 2006, stronger results on the stabilization of systems' interconnections developed by S. Hirche and co-authors [15,28]. In the papers, authors study a nonlinear (Q, S, R)-dissipative systems of the form

$$
\Sigma:\left\{\begin{array}{l}
\dot{x}=f(x, \eta)  \tag{1.14}\\
y=h(x, \eta)
\end{array}\right.
$$

where $x \in \mathbb{R}^{n}$ is the state, $\eta, y \in \mathbb{R}^{m}$ are the input and the output, respectively, and $f(\cdot, \cdot)$, $h(\cdot, \cdot)$ are locally Lipschitz maps of the corresponding dimensions. A system (1.14) is said to be dissipative with respect to supply rate $w: \mathbb{R}^{m} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ if there exists a storage function $V: \mathbb{R}^{n} \rightarrow \mathbb{R}_{+}$such that the inequality

$$
\begin{equation*}
V\left(x\left(t_{1}\right)\right)-V\left(x\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t_{1}} w(\eta(\tau), y(\tau)) d \tau \tag{1.15}
\end{equation*}
$$

holds along the trajectories of the system (1.14) for any $t_{1} \geq t_{0}$, any initial state $x\left(t_{0}\right)$, and any admissible control input $\eta(\tau), \tau \in\left[t_{0}, t\right)$. The notion of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ ) - dissipativity concerns the systems whose supply rate is a quadratic form of input-output variables, i.e.

$$
w(\eta, y)=\left[\begin{array}{l}
\eta  \tag{1.16}\\
y
\end{array}\right]^{T}\left[\begin{array}{cc}
R & S^{T} \\
S & Q
\end{array}\right]\left[\begin{array}{l}
\eta \\
y
\end{array}\right] .
$$

Papers [15,28] investigates a subclass of the (Q, S, R)-dissipative nonlinear systems. Specifically, an input-feedforward-output-feedback-passive (IF-OFP) system is a (Q, S, R)-dissipative system with the supply rate of the form

$$
\begin{equation*}
w(\eta, y)=2 v \cdot \eta^{T} y-\delta\|\eta\|^{2}-\varepsilon\|y\|^{2} . \tag{1.17}
\end{equation*}
$$

The [QSR] matrix of the quadratic supply rate (1.17) has the structure

$$
[\mathrm{QSR}]=\left[\begin{array}{cc}
R & S^{T}  \tag{1.18}\\
S & Q
\end{array}\right]=\left[\begin{array}{cc}
-\delta \mathbb{I} & v \mathbb{I} \\
v \mathbb{I} & -\varepsilon \mathbb{I}
\end{array}\right] .
$$

As was rightly noted [15], constraints imposed on the IF-OFP systems are less restrictive than in the passive case. Indeed, a specific values of the constant parameters $\varepsilon, \delta$, and $v$ determine well-known classes of systems, for example,

- if $\delta=\varepsilon=0, v=1 / 2$, the system is passive;
- if $\delta=0, \varepsilon>0, v=1 / 2$, the system is output-feedback strictly passive;
- if $\delta>0, \varepsilon=0, v=1 / 2$, the system is input-feedforward strictly passive;
- if $\delta=-\gamma^{2}, \varepsilon=1, v=0$, the system is finite gain $\mathcal{L}_{2}$ stable.

In addition, authors have established connection of the introduced class of IF-OFP systems with the notion of conicity introduced by G. Zames [45].

For the stabilization of a networked control system composed by the IF-OFP subsystems, the works [15,28] elaborate a more general version of the scattering-based approach. Namely, the authors have found out that the scattering transformation can be seen as the product of two matrices which perform rotation and scaling, respectively. Practically, the developed transformation turns a IF-OFP system into the $\mathcal{L}_{2}$ stable system with a finite gain, what have been also demonstrated graphically (see [15]) using analogy with conic systems. As a result, application of the small-gain approach to a IF-OFP system with scatterred input-output signals allows for deriving corresponding stability conditions in the presence of constant communication delays. From this point of view, works [15,28] extend the scattering-based stabilization approach to a more general class of systems in comparison with passive systems.

The next step in the development of the scattering-based approach belongs to the work by I.G. Polushin [35], where he gives a more precise and practically more convenient definition of conic systems, establishes stability conditions for their interconnections, and elaborates a scattering-based stabilization technique. Specifically, a system of the form (1.14) is said to
be interior conic with respect to the cone with center $\varphi_{c} \in \mathbb{R}$ and radius $\varphi_{r} \in(0, \pi / 2)$ (see Figure (1.4) if it is dissipative with the following supply rate

$$
\begin{align*}
w(y, \eta) & =\left[\begin{array}{ll}
y^{T} \eta^{T}
\end{array}\right] W\left(\varphi_{c}, \varphi_{r}\right)\left[\begin{array}{l}
y \\
\eta
\end{array}\right],
\end{align*} \quad \text { where } . ~\left(\begin{array}{cc}
\sin 2 \varphi_{c}  \tag{1.19}\\
W\left(\varphi_{c}, \varphi_{r}\right) & :=\frac{\lambda}{2}\left[\begin{array}{cc}
\left(\cos 2 \varphi_{c}-\cos 2 \varphi_{r}\right) & -\left(\cos 2 \varphi_{c}+\cos 2 \varphi_{r}\right)
\end{array}\right] \otimes I_{m} .
\end{array}\right.
$$

Representations of the supply rate in the forms (1.17) and (1.19) are equivalent, more precisely, for given parameters $\delta, \varepsilon$ and $v$ one can find corresponding center and radius of a cone in the sense of definition (1.19), and vise-versa. However, new representation of the conic systems reveals a geometrical structure of the supply rate, for example, a conic sector for a passive system with supply rate $w(y, \eta)=y^{T} \eta$ has center $\phi_{c}=\pi / 4$ and radius $\phi_{r}=\pi / 4$ (Figure 1.5), similarly, a conic sector of a finite gain $\mathcal{L}_{2}$-stable system with supply rate $w(y, \eta)=\gamma^{2} \cdot \eta^{T} \eta-$ $y^{T} y$ has center $\phi_{c}=0$ and radius $\phi_{r}=\arctan \gamma$ (Figure 1.6).




Figure 1.4: Example of planar Figure 1.5: Conic characteris- Figure 1.6: Conic characterisconic characteristics tics of a passive system tics of a finite $\mathcal{L}_{2}$ - gain stable system

Regarding the scattering-based stabilization approach, the definition of a planar conic system in terms of center and radius allows to illustrate the effect of the scattering transformation on the input-output characteristics of the system. As an example, consider the scattering (or wave) transformation $\mathbb{S}(1.12)$ developed for passive systems [2], which maps the power variables $\dot{x}, F$ into the wave variables $\mathbf{u}, \mathbf{v}$. For simplicity we suppose that the scale factor $b=1$, and then the scattering operator takes the form

$$
\mathbb{S}=\frac{1}{\sqrt{2}}\left[\begin{array}{rr}
1 & 1  \tag{1.20}\\
-1 & 1
\end{array}\right]=\left[\begin{array}{cc}
\cos (\pi / 4) & \sin (\pi / 4) \\
-\sin (\pi / 4) & \cos (\pi / 4)
\end{array}\right]
$$

The obtained matrix in 1.20 is a rotation by the angle of $\Phi=\pi / 4$. Thus, the conventional scattering/wave transformation $\mathbb{S}$ rotates the input-output characteristics of a passive system by the angle of $\Phi=\phi_{r}=\pi / 4$ and thereby turns it into a system with $\mathcal{L}_{2}$-gain less than or equal to one; this is illustrated in Figure 1.7. Stability of interconnections of subsystems with $\mathcal{L}_{2}$-gains less than or equal to one in the presence of constant communication delays then follows from an appropriate version of the small-gain theorem (more precisely, the interconnection is robustly stable if the product of gains $<1$, and is marginally stable if the product of gains $=1$ ). Based


Figure 1.7: Scattering transformation as a rotation in the space of input-output variables [35]
on the nature of the scattering transformation as a rotational operator, the paper [35] proposes a more general transformation that involves both a rotation by an arbitrary angle $\Phi \in(-\pi, \pi]$ with respect to cone's center $\phi_{c}$ (1.19), and change of cone's radius by incorporating gains $\gamma_{s}>0$ into the transformation.

$$
\left[\begin{array}{l}
\mathbf{u}  \tag{1.21}\\
\mathbf{v}
\end{array}\right]:=\mathbb{S}\left(\Phi, \gamma_{s}\right)\left[\begin{array}{l}
\eta \\
y
\end{array}\right],
$$

where $\mathbf{u}, \mathbf{v}$ are generalized scattering/wave variables, and

$$
\mathbb{S}\left(\Phi, \gamma_{s}\right):=\left[\begin{array}{cc}
\gamma_{s}^{-1 / 2} \cos \Phi & \gamma_{s}^{1 / 2} \sin \Phi  \tag{1.22}\\
-\gamma_{s}^{-1 / 2} \sin \Phi & \gamma_{s}^{1 / 2} \cos \Phi
\end{array}\right] \cdot\left[\begin{array}{cc}
\cos \varphi_{c} & -\sin \varphi_{c} \\
\sin \varphi_{c} & \cos \varphi_{c}
\end{array}\right] \otimes \mathbb{I} .
$$

In view of the "graph separation" condition for stability [36, 38], the ability to change the center and the radius of a dynamic conic sector can be used for stabilization of the interconnected systems. Specifically, a feedback interconnection of two planar conic systems $\Sigma_{i} \in \operatorname{Cone}\left(\varphi_{c i}, \varphi_{r i}\right), i=1,2$, is $\mathcal{L}_{2}$-gain stable if the corresponding cones are separated; the latter condition can be written in the form

$$
\varphi_{r 1}+\varphi_{r 2}<\left|\varphi_{c 1}-\varphi_{c}\right| .
$$

Besides, the stabilization problem of conic systems interconnections in the presence of constant delays has been also solved by the usage of the scattering-based algorithms elaborated in [35] together with small-gain approach.

Therefore, the approaches developed in [15, 28, 35] extend the scattering-based stabilization techniques to the class of planar conic (IF-OFP) nonlinear systems, and established stronger stability results. Moreover, the definition of a planar conic systems in terms of center and radius introduced in [35] gives a new look at the notion of conicity and reveals the way for expanding it to the non-planar case developed in the present research.

### 1.1.5 Complex interconnection of the ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems

The paper [29] considers the stabilization problem of complex systems interconnections without delays. In particular, the authors have studied a linear interconnection of dissipative subsystems and derived a matrix condition (see [29, Theorem 1]) guaranteeing both input-output stability and Lyapunov stability. Specifically, let a linear interconnection of $N(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative systems with supply rates $w_{i}\left(\eta_{i}, y_{i}\right)$ of the form (1.16) be determined by the equalities

$$
\begin{equation*}
\eta_{i}=\delta_{i}-\sum_{j=1}^{N} \mathcal{A}_{i j} y_{j} \tag{1.23}
\end{equation*}
$$

where $\eta_{i} \in \mathbb{R}^{m_{i}}$ is the input to the subsystem $i, y_{j} \in \mathbb{R}^{p_{j}}$ is the output of the subsystem $j, \delta_{i}$ is an external input, and $\mathcal{A}_{i j}$ are constant matrices. Denoting $\eta^{T}=\left[\eta_{1}^{T}, \ldots, \eta_{N}^{T}\right]^{T}, y^{T}=\left[y_{1}^{T}, \ldots, y_{N}^{T}\right]^{T}$ and $\delta^{T}=\left[\delta_{1}^{T}, \ldots, \delta_{N}^{T}\right]^{T}$, the interconnection (1.23) can be represented in the form

$$
\begin{equation*}
\eta=\delta-\mathcal{A} y \tag{1.24}
\end{equation*}
$$

where matrix $\mathcal{A}$ consists of the blocks $\mathcal{A}_{i j}$. Using [QSR] $]_{i}$ matrices (1.16) of the quadratic supply rates $w\left(\eta_{i}, y_{i}\right)(i=1, \ldots, N)$, introduce aggregated matrices $Q=\operatorname{diag}\left\{Q_{1}, \ldots, Q_{N}\right\} \in$ $\mathbb{R}^{p \times p}, S=\operatorname{diag}\left\{S_{1}, \ldots, S_{N}\right\} \in \mathbb{R}^{p \times m}$ and $R=\operatorname{diag}\left\{R_{1}, \ldots, R_{N}\right\} \in \mathbb{R}^{m \times m}$, where $m=\sum_{i=1}^{N} m_{i}$ and $p=\sum_{i=1}^{N} p_{i}$. As a result, the stability [29, Theorem 1] of the linearly interconnected system (1.24) with respect to the input $\delta$ and output $y$ is ensured by the positive definiteness of the matrix $\widehat{Q}$, where

$$
\widehat{Q}=S \mathcal{A}+\mathcal{A}^{T} S^{T}-\mathcal{A}^{T} R \mathcal{A}-Q
$$

Authors have adapted the obtained sufficient stability condition to the interconnections of passive, conic and finite gain systems without time delays.

A popular technique for stabilization of systems interconnection in the presence of constant time delay is the small-gain approach [1,44,45]. This approach deals with the interconnection of finite gain systems and ensures delay-independent stability [15, 32, 40]. Network version
of the small-gain approach is represented in [40, Section 8.3] and uses results obtained in [9]. These works consider a feedback interconnection of finite gain $\mathcal{L}_{2}$ stable subsystems with the gains not exceeding $\gamma_{i}(i \in\{1, \ldots, N\}$, where $N$ is the number of subsystems). The interconnection is determined by the network graph that is described by an adjacency matrix $\mathcal{A}$. The main result claims that a networked control system is finite gain $\mathcal{L}_{2}$ stable if the gain matrix $\Gamma:=\operatorname{diag}\left\{\gamma_{1}^{2}, \ldots, \gamma_{N}^{2}\right\} \cdot \mathcal{A}$ with nonnegative elements has the spectral radius less than or equal to 1 . Thus, this study have established the sufficient stability criterion for a feedback interconnection of finite gain $\mathcal{L}_{2}$ stable systems in the presence of constant time delays.

Small-gain approach together with the scattering-based stabilization technique allows for achieving delay-independent stability of interconnections of passive systems [5,22,33] and planar conic (or IF-OFP) systems [15, 28, 35]. Application of the scattering transformation turns a passive (or planar conic) subsystem into a finite gain $\mathcal{L}_{2}$ stable subsystem, which ensures stability of the interconnected system if the product of the subsystems' gains is less than 1. Moreover, this technique allows for generalization to the case of time-variable delays [5, 15, 33]. Modified version of the scattering transformation proposed in [5, 15, 27] incorporates time-dependent gains for scaling the scattering variables transmitted through the communication channel in the presence of time variable delays. The mentioned results have been mainly developed for teleoperation systems, where the delayed interconnection of only two subsystems (master and slave) must be stabilized. For more general networks of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems, as mentioned above, the input-output stability results has been obtained only for interconnections without time delays.

### 1.1.6 Coupled stability problem

Physical interaction of a robot with an environment affects the robot behavior and may lead to instability even when a simple system contacting a simple environment. Illustrative examples of this phenomenon are provided in [21, Chapter 19], and demonstrate that stable behavior of the isolated subsystems does not guarantee the stability of the interaction.

In 1996 [21, Section 19.1.2], it was proposed to investigate an interaction within a disturbance rejection approach that considers environment's dynamics as an external disturbance forces. This method imposes strong restrictions on bounding the disturbance forces, however, in practice, the environment may generate the forces exceeding the robot's nominal capacity; moreover, environmental forces may depend on robot's position and/or velocity. Therefore, analysis of interaction as external disturbances independent of the robot and the environment dynamics, in general, does not allow for solving the contact stability problem.

Another approach [21, Section 19.1.3] to the coupled stability problem proposes to model
the environment as an uncertain part of the robot, and uses robust control algorithms. Applicability of this approach is restricted to the case where the interaction does not affect the structure of the dynamical model. For example, if the effect of the interconnection changes robot's parameters (e.g. mass of the end-effector), then this approach can be successfully applied. However, if robot's end-effector contacts an elastic objects, it may lead to new type of behavior of the system due to the interaction between the robot inertia and the environment elasticity; as a result, this method does not guarantee the stable contact.

Series of the papers [16-18] presents the so-called impedance control approach which considers the contact between the robot and the environment as a dynamic bidirectional physical phenomenon rather than an exogenous signal/disturbance. For instance, the manipulator interacting with a soft obstacle may deform the latter, while the obstacle slows down the velocity of the robot or even stops it. In other words, robot's behavior is changed during the interaction with the environment. Reaction of the robot to interaction with an environment determines its response to a specific reference trajectory. Dynamic properties of the contact play significant role in the controller design for coupled stability problem. The impedance control approach became the starting point for the development of a port-Hamiltonian passivity-based technique for the contact stability problem [37, Section 3]. The information passed through the interaction ports can be expressed in terms of velocities and forces. In fact, velocity and force at the interaction port depend on known manipulator dynamics and often poorly characterized environment dynamics. The only thing which is independent of the environment dynamics is the dynamic relation between the force and the velocity at the port. It allows for designing the interaction port by means of a power port, and interpreting the interaction of the systems in terms of energy exchange. From this point of view, for stability of interconnection, the controller should regulate the energy exchange. Therefore, conventionally this problem is analyzed and solved within the passivity-based framework [6-8, 19, 21, 37] under the assumption of passivity of the environment.

Nevertheless, the study presented in [6] shows that the passivity-based approach can be partially extended to a limited class of active environments, specifically to those where the environmental behavior can be decomposed into passive dynamics and an active external force independent on the robots/environments states. Further, in the papers [25, 26], authors investigated the slippage phenomenon for wheeled mobile robots remotely controlled by a human operator. In these works, slippage is modelled as the environment termination for the wheeledmobile robot, and its fluctuations may cause passivity violation of the environment termination. Specifically, contact force experiencing by the robot includes a negative damping term. To deal with the problem, authors propose a decomposition of the non-passive contact into two components: passive and active (see [23, 26]), and design a controller which compensates for the
shortage of passivity induced by the slippage. This approach allows to make the contact passive and, therefore, stabilizes the interaction of the wheeled-mobile robot with a soft terrain.

Although the passivity-based approach solves the coupled stability problem for passive systems, an application of the scattering approach together with the graph separation stability condition allows for stabilization of robot-environment interaction even in the case where the behaviors of the environment and/or manipulator are essentially not passive. Corresponding extensions of the conventional passivity-based approach are discussed in [35] for planar conic systems. The paper provides a scattering transformation of the special form that changes the cone of the manipulator in a such way that the graph separation stability condition elaborated specifically for the planar conic systems is satisfied. It results in stability of the overall interconnected system. This approach deserves attention also because it allows to establish stability conditions for interconnections of arbitrary planar conic (or IF-OFP) systems under less conservative assumptions on behavior of the subsystems than the passivity-based methods.

### 1.2 Thesis contribution

In this thesis, the scattering-based stabilization technique has been extended for classes of systems which are more general than planar conic. Specifically, we address the class of (Q, S, R)-dissipative systems [43], which include passive, planar conic (or input-feedforward-output-feedback-passive, IF-OFP) [15, 28, 35, 45], finite gain $\mathcal{L}_{2}$ stable systems, and many others. In contrast with the planar conicity, $(\mathrm{Q}, \mathrm{S}, \mathrm{R})-$ dissipativity does not require a system to have equal number of inputs and outputs, and does not impose uniform constraints on every input-output pair in the supply rate. For stability analysis, the only condition that must be satisfied is that the number of nonnegative eigenvalues of the [QSR] matrix in the quadratic supply rate coincides with the dimension of system's input. We introduce a notion of non-planar conicity which allows for parameterization of the quadratic supply rate in terms of a central subspace $\Omega$ and a radius $\phi_{r}$. For a given quadratic supply rate, an algorithm for constructing a non-planar cone of a (Q, S, R)-dissipative system is presented. Using the notion of non-planar conicity, we conduct the stability analysis of systems interconnections with and without time delays. In both these cases, the stability conditions are established and expressed in terms of the centers and radii of the cones of interconnected subsystems. Therefore, to achieve these conditions, it might be necessary to transform cone(s) at least one of the subsystem(s). The procedure that transforms a cone is a generalized scattering transformation which is a combination of elementary linear transformations applied to the input-output variables of the system. This procedure can be geometrically interpreted using the concept of conic systems, as two of these transformations rotates the center by means of an orthogonal operator, while the third
one changes the radius using a diagonal scaling matrix. As a result, the developed scatteringbased approach allows for stabilization of interconnections of non-planar conic systems with and without time delays.

It is worth to mention that a scattering transformation that separates the interconnected systems' cones is not unique. On the contrary, there exists a continuum of transformations resolving this issue, for instance, the gap that separates the cones can be chosen arbitrary. This fact can be used for improving performance of control algorithms. In particular, some blocks in the scattering operator for solving the trajectory tracking problem can be predefined. If the goal is to minimize the change of signals, one can select such a scattering transformation that solves stabilization problem and, simultaneously, has the minimum possible deviation from the identity operator, i.e., $\|\mathbb{S}-\mathbb{I}\| \underset{\mathbb{S}}{ }$ min. In other words, non-uniqueness of the scattering transformations creates a basis for implementing additional requirements imposed on these transformations in order to improve performance of the control schemes.

As an application of the generalized scattering-based stabilization approach, a solution of the coupled stability problem is developed. The contact (or coupled) stability problem [24,42] is one of the fundamental problems in robotics, which is conventionally solved using the passivity-based approach. However, the research presented in this thesis addresses the contact stability problem under weaker assumptions on the dynamics of the robot manipulator and the environment. Starting from general models of robot and environment dynamics determined by Euler-Lagrange equations, we construct storage functions for both subsystems, derive corresponding quadratic supply rates, and then estimate the non-planar cones using the developed algorithm. To guarantee stable contact, the graph separation stability condition requires separation between the robot cone and the inverse environment cone. This problem is solved in this thesis by means of the generalized scattering transformation. In addition, we pursue the goal to preserve trajectory tracking performance in free space. This imposes constraints on the structure of the scattering operator, which generally speaking does not allow for direct application of the approach developed in the theoretical part of the work. To solve the coupling stability problem without jeopardizing the trajectory tracking performance in free space, we develop a numerical algorithm for calculation of a scattering transformation from the prescribed class that separates the cones with a prescribed gap. We present a detailed design example where a manipulator controlled by a trajectory tracking control algorithm experiences non-passive contact with an environment which results in coupled instability, while application of the proposed scattering based methods stabilizes the robot-environment interaction.

Further generalization of the scattering-based stabilization approach developed in this thesis deals with complex interconnections of $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative systems with communication delays. In this thesis, we introduce the notion of a finite $\mathcal{L}_{2}$-gain $(A, B)$-stable system and
establish conditions for $\mathcal{L}_{2}$-gain ( $A, B$ )-stability for ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems. The conventional $\mathcal{L}_{2}$ stable systems with the finite gain $\gamma$ satisfy the new definition as a special case for $A=\gamma^{2} \mathbb{I}_{m}$ and $B=\mathbb{I}_{p}$, where $m$ and $p$ are dimensions of system's input and output respectively. We subsequently present a scattering-based design approach for stabilization of complex interconnections with communication delays. A numerical design example and simulation results presented which illustrate the capabilities of the proposed method.

### 1.3 List of publications

The material presented in this work is based on the following publications, including submitted papers:

## Journal articles

1. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel. Scattering-Based Stabilization of Non-Planar Conic Systems. Automatica, Volume 93, July 2018, Pages 1-11.
DOI: https://doi.org/10.1016/j.automatica.2018.03.028.
2. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel. Scattering-based stabilization of complex interconnections of (Q,S,R)-dissipative systems with time delays. The IEEE Control Systems Letters. (Submission number: 18-0351), 2018.
3. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel. Stabilization of robot-environment interaction through generalized scattering techniques. The IEEE Transactions on Robotics (T-RO). (Submission number: 18-0373), 2018.

## Peer-reviewed conference proceedings

1. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel "A Graph Separation Stability Condition for Non-Planar Conic Systems," 10th IFAC Symposium on Nonlinear Control Systems, Monterey, CA, USA, August 23-25, 2016, pp. 945-950.
DOI: https://doi.org/10.1016/j.ifacol.2016.10.286.
2. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel "Scattering Transformation for Non-Planar Conic Systems," 20th IFAC World Congress, Toulouse, France, 9-14 July 2017, pp. 8808-8813.
DOI: https://doi.org/10.1016/j.ifacol.2017.08.819.

### 1.4 Thesis outline

This thesis is organized as follows:
Chapter 2 presents the theoretical framework for the generalized scattering-based stabilization approach. In this chapter, we introduce the notion of non-planar conicity and prove that any ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative system also belongs to the class of non-planar conic systems. Advantages of the introduced notion are also discussed in comparison with the existing classes of passive and conic (planar conic) systems. Also in this chapter, the problem of stabilization of interconnections of non-planar conic systems with and without time delays is addressed. For interconnections without time delays, we develop the graph separation stability condition in terms of centers and radii of the subsystems' cones. In the case of interconnections with communication delays, the delay-independent stability conditions are derived within the small gain framework. To guarantee the fulfilment of these conditions, the generalized scattering-based methods are proposed. Finally, applications of the developed scattering-based stabilization technique to the problems of stable robot-environment interaction and bilateral teleoperation with multiple heterogeneous communication delays are briefly discussed.

In Chapter 3, we investigate the coupled stability problem in robotics and propose its solution using the generalized scattering-based techniques. In the beginning of the chapter, the necessary theoretical background is provided, including results on stabilizing systems interconnection without delays developed in Chapter 2. However, we found that direct application of the theory developed in Chapter 2 to contact stability problem results in a scattering-based controller which interferes with the manipulator's tracking performance in free space. To avoid the negative effect on the trajectory tracking performance, we restrict the admissible set of scattering transformations to those that do not result in such interference. Subsequently, we develop a constrained optimization based numerical algorithm for calculating the parameters of the scattering transformation that guarantees coupled stability through graph separation condition while at the same time does not interfere with the free space tracking. A detailed design example is presented, which begins with calculation of supply rates for the manipulator controlled by a tracking control law and a non-passive environment. It is shown that the graph separation stability condition is not satisfied, which is confirmed by simulation results which demonstrate contact instability. Subsequently, a scattering transformation is designed that stabilizes the robot-environment interconnection, which is confirmed by numerical simulations. All steps in the scattering-based controller design are explained in details, and numerical implementation of the algorithm is provided in the MATLAB scripts. Demonstration and analysis of simulation results obtained with and without application of the generalized scattering-based stabilization approach conclude the chapter.

Chapter 4 deals with stabilization of complex interconnections of $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative systems with multiple communication delays. The chapter starts with the studies of properties of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems that play an important role in the problem of stabilization of their interconnections. Specifically, we analyze the relationship between the input's dimension and the number of nonnegative eigenvalues or, equivalently, between the output's dimension and the number of negative eigenvalues of [QSR] matrix in the quadratic supply rate. Another theoretical aspect of the chapter is a novel notion of finite $\mathcal{L}_{2}$-gain $(A, B)$ stability that slightly generalizes the notion of finite gain $\mathcal{L}_{2}$ stable systems. Subsequently, we design a scattering transformation that transforms a given ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative system whose output's dimension coincides with the number of negative eigenvalues of [QSR] matrix in the quadratic supply rate into a finite $\mathcal{L}_{2}$ gain $(A, B)$ stable system. Using on the above described results, we prove the main result of this chapter which contains a design procedure for delay-independent stabilization of complex interconnections of $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative systems with communication delays. At the end of the chapter, a numerical design example and simulation results are presented in support of the theoretical developments.

Chapter 5 summarizes the main results of the thesis, and discusses future research directions.

Appendix $\boldsymbol{A}$ deals with a particular case where the robot dynamics are determined by the Euler-Lagrange equations and locally controlled by the Lyapunov-based algorithm that ensures solution of the trajectory tracking problem in the absence of external forces. As is demonstrated, Euler-Lagrange dynamics together with the Lyapunov-based local controller generate a (Q, S, R)-dissipative system, where the external forces play role of an input, and the full state (robot's position and velocity) is an output. In addition, we prove that inclusion of the parameter adaptation mechanism in the Lyapunov-based local controller does not affect the supply rate derived in the non-adaptive version of this algorithm.

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## Chapter 2

## Scattering-Based Stabilization of Non-Planar Conic Systems

The material presented in this chapter is published in Automatica, vol. 93, 2018, pp. 1-11. Parts of this work have also been published in

- the Proceeding of the 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS), pp. 945-950, Monterey, CA, USA, August 23-25, 2016;
- the Proceeding of the 20th IFAC World Congress, pp. 8808-8813, Toulouse, France, 9-14 July 2017.

Methods for scattering-based stabilization of interconnections of nonlinear systems are developed for the case where the subsystems are non-planar conic. The notion of non-planar conicity is a generalization of the conicity notion to the case where the cone's center is a subspace with dimension greater than one. For a feedback interconnection of non-planar conic systems, a graph separation condition for finite-gain $\mathcal{L}_{2}$-stability is derived in terms of relationship between the maximal singular value of the product of projection operators onto the subsystems' central subspaces and the radii of the corresponding cones. Furthermore, a new generalized scattering transformation is developed that allows for rendering the dynamic characteristics of a non-planar conic system into an arbitrary prescribed cone with compatible dimensions. The new scattering transformation is subsequently applied to the problem of stabilization of interconnections of non-planar conic systems, with and without communication delays. Applications of the developed scattering-based stabilization methods to the problems of stable robot-environment interaction and bilateral teleoperation with multiple heterogeneous communication delays are discussed.

### 2.1 Introduction

Scattering transformation techniques have been used in the theory of electric networks, particularly transmission lines and networks with delays, since the middle of twentieth century [41]. In the control systems area, applications of the scattering transformation can be traced back to work [1] where a similar construction was used to establish relationship between passivity and small-gain theorems. In [2], the scattering transformation was applied to the problem of stabilization of force reflecting teleoperators in the presence of communication delays. The latter work, together with parallel developments presented in [23], have made a very substantial impact on the bilateral teleoperation area, where the scattering-based stabilization is currently among the most popular methods to deal with instabilities caused by force reflection in the presence of communication delays [15, 24, 25, 29, 33, 34]. The stabilizing effect of the scattering transformation is based on the fact that a conventional scattering operator transforms a passive system into a system with $\mathcal{L}_{2}$-gain less than or equal to one [2, Theorem 3.1]. Assuming all involved subsystems are passive, scattering transformations implemented on both sides of a communication channel transform the corresponding subsystems into those with gain less than or equal to one; stability of the overall system then follows from the small-gain arguments.

Extensions of the scattering transformation techniques to the case of interconnections of not necessarily passive systems were recently proposed in [13, 27]. These extensions are based on the observation that the conventional scattering transformation is essentially an operator of rotation by $\pi / 4$ in the space of input-output variables. Introduction of more general scattering operators that include arbitrary rotations and input-output gains results in substantial generalizations of the scattering-based stabilization techniques. In particular, the methods developed in [27] allow for stabilization of interconnections of arbitrary planar conic systems, with and without communication delays. The notion of a conic system was introduced and originally studied in 1960s by G. Zames [43]; extensions to the case of nonlinear conic sectors were subsequently developed in [28, 35]. Conic systems are nonlinear dynamical systems whose input-output behavior belongs to a dynamic cone. The notion of conicity studied in [43] was essentially planar in the sense that the dynamic cones were characterized by two scalar parameters which represent a conic sector on a plane. Even in this planar case, the notion of conicity is fairly general; in particular, it includes different versions of passivity, finite-gain $\mathcal{L}_{2}$-stability, etc., as special cases. The stabilization methods developed in [27] were based on a new generalized version of the scattering transformation which allows for rendering the dynamic input-output characteristics of an arbitrary planar conic system into a prescribed conic sector. Stability of interconnections can consequently be achieved by designing scattering transformation(s) that render the subsystems's cones in such a way that an appropriate stability
condition (i.e., a graph separation condition in the non-delayed case, or a small-gain condition in the presence of communication delays) is satisfied.

The class of planar conic systems, however, is quite limited in certain aspects. One particularly significant limitation is that, with the exception of systems with finite $\mathcal{L}_{2}$-gain, planar conic systems are required to have an equal number of inputs and outputs. The corresponding methods, including scattering-based design, are therefore limited to those systems where the number of inputs matches the number of outputs. Even in the latter case, description of a multi-input-multi-output system's cone in terms of two scalar parameters is typically overly crude; as a result, the methods that use such a parameterization lack flexibility, which in turn leads to limited applicability and analysis/design conservatism. Another substantial limitation of the planar conicity is that a feedback interconnection of two planar conic systems is, generally speaking, not a planar conic system. The latter fact makes it difficult to use the notion of planar conicity for analysis of complex interconnections. All the above, in turn, limits the applicability of the existing scattering-based methods to stabilization of interconnections of general nonlinear systems.

In this chapter, we develop an approach to scattering-based stabilization that removes all the limitations described above. The approach is based on an extension of the conicity notion to non-planar case, and subsequent development of a new generalized scattering transformation applicable to non-planar conic systems. The notion of non-planar conicity is based on an appropriate generalization of the planar conicity to the case where the cone's center is a subspace with dimension that can be greater than one. This generalization is quite substantial; in fact, the class of non-planar conic systems coincides with that of dissipative systems with quadratic supply rates (or ( $Q, S, R$ )-dissipative systems [12]). In particular, for a given quadratic supply rate, the parameters of the corresponding non-planar cone can be calculated using the procedure presented below in Section 2.2.1. For a feedback interconnection of two non-planar conic systems, a graph separation condition for finite-gain $\mathcal{L}_{2}$-stability is derived in terms of relationship between the maximal singular value of the product of projection operators onto the subsystems' central subspaces and the radii of the corresponding cones. Subsequently, a new generalized scattering transformation is developed that allows for rendering the dynamic characteristics of a non-planar conic system into an arbitrary prescribed cone with compatible dimensions. This property of the new scattering transformation, in turn, allows for its effective use in the problem of stabilization of interconnections of non-planar conic systems, with and without communication delays. Applications of the developed scattering-based stabilization methods to the problems of stable robot-environment interaction and bilateral teleoperation with multiple heterogeneous communication delays are also described.

The chapter has the following structure. In Section 2.2, the notion of non-planar conicity
is introduced, and a procedure for calculation of the parameters of the (non-planar) dynamic cone for a dissipative system with a quadratic supply rate is described. In Section 2.3, a graph separation condition for finite-gain $\mathcal{L}_{2}$-stability of interconnection of two non-planar conic systems is presented. In Section 2.4, a new generalized scattering transformation is developed that allows for rendering the input-output dynamics of a non-planar conic system into an arbitrary prescribed cone. The scattering-based stabilization of interconnections of non-planar conic systems in the absence of communication delays is addressed in Section 2.5, application of this method to the problem of stable robot-environment interaction is discussed in Section 2.5.1. In Section 2.6, a scattering-based method for stabilization of non-planar conic systems' interconnections in the presence of multiple heterogeneous communication delays is developed; application of this method to bilateral teleoperation with communication delays is described in Section 2.6.1. Concluding remarks are given in Section 2.7. Preliminary versions of some of the results presented in Sections 2.2, 2.3 were reported in the conference paper [37], while preliminary versions of some of the results in Sections 2.4, 2.5] were presented in [38].

### 2.2 Non-Planar Conicity

Consider a nonlinear system of the form

$$
\Sigma:\left\{\begin{array}{l}
\dot{x}=f(x, \eta)  \tag{2.1}\\
y=h(x, \eta)
\end{array}\right.
$$

where $x \in \mathbb{R}^{n}$ is the state, $\eta \in \mathbb{R}^{m}$ the input, and $y \in \mathbb{R}^{p}$ the output of system (2.1). The functions $f(\cdot, \cdot), h(\cdot, \cdot)$ are locally Lipschitz continuous in their arguments. A system (2.1) is said to be dissipative with respect to supply rate $w: \mathbb{R}^{p} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ if there exists a storage function $V: \mathbb{R}^{n} \rightarrow \mathbb{R}_{+}$such that the inequality

$$
V\left(x\left(t_{1}\right)\right)-V\left(x\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t_{1}} w(y(\tau), \eta(\tau)) d \tau
$$

holds along the trajectories of the system (2.1) for any $t_{1} \geq t_{0}$, any initial state $x\left(t_{0}\right)$, and an arbitrary admissible control input $\eta(t), t \in\left[t_{0}, t_{1}\right)$. In the definition below, $\mathbb{R} / \tilde{\pi}$ denotes the quotient set (i.e., the set of equivalence classes) of $\mathbb{R}$ with respect to equivalence relation $\tilde{\pi}:=$ $\left\{\phi_{1} \sim \phi_{2}\right.$ iff $\left.\phi_{1}-\phi_{2}=k \pi, k \in \mathbb{Z}\right\}$, where $\mathbb{Z}:=\{\ldots,-1,0,1, \ldots\}$ is the set of integer numbers.

Definition 2.1. A system $\Sigma$ of the form (2.1) with $m=p$ is said to be (planar) interior conic with respect to the cone with center $\phi_{c} \in \mathbb{R} / \tilde{\pi}$ and radius $\phi_{r} \in(0, \pi / 2)$ if it is dissipative with
supply rate

$$
w(y, \eta)=\left[\begin{array}{l}
\eta  \tag{2.2}\\
y
\end{array}\right]^{T} W\left(\phi_{c}, \phi_{r}\right)\left[\begin{array}{l}
\eta \\
y
\end{array}\right],
$$

where matrix $W\left(\phi_{c}, \phi_{r}\right)$ is determined by the formula

$$
W\left(\phi_{c}, \phi_{r}\right):=\frac{\lambda}{2} \cdot\left[\begin{array}{cc}
\cos 2 \phi_{c}-\cos 2 \phi_{r} & \sin 2 \phi_{c}  \tag{2.3}\\
\sin 2 \phi_{c} & -\cos 2 \phi_{c}-\cos 2 \phi_{r}
\end{array}\right] \otimes \mathbb{I}_{m},
$$

where $\otimes$ denotes the Kronecker product, and $\lambda>0$.
Representation (2.2), (2.3) of the supply rate of a conic system in terms of the cone's center $\phi_{c}$ and its radius $\phi_{r}$ is from [27]. There also exists a somewhat more conventional representation of the supply rate in terms of the cone's boundaries $a, b \in \mathbb{R} \cup\{ \pm \infty\}$, $a \leq b$, which was used for example in the classical work [43]. Specifically, a system of the form (2.1) with $m=p$ is interior $[a, b]$-conic if it is dissipative with respect to the supply rate

$$
\begin{equation*}
w(y, \eta):=(b \eta-y)^{T}(y-a \eta) . \tag{2.4}
\end{equation*}
$$

For any finite parameters $a, b \in \mathbb{R}, a \leq b$, the supply rate (2.4) can be represented in the form (2.2), (2.3) by choosing

$$
\begin{aligned}
\lambda & =\sqrt{\left(a^{2}+1\right)\left(b^{2}+1\right)} \\
\phi_{c} & =\frac{1}{2} \arg (1-a b+j(a+b)), \quad \text { and } \\
\phi_{r} & =\frac{1}{2} \cos ^{-1}\left(\frac{1+a b}{\sqrt{\left(a^{2}+1\right)\left(b^{2}+1\right)}}\right)
\end{aligned}
$$

The matrix $W\left(\phi_{c}, \phi_{r}\right)$ of the quadratic supply rate (2.2) can also be written in the form

$$
\begin{equation*}
W\left(\phi_{c}, \phi_{r}\right):=\lambda \cdot\left[l_{c} l_{c}^{T}-\cos ^{2} \phi_{r} \mathbb{I}_{2}\right] \otimes \mathbb{I}_{m}, \tag{2.5}
\end{equation*}
$$

where $l_{c}:=\left[\begin{array}{ll}\cos \phi_{c} \sin \phi_{c}\end{array}\right]^{T}$ is the unit vector that belongs to a center of the cone. Representation (2.5) is of special interest for our work as it allows for an extension to a non-planar case, as follows. For simplicity of exposition, consider the case $\lambda=1$ and $m=1$. Combining (2.2) and (2.5), one can write the supply rate in the form

$$
w(y, \eta)=\left[\begin{array}{ll}
\eta^{T} & y^{T}
\end{array}\right] l_{c} l_{c}^{T}\left[\begin{array}{l}
\eta  \tag{2.6}\\
y
\end{array}\right]-\cos ^{2} \phi_{r} \cdot\left|\begin{array}{l}
\eta \\
y
\end{array}\right|^{2}
$$

Since $l_{c}$ is the unit vector that belongs to the center of the cone, the scalar product of $l_{c}$ and $\left[\eta^{T} y^{T}\right]^{T}$ represents the length of the projection of the input-output vector $\left[\eta^{T} y^{T}\right]^{T}$ onto the
center of the cone. Based on these considerations, the supply rate 2.6 can be equivalently written in the form

$$
w(y, \eta)=\left[\begin{array}{l}
\eta  \tag{2.7}\\
y
\end{array}\right]^{T}\left[\Pi_{c}^{T} \Pi_{c}-\cos ^{2} \phi_{r} \mathbb{I}_{2}\right]\left[\begin{array}{l}
\eta \\
y
\end{array}\right]
$$

where

$$
\Pi_{c}:=\left[\begin{array}{cc}
\cos ^{2} \phi_{c} & \sin \phi_{c} \cos \phi_{c} \\
\sin \phi_{c} \cos \phi_{c} & \sin ^{2} \phi_{c}
\end{array}\right]
$$

is the matrix of projection onto the center of the cone. Since projection matrices are symmetric $\left(\Pi_{c}^{T}=\Pi_{c}\right)$ and idempotent $\left(\Pi_{c}^{2}=\Pi_{c}\right)$, we see that $\Pi_{c}^{T} \Pi_{c}=\Pi_{c}$, and the supply rate (2.7) can be rewritten in the form

$$
w(y, \eta)=\left[\begin{array}{l}
\eta  \tag{2.8}\\
y
\end{array}\right]^{T}\left[\Pi_{c}-\cos ^{2} \phi_{r} \mathbb{I}_{2}\right]\left[\begin{array}{l}
\eta \\
y
\end{array}\right] .
$$

The importance of the formula (2.8) stems from the fact that it allows for generalization in at least two important directions: i) the case where the conic sector is no longer planar, in particular, where the dimension of the center of the conic sector can be higher than one, and ii) where the dimensions of the input and the output are not equal.

Consider now a system (2.1) where, generally speaking, $m \neq p$. The following is a generalization of the notion of interior conicity to a non-planar case.

Definition 2.2. Given a subspace $\Omega \subset \mathbb{R}^{m+p}, \operatorname{dim} \Omega=l \in\{0, \ldots, m+p\}$, and $\phi_{r} \in[0, \pi / 2)$, a system $\Sigma$ of the form (2.1) is said to be interior conic with respect to the cone with centre $\Omega$ and radius $\phi_{r}\left(\Sigma \in \operatorname{Int}\left(\Omega, \phi_{r}\right)\right)$ if it is dissipative with supply rate

$$
w(y, \eta)=\left[\begin{array}{l}
\eta  \tag{2.9}\\
y
\end{array}\right]^{T} W\left(\Omega, \phi_{r}\right)\left[\begin{array}{l}
\eta \\
y
\end{array}\right],
$$

where matrix $W\left(\Omega, \phi_{r}\right)$ has the form

$$
\begin{equation*}
W\left(\Omega, \phi_{r}\right):=\Pi_{\Omega}-\cos ^{2} \phi_{r} \mathbb{I}_{m+p} \tag{2.10}
\end{equation*}
$$

where $\Pi_{\Omega}$ is the matrix of projection onto the subspace $\Omega$.
Remark To illustrate a substantially more general nature of the notion of conicity given by Definition 2.2 in comparison with that of Definition 2.1, let us begin by pointing out that the notion of planar conicity (Definition 2.1) is not well-defined if $m \neq p$. Moreover, even in the case where $m=p$, the notion of conicity given by Definition 2.2 allows for much higher flexibility in the choice of the central subspace in comparison with that of Definition 2.1 . Indeed, suppose a system $\Sigma$ is (planar) interior conic in the sense of Definition 2.1 with center
$\phi_{c} \in(-\pi / 2, \pi / 2]$. In this case, it is also interior conic in the sense of Definition 2.2 with a central subspace

$$
\Omega=\operatorname{span}\left\{\left[\begin{array}{ll}
\cos \phi_{c} & \sin \phi_{c} \tag{2.11}
\end{array}\right]^{T} \otimes \mathbb{I}_{m}\right\} .
$$

For any $\phi_{c} \in(-\pi / 2, \pi / 2]$, the central subspace (2.11) is an $m$-dimensional subspace in the $2 m$-dimensional space of input-output signals. In general, a set of $m$-dimensional subspaces in an $n$-dimensional linear space forms a Grassmanian manifold $\mathbf{G r}(m, n)$, which has a dimension $m(n-m)$. In our case, the set of all possible $m$-dimensional subspaces in the $2 m$-dimensional space of input-output signals is a manifold with dimension $m^{2}$. For planar conic systems (Definition 2.1), however, the set of all allowed central subspaces (2.11) is parameterized by a single scalar parameter $\phi_{c} \in(-\pi / 2, \pi / 2]$ and, therefore, forms a one-dimensional submanifold in the $m^{2}$-dimensional manifold of all central subspaces allowed by Definition 2.2. Thus, even in the case $m=p, m>1$, Definition 2.2 allows for fundamentally higher flexibility in the choice of central subspace in comparison with the planar case (Definition 2.1) and, consequently, potentially much more precise description of system's dynamics.

### 2.2.1 Relationship to ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipativity

To further illustrate applicability and usefulness of the above defined notion of non-planar conicity (Definition 2.2), it is beneficial to explore its relationship with a well-known notion of ( $Q, S, R$ )-dissipativity, see [12]. Given matrices $Q=Q^{T} \in \mathbb{R}^{p \times p}, R=R^{T} \in \mathbb{R}^{m \times m}$, and $S \in \mathbb{R}^{p \times m}$, a system of the form (2.1) is said to be ( $Q, S, R$ )-dissipative if it is dissipative with supply rate

$$
w(y, \eta)=y^{T} Q y+2 y^{T} S \eta+\eta^{T} R \eta .
$$

An interior conic system is obviously ( $Q, S, R$ )-dissipative. Conversely, if a system (2.1) is ( $Q, S, R$ )-dissipative, it is also interior conic in the sense of Definition 2.2. To show this, first denote

$$
[\mathrm{QSR}]:=\left[\begin{array}{cc}
R & S^{T} \\
S & Q
\end{array}\right] \in \mathbb{R}^{(m+p) \times(m+p)}
$$

Matrix [QSR] is real symmetric and, therefore, real orthogonal equivalent to a diagonal matrix; specifically,

$$
\begin{equation*}
G^{T} \cdot[\mathrm{QSR}] \cdot G=\operatorname{diag}\left[\mu_{1}, \ldots, \mu_{p+m}\right], \tag{2.12}
\end{equation*}
$$

where $\mu_{1}, \ldots, \mu_{m+p}$ are eigenvalues (all real) of [QSR] written in an arbitrary prescribed order, and $G$ is a real orthogonal matrix such that $i$-th column of $G$ is an eigenvector of [QSR] corresponding to $\mu_{i}, i=1, \ldots, m+p$. Let $\lambda(Q S R):=\left\{\mu_{1}, \ldots, \mu_{m+p}\right\}$ denote the set of eigenvalues of [QSR], $\lambda^{-}(Q S R) \subset \lambda(Q S R)$ denote the set of strictly negative $(<0)$ eigenvalues of [QSR], and
$\lambda^{+}(Q S R):=\lambda(Q S R) \backslash \lambda^{-}(Q S R)$ be the set of nonnegative $(\geq 0)$ eigenvalues of [QSR]. Let us also introduce the following notation

$$
\begin{align*}
l & :=\operatorname{card}\left\{\lambda^{+}(Q S R)\right\},  \tag{2.13}\\
\mu^{-} & :=\min \left\{\left|\mu_{i}\right|: \mu_{i} \in \lambda^{-}(Q S R)\right\},  \tag{2.14}\\
\mu^{+} & :=\max \left\{\left|\mu_{i}\right|: \mu_{i} \in \lambda^{+}(Q S R)\right\} . \tag{2.15}
\end{align*}
$$

Expression (2.13) defines $l \in\{0, \ldots, m+p\}$ as the number of nonnegative eigenvalues of [QSR]. The value of $\mu^{-}$is well-defined by expression (2.14) if $\lambda^{-}(Q S R) \neq \emptyset$ (equivalently, if $l<m+p$ ). Similarly, $\mu^{+}$is well-defined by expression (2.15) if $\lambda^{+}(Q S R) \neq \emptyset$ (equivalently, if $l>0$ ). The following statement is valid.

Lemma 2.2.1. Suppose the system (2.1) is $(Q, S, R)$-dissipative. Then it is interior conic in the sense of Definition 2.2 with center $\Omega \subset \mathbb{R}^{m+p}$, $\operatorname{dim} \Omega=l$, and radius $\phi_{r} \in[0, \pi / 2)$. Specifically, $\Omega:=\operatorname{span}\left\{g_{1}^{+}, \ldots, g_{l}^{+}\right\}$is the subspace spanned by those eigenvectors $g_{1}^{+}, \ldots, g_{l}^{+}$of matrix [QSR] that correspond to its nonnegative eigenvalues $\mu_{i} \in \lambda^{+}(Q S R)$. If $0<l<m+p$, then

$$
\begin{equation*}
\phi_{r}:=\tan ^{-1}\left(\sqrt{\mu^{+} / \mu^{-}}\right) . \tag{2.16}
\end{equation*}
$$

Otherwise (i.e., if $l=0$ or $l=m+p$ ), radius $\phi_{r} \in(0, \pi / 2)$ can be chosen arbitrarily.

Proof Suppose the system (2.1) is ( $Q, S, R$ )-dissipative with a storage function $V$. It is straightforward to check that, in this case, the system is also interior conic with central subspace $\Omega \subset \mathbb{R}^{m+p}$ and radius $\phi_{r} \in[0, \pi / 2)$ if there exists $\varepsilon>0$ such that the matrix

$$
\begin{equation*}
\Delta:=\Pi_{\Omega}-\cos ^{2} \phi_{r} \mathbb{I}-\varepsilon \cdot[\mathrm{QSR}] \tag{2.17}
\end{equation*}
$$

is non-negative definite $(\Delta \geq 0)$. Without loss of generality, let the eigenvalues in (2.12) be ordered such that $\mu_{1} \leq \ldots \leq \mu_{m+p}$. Consider first the case where $0<l<m+p$. In this case, the projection matrix $\Pi_{\Omega}$ has a form

$$
\Pi_{\Omega}=G \cdot\left[\begin{array}{cc}
\mathbb{O} & \mathbb{O} \\
\mathbb{O} & \mathbb{I}_{l}
\end{array}\right] \cdot G^{T}
$$

Consider a matrix $\bar{\Delta}:=G^{T} \Delta G$, where $\Delta$ is defined by 2.17). Taking into account (2.12), one sees that

$$
\bar{\Delta}=\left[\begin{array}{cc}
\mathbb{O} & \mathbb{O} \\
\mathbb{O} & \mathbb{I}_{l}
\end{array}\right]-\cos ^{2} \phi_{r} \mathbb{I}-\varepsilon \cdot \operatorname{diag}\left[\mu_{1}, \ldots, \mu_{p+m}\right]
$$

Thus, the matrix $\bar{\Delta}$ is diagonal; its diagonal elements $\delta_{1}, \ldots, \delta_{m+p}$ are the eigenvalues of $\Delta$. For $\Delta \geq 0$, all eigenvalues must be nonnegative. We have

$$
\begin{array}{rrrr}
\delta_{1}= & -\cos ^{2} \phi_{r}-\varepsilon \mu_{1} & \geq & -\cos ^{2} \phi_{r}+\varepsilon \mu^{-} \\
\vdots & \vdots & \\
\delta_{m+p-l}= & -\cos ^{2} \phi_{r}-\varepsilon \mu_{m+p-l} \geq & -\cos ^{2} \phi_{r}+\varepsilon \mu^{-}
\end{array}
$$

and

$$
\begin{array}{rrr}
\delta_{m+p-l+1}= & \sin ^{2} \phi_{r}-\varepsilon \mu_{m+p-l+1} & \geq \\
\vdots & \vdots & \sin ^{2} \phi_{r}-\varepsilon \mu^{+} \\
\delta_{m+p}= & \sin ^{2} \phi_{r}-\varepsilon \mu_{m+p} \geq & \\
& \sin ^{2} \phi_{r}-\varepsilon \mu^{+}
\end{array}
$$

Choosing $\phi_{r}$ as in (2.16), and $\varepsilon=1 /\left(\mu^{+}+\mu^{-}\right)$, we see that $\delta_{1}, \ldots, \delta_{m+p} \geq 0$, and the matrix $\Delta$ is nonnegative definite. If $l=0$, then $\Pi_{\Omega}=\mathbb{O}$, and

$$
\bar{\Delta}=G^{T} \Delta G=-\cos ^{2} \phi_{r} \mathbb{I}-\varepsilon \cdot \operatorname{diag}\left[\mu_{1}, \ldots, \mu_{p+m}\right]
$$

where all $\mu_{1}, \ldots, \mu_{p+m}<0$. The eigenvalues of $\Delta$ are

$$
\begin{aligned}
\delta_{1} & =-\cos ^{2} \phi_{r}-\varepsilon \mu_{1} & \geq-\cos ^{2} \phi_{r}+\varepsilon \mu^{-}, \\
& \vdots & \vdots \\
\delta_{m+p} & \left.=-\cos ^{2} \phi_{r}-\varepsilon \mu_{m+p}\right] & \geq-\cos ^{2} \phi_{r}+\varepsilon \mu^{-} .
\end{aligned}
$$

Therefore, choosing arbitrary $\varepsilon \geq 1 / \mu^{-}$guarantees that $\Delta$ is nonnegative definite regardless of the choice of $\phi_{r} \in(0, \pi / 2)$. Similarly, if $l=m+p$, we have $\Pi_{\Omega}=\mathbb{I}$, and

$$
\bar{\Delta}=G^{T} \Delta G=\sin ^{2} \phi_{r} \mathbb{I}-\varepsilon \cdot \operatorname{diag}\left[\mu_{1}, \ldots, \mu_{p+m}\right],
$$

where all $\mu_{1}, \ldots, \mu_{p+m} \geq 0$. In this case, the eigenvalues of $\Delta$ satisfy

$$
\begin{array}{rlr}
\delta_{1} & =\sin ^{2} \phi_{r}-\varepsilon \mu_{1} & \geq \sin ^{2} \phi_{r}-\varepsilon \mu^{+} \\
& \vdots & \vdots \\
\delta_{m+p} & =\sin ^{2} \phi_{r}-\varepsilon \mu_{m+p} & \geq \sin ^{2} \phi_{r}-\varepsilon \mu^{+} .
\end{array}
$$

Picking arbitrary $\phi_{r} \in(0, \pi / 2]$ and choosing $\varepsilon>0$ such that $\sin ^{2} \phi_{r} \geq \varepsilon \cdot \mu^{+}$guarantees that $\Delta$ is nonnegative definite. The proof of Lemma 2.2.1 is complete.

### 2.3 Graph Separation Stability Condition

In this section, we present a graph separation condition for finite gain $\mathcal{L}_{2}$-stability of a feedback interconnection of two non-planar conic subsystems. A system of the form (2.1) is said to be finite gain $\mathcal{L}_{2}$-stable if it is dissipative with supply rate $w(y, \eta):=\gamma^{2}|\eta|^{2}-|y|^{2}$, where $\gamma \geq 0$ is the $\mathcal{L}_{2}$-gain, see [39]. To formulate the graph separation condition, it is convenient to use a notion similar to the one of inverse graph in [36]. Informally, we will call a system $\Sigma$ inverse interior conic (with some centre $\Omega$ and radius $\phi_{r}$ ) if the same system with inverse causality (i.e., where $y$ considered an input and $\eta$ an output) is $\operatorname{Int}\left(\Omega, \phi_{r}\right)$. A more formal definition goes as follows. Given a central subspace $\Omega \subset \mathbb{R}^{m+p}, \operatorname{dim} \Omega=m$, let vectors $\omega_{1}, \ldots \omega_{m} \in \mathbb{R}^{m+p}$ form a basis in $\Omega$, i.e., $\operatorname{span}\left\{\omega_{1}, \ldots, \omega_{m}\right\}=\Omega$. Define

$$
\begin{equation*}
\bar{\Omega}:=\operatorname{span}\left\{P_{(m, p)} \omega_{1}, \ldots, P_{(m, p)} \omega_{m}\right\}, \tag{2.18}
\end{equation*}
$$

where $P_{(m, p)} \in \mathbb{R}^{m+p}$ is a permutation matrix of the form

$$
P_{(m, p)}:=\left[\begin{array}{cc}
\mathbb{O} & \mathbb{I}_{p}  \tag{2.19}\\
\mathbb{I}_{m} & \mathbb{O}
\end{array}\right]
$$

A system $\Sigma$ of the form (2.1) is called inverse interior conic with respect to the cone with centre $\Omega$ and radius $\phi_{r}$ (we will use notation $\Sigma \in \overline{\operatorname{Int}}\left(\Omega, \phi_{r}\right)$ ) iff $\Sigma \in \operatorname{Int}\left(\bar{\Omega}, \phi_{r}\right)$, where $\bar{\Omega}$ is defined by (2.18), (2.19). Clearly, left multiplication of the input-output vector by $P_{(m, p)}$ simply change the order of inputs and outputs, $P_{(m, p)} \cdot\left[\begin{array}{ll}\eta^{T} & y^{T}\end{array}\right]^{T}=\left[\begin{array}{ll}y^{T} & \eta^{T}\end{array}\right]^{T}$. Also, projection matrices $\Pi_{\Omega}, \Pi_{\bar{\Omega}}$ are related according to the formula $\Pi_{\bar{\Omega}}=P_{(m, p)} \Pi_{\Omega} P_{(m, p)}^{T}$.

Consider now two subsystems of the form

$$
\Sigma_{i}:\left\{\begin{array}{l}
\dot{x}_{i}=f_{i}\left(x_{i}, \eta_{i}\right),  \tag{2.20}\\
y_{i}=h_{i}\left(x_{i}, \eta_{i}\right)
\end{array} \quad i \in\{1,2\}\right.
$$

where $y_{2}, \eta_{1} \in \mathbb{R}^{m}, y_{1}, \eta_{2} \in \mathbb{R}^{p}$, interconnected according to the formulas

$$
\begin{equation*}
\eta_{1}=y_{2}+\chi_{1}, \quad \eta_{2}=y_{1}+\chi_{2}, \tag{2.21}
\end{equation*}
$$

where $\chi_{1} \in \mathbb{R}^{m}, \chi_{2} \in \mathbb{R}^{p}$ are external inputs, see Figure 2.1. The closed-loop system (2.20), (2.21) has the input $\left[\chi_{1}^{T}, \chi_{2}^{T}\right]^{T} \in \mathbb{R}^{m+p}$, and the output $\left[y_{1}^{T}, y_{2}^{T}\right]^{T} \in \mathbb{R}^{m+p}$. The following result is valid.

Theorem 2.3.1. (Graph separation condition for stability). Consider an interconnected system of the form (2.20), (2.21). Suppose $\Sigma_{1} \in \overline{\operatorname{Int}}\left(\Omega_{1}, \phi_{r 1}\right), \Sigma_{2} \in \operatorname{Int}\left(\Omega_{2}, \phi_{r 2}\right)$, where $\bar{\Omega}_{1} \cap \Omega_{2}=\{0\}$, $\operatorname{dim} \Omega_{1}=m, \operatorname{dim} \Omega_{2}=p$. If the following "graph separation" condition is satisfied

$$
\begin{equation*}
\sigma_{\max }\left(\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}\right)<\cos \left(\phi_{r 1}+\phi_{r 2}\right), \tag{2.22}
\end{equation*}
$$

then the interconnected system (2.20), (2.21) is finite gain $\mathcal{L}_{2}$-stable.


Figure 2.1: Feedback interconnection of $\Sigma_{1}$ and $\Sigma_{2}$.

The proof of Theorem 2.3.1 can be found in the Appendix B (see Proof B.1).

Remark In Theorem 2.3.1, the dimensions of the central subspaces of subsystems $\Sigma_{1}, \Sigma_{2}$ are equal to the corresponding inputs' dimensions, i.e., $\operatorname{dim} \Omega_{1}=\operatorname{dim} \eta_{1}=m, \operatorname{dim} \Omega_{2}=\operatorname{dim} \eta_{2}=$ $p$. This requirement is apparently necessary to exclude meaningless and/or overly conservative cases. Indeed, assumption $\operatorname{dim} \Omega_{1}<\operatorname{dim} \eta_{1}\left(\operatorname{dim} \Omega_{2}<\operatorname{dim} \eta_{2}\right)$ would impose restrictions on instantaneous values of the input $\eta_{1}(t)\left(\eta_{2}(t)\right)$, while any situation where $\operatorname{dim} \Omega_{1}+\operatorname{dim} \Omega_{2}>$ $m+p$ makes graph separation impossible. These issues will be studied in detail in our future research.

### 2.4 Scattering Transformation for Non-Planar Conic Systems

In [27], a general form of the scattering operator for planar conic systems was proposed. The scattering operator defined in [27] is essentially a combination of a planar rotation and an inputoutput scaling. When applied to the input-output pair of a planar conic system, the scattering operator transforms the system's input-output characteristics from the original cone into a new cone. More specifically, given a planar conic system and a target (desired) cone, there exists a scattering operator that renders the input-output characteristics of the system into the target cone. In this section, an operator with similar properties is constructed for a general non-planar conic system.

Consider a system $\Sigma$ of the form (2.1). Suppose this system is interior conic with respect to the cone with centre $\Omega \subset \mathbb{R}^{m+p}, \operatorname{dim} \Omega=m$, and radius $\phi_{r} \in(0, \pi / 2)$ (i.e., $\Sigma \in \operatorname{Int}\left(\Omega, \phi_{r}\right)$ ). Given a desired centre subspace $\Omega_{d} \subset \mathbb{R}^{m+p}$, $\operatorname{dim} \Omega_{d}=m$, and desired radius $\phi_{r d} \in(0, \pi / 2)$, we are looking for a transformation of the input-output variables of the form

$$
\left[\begin{array}{l}
\mathbf{u}  \tag{2.23}\\
\mathbf{v}
\end{array}\right]:=\mathbb{S}\left(\Omega, \Omega_{d}, \phi_{r}, \phi_{r d}\right)\left[\begin{array}{l}
\eta \\
y
\end{array}\right],
$$

such that the transformed system (2.1), (2.23) with new input-output variables ( $\mathbf{u}, \mathbf{v}$ ), $\mathbf{u} \in \mathbb{R}^{m}$, $\mathbf{v} \in \mathbb{R}^{p}$, is interior conic w.r.t. the cone with centre $\Omega_{d}$ and radius $\phi_{r d}$ (we will use notation $\left.\Sigma_{(\mathbf{u}, \mathrm{v})} \in \operatorname{Int}\left(\Omega_{d}, \phi_{r d}\right)\right)$.

A scattering transformation with the above described properties can be constructed using the following process. Let vectors $g_{1}, g_{2}, \ldots, g_{m}$ form an orthonormal basis in $\Omega$. The set of vectors $\left\{g_{1}, g_{2}, \ldots, g_{m}\right\} \in \Omega$ can be augmented with additional vectors $g_{m+1}, \ldots, g_{m+p} \in \Omega^{\perp}$ such that the columns of

$$
G:=\left[\begin{array}{llllll}
g_{1} & \ldots & g_{m} & g_{m+1} & \ldots & g_{m+p} \tag{2.24}
\end{array}\right]
$$

form an orthonormal basis in $\mathbb{R}^{m+p}$. Similarly, a matrix $G_{d}$ can be constructed such that its first $m$ columns form an orthonormal basis in $\Omega_{d}$, while the whole set of its columns forms an orthonormal basis in $\mathbb{R}^{m+p}$. Consider a scattering transformation

$$
\begin{equation*}
\mathbb{S}\left(\Omega, \Omega_{d}, \phi_{r}, \phi_{r d}\right):=G_{d} \cdot \Gamma\left(\phi_{r}, \phi_{r d}\right) \cdot G^{T} \tag{2.25}
\end{equation*}
$$

where

$$
\Gamma\left(\phi_{r}, \phi_{r d}\right):=\left(\frac{\cos \phi_{r_{d}}}{\cos \phi_{r}}\right)^{\alpha} \cdot\left(\frac{\sin \phi_{r_{d}}}{\sin \phi_{r}}\right)^{-\beta} \times\left[\begin{array}{cc}
\left(\frac{\tan \phi_{r d}}{\tan \phi_{r}}\right)^{\alpha} \mathbb{I}_{m} & \mathbb{O}_{m p}  \tag{2.26}\\
\mathbb{O}_{p m} & \left(\frac{\tan \phi_{r d}}{\tan \phi_{r}}\right)^{\beta} \mathbb{I}_{p}
\end{array}\right]
$$

and $\alpha:=-p /(m+p), \beta:=m /(m+p)$. The following lemma is valid.
Lemma 2.4.1. Suppose a system $\Sigma$ of the form (2.1) is such that $\Sigma \in \operatorname{Int}\left(\Omega, \phi_{r}\right)$, where $\Omega \subset$ $\mathbb{R}^{m+p}, \operatorname{dim} \Omega=m$, and $\phi_{r} \in(0, \pi / 2)$. Then the transformed system (2.1), (2.23), (2.25), (2.26) with new input-output variables $(\mathbf{u}, \mathbf{v})$ satisfies $\Sigma_{(\mathbf{u}, \mathbf{v})} \in \operatorname{Int}\left(\Omega_{d}, \phi_{r d}\right)$.

Proof By assumption, $\Sigma \in \operatorname{Int}\left(\Omega, \phi_{r}\right)$, i.e., it is dissipative with supply rate (2.9), (2.10). By construction of the basis $G$ (2.24), one has

$$
\Pi_{\Omega}=G\left[\begin{array}{cc}
\mathbb{I}_{m} & \mathbb{O} \\
\mathbb{O} & \mathbb{O}_{p}
\end{array}\right] G^{T}
$$

and therefore, matrix $W\left(\Omega, \phi_{r}\right)$ defined by 2.10 ) can be written in the form

$$
W\left(\Omega, \phi_{r}\right)=G\left[\begin{array}{cc}
\sin ^{2} \phi_{r} \mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & -\cos ^{2} \phi_{r} \mathbb{I}_{p}
\end{array}\right] G^{T}
$$

Taking into account (2.23), (2.25), we see that

$$
G^{T}\left[\begin{array}{l}
\eta \\
y
\end{array}\right]=\Gamma^{-1}\left(\phi_{r}, \phi_{r d}\right) \cdot G_{d}^{T}\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right],
$$

and, therefore, in the new coordinates $(\mathbf{u}, \mathbf{v})$ the supply rate (2.9) becomes

$$
w(\mathbf{u}, \mathbf{v})=\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right]^{T} G_{d} \Gamma^{-1}\left[\begin{array}{cc}
\sin ^{2} \phi_{r} \mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & -\cos ^{2} \phi_{r} \mathbb{I}_{p}
\end{array}\right] \Gamma^{-1} G_{d}^{T}\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right] .
$$

Taking into account that $\beta-\alpha=1$ and performing calculations, one obtains

$$
\Gamma^{-1}\left[\begin{array}{cc}
\sin ^{2} \phi_{r} \mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & -\cos ^{2} \phi_{r} \mathbb{I}_{p}
\end{array}\right] \Gamma^{-1}=\left[\begin{array}{cc}
\sin ^{2} \phi_{r d} \mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & -\cos ^{2} \phi_{r d} \mathbb{I}_{p}
\end{array}\right]
$$

which implies

$$
\begin{aligned}
w(\mathbf{u}, \mathbf{v}) & =\left[\begin{array}{c}
\mathbf{u} \\
\mathbf{v}
\end{array}\right]^{T} G_{d}\left[\begin{array}{cc}
\sin ^{2} \phi_{r d} \mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & -\cos ^{2} \phi_{r d} \mathbb{I}_{p}
\end{array}\right] G_{d}^{T}\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right] \\
& =\left[\begin{array}{c}
\mathbf{u} \\
\mathbf{v}
\end{array}\right]^{T}\left(G_{d}\left[\begin{array}{cc}
\mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & \mathbb{O}_{p}
\end{array}\right] G_{d}^{T}-\cos ^{2} \phi_{r d} \mathbb{I}_{p+m}\right)\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right] .
\end{aligned}
$$

Finally, by construction of basis $G_{d}$,

$$
w(\mathbf{u}, \mathbf{v})=\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right]^{T}\left(\Pi_{\Omega_{d}}-\cos ^{2} \phi_{r d} \mathbb{I}_{p+m}\right)\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right],
$$

i.e., $\Sigma_{(\mathbf{u}, \mathbf{v})} \in \operatorname{Int}\left(\Omega_{d}, \phi_{r d}\right)$. The proof is complete.

One special case of Lemma 2.4.1, which is of particular interest for the problem of stabilization of nonplanar conic systems interconnections in the presence of communication delays (addressed below in Section 2.6), is where the transformed system $\Sigma_{(\mathbf{u}, \mathbf{v})}$ is finite gain $\mathcal{L}_{2^{-}}$ stable (see Section 2.3). Finite gain $\mathcal{L}_{2}$-stability can be obtained from a more general notion of nonplanar conicity (Definition 2.2) by choosing the central subspace $\Omega$ to coincide with the input space $\mathcal{U}:=\left\{(\eta, \mathbf{0}), \eta \in \mathbb{R}^{m}, \mathbf{0} \in \mathbb{R}^{p}\right\}$. In this case, the projection matrix in (2.10) becomes

$$
\Pi_{\Omega}=\left[\begin{array}{cc}
\mathbb{I}_{m} & \mathbb{O} \\
\mathbb{O} & \mathbb{O}_{p}
\end{array}\right]
$$

and straightforward calculations then show that the system is finite $\mathcal{L}_{2}$-gain stable with $\mathcal{L}_{2^{-}}$ gain $\gamma=\tan \phi_{r}$. To construct a scattering transformation (2.23), 2.25), (2.26) that makes the transformed system finite $\mathcal{L}_{2}$-gain stable, it is therefore sufficient to choose $G_{d}=\mathbb{I}_{m+p}$. The following statement is a special case of Lemma 2.4.1.

Corollary 2.4.2. Suppose a system $\Sigma$ of the form (2.1) is such that $\Sigma \in \operatorname{Int}\left(\Omega, \phi_{r}\right)$, where $\Omega \subset \mathbb{R}^{m+p}, \operatorname{dim} \Omega=m$, and $\phi_{r} \in(0, \pi / 2)$. Given $\gamma_{d}>0$, consider a scattering transformation (2.23), (2.25), (2.26) with $G_{d}=\mathbb{I}_{m+p}$ and $\phi_{r d}=\tan ^{-1} \gamma_{d}$. Then the transformed system (2.1), (2.23), 2.25), (2.26) with input-output variables $(\mathbf{u}, \mathbf{v})$ is finite $\mathcal{L}_{2}$-gain stable with $\mathcal{L}_{2}$-gain less than or equal to $\gamma_{d}$.

Remark The above defined scattering transformation (2.25) depends on the choice of orthonormal bases $G, G_{d}$. Thus, generally speaking, there exist an infinite number of transformations that render dynamics of a given system into a prescribed non-planar cone. The specific choice of bases $G, G_{d}$ may depend on the particular task.

### 2.5 Scattering-based stabilization of systems' interconnection

A scattering-based approach to stabilization of feedback interconnections was introduced in [27], where it was developed for the case of planar conic subsystems. The basic idea of this approach is to design a scattering transformation that transforms the input-output characteristics (specifically, the dynamic cone) of one of the subsystems in a way that guarantees stability of the overall interconnection. In this Section, the scattering-based stabilization method is generalized to the case of non-planar conic systems interconnections. Consider two subsystems $\Sigma_{i}$, $i=1,2$ of the form (2.20), where $y_{2}, \eta_{1} \in \mathbb{R}^{m}, y_{1}, \eta_{2} \in \mathbb{R}^{p}$. Suppose both these subsystems are (non-planar) conic, i.e., $\Sigma_{i} \in \operatorname{Int}\left(\Omega_{i}, \phi_{r i}\right)$, where $\operatorname{dim} \Omega_{1}=m, \operatorname{dim} \Omega_{2}=p$, and $\phi_{r i} \in(0, \pi / 2)$, $i=1,2$. Our goal is to find a scattering transformation $\mathbb{S}_{2 d}: \mathbb{R}^{m+p} \rightarrow \mathbb{R}^{m+p}$ such that the interconnection defined by the constraints

$$
\left[\begin{array}{c}
y_{1}  \tag{2.27}\\
\eta_{1}-\chi_{1}
\end{array}\right]=\mathbb{S}_{2 d}\left[\begin{array}{c}
\eta_{2}-\chi_{2} \\
y_{2}
\end{array}\right]
$$

is finite-gain $\mathcal{L}_{2}$-stable with respect to external disturbances $\left[\chi_{1}^{T}(t), \chi_{2}^{T}(t)\right]^{T} \in \mathbb{R}^{m+p}$.
The block diagram of the interconnected system (2.20), 2.27) is shown in Figure 2.2 As can be seen, the scattering transformation is placed between the subsystems, and essentially plays a role of a controller that stabilizes the interconnection. The interconnection constraints 2.27) can also be rewritten in the form

$$
\left[\begin{array}{c}
\mathbf{u}_{2}  \tag{2.28}\\
\mathbf{v}_{2}
\end{array}\right]:=\left[\begin{array}{c}
y_{1}+\hat{\chi}_{2} \\
\eta_{1}-\chi_{1}+\hat{\chi}_{1}
\end{array}\right]=\mathbb{S}_{2 d}\left[\begin{array}{l}
\eta_{2} \\
y_{2}
\end{array}\right],
$$

where $\left[\hat{\chi}_{2}^{T} \hat{\chi}_{1}^{T}\right]^{T}:=\mathbb{S}_{2 d} \cdot\left[\chi_{2}^{T} 0 \ldots 0\right]^{T}$. The expression (2.28) allows for an equivalent representation of the system's block diagram as shown in Figure 2.3. The block diagram in Figure 2.3 makes it intuitively clear how one can design a scattering transformation $\mathbb{S}_{2 d}$ that stabilizes the interconnection (2.20), (2.27). Specifically, using Lemma 2.4.1, the scattering transformation $\mathbb{S}_{2 d}$ should be designed to render the input-output characteristics of the system $\Sigma_{2}^{(\mathbf{u}, \mathbf{v})}$ (see Figure 2.3) into a (non-planar) cone that satisfies the graph separation stability condition


Figure 2.2: Scattering-based stabilization of interconnections


Figure 2.3: An equivalent representation of the interconnection
given in Theorem 2.3.1. This can be done using the following procedure. By assumption, $\Sigma_{i} \in \operatorname{Int}\left(\Omega_{i}, \phi_{r i}\right)$, where $\operatorname{dim} \Omega_{1}=m, \operatorname{dim} \Omega_{2}=p$, and $\phi_{r i} \in(0, \pi / 2), i=1,2$. Let vectors $\omega_{1}^{1}, \ldots, \omega_{m}^{1}$ form an orthonormal basis in $\Omega_{1}$, and vectors $\omega_{1}^{2}, \ldots, \omega_{p}^{2}$ an orthonormal basis in $\Omega_{2}$. The set of vectors $\left\{\omega_{1}^{1}, \ldots, \omega_{m}^{1}\right\} \in \Omega_{1}$ can be augmented with additional vectors $\omega_{m+1}^{1}, \ldots, \omega_{m+p}^{1} \in \Omega_{1}^{\perp}$ such that the columns of

$$
G_{1}:=\left[\begin{array}{llllll}
\omega_{1}^{1} & \ldots & \omega_{m}^{1} & \omega_{m+1}^{1} & \ldots & \omega_{m+p}^{1} \tag{2.29}
\end{array}\right]
$$

form an orthonormal basis in $\mathbb{R}^{m+p}$. Similarly, the set $\left\{\omega_{1}^{2}, \ldots, \omega_{p}^{2}\right\} \in \Omega_{2}$ can be augmented with additional vectors $\omega_{p+1}^{2}, \ldots, \omega_{p+m}^{2} \in \Omega_{2}^{\perp}$ such that the columns of

$$
G_{2}:=\left[\begin{array}{llllll}
\omega_{1}^{2} & \ldots & \omega_{p}^{2} & \omega_{p+1}^{2} & \ldots & \omega_{p+m}^{2} \tag{2.30}
\end{array}\right]
$$

also form an orthonormal basis in $\mathbb{R}^{m+p}$. Now, let $\bar{\Omega}_{1}$ be defined as in (2.18). Since the permutation matrix $P_{(m, p)}$ is nonsingular and unitary, we see that vectors $P_{(m, p)} \omega_{1}^{1}, \ldots, P_{(m, p)} \omega_{m}^{1}$ form an orthonormal basis in $\bar{\Omega}_{1}$, while the columns of

$$
P_{(m, p)} G_{1}:=\left[\begin{array}{lll}
P_{(m, p)} \omega_{1}^{1} & \ldots & P_{(m, p)} \omega_{m+p}^{1} \tag{2.31}
\end{array}\right]
$$

form an orthonormal basis in $\mathbb{R}^{m+p}$. By construction, $P_{(m, p)} \omega_{m+1}^{1}, \ldots, P_{(m, p)} \omega_{m+p}^{1} \in \bar{\Omega}_{1}^{\perp}$. Choosing

$$
\begin{equation*}
\Omega_{2 d}:=\operatorname{span}\left\{P_{(m, p)} \omega_{m+1}^{1}, \ldots, P_{(m, p)} \omega_{m+p}^{1}\right\} \tag{2.32}
\end{equation*}
$$

one has $\Omega_{2 d}=\bar{\Omega}_{1}^{\perp}$ and therefore

$$
\begin{equation*}
\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2 d}}=\mathbb{O} . \tag{2.33}
\end{equation*}
$$

Now, define $G_{2 d}:=P_{(m, p)} \cdot\left[\begin{array}{llllll}\omega_{m+1}^{1} & \ldots & \omega_{m+p}^{1} & \omega_{1}^{1} & \ldots & \omega_{m}^{1}\end{array}\right]=P_{(m, p)} G_{1} P_{(m, p)}^{T}$, and consider the scattering transformation of the form

$$
\begin{equation*}
\mathbb{S}_{2 d}:=G_{2 d} \cdot \Gamma\left(\phi_{r 2}, \phi_{r 2 d}\right) \cdot G_{2}^{T}, \tag{2.34}
\end{equation*}
$$

where $\Gamma\left(\phi_{r 2}, \phi_{r 2 d}\right)$ is defined according to (2.26), and $\phi_{r 2 d} \in(0, \pi / 2)$ is such that

$$
\begin{equation*}
\phi_{r 1}+\phi_{r 2 d}<\pi / 2 . \tag{2.35}
\end{equation*}
$$

The following statement is valid.
Theorem 2.5.1. Consider an interconnected system (2.20), 2.27), where $\Sigma_{i} \in \operatorname{Int}\left(\Omega_{i}, \phi_{r i}\right)$, $\operatorname{dim} \Omega_{1}=m, \operatorname{dim} \Omega_{2}=p$, and $\phi_{r i} \in(0, \pi / 2), i=1,2$. Suppose $\mathbb{S}_{2 d}: \mathbb{R}^{m+p} \rightarrow \mathbb{R}^{m+p}$ is designed according to (2.34), and $\phi_{r 2 d} \in(0, \pi / 2)$ is chosen such that 2.35) holds. Then the interconnection (2.20), 2.27) is finite-gain $\mathcal{L}_{2}$-stable with respect to external disturbances $\left[\chi_{1}^{T}, \chi_{2}^{T}\right]^{T} \in \mathbb{R}^{m+p}$.

Proof By construction of the scattering transformation $\mathbb{S}_{2 d}$, it follows from Lemma 2.4.1 that $\Sigma_{2}^{(\mathbf{u}, \mathbf{v})} \in \operatorname{Int}\left(\Omega_{2 d}, \phi_{r 2 d}\right)$. Combining (2.33) and (2.35), one concludes that the graph separation condition (2.22) is satisfied for the system (2.20), (2.27). The statement of Theorem 2.5.1 now follows from Theorem 2.3.1.

### 2.5.1 Example: robot-environment interaction

In this subsection, an application of the scattering based control design to the problem of stable robot-environment interaction is motivated and outlined. The problem of stability of robotenvironment interaction, also known as coupled stability, is one of the fundamental problems in robotics [40]. A conventional approach to this problem is based on passivity considerations [7, 14]. Specifically, a necessary and sufficient condition for stability of a robot when coupled with an arbitrary passive environment is that the robot itself is passive. This approach, however, has a number of limitations. First, some environments demonstrate nonpassive behavior (examples of interaction with non-passive environments include robot performing surgery on a beating heart [42], robotic rehabilitation systems [5], mobile robotics
applications [20], haptic interaction with digitally implemented virtual environments [9, 21], and others [16]), in which case the passivity-based approach is not applicable, at least directly. Second, behavior of some passive environments may actually form a small subset of all possible passive behaviors, in which case the passivity-based design can be "arbitrarily conservative" [8]. In addition, passivity requirement imposed on the robot's closed-loop dynamics is frequently in contradiction with the trajectory tracking performance. Specifically, a conventional mechanical environment without inner source of energy is passive with respect to the velocity-force pair. For stable coupling with such an environment, the closed-loop robot's dynamics must be passive with respect to the same velocity-force pair, which is not the case for many existing tracking control algorithms, as demonstrated in the example below. Consider a robot manipulator whose dynamics are described in the task space as follows:

$$
\begin{equation*}
H_{\mathbf{x}}(q) \ddot{\mathbf{x}}+C_{\mathbf{x}}(q, \dot{q}) \dot{\mathbf{x}}+G_{\mathbf{x}}(q)=f_{e n v}+u \tag{2.36}
\end{equation*}
$$

where $q, \dot{q} \in \mathbb{R}^{n}$ are the robot's position and velocity, respectively, represented in the joint space coordinates, $\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}} \in \mathbb{R}^{m}$ are position, velocity, and acceleration, respectively, of the robot's end-effector represented in the task space coordinates, $H_{\mathbf{x}}, C_{\mathbf{x}}(q, \dot{q}), G_{\mathbf{x}}(q)$ are matrices of inertia, Coriolis/centrifugal forces, and a vector of gravitational forces represented in the task-space coordinates, $f_{\text {env }}$ is the environmental contact forces applied to the end-effector, and $u$ is the task-space control input (for more details of the task-space dynamic equations (2.36) and their relationship to joint-space dynamics the reader is referred, for example, to [10, Chapter 4]). Let the manipulator (2.36) be controlled by the following task-space algorithm:

$$
\begin{equation*}
u=H_{\mathbf{x}}(q) \dot{\mathbf{r}}+C_{\mathbf{x}}(q, \dot{q}) \mathbf{r}+G_{\mathbf{x}}(q)-K \sigma+f_{r}, \tag{2.37}
\end{equation*}
$$

where $\sigma:=\dot{\tilde{\mathbf{x}}}+\boldsymbol{\Lambda} \tilde{\mathbf{x}}, \tilde{\mathbf{x}}=\mathbf{x}-\mathbf{x}_{r}, \mathbf{r}:=\dot{\mathbf{x}}-\sigma=\dot{\mathbf{x}}_{r}-\Lambda \cdot \tilde{\mathbf{x}}$, and $\Lambda=\Lambda^{T}>0, K=K^{T}>0$ are matrices of feedback law parameters. Signals $\mathbf{x}_{r}(t), \dot{\mathbf{x}}_{r}(t)$ represent reference position and velocity, respectively, while $f_{r}(t)$ represents a reference force trajectory $\left(f_{r}(t) \equiv 0\right.$ in the case of position tracking). The control algorithm (2.37) is an augmented version of the passivity-based tracking control algorithm (see for example [10, 30]). Substituting the control algorithm (2.37) into the equations of the manipulator dynamics (2.36), the following closed-loop dynamics can be obtained:

$$
\begin{align*}
\dot{\tilde{\mathbf{x}}} & =-\Lambda \tilde{\mathbf{x}}+\sigma  \tag{2.38}\\
\dot{\sigma} & =H_{\mathbf{x}}^{-1}(q)\left[-C_{\mathbf{x}}(q, \dot{q}) \sigma-K \sigma+f_{e n v}+f_{r}\right] . \tag{2.39}
\end{align*}
$$

A well-known fact (which can be checked directly by choosing a storage function of the form $V=\frac{1}{2} \sigma^{T} H_{\mathbf{x}}(q) \sigma$ and calculating its derivative along the trajectories of (2.38, (2.39) assuming $f_{r}=0$ ) is that the closed-loop system is passive with respect to the pair $\left(f_{e n v}, \sigma\right)$ but not
with respect to $\left(f_{\text {env }}, \dot{\mathbf{x}}\right)$; coupled stability of the closed-loop robot (2.38), (2.39) with a passive environment is therefore not guaranteed.

The problem of stability of a controlled manipulator (2.36), (2.37) coupled with an environment which is not necessarily passive but satisfies a more general assumption of (non-planar) conicity can be solved using the scattering based design method developed above. Our approach to this problem is illustrated in Figure 2.4, where a generalized scattering transformation is placed between the robot and the environmental dynamics. As there is direct physical interaction between the robot and the environment, the scattering transformation between them cannot be implemented directly; however, it is implemented indirectly using appropriately designed reference signals $f_{r}, \mathbf{x}_{r}$, and $\dot{\mathbf{x}}_{r}$, as shown in Figure 2.4. The notation used in this figure is as follows: $\mathbf{x}_{d}(t)$ and $\dot{\mathbf{x}}_{d}(t)$ represent the desired position and the desired velocity of the endeffector, respectively, $\mathbf{v}_{x}:=\mathbf{x}-\mathbf{x}_{d}, \mathbf{v}_{\dot{x}}:=\dot{\mathbf{x}}-\dot{\mathbf{x}}_{d}$, and $\mathbf{v}_{f}:=f_{\text {env }}+f_{r}$. To design the scattering


Figure 2.4: Scattering-based stabilization of robot-environment interaction
transformation, the parameters of the system's (non-planar) cone can be determined as follows. First, the dynamical equations (2.38, (2.39) can be rewritten in terms of state variables $\tilde{\mathbf{x}}$ and $\dot{\tilde{\mathbf{x}}}=\sigma-\boldsymbol{\Lambda} \tilde{\mathbf{x}}$. Next, pick a storage function candidate $V=\frac{1}{2} \sigma^{T} H_{\mathbf{x}}(q) \sigma+\frac{1}{2} \tilde{\mathbf{x}}^{T} M \tilde{\mathbf{x}}$, where $M=M^{T}>0$ is an arbitrary parameter matrix. The time derivative of $V$ along the trajectories of (2.38), 2.39) is

$$
\dot{V}=-\sigma K \sigma-\sigma^{T}\left[(1 / 2) \cdot \dot{H}_{\mathbf{x}}(q)-C_{\mathbf{x}}(q, \dot{q})\right] \sigma+\sigma^{T} \mathbf{v}_{f}-\tilde{\mathbf{x}}^{T} M \Lambda \tilde{\mathbf{x}}+\tilde{\mathbf{x}}^{T} M \sigma=\left[\begin{array}{c}
\mathbf{v}_{f} \\
\tilde{\mathbf{x}} \\
\dot{\tilde{\mathbf{x}}}
\end{array}\right]^{T} \mathbf{W}\left[\begin{array}{c}
\mathbf{v}_{f} \\
\tilde{\mathbf{x}} \\
\dot{\tilde{\mathbf{x}}}
\end{array}\right]
$$

where

$$
\mathbf{W}:=\left[\begin{array}{ccc}
0 & \frac{1}{2} \Lambda & \frac{1}{2} \mathbb{I}  \tag{2.40}\\
\frac{1}{2} \Lambda & \frac{1}{2}(\Lambda M-M \Lambda)-\Lambda K \Lambda & \frac{1}{2} M-\Lambda K \\
\frac{1}{2} \mathbb{I} & \frac{1}{2} M-K \Lambda & -K
\end{array}\right]
$$

From expression (2.40), one sees that the system (2.38), (2.39) with input $\mathbf{v}_{f}$ and output $\left[\tilde{\mathbf{x}}^{T}, \dot{\mathbf{x}}^{T}\right]^{T}$ is non-planar conic; the parameters of the system's cone can be calculated using the method described in Section 2.2.1.

Now, suppose the environmental dynamics is dissipative with a quadratic supply rate. Using the method of Section 2.2.1, the environmental (non-planar) cone can be calculated. Given (non-planar) cones of the controlled manipulator and the environment, a scattering transformation $\mathbb{S}$ that stabilizes the robot-environment interaction can be designed using the method described above in Section 2.5. Once designed, such transformation can be implemented using the following line of reasoning. As shown in Figure 2.4 (right), the scattering transformation $\mathbb{S}$ defines the following relationship between signals:

$$
\left[\begin{array}{c}
\mathbf{v}_{f}  \tag{2.41}\\
\tilde{\mathbf{x}} \\
\dot{\tilde{\mathbf{x}}}
\end{array}\right]:=\mathbb{S}^{-1}\left[\begin{array}{c}
f_{e n v} \\
\mathbf{v}_{x} \\
\mathbf{v}_{\dot{x}}
\end{array}\right] .
$$

The signals that form the vector in the right-hand side of 2.41) are all known: $f_{\text {env }}$ is the environmental contact force applied to the end-effector which is assumed to either be measured directly or estimated using an input/disturbance observer [22,32], while the error signals $\mathbf{v}_{\tilde{x}}$ := $\mathbf{x}-\mathbf{x}_{d}, \mathbf{v}_{\dot{x}}:=\dot{\mathbf{x}}-\dot{\mathbf{x}}_{d}$ are directly calculated from the position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$ of the end-effector and known desired trajectory $\mathbf{x}_{d}, \dot{\mathbf{x}}_{d}$. Therefore, the signals $\mathbf{v}_{f}, \tilde{\mathbf{x}}, \dot{\mathbf{x}}$ can be determined from $f_{e n v}, \mathbf{v}_{\tilde{x}}, \mathbf{v}_{\dot{x}}$ according to (2.41). From here, it follows that the scattering transformation can be realized by the following choice of reference signals: $\mathbf{x}_{r}:=\mathbf{x}-\tilde{\mathbf{x}}, \dot{\mathbf{x}}_{r}:=\dot{\mathbf{x}}-\dot{\tilde{\mathbf{x}}}$, and $f_{r}:=\mathbf{v}_{f}-f_{\text {env }}$. Finally, signal $\ddot{\mathbf{x}}_{r}$ (which is required for implementation of the control algorithm (2.37)) can be obtained from $\mathbf{x}_{r}, \dot{\mathbf{x}}_{r}$ using, for example, an exact sliding mode differentiator [3, 18]. A more detailed development of the scattering-based approach to robot-environment interaction outlined here is a topic for future research.

### 2.6 Interconnections with heterogeneous communication delays

The scattering-based stabilization method developed above can also be extended to the case of interconnections of non-planar conic systems in the presence of constant heterogeneous


Figure 2.5: Block diagram of the scattering-based interconnected system with heterogeneous communication delays
communication delays. Consider an interconnected system shown in Figure 2.5. The system consists of subsystems $\Sigma_{1}, \Sigma_{2}$ of the form (2.20) which are interconnected according to the formulas

$$
\begin{align*}
& {\left[\begin{array}{l}
\tilde{\mathbf{u}}_{i} \\
\tilde{\mathbf{v}}_{i}
\end{array}\right]=\mathbb{S}_{i d}\left[\begin{array}{c}
\left(\eta_{i}-\chi_{i}\right) \\
y_{i}
\end{array}\right], \quad i=1,2,}  \tag{2.42}\\
& \tilde{\mathbf{u}}_{2}=\tilde{\mathbf{v}}_{1}^{d}+\delta_{1}, \quad \tilde{\mathbf{u}}_{1}=\tilde{\mathbf{v}}_{2}^{d}+\delta_{2}, \tag{2.43}
\end{align*}
$$

where $\tilde{\mathbf{v}}_{1}^{d}, \tilde{\mathbf{v}}_{2}^{d}$ are signals $\tilde{\mathbf{v}}_{1}, \tilde{\mathbf{v}}_{2}$ subjected to heterogeneous constant communication delays, specifically

$$
\begin{align*}
\tilde{\mathbf{v}}_{1}^{d}(t) & :=\left[\begin{array}{lll}
\tilde{\mathbf{v}}_{1_{1}}^{T}\left(t-T_{1}^{(1)}\right) & \ldots & \tilde{\mathbf{v}}_{1_{p}}^{T}\left(t-T_{1}^{(p)}\right)
\end{array}\right]^{T},  \tag{2.44}\\
\tilde{\mathbf{v}}_{2}^{d}(t) & :=\left[\begin{array}{lll}
\tilde{\mathbf{v}}_{2_{1}}^{T}\left(t-T_{2}^{(1)}\right) & \ldots & \tilde{\mathbf{v}}_{2_{m}}^{T}\left(t-T_{2}^{(m)}\right)
\end{array}\right]^{T}, \tag{2.45}
\end{align*}
$$

$T_{1}^{(1)}, \ldots T_{1}^{(p)} \geq 0$ are communication delays in the communication channels from $\Sigma_{1}$ to $\Sigma_{2}$, and $T_{2}^{(1)}, \ldots T_{2}^{(m)} \geq 0$ are the communication delays in the communication channels from $\Sigma_{2}$ to $\Sigma_{1}$. Signals $\delta_{1}(t) \in \mathbb{R}^{p}, \delta_{2}(t) \in \mathbb{R}^{m}$ represent communication errors in their respective communication channels, and $\chi_{1}(t) \in \mathbb{R}^{m}, \chi_{2}(t) \in \mathbb{R}^{p}$ are external additive disturbances applied to the inputs of $\Sigma_{1}$ and $\Sigma_{2}$, respectively.

Suppose the subsystems $\Sigma_{1}, \Sigma_{2}$ are arbitrary non-planar conic. Our goal is to design scattering transformations $\mathbb{S}_{1 d}, \mathbb{S}_{2 d}$ such that the overall interconnection is finite gain $\mathcal{L}_{2}$-stable with respect to inputs $\delta_{1}, \delta_{2}, \chi_{1}, \chi_{2}$. Note that, since the system under consideration contains communication delays, the definition of finite gain $\mathcal{L}_{2}$-stability used above in Sections 2.3 2.5 is no longer directly applicable. We will instead use the notion of weak finite gain $\mathcal{L}_{2}$-stability defined as follows: system with input $\eta$ and output $y$ is weakly finite gain $\mathcal{L}_{2}$-stable if there
exists $\gamma>0$ such that

$$
\int_{t_{0}}^{t_{1}}|y(s)|^{2} d s \leq \gamma^{2} \int_{t_{0}}^{t_{1}}|\eta(s)|^{2} d s+\beta_{0}
$$

holds for all $t_{1} \geq t_{0}$, where $\beta_{0} \geq 0$ may depend on system's trajectories before $t_{0}$ (i.e., for $t \leq t_{0}$ ). Clearly, the notion of weak finite gain $\mathcal{L}_{2}$-stability includes the finite gain $\mathcal{L}_{2}$-stability as a special case where $\beta_{0}=V\left(t_{0}\right)$.

In order to approach the above described stabilization problem, let us first define a new set of scattering variables $\mathbf{u}_{i}, \mathbf{v}_{i}, i=1,2$, according to the formulas

$$
\left[\begin{array}{c}
\mathbf{u}_{i}  \tag{2.46}\\
\mathbf{v}_{i}
\end{array}\right]=\mathbb{S}_{i d}\left[\begin{array}{l}
\eta_{i} \\
y_{i}
\end{array}\right], \quad i=1,2
$$

Comparing (2.46) with 2.42, one concludes that

$$
\left[\begin{array}{c}
\mathbf{u}_{i}  \tag{2.47}\\
\mathbf{v}_{i}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\mathbf{u}}_{i}+\hat{\chi}_{i 1} \\
\tilde{\mathbf{v}}_{i}+\hat{\chi}_{i 2},
\end{array}\right], \quad i=1,2
$$

where $\left[\begin{array}{cc}\hat{\chi}_{i 1}^{T} & \hat{\chi}_{i 2}^{T},\end{array}\right]^{T}:=\mathbb{S}_{i d}\left[\begin{array}{lll}\chi_{i}^{T} & 0 \ldots 0\end{array}\right]^{T}, i=1,2$. Using (2.47), the interconnection constraints (2.43), (2.44) can be rewritten in the form

$$
\begin{equation*}
\mathbf{u}_{2}=\mathbf{v}_{1}^{d}+\bar{\delta}_{1}, \quad \mathbf{u}_{1}=\mathbf{v}_{2}^{d}+\bar{\delta}_{2} \tag{2.48}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mathbf{v}_{1}^{d}(t):=\left[\begin{array}{lll}
\mathbf{v}_{1_{1}}\left(t-T_{1}^{(1)}\right) & \ldots & \mathbf{v}_{1_{p}}\left(t-T_{1}^{(p)}\right)
\end{array}\right]^{T}, \\
& \mathbf{v}_{2}^{d}(t):=\left[\begin{array}{lll}
\mathbf{v}_{2_{1}}\left(t-T_{2}^{(1)}\right) & \ldots & \mathbf{v}_{2_{m}}\left(t-T_{2}^{(m)}\right)
\end{array}\right]^{T}
\end{aligned}
$$

and $\bar{\delta}_{1}, \bar{\delta}_{2}$ are new external signals which are related to $\delta_{1}, \delta_{2}, \chi_{1}, \chi_{2}$ according to the formulas

$$
\begin{equation*}
\bar{\delta}_{1}:=\delta_{1}+\hat{\chi}_{21}-\hat{\chi}_{11}^{d}, \quad \bar{\delta}_{2}:=\delta_{2}+\hat{\chi}_{12}-\hat{\chi}_{22}^{d}, \tag{2.49}
\end{equation*}
$$

where

$$
\begin{aligned}
& \hat{\chi}_{11}^{d}(t):=\left[\begin{array}{lll}
\hat{\chi}_{11_{1}}\left(t-T_{1}^{(1)}\right) & \ldots & \hat{\chi}_{11_{p}}\left(t-T_{1}^{(p)}\right)
\end{array}\right]^{T}, \quad \text { and } \\
& \hat{\chi}_{22}^{d}(t):=\left[\begin{array}{lll}
\hat{\chi}_{22_{1}}\left(t-T_{2}^{(1)}\right) & \ldots & \hat{\chi}_{22_{m}}\left(t-T_{2}^{(m)}\right)
\end{array}\right]^{T} .
\end{aligned}
$$

Formulas (2.46)-2.49) define an equivalent representation of the interconnection (2.42)-2.44); the corresponding block diagram is shown in Figure 2.6. Weak finite gain $\mathcal{L}_{2}$-stability of the system (2.20), (2.42)-(2.44) with respect to external signals $\delta_{1}, \delta_{2}, \chi_{1}, \chi_{2}$ is equivalent to that of the system (2.20), 2.46, (2.48) with respect to the signals $\bar{\delta}_{1}, \bar{\delta}_{2}$. Weak finite gain $\mathcal{L}_{2}$-stability of the system (2.20), 2.46, (2.48), on the other hand, can be guaranteed by the small gain arguments. Specifically, the following lemma is valid.


Figure 2.6: An equivalent block diagram of the scattering-based interconnection with communication delays

Lemma 2.6.1. Suppose each subsystem $\Sigma_{i}^{(\mathbf{u}, \mathbf{v})}$ defined by (2.20), (2.46) with input $\mathbf{u}_{i}$ and output $\mathbf{v}_{i}, i=1,2$, is finite gain $\mathcal{L}_{2}$-stable, and the corresponding $\mathcal{L}_{2}$-gains satisfy the small gain condition $\gamma_{1} \cdot \gamma_{2}<1$. Then the interconnected system (2.20), (2.46), (2.48) with input $\left[\bar{\delta}_{1}^{T}, \bar{\delta}_{2}^{T}\right]^{T}$ and output $\left[y_{1}^{T}, y_{2}^{T}\right]^{T}$ is weakly finite gain $\mathcal{L}_{2}$-stable.

The proof of Lemma 2.6 .1 follows the same line of reasoning as the proof of Lemma 4 in [27] and can be found in the Appendix B (see Proof B.2).

Lemma 2.6.1 together with Corollary 2.4 .2 of Section 2.4 provide a recipe for scatteringbased stabilization of interconnections of non-planar conic systems with communication delays (2.20), 2.42)-(2.44). Specifically, using Corollary 2.4.2, the scattering transformations $\mathbb{S}_{1 d}, \mathbb{S}_{2 d}$ in (2.46) should be designed to make the subsystems $\Sigma_{1}^{(\mathbf{u}, \mathbf{v})}, \Sigma_{2}^{(\mathbf{u}, \mathbf{v})}$ finite gain $\mathcal{L}_{2}$-stable with gains that satisfy the small-gain condition of Lemma 2.6.1. The weak finite gain $\mathcal{L}_{2}{ }^{-}$ stability of the system (2.20), (2.46), (2.48) (and, therefore, that of the equivalent system (2.20), (2.42)-(2.44)) then follows from Lemma 2.6.1. The design procedure is as follows. Let the systems $\Sigma_{1}, \Sigma_{2}$ be arbitrary nonplanar conic, $\Sigma_{i} \in \operatorname{Int}\left(\Omega_{i}, \phi_{r i}\right), \operatorname{dim} \Omega_{1}=m, \operatorname{dim} \Omega_{2}=p$, and $\phi_{r i} \in(0, \pi / 2), i=1,2$. Similarly to the procedure described in Section 2.5, let vectors $\omega_{1}^{1}, \ldots$, $\omega_{m}^{1}$ form an orthonormal basis in $\Omega_{1}$, vectors $\omega_{1}^{2}, \ldots, \omega_{p}^{2}$ form an orthonormal basis in $\Omega_{2}$, and consider the matrices

$$
\left.\begin{array}{rl}
G_{1} & :=\left[\begin{array}{llllll}
\omega_{1}^{1} & \ldots & \omega_{m}^{1} & \omega_{m+1}^{1} & \ldots & \omega_{m+p}^{1}
\end{array}\right], \\
G_{2} & :=\left[\begin{array}{lllll}
\omega_{1}^{2} & \ldots & \omega_{p}^{2} & \omega_{p+1}^{2} & \ldots
\end{array} \omega_{p+m}^{2}\right.
\end{array}\right],
$$

whose columns form orthonormal bases in $\mathbb{R}^{m+p}$. Consider the scattering transformations of
the form

$$
\begin{equation*}
\mathbb{S}_{1 d}:=\Gamma\left(\phi_{r 1}, \phi_{r d 1}\right) G_{1}^{T}, \mathbb{S}_{2 d}:=\Gamma\left(\phi_{r 2}, \phi_{r d 2}\right) G_{2}^{T} \tag{2.50}
\end{equation*}
$$

where $\Gamma\left(\phi_{r i}, \phi_{r d i}\right), i=1,2$, are defined by $(2.26)$, and the desired radii $\phi_{r d i} \in(0, \pi / 2)$ are chosen to satisfy

$$
\begin{equation*}
\phi_{r d 1}+\phi_{r d 2}<\pi / 2 . \tag{2.51}
\end{equation*}
$$

The following statement summarizes the developed stabilization method.
Theorem 2.6.2. Consider a scattering-based interconnection with delays (2.20), (2.42)-(2.44). Suppose the subsystems $\Sigma_{1}, \Sigma_{2}$ are arbitrary non-planar conic, $\Sigma_{i} \in \operatorname{Int}\left(\Omega_{i}, \phi_{r i}\right), \operatorname{dim} \Omega_{1}=m$, $\operatorname{dim} \Omega_{2}=p, \phi_{r i} \in(0, \pi / 2), i=1,2$. Suppose the scattering transformations $\mathbb{S}_{1 d}, \mathbb{S}_{2 d}$ are designed according to (2.50), where $\phi_{r d 1}, \phi_{r d 2} \in(0, \pi / 2)$ are chosen to satisfy (2.51). Then the interconnection (2.20), (2.42)-(2.44) is weakly finite gain $\mathcal{L}_{2}$-stable.

Proof According to Corollary 2.4.2, scattering transformations (2.50) make the subsystems $\Sigma_{1}^{(\mathbf{u}, \mathbf{v})}, \Sigma_{2}^{(\mathbf{u}, \mathbf{v})}$ finite gain $\mathcal{L}_{2}$-stable with gains $\gamma_{1}:=\tan \left(\phi_{r d 1}\right), \gamma_{2}:=\tan \left(\phi_{r d 2}\right)$, respectively. Since $\phi_{r d 1}, \phi_{r d 2} \in(0, \pi / 2)$, the condition (2.51) is equivalent to $\tan \left(\phi_{r d 1}\right) \cdot \tan \left(\phi_{r d 2}\right)<1$. Lemma 2.6.1 then implies that the system (2.20), (2.46), 2.48) is weakly finite gain $\mathcal{L}_{2}$-stable. The latter is equivalent to the weak finite-gain $\mathcal{L}_{2}$-stability of the interconnection (2.20), (2.42)-(2.44). The proof is complete.

### 2.6.1 Example: bilateral teleoperation with communication delays

In this subsection, an application of the generalized scattering-based stabilization technique to bilateral teleoperators with communication delays is motivated, and one possible design approach is outlined. A teleoperator system consists of master and slave manipulators, where the master is controlled by the human operator's hand, while the slave executes a task in contact with the remote environment. The master and the slave sites exchange position, velocity, and force information, which allows for coordination of the motions of the master and the slave manipulators, as well as for providing the human operator with the haptic information which represents the slave-environment interaction. In the conventional approach to scattering-based master-slave teleoperator systems [2, 24], behaviors of both the environment and the human operator are assumed passive, and the local master and slave control laws must be chosen to guarantee/preserve passivity of the closed-loop master and slave subsystems. Under these conditions, the conventional scattering transformations on both sides of communication channel stabilize the system in the presence of constant communication delays. In practice, behavior of the human operator is not always passive [11], and so are the dynamics of the environment [4,19]. The choice of local master and slave control laws, on the other hand, is drastically
limited by the requirement of passivity imposed on the closed-loop master/slave dynamics. In addition, mechanical systems such as robots are naturally passive with respect to "velocityforce" input-output pair; adding position information to the vector of input-output signals is typically not possible, particularly because passivity requires equal number of inputs and outputs. The absence of position information in the signals transmitted between the master and the slave leads to well-documented problems with position tracking in the scattering-based teleoperation, where only partial results exist [6, 26].

All the above mentioned difficulties, however, can be avoided by using the framework of non-planar conicity and generalized scattering developed in our work. One possible design approach can be outlined as follows. The scattering-based bilateral teleoperator system has a structure shown in Figure 2.7. The leftmost block of the diagram in Figure 2.7 represents


Figure 2.7: Block diagram of the scattering-based bilateral teleoperator system with communication delays
an interconnection of the controlled master manipulator and the human operator dynamics. Similarly to the example in Section 2.5.1, the dynamics of the master manipulator in the task space are described by Euler-Lagrange equations of the form:

$$
\begin{equation*}
H_{\mathbf{m}}\left(q_{m}\right) \ddot{\mathbf{X}}_{m}+C_{\mathbf{m}}\left(q_{m}, \dot{q}_{m}\right) \dot{\mathbf{X}}_{m}+G_{\mathbf{m}}\left(q_{m}\right)=u_{m}-f_{h}+\hat{f_{r}}, \tag{2.52}
\end{equation*}
$$

where $\mathbf{x}_{m}, \dot{\mathbf{x}}_{m}, \ddot{\mathbf{x}}_{m} \in \mathbb{R}^{m}$ are position, velocity, and acceleration, respectively, of the master's end-effector in the task space coordinates, $q_{m}, \dot{q}_{m} \in \mathbb{R}^{n}$ are position and velocity of the master's joints, $H_{\mathbf{m}}\left(q_{m}\right), C_{\mathbf{m}}\left(q_{m}, \dot{q}_{m}\right)$ are matrices of inertia and Coriolis/centrifugal forces of the master manipulator, $G_{\mathbf{m}}\left(q_{m}\right)$ is the vector of potential forces, $u_{m}$ is the control input, $f_{h}$ is the human operator force applied to the end-effector of the master, and $\hat{f}_{r}$ is the force reflection signal. Let the master control algorithm have a simple "damping+gravity compensation" form:

$$
\begin{equation*}
u_{m}=G_{\mathbf{m}}\left(q_{m}\right)-K_{m}^{d} \dot{\mathbf{x}}_{m}, \tag{2.53}
\end{equation*}
$$

where $K_{m}^{d}=\left(K_{m}^{d}\right)^{T}>0$ is a matrix of control (damping) gains. The dynamics of the human operator in contact with the master are assumed to have the form:

$$
\begin{equation*}
H_{\mathbf{h}}\left(q_{h}\right) \ddot{\mathbf{x}}_{m}+\left[C_{\mathbf{h}}\left(q_{h}, \dot{q}_{h}\right)+K_{h}^{d}\right] \dot{\mathbf{x}}_{m}+K_{h}^{s} \mathbf{x}_{m}+f_{h}^{v}=f_{h}, \tag{2.54}
\end{equation*}
$$

where $H_{\mathbf{h}}\left(q_{h}\right), C_{\mathbf{h}}\left(q_{h}, \dot{q}_{h}\right)$ are the matrices of inertia and Coriolis/centrifugal forces experienced by the human arm, which are functions of the human arm configuration $q_{h}$ and its joint velocity $\dot{q}_{h}$. Also, $K_{h}^{d}, K_{h}^{s}$ are symmetric positive definite matrices that represent damping and stiffness of the human hand, and $f_{h}^{v}$ are additional forces that are voluntarily generated by the human muscles. Model (2.54) represents the dynamics of a human hand in the form of Euler-Lagrange equations similar to those used for the dynamics of a manipulator, with gravity forces assumed to be compensated by the human muscular system. The damping and particularly the end-point stiffness terms are known to play substantial and, in many cases, dominant role in the human hand dynamics [17,31]. Substituting (2.53) into (2.52) and combining with (2.54), one gets

$$
\begin{equation*}
\left[H_{\mathbf{m}}\left(q_{m}\right)+H_{\mathbf{h}}\left(q_{h}\right)\right] \ddot{\mathbf{x}}_{m}+\left[C_{\mathbf{m}}\left(q_{m}, \dot{q}_{m}\right)+C_{\mathbf{h}}\left(q_{h}, \dot{q}_{h}\right)\right] \dot{\mathbf{x}}_{m}+\left[K_{m}^{d}+K_{h}^{d}\right] \dot{\mathbf{x}}_{m}+K_{h}^{S} \mathbf{x}_{m}=\hat{f}_{r}-f_{h}^{v} \tag{2.55}
\end{equation*}
$$

The above equation (2.55) describes the dynamics of the controlled master device in contact with the human hand. To determine the parameters of a (non-planar) cone that represents the dynamics 2.55, one can proceed as follows. Consider a storage function candidate

$$
V=\frac{1}{2} \dot{\mathbf{x}}_{m}^{T}\left[H_{\mathbf{m}}\left(q_{m}\right)+H_{\mathbf{h}}\left(q_{h}\right)\right] \dot{\mathbf{x}}_{m}+\frac{1}{2} \mathbf{x}_{m}^{T} \mathbf{M} \mathbf{x}_{m}
$$

where $\mathbf{M}=\mathbf{M}^{T}>0$ is a matrix of constant design parameters. The derivative of $V$ along the trajectories of (2.55) can be calculated as

$$
\dot{V}=\left[\begin{array}{l}
f^{*}  \tag{2.56}\\
\mathbf{x}_{m} \\
\dot{\mathbf{x}}_{m}
\end{array}\right]^{T}\left[\begin{array}{ccc}
\mathbb{O} & 0 & \frac{1}{2} \mathbb{I} \\
\mathbb{O} & \mathbb{O} & \frac{1}{2}\left[\mathbf{M}-K_{h}^{s}\right] \\
\frac{1}{2} \mathbb{I} & \frac{1}{2}\left[\mathbf{M}-K_{h}^{s}\right] & {\left[K_{m}^{d}+K_{h}^{d}\right]}
\end{array}\right]\left[\begin{array}{l}
f^{*} \\
\mathbf{x}_{m} \\
\dot{\mathbf{x}}_{m}
\end{array}\right],
$$

where $f^{*}:=\hat{f}_{r}-f_{h}^{v}$. The parameters of a non-planar cone that represents the closed-loop master-human dynamics (2.55) can be calculated from (2.56) using the procedure described in Section 2.2.1. Note that some of the parameters in the quadratic form 2.56) (specifically, matrices $\mathbf{M}$ and $K_{m}^{d}$ ) are design parameters which can be chosen freely. This already provides the designer with certain control over the parameters of the corresponding dynamic cone even before the scattering-based design is applied. The rightmost block in Figure 2.7represents the dynamics of a controlled slave manipulator in contact with the environment. The stability of such an interconnection can be guaranteed, and the parameters of the corresponding cone can be calculated using the method described in Section 2.5.1. Once the parameters of the dynamic cones of both master-human and slave-environment interconnections are obtained, the
scattering transformations $\mathbb{S}_{m}$ and $\mathbb{S}_{s}$ can be designed using the method presented earlier in this section which is summarized in Theorem 2.6.2. As a result, stability of the bilateral teleoperator system in the presence of multiple communication delays can be guaranteed. Detailed development of this approach is a topic for future research.

### 2.7 Conclusions

In this chapter, the notion of non-planar conicity is introduced, and scattering-based methods for stabilization of interconnections of non-planar conic systems are developed. The notion of non-planar conicity extends the conventional (planar) conicity notion to the case where the dimension of the central subspace is, generally speaking, greater than one. This extension allows for capturing a substantially larger class of systems as compared to the case of planar conicity, including systems with non-equal number of inputs and outputs, and also for more precise description of system's dynamics due to higher flexibility in the choice of the central subspace. A procedure for calculation of the parameters of the (non-planar) cone for an arbitrary system dissipative with a quadratic supply rate is presented. Furthermore, a generalized scattering transformation is developed that allows for rendering the input-output dynamic characteristics of a non-planar conic system into an arbitrary prescribed cone of compatible dimensions. Consequently, scattering-based methods for stabilization of interconnections of non-planar conic systems are developed, including the case of interconnections with multiple heterogeneous communication delays. Applications of the developed scattering-based techniques to coupled stability problem in robotics and bilateral teleoperation with communication delays are outlined. Overall, the approach developed in the chapter offers direct extension of the existing scattering-based stabilization methods to a fundamentally larger class of dynamical systems, while the new scattering transformation makes these methods substantially more flexible and powerful as compared to the existing methods. Future research directions, in particular, include detailed development of performance-oriented design procedures for coupled stability and bilateral teleoperation with communication delays along the lines described in Sections 2.5.1 and 2.6.1, respectively. One particularly important issue is related to the choice of bases $G$ and $G_{d}$ in the expression for the scattering transformation 2.25). There is obviously a continuum of possible choices of $G$ and $G_{d}$ and, while stability is guaranteed for any $G$ and $G_{d}$, however performance would drastically depend on the specific choice of these bases. Since the definition of performance may substantially depend on the particular task, any recipe for the choice of $G, G_{d}$ would also be task-dependent. Development of performance-oriented design procedures for scattering-based stabilization in different applications is an important issue for future research.

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## Chapter 3

## Stabilization of Robot-Environment Interaction Through Generalized Scattering Techniques

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A framework for the coupled stability problem is presented which is based on the nonplanar conic systems formalism and generalized scattering-based stabilization methods. The proposed framework fundamentally generalizes the conventional passivity-based approaches to the coupled stability problem. In particular, it allows for stabilization of not necessarily passive robot-environment interaction where both the manipulator and the environment are general dissipative systems with quadratic supply rates. Also, it can be used in combination with an arbitrary robot's tracking control algorithm and does not affect the trajectory tracking performance in free space. A detailed design example is presented which illustrates the capabilities of the proposed method.

### 3.1 Introduction

The problem of stability of robot environment interaction, also known as contact or coupled stability, is a fundamental problem in robotics [14, 17, 34]. Conventional results in this area are based on the passivity framework [4,6,7,13]. Essentially, in order to form a stable contact with a passive environment, the closed-loop manipulator dynamics must be passive. The passivitybased approach can be partially extended to a limited class of active environments, specifically
to those where the environmental behavior can be decomposed into passive dynamics and an active external force independent of the robot's/environment's states [4]. The approaches based on passivity, however, suffer from at least two limitations. First, examples of non-passive environmental dynamics do exist [3, 18], to which passivity-based design is not applicable, at least directly. Second, even for passive environments, a more detailed description of the environmental behavior can frequently be obtained which forms a (possibly small) subset of a general passive behavior. In these cases, design based solely on passivity considerations can be overly conservative and may impose unnecessary constraints on the interaction control algorithms. In particular, the requirement of passivity imposed on the closed-loop manipulator dynamics appears to be in contradiction with the manipulator's position tracking performance. Extensions of coupled stability criteria that go beyond the passivity framework are pursued in [5,8]. In fact, the latter works implement loop transformations that, for linear time-invariant systems, expand the passivity and the small-gain criteria to more general cases of graph separation stability conditions.

In this work, we propose a comprehensive approach to the problems of coupled stability and stabilization of robot-environment interaction. The approach is based on the non-planar conic systems formalism as well as the generalized version of the scattering transformation applicable to non-planar conic systems, which were recently developed by the authors [30-32]. The notion of non-planar conicity is an extension of the conventional (planar) conicity [35] to the case where the cone's center is a subspace of dimension, generally speaking, higher than 1. As shown in [30, 32], the class of non-planar conic systems essentially coincides with that of dissipative systems with quadratic supply rate; the method for calculation of parameters of a non-planar cone for an arbitrary (Q,S,R)-dissipative system is given in [30, Lemma 3] (see also [32, Lemma 4]). The generalized version of the scattering transformation pursued in [31,32], on the other hand, allows for rendering of input-output characteristics of a nonplanar conic system into a prescribed cone of compatible dimensions. In view of the graph separation stability condition [30,32], the ability to change the parameters of a system's cone can be used for stabilization of interconnections of systems; the corresponding methods were developed in [31,32], including the case of interconnections with communication delays.

When it comes to the problem of coupled stability, the use of the non-planar conic systems framework and generalized scattering transformations may lead to fundamental extensions of the existing coupled stability criteria and methods for stabilization of robot-environment interaction. Direct application of the methods developed in $[30-32]$ to the coupled stability problem, however, is not preferable, partially because straightforward design based on the methods from the above cited works would interfere with a robot's tracking performance in free space. In this chapter, we present a design framework for coupled stability problem which is compatible with
an arbitrary trajectory tracking control algorithm and does not affect the trajectory tracking performance. The framework is based on the non-planar conic systems formalism and scattering based stabilization, where the scattering transformation is designed to guarantee coupled stability while satisfying specially formulated constraints that preclude its interference with free space tracking. A detailed design example is presented where a manipulator controlled by a trajectory tracking control algorithm experiences non-passive contact with an environment which results in coupled instability, while the application of the proposed scattering based methods stabilizes the robot-environment interaction.

The chapter is organized as follows. In Section 3.2, the necessary background material related to recent developments of non-planar conicity and generalized scattering transformations is described following [32]. The scattering-based design approach to the coupled stability problem is presented in Section 3.3. A procedure for constrained scattering-based design is described in Section 3.4. An example of scattering-based design for coupled stability is presented in detail in Section 3.5 , including a theoretical rationale as well as results of simulations. Conclusions are given in Section 3.6.

### 3.2 Non-planar conicity and scattering-based stabilization

In this section, a brief overview of some recent developments related to the notion of nonplanar conicity and generalized scattering transformations is presented. Further details can be found in $[30-32]$.

### 3.2. 1 Non-planar conicity

Consider a nonlinear system of the form

$$
\Sigma:\left\{\begin{array}{l}
\dot{x}=f(x, \eta)  \tag{3.1}\\
y=h(x, \eta)
\end{array}\right.
$$

where $x \in \mathbb{R}^{n}$ is the state, $\eta \in \mathbb{R}^{m}$ the input, and $y \in \mathbb{R}^{p}$ the output of system (3.1), respectively. The functions $f(\cdot, \cdot), h(\cdot, \cdot)$ are locally Lipschitz continuous in their arguments. A system (3.1) is said to be dissipative with respect to supply rate $w: \mathbb{R}^{p} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ if there exists a storage function $V: \mathbb{R}^{n} \rightarrow \mathbb{R}_{+}$such that the inequality

$$
V\left(x\left(t_{1}\right)\right)-V\left(x\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t_{1}} w(y(\tau), \eta(\tau)) d \tau
$$

holds along the trajectories of the system (3.1) for any $t_{1} \geq t_{0}$, any initial state $x\left(t_{0}\right)$, and an arbitrary admissible control input $\eta(t), t \in\left[t_{0}, t_{1}\right)$.

Definition 3.1. [30] Given a subspace $\Omega \subset \mathbb{R}^{m+p}, \operatorname{dim} \Omega=l \in\{0, \ldots, m+p\}$, and $\phi_{r} \in[0, \pi / 2)$, a system $\Sigma$ of the form (3.1) is said to be interior conic with respect to the cone with centre $\Omega$ and radius $\phi_{r}\left(\Sigma \in \operatorname{Int}\left(\Omega, \phi_{r}\right)\right)$ if it is dissipative with supply rate

$$
w(y, \eta)=\left[\begin{array}{ll}
\eta^{T} & y^{T}
\end{array}\right] W\left(\Omega, \phi_{r}\right)\left[\begin{array}{l}
\eta  \tag{3.2}\\
y
\end{array}\right],
$$

where matrix $W\left(\Omega, \phi_{r}\right)$ has the form

$$
\begin{equation*}
W\left(\Omega, \phi_{r}\right):=\Pi_{\Omega}-\cos ^{2} \phi_{r} \cdot \mathbb{I}_{m+p}, \tag{3.3}
\end{equation*}
$$

where $\Pi_{\Omega}$ is the matrix of projection onto subspace $\Omega$.
For a dissipative system with a given quadratic supply rate, a parametrization in terms of the cone's center $\Omega$ and radius $\phi_{r}$ can be obtained as follows. Given matrices $Q=Q^{T} \in \mathbb{R}^{p \times p}$, $R=R^{T} \in \mathbb{R}^{m \times m}$, and $S \in \mathbb{R}^{p \times m}$, a system of the form (3.1) is said to be ( $Q, S, R$ )-dissipative [11] if it is dissipative with supply rate

$$
w(y, \eta):=y^{T} Q y+2 y^{T} S \eta+\eta^{T} R \eta=\left[\begin{array}{ll}
\eta^{T} & y^{T}
\end{array}\right][\mathrm{QSR}]\left[\begin{array}{l}
\eta \\
y
\end{array}\right],
$$

where

$$
[\mathrm{QSR}]:=\left[\begin{array}{cc}
R & S^{T} \\
S & Q
\end{array}\right] \in \mathbb{R}^{(m+p) \times(m+p)}
$$

Matrix [QSR] is real symmetric and, therefore, its eigenvalues $\mu_{1}, \ldots, \mu_{m+p}$ are all real. Let $\lambda(Q S R):=\left\{\mu_{1}, \ldots, \mu_{m+p}\right\}$ denote the set of eigenvalues of $[Q S R], \lambda^{-}(Q S R) \subset \lambda(Q S R)$ the set of strictly negative $(<0)$ eigenvalues of [ $Q S R$ ], and $\lambda^{+}(Q S R):=\lambda(Q S R) \backslash \lambda^{-}(Q S R)$ the set of nonnegative $(\geq 0)$ eigenvalues of $[Q S R]$. Let $l:=\operatorname{card}\left\{\lambda^{+}(Q S R)\right\}$ be the number of nonnegative eigenvalues of [QSR], and

$$
\begin{align*}
& \mu^{-}:=\min \left\{\left|\mu_{i}\right|: \mu_{i} \in \lambda^{-}(Q S R)\right\},  \tag{3.4}\\
& \mu^{+}:=\max \left\{\left|\mu_{i}\right|: \mu_{i} \in \lambda^{+}(Q S R)\right\} \tag{3.5}
\end{align*}
$$

The value of $\mu^{-}$is well-defined if $\lambda^{-}(Q S R) \neq \emptyset$ (equivalently, if $l<m+p$ ). Similarly, $\mu^{+}$is well-defined if $\lambda^{+}(Q S R) \neq \emptyset$ (equivalently, if $l>0$ ). The following statement is valid.

Lemma 3.2.1. [30] Suppose the system (3.1) is $(Q, S, R)$-dissipative. Then it is interior conic in the sense of Definition 3.1 with center $\Omega \subset \mathbb{R}^{m+p}$, $\operatorname{dim} \Omega=l$, and radius $\phi_{r} \in[0, \pi / 2)$. Specifically,

$$
\Omega:=\operatorname{span}\left\{g_{1}^{+}, \ldots, g_{l}^{+}\right\}
$$

is the subspace spanned by those eigenvectors $g_{1}^{+}, \ldots, g_{l}^{+}$of matrix $[Q S R]$ that correspond to its nonnegative eigenvalues $\mu_{i} \in \lambda^{+}(Q S R)$. If $0<l<m+p$, then

$$
\phi_{r}:=\tan ^{-1}\left(\sqrt{\frac{\mu^{+}}{\mu^{-}}}\right)
$$

Otherwise (i.e., if $l=0$ or $l=m+p$ ), radius $\phi_{r} \in(0, \pi / 2)$ can be chosen arbitrarily.
Lemma 3.2.1 gives a method for calculation of the dynamic cone's parameters (i.e., the central subspace and the radius) of a system of the form (3.1) dissipative with a given quadratic supply rate. Conditions for finite gain $\mathcal{L}_{2}$-stability of a feedback interconnection of two nonplanar conic systems based on the parameters of their dynamic cones were developed in [30, 32]. A system of the form (3.1) is said to be finite gain $\mathcal{L}_{2}$-stable if it is dissipative with supply rate $w(y, \eta):=\gamma^{2}|\eta|^{2}-|y|^{2}$, where $\gamma \geq 0$ is the $\mathcal{L}_{2}$-gain, see [33]. Finite gain $\mathcal{L}_{2^{-}}$ stability of a feedback interconnection of two non-planar conic subsystems shown in Figure 3.1 can be guaranteed by a "graph separation" condition given below in Theorem 3.2.2. To for-


Figure 3.1: Feedback interconnection of $\Sigma_{1}$ and $\Sigma_{2}$.
mulate the graph separation condition, it is convenient to use a notion similar to the one of the inverse graph in [27]. Informally, a system $\Sigma$ is inverse interior conic (with some centre $\Omega$ and radius $\phi_{r}$ ) if the same system with inverse causality (i.e., with $y$ considered an input and $\eta$ an output) is $\operatorname{Int}\left(\Omega, \phi_{r}\right)$. Formally, given a central subspace $\Omega \subset \mathbb{R}^{m+p}, \operatorname{dim} \Omega=m$, let vectors $\omega_{1}, \ldots \omega_{m} \in \mathbb{R}^{m+p}$ form a basis in $\Omega$, i.e., $\operatorname{span}\left[\begin{array}{lll}\omega_{1} & \ldots & \omega_{m}\end{array}\right]=\Omega$. Define $\bar{\Omega}:=\operatorname{span}\left(P_{(m, p)} \cdot\left[\begin{array}{lll}\omega_{1} & \ldots & \omega_{m}\end{array}\right]\right)$, where $P_{(m, p)} \in \mathbb{R}^{m+p}$ is a permutation matrix of the form

$$
P_{(m, p)}:=\left[\begin{array}{cc}
\mathbb{O} & \mathbb{I}_{p}  \tag{3.6}\\
\mathbb{I}_{m} & \mathbb{O}
\end{array}\right]
$$

A system $\Sigma$ of the form (3.1) is called inverse interior conic with respect to the cone with centre $\Omega$ and radius $\phi_{r}$ (we will use notation $\Sigma \in \overline{\operatorname{Int}}\left(\Omega, \phi_{r}\right)$ ) iff $\Sigma \in \operatorname{Int}\left(\bar{\Omega}, \phi_{r}\right)$.

Consider now two subsystems of the form

$$
\Sigma_{i}:\left\{\begin{array}{l}
\dot{x}_{i}=f_{i}\left(x_{i}, \eta_{i}\right),  \tag{3.7}\\
y_{i}=h_{i}\left(x_{i}, \eta_{i}\right),
\end{array} \quad i \in\{1,2\}\right.
$$

where $y_{2}, \eta_{1} \in \mathbb{R}^{m}, y_{1}, \eta_{2} \in \mathbb{R}^{p}$, interconnected as follows

$$
\begin{equation*}
\eta_{1}=y_{2}+\chi_{1}, \quad \eta_{2}=y_{1}+\chi_{2} \tag{3.8}
\end{equation*}
$$

where $\chi_{1} \in \mathbb{R}^{m}, \chi_{2} \in \mathbb{R}^{p}$ are external inputs, as shown in Figure 3.1. The closed-loop system (3.7), (3.8) has the input $\left[\chi_{1}^{T}, \chi_{2}^{T}\right]^{T} \in \mathbb{R}^{m+p}$, and the output $\left[y_{1}^{T}, y_{2}^{T}\right]^{T} \in \mathbb{R}^{m+p}$. The following result is valid.

Theorem 3.2.2. [30] Consider an interconnected system of the form 3.7, (3.8). Suppose $\Sigma_{1} \in \overline{\operatorname{Int}}\left(\Omega_{1}, \phi_{r 1}\right), \Sigma_{2} \in \operatorname{Int}\left(\Omega_{2}, \phi_{r 2}\right)$, where $\bar{\Omega}_{1} \cap \Omega_{2}=\{0\}, \operatorname{dim} \Omega_{1}=m$, $\operatorname{dim} \Omega_{2}=p$. If the following "graph separation" condition is satisfied

$$
\begin{equation*}
\sigma_{\max }\left(\Pi_{\bar{\Omega}_{1}} \cdot \Pi_{\Omega_{2}}\right)<\cos \left(\phi_{r 1}+\phi_{r 2}\right) \tag{3.9}
\end{equation*}
$$

then the interconnected system (3.7), (3.8) is finite gain $\mathcal{L}_{2}$-stable.
Remark Condition (3.9) is equivalent to the existence of a constant $\delta_{0}>0$ such that

$$
\begin{equation*}
\cos ^{-1}\left(\sigma_{\max }\left(\Pi_{\bar{\Omega}_{1}} \cdot \Pi_{\Omega_{2}}\right)\right)-\phi_{r 1}-\phi_{r 2}=\delta_{0} \tag{3.10}
\end{equation*}
$$

Since $\cos ^{-1}\left(\sigma_{\max }\left(\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}\right)\right)$ represents the angle between subspaces $\bar{\Omega}_{1}$ and $\Omega_{2}$, condition (3.10) (equivalently, (3.9) implies that the subsystems' cones are separated by a gap $\delta_{0}$. The size of gap $\delta_{0}$, in particular, represents the amount of robustness of stability in an interconnected system [10].

### 3.2.2 Scattering-based stabilization of interconnections of non-planar conic systems

The scattering (wave) transformation was first used for stabilization purposes in [1, 2, 21], where it was implemented to overcome delay induced instability in force reflecting teleoperator systems. Since then, scattering/wave based stabilization has become one of the most popular techniques in bilateral teleoperation with communication delays [15, 20, 22, 26]. One possible interpretation of the effect of the scattering transformation used in the above cited works is that it transforms a passive system into a system with gain less than or equal to one [1]. In recent years, progressively more powerful scattering transformation techniques were developed in [12, 24, 31, 32]. In particular, the scattering transformation presented in [31, 32] allows for rendering of the input-output characteristics of a non-planar conic system into an arbitrary prescribed cone with equal dimension of the central subspace. Specifically, suppose a system $\Sigma \in \operatorname{Int}\left(\Omega, \phi_{r}\right)$, where $\Omega \subset \mathbb{R}^{m+p}$ is the central subspace, $\operatorname{dim} \Omega=m$, and $\phi_{r} \in(0, \pi / 2)$

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is a radius. Given a desired central subspace $\Omega_{d} \subset \mathbb{R}^{m+p}$, $\operatorname{dim} \Omega_{d}=m$, and desired radius $\phi_{r d} \in(0, \pi / 2)$, one would like to construct a transformation of the form

$$
\left[\begin{array}{l}
\mathbf{u}  \tag{3.11}\\
\mathbf{v}
\end{array}\right]:=\mathbb{S}\left(\Omega, \Omega_{d}, \phi_{r}, \phi_{r d}\right)\left[\begin{array}{l}
\eta \\
y
\end{array}\right],
$$

such that the transformed system $\Sigma_{(\mathbf{u}, \mathbf{v})}$ with new input-output variables ( $\left.\mathbf{u}, \mathbf{v}\right), \mathbf{u} \in \mathbb{R}^{m}, \mathbf{v} \in$ $\mathbb{R}^{p}$, is interior conic with central subspace $\Omega_{d}$ and radius $\phi_{r d}$ (i.e., $\Sigma_{(\mathbf{u}, \mathbf{v})} \in \operatorname{Int}\left(\Omega_{d}, \phi_{r d}\right)$ ). A transformation with the above described properties can be constructed as follows. Let vectors $g_{1}, g_{2}, \ldots, g_{m}$ form an orthonormal basis in $\Omega$. The set of vectors $\left\{g_{1}, g_{2}, \ldots, g_{m}\right\} \in \Omega$ can be augmented with additional vectors $g_{m+1}, \ldots, g_{m+p} \in \Omega^{\perp}$ such that the columns of

$$
G:=\left[\begin{array}{llllll}
g_{1} & \ldots & g_{m} & g_{m+1} & \ldots & g_{m+p} \tag{3.12}
\end{array}\right]
$$

form an orthonormal basis in $\mathbb{R}^{m+p}$. Similarly, a matrix $G_{d}$ can be constructed such that its first $m$ columns form an orthonormal basis in $\Omega_{d}$, while the whole set of its columns forms an orthonormal basis in $\mathbb{R}^{m+p}$. Consider a scattering transformation

$$
\begin{equation*}
\mathbb{S}\left(\Omega, \Omega_{d}, \phi_{r}, \phi_{r d}\right):=G_{d} \cdot \Gamma\left(\phi_{r}, \phi_{r d}\right) \cdot G^{T}, \tag{3.13}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma\left(\phi_{r}, \phi_{r d}\right):=\left(\frac{\cos \phi_{r_{d}}}{\cos \phi_{r}}\right)^{\alpha} \cdot\left(\frac{\sin \phi_{r_{d}}}{\sin \phi_{r}}\right)^{-\beta} \cdot \operatorname{diag}\left\{\left(\frac{\tan \phi_{r d}}{\tan \phi_{r}}\right)^{\alpha} \mathbb{I}_{m},\left(\frac{\tan \phi_{r d}}{\tan \phi_{r}}\right)^{\beta} \mathbb{I}_{p}\right\} \tag{3.14}
\end{equation*}
$$

and $\alpha:=-p /(m+p), \beta:=m /(m+p)$. The following lemma is valid.
Lemma 3.2.3. [31] 32] Suppose a system $\Sigma$ of the form (3.1) is such that $\Sigma \in \operatorname{Int}\left(\Omega, \phi_{r}\right)$, where $\Omega \subset \mathbb{R}^{m+p}, \operatorname{dim} \Omega=m$, and $\phi_{r} \in(0, \pi / 2)$. Then the transformed system (3.1), (3.11), (3.13), (3.14) with new input-output variables $(\mathbf{u}, \mathbf{v})$ satisfies $\Sigma_{(\mathbf{u}, \mathbf{v})} \in \operatorname{Int}\left(\Omega_{d}, \phi_{r d}\right)$.

One important application of Lemma 3.2 .3 is for stabilization of interconnections of nonplanar conic systems. Suppose subsystems $\Sigma_{i}, i=1,2$ are non-planar conic. To guarantee stability of the feedback interconnection of $\left(\Sigma_{1}, \Sigma_{2}\right)$, one can implement a scattering transformation for one of the subsystem which renders its input-output characteristics into a desired dynamic cone. If the parameters of the desired cone are chosen in a way that guarantees the fulfilment of the graph separation stability condition (Theorem 3.2.2), then the interconnection is guaranteed to be finite gain $\mathcal{L}_{2}$-stable. A detailed description of the design methods that use scattering transformation of the form (3.13), (3.14), including the case of interconnections with multiple communication delays, can be found in [32].

### 3.3 Scattering-based approach to the coupled stability problem

The purpose of this work is to develop a scattering-based technique for stabilization of robotenvironment interactions. We assume that the robot is controlled such that the trajectory of its end-effector $\Psi(t):=\left[\mathbf{x}^{T}(t), \dot{\mathbf{x}}^{T}(t)\right]^{T}$ follows some sufficiently smooth desired trajectory $\Psi_{d}(t):=$ $\left[\mathbf{x}_{d}^{T}(t), \dot{\mathbf{x}}_{d}^{T}(t)\right]^{T}$, where $\mathbf{x}$ and $\mathbf{x}_{d}$ represent the actual and the desired positions of the end-effector, respectively, while $\dot{\mathbf{x}}$ and $\dot{\mathbf{x}}_{d}$ represent the actual and the desired velocities of the end-effector. When the robot's end-effector encounters an environment, an interaction force $f_{e}$ is generated which is applied to the robot's end-effector, thus forming the closed-loop robot-environment dynamics. The problem of stability of this closed-loop system is known as the coupled stability problem [8].

Our approach to the coupled stability problem is illustrated in Figure 3.2. In order to guarantee stability of the robot-environment interaction, a scattering transformation is inserted between the robot and the environment subsystems, as shown in the left side of Figure 3.2. The scattering transformation is designed in a way that guarantees the fulfilment of the graph separation stability condition (described by Theorem 3.2.2) between the robot's and the environment dynamics. As can be seen in Figure 3.2, the scattering transformation defines a relationship between the contact force $f_{e}$, the tracking error $\mathcal{V}:=\Psi-\Psi_{d}$, and two new "inner" signals $\mathbf{v}_{f}$ and $\mathcal{E}$, according to the formula

$$
\left[\begin{array}{c}
f_{e}  \tag{3.15}\\
\mathcal{V}
\end{array}\right]=\mathbb{S}\left[\begin{array}{c}
\mathbf{v}_{f} \\
\mathcal{E}
\end{array}\right],
$$

where $\mathbb{S}$ is the matrix of scattering transformation. The new signals $\mathbf{v}_{f}$ and $\mathcal{E}$ represent the robot force input and a new tracking error, respectively. Obviously, scattering transformation (3.15) cannot be directly implemented between the robot and the environment as there is mechanical interaction with energy exchange between these rather than simply an exchange of information signals. However, the scattering transformation can be implemented indirectly through introduction of auxiliary reference signals $f_{r}:=\mathbf{v}_{f}-f_{e}, \tilde{\Psi}_{r}:=\mathcal{V}-\mathcal{E}$, as shown in the right part of Figure 3.2.

One additional design consideration, which is specific for the coupled stability problem, is that one does not wish the designed scattering transformation to affect the robot's tracking performance in free space. More precisely, in the absence of contact between the robot and the environment, i.e., when $f_{e}=0$, it makes sense to require that the transformed force $\mathbf{v}$ is also equal to zero and the transformed tracking error $\mathcal{E}$ is equal to the actual tracking error $\mathcal{V}$. This requirement imposes constraints on the structure of the scattering matrix $\mathbb{S}$ in (3.15);
specifically, its inverse $\mathbb{S}^{-1}$ must be of the form

$$
\mathbb{S}^{-1}=\left[\begin{array}{ll}
\mathbb{S}_{1} & \mathbb{O}  \tag{3.16}\\
\mathbb{S}_{2} & \mathbb{I}
\end{array}\right]
$$

where $\mathbb{O}, \mathbb{I}$ are zero and unit matrices of appropriate dimensions, and the matrices $\mathbb{S}_{1}, \mathbb{S}_{2}$ are arbitrary with nonsingular $\mathbb{S}_{1}$. It is worth mentioning that, in this case, the transform $\mathbb{S}$ has a structure similar to that of its inverse (3.16), specifically,

$$
\mathbb{S}=\left[\begin{array}{cc}
\mathbb{S}_{1}^{-1} & \mathbb{O}  \tag{3.17}\\
-\mathbb{S}_{2} \mathbb{S}_{1}^{-1} & \mathbb{I}
\end{array}\right]
$$

From implementation point of view, however, it is more convenient to work with the inverse transform $\mathbb{S}^{-1}$ rather than with $\mathbb{S}$, partially because the signals in the left-hand side of (3.15) (i.e., $f_{e}, \mathcal{V}$ ) are readily available, while the signals in the right-hand side of (3.15) (i.e., $\mathbf{v}_{f}$ and $\mathcal{E})$ are to be determined. The requirement for the scattering transformation to satisfy constraints (3.16) (equivalently, (3.17)) effectively makes the design methods developed in [31, 32] (i.e., those based on the scattering transformation of the form (3.13), (3.14) inapplicable to the coupled stability problem. In the next section, we describe a procedure for the design of a scattering transformation that guarantees stability through graph separation while satisfying the constraints (3.16).


Figure 3.2: Scattering-based stabilization of robot-environment interaction

### 3.4 A procedure for constrained scattering-based design

Consider a scattering based robot-environment interconnection shown in Figure 3.2. Suppose the environment with input $\mathcal{V} \in \mathbb{R}^{m}$ and output $f_{e} \in \mathbb{R}^{p}$ is a non-planar conic system with a central subspace $\Omega_{e}, \operatorname{dim} \Omega_{e}=m$, and radius $\phi_{e} \in(0, \pi / 2)$, which is denoted by
$\Sigma_{\left(V, f_{e}\right)}^{e} \in \operatorname{Int}\left(\Omega_{e}, \phi_{e}\right)$. Suppose the robot with input $\mathbf{v}_{f} \in \mathbb{R}^{p}$ and output $\mathcal{E} \in \mathbb{R}^{m}$ is also non-planar conic with central subspace $\Omega_{r}, \operatorname{dim} \Omega_{r}=p$ and radius $\phi_{r} \in(0, \pi / 2)$, i.e., $\Sigma_{\left(v_{f}, \mathcal{E}\right)}^{r} \in \operatorname{Int}\left(\Omega_{r}, \phi_{r}\right)$. Our goal is to find a scattering transformation $\mathbb{S} \in \mathbb{R}^{(p+m) \times(p+m)}$ of the form (3.15) that renders the dynamic characteristics of the robot with new input-output pair $\left(f_{e}, \mathcal{V}\right)$ into a dynamic cone $\operatorname{Int}\left(\Omega_{r}^{s}, \phi_{r}^{s}\right)$, where $\Omega_{r}^{s}$ and $\phi_{r}^{s} \in(0, \pi / 2)$ are the transformed center and radius, respectively, with the following property: $\Sigma_{\left(f_{e}, \mathcal{V}\right)}^{r} \in \operatorname{Int}\left(\Omega_{r}^{s}, \phi_{r}^{s}\right)$ and $\Sigma_{\left(\mathcal{V}, f_{e}\right)}^{e} \in \operatorname{Int}\left(\Omega_{e}, \phi_{e}\right)$ satisfy the graph separation stability condition of Theorem 3.2.2 with a prescribed gap $\delta_{0}>0$ (see Remark 3.2.1). In addition, thus designed scattering transformation $\mathbb{S}$ must satisfy the constraints (3.16), (3.17). Since there may exist many (generally speaking, a continuum of) transformations with the above described properties, one may like to choose a transformation that results in the minimum deviation from the tracking control law that controls the manipulator's motion in free space. Such a deviation can be measured as a norm of a (possibly weighted) difference between the inverse scattering transformation matrix $\mathbb{S}^{-1}$ and the unit matrix $\mathbb{I}$. One particular way to describe a difference between $\mathbb{S}^{-1}$ and $\mathbb{I}$, which is utilized in our work, is to consider a functional of the form

$$
\begin{equation*}
F_{\Delta}(\mathbb{S}):=\operatorname{tr}\left[\left[\mathbb{S}^{-1}-\mathbb{I}\right]^{T} \cdot \Delta \cdot\left[\mathbb{S}^{-1}-\mathbb{I}\right]\right] \tag{3.18}
\end{equation*}
$$

where $\Delta$ is a diagonal weighting matrix with positive diagonal elements such that $\operatorname{tr} \Delta=1$. Different diagonal elements in $\Delta$ assign different weights to rows of $\left[\mathbb{S}^{-1}-\mathbb{I}\right]$. Based on the above description, our goal is to solve an optimization problem of the form:

$$
\begin{equation*}
\mathbb{S}_{*}=\underset{\mathbb{S}^{-1} \text { of the form }}{\arg \min } F_{\Delta}(\mathbb{S}) \tag{3.19}
\end{equation*}
$$

subject to constraint

$$
\begin{equation*}
\cos ^{-1}\left(\sigma_{\max }\left(\Pi_{\overline{\Omega_{r}^{s}}} \cdot \Pi_{\Omega_{e}}\right)\right)-\phi_{e}-\phi_{r}^{s}-\delta_{0} \geq 0 . \tag{3.20}
\end{equation*}
$$

The transformed center $\Omega_{r}^{s}$ and radius $\phi_{r}^{s}$ that enter the constraint 3.20 can be calculated by applying Lemma 3.2 .1 to the transformed supply rate matrix $W_{r}^{s}:=\mathbb{S}^{-T} \cdot\left[\Pi_{\Omega_{r}}-\cos ^{2} \phi_{r} \mathbb{I}\right]$. $\mathbb{S}^{-1}$. The Matlab code that solves the optimization problem (3.19), 3.20) for a specific robotenvironment interaction task addressed below in Section 3.5] can be downloaded from [29].

### 3.5 Example of scattering-based design for coupled stability

### 3.5.1 Mathematical models of the controlled manipulator and the environment

We address a problem of coupled stability between a robot manipulator controlled by a trajectory tracking control algorithm and an environment where the interaction between the two is
characterized by a damping matrix with some negative eigenvalues and therefore non-passive. The problem is illustrated in Figure 3.3. Consider a robot manipulator whose dynamics are


Figure 3.3: Coupled stability problem
described in the task space as follows:

$$
\begin{equation*}
H_{\mathbf{x}}(q) \ddot{\mathbf{x}}+C_{\mathbf{x}}(q, \dot{q}) \dot{\mathbf{x}}+G_{\mathbf{x}}(q)=f_{e}+u \tag{3.21}
\end{equation*}
$$

where $q, \dot{q} \in \mathbb{R}^{n}$ are robot's position and velocity vectors represented in the joint space coordinates, $\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}} \in \mathbb{R}^{m}$ are position, velocity, and acceleration, respectively, of the robot's end-effector represented in the task space coordinates, $H_{\mathbf{x}}(q), C_{\mathbf{x}}(q, \dot{q}), G_{\mathbf{x}}(q)$ are matrices of inertia, Coriolis/centrifugal forces, and a vector of gravitational forces represented in the taskspace coordinates, $f_{e}$ denotes the environmental contact forces applied to the end-effector, and $u$ is the task-space control input (for more details of the task-space dynamic equations (3.21) and their relationship to the joint-space dynamics the reader is referred, for example, to [9, Chapter 4]). Consider a control algorithm

$$
\begin{equation*}
u=H_{\mathbf{x}}(q) \dot{\mathbf{r}}+C_{\mathbf{x}}(q, \dot{q}) \mathbf{r}+G_{\mathbf{x}}(q)-K \sigma+f_{r}, \tag{3.22}
\end{equation*}
$$

where $\sigma:=\dot{\tilde{\mathbf{x}}}+\boldsymbol{\Lambda} \tilde{\mathbf{x}}, \tilde{\mathbf{x}}=\mathbf{x}-\mathbf{x}_{r}, \mathbf{r}:=\dot{\mathbf{x}}-\sigma=\dot{\mathbf{x}}_{r}-\Lambda \cdot \tilde{\mathbf{x}}$, and $\Lambda=\Lambda^{T}>0, K=K^{T}>0$ are matrices of feedback law parameters. Signals $\mathbf{x}_{r}(t), \dot{\mathbf{x}}_{r}(t)$ represent the reference position and velocity, respectively. When $f_{r}(t) \equiv 0$ and $\mathbf{x}_{r}(t), \dot{\mathbf{x}}_{r}(t)$ are equal to the desired position $\mathbf{x}_{d}(t)$ and velocity $\dot{\mathbf{x}}_{d}(t)$ of the end-effector in task space, respectively, the control algorithm (3.22) becomes a nonadaptive task space version of the Slotine-Li tracking control algorithm [16], see also [9.25]. It
can be readily established that in this case the controlled manipulator (3.21), (3.22) is strictly output passive with respect to the input-output pair $\left(f_{e}, \sigma\right)$; specifically, the time derivative of a storage function candidate $V=(1 / 2) \sigma^{T} H_{\mathbf{x}}(q) \sigma$ along the trajectories of (3.21), (3.22) with $f_{r e f} \equiv 0$ is $\dot{V}=-\sigma K \sigma-\sigma^{T} f_{e}$ (this fact is originally due to [23]). It is worth emphasizing, however, that the system (3.21), (3.22) is not passive with respect to the conventional pair of power variables $\left(f_{e}, \dot{\mathbf{x}}\right)$. Indeed, the passive output $\sigma:=\dot{\mathbf{x}}-\dot{\mathbf{x}}_{r}+\Lambda \mathbf{x}-\Lambda \mathbf{x}_{r}$ in this case contains three other terms in addition to $\dot{\mathbf{x}}$, including terms that depend on reference trajectory and the end-effector position. The situation is typical for tracking control algorithms where the necessity to force the robot to converge to the desired trajectory is in contradiction with (conventional) passivity. Therefore, contact stability of the controlled robot (3.21), (3.22) with even a passive environment is not automatically guaranteed.

On the other hand, consider the environmental dynamics described by an Euler-Lagrange equation of the form:

$$
\begin{equation*}
H_{\mathbf{e}}\left(q_{e}\right) \ddot{\mathbf{x}}_{e}+C_{\mathbf{e}}\left(q_{e}, \dot{q}_{e}\right) \dot{\mathbf{x}}_{e}+\frac{\partial P\left(q_{e}\right)}{\partial \mathbf{x}_{e}}+D_{e} \dot{\mathbf{x}}_{e}+f_{e}=0 \tag{3.23}
\end{equation*}
$$

where $\mathbf{x}_{e}, \dot{\mathbf{x}}_{e}, \ddot{\mathbf{x}}_{e} \in \mathbb{R}^{m}$ are the environmental position, velocity, and acceleration, respectively, $H_{\mathbf{e}}\left(q_{e}\right), C_{\mathbf{e}}\left(q_{e}, \dot{q}_{e}\right), D_{e}$ are matrices of inertia, Coriolis/centrifugal forces, and environmental damping, respectively, and $P\left(q_{e}\right)$ is the potential energy. Denote $\tilde{\mathbf{x}}_{e}:=\mathbf{x}_{e}-\mathbf{x}$. Let the robotenvironment interaction be described by equation of the form

$$
f_{e}:= \begin{cases}0 & \text { if } e_{s e}^{T} \tilde{\mathbf{x}}_{e} \leq 0  \tag{3.24}\\ K_{s e} \tilde{\mathbf{x}}_{e}+D_{s e} \dot{\mathbf{x}}_{e} & \text { if } e_{s e}^{T} \tilde{\mathbf{x}}_{e}>0\end{cases}
$$

where $K_{s e}=K_{s e}^{T} \geq 0$ is a stiffness matrix of rank $1, e_{s e}$ is a fixed eigenvector of $K_{s e}$ that corresponds to its positive eigenvalue, and $D_{s e}=D_{s e}^{T}$ is a damping matrix. Nonnegative definite damping matrix $D_{s e}$ would result in a passive environment. To make the problem more interesting, let's assume that $D_{s e}$ is not sign definite, i.e., some of its eigenvalues may be strictly negative, which implies negative damping in certain directions. Negative contact damping, which may describe different mechanical phenomena such as slippage [19], results in nonpassivity of robot-environment interaction.

In our simulations presented below, we use a mathematical model of a 3-DOF manipulator described in detail in Appendix 3.7. For modeling an environment, we use a 2-DOF manipulandum whose mathematical model together with its parameters is presented in Appendix 3.8 .

### 3.5.2 Quadratic supply rates for the controlled manipulator and the environment

We begin by analyzing the dissipativity properties of the controlled manipulator (3.21), (3.22) and the environment (3.23), (3.24). Substituting (3.22) into (3.21), the following closed-loop dynamics of the controlled manipulator can be obtained

$$
\begin{align*}
\dot{\tilde{\mathbf{x}}} & =-\Lambda \tilde{\mathbf{x}}+\sigma  \tag{3.25}\\
\dot{\sigma} & =H_{\mathbf{x}}^{-1}(q)\left[-C_{\mathbf{x}}(q, \dot{q}) \sigma-K \sigma+f_{e}+f_{r}\right] \tag{3.26}
\end{align*}
$$

The dynamical equations (3.25), (3.26) represent the dynamics in terms of state variables $\tilde{\mathbf{x}}, \sigma$; they can be rewritten in terms of state variables $\tilde{\mathbf{x}}, \dot{\tilde{\mathbf{x}}}$ using the following coordinate transformation:

$$
\left[\begin{array}{c}
\tilde{\mathbf{x}}  \tag{3.27}\\
\dot{\tilde{\mathbf{x}}}
\end{array}\right]=\mathbf{T}_{\Lambda}^{-1}\left[\begin{array}{c}
\tilde{\mathbf{x}} \\
\sigma
\end{array}\right], \text { where } \mathbf{T}_{\Lambda}:=\left[\begin{array}{cc}
\mathbb{I} & \mathbb{O} \\
\Lambda & \mathbb{I}
\end{array}\right] \in \mathbb{R}^{2 m \times 2 m} .
$$

Pick a storage function candidate for the robot of the form

$$
V_{r}=\frac{1}{2} \sigma^{T} H_{\mathbf{x}}(q) \sigma+\frac{\mu}{2} \tilde{\mathbf{x}}^{T} \tilde{\mathbf{x}}=\left[\begin{array}{c}
\tilde{\mathbf{x}} \\
\dot{\tilde{\mathbf{x}}}
\end{array} \mathbf{c}^{T} \mathbf{T}_{\Lambda}^{T} \cdot\left[\begin{array}{cc}
\frac{1}{2} \mu & 0 \\
\mathcal{O} & \frac{1}{2} H_{\mathbf{x}}(q)
\end{array}\right] \cdot \mathbf{T}_{\Lambda}\left[\begin{array}{c}
\tilde{\mathbf{x}} \\
\dot{\mathbf{x}}
\end{array}\right],\right.
$$

where $\mu>0$ is a parameter. Using notation $\mathbf{v}_{f}:=f_{e}+f_{r}$, the time derivative of $V_{r}$ along the trajectories of (3.25), (3.26), (3.27) is

$$
\dot{V}_{r}=-\sigma K \sigma-\sigma^{T}\left[\frac{1}{2} \dot{H}_{\mathbf{x}}(q)-C_{\mathbf{x}}(q, \dot{q})\right] \sigma+\sigma^{T} \mathbf{v}_{f}-\mu \cdot \tilde{\mathbf{x}}^{T} \Lambda \tilde{\mathbf{x}}+\mu \cdot \tilde{\mathbf{x}}^{T} \sigma=\left[\begin{array}{c}
\mathbf{v}_{f} \\
\tilde{\mathbf{x}} \\
\dot{\tilde{\mathbf{x}}}
\end{array}\right]^{T} \mathbf{W}_{r}\left[\begin{array}{c}
\mathbf{v}_{f} \\
\tilde{\mathbf{x}} \\
\dot{\mathbf{x}}
\end{array}\right],
$$

where the matrix of the quadratic supply rate of the controlled robot has a form

$$
\mathbf{W}_{r}:=\left[\begin{array}{cc}
\mathbb{I} & \mathbb{O}  \tag{3.28}\\
\mathbb{O} & \mathbf{T}_{\Lambda}^{T}
\end{array}\right] \cdot\left[\begin{array}{ccc}
\mathbb{O} & \mathbb{O} & \frac{1}{2} \mathbb{I} \\
\mathbb{O} & -\mu \Lambda & \frac{1}{2} \mu \mathbb{I} \\
\frac{1}{2} \mathbb{I} & \frac{1}{2} \mu \mathbb{I} & -K
\end{array}\right] \cdot\left[\begin{array}{cc}
\mathbb{I} & \mathbb{O} \\
\mathbb{O} & \mathbf{T}_{\Lambda}
\end{array}\right]
$$

On the other hand, consider the environmental dynamics (3.23), (3.24). Pick a storage function candidate of the form

$$
V_{e}=\frac{1}{2} \dot{\mathbf{x}}_{e}^{T} H_{\mathbf{e}}\left(q_{e}\right) \dot{\mathbf{x}}_{e}+P\left(q_{e}\right)+\frac{1}{2} \tilde{\mathbf{x}}_{e}^{T} K_{s e} \tilde{\mathbf{x}}_{e} .
$$

The time derivative of $V_{e}$ along the trajectories of (3.23), (3.24) is

$$
\dot{V}_{e}=-\dot{\mathbf{x}}_{e}^{T} D_{e} \dot{\mathbf{x}}_{e}-\dot{\mathbf{x}}_{e}^{T} f_{e}+\dot{\tilde{\mathbf{x}}}_{e}^{T} K_{s e} \tilde{\mathbf{x}}_{e}=-\dot{\mathbf{x}}_{e}^{T} D_{e} \dot{\mathbf{x}}_{e}-\dot{\mathbf{x}}^{T} f_{e}-\dot{\tilde{\mathbf{x}}}_{e}^{T} D_{s e} \dot{\tilde{\mathbf{x}}}_{e} \leq-\dot{\mathbf{x}}_{e}^{T} D_{e} \dot{\mathbf{x}}_{e}-\dot{\mathbf{x}}^{T} f_{e}+\dot{\tilde{\mathbf{x}}}_{e}^{T} D_{s e}^{*} \dot{\tilde{\mathbf{x}}}_{e},
$$

where $D_{s e}^{*}=D_{s e}^{* T} \geq 0$ is the nonnegative definite component of $-D_{s e}$. Picking an arbitrary $\epsilon>0$ such that

$$
\begin{equation*}
D^{\epsilon}:=D_{e}-(1+\epsilon) D_{s e}^{*} \geq 0, \tag{3.29}
\end{equation*}
$$

and using Young's quadratic inequality, one can write

$$
\dot{V}_{e} \leq-\dot{\mathbf{x}}_{e}^{T} D^{\epsilon} \dot{\mathbf{x}}_{e}-\dot{\mathbf{x}}^{T} f_{e}+\left(\frac{\epsilon+1}{\epsilon}\right) \dot{\mathbf{x}}^{T} D_{s e}^{*} \dot{\mathbf{x}} \leq-\dot{\mathbf{x}}^{T} f_{e}+\left(\frac{\epsilon+1}{\epsilon}\right) \dot{\mathbf{x}}^{T} D_{s e}^{*} \dot{\mathbf{x}}=\left[\begin{array}{c}
f_{e} \\
\mathbf{x} \\
\dot{\mathbf{x}}
\end{array}\right]^{T} \mathbf{W}_{e}\left[\begin{array}{c}
f_{e} \\
\mathbf{x} \\
\dot{\mathbf{x}}
\end{array}\right],
$$

where

$$
\mathbf{W}_{e}:=\left[\begin{array}{ccc}
\mathbb{O} & \mathbb{O} & -\frac{1}{2} \mathbb{I}  \tag{3.30}\\
\mathbb{O} & \mathbb{O} & \mathbb{O} \\
-\frac{1}{2} \mathbb{I} & \mathbb{O} & \left(\frac{\epsilon+1}{\epsilon}\right) D_{s e}^{*}
\end{array}\right]
$$

is the matrix of quadratic supply rate of the environment.

### 3.5.3 Dynamic cone analysis

As the next step, the parameters of (non-planar) cones that characterize the dynamics of the controlled manipulator (3.21), (3.22) and the environment (3.23), (3.24) are to be determined. This can be done based on the corresponding expressions for [QSR]-matrices (i.e., matrices of the quadratic supply rates (3.28) and (3.30) using the algorithm described in Lemma 3.2.1. The Matlab code that implements the algorithm of Lemma 3.2.1] can be downloaded from [28]. In order to determine the parameters of the robot's cone, we pick specific values of the design coefficients that comprise the matrix of the quadratic supply rate (3.28). For the 3-DOF manipulator described in Appendix 3.7, the matrices $\Lambda, K$ of the feedback coefficients in the tracking control algorithm (3.22) are chosen as follows:

$$
\Lambda=\left[\begin{array}{ccc}
2.25 & 0 & 0  \tag{3.31}\\
0 & 2 & 0 \\
0 & 0 & 2
\end{array}\right], \quad K=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

This choice of feedback matrices ensures that the controlled manipulator demonstrates acceptable tracking performance in free space. The value of the weighting coefficient $\mu>0$ in the storage function (3.5.2) is then chosen to minimize the dynamic cone radius $\phi_{r}$. The cone radius $\phi_{r}$ as a function of $\mu>0$ is shown in Figure 3.4; the minimum value of $\phi_{r} \approx 0.54 \mathrm{rad}$ $\left(\phi_{r}=30.97^{\circ}\right)$ is achieved at $\mu \approx 3.6$. Finally, the center of the robot's cone is calculated to be


Figure 3.4: Radius $\phi_{r}$ of the manipulator's dynamic cone as a function of $\mu>0$.
a three-dimensional subspace spanned by the following vectors

$$
\Omega_{r}=\operatorname{span}\left\{\left[\begin{array}{c}
0  \tag{3.32}\\
0.9303 \\
0 \\
0 \\
0.1984 \\
0 \\
0 \\
0.3085 \\
0
\end{array}\right],\left[\begin{array}{c}
0 \\
0 \\
0.9303 \\
0 \\
0 \\
0.1984 \\
0 \\
0 \\
0.3085
\end{array}\right],\left[\begin{array}{c}
-0.9416 \\
0 \\
0 \\
-0.1714 \\
0 \\
0 \\
-0.2898 \\
0 \\
0
\end{array}\right]\right\} .
$$

For the environment (3.23), (3.24), in our design example the values of environmental damping in (3.23) and the contact damping in (3.24) are chosen as follows

$$
D_{e}:=\left[\begin{array}{ccc}
10 & 0 & 0  \tag{3.33}\\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \quad D_{s e}:=\left[\begin{array}{ccc}
-2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right](\mathrm{N} \cdot \mathrm{~s} / \mathrm{m})
$$

For the above choice of contact damping matrix $D_{s e}$, the matrix $D_{s e}^{*}$ (i.e, the nonnegative defi-
nite component of $-D_{s e}$ ) is

$$
D_{s e}^{*}:=\left[\begin{array}{lll}
2 & 0 & 0  \tag{3.34}\\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right](\mathrm{N} \cdot \mathrm{~s} / \mathrm{m})
$$

Taking into account the specific structure of the matrix $D_{s e}^{*}$, it is easy to see that the set of eigenvalues of $\mathbf{W}_{e}$ consists of the following subsets:

$$
\begin{aligned}
& \lambda^{+}\left(\mathbf{W}_{e}\right)=\left\{0,0,0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\left(\frac{2(1+\epsilon)}{\epsilon}+\sqrt{\left(\frac{2(1+\epsilon)}{\epsilon}\right)^{2}+1}\right)\right\}, \quad \text { and } \\
& \lambda^{-}\left(\mathbf{W}_{e}\right)=\left\{-\frac{1}{2},-\frac{1}{2}, \frac{1}{2}\left(\frac{2(1+\epsilon)}{\epsilon}-\sqrt{\left(\frac{2(1+\epsilon)}{\epsilon}\right)^{2}+1}\right)\right\}
\end{aligned}
$$

Therefore, parameters $\mu^{-}$and $\mu^{+}$defined by (3.4), (3.5), can be calculated as follows:

$$
\begin{aligned}
& \mu^{+}:=\max \left\{\left|\mu_{i}\right|: \mu_{i} \in \lambda^{+}\left(\mathbf{W}_{e}\right)\right\}=\frac{1}{2}\left(\sqrt{\left(\frac{2(1+\epsilon)}{\epsilon}\right)^{2}+1}+\frac{2(1+\epsilon)}{\epsilon}\right), \\
& \mu^{-}:=\min \left\{\left|\mu_{i}\right|: \mu_{i} \in \lambda^{-}\left(\mathbf{W}_{e}\right)\right\}=\frac{1}{2}\left(\sqrt{\left(\frac{2(1+\epsilon)}{\epsilon}\right)^{2}+1}-\frac{2(1+\epsilon)}{\epsilon}\right) .
\end{aligned}
$$

Applying Lemma 3.2.1, we conclude that the environmental cone's radius $\phi_{e}$ satisfies

$$
\tan \phi_{e}:=\sqrt{\frac{\mu^{+}}{\mu^{-}}}=\sqrt{\frac{4(1+\epsilon)^{2}}{\epsilon^{2}}+1}+\frac{2(1+\epsilon)}{\epsilon}
$$

From (3.5.3), it is easy to see that $\tan \phi_{e}$ (and therefore radius $\phi_{e} \in(0, \pi / 2)$ ) is a decreasing function of $\epsilon>0$. Taking into account the choice of $D_{e}$ and $D_{s e}^{*}$ as in (3.33) and (3.34), respectively, the maximum value of $\epsilon>0$ such that (3.29) holds is $\epsilon=4$. In order to achieve the minimum possible upper bound for $\phi_{e}$, we should therefore choose $\epsilon=4$, which results in

$$
\left.\tan \phi_{e}\right|_{\epsilon=4} \approx 5.1926,
$$

which corresponds to $\phi_{e}=1.38 \mathrm{rad}$ or $\phi_{e}=79.0993^{\circ}$. Finally, at $\epsilon=4$, the center of the
environmental cone is calculated as a 6-dimensional subspace as follows:

$$
\Omega_{e}=\operatorname{span}\left\{\left[\begin{array}{c}
0.189  \tag{3.35}\\
0 \\
0 \\
0 \\
0 \\
0 \\
-0.982 \\
0 \\
0
\end{array}\right],\left[\begin{array}{c}
0 \\
0 \\
0.707 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-0.707
\end{array}\right],\left[\begin{array}{c}
0 \\
-0.707 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0.707 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
0 \\
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{array}\right]\right\} .
$$

Once the dynamic cones of the controlled manipulators and the environment are determined, the next step is to check the fulfilment of the graph separation stability condition 3.9). Matrices of projections onto the central subspaces $\Pi_{\Omega_{r}}$ and $\Pi_{\Omega_{e}}$ can be calculated in a straightforward manner from (3.32) and (3.35), respectively (specifically, let $\hat{\Omega}_{r}$ denote the matrix whose columns are the unit vectors that span $\Omega_{r}$ in (3.32), then $\Pi_{\Omega_{r}}:=\hat{\Omega}_{r} \hat{\Omega}_{r}^{T} ; \Pi_{\Omega_{e}}$ is calculated similarly). Further calculations indicate that $\sigma\left(\Pi_{\bar{\Omega}_{r}} \Pi_{\Omega_{e}}\right) \approx 0.4823$, while $\cos \left(\phi_{r}+\phi_{e}\right)=$ $\cos (0.54+1.38) \approx-0.342$. We see that $\sigma\left(\Pi_{\bar{\Omega}_{r}} \Pi_{\Omega_{e}}\right) \nless \cos \left(\phi_{r}+\phi_{e}\right)$, i.e., the interconnection of the controlled manipulator (3.21), (3.22) and the environment (3.23), (3.24) fails to satisfy the graph separation stability condition (3.9). This theoretical result is in complete accordance with our simulations (Section 3.5.5) that demonstrate contact instability of the robot-environment interaction. Below the problem is solved using scattering-based stabilization methods.

### 3.5.4 Design of scattering transformation

The dynamic cone analysis presented above corresponds to the case where $\mathbf{x}_{r}(t) \equiv \mathbf{x}_{d}(t)$ and $f_{r}(t) \equiv 0$. In this subsection, we design scattering transformation(s) that solve the coupled stability problem through generation of new reference signals $\mathbf{x}_{r}(t)$, $\dot{\mathbf{x}}_{r}(t)$, and $f_{r}(t)$. Based on the general description given in Section 3.3, the scattering transformation $\mathbb{S}$ establishes relationship between the system's variables according to the formula

$$
\left[\begin{array}{c}
f_{e}+f_{r}  \tag{3.36}\\
\mathbf{x}-\mathbf{x}_{r} \\
\dot{\mathbf{x}}-\dot{\mathbf{x}}_{r}
\end{array}\right]=\mathbb{S}^{-1}\left[\begin{array}{c}
f_{e} \\
\mathbf{x}-\mathbf{x}_{d} \\
\dot{\mathbf{x}}-\dot{\mathbf{x}}_{d}
\end{array}\right] .
$$

Equation (3.36) is equivalent to

$$
\left[\begin{array}{c}
f_{r} \\
\mathbf{x}_{d}-\mathbf{x}_{r} \\
\dot{\mathbf{x}}_{d}-\dot{\mathbf{x}}_{r}
\end{array}\right]:=\left[\mathbb{S}^{-1}-\mathbb{I}\right]\left[\begin{array}{c}
f_{e} \\
\mathbf{x}-\mathbf{x}_{d} \\
\dot{\mathbf{x}}-\dot{\mathbf{x}}_{d}
\end{array}\right],
$$

the latter gives an explicit formula for the reference force and the correction to the desired trajectory that implement the designed scattering transformation $\mathbb{S}$. As described in Section 3.3 , we are looking for a scattering transformation of the form (3.16), which result in a control law that does not affect the system's trajectory tracking performance in free space. More specifically, we restrict our search to the scattering transformations 3.16 where $\mathbb{S}_{1}$ is diagonal, and $\mathbb{S}_{2}$ double-diagonal, i.e.,

$$
\mathbb{S}^{-1}:=\left[\begin{array}{ccc}
\mathbb{S}_{1} & \mathbb{O} & \mathbb{O}  \tag{3.37}\\
\mathbb{S}_{21} & \mathbb{I}_{3} & \mathbb{O} \\
\mathbb{S}_{22} & \mathbb{O} & \mathbb{I}_{3}
\end{array}\right] \in \mathbb{R}^{9 \times 9}
$$

where $\mathbb{S}_{1}:=\operatorname{diag}\left\{a_{1}, a_{2}, a_{3}\right\}, \mathbb{S}_{21}:=\operatorname{diag}\left\{a_{4}, a_{5}, a_{6}\right\}, \mathbb{S}_{22}:=\operatorname{diag}\left\{a_{7}, a_{8}, a_{9}\right\}$. The scattering transformation (3.37) therefore is a function of nine parameters which comprise a vector $\mathbf{a}:=$ $\left[a_{1} \ldots a_{9}\right]^{T} \in \mathbb{R}^{9}$. In this case, the functional (3.18) becomes

$$
\begin{equation*}
F_{\Delta}(\mathbb{S}(\mathbf{a})):=\left[\mathbf{a}-\mathbf{a}_{0}\right]^{T} \cdot \Delta \cdot\left[\mathbf{a}-\mathbf{a}_{0}\right], \tag{3.38}
\end{equation*}
$$

where $\mathbf{a}_{0}:=\left[\begin{array}{lllll}1 & 1 & 1 & 0 & \ldots\end{array}\right]^{T} \in \mathbb{R}^{9}$, and $\Delta:=\operatorname{diag}\left\{\delta_{1}, \ldots, \delta_{9}\right\} \in \mathbb{R}^{9 \times 9}$ is a diagonal matrix with $\delta_{i}>0, i=1, \ldots 9$, such that $\operatorname{tr} \Delta=\sum \delta_{i}=1$. Consequently, the optimization problem (3.19) becomes

$$
\begin{equation*}
\mathbf{a}^{*}:=\underset{\mathbf{a} \in \mathbb{R}^{9}}{\arg \min } F_{\Delta}(\mathbb{S}(\mathbf{a})) \tag{3.39}
\end{equation*}
$$

subject to the same constraints (3.20). The Matlab code that solves the problem (3.39), (3.20) can be obtained from [29].

In our design example, we set the minimum gap $\delta_{0}=4^{\circ}$, and consider four different sets of weighting coefficients $\Delta$, as follows:

- Case 1: $\Delta=\Delta_{1}:=(1 / 9) \cdot \mathbb{I}_{9}$.
- Case 2: $\Delta=\Delta_{2}:=(1 / 6.3) \cdot \operatorname{diag}\left\{0.1 \cdot \mathbb{I}_{3}, \mathbb{I}_{3}, \mathbb{I}_{3}\right\}$.
- Case 3: $\Delta=\Delta_{3}:=(1 / 6.3) \cdot \operatorname{diag}\left\{\mathbb{I}_{3}, 0.1 \cdot \mathbb{I}_{3}, \mathbb{I}_{3}\right\}$.
- Case 4: $\Delta=\Delta_{4}:=(1 / 6.3) \cdot \operatorname{diag}\left\{\mathbb{I}_{3}, \mathbb{I}_{3}, 0.1 \cdot \mathbb{I}_{3}\right\}$.

It is easy to see that the Case 1 corresponds to a uniform assignment of weighting coefficients. In Case 2, the weighting coefficients corresponding to the force component are decreased 10
times, which essentially decrease penalty for the reference force component. Cases 3 and 4 correspond to decreased penalties for position and velocity correction components, respectively. The optimization problem (3.39), (3.20) for the above Cases 1-4 are then solved using the Matlab code which can be downloaded at [28]. The results are summarized in Table 3.1. In this table, the columns correspond to the minimum gap $\delta_{0}$, the choice of weighting coefficients $\Delta$, the parameters $\mathbf{a}^{*}$ of the resulting scattering transformation (3.37), the parameter $\Phi_{r e}:=\cos ^{-1}\left(\sigma_{\max }\left(\Pi_{\Omega_{r}^{s}} \cdot \Pi_{\Omega_{e}}\right)\right)$ which represents the angle between the inverse transformed center subspace of the robot dynamics $\overline{\Omega_{r}^{s}}$ and the center subspace of the environmental dynamics $\Omega_{e}$, the radius $\phi_{r}^{s}$ of the transformed robot's cone, and the actual gap between the transformed robot and the environment cones. As can be seen from this table, the design procedure is successful in all four cases; in particular, the actual gap achieved is always greater than the minimum required gap $\delta_{0}$.

|  | Case 1 | Case 2 | Case 3 | Case 4 |
| :---: | :---: | :---: | :---: | :---: |
| $\delta_{0}$ | $4^{\circ}$ | $4^{\circ}$ | $4^{\circ}$ | $4^{\circ}$ |
| $\Delta$ | $\Delta_{1}$ | $\Delta_{2}$ | $\Delta_{3}$ | $\Delta_{4}$ |
| $\mathbf{a}^{*}$ | $\left[\begin{array}{c}0.1630 \\ 0.2231 \\ 0.2231 \\ 0.0421 \\ 0.0674 \\ 0.0674 \\ -0.1234 \\ -0.9060 \\ -0.9060\end{array}\right]$ | $\left[\begin{array}{c}0.0019 \\ 0.0028 \\ 0.0028 \\ 0.0015 \\ 0.0164 \\ -0.0029 \\ -0.0918 \\ -0.8152 \\ -0.814\end{array}\right]$ | $\left[\begin{array}{c}0.1632 \\ 0.2234 \\ 0.2234 \\ 0.0422 \\ 0.0675 \\ 0.0675 \\ -0.1233 \\ -0.9059 \\ -0.9059\end{array}\right]$ | $\left[\begin{array}{c}0.1621 \\ 0.2219 \\ 0.2219 \\ 0.0419 \\ 0.0671 \\ 0.0671 \\ -0.1238 \\ -0.9064 \\ -0.9064\end{array}\right]$ |
| $\Phi_{r e}$ | $\approx 90^{\circ}$ | $\approx 84.19^{\circ}$ | $\approx 90^{\circ}$ | $\approx 90^{\circ}$ |
| $\phi_{r}^{s}$ | $\approx 6.4^{\circ}$ | $\approx 0.09^{\circ}$ | $\approx 6.4^{\circ}$ | $\approx 6.36{ }^{\circ}$ |
| Gap | $\approx 4.5^{\circ}$ | $\approx 5^{\circ}$ | $\approx 4.49^{\circ}$ | $\approx 4.54^{\circ}$ |

Table 3.1: Design of the scattering transformations for Cases 1-4.

### 3.5.5 Simulation results

In this subsection, we present examples of simulations of the robot-environment interaction problem. In every simulation presented below, the feedback matrices $K, \Lambda$ of the robot's tracking control algorithm are given by (3.31), and the environmental damping matrices $D_{e}, D_{\text {se }}$
are as in (3.33). We present simulation results for 2 different sets of environmental parameters, specifically the environmental stiffness matrix $K_{e}$, and the contact stiffness matrix $K_{s e}$. For each of these two sets of environmental parameters, we simulate the contact stability problem for five different robot control/stabilization algorithms: the tracking control algorithm (3.22) without scattering-based component (i.e., with $\mathbf{x}_{r}(t) \equiv \mathbf{x}_{d}(t)$ and $f_{r}(t) \equiv 0$ ), as well as the four cases of algorithm (3.22) with scattering-based stabilization component (3.5.4) designed above in Section 3.5.4 and summarized in Table 3.1.

Parameter set 1. In this set, the environment matrices $K_{e}$ and $K_{s e}$ are chosen as follows:

$$
K_{e}=\left[\begin{array}{ccc}
100 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right] \quad(\mathrm{N} / \mathrm{m}), \quad K_{s e}=\left[\begin{array}{ccc}
20 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \quad(\mathrm{N} / \mathrm{m})
$$

This choice corresponds to environment with relatively low stiffness. Simulation results of the five control algorithms are shown in Figures 3.5, 3.9.


Figure 3.5: Parameter set 1, tracking control algorithm (3.22) without a scattering-based component: x-coordinates of the robot's end-effector and the environment (left), contact forces (right).

Parameter set 2. In this set, the environment matrices $K_{e}$ and $K_{s e}$ are increased 10 times as compared to Case 1, specifically:

$$
K_{e}=\left[\begin{array}{ccc}
1000 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right] \quad(\mathrm{N} / \mathrm{m}), \quad K_{s e}=\left[\begin{array}{ccc}
200 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \quad(\mathrm{N} / \mathrm{m})
$$

This choice of $K_{e}, K_{s e}$ corresponds to environment with high stiffness. Simulation results of the five control algorithms are shown in Figures 3.10.3.14.


Figure 3.6: Parameter set 1, scattering-based design (case 1): x-coordinates of the robot's end-effector and the environment (left), contact forces (right).


Figure 3.7: Parameter set 1, scattering-based design (case 2): x-coordinates of the robot's end-effector and the environment (left), contact forces (right).


Figure 3.8: Parameter set 1, scattering-based design (case 3): x-coordinates of the robot's end-effector and the environment (left), contact forces (right).


Figure 3.9: Parameter set 1, scattering-based design (case 4): x-coordinates of the robot's end-effector and the environment (left), contact forces (right).


Figure 3.10: Parameter set 2, tracking control algorithm (3.22) without a scattering-based component: x-coordinates of the robot's end-effector and the environment (left), contact forces (right).

It can be seen from the results of simulations that, in both cases of the environment with relatively low stiffness (parameter set 1) and the environment with high stiffness (parameter set 2 ), the tracking control algorithm without a scattering-based component results in contact instability when coupled with the environment. All four cases of scattering-based design, on the other hand, successfully stabilize the robot-environment interaction. Among these four cases, case 2 (which corresponds to decreased penalty for the reference force component) seems to demonstrate lower performance and, in particular, results in higher contact forces as compared to the other three cases of scattering-based design (cases 1, 3, and 4).


Figure 3.11: Parameter set 2, scattering-based design (case 1): x-coordinates of the robot's end-effector and the environment (left), contact forces (right).


Figure 3.12: Parameter set 2, scattering-based design (case 2): x-coordinates of the robot's end-effector and the environment (left), contact forces (right).


Figure 3.13: Parameter set 2, scattering-based design (case 3): x-coordinates of the robot's end-effector and the environment (left), contact forces (right).


Figure 3.14: Parameter set 2, scattering-based design (case 4): x-coordinates of the robot's end-effector and the environment (left), contact forces (right).

### 3.6 Conclusion

In this work, a non-planar conic system formalism and generalized scattering transformation techniques developed previously in [32] were applied to the problem of stable robot environment interaction. The conventional passivity-based approaches to the coupled stability problem are limited to the case of passive interaction and not compatible with majority of the trajectory tracking control algorithms. Following the general approach of [32], we develop a design method for coupled stability applicable to arbitrary ( $Q, S, R$ )-dissipative environments, which can be used in combination with an arbitrary robot's tracking control algorithm and does not affect the trajectory tracking performance in free space. A detailed design example is presented that illustrates the capabilities of the proposed design method. A complete analytical solution of the scattering-based design problem for coupled stability subject to constraints such as (3.16) is a topic for future research.

### 3.7 Appendix: Mathematical model of the manipulator

In our simulations, we use a mathematical model of a 3-DOF manipulator whose kinematic structure and frames are shown in Figure 3.15. The following physical parameters were chosen for our simulations: $l_{1}=0.6731 \mathrm{~m}, l_{2}=0.432 \mathrm{~m}, l_{3}=0.434 \mathrm{~m}, l_{c_{1}}=0.216 \mathrm{~m}, l_{c_{2}}=0.164 \mathrm{~m}$, $m_{2}=3.092 \mathrm{~kg}, m_{3}=1.91 \mathrm{~kg}, J=0.0151 \mathrm{~kg} \cdot \mathrm{~m}^{2}$. The Denavit-Hartenberg (DH) parameters of the robot are shown in the Table 3.2. The forward kinematics are given by

$$
\begin{aligned}
& x=c_{1}\left(l_{2} c_{2}+l_{3} s_{23}\right), \\
& y=s_{1}\left(l_{2} c_{2}+l_{3} s_{23}\right), \\
& z=l_{1}-l_{2} s_{2}+l_{3} c_{23},
\end{aligned}
$$



Figure 3.15: Manipulator structure
where we use notation $s_{i}:=\sin \left(q_{i}\right), c_{i}:=\cos \left(q_{i}\right), s_{i j}:=\sin \left(q_{i}+q_{j}\right)$, and $c_{i j}:=\cos \left(q_{i}+q_{j}\right)$, $i, j=1,2,3$. Based on the forward kinematics, the following inverse kinematics were obtained

$$
\begin{aligned}
q_{1} & =\operatorname{atan} 2(y ; x), \\
q_{2} & = \pm \alpha-\operatorname{atan} 2\left(R^{2}+l_{2}^{2}-l_{3}^{2} ; \pm \sqrt{\left(\left(l_{2}+l_{3}\right)^{2}-R^{2}\right)\left(R^{2}-\left(l_{2}-l_{3}\right)^{2}\right)}\right) \\
q_{3} & =\operatorname{atan} 2\left(R^{2}-\left(l_{2}^{2}+l_{3}^{2}\right) ; \pm \sqrt{\left(\left(l_{2}+l_{3}\right)^{2}-R^{2}\right)\left(R^{2}-\left(l_{2}-l_{3}\right)^{2}\right)}\right), \quad \text { where } \\
\alpha & =\operatorname{atan} 2\left(r ; z-l_{1}\right), \quad R^{2}:=r^{2}+\left(z-l_{1}\right)^{2}, \quad r^{2}:=x^{2}+y^{2} . \\
& \# \left\lvert\, \begin{array}{lllll}
\alpha_{i-1} & a_{i-1} & d_{i} & q_{i} \\
\hline 1 & 0 & 0 & l_{1} & q_{1} \\
2 & -\pi / 2 & 0 & 0 & q_{2} \\
3 & 0 & l_{2} & 0 & q_{3} \\
4 & \pi / 2 & 0 & l_{3} & 0
\end{array}\right.
\end{aligned}
$$

Table 3.2: The Denavit-Hartenberg parameters

The manipulator's Jacobian is

$$
J=\left[\begin{array}{ccc}
-s_{1}\left(l_{2} c_{2}+l_{3} s_{23}\right) & c_{1}\left(-l_{2} s_{2}+l_{3} c_{23}\right) & l_{3} c_{1} c_{23} \\
c_{1}\left(l_{2} c_{2}+l_{3} s_{23}\right) & s_{1}\left(-l_{2} s_{2}+l_{3} c_{23}\right) & l_{3} s_{1} c_{23} \\
0 & -l_{2} c_{2}-l_{3} s_{23} & -l_{3} s_{23}
\end{array}\right] .
$$

The manipulator's dynamics are described by the Euler-Lagrange equations of the form

$$
H(q) \ddot{q}+C(q, \dot{q}) \dot{q}+G(q)=\tau+\tau_{e}
$$

where $\tau_{e}$ is an external torque due to interaction with environment, $H(\cdot)$ is the inertia matrix of the form

$$
H=\left[\begin{array}{ccc}
J+J_{1} s_{23}^{2}+J_{12} c_{2}^{2}+2 J_{13} c_{2} s_{23} & 0 & 0 \\
0 & J_{1}+J_{12}+2 J_{13} s_{3} & J_{1}+J_{13} s_{3} \\
0 & J_{1}+J_{13} s_{3} & J_{1}
\end{array}\right]
$$

where $J_{1}=l_{c_{3}}^{2} m_{3}, J_{12}=l_{c_{2}}^{2} m_{2}+l_{2}^{2} m_{3}, J_{13}=l_{2} l_{c_{3}} m_{3}, C(\cdot)$ is the matrix of Coriolis and centrifugal forces,

$$
C(q, \dot{q}) \dot{q}=\left[\begin{array}{ccc}
0 & c_{12}^{(1)} \dot{q}_{1} & c_{13}^{(1)} \dot{q}_{3} \\
c_{11}^{(2)} \dot{q}_{1} & 0 & \left(c_{22}^{(2)} \dot{q}_{2}+c_{23}^{(2)} \dot{q}_{3}\right) \\
c_{11}^{(3)} \dot{q}_{1} & c_{22}^{(3)} \dot{q}_{2} & 0
\end{array}\right]\left[\begin{array}{l}
\dot{q}_{1} \\
\dot{q}_{2} \\
\dot{q}_{3}
\end{array}\right],
$$

where

$$
\begin{aligned}
& c_{12}^{(1)}=J_{1} \sin 2\left(q_{1}+q_{2}\right)-J_{12} \sin 2 q_{2}+2 J_{13} \cos \left(2 q_{2}+q_{3}\right), \\
& c_{13}^{(1)}=J_{1} \sin 2\left(q_{2}+q_{3}\right)+2 J_{13} c_{2} c_{23}, \\
& c_{11}^{(2)}=\frac{1}{2}\left(-J_{1} \sin 2\left(q_{2}+q_{3}\right)+J_{12} \sin 2 q_{2}-2 J_{13} \cos \left(2 q_{2}+q_{3}\right)\right), \\
& c_{22}^{(2)}=2 J_{13} c_{3}, \quad c_{33}^{(2)}=J_{13} c_{3}, \quad c_{22}^{(3)}=-J_{13} c_{3} \dot{q}_{2}, \\
& c_{11}^{(3)}=-\frac{1}{2}\left(J_{1} \sin 2\left(q_{2}+q_{3}\right)+J_{13} c_{2} c_{23}\right),
\end{aligned}
$$

and $G(\cdot)$ is the vector of potential (gravity) forces,

$$
G(q)=-\left[\begin{array}{c}
0 \\
l_{c_{3}} m_{3} s_{23}+l_{c_{2}} m_{2} c_{2}+l_{2} m_{3} c_{2} \\
l_{c_{3}} m_{3} s_{23}
\end{array}\right] \cdot g
$$

where $g=9.81 \mathrm{~m} / \mathrm{s}^{2}$ is the acceleration due to gravity. The dynamic equations in the Cartesian space are

$$
H_{\mathbf{x}}(q) \ddot{\mathbf{x}}+C_{\mathbf{x}}(q, \dot{q}) \dot{\mathbf{x}}+G_{\mathbf{x}}(q)=u+f_{e}
$$

where

$$
\begin{array}{ll}
H_{\mathbf{x}}(q) & =J^{-T}(q) H(q) J^{-1}(q), \\
C_{\mathbf{x}}(q, \dot{q}) & =J^{-T}\left(C(q, \dot{q})-H(q) J^{-1}(q) \dot{J}(q)\right) J^{-1}(q) \\
G_{\mathbf{x}}(q) & =J^{-T}(q) G(q) \\
u & =J^{-T}(q) \tau, \quad f_{e}=J^{-T}(q) \tau_{e} .
\end{array}
$$

### 3.8 Mathematical Model of the Environment



Figure 3.16: Environment model

The environment is a manipulandum with the kinematic structure shown in Figure 3.16, The forward kinematics of the environment are described by the following equations

$$
\mathbf{x}_{e}=\left[\begin{array}{l}
x_{e} \\
y_{e} \\
z_{e}
\end{array}\right]=\left[\begin{array}{c}
x_{e_{\text {base }}}-l_{1 e} s_{1}-l_{2 e} c_{12} \\
0 \\
l_{1 e} c_{1}-l_{2 e} s_{12}
\end{array}\right],
$$

where $s_{i}=\sin \theta_{i}, c_{i}=\cos \theta_{i}(i=1,2)$ and $s_{12}=\sin \left(\theta_{1}+\theta_{2}\right), c_{12}=\cos \left(\theta_{1}+\theta_{2}\right)$. The dynamics of the environment in the joint space are given by

$$
H_{e}(\theta) \ddot{\theta}+C_{e}(\theta, \dot{\theta}) \dot{\theta}+\frac{\partial P_{s}(\theta)}{\partial \theta}+D_{e}^{*}(\theta) \dot{\theta}+\tau_{\mathrm{env}}=0
$$

where the inertia matrix $H_{e}(\cdot)$ is

$$
H_{e}(\theta)=\left[\begin{array}{cc}
m_{2 e} l_{2 e}^{2}-2 m_{2 e} l_{1 e} l_{2 e} s_{2}+\left(m_{1 e}+m_{2 e}\right) l_{1 e}^{2} & m_{2 e} l_{2 e}\left(l_{2 e}-l_{1 e} s_{2}\right) \\
m_{2 e} l_{2 e}\left(l_{2 e}-l_{1 e} s_{2}\right) & m_{2 e} l_{2 e}^{2}
\end{array}\right] .
$$

The matrix of Coriolis and centrifugal forces are described by the equation

$$
C_{e}(\theta, \dot{\theta})=m_{2 e} l_{1 e} l_{2 e} c_{2}\left[\begin{array}{cc}
-2 \dot{\theta}_{2} & -\dot{\theta}_{2} \\
\dot{\theta}_{1} & 0
\end{array}\right] .
$$

The stiffness term $\partial P_{s}(\theta) / \partial \theta$ is the gradient of a quadratic potential function

$$
P_{s}(\theta):=\left(\mathbf{x}_{e}-\mathbf{x}_{e_{0}}\right)^{T} K_{e}\left(\mathbf{x}_{e}-\mathbf{x}_{e_{0}}\right), \mathbf{x}_{e_{0}}=\left[\begin{array}{c}
x_{e_{\text {base }}}-l_{2 e} \\
0 \\
l_{1 e}
\end{array}\right]
$$

which gives in the following components of the stiffness vector

$$
\begin{aligned}
& \frac{\partial P_{s}(\theta)}{\partial \theta_{1}}=2\left(\mathbf{x}_{e}-\mathbf{x}_{e_{0}}\right)^{T} K_{e}\left[\begin{array}{c}
l_{2 e} s_{12}-l_{1 e} c_{1} \\
0 \\
-l_{1 e} s_{1}-l_{2 e} c_{12}
\end{array}\right], \\
& \frac{\partial P_{s}(\theta)}{\partial \theta_{2}}=2 l_{2 e}\left(\mathbf{x}_{e}-\mathbf{x}_{e_{0}}\right)^{T} K_{e}\left[\begin{array}{c}
s_{12} \\
0 \\
-c_{12}
\end{array}\right]
\end{aligned}
$$

where the stiffness matrix $K_{e}$ has the following diagonal form

$$
K_{e}=\left[\begin{array}{ccc}
k_{e_{1}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 10
\end{array}\right] \quad(\mathrm{N} / \mathrm{m})
$$

In our simulations, the parameter $k_{e_{1}}$ was varied. The damping term $D_{e}^{*}(\theta)$ of the environment dynamics has the structure

$$
D_{e}^{*}(\theta)=J_{e}^{T}(\theta) D_{e} J_{e}(\theta), \quad D_{e}=\left[\begin{array}{ccc}
d_{e_{1}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right](\mathrm{N} \cdot \mathrm{~s} / \mathrm{m})
$$

In our simulations, we used $d_{e_{1}}=10(\mathrm{~N} \cdot \mathrm{~s} / \mathrm{m})$. The matrix $J_{e}$ in the representation of $D_{e}^{*}$ is the Jacobian matrix

$$
J_{e}(\theta)=\left[\begin{array}{cc}
-l_{1 e} c_{1}+l_{2 e} s_{12} & l_{2 e} s_{12} \\
0 & 0 \\
-l_{1 e} s_{1}-l_{2 e} c_{12} & -l_{2 e} c_{12}
\end{array}\right]
$$

The remaining parameters of the environment were chosen as $l_{e_{1}}=0.8901 \mathrm{~m}, l_{e_{2}}=0.4320 \mathrm{~m}$, $x_{e_{\text {base }}}=1.0810 \mathrm{~m}, m_{e_{1}}=m_{e_{2}}=2.5 \mathrm{~kg}$.

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## Chapter 4

## Scattering-based stabilization of complex interconnections of (Q, S, R)-dissipative systems with time delays

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A method for scattering-based stabilization of networks of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ ) - dissipative systems in the presence of multiple heterogeneous communication delays is presented. It is demonstrated that, for a wide class of dissipative systems with quadratic supply rates, the finite-gain stability of complex interconnections with multiple time delays can be achieved through an appropriate design of local scattering transformations. A numerical example and simulation results are presented in support of the theoretical developments.

### 4.1 Introduction

Stability and stabilization of networks of dynamical systems, particularly in the presence of communication constraints, is a topic of long-standing research interest for the control community $[2-5,8,8,10,13]$. One specific approach to stabilization of systems interconnected with communication delays is based on implementation of the so-called scattering or wave transformations. The approach was originally developed for interconnections of passive systems, and was particularly successful in applications to bilateral teleoperators with communication delays [1, 11, 14]. The stabilizing effect of the scattering transformations in the presence of communication delays is based on the fact that it transforms a passive system into a system
with gain less than or equal to one, which allows for eliminating the destabilizing effect of the phase shift in the delayed communication channel. Extensions of this approach to interconnections of not necessarily passive systems were pursued in [6, 12, 15]. In [15], a general form of the scattering transformation was developed which is applicable to a class of so-called nonplanar conic systems and, in particular, allows for rendering the input-output characteristics of a system into a prescribed non-planar cone of compatible dimensions. This is subsequently used for stabilization of system's feedback interconnections, with and without communication delays.

In this chapter, we develop a scattering-based approach to stabilization of interconnections of nonlinear dissipative systems with communication delays that, in particular, does not refer to any kind of conicity notion. Essentially, the approach developed in this chapter allows for stabilization of arbitrarily complex interconnections of (Q, S, R)-dissipative systems (i.e., systems dissipative with quadratic supply rates) with communication delays through an appropriate design of local input-output transformations. Specifically, following some preliminary developments, the main result of the chapter (Theorem 4.4.1) presents a construction of local scattering transformations that guarantee finite $\mathcal{L}_{2}$-gain stabilization of complex interconnections of (Q, S, R)-dissipative systems with respect to external disturbances in the presence of multiple heterogeneous communication delays, which fundamentally generalizes the existing results in this area [1, 6, 15].A numerical example and results of simulations are presented in support of the theory developed. The chapter is organized as follows. In Section 4.2, we give definitions and discuss some preliminary considerations regarding the eigenvalues of the matrix of quadratic supply rate in dissipative systems. In Section 4.3, we introduce the scattering transformation and present basic results that describe its effect on $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative systems. The main result of the chapter is presented in Section 4.4. A numerical example and the results of simulations are described in Section4.5.

## 4.2 (Q, S, R)-Dissipativity

Consider a nonlinear system of the form

$$
\Sigma:\left\{\begin{array}{l}
\dot{x}=f(x, \eta)  \tag{4.1}\\
y=h(x, \eta)
\end{array}\right.
$$

where $x \in \mathbb{R}^{n}$ is the state, $\eta \in \mathbb{R}^{m}$ the input, and $y \in \mathbb{R}^{p}$ the output of system (4.1), respectively. Functions $f(\cdot, \cdot), h(\cdot, \cdot)$ are locally Lipschitz continuous in their arguments. A system (4.1) is said to be dissipative with respect to supply rate $w: \mathbb{R}^{p} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ if there exists a storage
function $V: \mathbb{R}^{n} \rightarrow \mathbb{R}_{+}:=[0,+\infty)$ such that the inequality

$$
\begin{equation*}
V\left(x\left(t_{1}\right)\right)-V\left(x\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t_{1}} w(y(\tau), \eta(\tau)) d \tau \tag{4.2}
\end{equation*}
$$

holds along the trajectories of the system (4.1) for any $t_{1} \geq t_{0}$, any initial state $x\left(t_{0}\right)$, and any input $\eta(t), t \in\left[t_{0}, t_{1}\right)$ such that $x(t)$ is well-defined for $t \in\left[t_{0}, t_{1}\right]$.

Definition 4.1. [10] The system (4.1) is called ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative if it is dissipative with a quadratic supply rate of the form

$$
w(\eta, y)=\left[\begin{array}{l}
\eta  \tag{4.3}\\
y
\end{array}\right]^{T}[\mathrm{QSR}]\left[\begin{array}{l}
\eta \\
y
\end{array}\right], \quad[\mathrm{QSR}]:=\left[\begin{array}{cc}
R & S^{T} \\
S & Q
\end{array}\right]
$$

where $Q=Q^{T} \in \mathbb{R}^{p \times p}, R=R^{T} \in \mathbb{R}^{m \times m}, S \in \mathbb{R}^{p \times m}$.

### 4.2.1 A note on eigenvalues of [QSR]

Matrix [QSR] is real symmetric therefore its eigenvalues are real. As noticed for example in [9], not all [QSR] matrices result in meaningful dissipativity properties. For example, if all eigenvalues of [QSR] are nonnegative then ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ ) - dissipativity property becomes trivial. Indeed, in this case $w(\eta, y) \geq 0$ and choosing $V(x) \equiv$ const we see that inequality (4.2) always holds, which means that any system is ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative for this choice of supply rate. On the other hand, if all eigenvalues of [QSR] are negative then applying any constant nonzero input $\eta(t) \equiv \eta^{0} \neq 0$ results in supply rate $w\left(\eta^{0}, y\right) \leq-\epsilon_{0}$ for some $\epsilon_{0}>0$. Since a storage function is by definition bounded from below, it is easy to see that under very mild technical assumptions dissipation inequality (4.2) is impossible to satisfy. Below, we use similar considerations to establish a somewhat more refined result of this type. Consider a system of the form

$$
\Sigma_{a}:\left\{\begin{array}{l}
\dot{x}=f(x, \eta)  \tag{4.4}\\
y=h_{a}(x)
\end{array}\right.
$$

which is a special case of (4.1) where the output map is independent of the input $\eta, h(x, \eta) \equiv$ $h_{a}(x)$. The following result is valid.

Lemma 4.2.1. Suppose a system of the form (4.4) is (Q, S, R)-dissipative and its storage function $V(\cdot)$ achieves a local (non-strict) minima at some point $x_{0} \in \mathbb{R}^{n}$, i.e., there exists a neighbourhood $\mathcal{B}\left(x_{0}\right)$ such that $V\left(x_{0}\right) \leq V(x)$ for all $x \in \mathcal{B}\left(x_{0}\right)$. Then the number of nonnegative eigenvalues of $[Q S R]$ is greater than or equal to the number of inputs $m$.

Proof Denote $\omega:=\left[\eta^{T}, y^{T}\right]^{T} \in \mathbb{R}^{m+p}$. It is known (see for example [7. Corollary 4.2.12]) that a symmetric matrix [QSR] has at least $m$ nonnegative eigenvalues if there exists a subspace $S \subset \mathbb{R}^{m+p}, \operatorname{dim} S=m$, such that

$$
\begin{equation*}
\omega^{T}[\mathrm{QSR}] \omega \geq 0 \quad \text { for all } \omega \in S \tag{4.5}
\end{equation*}
$$

To prove the lemma, assume the converse, i.e., no such a subspace exists. More precisely, for any $m$-dimensional subspace $S$ there exists a vector $\omega_{*} \in S$ such that $\omega_{*}^{T}[\mathrm{QSR}] \omega_{*}<0$. Without loss of generality, assume that $\left|\omega_{*}\right|=1$. Let $S_{\eta}$ be a subspace spanned by inputs $\omega:=\left[\eta^{T}, 0^{T}\right]^{T}$ (satisfying $y=0$ ). By assumption, there exists $\eta_{*} \in \mathbb{R}^{m},\left|\eta_{*}\right|=1$ such that

$$
\left[\begin{array}{c}
\eta_{*} \\
0
\end{array}\right]^{T}[\mathrm{QSR}]\left[\begin{array}{c}
\eta_{*} \\
0
\end{array}\right]=\eta_{*}^{T} R \eta_{*}=-\epsilon_{*}<0
$$

Let $\eta_{0}:=\kappa_{1} \kappa_{2} \eta_{*}$, where $\kappa_{2}:=\sqrt{\frac{1}{\epsilon_{*}} h^{T}\left(x_{0}\right) Q h\left(x_{0}\right)+1}$, and $\kappa_{1} \in\{ \pm 1\}$ be such that $\kappa_{1} \eta_{*}^{T} S^{T} h\left(x_{0}\right) \leq$ 0 . It follows that $\left[\eta_{0}^{T}, h^{T}\left(x_{0}\right)\right][\mathrm{QSR}]\left[\eta_{0}^{T}, h^{T}\left(x_{0}\right)\right]^{T} \leq-\epsilon_{*}$. Now suppose $x\left(t_{0}\right)=x_{0}$, and consider the corresponding trajectory $x(t)$ of the system (4.4) under the constant input $\eta(t) \equiv \eta_{0}$. Such a trajectory is well-defined at least on an interval $\left[t_{0}, t_{0}+\tau\right)$ for some $\tau>0$. Choosing $\tau_{1} \in(0, \tau)$ sufficiently small, by continuity of trajectories, we guarantee that

$$
x(t) \in \mathcal{B}\left(x_{0}\right) \quad \text { and } \quad\left[\begin{array}{c}
\eta_{0} \\
h(x(t))
\end{array}\right]^{T}[\mathrm{QSR}]\left[\begin{array}{c}
\eta_{0} \\
h(x(t))
\end{array}\right] \leq-\epsilon_{*} / 2 \quad \text { hold for all } \quad t \in\left[t_{0}, t_{0}+\tau_{1}\right]
$$

Since the system is $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative, one has $V\left(x\left(t_{0}+\tau_{1}\right)\right)-V\left(x_{0}\right) \leq-\epsilon_{*} \tau / 2<0$, which implies $V\left(x\left(t_{0}+\tau_{1}\right)\right)<V\left(x_{0}\right)$. However, $x\left(t_{0}+\tau_{1}\right) \in \mathcal{B}\left(x_{0}\right)$, and therefore by assumption $V\left(x\left(t_{0}+\tau_{1}\right)\right) \geq V\left(x_{0}\right)$. This contradiction proves (4.5). The statement of lemma follows.

Corollary 4.2.2. Suppose a system of the form (4.4) is (Q, S, R)-dissipative and its storage function satisfies $V\left(x_{0}\right)=0$ for some $x_{0} \in \mathbb{R}^{n}$. Then the number of nonnegative eigenvalues of [QSR] is greater than or equal to the number of inputs $m$.

Proof The statement follows from Lemma 4.2 .1 due to the fact that a storage function is by definition nonnegative, and therefore achieves its (global) minima at $x_{0}$.

Lemma 4.2.1 and Corollary 4.2.2 indicate that for (Q, S, R)-dissipative systems the number of nonnegative eigenvalues of [QSR] matrix is typically equal to or greater than the number of inputs. In the developments below, we address the case where the number of nonnegative eigenvalues of [QSR] matrix is equal to the number of system's inputs $m$.

### 4.3 Scattering transformation for finite-gain stability

Consider a system of the form (4.1). The following definition is a specification of the notion of finite gain $\mathcal{L}_{2}$-stability in the case of systems with multiple inputs and/or outputs.

Definition 4.2. Given $A=A^{T} \in \mathbb{R}^{m \times m}, A \geq 0, B=B^{T} \in \mathbb{R}^{p \times p}, B>0$, the system (4.1) is said to be finite $\mathcal{L}_{2}$-gain $(A, B)$-stable if it is dissipative with supply rate

$$
\begin{equation*}
w(\eta, y):=\eta^{T} A \eta-y^{T} B y . \tag{4.6}
\end{equation*}
$$

Suppose a system (4.1) is (Q, S, R)-dissipative. Since [QSR] is symmetric, its eigenvalues are all real, and there exists a basis in $\mathbb{R}^{m+p}$ that consists of orthonormal eigenvectors of QSR. Write the eigenvalues of [QSR] in the descending order, i.e., $\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{m+p}$, and define

$$
\mathrm{G}:=\left[\begin{array}{llll}
g_{1} & g_{2} & \ldots & g_{m+p} \tag{4.7}
\end{array}\right] \in \mathbb{R}^{(m+p) \times(m+p)},
$$

where $g_{1}, \ldots, g_{m+p} \in \mathbb{R}^{m+p}$ are the orthonormal eigenvectors of [QSR] such that [QSR] $\cdot g_{i}=$ $\lambda_{i} g_{i}, i=1, \ldots, m+p$. Let $\Gamma:=\operatorname{diag}\left\{\gamma_{1}, \ldots, \gamma_{m+p}\right\}>0$ be a diagonal positive definite matrix. Consider an input-output transformation

$$
\left[\begin{array}{l}
\mathbf{u}  \tag{4.8}\\
\mathbf{v}
\end{array}\right]=\Gamma \cdot \mathbf{G}^{T}\left[\begin{array}{l}
\eta \\
y
\end{array}\right],
$$

where $\mathbf{u} \in \mathbb{R}^{m}$ and $\mathbf{v} \in \mathbb{R}^{p}$ are new input and output signals, respectively. The following result is valid.

Lemma 4.3.1. Suppose a system (4.1) is (Q, S, R)-dissipative, and its [QSR] matrix has exactly p negative eigenvalues, i.e., the eigenvalues of $[Q S R]$ are such that $\lambda_{1} \geq \ldots \geq \lambda_{m} \geq 0>\lambda_{m+1} \geq$ $\lambda_{m+p}$. Then the transformed system (4.1), (4.8) with input $\mathbf{u}$ and output $\mathbf{v}$ is finite $\mathcal{L}_{2}$-gain $(\hat{A}, \hat{B})$ stable, where $\hat{A}:=\operatorname{diag}\left\{\lambda_{1} / \gamma_{1}^{2}, \ldots, \lambda_{m} / \gamma_{m}^{2}\right\} \geq 0$, and $\hat{B}:=\operatorname{diag}\left\{-\lambda_{m+1} / \gamma_{m+1}^{2}, \ldots,-\lambda_{m+p} / \gamma_{m+p}^{2}\right\}>$ 0 .

Proof Using (4.8), and taking into account that $G$ is orthogonal $\left(\mathrm{G}^{-1}=\mathrm{G}^{T}\right)$, one gets

$$
\begin{aligned}
& w=\left[\begin{array}{l}
\eta \\
y
\end{array}\right]^{T}[\mathrm{QSR}]\left[\begin{array}{l}
\eta \\
y
\end{array}\right]=\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right]^{T} \Gamma^{-1} \mathrm{G}^{T}[\mathrm{QSR}] \mathrm{G} \Gamma^{-1}\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right] \\
= & {\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right]^{T} \Gamma^{-2} \operatorname{diag}\left\{\lambda_{1}, \ldots, \lambda_{m+p}\right\}\left[\begin{array}{l}
\mathbf{u} \\
\mathbf{v}
\end{array}\right]=\mathbf{u}^{T} \hat{A} \mathbf{u}-\mathbf{v}^{T} \hat{B} \mathbf{v} . }
\end{aligned}
$$

The following result is a direct consequence of Lemma 4.3.1.

Corollary 4.3.2. Suppose a system (4.1) is $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative, and matrix $[Q S R] \in \mathbb{R}^{(m+p) \times(m+p)}$ has exactly $p$ negative eigenvalues. Given diagonal matrices $A^{d}:=\operatorname{diag}\left\{a_{1}^{d}, \ldots a_{m}^{d}\right\}>0$, and $B_{d}:=\operatorname{diag}\left\{b_{1}^{d}, \ldots b_{p}^{d}\right\}>0$, there exists a transformation of the form (4.8), such that the transformed system (4.1), (4.8) with input $\mathbf{u}$ and output $\mathbf{v}$ is finite $\mathcal{L}_{2}$-gain $\left(A^{d}, B^{d}\right)$-stable. Specifically, the finite $\mathcal{L}_{2}$-gain $\left(A^{d}, B^{d}\right)$-stability of the transformed system (4.1), (4.8) is achieved by choosing diagonal elements of $\Gamma:=\operatorname{diag}\left\{\gamma_{1}, \ldots, \gamma_{m+p}\right\}$ positive and such that $\gamma_{i} \geq \sqrt{\lambda_{i} / a_{i}^{d}}$ for $i=1, \ldots, m$, and $\gamma_{i} \leq \sqrt{-\lambda_{i} / b_{i}^{d}}$ for $i=m+1, \ldots, m+p$.

### 4.4 Stabilization of complex interconnections with delays



Figure 4.1: A network of dissipative systems with input-output transformations and communication constraints.

In this section, we present the main result of this chapter which deals with scattering-based stabilization of networked (Q, S, R)-dissipative systems with communication delays as shown in Figure 4.1. Consider a set of nonlinear systems of the form

$$
\Sigma_{i}:\left\{\begin{array}{rl}
\dot{x}_{i} & =f_{i}\left(x_{i}, \eta_{i}\right),  \tag{4.9}\\
y_{i} & =h_{i}\left(x_{i}, \eta_{i}\right),
\end{array} \quad i=1, \ldots, N\right.
$$

where $x_{i} \in \mathbb{R}^{n_{i}}, \eta_{i} \in \mathbb{R}^{m_{i}}, y_{i} \in \mathbb{R}^{p_{i}}$. The systems $\Sigma_{1}, \ldots, \Sigma_{N}$ are interconnected through a set of input-output transformations and an interconnection network with communication constraints as illustrated in Figure 4.1 and mathematically described as follows. The local input-output scattering transformations have the form

$$
\left[\begin{array}{l}
\mathbf{u}_{i}  \tag{4.10}\\
\mathbf{v}_{i}
\end{array}\right]=\mathbb{S}_{i}\left[\begin{array}{l}
\eta_{i} \\
y_{i}
\end{array}\right], \quad i=1, \ldots, N,
$$

where $\mathbf{u}_{i} \in \mathbb{R}^{m_{i}}, \mathbf{v}_{i} \in \mathbb{R}^{p_{i}}$ are the transformed input and output, respectively. To formulate the constraints imposed by a network of communication channels between the subsystems (4.9), (4.10), let us denote $\mathbf{U}:=\left[\mathbf{u}_{1}^{T}, \ldots, \mathbf{u}_{N}^{T}\right] \in \mathbb{R}^{\mathbf{m}}, \mathbf{V}:=\left[\mathbf{v}_{1}^{T} \ldots \mathbf{v}_{N}^{T}\right] \in \mathbb{R}^{\mathbf{p}}$, where $\mathbf{m}:=\sum_{i=1}^{N} m_{i}$, and $\mathbf{p}:=\sum_{i=1}^{N} p_{i}$. The communication constraints are described in terms of a gain matrix $\Psi \in \mathbb{R}_{+}^{\mathbf{m} \times \mathbf{p}}$ and a matrix of communication delays $\mathrm{T} \in \mathbb{R}_{+}^{\mathbf{m} \times \mathbf{p}}$, where $\mathbb{R}_{+}^{\mathbf{m} \times \mathbf{p}}$ denotes the set of $\mathbf{m} \times \mathbf{p}$ matrices with nonnegative elements. In particular, an element $\mathrm{T}_{i j}$ of matrix T describes a (constant) communication delay inside the network between the signals $\mathbf{v}_{j}$ and $\mathbf{u}_{i}$. In addition, all disturbances acting on signals in the communication network are aggregated in an external signal $\Delta(t) \in \mathbb{R}^{\mathbf{q}}$, and its effect on signals in the network is described by a matrix $\mathcal{D} \in \mathbb{R}_{+}^{\mathbf{m} \times \mathbf{q}}$. Overall, the interconnection and communication constraints $\mathcal{M}(\Psi, T, \mathcal{D})$ imposed by the network of communication channels between the subsystems (4.9), (4.10) are described by a set of inequalities of the following form: for $i=1, \ldots, \mathbf{m}$,

$$
\begin{equation*}
\left|\mathbf{U}_{i}(t)\right| \leq \max \left\{\max _{j \in\{1, \ldots, \mathbf{p}\}} \Psi_{i j} \cdot\left|\mathbf{V}_{j}\left(t-\mathrm{T}_{j i}\right)\right|, \max _{k \in\{1, \ldots, \mathbf{q}\}} \mathcal{D}_{i k} \cdot\left|\Delta_{k}(t)\right|\right\} \tag{4.11}
\end{equation*}
$$

Also, denote $\mathbf{x}:=\left[\mathbf{x}_{1}^{T} \ldots \mathbf{x}_{N}^{T}\right] \in \mathbb{R}^{\mathbf{p}}$, where $\mathbf{p}:=\sum_{i=1}^{N} n_{i}$. Assuming each subsystem $\Sigma_{i}, i=$ $1, \ldots, N$ is $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative, our goal is to find scattering transformations $\mathbb{S}_{i}, i=1, \ldots, N$, such that the interconnection (4.9)-(4.11) is finite-gain $\mathcal{L}_{2}$-stable with respect to the disturbance input $\Delta(t)$ for any set of constant communication delays T. Since the interconnected system (4.9)-(4.11) contains multiple communication delays, the definition of finite gain $\mathcal{L}_{2^{-}}$ stability (Definition 4.2) needs to be adjusted, as follows.

Definition 4.3. Given $A=A^{T} \in \mathbb{R}^{m \times m}, A \geq 0, B=B^{T} \in \mathbb{R}^{p \times p}, B>0$, a system with delays (4.9)-(4.11) is said to be weakly finite $\mathcal{L}_{2}$-gain $(A, B)$-stable if there exists a storage function $\mathcal{V}: \mathbb{R}^{\mathbf{n}} \rightarrow \mathbb{R}_{+}$such that the inequality

$$
\mathcal{V}(\mathbf{x}(t))-\mathcal{V}\left(\mathbf{x}\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t}\left(\Delta^{T}(\tau) A \Delta(\tau)-\mathbf{V}^{T}(\tau) B \mathbf{V}(\tau)\right) d \tau+a\left(t_{0}\right)
$$

holds along the trajectories of (4.9)-(4.11), where $a\left(t_{0}\right) \geq 0$ may depend on the system's trajectories during the time interval $\left[t_{0}-\mathrm{T}_{\max }, t_{0}\right]$, where $\mathrm{T}_{\max }:=\underset{i \in\{1, \ldots, \mathbf{m}\}, j \in\{1, \ldots, \mathbf{p}\}}{\operatorname{Tax}} \mathrm{T}_{j i}$.

Finite $\mathcal{L}_{2}$-gain stability (Definition 4.2 is a special case of weak finite $\mathcal{L}_{2}$-gain stability (Definition 4.3) where $a\left(t_{0}\right) \equiv 0$. Nonzero $a\left(t_{0}\right)$ summarizes the effect of previous trajectories on the current value of the storage function which may exist due to communication delays in the interconnected system.

Suppose now each subsystem $\Sigma_{1}, \ldots, \Sigma_{N}$ is (Q, S, R)-dissipative, and the corresponding supply rate matrices are denoted by $[\mathrm{QSR}]_{i}, i=1, \ldots, N$. For each $\Sigma_{i}$, let the eigenvalues of $[\mathrm{QSR}]_{i}$ be written in descending order, i.e., $\lambda_{1}^{i} \geq \lambda_{2}^{i} \geq \ldots \geq \lambda_{m_{i}+p_{i}}^{i}$, and define $G_{i}:=$
$\left[g_{1}^{i} g_{2}^{i} \ldots g_{m_{i}+p_{i}}^{i}\right] \in \mathbb{R}^{\left(m_{i}+p_{i}\right) \times\left(m_{i}+p_{i}\right)}$, where $g_{1}^{i}, \ldots, g_{m_{i}+p_{i}}^{i} \in \mathbb{R}^{m_{i}+p_{i}}$ are the orthonormal eigenvectors of $[\mathrm{QSR}]_{i}$ such that $[\mathrm{QSR}]_{i} \cdot g_{j}^{i}=\lambda_{j}^{i} j_{j}^{i}, i=1, \ldots, m_{i}+p_{i}$. Furthermore, for each $i=1, \ldots, N$, denote $A^{i}:=\operatorname{diag}\left\{\lambda_{1}^{i}, \ldots, \lambda_{m_{i}}^{i}\right\} \in \mathbb{R}^{m_{i} \times m_{i}}$, and $B^{i}:=\operatorname{diag}\left\{-\lambda_{m_{i}+1}^{i}, \ldots,-\lambda_{m_{i}+p_{i}}^{i}\right\} \in \mathbb{R}^{p_{i} \times p_{i}}$. Also, define diagonal positive definite matrices of coefficients $\Gamma_{1}^{i} \in \mathbb{R}^{m_{i} \times m_{i}}, \Gamma_{2}^{i} \in \mathbb{R}^{p_{i} \times p_{i}}, \Gamma_{1}^{i}>0$, $\Gamma_{2}^{i}>0$. Finally, define $\mathbf{A}:=\operatorname{diag}\left\{A^{1}, \ldots, A^{N}\right\} \in \mathbb{R}^{\mathbf{m} \times \mathbf{m}}, \mathbf{B}:=\operatorname{diag}\left\{B^{1}, \ldots, B^{N}\right\} \in \mathbb{R}^{\mathbf{p} \times \mathbf{p}}$, $\Gamma_{1}:=\operatorname{diag}\left\{\Gamma_{1}^{1}, \ldots, \Gamma_{1}^{N}\right\} \in \mathbb{R}^{\mathbf{m} \times \mathbf{m}}$, and $\Gamma_{2}:=\operatorname{diag}\left\{\Gamma_{2}^{1}, \ldots, \Gamma_{2}^{N}\right\} \in \mathbb{R}^{\mathbf{p} \times \mathbf{p}}$. The following theorem is the main result of this work.

Theorem 4.4.1. Suppose each subsystem $\Sigma_{1}, \ldots, \Sigma_{N}$ is $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative, and each matrix $[Q S R]_{i}$ has exactly $p_{i}$ negative eigenvalues. If the matrices of coefficients $\Gamma_{1}$ and $\Gamma_{2}$ are chosen such that

$$
\begin{equation*}
\sigma_{\max }\left(\Gamma_{1}^{-1} \mathbf{A}^{1 / 2} \Psi \mathbf{B}^{-1 / 2} \Gamma_{2}\right)<1 \tag{4.12}
\end{equation*}
$$

then the scattering transformations of the form (4.10), where

$$
\mathbb{S}_{i}:=\left[\begin{array}{cc}
\Gamma_{1}^{i} & \mathbb{O}  \tag{4.13}\\
\mathbb{O} & \Gamma_{2}^{i}
\end{array}\right] \cdot G_{i}^{T}, \quad i=1, \ldots, N,
$$

make the interconnected system (4.9), (4.10), (4.11) weakly finite-gain $\mathcal{L}_{2}$-stable with respect to the disturbance input $\Delta(t)$.

Proof By assumption, each subsystem $\Sigma_{1}, \ldots, \Sigma_{N}$ is (Q, S, R)-dissipative, and each matrix $[\mathrm{QSR}]_{i}, i=1, \ldots, N$, has exactly $p_{i}$ negative eigenvalues. Applying Lemma 4.3.1, one concludes that, for each $i=1, \ldots, N$, the $i$-th transformed subsystem (4.9), (4.10), (4.13) with input $\mathbf{u}_{i}$ and output $\mathbf{v}_{i}$ is finite $\mathcal{L}_{2}$-gain $\left(\hat{A}^{i}, \hat{B}^{i}\right)$-stable, where $\hat{A}^{i}=\left(\Gamma_{1}^{i}\right)^{-1} A^{i}\left(\Gamma_{1}^{i}\right)^{-1} \geq 0$ and $\hat{B}^{i}=\left(\Gamma_{2}^{i}\right)^{-1} B^{i}\left(\Gamma_{2}^{i}\right)^{-1}>0$ are diagonal matrices. Using notation $\hat{\mathbf{A}}:=\operatorname{diag}\left\{\hat{A}^{1}, \ldots, \hat{A}^{N}\right\} \in \mathbb{R}^{\mathbf{m} \times \mathbf{m}}$, $\hat{\mathbf{B}}:=\operatorname{diag}\left\{\hat{B}^{1}, \ldots, \hat{B}^{N}\right\} \in \mathbb{R}^{\mathbf{p} \times \mathbf{p}}$, one has

$$
\hat{\mathbf{A}}=\Gamma_{1}^{-1} \mathbf{A} \Gamma_{1}^{-1}, \quad \text { and } \quad \hat{\mathbf{B}}=\Gamma_{2}^{-1} \mathbf{B} \Gamma_{2}^{-1}
$$

Furthermore, taking into account that matrices $\mathbf{A}, \mathbf{B}, \Gamma_{1}, \Gamma_{2}$ are all diagonal with nonnegative elements, one can write

$$
\hat{\mathbf{A}}^{1 / 2}=\Gamma_{1}^{-1} \mathbf{A}^{1 / 2}, \quad \text { and } \quad \hat{\mathbf{B}}^{-1 / 2}=\mathbf{B}^{-1 / 2} \Gamma_{2},
$$

and therefore

$$
\Gamma_{1}^{-1} \mathbf{A}^{1 / 2} \Psi \mathbf{B}^{-1 / 2} \Gamma_{2}=\hat{\mathbf{A}}^{1 / 2} \Psi \hat{\mathbf{B}}^{-1 / 2}
$$

Using the last equality, condition (4.12) can be rewritten in the form

$$
\begin{equation*}
\mathbb{I}_{\mathbf{p}}-\left(\hat{\mathbf{A}}^{1 / 2} \Psi \hat{\mathbf{B}}^{-1 / 2}\right)^{T}\left(\hat{\mathbf{A}}^{1 / 2} \Psi \hat{\mathbf{B}}^{-1 / 2}\right)>0, \quad \text { or } \quad \mathbb{I}_{\mathbf{p}}-\hat{\mathbf{B}}^{-1 / 2} \Psi^{T} \hat{\mathbf{A}} \Psi \hat{\mathbf{B}}^{-1 / 2}>0, \tag{4.14}
\end{equation*}
$$

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and taking into account that $\hat{\mathbf{B}}$ is positive definite, (4.12) becomes $\hat{\mathbf{B}}-\Psi^{T} \hat{\mathbf{A}} \Psi>0$. Finally, by continuity, the last inequality implies that

$$
\begin{equation*}
\hat{\mathbf{B}}-(1+\epsilon) \Psi^{T} \hat{\mathbf{A}} \Psi>0 \tag{4.15}
\end{equation*}
$$

holds for sufficiently small $\epsilon>0$.
Now, for each $i=1, \ldots, N$, let $V_{i}\left(x_{i}\right)$ be a storage function of the $i$-th subsystem (4.9). Consider the function $\mathcal{V}(\mathbf{x}):=\sum_{i=1}^{N} V_{i}\left(x_{i}\right)$, which is a storage function candidate for the interconnection (4.9), (4.10), (4.11), (4.13). Since each $i$-th transformed subsystem (4.9), (4.10), (4.13) with input $\mathbf{u}_{i}$ and output $\mathbf{v}_{i}$ is finite $\mathcal{L}_{2}$-gain $\left(\hat{A}^{i}, \hat{B}^{i}\right)$-stable, it follows that the dissipation inequality

$$
\begin{equation*}
\mathcal{V}(\mathbf{x}(t))-\mathcal{V}\left(\mathbf{x}\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t}\left(\mathbf{U}^{T} \hat{\mathbf{A}} \mathbf{U}-\mathbf{V}^{T} \hat{\mathbf{B}} \mathbf{V}\right) d \tau \tag{4.16}
\end{equation*}
$$

holds along the trajectories of the aggregated system which consists of $N$ subsystems (4.9), (4.10), (4.13) in parallel. Consider the first term under the integral in the right-hand side of (4.16). Since the matrix $\hat{\mathbf{A}}$ is diagonal with nonnegative elements, we have

$$
\mathbf{U}^{T} \hat{\mathbf{A}} \mathbf{U}=\sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \mathbf{U}_{i}^{2} .
$$

Taking into account the communication constraints 4.11, one obtains

$$
\begin{equation*}
\mathbf{U}^{T}(t) \hat{\mathbf{A}} \mathbf{U}(t) \leq \sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \max _{j \in\{1, \ldots, \mathbf{p}\}} \Psi_{i j}^{2} \cdot \mathbf{V}_{j}^{2}\left(t-\mathrm{T}_{j i}\right)+\sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \max _{k \in\{1, \ldots, \mathbf{q}\}} \mathcal{D}_{i k}^{2} \Delta_{k}^{2}(t) \tag{4.17}
\end{equation*}
$$

We have

$$
\begin{equation*}
\sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \max _{k \in\{1, \ldots, \mathbf{q}\}} \mathcal{D}_{i k}^{2} \Delta_{k}^{2} \leq \sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i}\left(\sum_{k=1}^{\mathbf{q}} \mathcal{D}_{i k}^{2} \Delta_{k}^{2}\right)=\sum_{k=1}^{\mathbf{q}}\left(\sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \mathcal{D}_{i k}^{2}\right) \Delta_{k}^{2}=\Delta^{T} \hat{\mathbf{D}} \Delta \tag{4.18}
\end{equation*}
$$

where $\hat{\mathbf{D}} \in \mathbb{R}^{\mathbf{q} \times \boldsymbol{q}}$ is a diagonal matrix with nonnegative elements $\hat{\mathbf{D}}_{k k}:=\sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \mathcal{D}_{i k}^{2} \geq 0$. On the other hand,

$$
\begin{align*}
\sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \max _{j \in\{1, \ldots, \mathbf{p}\}} \Psi_{i j}^{2} \cdot \mathbf{V}_{j}^{2} & \leq \sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i}\left(\sum_{j=1}^{\mathbf{p}} \Psi_{i j} \cdot\left|\mathbf{V}_{j}\right|\right)^{2} \\
& =\left[\left|\mathbf{V}_{1}\right| \ldots\left|\mathbf{V}_{\mathbf{p}}\right|\right] \cdot \Psi^{T} \hat{\mathbf{A}} \Psi \cdot\left[\left|\mathbf{V}_{1}\right| \ldots\left|\mathbf{V}_{\mathbf{p}}\right|\right]^{T} \tag{4.19}
\end{align*}
$$

Now, substituting (4.17) into the dissipation inequality (4.16) and using (4.18), one obtains

$$
\begin{equation*}
\mathcal{V}(\mathbf{x}(t))-\mathcal{V}\left(\mathbf{x}\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t}\left(\Delta^{T} \hat{\mathbf{D}} \Delta-\mathbf{V}^{T} \hat{\mathbf{B}} \mathbf{V}\right) d \tau+\int_{t_{0}}^{t} \sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \max _{j \in\{1, \ldots, \mathbf{p}\}} \Psi_{i j}^{2} \mathbf{V}_{j}^{2}\left(\tau-\mathrm{T}_{j i}\right) d \tau \tag{4.20}
\end{equation*}
$$

## However,

$$
\begin{align*}
\int_{t_{0}}^{t} \sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \max _{j \in\{1, \ldots, \mathbf{p}\}} \Psi_{i j}^{2} \mathbf{V}_{j}^{2}\left(\tau-\mathrm{T}_{j i}\right) d \tau & \leq \int_{t_{0}-\mathrm{T}_{\max }}^{t} \sum_{i=1}^{\mathbf{m}} \hat{\mathbf{A}}_{i i} \max _{j \in\{1, \ldots, \mathbf{p}\}} \Psi_{i j}^{2} \mathbf{V}_{j}^{2}(\tau) d \tau \\
& \leq \int_{t_{0}-\mathrm{T}_{\max }}^{t}\left[\begin{array}{c}
{\left[\mathbf{V}_{1}(\tau) \mid\right.} \\
\vdots \\
\left|\mathbf{V}_{\mathbf{p}}(\tau)\right|
\end{array}\right]^{T} \Psi^{T} \hat{\mathbf{A}} \Psi\left[\begin{array}{c}
\left|\mathbf{V}_{1}(\tau)\right| \\
\vdots \\
\left|\mathbf{V}_{\mathbf{p}}(\tau)\right|
\end{array}\right] d \tau \tag{4.21}
\end{align*}
$$

where the first inequality is valid because the term under the integral is non-negative and all delays $\mathrm{T}_{j i}$ are bounded by $\mathrm{T}_{\text {max }}$, and second inequality is due to (4.19). Denote

$$
a_{0}\left(t_{0}\right):=\int_{t_{0}-T_{\max }}^{t_{0}}\left[\begin{array}{c}
\left|\mathbf{V}_{1}\right|  \tag{4.22}\\
\vdots \\
\left|\mathbf{V}_{\mathbf{p}}\right|
\end{array}\right]^{T} \Psi^{T} \hat{\mathbf{A}} \Psi\left[\begin{array}{c}
\left|\mathbf{V}_{1}\right| \\
\vdots \\
\left|\mathbf{V}_{\mathbf{p}}\right|
\end{array}\right] d \tau
$$

Also,

$$
\left[\begin{array}{c}
\left|\mathbf{V}_{1}\right|  \tag{4.23}\\
\vdots \\
\left|\mathbf{V}_{\mathbf{p}}\right|
\end{array}\right]^{T} \Psi^{T} \hat{\mathbf{A}} \Psi\left[\begin{array}{c}
\left|\mathbf{V}_{1}\right| \\
\vdots \\
\left|\mathbf{V}_{\mathbf{p}}\right|
\end{array}\right] \leq \frac{1}{1+\epsilon} \mathbf{V}^{T} \hat{\mathbf{B}} \mathbf{V}
$$

which follows from the small gain condition (4.15) and the fact that $\hat{\mathbf{B}}$ is diagonal and positive definite. Substituting (4.21)-(4.23) into (4.20), one gets

$$
\mathcal{V}(\mathbf{x}(t))-\mathcal{V}\left(\mathbf{x}\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t}\left(-\frac{\epsilon}{1+\epsilon} \mathbf{V}^{T} \hat{\mathbf{B}} \mathbf{V}+\Delta^{T} \hat{\mathbf{D}} \Delta\right) d \tau+a_{0}\left(t_{0}\right)
$$

The statement of Theorem 4.4.1 then follows.

### 4.5 Numerical example

In this section, we present a numerical design example and the results of simulations that support the theoretical developments presented above. Consider an interconnected system of the general structure shown in Figure 4.1, which consists of three subsystems $\Sigma_{i}, i=1,2,3$, described as follows. Subsystem $\Sigma_{1}$ has a form

$$
\Sigma_{1}:\left\{\begin{array}{l}
\dot{y}_{11}=-\left(2+\cos ^{2} y_{12}\right) y_{12}-\eta_{1}  \tag{4.24}\\
\dot{y}_{12}=\left(\cos ^{2} y_{12}\right) y_{11}-y_{12}+2 \eta_{1}
\end{array}\right.
$$

where $\eta_{1} \in \mathbb{R}^{1}$ is the input, and $y_{1}:=\left[\begin{array}{ll}y_{11} & y_{12}\end{array}\right]^{T} \in \mathbb{R}^{2}$ is the output of $\Sigma_{1}$. Subsystem $\Sigma_{2}$ is described as follows:

$$
\begin{equation*}
\Sigma_{2}: \quad \dot{y}_{2}=-y_{2}+\eta_{21}-\eta_{22} . \tag{4.25}
\end{equation*}
$$

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Its output is $y_{2} \in \mathbb{R}^{1}$, and the input is $\eta_{2}:=\left[\begin{array}{ll}\eta_{21} & \eta_{22}\end{array}\right]^{T} \in \mathbb{R}^{2}$. The third subsystem is

$$
\Sigma_{3}:\left\{\begin{array}{l}
\dot{y}_{31}=\left(1+e^{-\left|y_{31}\right|}\right) y_{32}+\eta_{31}  \tag{4.26}\\
\dot{y}_{32}=-\left(1+e^{-\left|y_{31}\right|}\right) y_{31}-0.5 y_{32}-\eta_{32}
\end{array}\right.
$$

The input of $\Sigma_{3}$ is $\eta_{3}:=\left[\begin{array}{ll}\eta_{31} & \eta_{32}\end{array}\right]^{T} \in \mathbb{R}^{2}$, and the output is $y_{3}:=\left[\begin{array}{ll}y_{31} & y_{32}\end{array}\right]^{T} \in \mathbb{R}^{2}$. Each subsystem $\Sigma_{i}, i \in\{1,2,3\}$, is equipped with a scattering transformation of the form (4.10), where $\mathbb{S}_{1}, \mathbb{S}_{2} \in \mathbb{R}^{3 \times 3}, \mathbb{S}_{3} \in \mathbb{R}^{4 \times 4}$, and $\mathbf{u}_{1}, \mathbf{v}_{2} \in \mathbb{R}^{1}, \mathbf{u}_{2}:=\left[\mathbf{u}_{21}, \mathbf{u}_{22}\right] \in \mathbb{R}^{2}, \mathbf{u}_{3}:=\left[\mathbf{u}_{31}, \mathbf{u}_{32}\right] \in \mathbb{R}^{2}$, $\mathbf{v}_{1}:=\left[\mathbf{v}_{11}, \mathbf{v}_{12}\right] \in \mathbb{R}^{2}, \mathbf{v}_{3}:=\left[\mathbf{v}_{31}, \mathbf{v}_{32}\right] \in \mathbb{R}^{2}$. The subsystems are subsequently interconnected through a network with communication delays described as follows

$$
\left[\begin{array}{c}
\mathbf{u}_{1}(t)  \tag{4.27}\\
\mathbf{u}_{21}(t) \\
\mathbf{u}_{22}(t) \\
\mathbf{u}_{31}(t) \\
\mathbf{u}_{32}(t)
\end{array}\right]=\frac{1}{4}\left[\begin{array}{cccccc}
0 & 0 & +1 & -1 & +1 & +1 \\
+1 & +1 & 0 & -1 & 0 & +1 \\
-1 & -1 & 0 & +1 & 0 & -1 \\
+1 & +1 & -1 & 0 & 0 & +1 \\
+1 & +1 & -1 & 0 & 0 & +1
\end{array}\right]\left[\begin{array}{c}
\mathbf{v}_{11}\left(t-\mathrm{T}_{1}\right) \\
\mathbf{v}_{12}\left(t-\mathrm{T}_{2}\right) \\
\mathbf{v}_{2}\left(t-\mathrm{T}_{3}\right) \\
\mathbf{v}_{31}\left(t-\mathrm{T}_{4}\right) \\
\mathbf{v}_{32}\left(t-\mathrm{T}_{5}\right) \\
\Delta(t)
\end{array}\right]
$$

where $\mathrm{T}_{1}, \ldots, \mathrm{~T}_{5}$ are communication delays, and $\Delta(t)$ is an external input signal. In our simulations, we use $\mathrm{T}_{1}=0.5 \mathrm{~s}, \mathrm{~T}_{2}=0.3 \mathrm{~s}, \mathrm{~T}_{3}=0.4 \mathrm{~s}, \mathrm{~T}_{4}=0.6 \mathrm{~s}, \mathrm{~T}_{5}=0.7 \mathrm{~s}$, and $\Delta(t)=10 \cdot \sin (t)$. Our simulation results indicate that without scattering-based design (i.e., where $\mathbb{S}_{1}, \mathbb{S}_{2}, \mathbb{S}_{3}$ are all unit matrices of the corresponding dimensions) the interconnected system (4.10), (4.24)-(4.27) is unstable; specifically, it exhibits an unbounded response to the input $\Delta(t)$. The corresponding plots are shown in Figure 4.2 (top).

In order to stabilize the interconnection (4.10), (4.24)-(4.27), we proceed with the scatteringbased design as developed above. We begin by establishing (Q, S, R) - dissipativity properties of each of the subsystems $\Sigma_{i}, i=1,2,3$. For subsystem $\Sigma_{1}$, choosing a storage function candidate $V_{1}=\left(y_{11}^{2}+y_{12}^{2}\right) / 2$, and calculating its time derivative along the trajectories of (4.24), one obtains

$$
\dot{V}_{1}=\left[\begin{array}{l}
\eta_{1} \\
y_{1}
\end{array}\right]^{T} \cdot[\mathrm{QSR}]_{1} \cdot\left[\begin{array}{l}
\eta_{1} \\
y_{1}
\end{array}\right], \quad \text { where } \quad[\mathrm{QSR}]_{1}=\left[\begin{array}{ccc}
0 & -1 / 2 & 1 \\
-1 / 2 & 0 & -1 \\
1 & -1 & -1
\end{array}\right] .
$$

The eigenvalues of $[\mathrm{QSR}]_{1}$ in descending order are $\lambda_{1}=1.3508, \lambda_{2}=-0.5$, and $\lambda_{3}=-1.8508$, and

$$
G_{1}=\left[\begin{array}{ccc}
0.6059 & 0.7071 & 0.3645 \\
-0.6059 & 0.7071 & -0.3645 \\
0.5155 & 0 & -0.8569
\end{array}\right]
$$

is a matrix whose columns are orthonormal eigenvectors of $[\mathrm{QSR}]_{1}$ corresponding to the above eigenvalues. For subsystem $\Sigma_{2}$, we choose $V_{2}=y_{2}^{2} / 2$, which results in dissipation equality

$$
\dot{V}_{2}=\left[\begin{array}{l}
\eta_{2} \\
y_{2}
\end{array}\right]^{T} \cdot[\mathrm{QSR}]_{2} \cdot\left[\begin{array}{l}
\eta_{2} \\
y_{2}
\end{array}\right], \quad \text { where } \quad[\mathrm{QSR}]_{2}=\left[\begin{array}{ccc}
0 & 0 & 0.5 \\
0 & 0 & -0.5 \\
0.5 & -0.5 & -1
\end{array}\right] .
$$

The eigenvalues of $[\mathrm{QSR}]_{2}$ in descending order are $\lambda_{1}=0.366, \lambda_{2}=0, \lambda_{3}=-1.366$, and the corresponding matrix of orthonormal eigenvectors is

$$
G_{2}=\left[\begin{array}{ccc}
-0.6280 & 0.7071 & -0.3251 \\
0.6280 & 0.7071 & 0.3251 \\
-0.4597 & 0 & 0.8881
\end{array}\right]
$$

Finally, for subsystem $\Sigma_{3}$, one can choose a storage function candidate of the form $V_{3}:=$ $\left(y_{31}^{2}+y_{32}^{2}\right) / 2$, whose derivative along the trajectories of (4.26) is

$$
\dot{V}_{3}=\left[\begin{array}{l}
\eta_{3} \\
y_{3}
\end{array}\right]^{T} \cdot[\mathrm{QSR}]_{3} \cdot\left[\begin{array}{l}
\eta_{3} \\
y_{3}
\end{array}\right], \quad \text { where } \quad[\mathrm{QSR}]_{3}:=\left[\begin{array}{cccc}
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & -0.5 \\
0.5 & 0 & 0 & 0 \\
0 & -0.5 & 0 & -0.5
\end{array}\right] .
$$

The eigenvalues of $[\mathrm{QSR}]_{3}$ are $\lambda_{1}=0.5, \lambda_{2}=0.309, \lambda_{3}=-0.5, \lambda_{4}=-0.809$, and the matrix of orthonormal eigenvectors corresponding to these eigenvalues is

$$
G_{3}=\left[\begin{array}{cccc}
0.7071 & 0 & -0.7071 & 0 \\
0 & -0.8507 & 0 & 0.5257 \\
0.7071 & 0 & 0.7071 & 0 \\
0 & 0.5257 & 0 & 0.8507
\end{array}\right]
$$

The next step is to design the gain matrices $\Gamma_{1}^{i}, \Gamma_{2}^{i}, i=1,2,3$. This can be done by choosing the aggregate matrices $\Gamma_{1}:=\operatorname{diag}\left\{\Gamma_{1}^{1}, \Gamma_{1}^{2}, \Gamma_{1}^{3}\right\}, \Gamma_{2}:=\operatorname{diag}\left\{\Gamma_{2}^{1}, \Gamma_{2}^{2}, \Gamma_{2}^{3}\right\}$ such that the small-gain condition (4.12) is satisfied with a prescribed margin. From (4.27), it is straightforward to evaluate the matrix $\Psi \in \mathbb{R}^{5 \times 5}$ in (4.11) as follows

$$
\Psi=\left[\begin{array}{lllll}
0 & 0 & 1 & 1 & 1  \tag{4.28}\\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0
\end{array}\right] .
$$

Matrices $\mathbf{A}$ and $\mathbf{B}$ are formed from the eigenvalues of $[\mathrm{QSR}]_{i}, i=1,2,3$, as follows:

$$
\begin{aligned}
& \mathbf{A}:=\operatorname{diag}\{1.3508,0.366,0,0.5,0.309\} \\
& \mathbf{B}:=\operatorname{diag}\{0.5,1.8508,1.366,0.5,0.809\}
\end{aligned}
$$

One would like to choose diagonal matrices $\Gamma_{1}, \Gamma_{2}$ such that the closed-loop gain satisfies

$$
\begin{equation*}
g_{\min } \leq \sigma_{\max }\left(\Gamma_{1}^{-1} \mathbf{A}^{1 / 2} \Psi \mathbf{B}^{-1 / 2} \Gamma_{2}\right) \leq g_{\max } \tag{4.29}
\end{equation*}
$$

for some $0<g_{\text {min }}<g_{\text {max }}<1$. For our design example, we choose $g_{\text {min }}=0.9, g_{\max }=0.95$. Additionally, we would like to guarantee the fulfilment of (4.29) such that $\Gamma_{1}, \Gamma_{2}$ are as close to unit matrices as possible; specifically, we would like to achieve that

$$
\max \left\{\left\|\Gamma_{1}-\mathbb{I}\right\|_{\infty},\left\|\Gamma_{2}-\mathbb{I}\right\|_{\infty}\right\} \rightarrow \min
$$

This is a constrained optimization problem, which we solved using fmincon function of Matlab. The obtained set of gain matrices $\Gamma_{1}^{i}, \Gamma_{2}^{i}, i=1,2,3$ is as follows:

$$
\begin{array}{ll}
\Gamma_{1}^{1}=[1.2802], & \Gamma_{2}^{1}=\left[\begin{array}{cc}
0.4672 & 0 \\
0 & 0.4651
\end{array}\right], \\
\Gamma_{1}^{2}=\left[\begin{array}{cc}
1.5141 & 0 \\
0 & 0.7263
\end{array}\right], & \Gamma_{2}^{2}=\left[\begin{array}{cc}
0.4655
\end{array}\right], \\
\Gamma_{1}^{3}=\left[\begin{array}{cc}
1.0122 & 0 \\
0 & 1.0098
\end{array}\right], & \Gamma_{2}^{3}=\left[\begin{array}{cc}
0.4580 & 0 \\
0 & 0.4678
\end{array}\right] . \tag{4.32}
\end{array}
$$

This choice of gain matrices results in the closed-loop gain $\sigma_{\max }\left(\Gamma_{1}^{-1} \mathbf{A}^{1 / 2} \Psi \mathbf{B}^{-1 / 2} \Gamma_{2}\right)=0.925$. Now, using the obtained values of $G_{i}, \Gamma_{1}^{i}, \Gamma_{2}^{i}, i=1,2,3$, the matrices of scattering transformations are calculated based on (4.13), as follows:

$$
\begin{align*}
& \mathbb{S}_{1}=\left[\begin{array}{cc}
\Gamma_{1}^{1} & \mathbb{O} \\
\mathbb{O} & \Gamma_{2}^{1}
\end{array}\right] G_{1}^{T}=\left[\begin{array}{ccc}
0.7757 & -0.7757 & 0.6599 \\
0.3304 & 0.3304 & 0 \\
0.1695 & -0.1695 & -0.3985
\end{array}\right],  \tag{4.33}\\
& \mathbb{S}_{2}=\left[\begin{array}{cc}
\Gamma_{1}^{2} & \mathbb{O} \\
\mathbb{O} & \Gamma_{2}^{2}
\end{array}\right] G_{2}^{T}=\left[\begin{array}{ccc}
-0.9509 & 0.9509 & -0.6960 \\
0.5136 & 0.5136 & 0 \\
-0.1513 & 0.1513 & 0.4134
\end{array}\right],  \tag{4.34}\\
& \mathbb{S}_{3}=\left[\begin{array}{cc}
\Gamma_{1}^{3} & \mathbb{O} \\
\mathbb{O} & \Gamma_{2}^{3}
\end{array}\right] G_{3}^{T}=\left[\begin{array}{cccc}
0.7157 & 0 & 0.7157 & 0 \\
0 & -0.8590 & 0 & 0.5309 \\
-0.3239 & 0 & 0.3239 & 0 \\
0 & 0.2459 & 0 & 0.3980
\end{array}\right] . \tag{4.35}
\end{align*}
$$

Finally, matrices of scattering transformations 4.33-4.35) allow for calculations of $\mathbf{u}_{i}, \mathbf{v}_{i}$ based on $\eta_{i}, y_{i}$ according to the formula (4.10). For implementation purposes, however, we need to calculate $\eta_{i}, \mathbf{v}_{i}$ based on known $\mathbf{u}_{i}, y_{i}$. It is straightforward to see that

$$
\left[\begin{array}{l}
\mathbf{u}  \tag{4.36}\\
\mathbf{v}
\end{array}\right]=\left[\begin{array}{ll}
\mathrm{S}_{11} & \mathrm{~S}_{12} \\
\mathrm{~S}_{21} & \mathrm{~S}_{22}
\end{array}\right]\left[\begin{array}{l}
\eta \\
y
\end{array}\right] \Leftrightarrow\left[\begin{array}{l}
\eta \\
\mathbf{v}
\end{array}\right]=\left[\begin{array}{cc}
\mathrm{S}_{11}^{-1} & -\mathrm{S}_{11}^{-1} \mathrm{~S}_{12} \\
\mathrm{~S}_{21} \mathrm{~S}_{11}^{-1} & \mathrm{~S}_{22}-\mathrm{S}_{21} \mathrm{~S}_{11}^{-1} \mathrm{~S}_{12}
\end{array}\right]\left[\begin{array}{l}
\mathbf{u} \\
y
\end{array}\right],
$$

provided that $S_{11}$ is full rank. Based on (4.36), the following relationships follow from (4.33)(4.35):

$$
\begin{align*}
& {\left[\begin{array}{l}
\eta_{1} \\
\mathbf{v}_{11} \\
\mathbf{v}_{12}
\end{array}\right]=\left[\begin{array}{llll}
1.2892 & 1 & -0.8507 \\
0.4259 & 0.6608 & -0.2811 \\
0.2185 & 0 & -0.5427
\end{array}\right]\left[\begin{array}{l}
\mathbf{u}_{1} \\
y_{11} \\
y_{12}
\end{array}\right],}  \tag{4.37}\\
& {\left[\begin{array}{l}
\eta_{21} \\
\eta_{22} \\
\mathbf{v}_{21}
\end{array}\right]=\left[\begin{array}{cccc}
-0.5258 & 0.9735 & -0.3660 \\
0.5258 & 0.9735 & 0.3660 \\
0.1591 & 0 & 0.5241
\end{array}\right]\left[\begin{array}{l}
\mathbf{u}_{21} \\
\mathbf{u}_{22} \\
y_{2}
\end{array}\right],}  \tag{4.38}\\
& {\left[\begin{array}{l}
\eta_{31} \\
\eta_{32} \\
\mathbf{v}_{31} \\
\mathbf{v}_{32}
\end{array}\right]=\left[\begin{array}{cccc}
1.397 & 0 & -1 & 0 \\
0 & -1.164 & 0 & 0.618 \\
-0.453 & 0 & 0.648 & 0 \\
0 & -0.286 & 0 & 0.55
\end{array}\right]\left[\begin{array}{l}
\mathbf{u}_{31} \\
\mathbf{u}_{32} \\
y_{31} \\
y_{32}
\end{array}\right] .} \tag{4.39}
\end{align*}
$$

The above formulas 4.37)-4.39) are used for implementation of the scattering-based stabilization scheme in our simulations. Examples of simulations of the interconnected system (4.10), (4.24)-(4.27) with scattering transformations (4.37)-(4.39) are shown in Figure 4.2 (bottom). These simulations have been performed for the same set of communication delays ( $\mathrm{T}_{1}=0.5$ $\mathrm{s}, \mathrm{T}_{2}=0.3 \mathrm{~s}, \mathrm{~T}_{3}=0.4 \mathrm{~s}, \mathrm{~T}_{4}=0.6 \mathrm{~s}, \mathrm{~T}_{5}=0.7 \mathrm{~s}$ ) and the same external input signal $(\Delta(t)=10 \sin (t))$ as those in the case without scattering-based stabilization (shown in Figure 4.2 (top)). It can be seen that the scattering-based design successfully stabilizes the interconnection, which is in accordance with the theory developed above.

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Figure 4.2: Response of the interconnected system (4.10), (4.24)-(4.27) with communication delays to the input signal $\Delta(t)=10 \cdot \sin (t)$.
Trajectories $\mathbf{v}_{11}(t), \mathbf{v}_{12}(t), \mathbf{v}_{2}(t), \mathbf{v}_{31}(t), \mathbf{v}_{32}(t)$ are shown. Top plot: without scatteringbased design, i.e., i.e., $\mathbb{S}_{1}, \mathbb{S}_{2}, \mathbb{S}_{3}$ are unit matrices. Bottom plot: with scattering transformations (4.37)-(4.39). Notice the substantially different scales along y-axes of the two figures.

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## Chapter 5

## Conclusion

### 5.1 Summary

This dissertation focuses on generalization of the scattering-based approach to stabilization of interconnections of non-planar conic systems. Chapter 1 describes research objectives and outlines existing results to date.

Chapter 2 presents a novel notion of non-planar conic systems. The proposed class of nonlinear systems involves the ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems and, consequently, the conventional passive and conic [18] (planar conic [13]) systems. In order to represent a (Q, S, R)-dissipative system as a non-planar conic system, we have developed the cone construction algorithm. The main advantages of the non-planar conic systems are as follows: dimensions of system's input and output can be different, restrictions on any input-output pair are not required to be uniform, and an interconnection of non-planar conic systems is a non-planar conic system. Therefore the class of non-planar conic systems allows to overcome limitations imposed by the notion(-s) of passivity and/or planar conicity.

Next, we have specified stability conditions for the interconnections with and without time delay of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems whose dimension of the output coincides with the number of negative eigenvalues of the [QSR] matrix in the quadratic supply rate. The interconnected systems can be represented as non-planar conic systems by estimating their centers and radii using the developed algorithm. In the case of non-delayed interconnection, we have established a graph separation stability condition in terms of the radii and the angle between centers of subsystems' cones. In the presence of time delay, stability results were obtained within the small-gain framework. To satisfy these stability conditions, we propose scattering-based controller design. A scattering transformation is essentially a combination of rotational and scale operators that allows for rendering the dynamics of a non-planar conic system into a prescribed
cone of compatible dimensions. For interconnections without delays, the generalized scattering transformation is designed to separate the cones of the interacted subsystems. An advantage of the scattering transformation is its relatively flexible structure that potentially may stabilize an interconnected system and simultaneously improve performance of the interconnected system. Regarding the delay independent stability, it can be achieved by applying appropriate scattering transformations at the both sides of the communication channel. In particular, the rotational operator aligns the center of the system's cone along the input subspace, whereas the scaling operator allows for tuning the gain of the system. As a result, the designed scattering transformation transforms a non-planar conic system into a finite gain $\mathcal{L}_{2}$ stable system with an appropriate gain. Overall, the proposed technique expands scattering-based stabilization approach to the class of non-planar conic systems.

Chapter 3 is devoted to the coupled stability problem which is solved within the generalized scattering-based approach developed in Chaper 2. The main goal of this chapter is to demonstrate that the proposed technique can be applied to stabilization of an interconnection of essentially non-passive systems. Conventional results on the coupled stability problem have been dominantly developed under the assumption of passivity of the systems involved in the interaction. However, there are examples of robot-environment interaction where the requirement of passivity is violated, for example, in the presence of the so-called slippage phenomena [7, 8]. In addition, passivity-based methods do not allow for solving the stability problem without affecting the robot's trajectory tracking performance in free space. The design example presented in Chapter 3 illustrates the main advantages of the developed stabilization approach. In this example, stabilization is achieved under rather general assumptions on the robot and the environment subsystems. The example outlines an approach that can be applied for many other coupled stability problems. In particular, the robot manipulator dynamics are described by Euler-Lagrange equation and controlled by the Lyapunov-based algorithm that provides convergence of the robot trajectory to a reference trajectory. The Euler-Lagrange dynamic equations of the environment also include positive definite stiffness and damping terms. The contact force is represented as a linear combination of stiffness and damping components, where the damping term is not sign definite, which results in non-passivity of the contact between the robot and the environment. In order to satisfy the graph separation stability condition, we designed the scattering-based controller that does not interfere the trajectory tracking in free space. In the considered model, the existence of such a controller is guaranteed by the specially developed scattering transformation that is constructed using numerical algorithms. As a result, we have demonstrated that for any positive stiffness of the environment, and for the environmental and contact damping terms, whose components satisfy some reasonable constraints, the designed controller provides the stable interaction and does not affect the trajectory track-
ing performance in free space. Thus, the developed approach successfully solves the coupled stability problem.

Chapter 4 presents an extension of the scattering-based stabilization technique to the case of complex interconnections of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems in the presence of multiple communication delays. The problem of stability of complex interconnections of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative systems has been addressed in [10] in the absence of time delay. The existing scattering-based stabilization results are largely limited to the case of feedback interconnections of passive systems with time delays [1, 3, 6, 11]. In this thesis, the aforementioned results have been substantially expanded using the generalized scattering-based stabilization technique. First, we show that under mild technical assumptions [QSR] matrix in the quadratic supply rate of a ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative system with $m$-dimensional input must have the number of nonnegative eigenvalues greater than or equal to $m$. This motivates the assumption made in this chapter where we address ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ ) - dissipative systems whose [QSR] matrix has exactly $m$ nonnegative eigenvalues, or, equivalently, whose output dimension coincides with the number of negative eigenvalues of the [QSR] matrix. Next, we introduced the notion of finite $\mathcal{L}_{2}$-gain $(A, B)$ stability that is a specification of the notion of finite gain $\mathcal{L}_{2}$ stability in the case of systems with multiple inputs and/or outputs. We then present a scattering transformation that transforms a ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative system into a finite $\mathcal{L}_{2}$-gain $(A, B)$-stable system, where $A(A \geq 0)$ and $B$ ( $B>0$ ) are any predefined diagonal gain matrices of admissible orders, specifically, the orders of matrices $A$ and $B$ coincide with the number of system's inputs and outputs, respectively. The algorithm uses the generalized version of the scattering transformation which is a combination of two linear transformations. The first transformation is an orthogonal operator that maps the subspace generated by the eigenvectors corresponding to nonnegative eigenvalues into the input subspace. The second transformation performs scaling and is used fora assigning arbitrary admissible gains $(A, B)$. The main result of the chapter describes the stabilization procedure for complex interconnections of (Q, S, R)-dissipative systems which guarantees a delay independent weak finite $\mathcal{L}_{2}$-gain $(A, B)$-stability of the overall system with respect to external disturbances in the presence of multiple heterogeneous communication delays. The developed technique substantially generalizes the existing results in this area [1,5, 15].

### 5.2 Future work

The final section discusses possible improvements, extensions and development directions of the presented results. Throughout the work, we dealt with the notion of $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipativity that requires an existence of a quadratic supply rate where the system's output dimension is equal to the number of negative eigenvalues of the [QSR] matrix. However, it is known that
a storage function is generally not unique [2, 4, 17]. Instead, the set of storage functions is bounded from below by the available storage $V_{a}(\cdot)$ and from above by the required supply $V_{r}(\cdot)$ [2, Section 4.4.3]. In many applications, a storage function represents total energy stored in the system and may depend on wide range of model parameters and the system's control algorithm. Varying these components of the model may result in different storage functions. Therefore, one possible direction is the development of methods for constructing a storage function with a quadratic supply rate whose [QSR] matrix has the same number of negative eigenvalues as the dimension of the system's output.

Next, in Chapter3, the coupled stability problem has been investigated. As we have demonstrated, the scattering-based controller allows for achieving the stable interaction of the manipulator and the environment while at the same time does not affect the robot's trajectory tracking performance in free space. However, the theoretical methods developed in Chapter 2 for the generalized scattering-based technique do not guarantee, at least directly, the desired structure of the scattering transformation. Thus, a complete analytical solution of the scattering-based design problem for coupled stability subject to constraints is the topic for future research.

Another direction for future work is an application of the proposed technique to teleoperator systems. Theoretical methods for systems of this type were developed in the Chapter 2 . However in practice, as mentioned above, there can be additional design requirements imposed on the scattering-based controller in order to guarantee the control scheme performance. An example is a requirement imposed on the slave tracking performance in free space; i.e., the slave manipulator must follow the trajectory generated by the master. Another example is the requirement of avoiding "wave-reflection" phenomena. It is expected that these requirements can be fulfilled by a special choice of scattering transformations applied on the both sides of the communication channel. Therefore, development of the analytical and numerical algorithms that are capable to perform both stabilization and improvement of performance issues is one of the most important questions for future research.

Also, some extensions of the conventional scattering-based approach to stabilization of the teleoperator systems in the presence of time-variable delays have been reported [9, 11]. The idea behind this extension is to implement time-variable gains at the local and remote sides of the teleoperator. This approach can potentially be combined with the proposed generalized scattering-based controller design.

Finally, one more possible direction for future research is related to the extension of the generalized scattering-based stabilization methods to the case of systems modeled within the so-called behavioral framework. The behavioral approach to modeling of interactive systems was developed in the works of J.C. Willems and summarized in [16]; its application to control of geometrically nontrivial interactive robot behavior is described in [14]. According to [12], a
dynamical system is defined within the behavioral framework as a triple $\Sigma=(\mathbb{T}, \mathbb{W}, \mathcal{B})$, where $\mathbb{T}$ is the set of time instants, $\mathbb{W}$ is a set called the signal space, and $\mathcal{B}$ a subset of $\mathbb{T} \rightarrow \mathbb{W}$ called the behavior. It is worth to mention that the conventional "input-output" description of system's behavior is a special case in the above definition, where $\mathbb{W}$ is a direct product of two sets: the set of inputs $\mathbb{U}$ and the set of outputs $\mathbb{Y}$, i.e., $\mathbb{W}:=\mathbb{U} \times \mathbb{Y}$. For interactive applications such as teleoperator systems, the partition of the external signals into inputs and outputs is frequently unnatural and may lead to difficulties related to the fact that such a partition imposes a causality structure (inputs are "cause", the outputs are "effect") that does not hold in physical reality. For example, in mechanical systems, "force" is related to "motion", but neither of them necessarily causes the other. In the case of linear time invariant systems, the multi-port network modeling of teleoperators does not impose such a causality structure; however, in the nonlinear case, the existing theory is based on the "input-state-output" paradigm. Therefore, a possible future direction of research is development of a nonlinear counterpart to multi-port network description of teleoperator systems within the behavioral framework.

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## Appendix A

## Adaptive schemes

Appendix A considers a dynamic system that is described by the Euler-Lagrange equations and locally controlled by the Lyapunov-based control algorithm. In this case, we show that the system is ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative, and furthermore, application of the adaptive version of the Lyapunov-based controller does not change the [QSR] matrix in the supply rate constructed for the non-adaptive version of the controller. As a result, it implies that the parameter adjustment algorithm in the adaptive Lyapunov-based controller does not affect the scattering transformation constructed by the [QSR] matrix of the $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative system.

This development is motivated by the adaptive control schemes proposed within the conventional passivity-based approach. Particularly, an inclusion of the parameter adaptation mechanism into the local passivity-based control algorithm does not interfere with the stabilization procedure Therefore, this analysis is aimed at establishing similar results for the more general case of ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ ) - dissipative and non-planar conic systems.

Let the robot manipulator dynamics be determined by the Euler-Lagrange equations in the task space

$$
\begin{equation*}
\Sigma_{R}: H_{\mathbf{x}}(q) \ddot{\mathbf{x}}+C_{\mathbf{x}}(q, \dot{q}) \dot{\mathbf{x}}+G_{\mathbf{x}}(q)=f+u, \tag{A.1}
\end{equation*}
$$

where $f$ is an external force, vectors $\mathbf{x}, \dot{\mathbf{x}}$ and $q, \dot{q}$ are position and velocity of the manipulator in the task and joint space, respectively. The control $u$ in (A.1) realizes the Lyapunov-based control algorithm that solves the trajectory tracking problem in free space. Namely, this controller ensures the convergence of the robot trajectory to a reference trajectory $\mathbf{x}_{r}$.

$$
\begin{align*}
u & =H_{\mathbf{x}}(q) \dot{r}+C_{\mathbf{x}}(q, \dot{q}) r+G_{\mathbf{x}}(q)-K \sigma, \quad K=K^{T}>0  \tag{A.2}\\
r & =\dot{\mathbf{x}}_{r}-\Lambda \widetilde{\mathbf{x}}, \quad \sigma=\dot{\mathbf{x}}+\Lambda \widetilde{\mathbf{x}}, \quad \widetilde{\mathbf{x}}=\mathbf{x}-\mathbf{x}_{r}, \quad \Lambda=\Lambda^{T}>0 \tag{A.3}
\end{align*}
$$

[^1]where positive definite matrices $K$ and $\Lambda$ are control gains. Substituting the control $u$ A.2) to the system A.1, we derive the following dynamics
\[

\left\{$$
\begin{align*}
\dot{\mathbf{x}} & =-\Lambda \widetilde{\mathbf{x}}+\sigma  \tag{A.4}\\
\dot{\sigma} & =\left[H_{\mathbf{x}}(q)\right]^{-1}\left(-C_{\mathbf{x}}(q, \dot{q}) \sigma-K \sigma+f\right)
\end{align*}
$$\right.
\]

Proposition A.1. The dynamic system (A.4) is $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative with respect to a quadratic supply rate depending on the external force $f$ as an input and position-velocity error $\left[\widetilde{\mathbf{x}}^{T}, \dot{\overrightarrow{\mathbf{x}}}^{T}\right]^{T}$ (A.3) as an output.

Proof In order to prove ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipativity of the system $(\mathrm{A} .4$, we need to find a nonnegative storage function $V(\mathbf{x})$ and a quadratic supply rate $w(f ; \widetilde{\mathbf{x}}, \dot{\mathbf{x}})$, such that the inequality

$$
\begin{equation*}
V\left(\mathbf{x}\left(t_{1}\right)\right)-V\left(\mathbf{x}\left(t_{0}\right)\right) \leqslant \int_{t_{0}}^{t_{1}} w(f(t) ; \widetilde{\mathbf{x}}(t), \dot{\overrightarrow{\mathbf{x}}}(t)) d t \tag{A.5}
\end{equation*}
$$

holds along the trajectories of the system (A.4) for any $t_{1} \geqslant t_{0}$, any initial state $\mathbf{x}\left(t_{0}\right)$.
Consider a storage function $V$ of the form

$$
\begin{equation*}
V=\frac{1}{2}\left(\sigma^{T} H_{\mathbf{x}}(q) \sigma+\widetilde{\mathbf{x}}^{T} M \widetilde{\mathbf{x}}\right), \tag{A.6}
\end{equation*}
$$

where $M$ is a symmetric positive definite matrix that can be chosen freely. The function $V(\cdot)$ is nonnegative, since the inertia matrix $H_{\mathbf{x}}(\cdot)$ is positive definite.

The next series of equalities computes the time derivative of the storage function $V$ A.6) along the trajectories of the system (A.4), using the fact that $\left[\dot{H}_{\mathbf{x}}(q)-2 C_{\mathbf{x}}(q, \dot{q})\right]$ is a skewsymmetric matrix.

$$
\begin{aligned}
& \dot{V}=\sigma^{T} H_{\mathbf{x}}(q) \dot{\sigma}+\frac{1}{2} \sigma^{T} \dot{H}_{\mathbf{x}}(q) \sigma+\widetilde{\mathbf{x}}^{T} M \dot{\mathbf{x}}= \\
& \sigma^{T}\left(\frac{1}{2} \dot{H}_{\mathbf{x}}(q)-C_{\mathbf{x}}(q, \dot{q})\right) \sigma+\sigma^{T}(-K \sigma+f)+\widetilde{\mathbf{x}}^{T} M \dot{\mathbf{x}}=-\sigma^{T} K \sigma+\sigma^{T} f+\widetilde{\mathbf{x}}^{T} M \dot{\tilde{\mathbf{x}}}= \\
& f^{T} \Lambda \widetilde{\mathbf{x}}+f^{T} \dot{\mathbf{x}}-\widetilde{\mathbf{x}}^{T} \Lambda K \Lambda \widetilde{\mathbf{x}}+\widetilde{\mathbf{x}}^{T}\left(\frac{1}{2} M-\Lambda K\right) \dot{\mathbf{x}}+\dot{\tilde{\mathbf{x}}}^{T}\left(\frac{1}{2} M-K \Lambda\right) \widetilde{\mathbf{x}}-\dot{\widetilde{\mathbf{x}}}^{T} K \dot{\mathbf{x}}
\end{aligned}
$$

Finally, the derivative of the storage function $V$ A.6 can be represented as a quadratic form in variables $f, \widetilde{\mathbf{x}}$ and $\dot{\widetilde{\mathbf{x}}}$, namely

$$
\begin{align*}
\dot{V} & =\left[\begin{array}{c}
f \\
\widetilde{\mathbf{x}} \\
\dot{\overrightarrow{\mathbf{x}}}
\end{array}\right]^{T}[\mathrm{QSR}]\left[\begin{array}{c}
f \\
\widetilde{\mathbf{x}} \\
\dot{\overrightarrow{\mathbf{x}}}
\end{array}\right]=: w(f ; \widetilde{\mathbf{x}}, \dot{\mathbf{x}}), \quad \text { where }  \tag{A.7}\\
{[\mathrm{QSR}] } & =\left[\begin{array}{ccc}
0 & 1 / 2 \Lambda & 1 / 2 \mathbb{I} \\
1 / 2 \Lambda & -\Lambda K \Lambda & 1 / 2 M-\Lambda K \\
1 / 2 \mathbb{I} & 1 / 2 M-K \Lambda & -K
\end{array}\right] \tag{A.8}
\end{align*}
$$

Obtained quadratic supply rate $w(f ; \widetilde{\mathbf{x}}, \dot{\tilde{\mathbf{x}}})$ A.7) together with the storage function A.6) satisfies the dissipation inequality (A.5), that proves $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipativity of the system (A.4).

Further, we consider an adaptive version of the Lyapunov-based controller and show that the application of the adaptive control algorithm to the system A.1] results in the same [QSR] matrix (A.8) as obtained in Proposition A.1.

According the adaptive control approach, the left-hand part of the dynamic equation (A.1) can be equivalently rewritten using the regressor $Y(q, \dot{q}, \ddot{q})$, i.e.

$$
\begin{equation*}
H_{\mathbf{x}}(q) \ddot{\mathbf{x}}+C_{\mathbf{x}}(q, \dot{q}) \dot{\mathbf{x}}+G_{\mathbf{x}}(q)=Y(q, \dot{q}, \ddot{q}) \theta \tag{A.9}
\end{equation*}
$$

where $\theta$ is a vector of unknown parameters of the manipulator. Substitution in the equation A.9) of a specific admissible value $\widehat{\theta}$ of the model parameters $\theta$ provides the following equality

$$
\begin{equation*}
\widehat{H}_{\mathbf{x}}(q) \ddot{\mathbf{x}}+\widehat{C}_{\mathbf{x}}(q, \dot{q}) \dot{\mathbf{x}}+\widehat{G}_{\mathbf{x}}(q)=Y(q, \dot{q}, \ddot{q}) \widehat{\theta} \tag{A.10}
\end{equation*}
$$

where inertia matrix $\widehat{H}_{\mathbf{x}}(\cdot)$, Coriolis and centrifugal forces $\widehat{C}_{\mathbf{x}}(\cdot)$, and gravitation term $\widehat{G}_{\mathbf{x}}(\cdot)$ are computed for $\theta=\widehat{\theta}$.

Using the adaptive Lyapunov-based control law, we establish a local manipulator controller $u$ in the following way

$$
\begin{equation*}
u=\widehat{H}_{\mathbf{x}}(q) \dot{r}+\widehat{C}_{\mathbf{x}}(q, \dot{q}) r+\widehat{G}_{\mathbf{x}}(q)-K \sigma, \quad K=K^{T}>0, \tag{A.11}
\end{equation*}
$$

where variable $\widetilde{\mathbf{x}}, r$, and $\sigma$ are determined in (A.3), and gain matrix $K$ is the same as in the non-adaptive controller A.2).

An equivalent representation of the controller $u$ A.11) reveals the connection between this controller and the regressor $Y(\cdot)$ A.10), specifically

$$
\begin{align*}
& u=\widehat{H}_{\mathbf{x}}(q)(\dot{\mathbf{x}}-\dot{\sigma})+\widehat{C}_{\mathbf{x}}(q, \dot{q})(\dot{\mathbf{x}}-\sigma)+\widehat{G}_{\mathbf{x}}(q)-K \sigma= \\
& Y(q, \dot{q}, \ddot{q}) \widehat{\theta}-\left(\widehat{H}_{\mathbf{x}}(q) \dot{\sigma}+\widehat{C}_{\mathbf{x}}(q, \dot{q}) \sigma+K \sigma\right) . \tag{A.12}
\end{align*}
$$

Returning to the original dynamic relations (A.1)-(A.9), we have

$$
\begin{equation*}
Y(q, \dot{q}, \ddot{q})=f+u \quad \Rightarrow \quad u=Y(q, \dot{q}, \ddot{q})-f . \tag{A.13}
\end{equation*}
$$

Obtained equalities (A.12)-(A.13) provides the following dynamics

$$
\begin{equation*}
\widehat{H}_{\mathbf{x}}(q) \dot{\sigma}+\widehat{C}_{\mathbf{x}}(q, \dot{q}) \sigma+K \sigma=\bar{Y}(q, \dot{q}, \dot{r}, \dot{\sigma}) \widetilde{\theta}+f \tag{A.14}
\end{equation*}
$$

where $\widetilde{\theta}$ is the parameter estimation error, i.e. $\widetilde{\theta}=\widehat{\theta}-\theta$, and $\bar{Y}(q, \dot{q}, \dot{r}, \dot{\sigma})$ is the regressor $Y(q, \dot{q}, \ddot{q})$, where acceleration $\ddot{q}$ is expressed through the new variables A.3). The following
chain of equalities explains the transfer from $Y(q, \dot{q}, \ddot{q})$ to $\bar{Y}(q, \dot{q}, \dot{r}, \dot{\sigma})$ in details using manipulator Jacobian $J$. First, formulae (A.3) imply that $\ddot{\mathbf{x}}=\dot{\sigma}+\dot{r}$, and then we express $\ddot{q}$ through mentioned variables $q, \dot{q}, \dot{r}$, and $\dot{\sigma}$

$$
\dot{\mathbf{x}}=J(q) \dot{q} \Rightarrow \quad \ddot{\mathbf{x}}=\dot{J}(q) \dot{q}+J(q) \ddot{q} \Rightarrow \quad \ddot{q}=J^{-1}(q)(\ddot{\mathbf{x}}-\dot{J}(q) \dot{q})=J^{-1}(q)(\dot{\sigma}+\dot{r}-\dot{J}(q) \dot{q}) .
$$

A parameter adaptation algorithm is borrowed from the adaptive control scheme

$$
\begin{equation*}
\dot{\bar{\theta}}=\dot{\bar{\theta}}=-\Gamma \bar{Y}^{T}(q, \dot{q}, \dot{r}, \dot{\sigma}) \sigma \tag{A.15}
\end{equation*}
$$

where $\Gamma$ is a symmetric positive definite matrix, i.e. $\Gamma=\Gamma^{T}>0$. Combining robot dynamics (A.14) and the parameter adaptation algorithm(A.15), we derive the dynamic system of the form

$$
\left\{\begin{align*}
\dot{\mathbf{x}} & =-\Lambda \widetilde{\mathbf{x}}+\sigma  \tag{A.16}\\
\dot{\sigma} & =\left[\widehat{H}_{\mathbf{x}}(q)\right]^{-1}\left(-\widehat{C}_{\mathbf{x}}(q, \dot{q}) \sigma-K \sigma+\bar{Y}\left(q_{R}, \dot{q}_{R}, \dot{r}, \dot{\sigma}\right) \widetilde{\theta}+f_{\mathrm{env}}+f_{r}\right) \\
\dot{\vec{\theta}} & =-\Gamma \bar{Y}^{T}(q, \dot{q}, \dot{r}, \dot{\sigma}) \sigma
\end{align*}\right.
$$

The next proposition constructs the [QSR] matrix of the dynamic system A.16) and establishes its connection with the [QSR] matrix (A.8) derived in the non-adaptive case.

Proposition A.2. The dynamic system (A.16) is $(\mathrm{Q}, \mathrm{S}, \mathrm{R})$-dissipative with respect to the input $f$ and output $\left[\widetilde{\mathbf{x}}^{T}, \dot{\mathbf{x}}^{T}\right]^{T}$. Moreover, its quadratic supply rate coincides with the supply rate (A.7) computed within the application of the non-adaptive Lyapunov-based controller to the system (A.1). Here we suppose that the control gains $K, \Lambda$ are the same in both controllers (A.2) and A.11.

Proof Consider a storage function candidate of the form

$$
\begin{equation*}
V_{A}=\frac{1}{2}\left(\sigma^{T} \widehat{H}_{\mathbf{x}}(q) \sigma+\widetilde{\mathbf{x}}^{T} M \widetilde{\mathbf{x}}+\widetilde{\theta}^{T} \Gamma^{-1} \widetilde{\theta}\right), \tag{A.17}
\end{equation*}
$$

where $M$ is a symmetric positive definite matrix that can be chosen freely. Here we use the same matrix $M$ as in the storage function (A.6) for the non-adaptive case.

Computation of the time derivative of the storage function $V_{A}$ A.17) along the trajectories of the system A.16 provides the following

$$
\begin{aligned}
\dot{V}_{A} & =\sigma^{T} \widehat{H}_{\mathbf{x}}(q) \dot{\sigma}+\frac{1}{2} \sigma^{T} \dot{\vec{H}}_{\mathbf{x}}(q) \sigma+\widetilde{\mathbf{x}}^{T} M \dot{\mathbf{x}}+\widetilde{\theta}^{T} \Gamma^{-1} \dot{\vec{\theta}}=\sigma^{T}\left(\frac{1}{2} \dot{\vec{H}}_{\mathbf{x}}(q)-\widehat{C}_{\mathbf{x}}(q, \dot{q})\right) \sigma+ \\
& \sigma^{T}(-K \sigma+\bar{Y}(q, \dot{q}, \dot{r}, \dot{\sigma}) \widetilde{\theta}+f)+\widetilde{\mathbf{x}}^{T} M \dot{\overrightarrow{\mathbf{x}}}+\widetilde{\theta}^{T} \Gamma^{-1}\left(-\Gamma \bar{Y}^{T}(q, \dot{q}, \dot{r}, \dot{\sigma}) \sigma\right)= \\
& -\sigma^{T} K \sigma+\sigma^{T} \bar{Y}(q, \dot{q}, \dot{r}, \dot{\sigma}) \widetilde{\theta}+\sigma^{T} f+\widetilde{\mathbf{x}}^{T} M \dot{\mathbf{x}}-\widetilde{\theta}^{T} \bar{Y}^{T}(q, \dot{q}, \dot{r}, \dot{\sigma}) \sigma= \\
& -\sigma^{T} K \sigma+\sigma^{T} f+\widetilde{\mathbf{x}}^{T} M \dot{\mathbf{x}}= \\
& \left(f^{*}\right)^{T} \Lambda \widetilde{\mathbf{x}}+\left(f^{*}\right)^{T}{\dot{\tilde{\mathbf{x}}}-\widetilde{\mathbf{x}}^{T} \Lambda K \Lambda \widetilde{\mathbf{x}}+\widetilde{\mathbf{x}}^{T}\left(\frac{1}{2} M-\Lambda K\right) \dot{\tilde{\mathbf{x}}}+\dot{\tilde{\mathbf{x}}}^{T}\left(\frac{1}{2} M-K \Lambda\right) \widetilde{\mathbf{x}}-\dot{\tilde{\mathbf{x}}}^{T} K \dot{\tilde{\mathbf{x}}}}^{2}
\end{aligned}
$$

The obtained derivative of the storage function $V_{A}$ (A.17) is a quadratic form in variables $f^{*}, \widetilde{\mathbf{x}}$ and $\dot{\mathbf{x}}$ that can be expressed through the matrix as follows

$$
\begin{align*}
\dot{V}_{A} & =\left[\begin{array}{c}
f \\
\widetilde{\mathbf{x}} \\
\dot{\mathbf{x}}
\end{array}\right]^{T}[\mathrm{QSR}]_{A}\left[\begin{array}{c}
f \\
\widetilde{\mathbf{x}} \\
\dot{\mathbf{x}}
\end{array}\right]=: w_{A}(f ; \widetilde{\mathbf{x}}, \dot{\mathbf{x}}), \quad \text { where }  \tag{A.18}\\
{[\mathrm{QSR}]_{A} } & =\left[\begin{array}{ccc}
0 & 1 / 2 \Lambda & 1 / 2 \mathbb{I} \\
1 / 2 \Lambda & -\Lambda K \Lambda & 1 / 2 M-\Lambda K \\
1 / 2 \mathbb{I} & 1 / 2 M-K \Lambda & -K
\end{array}\right] \tag{A.19}
\end{align*}
$$

Obviously, non-adaptive and adaptive versions of the Lyapunov-based controller applied to the system (A.1) result in the same [QSR] matrices (A.8) and A.19). Consequently, the supply rates $w(\cdot)$ A.7) and $w_{A}(\cdot)$ A.18) coincide with each other.

The present proposition reveals that both the Lyapunov-based controller and the adaptive Lyapunov-based control algorithm implemented as the local manipulator controller result in the same supply rate. Regarding the scattering-based approach, this implies that inclusion of the parameter adjustment algorithm into the Lyapunov-based local controller does not affect the scattering transformation constructed by the[QSR] matrices (or cones) of the ( $\mathrm{Q}, \mathrm{S}, \mathrm{R}$ )-dissipative (non-planar conic) systems involved into the interconnections.

## Appendix B

## Proofs of Theorems and Lemmas

## B. 1 Proof of Theorem 2.3.1

Consider the system (2.20), 2.21), where $\Sigma_{1} \in \overline{\operatorname{Int}}\left(\Omega_{1}, \phi_{r 1}\right), \Sigma_{2} \in \operatorname{Int}\left(\Omega_{2}, \phi_{r 2}\right)$. For brevity, throughout the proof we denote $W_{1}:=W\left(\Omega_{1}, \phi_{r 1}\right), W_{2}:=W\left(\Omega_{2}, \phi_{r 2}\right)$. Let vectors $g_{1}, g_{2}, \ldots, g_{m}$ form an orthonormal basis in $\Omega_{1}$, and $e_{1}, e_{2}, \ldots, e_{p}$ form an orthonormal basis in $\Omega_{2}$. Since $\bar{\Omega}_{1} \cap \Omega_{2}=\{0\}$, the vectors $g_{1}, \ldots, g_{m}, e_{1}, \ldots, e_{p}$ are linearly independent and therefore form a basis in $\mathbb{R}^{m+p}$. Define

$$
\begin{aligned}
P & :=\left[\begin{array}{llllll}
g_{1} & \ldots & g_{m} & e_{1} & \ldots & e_{p}
\end{array}\right] \in \mathbb{R}^{(m+p) \times(m+p)}, \\
Q & :=\left[\begin{array}{llllll}
e_{1} & \ldots & e_{p} & g_{1} & \ldots & g_{m}
\end{array}\right] \in \mathbb{R}^{(m+p) \times(m+p)} .
\end{aligned}
$$

One sees that columns of both $P$ and $Q$ form bases in $\mathbb{R}^{m+p}$. In the following, we will identify ordered bases in $\mathbb{R}^{m+p}$ with the corresponding $(m+p) \times(m+p)$-matrices whose columns are the vectors of these bases. Clearly, $P$ and $Q$ are related according to the following formulas

$$
Q=P \cdot\left[\begin{array}{cc}
\mathbb{O}_{m p} & \mathbb{I}_{m}  \tag{B.1}\\
\mathbb{I}_{p} & \mathbb{O}_{p m}
\end{array}\right], \quad P=Q \cdot\left[\begin{array}{cc}
\mathbb{O}_{p m} & \mathbb{I}_{p} \\
\mathbb{I}_{m} & \mathbb{O}_{m p}
\end{array}\right]
$$

Both bases $P$ and $Q$ consist of unit vectors, however, they are not necessarily orthonormal. For our purposes, it is convenient to introduce another two bases, denoted by $G$ and $E$, which are orthonormal. Specifically, the set of vectors $\left\{g_{1}, g_{2}, \ldots, g_{m}\right\} \in \Omega_{1}$ can be augmented with additional vectors $g_{m+1}, \ldots, g_{m+p} \in \Omega_{1}^{\perp}$ such that the columns of

$$
G:=\left[\begin{array}{llllll}
g_{1} & \ldots & g_{m} & g_{m+1} & \ldots & g_{m+p}
\end{array}\right]
$$

form an orthonormal basis in $\mathbb{R}^{m+p}$. Similarly, the set $\left\{e_{1}, e_{2}, \ldots, e_{p}\right\} \in \Omega_{2}$ can be augmented with additional vectors $e_{p+1}, \ldots, e_{p+m} \in \Omega_{2}^{\perp}$ such that the columns of

$$
E:=\left[\begin{array}{llllll}
e_{1} & \ldots & e_{p} & e_{p+1} & \ldots & e_{p+m}
\end{array}\right]
$$

form another orthonormal basis in $\mathbb{R}^{m+p}$. Since both $G$ and $E$ are orthonormal bases, one can find an orthogonal transformation that relates these two bases. Note that, by construction,

$$
P=G \cdot\left[\begin{array}{ll}
\mathbb{I}_{m} & A  \tag{B.2}\\
\mathbb{O}_{p m} & C
\end{array}\right]
$$

where

$$
A:=\left[\begin{array}{cccc}
e_{1}^{T} g_{1} & e_{2}^{T} g_{1} & \cdots & e_{p}^{T} g_{1} \\
e_{1}^{T} g_{2} & e_{2}^{T} g_{2} & \cdots & e_{p}^{T} g_{2} \\
\vdots & \vdots & \ddots & \vdots \\
e_{1}^{T} g_{m} & e_{2}^{T} g_{m} & \cdots & e_{p}^{T} g_{m}
\end{array}\right] \in \mathbb{R}^{m \times p}, \quad C:=\left[\begin{array}{cccc}
e_{1}^{T} g_{m+1} & e_{2}^{T} g_{m+1} & \cdots & e_{p}^{T} g_{m+1} \\
e_{1}^{T} g_{m+2} & e_{2}^{T} g_{m+2} & \cdots & e_{p}^{T} g_{m+2} \\
\vdots & \vdots & \ddots & \vdots \\
e_{1}^{T} g_{m+p} & e_{2}^{T} g_{m+p} & \cdots & e_{p}^{T} g_{m+p}
\end{array}\right] \in \mathbb{R}^{p \times p} .
$$

Similarly,

$$
Q=E \cdot\left[\begin{array}{cc}
\mathbb{I}_{p} & A^{T}  \tag{B.3}\\
\mathbb{O}_{m p} & D
\end{array}\right]
$$

where

$$
D:=\left[\begin{array}{cccc}
e_{p+1}^{T} g_{1} & e_{p+1}^{T} g_{2} & \cdots & e_{p+1}^{T} g_{m} \\
e_{p+2}^{T} g_{1} & e_{p+2}^{T} g_{2} & \cdots & e_{p+2}^{T} g_{m} \\
\vdots & \vdots & \ddots & \vdots \\
e_{p+m}^{T} g_{1} & e_{p+m}^{T} g_{2} & \cdots & e_{p+m}^{T} g_{m}
\end{array}\right] \in \mathbb{R}^{m \times m}
$$

Substituting (B.3) and (B.2) into (B.1), one obtains $G=E \cdot T_{E G}$, where

$$
T_{E G}=\left[\begin{array}{cc}
I_{p} & A^{T} \\
\mathbb{O}_{m p} & D
\end{array}\right]\left[\begin{array}{cc}
\mathbb{O}_{p m} & \mathbb{I}_{p} \\
\mathbb{I}_{m} & \mathbb{O}_{m p}
\end{array}\right]\left[\begin{array}{cc}
\mathbb{I}_{m} & -A C^{-1} \\
\mathbb{O}_{p m} & C^{-1}
\end{array}\right]=\left[\begin{array}{cc}
A^{T} & \left(\mathbb{I}_{p}-A^{T} A\right) C^{-1} \\
D & -D A C^{-1}
\end{array}\right]
$$

Similarly,

$$
E=G \cdot T_{G E}, \quad \text { where } \quad T_{G E}=\left[\begin{array}{cc}
A_{m p} & \left(\mathbb{I}_{m}-A A^{T}\right) D^{-1} \\
C & -C A^{T} D^{-1}
\end{array}\right]
$$

Since both $E$ and $G$ are orthonormal bases, we see that $T_{E G}, T_{G E}$ are real orthogonal, and $T_{E G}=T_{G E}^{T}$, or

$$
\left[\begin{array}{cc}
A^{T} & \left(\mathbb{I}_{p}-A^{T} A\right) C^{-1}  \tag{B.4}\\
D & -D A C^{-1}
\end{array}\right]=\left[\begin{array}{cc}
A^{T} & C^{T} \\
D^{-T}\left(\mathbb{I}_{m}-A A^{T}\right) & -D^{-T} A C^{T}
\end{array}\right]
$$

From (B.4), it follows that

$$
\begin{align*}
\mathbb{I}_{p}-A^{T} A & =C^{T} C, \quad \text { and }  \tag{B.5}\\
\mathbb{I}_{m}-A A^{T} & =D^{T} D \tag{B.6}
\end{align*}
$$

Taking into account (B.5), (B.6), the expressions for $T_{E G}, T_{G E}$ can be simplified, as follows,

$$
T_{E G}=\left[\begin{array}{cc}
A^{T} & C^{T} \\
D & -D A C^{-1}
\end{array}\right], \quad T_{G E}=\left[\begin{array}{cc}
A & D^{T} \\
C & -C A^{T} D^{-1}
\end{array}\right]
$$

Now note that, by construction of the basis $G$, the expression for projection matrix $\Pi_{\bar{\Omega}_{1}}$ in this basis has a particularly simple form, as follows:

$$
\left(\Pi_{\bar{\Omega}_{1}}\right)_{G}=\left[\begin{array}{cc}
\mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & \mathbb{O}_{p p}
\end{array}\right]
$$

Therefore,

$$
\left(W_{1}\right)_{G}=\left[\begin{array}{cc}
\mathbb{I}_{m} & \mathbb{O}_{m p}  \tag{B.7}\\
\mathbb{O}_{p m} & \mathbb{O}_{p p}
\end{array}\right]-\cos ^{2} \phi_{r 1} \mathbb{I}_{m+p}
$$

Similarly, the expression for projection matrix $\Pi_{\Omega_{2}}$ has a particularly simple form in the basis $E$, that implies the canonical form $\left(W_{2}\right)_{E}$ of $W_{2}$, namely

$$
\left(\Pi_{\Omega_{2}}\right)_{E}=\left[\begin{array}{cc}
\mathbb{I}_{p} & \mathbb{O}_{p m} \\
\mathbb{O}_{m p} & \mathbb{O}_{p p}
\end{array}\right] \Rightarrow \quad\left(W_{2}\right)_{E}=\left[\begin{array}{cc}
\mathbb{I}_{p} & \mathbb{O}_{p m} \\
\mathbb{O}_{m p} & \mathbb{O}_{p p}
\end{array}\right]-\cos ^{2} \phi_{r 2} \cdot \mathbb{I}_{m+p}
$$

For the purposes of our subsequent analysis, however, it is convenient to represent both $W_{1}$ and $W_{2}$ in the basis $Q$. Combining (B.1) and (B.2), one obtains $Q=G \cdot T_{G Q}$, where

$$
T_{G Q}=\left[\begin{array}{cc}
\mathbb{I}_{m} & A  \tag{B.8}\\
\mathbb{O}_{p m} & C
\end{array}\right]\left[\begin{array}{cc}
\mathbb{O}_{m p} & \mathbb{I}_{m} \\
\mathbb{I}_{p} & \mathbb{O}_{p m}
\end{array}\right]=\left[\begin{array}{cc}
A & \mathbb{I}_{m} \\
C & \mathbb{O}_{p m}
\end{array}\right]
$$

Using (B.7) and (B.8), one obtains the expression for $W_{1}$ in basis $Q$, as follows:

$$
\begin{align*}
\left(W_{1}\right)_{Q} & =T_{G Q}^{T} \cdot\left(W_{1}\right)_{G} \cdot T_{G Q}=\left[\begin{array}{cc}
A & \mathbb{I}_{m} \\
C & \mathbb{O}_{p m}
\end{array}\right]^{T}\left(\left[\begin{array}{cc}
\mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & \mathbb{O}_{p p}
\end{array}\right]-\cos ^{2} \phi_{r 1} \mathbb{I}\right)\left[\begin{array}{cc}
A & \mathbb{I}_{m} \\
C & \mathbb{O}_{p m}
\end{array}\right] \\
& =\left[\begin{array}{cc}
A^{T} A & A^{T} \\
A & \mathbb{I}_{m}
\end{array}\right]-\cos ^{2} \phi_{r 1}\left[\begin{array}{cc}
\mathbb{I}_{p} & A^{T} \\
A & \mathbb{I}_{m}
\end{array}\right]=\left[\begin{array}{cc}
\left(A^{T} A-\cos ^{2} \phi_{r 1} \mathbb{I}_{p}\right) & \sin ^{2} \phi_{r 1} A^{T} \\
\sin ^{2} \phi_{r 1} A & \sin ^{2} \phi_{r 1} \mathbb{I}_{m}
\end{array}\right] . \tag{B.9}
\end{align*}
$$

On the other hand, using (B.3), one can obtain the following expression for $W_{2}$ in basis $Q$ :

$$
\begin{align*}
\left(W_{2}\right)_{Q} & =T_{E Q}^{T} \cdot\left(W_{2}\right)_{E} \cdot T_{E Q}=\left[\begin{array}{cc}
\mathbb{I}_{p} & A^{T} \\
\mathbb{O}_{m p} & D
\end{array}\right]^{T}\left(\left[\begin{array}{cc}
\mathbb{I}_{p} & \mathbb{O}_{p m} \\
\mathbb{O}_{m p} & \mathbb{O}_{p p}
\end{array}\right]-\cos ^{2} \phi_{r 2} \mathbb{I}\right)\left[\begin{array}{cc}
\mathbb{I}_{p} & A^{T} \\
\mathbb{O}_{m p} & D
\end{array}\right] \\
& =\left[\begin{array}{cc}
\mathbb{I}_{p} & A^{T} \\
A & A A^{T}
\end{array}\right]-\left[\begin{array}{cc}
\mathbb{I}_{p} & A^{T} \\
A & \mathbb{I}_{m}
\end{array}\right] \cos ^{2} \phi_{r 2}=\left[\begin{array}{cc}
\sin ^{2} \phi_{r 2} \mathbb{I}_{p} & \sin ^{2} \phi_{r 2} A^{T} \\
\sin ^{2} \phi_{r 2} A & \left(A A^{T}-\cos ^{2} \phi_{r 2} \mathbb{I}_{m}\right)
\end{array}\right] . \tag{B.10}
\end{align*}
$$

By means of the obtained representation of $W_{1}(\overline{\mathrm{~B} .9})$ and $W_{2}(\overline{\mathrm{~B} .10})$ in the basis $Q$, the following lemma calculates the maximum singular values of the product of projectors $\Pi_{\bar{\Omega}_{1}}$ and $\Pi_{\Omega_{2}}$.

Lemma B.1.1. The maximum singular value of $\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}$ satisfies

$$
\sigma_{\max }\left(\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}\right)=\sigma_{\max }(A):=\sqrt{\max _{i=1, \ldots, k} \lambda_{i}\left(A^{T} A\right)}
$$

where $\lambda_{i}\left(A^{T} A\right), i=1, \ldots, k$ denote the eigenvalues of $A^{T} A$.
Proof The product of projection matrices $\Pi_{\Omega_{1}} \Pi_{\Omega_{2}}$ in the basis $E$ has a form

$$
\begin{align*}
\left(\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}\right)_{E} & =T_{E G} \cdot\left(\Pi_{\bar{\Omega}_{1}}\right)_{G} \cdot T_{G E} \cdot\left(\Pi_{\Omega_{2}}\right)_{E} \\
& =\left[\begin{array}{cc}
A^{T} & C^{T} \\
D & -D A C^{-1}
\end{array}\right] \cdot\left[\begin{array}{cc}
\mathbb{I}_{m} & \mathbb{O}_{m p} \\
\mathbb{O}_{p m} & \mathbb{O}_{p p}
\end{array}\right] \cdot\left[\begin{array}{cc}
A & D^{T} \\
C & -C A^{T} D^{-1}
\end{array}\right] \cdot\left[\begin{array}{cc}
\mathbb{I}_{p} & \mathbb{O}_{p m} \\
\mathbb{O}_{m p} & \mathbb{O}_{m m}
\end{array}\right] \\
& =\left[\begin{array}{cc}
A^{T} A & \mathbb{O}_{p m} \\
D A & \mathbb{O}_{m m}
\end{array}\right] \tag{B.11}
\end{align*}
$$

The singular values of $\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}$ are square roots of the eigenvalues of the (symmetric and nonnegative definite) matrix $\left(\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}\right)^{T}\left(\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}\right)$. Using ( $\overline{\mathrm{B} .11}$ ), one obtains

$$
\begin{aligned}
\left(\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}\right)^{T}\left(\Pi_{\bar{\Omega}_{1}} \Pi_{\Omega_{2}}\right) & =\left[\begin{array}{cc}
A^{T} A & A^{T} D^{T} \\
\mathbb{O}_{m p} & \mathbb{O}_{m m}
\end{array}\right] \cdot\left[\begin{array}{cc}
A^{T} A & \mathbb{O}_{p m} \\
D A & \mathbb{O}_{m m}
\end{array}\right]=\left[\begin{array}{cc}
\left(A^{T} A\right)^{2}+A^{T} D^{T} D A & \mathbb{O}_{p m} \\
\mathbb{O}_{m p} & \mathbb{O}_{m m}
\end{array}\right] \\
& =\left[\begin{array}{cc}
\left(A^{T} A\right)^{2}+A^{T}\left(\mathbb{I}_{m}-A A^{T}\right) A & \mathbb{O}_{p m} \\
\mathbb{O}_{m p} & \mathbb{O}_{m m}
\end{array}\right]=\left[\begin{array}{cc}
A^{T} A & \mathbb{O}_{p m} \\
\mathbb{O}_{m p} & \mathbb{O}_{m m}
\end{array}\right]
\end{aligned}
$$

The statement of Lemma B.1.1 follows.
The next lemma establishes that the sets $\mathcal{W}_{1}=\left\{v_{1}: v_{1}^{T} W_{1} v_{1} \geq 0, v_{1} \in \mathbb{R}^{m+p},\left|v_{1}\right| \neq 0\right\}$ and $\mathcal{W}_{2}=\left\{v_{2}: v_{2}^{T} W_{2} v_{2} \geq 0, v_{2} \in \mathbb{R}^{m+p},\left|v_{2}\right| \neq 0\right\}$ are separated.

Lemma B.1.2. Suppose condition (2.22) holds. If some $v \in \mathbb{R}^{m+p}$ is such that $v \in \mathcal{W}_{1}$, then $v \notin \mathcal{W}_{2}$, or in other words, if $v^{T} W_{1} v \geq 0(|v| \neq 0)$, then $v^{T} W_{2} v<0$.

Proof Suppose $v^{T} W_{1} v \geq 0$ for some $v \in \mathbb{R}^{m+p},|v| \neq 0$. Without loss of generality, assume $|v|=1$. Consider the representation of the quadratic form $v^{T} W_{1} v$ in the basis $Q$ :

$$
\begin{align*}
\left(v^{T} W_{1} v\right)_{Q} & =\left[\begin{array}{l}
v_{p} \\
v_{m}
\end{array}\right]^{T}\left[\begin{array}{cc}
\left(A^{T} A-\cos ^{2} \phi_{r 1} \mathbb{I}_{p}\right) & \sin ^{2} \phi_{r 1} A^{T} \\
\sin ^{2} \phi_{r 1} A & \sin ^{2} \phi_{r 1} \mathbb{I}_{m}
\end{array}\right]\left[\begin{array}{l}
v_{p} \\
v_{m}
\end{array}\right] \\
& =v_{p}^{T} A^{T} A v_{p}-\cos ^{2} \phi_{r 1}\left|v_{p}\right|^{2}+\sin ^{2} \phi_{r 1}\left|v_{m}\right|^{2}+2 \sin ^{2} \phi_{r 1} v_{m}^{T} A v_{p} \geq 0 \tag{B.12}
\end{align*}
$$

where, by definition of basis $Q,\left[v_{p}^{T}, 0_{m}^{T}\right]^{T} \in \Omega_{2}$ and $\left[0_{p}^{T}, v_{m}^{T}\right]^{T} \in \bar{\Omega}_{1}$. Taking into account that

$$
\begin{equation*}
1=|v|^{2}=\left|v_{p}\right|^{2}+\left|v_{m}\right|^{2}+2 v_{m}^{T} A v_{p} \tag{B.13}
\end{equation*}
$$

one obtains from (B.12) that

$$
\begin{equation*}
0 \leq v^{T} W_{1} v \leq \sin ^{2} \phi_{r 1}-\left(1-\sigma_{\max }^{2}\right)\left|v_{p}\right|^{2} \tag{B.14}
\end{equation*}
$$

Similarly, the quadratic form $v^{T} W_{2} v$ in the in the basis $Q$ has a form

$$
\begin{aligned}
\left(v^{T} W_{2} v\right)_{Q} & =\left[\begin{array}{l}
v_{p} \\
v_{m}
\end{array}\right]^{T}\left[\begin{array}{cc}
\sin ^{2} \phi_{r 2} \mathbb{I}_{p} & \sin ^{2} \phi_{r 2} A^{T} \\
\sin ^{2} \phi_{r 2} A & \left(A A^{T}-\cos ^{2} \phi_{r 2} \mathbb{I}_{m}\right)
\end{array}\right]\left[\begin{array}{l}
v_{p} \\
v_{m}
\end{array}\right] \\
& =\sin ^{2} \phi_{r 2}\left|v_{p}\right|^{2}+v_{m}^{T} A^{T} A v_{m}-\cos ^{2} \phi_{r 2}\left|v_{m}\right|^{2}+2 \sin ^{2} \phi_{r 2} v_{m}^{T} A v_{p}
\end{aligned}
$$

and taking B.13 into account, one obtains

$$
v^{T} W_{2} v \leq \sin ^{2} \phi_{r 2}-\left(1-\sigma_{\max }^{2}\right)\left|v_{m}\right|^{2} .
$$

Further, we will show that

$$
\begin{equation*}
\sin ^{2} \phi_{r 2}-\left(1-\sigma_{\max }^{2}\right)\left|v_{m}\right|^{2}<0 \tag{B.15}
\end{equation*}
$$

and therefore $v^{T} W_{2} v<0$. Inequality (B.15) will follow from the following sequence of inequalities

$$
\begin{equation*}
\left|v_{m}\right| \geq \cos \phi_{r 1}-\sigma_{\max } \cdot \frac{\sin \phi_{r 1}}{\sqrt{1-\sigma^{2}}}>\frac{\sin \phi_{r 2}}{\sqrt{1-\sigma^{2}}} \tag{B.16}
\end{equation*}
$$

To prove the first inequality in (B.16), one can use (B.13) and (B.14) to obtain

$$
\begin{aligned}
1= & \left|v_{p}\right|^{2}+\left|v_{m}\right|^{2}+2 v_{m}^{T} A v_{p} \leq\left|v_{p}\right|^{2}+\left|v_{m}\right|^{2}+2 \sigma_{\max }\left|v_{m}\right|\left|v_{p}\right| \\
\leq & \frac{\sin ^{2} \phi_{r 1}}{1-\sigma_{\max }^{2}}+\left|v_{m}\right|^{2}+2 \sigma_{\max }\left|v_{m}\right| \frac{\sin \phi_{r 1}}{\sqrt{1-\sigma_{\max }^{2}}}, \quad \text { or } \\
& \left|v_{m}\right|^{2}+2 \sigma_{\max }\left|v_{m}\right| \frac{\sin \phi_{r 1}}{\sqrt{1-\sigma_{\max }^{2}}}+\frac{\sin ^{2} \phi_{r 1}}{1-\sigma_{\max }^{2}}+1 \geq 0 .
\end{aligned}
$$

Factoring the left-hand side of the above inequality, one gets

$$
\left(\left|v_{m}\right|+\sigma_{\max }\left|v_{m}\right| \frac{\sin \phi_{r 1}}{\sqrt{1-\sigma_{\max }^{2}}}-\cos \phi_{r 1}\right) \cdot\left(\left|v_{m}\right|+\sigma_{\max }\left|v_{m}\right| \frac{\sin \phi_{r 1}}{\sqrt{1-\sigma_{\max }^{2}}}+\cos \phi_{r 1}\right) \geq 0
$$

The expression inside the second bracket in the left hand side of the above inequality is positive $(>0)$, therefore the first bracket is non-negative, which implies that the first inequality in (B.16) is valid. To prove the second inequality in (B.16), note that it is equivalent to the following

$$
\begin{equation*}
\sigma_{\max }^{2} \sin ^{2} \phi_{r 1}+2 \sigma_{\max } \sin \phi_{r 1} \sin \phi_{r 2}+\sin ^{2} \phi_{r 2}-\cos ^{2} \phi_{r 1}\left(1-\sigma_{\max }^{2}\right)<0 . \tag{B.17}
\end{equation*}
$$

To prove $\overline{B .17}$ ) (equivalently, the second inequality in (B.16), one can factor the left hand side of the above inequality and use some basic trigonometric formulae to obtain

$$
\begin{aligned}
& \sigma_{\max }^{2} \sin ^{2} \phi_{r 1}+2 \sigma_{\max } \sin \phi_{r 1} \sin \phi_{r 2}+\sin ^{2} \phi_{r 2}-\cos ^{2} \phi_{r 1}\left(1-\sigma_{\max }^{2}\right) \\
= & \sigma_{\max }^{2}+2 \sigma_{\max } \sin \phi_{r 1} \sin \phi_{r 2}-\cos \left(\phi_{r 1}+\phi_{r 2}\right) \cos \left(\phi_{r 1}-\phi_{r 2}\right) \\
= & \sigma_{\max }^{2}+\sigma_{\max }\left(\cos \left(\phi_{r 1}-\phi_{r 2}\right)-\cos \left(\phi_{r 1}+\phi_{r 2}\right)\right)-\cos \left(\phi_{r 1}+\phi_{r 2}\right) \cos \left(\phi_{r 1}-\phi_{r 2}\right) \\
= & \left(\sigma_{\max }-\cos \left(\phi_{r 1}+\phi_{r 2}\right)\right)\left(\sigma_{\max }+\cos \left(\phi_{r 1}+\phi_{r 2}\right)\right)
\end{aligned}
$$

Condition (2.22), however, implies that

$$
\left(\sigma_{\max }-\cos \left(\phi_{r 1}+\phi_{r 2}\right)\right)\left(\sigma_{\max }+\cos \left(\phi_{r 1}+\phi_{r 2}\right)\right)<0
$$

Therefore, the second inequality in (B.16) holds, and so does B.15). Thus, $v^{T} W_{2} v<0$, or $v \notin \mathcal{W}_{2}$.

Lemma B.1.3. Suppose condition (2.22) holds. Then there exists $\rho \geq 0$ such that $\mathbb{W}:=$ $\rho W_{1}+W_{2}$ is negative definite.

Proof The statement of Lemma B.1.3 follows immediately from Lemma B.1.2 by application of the seminal theorem on the losslessness of S-procedure for two quadratic forms originally proven in [1] (see also [2, Theorem 12]).

Using Lemma B.1.3, the proof of Theorem 2.3.1 is conducted as follows.
Proof of Theorem 2.3.1 Consider the interconnection (2.20), (2.21), where $\Sigma_{1} \in \overline{\operatorname{Int}}\left(\Omega_{1}, \phi_{r 1}\right)$, $\Sigma_{2} \in \operatorname{Int}\left(\Omega_{2}, \phi_{r_{2}}\right)$. For this interconnection, consider a storage function candidate

$$
\mathbb{V}:=\rho \cdot V_{1}+V_{2},
$$

where $V_{1}, V_{2}$ are storage functions of $\Sigma_{1}, \Sigma_{2}$, and $\rho \geq 0$ is a constant given by Lemma B.1.3. Then the following inequality holds along the trajectories of $\Sigma_{1}, \Sigma_{2}$ :

$$
\mathbb{V}\left(t_{1}\right)-\mathbb{V}\left(t_{0}\right) \leq \int_{t_{0}}^{t_{1}}\left(\left[\begin{array}{l}
y_{1}  \tag{B.18}\\
\eta_{1}
\end{array}\right]^{T} \rho W_{1}\left[\begin{array}{l}
y_{1} \\
\eta_{1}
\end{array}\right]+\left[\begin{array}{l}
\eta_{2} \\
y_{2}
\end{array}\right]^{T} W_{2}\left[\begin{array}{l}
\eta_{2} \\
y_{2}
\end{array}\right]\right) d \tau
$$

Taking into account the formulae (2.21) describing the systems' interconnection, the integrand in (B.18) can be represented in the form

$$
\begin{align*}
{\left[\begin{array}{l}
y_{1} \\
\eta_{1}
\end{array}\right]^{T} \rho W_{1}\left[\begin{array}{l}
y_{1} \\
\eta_{1}
\end{array}\right]+\left[\begin{array}{l}
\eta_{2} \\
y_{2}
\end{array}\right]^{T} W_{2}\left[\begin{array}{l}
\eta_{2} \\
y_{2}
\end{array}\right] } & =\left[\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right]^{T} \mathbb{W}\left[\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right]+\left[\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right]^{T} 2 \rho W_{1}\left[\begin{array}{c}
0 \\
\chi_{1}
\end{array}\right]+\left[\begin{array}{c}
0 \\
\chi_{1}
\end{array}\right]^{T} \rho W_{1}\left[\begin{array}{c}
0 \\
\chi_{1}
\end{array}\right] \\
& +\left[\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right]^{T} 2 \cdot W_{2}\left[\begin{array}{c}
\chi_{2} \\
0
\end{array}\right]+\left[\begin{array}{c}
\chi_{2} \\
0
\end{array}\right]^{T} W_{2}\left[\begin{array}{c}
\chi_{2} \\
0
\end{array}\right], \tag{B.19}
\end{align*}
$$

where $\mathbb{W}:=\rho W_{1}+W_{2}$. According to Lemma B.1.3, W is negative definite; from here, combining (B.18) and (B.19) and using some simple matrix estimates as well as applying Young's quadratic inequality, one sees that

$$
\mathbb{V}\left(t_{1}\right)-\mathbb{V}\left(t_{0}\right) \leq \int_{t_{0}}^{t}\left(-\delta\left|\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right|^{2}+\sigma\left|\begin{array}{l}
\chi_{1} \\
\chi_{2}
\end{array}\right|^{2}\right) d \tau
$$

where $\delta>0, \sigma \geq 0$. The statement of Theorem 2.3.1 follows.

## B. 2 Proof of Lemma 2.6 .1

Proof Due to the s,all-gain condition $\gamma_{1} \cdot \gamma_{2}<1$, there exists a small $\varepsilon$ such that

$$
(1+\varepsilon) \cdot \gamma_{1} \cdot \gamma_{2}<1
$$

Consider inputs of the interconnected system

$$
\mathbf{u}_{1 i}(t)=\mathbf{v}_{2 i}\left(t-T_{2}^{(i)}\right)+\delta_{1 i}(t), \quad \mathbf{u}_{2 j}(t)=\mathbf{v}_{1 j}\left(t-T_{1}^{(j)}\right)+\delta_{2 j}(t), \quad i=1, \ldots, m, j=1, \ldots, p .
$$

Using Young's quadratic inequality, the following estimates can be written for the norms of the input components

$$
\begin{aligned}
& \left|\mathbf{u}_{1}(t)\right|^{2}=\sum_{i=1}^{m}\left|\mathbf{u}_{1 i}(t)\right|^{2} \leqslant(1+\varepsilon) \sum_{i=1}^{m}\left|\mathbf{v}_{2 i}\left(t-T_{2}^{(i)}\right)\right|^{2}+\left(1+\frac{1}{\varepsilon}\right)\left\|\delta_{1}(t)\right\|^{2} \\
& \left|\mathbf{u}_{2}(t)\right|^{2}=\sum_{j=1}^{p}\left|\mathbf{u}_{1 j}(t)\right|^{2} \leq(1+\varepsilon) \sum_{j=1}^{p}\left|\mathbf{v}_{1 j}\left(t-T_{1}^{(j)}\right)\right|^{2}+\left(1+\frac{1}{\varepsilon}\right)\left|\delta_{2}(t)\right|^{2} .
\end{aligned}
$$

Consider a function $V\left(x_{1}, x_{2}\right):=V_{1}\left(x_{1}\right)+V_{2}\left(x_{2}\right)$, where $V_{1}\left(x_{1}\right), V_{2}\left(x_{2}\right)$ are the storage functions of $\Sigma_{1}, \Sigma_{2}$, respectively. The change of $V(t):=V\left(x_{1}(t), x_{2}(t)\right)$ along the trajectories of the interconnected system can be estimated as follows

$$
\begin{align*}
V\left(x\left(t_{1}\right)\right)-V\left(x\left(t_{0}\right)\right) \leq & \left.\left.\int_{t_{0}}^{t_{1}}\left[\left.-\frac{1}{\gamma_{1}}\left|\mathbf{v}_{1}(\tau)\right|^{2}+\gamma_{1}\left|\mathbf{u}_{1}(\tau)\right|^{2}-\frac{1}{\gamma_{2}}\left|\mathbf{v}_{2}(\tau)\right|^{2}+\gamma_{2} \right\rvert\, \mathbf{u}\right)_{2}(\tau)\right|^{2}\right] \\
\leq & \int_{t_{0}}^{t_{1}}\left[-\frac{1}{\gamma_{1}}\left|\mathbf{v}_{1}(\tau)\right|^{2}-\frac{1}{\gamma_{2}}\left|\mathbf{v}_{2}(\tau)\right|^{2}+\gamma_{1}(1+\varepsilon) \sum_{i=1}^{m}\left|\mathbf{v}_{2}\left(\tau-T_{2}^{(i)}\right)\right|^{2}\right.  \tag{B.20}\\
& +\gamma_{1}\left(1+\frac{1}{\varepsilon}\right)\left|\delta_{1}(\tau)\right|^{2}+\gamma_{2}(1+\varepsilon) \sum_{j=1}^{p}\left|\mathbf{v}_{1}\left(\tau-T_{1}^{(j)}\right)\right|^{2} \\
& \left.+\gamma_{2}\left(1+\frac{1}{\varepsilon}\right)\left|\delta_{2}(\tau)\right|^{2}\right] d \tau .
\end{align*}
$$

Let $T_{1}$ and $T_{2}$ denote the maximal delays in the forward and the return path, respectively, i.e.

$$
T_{1}:=\max _{j=1, \ldots, p} T_{1}^{(j)}, \quad T_{2}:=\max _{i=1, \ldots, m} T_{2}^{(i)} .
$$

We have

$$
\begin{align*}
\int_{t_{0}}^{t_{1}}\left|\mathbf{v}_{1}\left(\tau-T_{1}^{(j)}\right)\right|^{2} d \tau & =\int_{t_{0}-T_{1}^{(j)}}^{t_{1}-T_{1}^{(j)}}\left|\mathbf{v}_{1}(\tau)\right|^{2} d \tau \leq \int_{t_{0}-T_{1}^{(j)}}^{t_{1}}\left|\mathbf{v}_{1}(\tau)\right|^{2} d \tau \\
& =\int_{t_{0}-T_{1}^{(j)}}^{t_{0}}\left|\mathbf{v}_{1}(\tau)\right|^{2} d \tau+\int_{t_{0}}^{t_{1}}\left|\mathbf{v}_{1}(\tau)\right|^{2} d \tau  \tag{B.21}\\
& \leq \int_{t_{0}-T_{1}}^{t_{0}}\left|\mathbf{v}_{1}(\tau)\right|^{2} d \tau+\int_{t_{0}}^{t_{1}}\left|\mathbf{v}_{1}(\tau)\right|^{2} d \tau, \quad j=1, \ldots, p
\end{align*}
$$

Similarly, one can estimate the second output vector $\mathbf{v}_{2}(\cdot)$

$$
\begin{equation*}
\int_{t_{0}}^{t_{1}}\left|\mathbf{v}_{2}\left(\tau-T_{2}^{(i)}\right)\right|^{2} d \tau \leq \int_{t_{0}-T_{2}}^{t_{0}}\left|\mathbf{v}_{2}(\tau)\right|^{2} d \tau+\int_{t_{0}}^{t_{1}}\left|\mathbf{v}_{2}(\tau)\right|^{2} d \tau, \quad i=1, \ldots, m \tag{B.22}
\end{equation*}
$$

Define parameters $\alpha_{\varepsilon}, \beta$ and $\gamma_{\varepsilon}$

$$
\begin{align*}
\alpha_{\varepsilon} & :=\min \left\{\frac{1}{\gamma_{2}}-\gamma_{1}(1+\varepsilon) ; \frac{1}{\gamma_{1}}-\gamma_{2}(1+\varepsilon)\right\}, \quad \gamma_{\varepsilon}:=\max \left\{\gamma_{1} ; \gamma_{2}\right\} \cdot\left(1+\frac{1}{\varepsilon}\right), \\
\beta & =\beta\left(t_{0}, T_{1}, T_{2}\right):=\int_{t_{0}-T_{1}}^{t_{0}}\left|\mathbf{v}_{1}(\tau)\right|^{2} d \tau+\int_{t_{0}-T_{2}}^{t_{0}}\left|\mathbf{v}_{2}(\tau)\right|^{2} d \tau \tag{B.23}
\end{align*}
$$

Substituting ( (B.21), ( $\overline{\mathrm{B} .22)}$ into (B.20), and using ( (B.23), we obtain

$$
V\left(x\left(t_{1}\right)\right)-V\left(x\left(t_{0}\right)\right) \leq \int_{t_{0}}^{t_{1}}\left(-\alpha_{\varepsilon}\|\mathbf{v}\|^{2}+\gamma_{\varepsilon}\|\delta\|^{2}\right) d \tau+\beta\left(t_{0}, T_{1}, T_{2}\right)
$$

The inequality implies that the interconnected system is weakly $\mathcal{L}_{2}$-gain stable with finite gain $\gamma:=\sqrt{\gamma_{\varepsilon} / \alpha_{\varepsilon}}$.

## Bibliography

[1] V. A. Yakubovich. S-procedure in the nonlinear control theory. Vestnik Leningradskogo Universiteta, 1(1), 62-77, 1971.
[2] S. V. Gusev, A. L. Likhtarnikov. Kalman-Popov-Yakubovich lemma and the S-procedure: A historical essay. Automation and Remote Control, 67(11), 1768-1810, 2006.

## Appendix C

## MATLAB scripts for the coupled stability problem

In the appendix the program realization of the Robot-Environment interaction model is represented, using the numerical computing environment MATLAB.

## C. 1 Algorithm for computation of the dynamic cone's parameters

```
classdef Cone < handle
```

properties
Basis
Center
Radius
Projector
end
methods
function obj = Cone ${ }^{(*)}$ (QSR)
[obj.Basis, obj.Center, obj.Radius, obj.Projector] =... ConeConstruction ${ }^{(0)}$ (QSR);
end
end
end

Function: ConeConstruction ${ }^{(0)}$
function [Basis, Center, Radius, Projector] = ... ConeConstructionWithSort(QSR)

$$
\mathrm{W}=\mathrm{QSR} ;
$$

[BG, V] = eig(W);
$[B G, r 1]=\operatorname{qr}(B G, 0)$;
[V, I] = sort(diag(V), 'descend');
$\mathrm{n}=\operatorname{size}(\mathrm{V}, 1)$;
G1 = BG;
\% number of nonnegative eigenvalues >=0
$\mathrm{m}=0$;
\% number negative eigenvalues < 0 ;
$\mathrm{p}=0$;
$m p=\max (\mathrm{V})$;
$\mathrm{mm}=\operatorname{Inf} ;$
\% Eigenvectors corresponding to negative eigenvalues $\mathrm{NB}=\operatorname{zeros}(\operatorname{size}(B G))$;
\% Eigenvectors corresponding to non-negative eigenvalues PB = zeros(size(BG));
for $i=1: n$
BG(:, i) = G1(:, I(i));
if $V(i)>=0$
$\mathrm{m}=\mathrm{m}+1$;
$\operatorname{PB}(:, m)=B G(:, i) ;$
else

$$
\mathrm{p}=\mathrm{p}+1 ;
$$

$$
\operatorname{NB}(:, p)=B G(:, i) ;
$$

if abs(V(i)) < mm

```
                mm = abs(V(i));
```

```
            end
    end
    end
    if nargout > 1
    Center = PB(:, 1:m);
    end
    if nargout > 2
    Radius = atan(sqrt(mp/mm));
    end
    if nargout > 3
    Projector = PB(:, 1:m)*PB(:, 1:m)';
    end
    Basis = BG;
end
```


## C. 2 Robot manipulator model

classdef Robot < handle
properties
TimeSpan
\% Physical parameters
Mass
LinkLen
CMLen
InertiaMoment
\% Model Properties
QSR
Mu
Lambda
Stiffness
rCone

ContactPos
DesiredPath
end
methods
function obj $=\operatorname{Robot}(T)$
obj.TimeSpan = T;
obj.setLinkLen;
obj.setMass;
obj.setCenterMassLen;
obj.setInertiaMoment;
obj.setLambda;
obj.setStiffness;
obj.setMu;
obj.setQSR;
obj.setCone;
obj.setDesiredPath;
end
function [] = setMass(obj)
\% In kg
obj.Mass $=[0 ; 3.092 ; 1.910]$;
end
function [] = setLinkLen(obj)
\% In meters
obj.LinkLen $=$ [0.6731; 0.432; 0.434];
obj. ContactPos $=[$ obj.LinkLen(2) $+0.5 * o b j . \operatorname{LinkLen(3);~} .$. 0; ... obj.LinkLen(1) + 0.5*obj.LinkLen(3)];
end
function [] = setCenterMassLen(obj)
\% In meters

```
    obj.CMLen = [0; 0.216; 0.164];
```

end

```
function [] = setInertiaMoment(obj)
    % In kg*m^2
    I0 = 0.0151;
    l = obj.LinkLen;
    lc = obj.CMLen;
    m = obj.Mass;
    obj.InertiaMoment(1:4, 1) = [I0; ...% ]
                                    lc(3)^2*m(3); ... % J1
                                    lc(2)^2*m(2) + l(2)^2*m(3);... % J12
                                    l(2)*lc(3)*m(3)]; % J13
```

end
function [] = setMu(obj)
obj.Mu = GetMu ${ }^{(1)}$ (obj.Lambda, obj.Stiffness, 100);
end
function [] = setLambda(obj)
obj.Lambda = diag([2.25; 2; 2]);
end
function [] = setStiffness(obj)
obj.Stiffness = diag([1; 1; 1]);
end
function [] = setQSR(obj)
I3 = eye(3);
03 = zeros $(3,3)$;
TLm $=$ [I3 03; obj.Lambda I3];
T = [I3, 03, 03; [03; 03], TLm];
obj.QSR $=T$ '*[03, 03, 0.5*I3; ...

03, -obj.Mu*obj.Lambda, 0.5*obj.Mu*I3; ...
Q.5*I3, 0.5*obj.Mu*I3, -obj.Stiffness]*T;

RobQSR $=$ obj.QSR
end
function [] = setCone(obj)
obj.rCone $=$ Cone ${ }^{(*)}$ (obj.QSR);
${ }^{(*)}$ MATLAB script for Cone() in Section C. 1
end
function [] = setDesiredPath(obj)
1 = obj.LinkLen;
Pc = obj.ContactPos;
PQ = [l(2); 0 ; Pc(3)];

Tc = 2; \% seconds
Offset = 0.07; \% meters.
[tx, x] = SmoothDesiredPath ${ }^{(2)}$ (obj.TimeSpan, PQ, Pc, Tc, Offset);
function $[p, d p, d 2 p]=\operatorname{Path}(t)$
$\mathrm{p}=$ zeros(length( t$), 3$ );
if nargout > 1
dp = zeros(length(t), 3);
if nargout > 2
d2p = zeros(length(t), 3);
end
end
$\mathrm{p}(:, 1)=$ interp1(tx, $x(:, 1), \mathrm{t}$, 'spline’);
$p(:, 3)=\operatorname{Pc}(3) *$ ones (length $(\mathrm{t}), 1)$;
if nargout > 1
$\mathrm{dp}(:, 1)=\operatorname{interp} 1(\mathrm{tx}, \mathrm{x}(:, 2), \mathrm{t}, \mathrm{spline})$;
if nargout > 2
$\mathrm{d} 2 \mathrm{p}(:, 1)=\operatorname{interp} 1(\mathrm{tx}, \mathrm{x}(:, 3), \mathrm{t}$, 'spline');
end
end
end
obj.DesiredPath = @Path;
end
\% Dynamic methods
function $[\mathrm{X}]=$ FwdKin(obj, Q)
s1 = $\sin (Q(1,:)) ;$
$c 1=\cos (Q(1,:)) ;$
s2 = $\sin (Q(2,:)) ;$
c2 = $\cos (Q(2,:))$;
s23 $=\sin (Q(2,:)+Q(3,:)) ;$
$c 23=\cos (Q(2,:)+Q(3,:)) ;$
l = obj.LinkLen;
$\mathrm{X}(1,:)=c 1 . *(1(2) * c 2+1(3) * s 23)$;
$X(2,:)=s 1 . *(1(2) * c 2+1(3) * s 23) ;$
$X(3,:)=1(1)-l(2) * s 2+l(3) * c 23 ;$
end
function [Q] = InvKin(obj, X)
function $F=S y s Q(q)$
$F(:)=$ obj.FwdKin(q) - X;
end
Q = fsolve(@SysQ, zeros(3,1));
end
function [J] = Jacobian(obj, Q)
1 = obj.LinkLen;
s1 = $\sin (Q(1)) ;$
$c 1=\cos (Q(1)) ;$
$\mathrm{s} 2=\sin (\mathrm{Q}(2)) ;$
$c 2=\cos (Q(2))$;
s23 $=\sin (Q(2)+Q(3)) ;$
$c 23=\cos (Q(2)+Q(3)) ;$

J = zeros $(3,3)$;

$$
\begin{aligned}
& J(1,1)=-s 1 *(1(2) * c 2+1(3) * s 23) ; \\
& J(1,2)=c 1 *(-1(2) * s 2+1(3) * c 23) ; \\
& J(1,3)=1(3) * c 1 * c 23 ; \\
& J(2,1)=c 1 *(1(2) * c 2+1(3) * s 23) ; \\
& J(2,2)=s 1 *(-1(2) * s 2+1(3) * c 23) ; \\
& J(2,3)=1(3) * s 1 * c 23 ; \\
& J(3,2)=-1(2) * c 2-1(3) * s 23 ; \\
& J(3,3)=-1(3) * s 23 ;
\end{aligned}
$$

end
function $H=$ InertiaMatrix(obj, Q)
I = obj.InertiaMoment;
s3 $=\sin (Q(3)) ;$
$\mathrm{c} 2=\cos (\mathrm{Q}(2))$;
s23 = $\sin (Q(2)+Q(3)) ;$
$\mathrm{H}=\operatorname{zeros}(3,3)$;
$H(1,1)=I(1)+I(2) * s 23^{\wedge} 2+I(3) * c 2^{\wedge} 2+2 * I(4) * c 2 * s 23 ;$
$H(2,2)=I(2)+I(3)+2 * I(4) * s 3$;
$H(2,3)=I(2)+I(4) * s 3$;
$H(3,2)=H(2,3)$;
$H(3,3)=I(2)$;
end
function C = CoriolisMatrix(obj, Q, dQ)
I = obj.InertiaMoment;
$c 2=\cos (Q(2))$;
$c 3=\cos (Q(3))$;
$c 23=\cos (Q(2)+Q(3)) ;$

```
C = zeros(3,3);
```

$C(1,2)=(I(2) * \sin (2 *(Q(1)+Q(2)))-I(3) * \sin (2 * Q(2))+\ldots$ $2 * I(4) * \cos (2 * Q(2)+Q(3))) * d Q(1) ;$
$C(1,3)=(I(2) * \sin (2 *(Q(1)+Q(2)))+2 * I(4) * c 2 * c 23) * d Q(3) ;$
$C(2,1)=0.5 *(-I(2) * \sin (2 *(Q(1)+Q(2)))+I(3) * \sin (2 * Q(2))-\ldots$
$2 * I(4) * \cos (2 * Q(2)+Q(3))) * d Q(1) ;$
$C(2,3)=2 * I(4) * c 3 * d Q(2)+I(4) * c 3 * d Q(3)$;
$C(3,1)=-0.5 *(I(2) * \sin (2 *(Q(2)+Q(3)))+I(4) * c 2 * c 23) * d Q(1) ;$
$C(3,2)=-I(4) * c 3 * d Q(2)$;
end
function $G=$ Gravity (obj, Q)
1 = obj.LinkLen;
lc = obj.CMLen;
m = obj.Mass;
$\mathrm{g}=9.81$;
c2 $=\cos (Q(2))$;
s23 $=\sin (Q(2)+Q(3)) ;$
$\mathrm{G}=\operatorname{zeros}(3,1)$;
$G(2)=-(l c(3) * m(3) * s 23+l c(2) * m(2) * c 2+l(2) * m(3) * c 2) * g$;
$G(3)=-1 c(3) * m(3) * s 23 * g$;
end
function $\mathrm{dJ}=\mathrm{dJacobian(obj}$,Q , dQ )
1 = obj.LinkLen;
s1 = $\sin (Q(1)) ;$
$\mathrm{c} 1=\cos (\mathrm{Q}(1))$;
s2 = $\sin (Q(2)) ;$
$c 2=\cos (Q(2))$;
s23 $=\sin (Q(2)+Q(3)) ;$
$c 23=\cos (Q(2)+Q(3)) ;$

```
    % dJi derivative of each element of Jacobian wrt i-th variable
    %--- 1 ---
    dJ1 = [-c1*(l(2)*c2 + l(3)*s23), ...
        -s1*(-1(2)*s2 + l(3)*c23), ...
        -l(3)*s1*c23; ...
        -s1*(l(2)*c2 + l(3)*s23), ...
        c1*(-1(2)*s2 + 1(3)*c23), ...
        l(3)*c1*c23;...
        0, O, 0];
    %--- 2 ---
        dJ2 = [-s1*(-1(2)*s2 + l(3)*c23), ...
            -c1*(l(2)*c2 + l(3)*s23), ...
        -l(3)*c1*s23;...
        c1*(-1(2)*s2 + l(3)*c23), ...
        -s1*(l(2)*c2 + l(3)*s23), ...
        -1(3)*s1*s23;...
        0, l(2)*s2 - l(3)*c23, -1(3)*c23];
            %--- 3 ---
            dJ3 = -l(3)*[ s1*c23, c1*s23, c1*s23;...
                -c1*c23, s1*s23, s1*s23;...
            0, c23, c23];
        %--- dJ ---
        dJ = dJ1*dQ(1) + dJ2*dQ(2) + dJ3*dQ(3);
```

    end
    end
end

Function: GetMu ${ }^{(1)}$
function Mu = GetMu(Lambda, Stiffness, MuMax)
\% Set up such value of parameter mu
\% that ensures the minimal radius of the robot's cone

```
if nargin < 0
    MuMax = 100;
```

end

```
    L = Lambda;
    K = Stiffness;
    I3 = eye(3);
    03 = zeros(3,3);
    TLm = [I3 03; L I3];
    T = [I3, 03, 03; [03; 03], TLm];
    N = 1000;
    mu = linspace(0.01, MuMax, N)';
    phi = zeros(N, 1);
    vmin = phi;
    vmax = phi;
    for i = 1:N
        QSR = T'*[03, 03, 0.5*I3; ...
            03, -mu(i)*L, 0.5*mu(i)*I3; ...
            0.5*I3, 0.5*mu(i)*I3, -K]*T;
        C = Cone (*) (QSR);
        (*) MATLAB script for Cone() in Section C.1
        phi(i) = C.Radius;
    end
    [Phi, I] = min(phi);
    Mu = mu(I(1));
end
```

Function: SmoothDesiredPath ${ }^{(2)}$
function [tx, $x$ ] = SmoothDesiredPath(T, InitialPos, ContactPos, ...
ContactTime, Offset)
\% Desired path has the following structure
$\% \mathrm{x}(\mathrm{t})=\mathrm{at}+\mathrm{b}$, if $0<=\mathrm{t}<=$ Toffset;
$\% \mathrm{x}(\mathrm{t})=$ ContactPos(1) + Offset, if $\mathrm{t}>$ Toffset;
$\%$ a and $b$ are found by the conditions:
$\% \mathrm{x}(\mathrm{Q})=$ InitialPos(1), $\mathrm{x}($ ContactTime) $=$ ContactPos(1)

```
% y(t) = ContactPos(2);
% z(t) = ContactPos(3);
% Tc in seconds
Tc = ContactTime;
% Offset in meters
eps = Offset;
x0 = InitialPos(1);
xc = ContactPos(1);
b = x0;
a = (xc - x0)/Tc;
Teps = (1 + eps/(xc - x0))*Tc;
% Construct an observer
A = [0 1 0 0 0; 0 0 1 0 0; 0 0 0 1 0; 0 0 0 0 1; 0 0 0 0 0];
C = [100 0 0 0 ];
pls = linspace(-3, -2, 5);
L = place(A', C', pls);
L = L';
function [x, dx] = xRefSignal(t)
    N = length(t);
    x = zeros(1, N);
    dx = zeros(1, N);
    for i=1:N
        if t(i) <= Teps
            x(i) = a*t(i) + b;
            dx(i) = a;
        else
            x(i) = xc + eps;
            dx(i) = 0;
        end
    end
end
```

```
    function \(d x=x S y s(t, x)\)
        \(\mathrm{xr}=\mathrm{xRefSignal(t)} \mathrm{;} \mathrm{f}\)
        \(\mathrm{A} 1=\mathrm{A}-\mathrm{L} * \mathrm{C}\);
    \(\mathrm{dx}(1: 5,1)=\mathrm{A} 1 * \mathrm{x}+\mathrm{L} *[\mathrm{xr}] ;\)
    end
```

    [x0, dx0] = xRefSignal(0);
    [tx, \(x]=\) ode45(@xSys, [ \(0, \mathrm{~T}],[\mathrm{x} \theta ; \mathrm{dx} \theta ; 0 ; 0 ; 0])\);
    end

## C. 3 Environment model

classdef Environment < handle

## properties

Base
ContactPos

LinkLen
Mass

Stiffness
Damping

QSR
eCone
end
methods

$$
\begin{aligned}
& \text { function obj = Environment(LinkLength, ContactPosition, } . \text {. } \\
& \text { ContactDamping) } \\
& \text { \% ContactPos - coordinates of the conact position } \\
& \% \text { in the robot base frame } \\
& \% \text { Base - coordinates of the environment base } \\
& \% \text { in the robot base frame }
\end{aligned}
$$

```
    obj.ContactPos = ContactPosition;
    obj.LinkLen = LinkLength;
    obj.Base = [ContactPosition(1) + LinkLength(2); 0; 0];
    obj.setMass;
    obj.setStiffness;
    obj.setDamping;
    obj.setQSR(ContactDamping);
    obj.setCone;
end
function [] = setMass(obj)
    % In kg
    obj.Mass = [2.5; 2.5];
end
function [] = setStiffness(obj)
    % In N/m
    % Parameter set 2
    obj.Stiffness = [1000 0 0; 0 0 0; 0 0 1];
    % Parameters set 1
    % obj.Stiffness = [100 0 0; 0 0 0; 0 0 1];
end
function [] = setDamping(obj)
    obj.Damping = [10 0 0; 0 0 0; 0 0 0];
end
function [] = setQSR(obj, ContactDamping)
%---------------------------------------------------------
% Compute contact damping term satisfying requirements
% De - (1+eps)*cDse >= 0 (1)
% cDse - for the environmental cone;
%----------------------------------------------------------
```

```
    I3 = eye(3);
    03 = zeros(3,3);
    cDse = diag([max(-ContactDamping(1,1), 0);...
        max(-ContactDamping(2,2), 0);...
        max(-ContactDamping(3, 3), 0)]);
    dc = cDse(1,1);
    de = obj.Damping(1,1);
    if dc > 0
        eps = de/dc - 1;
        cDse = (1 + eps)*cDse/eps;
    else
        cDse = zeros(3,3);
    end
    QSRenv = [ 03, 03, -0.5*I3; ...
        03, 03, 03; ...
        -0.5*I3, 03, cDse];
    obj.QSR = QSRenv;
```

end

```
function [] = setCone(obj)
    obj.eCone = Cone (*)}(\textrm{obj.QSR);
    (*) MATLAB script for Cone() in Section C.1
```

end

```
% Dynamic methods
function X = FwdKin(obj, Q)
    l = obj.LinkLen;
    s1 = sin(Q(1, :));
    c1 = cos(Q(1, :));
    s12 = sin(Q(1, :) + Q(2, :));
    c12 = cos(Q(1, :) + Q(2, :));
    X(1, :) = obj.Base(1) - l(1)*s1 - l(2)*c12;
```

$$
\begin{aligned}
& X(2,:)=0 * c 1 ; \\
& X(3,:)=1(1) * c 1-1(2) * s 12
\end{aligned}
$$

end
function J = Jacobian(obj, Q)
J = zeros (3, 2);
s1 $=\sin (Q(1)) ;$
$c 1=\cos (Q(1)) ;$
$s 12=\sin (Q(1)+Q(2)) ;$
$\mathrm{c} 12=\cos (\mathrm{Q}(1)+\mathrm{Q}(2))$;
l = obj.LinkLen;
$J(1,1)=-l(1) * c 1+l(2) * s 12$;
$J(1,2)=1(2) * s 12 ;$
$J(3,1)=-l(1) * s 1-l(2) * c 12 ;$
$J(3,2)=-l(2) * c 12 ;$
end
function $d J=d J a c o b i a n(o b j, ~ Q, ~ d Q)$
$s 1=\sin (Q(1)) ;$
$\mathrm{c} 1=\cos (\mathrm{Q}(1))$;
$s 12=\sin (Q(1)+Q(2)) ;$
$c 12=\cos (Q(1)+Q(2))$;
l = obj.LinkLen;
\% dJi derivative of each element of Jacobian wrt i-th variable
\%--- 1 ---
$\mathrm{dJ} 1=\operatorname{zeros}(3,2)$;
$\mathrm{dJ} 1(1,1)=1(1) * s 1+1(2) * c 12$;
dJ1 $(1,2)=1(2) * c 12$;
dJ1 $(3,1)=-1(1) * c 1+l(2) * s 12 ;$
dJ1 $(3,2)=1(2) * s 12 ;$
\%--- 2 ---
dJ2 = zeros $(3,2)$;

```
    dJ2(1,1) = l(2)*c12;
    dJ2(1,2) = l(2)*c12;
    dJ2(3,1) = l(2)*s12;
    dJ2(3,2) = l(2)*s12;
    dJ = dJ1*dQ(1) + dJ2*dQ(2);
end
function H = InertiaMatrix(obj, Q)
    l = obj.LinkLen;
    m = obj.Mass;
    s2 = sin(Q(2));
    H = zeros(2, 2);
    H(1, 1) = l(1)^2*(m(1) +m(2)) + l(2)^2*m(2) - ...
        2*l(1)*l(2)*m(2)*s2;
    H(1, 2) = l(2)*m(2)*(l(2) - l(1)*s2);
    H(2, 1) = H(1, 2);
    H(2, 2) = l(2)^2*m(2);
end
function C = CoriolisMatrix(obj, Q, dQ)
    l = obj.LinkLen;
    m = obj.Mass;
    c2 = cos(Q(2));
    C = l(1)*l(2)*m(2)*c2*[-2*dQ(2) -dQ(2); dQ(1) 0];
end
function G = Gravity(obj, Q)
    l = obj.LinkLen;
    m = obj.Mass;
```

```
        s1 = sin(Q(1));
        c12 = cos(Q(1) + Q(2));
        g = 9.81;
        G = -g*[l(1)*(m(1) + m(2))*s1 + l(2)*m(2)*c12; l(2)*m(2)*c12];
        end
        function dP = StiffnessTerm(obj, Q)
            %Returns stiffness term in Joint space
            K = obj.Stiffness;
            X = obj.FwdKin(Q);
            J = obj.Jacobian(Q);
            Xs0 = [obj.ContactPos(1); 0; obj.ContactPos(3)];
            dP = 2*J'*K*(X - Xs0);
        end
            function D = DampingTerm(obj, Q)
            J = obj.Jacobian(Q);
            D = J'*obj.Damping*J;
        end
    end
end
```


## C. 4 Scattering Transformation

In this section, we represent MATLAB code that solves the optimization problem (3.19)-(3.20).

Function: ScatteringTransformation(s)
function $S$ = ScatteringTransformation(Rob, Env, Gap, Weights)
\% Original cone of the robot subsystem
Cr = Rob.rCone;

```
% Original cone of the environment subsystem
Ce = Env.eCone;
0 = zeros(3, 3);
I3 = eye(3);
I6 = eye(6);
s = sin(Cr.Radius);
c = cos(Cr.Radius);
Wr = Cr.Basis*[[s^2*I3, 0, 0]; [[0; 0], -c^2*I6]]*Cr.Basis’;
function F = Functional(a)
    S1 = diag(a(1:3));
    S21 = diag(a(4:6));
    S22 = diag(a(7:9));
    W1 = Weights(1)*I3;
    W2 = Weights(2)*I3;
    W3 = Weights(3)*I3;
    F = trace(W1*(S1 - I3)^2 + W2*S21^2 + W3*S22^2);
end
function [c, ceq] = NonLinCond(a)
    SO = [[diag(a(1:3)); diag(a(4:6)); diag(a(7:9))]...
    [[0 0]; I6]...
    ];
QSRd = SQ'*Wr*SO;
% Construct a cone for the derived matrix QSRd
C = Cone(QSRd);
M = C.Projector*Ce.Projector;
sigma = sqrt(max(eig(M*M')));
% Cone separation condition with desired Gap between cones
c = [sigma - cos(Ce.Radius + C.Radius + Gap)];
ceq = [];
```

end
options = optimoptions(@fmincon, 'MaxFunctionEvaluations', 30000,...
'MaxIterations', 10000);
\% Starting guess
$a 0(1: 3)=\operatorname{ones}(3,1)$;
$\mathrm{a} Q(4: 9)=\operatorname{zeros}(6,1)$;
[a,fval,exitflag,output] = fmincon(@OptFun, a0, [], [], [], [], ... [], [], @NonLinCond, options);
\% Scattering transformation

$$
\begin{aligned}
& S=[[\operatorname{diag}(a(1: 3)) ; \operatorname{diag}(a(4: 6)) ; \operatorname{diag}(a(7: 9))] \ldots \\
& \quad[[00] ; 16] \ldots
\end{aligned}
$$

];
end

## C. 5 Main unit

## C.5.1 Contact forces

function Fenv = ContactForces(q)
global Rob Env g_ContactDamping g_ContactStiffness
\% Compute a vector column of the contact force fenv
\% based on current state of the robot Pqr and
\% the environment Pqe in their joint space
$\% \mathrm{q}$ is a 10 -th dimensional vector
$\% \mathrm{q}(1,2,3)=\mathrm{qr}$ (robot joint coordinates)
$\% \mathrm{q}(4,5,6)=$ dqr (robot joint velocities)
$\% \mathrm{q}(7,8)=$ qe (environment joint coordinates)
$\% \mathrm{q}(9,10)=$ dqe (environment joint velocities)

```
Pr = Rob.FwdKin }\mp@subsup{}{}{(4)}(q(1:3))
dPr = Rob.Jacobian }\mp@subsup{}{}{(4)}(\textrm{q}(1:3))*q(4:6)
```

```
(4) MATLAB scripts in Section C.2
Pe = Env.FwdKin }\mp@subsup{}{}{(5)}(q(7:8))
dPe = Env.Jacobian (5) (q(7:8))*q(9:10);
(5) MATLAB scripts in Section C.3
P = Pe - Pr;
dP = dPe - dPr;
Fenv = zeros(3,1);
if Pe(1) - Pr(1) < 0
    Fenv = (g_ContactStiffness*P + g_ContactDamping*dP);
end
end
```


## C.5.2 Reference signals

```
function Ref = ReferenceSignals(Fenv, Des, X)
global g_ScatteringTransformation
% Compute reference signals Ref = [Fref, Pref, dPref]:
% reference force Fref,
% reference position and velocity [Pref, dPref]
% using desired position, velocity (Des) and contact force (Fenv)
% Here q is a 10-th dimensional vector
% q(1,2,3) = qr (robot joint coordinates)
% q(4,5,6) = dqr (robot joint velocities)
% q(7,8) = qe (environment joint coordinates)
% q(9,10) = dqe (environment joint velocities)
Xd = Des(:, 1);
dXd = Des(:, 2);
S = g_ScatteringTransformation;
S1 = S(1:3, 1:3);
S21 = S(4:6, 1:3);
```

```
    S22 = S(7:9, 1:3);
    S3 = S(7:9, 4:6);
    Ref(:, 1) = S1*Fenv - Fenv;
    Ref(:, 2) = Xd - S21*Fenv;
    Ref(:, 3) = dXd - S22*Fenv - S3*(X - Xd);
end
```


## C.5.3 Local controller

function Utau $=$ Controller (q, Xr, dXr, d2Xr)
global Rob
\% Compute the controller in the joint space: tau = J'*u
$\% \mathrm{u}=\mathrm{Hx}(\mathrm{q}) * \mathrm{dr}+\mathrm{Cx}(\mathrm{q}, \mathrm{dq}) * r-\mathrm{K} *$ sigma;
\% Gravity term is omitted from the dynamics and the controller, \% since it is cancelled.
\% Reference force is added directly to the dynamic equation, \% not in the controller.
\% q(1..6) - vector of joint variables for the robot:
\% q(1..3) - position, q(4..6) - velocity.
$\% \mathrm{q}(7 . .10)$ - vector of joint variables for the environment:
\% q(7..8) - position, q(9..10) - velocity,
\% q(11..19) - estimates of the reference signal in the task space:
\% q(11..13) $=$ Pref, $q(14 . .16)=$ dPref, $q(17 . .19)=$ d2Pref)
\% Robot Jacobian
$\mathrm{J}=$ Rob.Jacobian ${ }^{(6)}(\mathrm{q}(1: 3))$;
iJ = eye(3)/J;
$\mathrm{dJ}=$ Rob.dJacobian ${ }^{(6)}(\mathrm{q}(1: 3), \mathrm{q}(4: 6))$;
$\mathrm{Hr}=$ Rob.InertiaMatrix ${ }^{(6)}(\mathrm{q}(1: 3))$;
$\mathrm{Cr}=$ Rob.CoriolisMatrix ${ }^{(6)}(\mathrm{q}(1: 3), \mathrm{q}(4: 6))$;
X $=$ Rob.FwdKin ${ }^{(6)}(q(1: 3))$;
${ }^{(6)}$ MATLAB scripts in Section $\mathrm{C.2}$
\% In Cartesian space

```
Hx = iJ'*Hr*iJ;
Cx = (iJ'*Cr - Hx*dJ)*iJ;
dX = J*q(4:6);
sigma = (dX - dXr) + Rob.Lambda*(X - Xr);
r = dX - sigma;
dr = d2Xr - Rob.Lambda*(dX - dXr);
U = Hx*dr + Cx*r - Rob.Stiffness*sigma;
Utau = J'*U;
end
```


## C.5.4 Closed-loop dynamics

function dq = System(t, q)
global Rob Env g_RefSysA g_RefSysB
\% q(1,2,3) - qr (robot joint coordinates)
\% q(4,5,6) - dqr (robot joint velocities)
\% q(7,8) - qe (environment joint coordinates)
$\% \mathrm{q}(9,10)$ - dqe (environment joint velocities)
\% q (11,22) - estimates for reference trajectory and its derivatives
\% Contact force
Fenv $=$ ContactForces ${ }^{(7)}(\mathrm{q}(1: 10))$;
(7) MATLAB script in Section C.5.1
\% Current robot information
[Pdes, dPdes] = Rob.DesiredPath ${ }^{(8)}(\mathrm{t})$;
Pdes = Pdes';
dPdes = dPdes';
$\mathrm{X}=$ Rob.FwdKin ${ }^{(8)}(\mathrm{q}(1: 3))$;
$\mathrm{dX}=$ Rob.Jacobian ${ }^{(8)}(\mathrm{q}(1: 3)) * q(4: 6)$;
$\mathrm{Hr}=$ Rob.InertiaMatrix ${ }^{(8)}(\mathrm{q}(1: 3))$;

```
Cr = Rob.CoriolisMatrix (8) (q(1:3), q(4:6));
Jr = Rob.Jacobian }\mp@subsup{}{}{(8)}(\textrm{q}(1:3))
iHr = eye(3)/Hr;
(8) MATLAB script in Section C.2
% Reference sigmals
Ref = ReferenceSignals (9)}\mp@subsup{}{}{(9)
Fref = Ref(:, 1);
Xr = Ref(:, 2);
dXr = Ref(:, 3);
d2Xr = q(17:19);
(9) MATLAB script in Section C.5.2
% Local manipulator controller
tau = Controller (10) (q(1:19), Xr, dXr, d2Xr);
(10) MATLAB script in Section C.5.3
% Current information about environment
He = Env.InertiaMatrix (11)(q(7:8));
Ce = Env.CoriolisMatrix (11) (q(7:8), q(9:10));
Se = Env.StiffnessTerm }\mp@subsup{}{}{(11)}(\textrm{q}(7:8))
De = Env.DampingTerm }\mp@subsup{}{(11)}{(q(7:8));
Je = Env.Jacobian }\mp@subsup{}{}{(11)}(q(7:8))
iHe = eye(2)/He;
(11) MATLAB script in Section C.3
dq = zeros(22, 1);
% Closed-loop dynamics
dq(1:3) = q(4:6);
dq(4:6) = iHr*(-Cr*q(4:6) + tau + Jr'*(Fenv + Fref));
dq(7:8) = q(9:10);
dq(9:10) = iHe*(-Ce*q(9:10) - Se - De*q(9:10) - Je'*(Fenv));
dq(11:22) = g_RefSysA*q(11:22) + g_RefSysB*[Xr; dXr];
```

end

## C.5.5 Function: main.m

```
function main(Tend, ApplyScattering, Gap, xWeights)
```

clear global;
clc;
global g_ContactStiffness g_ContactDamping g_Tend ...
g_ScatteringTransformation ...
Rob Env ...
g_RefSysA g_RefSysB...
g_Initial
\% Set global contact parameters
g_ContactStiffness = zeros(3,3);
\% If Parameter set 1: g_ContactStiffness(1,1) = 20;
\% If Parameter set 2: g_ContactStiffness(1,1) = 200;
g_ContactStiffness(1,1) = 200;
g_ContactDamping = diag([-2; 0; 0]);
\% Set intergartion time interval [0, Tend]
g_Tend = Tend;
\% Robot initialization
Rob $=\operatorname{Robot}{ }^{(12)}$ (Tend);
${ }^{(12)}$ MATLAB script in Section C. 2
\% Initialize parameters for Environment:
\% ContactPosition = [Xe(0), Ye(0), Ze(0)]
\% Links length of the environment:
\% LinkLength = [Rob.ContactPos(3), Rob.LinkLen(2)];
\% Environment initialization
Env = Environment ${ }^{(13)}$ (LinkLength, Rob.ContactPos, g_ContactDamping);
${ }^{(13)}$ MATLAB script in Section C. 3
\% Scattering operator
if ApplyScattering == 0
g_ScatteringTransformation = eye(9);
else
g_ScatteringTransformation $=$ ScatteringTransformation ${ }^{(s)}$ (...
Rob, Env, Gap, xWeights);
${ }^{(s)}$ MATLAB script for ScatteringTransformation() in Section C.4
end
\% System for the restoration of 2 nd derivative of the reference signals. [g_RefSysA, g_RefSysB] = d2RefSignal;
function [RefSysA, RefSysB] = d2RefSignal
03 = zeros (3,3); I3 = eye(3);
rA = [03 I3 03 03; 03 03 I3 03; 030303 I3; 030303 03];
rC = [I3 0303 03; 03 I3 03 03];
pls = linspace(-1.75, -1, 12);
RefSysB = place(rA', rC', pls);
RefSysB = RefSysB';
RefSysA = (rA - RefSysB*rC);
end
\% Setup initial state
g_Initial $=\operatorname{zeros}(22,1)$;
[Xd, dXd, d2Xd] = Rob.DesiredPath ${ }^{(14)}(0)$;
g_Initial(11:19) = [Xd, dXd, d2Xd]';
Qr0 = Rob. InvKin ${ }^{(14)}$ (Xd');
g_Initial(1:3) = Qr0;
(14) MATLAB script in Section C.2
\% Solve the system
options $=\operatorname{odeset}\left(' M a x S t e p ', 10^{\wedge}(-3)\right)$;
S = @System;
[t, q] = ode15s(S, [0, g_Tend], g_Initial, options);
end

## Appendix D

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DOI: https://doi.org/10.1016/j.automatica.2018.03.028.
2. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel

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3. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel

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The IEEE Transactions on Robotics (T-RO).
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## Peer-reviewed conference proceedings:

1. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel
"A Graph Separation Stability Condition for Non-Planar Conic Systems," 10th IFAC Symposium on Nonlinear Control Systems, Monterey, CA, USA, August 23-25, 2016, pp. 945-950.
DOI: https://doi.org/10.1016/j.ifacol.2016.10.286.
2. Anastasiia A. Usova, Ilia G. Polushin, Rajnikant V. Patel
"Scattering Transformation for Non-Planar Conic Systems," 20th IFAC World Congress, Toulouse, France, July 9-14, 2017, pp. 8808-8813.
DOI: https://doi.org/10.1016/j.ifacol.2017.08.819.

[^0]:    ${ }^{1}$ The spectral norm of a scattering matrix $S(s)$ is defined as $\|S\|:=\sup _{\omega} \sigma_{\max }(S(j \omega))$.

[^1]:    ${ }^{1}$ Nuno, E., Basanez, L. and Ortega, R. Passivity-based control for bilateral teleoperation: A tutorial, Automatica 47(3), 485-495, 2011

