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Selected Topics in Quantization and Renormalization of Gauge Fields

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Abstract

My thesis covers several topics in the quantization and renormalization of gauge fields, ranging from the application of Dirac constraint procedure on the light front, to the manipulation of Faddeev-Popov method to enable use of the transverse-traceless gauge in first order gravity. Last, I study renormalization group ambiguities and carry out a new characterization method for models with one, two and five couplings.

In chapter 2 we apply the Dirac constraint procedure to the quantization of gauge theories on the light front. The light cone gauge is used in conjunction with the first class constraints that arise and the resulting Dirac brackets are found. These gauge conditions are not used to eliminate degrees of freedom from the action prior to applying the Dirac constraint procedure. This approach is illustrated by considering Yang-Mills theory and the superparticle in a 2 + 1 dimensional target space.

We consider the first order form of the Einstein-Hilbert action and quantize it using the path integral in chapter 3. Two gauge fixing conditions are imposed so that the graviton propagator is both traceless and transverse. It is shown that these two gauge conditions result in two complex Fermionic vector ghost fields and one real Bosonic vector ghost field. All Feynman diagrams to any order in perturbation theory can be constructed from two real Bosonic fields, two Fermionic ghost fields and one real Bosonic ghost field that propagate. These five fields interact through just five three point vertices and one four point vertex.

Finally in chapter 4 we study the ambiguities inherent in renormalization when using mass independent renormalization in massless theories that involve two coupling constants. We review how unlike models in which there is just one coupling constant there is no renormalization scheme in which the β-functions can be chosen to vanish beyond a certain order in perturbation theory, and also the β-functions always contain ambiguities beyond first order. We examine how the coupling constants depend on the coefficients of the β-functions beyond one loop order. A way of characterizing renormalization schemes that doesn't use coefficients of the β-function is considered for models with one, two and five couplings. The renormalization scheme ambiguities of physical quantities computed to finite order in perturbation theory are also examined. The renormalization group equation makes it possible to sum the logarithms that have explicit dependence on the renormalization scale parameter μ in a physical quantity R and this leads to a cancellation with the implicit dependence of R on μ through the running couplings, thereby removing the ambiguity associated with the renormalization scale parameter μ. It is also shown that there exists a renormalization scheme in which all radiative contributions beyond lowest order to R are incorporated into the behavior of the running couplings and the perturbative expansion for R is a finite series.

Keywords: gauge theory, Dirac constraint formalism, first order gravity, transverse traceless gauge, renormalization scheme ambiguities, multiple couplings

Statement of Co-Authorship

This integrated-article thesis is based on the following papers:

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Chapter 2 is based on the first paper which is co-authored with Dr. D. G. C. McKeon. Dr. D. G. C. McKeon introduced this topic to me as the very first project in my Ph.D. research and provided essential guidance on various aspects of this work.

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Chapter 4 is based on the third and fourth paper which is co-authored with Dr. D. G. C. McKeon. Dr. D. G. C. McKeon formulated the problem and performed a portion of the analytical derivation. I did the rest of analytical derivation and performed the symbolic computation.

In existential mathematics that experience takes the form of two basic equations:
The degree of slowness is directly proportional to the intensity of memory;
the degree of speed is directly proportional to the intensity of forgetting.
Milan Kundera

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Contents

Chapter 1

Introduction

In this thesis we study three topics: the Dirac Constraint Formalism in quantum field theory, quantization of first order gravity using the transverse-traceless gauge, and the renormalization ambiguity. This chapter provides a review of all three topics.

1.1 Dirac Constraint Formalism

In this section I review the Dirac Constraint Formalism [1-7]. Dirac Constraint Formalism is a generalization of classical Hamiltonian formalism to treat systems with constraints. More specifically, when the definition of the canonical momentum gives rise to a constraint, it would be inadequate to quantize the system using Hamiltonian Mechanics. Paul Dirac introduced Dirac Brackets to fix the unphysical degrees of freedom contained by the constraints which allows the system to undergo canonical quantization.

In the Hamiltonian procedure, from the canonical momenta and the Lagrangian we can define the naive canonical Hamiltonian

$$H_0 = p_i\dot{q}_i - L(q_i, \dot{q}_i). \tag{1.1}$$

If the Lagrangian is at most linear in at least one coordinate, the canonical momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.2}$$

are not invertible to the velocities and are constrained to be functions of the coordinates, making the variable basis overcomplete. This makes it impossible to move to the Hamiltonian approach as velocities in the Lagrangian cannot be eliminated in favour of momenta. Such a canonical momentum condition would imply a "primary" constraint

$$\chi_i(q_i, p_i) = 0. \tag{1.3}$$

The primary constraints must hold regardless of time; this leads to the consistency condition

$$\frac{d}{dt}\chi_i(q_i, p_i) = \{\chi_i, H\}_{PB} = 0. \tag{1.4}$$

Here we use Poisson brackets

$$\{A, B\}_{PB} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right). \tag{1.5}$$

and H is the extended Hamiltonian

$$H = H_0 + c_i\chi_i(q_i, p_i). \tag{1.6}$$

The constraints coming from the definition of canonical momenta eq. (1.2) are called primary constraints. The consistency condition eq. (1.4) could lead to additional constraints. These additional constraints generated by the consistency conditions of primary constraints are called secondary constraints. We could incorporate secondary constraints into the extended Hamiltonian and use further extended Hamiltonian to check the consistency condition eq. (1.4) for secondary constraints, this could lead to tertiary constraints.

Constraints χ_i can be divided into first class constraints ϕ_i and second class constraints θ_i. First class constraints have weakly vanishing Poisson brackets with other constraints (i.e., they vanish if the constraints themselves vanish)

$$\{\phi_i, \chi_i\} \approx 0. \tag{1.7}$$

A constraint is second class if it is not first class.

After classifying constraints into first and second class, we can write our extended Hamiltonian as

$$H = H_0 + c_i\chi_i(q_i, p_i) = H_0 + a_i\phi_i + b_i\theta_i. \tag{1.8}$$

Therefore we have

$$\begin{aligned}
\frac{d}{dt}\phi_i(q_i, p_i) &= [\phi_i, H] \qquad (1.9)\\
&= [\phi_i, H_0 + a_j\phi_j + b_j\theta_j]\\
&= [\phi_i, H_0] + a_j[\phi_i, \phi_j] + b_j[\phi_i, \theta_j]\\
&\approx [\phi_i, H_0]\\
&\approx 0,
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\theta_i(q_i, p_i) &= [\theta_i, H] \\
&= [\theta_i, H_0 + a_j\phi_j + b_j\theta_j] \\
&= [\theta_i, H_0] + a_j[\theta_i, \phi_j] + b_j[\theta_i, \theta_j] \\
&\approx [\theta_i, H_0] + b_j[\theta_i, \theta_j] \\
&\approx 0,
\end{aligned}
\tag{1.10}
$$

From consistency condition eq. (1.10) we can fix b_j but we can not fix a_i. This means for each first class constraint there is an arbitrary a_i in Hamiltonian. In order to fix these arbitrariness we can introduce a gauge condition γ_i for each first class constraint ϕ_i. In fact, one can exploit constraint formalism to systematically find all the local gauge symmetries for any given theory [8]. Each primary first class constraint leads to a gauge symmetry.

Dirac introduced Dirac brackets as replacements of Poisson brackets to eliminate all constraints from the theory, namely

$$
[A, B]^* = [A, B] - \sum_{i,j}\{A, \theta_i\}d_{ij}^{-1}\{\theta_j, B\} \tag{1.11}
$$

where

$$
d_{ij}^{-1} = \{\theta_i, \theta_j\} = -\{\theta_j, \theta_i\} \tag{1.12}
$$

is an anti-symmetric matrix. Here i and j must be even. Therefore we always have an even number of second class constraints.

In fact, any pair of a first class constraint and its associated gauge condition make up a second class constraint while the intrisic arbitrariness in the first class constraint is fixed by the gauge condition. In this sense, we can treat

$$\Theta_i = \{\phi_i, \theta_i, \gamma_i\} \tag{1.13}$$

as a large, complete set of second class constraints, and if

$$D^{-1}_{ij} = [\Theta_i, \Theta_j], \tag{1.14}$$

then one can have the modified Dirac brackets

$$[A, B]^* = [A, B] - \sum_{i,j} \{A, \Theta_i\} D^{-1}_{ij} \{\Theta_j, B\}. \tag{1.15}$$

In Chapter 2 we apply Dirac Constraint Formalism to light front quantization of Yang-Mills theory and 2 + 1 dimensional superparticle.

1.2 First Order Gravity and Transverse Traceless Propagator

First order gravity has been of great interest to physicists working on the quantization of gravity. Employing the first order Einstein-Hilbert (1EH) action has the advantage over the second order form of the action (2EH) that the interaction vertices are simplified [9-14]. It has been shown that the first and second order forms of the EH action are equivalent both classically and quantum mechanically. In Chapter 3 of my thesis I consider the realization of transverse-traceless gauge in first order gravity. Having a propagator that is both transverse and traceless ensures that only the physical degrees of freedom associated with the tensor field propagate. It is analogous to the Landau gauge in quantum electrodynamics. To obtain such a traceless-transverse propagator, one must employ a non-quadratic gauge fixing Lagrangian [15-19] which is not encountered in the usual Faddeev-Popov procedure [20,21]. In this section I provide the context of first order gravity and non-quadratic gauge fixing.

1.2.1 First Order Gravity

The second order Einstein-Hilbert (2EH) action is

$$S = \int d^d x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma), \tag{1.16}$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \right), \tag{1.17}$$

and

$$R_{\mu\nu}(\Gamma) = \Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}. \tag{1.18}$$

The 1EH action has the form

$$\begin{aligned} \mathcal{L}_{1EH} &= h^{\mu\nu} \left(G^{\lambda}_{\mu\nu,\lambda} + \frac{1}{d-1} G^{\lambda}_{\mu\lambda} G^{\sigma}_{\nu\sigma} - G^{\lambda}_{\mu\sigma} G^{\sigma}_{\nu\lambda} \right) \\ &= G^{\lambda}_{\mu\nu} \left(-h^{\mu\nu}_{,\lambda} \right) + \frac{1}{2} M^{\mu\nu\pi\tau}_{\lambda\ \sigma}(h) G^{\lambda}_{\mu\nu} G^{\sigma}_{\pi\tau}, \end{aligned} \tag{1.19}$$

where

$$h^{\mu\nu} = \sqrt{-g} g^{\mu\nu}, \tag{1.20}$$

$$G^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \frac{1}{2} \left(\delta^{\lambda}_{\mu} \Gamma^{\sigma}_{\nu\sigma} + \delta^{\lambda}_{\nu} \Gamma^{\sigma}_{\mu\sigma} \right). \tag{1.21}$$

and

$$M^{\mu\nu\pi\tau}_{\lambda\ \sigma}(h) = \frac{1}{2}\left[\frac{1}{d-1}\left(\delta^{\nu}_{\lambda}\delta^{\tau}_{\sigma}h^{\mu\pi} + \delta^{\mu}_{\lambda}\delta^{\tau}_{\sigma}h^{\nu\pi} + \delta^{\nu}_{\lambda}\delta^{\pi}_{\sigma}h^{\mu\tau} + \delta^{\mu}_{\lambda}\delta^{\pi}_{\sigma}h^{\nu\tau}\right)\right.$$
$$\left. - \left(\delta^{\tau}_{\lambda}\delta^{\nu}_{\sigma}h^{\mu\pi} + \delta^{\tau}_{\lambda}\delta^{\mu}_{\sigma}h^{\nu\pi} + \delta^{\pi}_{\lambda}\delta^{\nu}_{\sigma}h^{\mu\tau} + \delta^{\pi}_{\lambda}\delta^{\mu}_{\sigma}h^{\nu\tau}\right)\right] \tag{1.22}$$

2EH action is equivalent to the first order Einstein-Hilbert (1EH) action at classical level. At quantum level, Fernando has derived a set of Feynman rules from 1EH action and computed the two point function to one loop order [13]. The computational result is in complete agreement with that of 2EH.

The classical equivalence can be shown by obtaining the equation of motion from eq. (1.19)

$$h^{\mu\nu}_{,\lambda} = M^{\mu\nu\pi\tau}_{\lambda\ \sigma}(h)G^{\sigma}_{\pi\tau} \tag{1.23}$$

from which we can use eq. (1.22) and $h_{\mu\lambda}h^{\lambda\nu} = \delta^{\nu}_{\mu}$ to derive

$$\begin{aligned} H_{\pi\tau,\lambda} &\equiv -h_{\pi\mu}h_{\tau\nu}h^{\mu\nu}_{,\lambda} + h_{\tau\mu}h_{\lambda\nu}h^{\mu\nu}_{,\pi} + h_{\lambda\mu}h_{\pi\nu}h^{\mu\nu}_{,\tau} \\ &= 2\left(\frac{1}{d-1}h_{\pi\tau}G^{\sigma}_{\lambda\sigma} - h_{\lambda\sigma}G^{\sigma}_{\pi\tau}\right). \end{aligned} \tag{1.24}$$

Contracting Eq. (1.24) with $h^{\tau\lambda}$ we have

$$G^{\sigma}_{\pi\sigma} = -\frac{d-1}{2(d-2)}h_{\mu\nu}h^{\mu\nu}_{,\pi} \tag{1.25}$$

and so by Eq. (1.24)

$$G^{\rho}_{\pi\tau} = \frac{1}{2}h^{\rho\lambda}\left(-\frac{1}{d-2}h_{\pi\tau}h_{\mu\nu}h^{\mu\nu}_{,\lambda} - H_{\pi\tau,\lambda}\right). \tag{1.26}$$

We can insert Eq. (1.26) into the Lagrangian of eq. (1.19) and obtain

$$\mathcal{L}_{1EH} = -\frac{1}{2}h^{\mu\nu}_{,\lambda}\left(M^{-1}\right)^{\lambda\ \sigma}_{\mu\nu\pi\tau}(h)h^{\pi\tau}_{,\sigma}. \tag{1.27}$$

This is just the second-order EH Lagrangian $\mathcal{L}_{2EH}$. This demonstrates that classically, $\mathcal{L}_{1EH}$ and $\mathcal{L}_{2EH}$ are equivalent.

The path integral associated with the 1EH action, when using conventional gauge fixing, is

$$Z_{1EH} = \int \mathcal{D}h^{\mu\nu}\mathcal{D}G^{\lambda}_{\mu\nu}\Delta_{FP}(h)\exp i\int d^dx\left[\mathcal{L}_{1EH} + \mathcal{L}_{gf}\right]. \tag{1.28}$$

If we make the shift

$$G^{\lambda}_{\mu\nu} \to G^{\lambda}_{\mu\nu} + \left(M^{-1}\right)^{\lambda\ \ \sigma}_{\mu\nu\pi\tau}(h)h^{\pi\tau}_{,\sigma}, \tag{1.29}$$

it is found that

$$Z_{1EH} = \int \mathcal{D}h^{\mu\nu}\mathcal{D}G^{\lambda}_{\mu\nu}\Delta_{FP}(h)\exp i\int d^dx\left[\frac{1}{2}G^{\lambda}_{\mu\nu}M^{\mu\nu\pi\tau}_{\lambda\ \ \sigma}(h)G^{\sigma}_{\pi\tau} + \frac{1}{2}h^{\mu\nu}_{,\lambda}\left(M^{-1}\right)^{\lambda\ \ \sigma}_{\mu\nu\pi\tau}(h)h^{\pi\tau}_{,\sigma} + \mathcal{L}_{gf}\right]. \tag{1.30}$$

To study the behavior of this path integral, it's convenient to break $h^{\mu\nu}(x)$ into the Minkowski metric $\eta^{\mu\nu}$ and a perturbation term $\phi^{\mu\nu}(x)$

$$h^{\mu\nu}(x) = \eta^{\mu\nu} + \phi^{\mu\nu}(x). \tag{1.31}$$

We now make the shift

$$G^{\lambda}_{\mu\nu} \to G^{\lambda}_{\mu\nu} + \left(M^{-1}\right)^{\lambda\ \ \sigma}_{\mu\nu\pi\tau}(h)h^{\pi\tau}_{,\sigma} \tag{1.32}$$

in the path integral of eq. (1.28). We then find that

$$Z_{1EH} = \int \mathcal{D}h^{\mu\nu}\mathcal{D}G^{\lambda}_{\mu\nu}\Delta_{FP}(h)\exp i\int d^dx\left[\frac{1}{2}G^{\lambda}_{\mu\nu}M^{\mu\nu\pi\tau}_{\lambda\ \ \sigma}(h)G^{\sigma}_{\pi\tau} + \frac{1}{2}h^{\mu\nu}_{,\lambda}\left(M^{-1}\right)^{\lambda\ \ \sigma}_{\mu\nu\pi\tau}(h)h^{\pi\tau}_{,\sigma} + \mathcal{L}_{gf}\right]. \tag{1.33}$$

The expansion of eq. (1.31) can now be made in eq. (1.30). Since M is linear in $h^{\mu\nu}$, it follows that

$$M^{\mu\nu\pi\tau}_{\lambda\ \ \sigma}(\eta+\phi) = M^{\mu\nu\pi\tau}_{\lambda\ \ \sigma}(\eta) + M^{\mu\nu\pi\tau}_{\lambda\ \ \sigma}(\phi). \tag{1.34}$$

Consequently, any Feynman diagrams contributing to Green's functions with only the field $\phi^{\mu\nu}$ on external legs and which involve the field $G^{\lambda}_{\mu\nu}$ on internal lines, necessarily will have the field $G^{\lambda}_{\mu\nu}$ appearing in a closed loop. But the propagator for the field $G^{\lambda}_{\mu\nu}$ is independent of momentum and hence the loop momentum integral associated with any loop coming from the field $G^{\lambda}_{\mu\nu}$ is of the form

$$\int d^d k P(k^\mu), \tag{1.35}$$

where $P(k^\mu)$ is a polynomial in the loop momentum k^μ. If we use dimensional regularization [51, 52] then such loop momentum integrals vanish.

Consequently, for Green's functions involving only the field $\phi^{\mu\nu}$ on external legs, the only contribution to Feynman diagrams come from the last two terms in the argument of the exponential in eq. (1.30); from eq. (1.27) we see that this is just the generating functional associated with $-\mathcal{L}_{2EH}$ and so these Green's functions can be derived by using either the first order or the second order form of the EH action.

The M^{-1} appeared in the action can be expanded as

$$\left(M^{-1}\right)(\eta+\phi) = M^{-1}(\eta) - M^{-1}(\eta)M(\phi)M^{-1}(\eta) + M^{-1}(\eta)M(\phi)M^{-1}(\eta)M(\phi)M^{-1}(\eta) - \dots. \tag{1.36}$$

After obtaining expansion eq. (1.36), instead of making the shift eq. (1.29), we can now make

$$G^{\lambda}_{\mu\nu} \to G^{\lambda}_{\mu\nu} + \left(M^{-1}\right)^{\lambda\ \ \ \sigma}_{\mu\nu\pi\tau}(\eta) h^{\pi\tau}_{,\sigma} \tag{1.37}$$

so our path integral now becomes

$$\begin{aligned} Z_{1EH} &= \int \mathcal{D}h^{\mu\nu}\mathcal{D}G^{\lambda}_{\mu\nu}\Delta_{FP}(h) \exp i \int d^d x \left[\frac{1}{2} G^{\lambda}_{\mu\nu} M^{\mu\nu\pi\tau}_{\lambda\ \ \ \sigma}(\eta) G^{\sigma}_{\pi\tau} - \frac{1}{2}\phi^{\mu\nu}_{,\lambda} M^{-1\lambda\ \ \ \sigma}_{\ \ \ \mu\nu\pi\tau}(\eta)\phi^{\pi\tau}_{,\sigma} \right. \\ &\left. + \frac{1}{2}\left(G^{\lambda}_{\mu\nu} + \phi^{\alpha\beta}_{,\rho}\left(M^{-1}\right)^{\rho\ \ \ \lambda}_{\alpha\beta\mu\nu}(\eta)\right)\left(M^{\mu\nu\pi\tau}_{\lambda\ \ \ \sigma}(\phi)\right)\left(G^{\sigma}_{\pi\tau} + \left(M^{-1}\right)^{\sigma\ \ \ \xi}_{\pi\tau\gamma\delta}(\eta)\phi^{\gamma\delta}_{,\xi}\right) + \mathcal{L}_{gf} \right]. \end{aligned} \tag{1.38}$$

This 1EH generating functional can be used to compute Green's functions with only the two propagators $<\phi\phi>$, $<GG>$ and the three point functions $<GG\phi>$, $<G\phi\phi>$ and $<\phi\phi\phi>$. In eq. (1.38), the $\mathcal{L}_{gf}$ and $\Delta_{FP}(h)$ are to be fixed altogether through a Faddeev–Popov procedure, which I will introduce in next subsection.

1.2.2 Faddeev–Popov Procedure in a Nutshell

We start by introducing the standard Faddeev–Popov procedure [20,21]. If we consider an ordinary generating functional

$$Z = \int \mathrm{d}\vec{h}\ \exp\left(-\vec{h}^T \underset{\sim}{M}\vec{h}\right) = \frac{\pi^{n/2}}{\det^{1/2} \underset{\sim}{M}} \tag{1.39}$$

If there exists a matrix $A^{(0)}$ such that

$$\underset{\sim}{M}\underset{\sim}{A}^{(0)}\vec{\theta} = 0 \tag{1.40}$$

for any vector $\vec{\theta}$, then $\underset{\sim}{M}$ has vanishing eigenvalues and the path integral eq. (1.39) is ill defined. Faddeev and Popov [24] proposed we insert

$$1 = \int \mathrm{d}\vec{\theta}\delta(\underset{\sim}{F}(\vec{h} + \underset{\sim}{A}^{(0)}\vec{\theta}) - \vec{p}) \det(\underset{\sim}{F}\underset{\sim}{A}^{(0)}) \tag{1.41}$$

into path integral eq. (1.39), and then make a change of variable

$$\vec{h} \to \vec{h} - \underset{\sim}{A}^{(0)}\vec{\theta} \tag{1.42}$$

so our path integral eq. (1.39) now becomes

$$Z = \int \mathrm{d}\vec{\theta} \int \mathrm{d}\vec{h}\delta(\underset{\sim}{F}\vec{h} - \vec{p}) \det(\underset{\sim}{F}\underset{\sim}{A}^{(0)}) \exp\left(-\vec{h}^T \underset{\sim}{M}\vec{h}\right). \tag{1.43}$$

To absorb the $\delta(\underset{\sim}{F}\vec{h}-\vec{p})$, we can further insert

$$1 = \pi^{-n/2}\int \mathrm{d}\vec{p}\,\mathrm{e}^{-\vec{p}^T\underset{\sim}{N}\vec{p}}\det{}^{1/2}(\underset{\sim}{N}) \tag{1.44}$$

so eq. (1.43) becomes

$$Z = \pi^{-n/2}\int \mathrm{d}\vec{\theta}\int \mathrm{d}\vec{h}\,\det(\underset{\sim}{F}\underset{\sim}{A}^{(0)})\det{}^{1/2}(\underset{\sim}{N})\exp\left[-\vec{h}^T\left(\underset{\sim}{M}+\underset{\sim}{F}^T\underset{\sim}{N}\underset{\sim}{F}\right)\vec{h}\right]. \tag{1.45}$$

We can further introduce Grassmann "ghost" fields $\vec{c}$ and $\vec{\bar{c}}$ [22-25] and a Nielsen-Kallosh ghost $\vec{k}$ [26,27] to absorb $\det(\underset{\sim}{F}\underset{\sim}{A}^{(0)})$ and $\det^{1/2}(\underset{\sim}{N})$

$$\begin{aligned} Z &= \pi^{-n/2}\int \mathrm{d}\vec{\theta}\int \mathrm{d}\vec{h}\int \mathrm{d}\vec{\bar{c}}\int \mathrm{d}\vec{c}\int \mathrm{d}\vec{k} \\ &\exp\left[-\vec{\bar{c}}\underset{\sim}{F}\underset{\sim}{A}^{(0)}\vec{c}-\vec{k}^T\underset{\sim}{N}\vec{k}-\vec{h}^T\left(\underset{\sim}{M}+\underset{\sim}{F}^T\underset{\sim}{N}\underset{\sim}{F}\right)\vec{h}\right]. \end{aligned} \tag{1.46}$$

As a result of $\det\underset{\sim}{M}$ vanishing, an "infinity" is incurred in eq. (1.39). However, this infinity is parametrized by the integral over the "gauge function" $\vec{\theta}$ which can be absorbed into a normalization factor.

1.2.3 Faddeev–Popov Procedure for Second Order Gravity

The second order Einstein-Hilbert action takes the form

$$S = -\int \mathrm{d}^d x\left(h^{\lambda\sigma}M_{\lambda\sigma,\mu\nu}h^{\mu\nu}\right) \tag{1.47}$$

where

$$\begin{aligned} M_{\lambda\sigma,\mu\nu} &= \frac{k^2}{2}\left[\frac{1}{2}\left(\eta_{\mu\lambda}\eta_{\nu\sigma}+\eta_{\nu\lambda}\eta_{\mu\sigma}\right)-\eta_{\mu\nu}\eta_{\lambda\sigma}\right] \\ &- \frac{1}{4}\left[k_\mu k_\lambda\eta_{\nu\sigma}+k_\nu k_\lambda\eta_{\mu\sigma}+k_\mu k_\sigma\eta_{\nu\lambda}+k_\nu k_\sigma\eta_{\mu\lambda}\right] \\ &+ \frac{1}{2}\left[k_\mu k_\nu\eta_{\lambda\sigma}+k_\lambda k_\sigma\eta_{\mu\nu}\right], \end{aligned} \tag{1.48}$$

if we restrict ourselves to terms quadratic in the quantum field $h_{\mu\nu}$.

Applying Faddeev–Popov procedure we have just introduced, according to eq. (1.46), the gauge fixing Lagrangian for second order Einstein-Hilbert gravity is

$$\mathcal{L}_{\mathrm{gf}} = -h_{\lambda\sigma} F_{\alpha}{}^{\lambda\sigma} N^{\alpha\beta} F_{\beta}{}^{\mu\nu} h_{\mu\nu} \tag{1.49}$$

where

$$\begin{aligned} \underset{\sim}{F}\vec{h} &= F_{\alpha}{}^{\lambda\sigma}\, h_{\lambda\sigma} \\ &= \left[\frac{1}{\alpha} k_\alpha \eta^{\lambda\sigma} + \frac{1}{\beta}\left(k^\lambda \delta^\sigma_\alpha + k^\sigma \delta^\lambda_\alpha\right) \right. \\ &+ \left. \frac{1}{\gamma}\frac{k_\alpha k^\lambda k^\sigma}{k^2}\right] h_{\lambda\sigma} \end{aligned} \tag{1.50}$$

and the “Nielsen-Kallosh” factor is

$$N^{\alpha\beta} = \xi \eta^{\alpha\beta} + \zeta \frac{k^\alpha k^\beta}{k^2}. \tag{1.51}$$

1.2.4 Pursuit of Transverse Traceless Propagator

A transverse traceless propagator $D^{\mathrm{TT}}_{\mu\nu,\lambda\sigma}(k)$ satisfies

$$\eta^{\mu\nu} D^{\mathrm{TT}}_{\mu\nu,\,\lambda\sigma}(k) = 0 \tag{1.52}$$

$$k^{\mu} D^{\mathrm{TT}}_{\mu\nu,\,\lambda\sigma}(k) = 0. \tag{1.53}$$

The propagator $D_{\lambda\sigma,\,\alpha\beta}$ for the spin-two field with this gauge fixing Lagrangian can be computed via

$$\begin{aligned} & D^{\lambda\sigma,\,\alpha\beta}\left(M_{\alpha\beta,\,\mu\nu} + F_{\rho,\alpha\beta} N^{\rho\delta} F_{\delta,\mu\nu}\right) \\ &\quad = \frac{1}{2}\left(\delta^\lambda_\mu \delta^\sigma_\nu + \delta^\lambda_\nu \delta^\sigma_\mu\right) \equiv \bar{\Delta}^{\lambda\sigma}_{\mu\nu}. \end{aligned} \tag{1.54}$$

In order to perform this computation, we can introduce basis in tensor space

$$T^1_{\lambda\sigma,\mu\nu} = \eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\nu\lambda}\eta_{\mu\sigma} \tag{1.55a}$$

$$T^2_{\lambda\sigma,\mu\nu} = \eta_{\mu\nu}\eta_{\lambda\sigma} \tag{1.55b}$$

$$T^3_{\lambda\sigma,\mu\nu} = \frac{1}{k^2}\left(k_\mu k_\lambda \eta_{\nu\sigma} + k_\mu k_\sigma \eta_{\nu\lambda}\right) + (\mu \leftrightarrow \nu) \tag{1.55c}$$

$$T^4_{\lambda\sigma,\mu\nu} = \frac{1}{k^2}\left(k_\mu k_\nu \eta_{\lambda\sigma} + k_\lambda k_\sigma \eta_{\mu\nu}\right) \tag{1.55d}$$

$$T^5_{\lambda\sigma,\mu\nu} = \frac{1}{k^4}\left(k_\mu k_\nu k_\lambda k_\sigma\right) \tag{1.55e}$$

so our gauge fixing Lagrangian eq. (1.49) can be described by the basis

$$\begin{aligned} L_{\text{gf}} = & -h^{\lambda\sigma}\left\{\frac{\xi+\zeta}{\alpha^2}T^2_{\lambda\sigma,\mu\nu} + \frac{\xi}{\beta^2}T^3_{\lambda\sigma,\mu\nu}\right. \\ & + \frac{\xi+\zeta}{\alpha}\left(\frac{2}{\beta}+\frac{1}{\gamma}\right)T^4_{\lambda\sigma,\mu\nu} \\ & \left. + \left[\frac{\xi+\zeta}{\gamma}\left(\frac{4}{\beta}+\frac{1}{\gamma}\right)+\frac{4\zeta}{\beta^2}\right]T^5_{\lambda\sigma,\mu\nu}\right\}k^2 h^{\mu\nu}. \end{aligned} \tag{1.56}$$

Explicit calculation is performed in d dimensions for propagator $D_{\mu\nu,\lambda\sigma}(k)$ in tensor space

$$D_{\mu\nu,\lambda\sigma}(k) = \frac{1}{k^2}\sum_{i=1}^{5}\mathbf{C}^i T^i_{\mu\nu,\lambda\sigma}, \tag{1.57}$$

The analytical result is

$$\mathbf{C}^1 = 1 \tag{1.58a}$$

$$\mathbf{C}^2 = -\frac{2}{d-2} \tag{1.58b}$$

$$\mathbf{C}^3 = \left(\frac{\beta^2}{4\xi} - 1\right) \tag{1.58c}$$

$$\mathbf{C}^4 = \frac{2}{d-2}\left[1 + \frac{\beta\gamma}{\alpha(\beta+\gamma)+\gamma(\alpha+\beta)}\right] \tag{1.58d}$$

$$\begin{aligned} \mathbf{C}^5 = & -\frac{\beta^2}{\xi} + \frac{1}{\xi+\zeta}\frac{(\alpha\beta\gamma)^2}{[\alpha(\beta+\gamma)+\gamma(\alpha+\beta)]^2} \\ & + \frac{2}{d-2}\frac{(d-3)\alpha(\beta+2\gamma)-2\beta\gamma}{\alpha(\beta+\gamma)+\gamma(\alpha+\beta)}. \end{aligned} \tag{1.58e}$$

The limit $\alpha \to 0$ leads to a traceless propagator which is however not transverse. The limit $\beta \to 0$ leads to a transverse propagator that is not traceless. These two limits do not commute. This means we cannot have a transverse traceless propagator via ordinary gauge fixing procedure.

1.2.5 Non-quadratic Gauge Fixing and Transverse Traceless Gauge

In this subsection we introduce non-quadratic gauge fixing and the resulting transverse traceless gauge.

Into eq. (1.39) we can insert two factors of "1"

$$1 = \int \mathrm{d}\vec{\theta}_1 \delta(\underset{\sim}{F}(\vec{h} + \alpha \underset{\sim}{A}\vec{\theta}_1) - \vec{p}) \det(\alpha \underset{\sim}{F}\underset{\sim}{A}^{(0)}) \tag{1.59a}$$

$$1 = \int \mathrm{d}\vec{\theta}_2 \delta(\underset{\sim}{G}(\vec{h} + \alpha \underset{\sim}{A}\vec{\theta}_2) - \vec{q}) \det(\alpha \underset{\sim}{G}\underset{\sim}{A}^{(0)}) \tag{1.59b}$$

as well as another "1" of the form

$$1 = \pi^{-n} \int \mathrm{d}\vec{p}\, \mathrm{d}\vec{q}\, \mathrm{e}^{-\frac{1}{\alpha}\vec{p}^T \underset{\sim}{N}\vec{q}} \det(\underset{\sim}{N}/\alpha). \tag{1.60}$$

Similar to eq. (1.45), now we have

$$\begin{aligned} Z &= \pi^{-n} \int \mathrm{d}\vec{\theta}_1 \mathrm{d}\vec{\theta}_2 \int \mathrm{d}\vec{h} \det(\alpha \underset{\sim}{F}\underset{\sim}{A}^{(0)}) \det(\alpha \underset{\sim}{G}\underset{\sim}{A}^{(0)}) \\ &\times \det\left(\frac{\underset{\sim}{N}}{\alpha}\right) \exp\left\{-\vec{h}^T \underset{\sim}{M}\vec{h} - \frac{1}{\alpha}\left[\underset{\sim}{F}(\vec{h} + \alpha \underset{\sim}{A}^{(0)}\vec{\theta}_1)\right]^T \right. \\ &\left. \underset{\sim}{N}\left[\underset{\sim}{G}(\vec{h} + \alpha \underset{\sim}{A}^{(0)}\vec{\theta}_2)\right]\right\}. \end{aligned} \tag{1.61}$$

We now complete the square and make the shift $\vec{h} \to \vec{h} - \alpha \underset{\sim}{A}^{(0)}\vec{\theta}_1$, then let $\vec{\theta} = \vec{\theta}_2 - \vec{\theta}_1$ and use $\vec{\theta}$

to replace $\vec{\theta}_2$, our generating functional becomes

$$\begin{aligned} Z &= \left(\frac{\alpha}{\pi}\right)^n \int \mathrm{d}\vec{\theta}_1 \int \mathrm{d}\vec{\theta} \int \mathrm{d}\vec{h} \det(\underset{\sim}{F}\underset{\sim}{A}^{(0)}) \det(\underset{\sim}{G}\underset{\sim}{A}^{(0)}) \\ &\times \det(\underset{\sim}{N}) \exp\left\{ -\vec{h}^T \left(\underset{\sim}{M} + \frac{1}{\alpha} \underset{\sim}{F}^T \underset{\sim}{N}\underset{\sim}{G} \right) \vec{h} \right. \\ &- \left. \vec{h}^T \underset{\sim}{F}^T \underset{\sim}{N}\underset{\sim}{G}\underset{\sim}{A}^{(0)} \vec{\theta} \right\}. \end{aligned} \tag{1.62}$$

We drop the infinite normalization factors and make the shift to diagonalize the exponential in $\vec{h}$ and $\vec{\theta}$

$$\vec{h} \to \vec{h} - \frac{1}{2}\left(\underset{\sim}{M} + \frac{1}{\alpha} \underset{\sim}{F}^T \underset{\sim}{N}\underset{\sim}{G} \right)^{-1} \left(\underset{\sim}{F}^T \underset{\sim}{N}\underset{\sim}{G}\underset{\sim}{A}^{(0)} \right) \vec{\theta}, \tag{1.63}$$

we can obtain

$$\begin{aligned} Z &= \int \mathrm{d}\vec{\theta} \int \mathrm{d}\vec{h} \det(\underset{\sim}{F}\underset{\sim}{A}^{(0)}) \det(\underset{\sim}{G}\underset{\sim}{A}^{(0)}) \det(\underset{\sim}{N}) \\ &\times \exp\left\{ -\vec{h}^T \left(\underset{\sim}{M} + \frac{1}{\alpha} \underset{\sim}{F}^T \underset{\sim}{N}\underset{\sim}{G} \right) \vec{h} \right. \\ &+ \frac{1}{4}\vec{\theta}^T \left(\underset{\sim}{A}^{(0)T} \underset{\sim}{G}^T \underset{\sim}{N}^T \underset{\sim}{F} \right) \left(\underset{\sim}{M} + \frac{1}{\alpha} \underset{\sim}{F}^T \underset{\sim}{N}\underset{\sim}{G} \right)^{-1} \\ &\times \left. \left(\underset{\sim}{F}^T \underset{\sim}{N}\underset{\sim}{G}\underset{\sim}{A}^{(0)} \right) \vec{\theta} \right\}, \end{aligned} \tag{1.64}$$

where

$$\vec{h}^T \underset{\sim}{F}^T \underset{\sim}{N}\underset{\sim}{G}\vec{h} = h^{\mu\nu} F^T_{\mu\nu,\alpha} N^{\alpha\beta} G_{\beta,\lambda\sigma} h^{\lambda\sigma} \tag{1.65}$$

and

$$F^T_{\mu\nu,\alpha} = g_1 \eta_{\mu\nu}\partial_\alpha + \eta_{\mu\alpha}\partial_\nu \tag{1.66a}$$

$$G_{\beta,\lambda\sigma} = g_2 \eta_{\lambda\sigma}\partial_\beta + \eta_{\lambda\beta}\partial_\sigma \tag{1.66b}$$

$$N^{\alpha\beta} = \eta^{\alpha\beta}. \tag{1.66c}$$

Now the non-quadratic gauge fixing has been performed, we can then invert the quadratic form $\underset{\sim}{M} + \frac{1}{\alpha}\underset{\sim}{F}^T\underset{\sim}{N}\underset{\sim}{G}$ and solve for the propagator in tensor space as in eq. (1.57), in this case the resulting coefficients are

$$\mathbf{C}^1 = 1 \tag{1.67a}$$

$$\mathbf{C}^2 = -2\frac{(g_2 - g_1)^2 + 2(g_1 + 1)(g_2 + 1)\alpha}{(d-1)(g_2 - g_1)^2 + 2(d-2)(g_1 + 1)(g_2 + 1)\alpha} \tag{1.67b}$$

$$\mathbf{C}^3 = \alpha - 1 \tag{1.67c}$$

$$\mathbf{C}^4 = 2\frac{(g_2 - g_1)^2 + \left[4(g_1 + 1)(g_2 + 1) - g_1 - g_2 - 2\right]\alpha}{(d-1)(g_2 - g_1)^2 + 2(d-2)(g_1 + 1)(g_2 + 1)\alpha} \tag{1.67d}$$

$$\begin{aligned}\mathbf{C}^5 \;&=\; \left[(d-1)(g_2 - g_1)^2 + 2(d-2)(g_1 + 1)(g_2 + 1)\alpha\right]^{-1} \\ &\times\; \Big\{4\alpha\left[(g_1 + g_2)(d-4) + (2g_1g_2 + 1)(d-3) - \left(g_1^2 + g_2^2\right)(d-1)\right] \\ &+\; 2(d-2)\left[(g_1 - g_2)^2 - \alpha^2(4(g_1 + 1)(g_2 + 1) - 1)\right]\Big\}\end{aligned} \tag{1.67e}$$

Fortunately, from eq. (1.67) it is found that if we take the limit $\alpha \to 0$, with $g_2 \neq g_1$, the propagator becomes transverse and traceless, and is independent of g_1 and g_2. However, the limits $g_2 \to g_1$ and $\alpha \to 0$ do not commute. If we set $g_2 = g_1$, the resulting propagator is not transverse nor traceless even for $\alpha = 0$. This is another verification of the impossibility of obtaining the transverse and traceless propagator using the quadratic gauge fixing where $g_1 = g_2$.

This concludes the introduction for Chapter 3. In Chapter 3 we will make the most out of our knowledge from this section to quantize first order gravity using a transverse traceless propagator.

1.3 Renormalization Scheme Dependence and Renormalization Group Summation

In quantum field theory, renormalization is the process that eliminates divergences arising in the computation of radiative effects. In perturbation theory the process of renormalization induces a dependence on arbitrary parameters that absorbs divergences. The requirement that physical processes have to be independent of these parameters leads to the renormalization group (RG) equations [28-30]. One of these arbitrary parameters is the renormalization mass scales parameter μ, irrespective of the renormalization scheme (RS) being used. Ambiguities in perturbative computation arise from the presence of both the unphysical parameter μ and finite renormalization. Especially in Quantum Chromodynamics (QCD), by varying renormalization scheme, one can widely vary the results of higher loop calculations. In this section I will introduce existing strategies physicists have developed to minimize the renormalization scale dependence. In principle there is no dependence on μ or the renormalization scheme, but this is true only for the exact result; at finite order there is explicit and implicit dependence on both sources of arbitrariness. There are attempts in the literature to reduce dependence on the arbitrary parameters that arise in perturbation theory [37-44]. We will show how the renormalization group equation can be used to completely cancel the implicit and explicit dependence on μ and to choose a renormalization scheme so that the perturbative expansion of the calculated value any physical quantity terminates at finite order.

1.3.1 Renormalization Scheme Dependence

In $\bar{MS}$ scheme, the form of QCD cross section $R_{e^+e^-}$ is given by

$$R_{e^+e^-} = 3\left(\sum_i q_i^2\right)(1 + R) \tag{1.68}$$

where R is given by a perturbative expansion

$$R = R_{\text{pert}} = \sum_{n=0}^{\infty} r_n a^{n+1} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} T_{n,m} L^m a^{n+1} \qquad (T_{0,0} = 1) \tag{1.69}$$

with

$$L = b \ln\left(\frac{\mu}{Q}\right) \tag{1.70}$$

and Q is the centre of mass momentum.

In the renormalization group equation, the explicit dependence of R on the renormalization scale parameter μ is compensated for by implicit dependence of the "running coupling" $a(\mu)$ on μ,

$$\mu\frac{\partial a}{\partial \mu} = \beta(a) = -ba^2\left(1 + ca + c_2 a^2 + \ldots\right). \tag{1.71}$$

Here b and c are scheme independent [31] while the $c_n (n \geq 2)$ are scheme dependent. For mass-independent renormalization [32,33], different renormalizations schemes have their couplings a and $\overline{a}$ related by [34]

$$\begin{aligned} \overline{a} &= a + x_2 a^2 + x_3 a^3 + \ldots && \text{(1.72a)} \\ &\equiv F(a) \end{aligned}$$

From the equation $\overline{\beta}(\overline{a}) = \beta(a)F'(a)$, we can solve for [35]

$$\overline{c}_2 = c_2 - cx_2 + x_3 - x_2^2 \tag{1.72b}$$

$$\overline{c}_3 = c_3 - 3cx_2^2 + 2(c_2 - 2\overline{c}_2)x_2 + 2x_4 - 2x_2x_3 \tag{1.72c}$$

etc.

Plenty of strategies have been developed to minimize the dependence of perturbative results on both μ and on general scheme dependency. It is worth noticing that if the exact result for R were known, all such dependency should disappear [36]. One of the most well known strategies is "principle of minimal sensitivity" (PMS) [37], in which the parameters μ and c_i are chosen to minimize the variations of $R_{e^+e^-}$ when these parameters themselves are altered. Another method involves the "principle of maximum conformality" (PMC) [38-40]. In PMC a different renormalization mass scale is introduced at each order of perturbation theory to absorb all dependence on the coefficients c_i. In the "fastest apparent convergence" (FAC) approach [41-44], it is proposed that one should introduce "effective charges" to minimize contributions beyond a given order in perturbation theory.

1.3.2 Renormalization Group Summation

An alternative approach to manage scheme dependence is "renormalization group summation" ($RG\sum$) [45-48]. In $RG\sum$ the RG equation with one loop RG functions permits summation of all "leading-log" (LL) contributions to the sum in eq. (1.69), two loop RG functions permits summation of all "next-to-leading-log" (NLL) contributions etc. As expected, $RG\sum$ reduces the dependence of any calculation on the scale parameter μ, which one might anticipate as upon including higher order logarithmic effects, one should be closer to the exact result, which is fully independent of μ. After all we should keep in mind that in perturbation theory any computation to finite order is scheme dependent.

In order to sum *LL*, *NLL* etc. contributions to *R* in eq. (1.69) we use the groupings

$$S_n(aL) = \sum_{k=0}^{\infty} T_{n+k,k}(aL)^k \tag{1.73}$$

so the *RG* equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a}\right)R = 0 \tag{1.74}$$

with $\beta(a)$ given by eq. (1.71) and *R* given by eq. (1.69) leading to a set of nested differential equations of $S_n(u)$

$$S_0' - (S_0 + uS_0') = 0 \tag{1.75a}$$

$$S_1' - (2S_1 + uS_1') - c(S_0 + uS_0') = 0 \tag{1.75b}$$

$$S_2' - (3S_2 + uS_2') - c(2S_1 + uS_1') - c_2(S_0 + uS_0') = 0 \tag{1.75c}$$

$$S_3' - (4S_3 + uS_3') - c(3S_2 + uS_2') - c_2(2S_1 + uS_1') - c_3(S_0 + uS_0') = 0 \tag{1.75d}$$

etc.

And the associated boundary conditions are

$$S_n(0) = T_{n,0} \equiv T_n. \tag{1.76}$$

With these boundry conditions, one can solve for $S_n(u)$ [46,47]

$$wS_0 = T_{00} \quad (w = 1 - u) \tag{1.77a}$$

$$w^2 S_1 = T_{10} - cT_{00} \ln |w| \tag{1.77b}$$

$$w^3 S_2 = T_{20} - (2cT_{10} + c^2 T_{00}) \ln |w| + (c^2 - c_2) T_{00}(w - 1) + c^2 T_{00} \ln^2 |w| \tag{1.77c}$$

$$\begin{aligned} w^4 S_4 = {} & T_{30} - c^3 T_{00} \ln^3 |w| + \frac{1}{2}(6c^2 T_{10} + 5c^3 T_{00}) \ln^2 |w| - 2c(c^2 - c_2) T_{00}(w \ln |w| - (w - 1)) \\ & - 3c(T_{20} - (c^2 - c_2) T_{00}) \ln |w| + (-2c_2 T_{10} - c(2c^2 - c_2) T_{00})(w - 1) + (-c^3 + 2cc_2 - c_3) T_{00}(\frac{w^2 - 1}{2}) \end{aligned} \tag{1.77d}$$

etc.

where the $S_i (i = 0, 1 \ldots 4)$ are the LL, NLL, N^2LL and N^3LL contributions to R.

Now I introduce another way of organizing the sum of eq. (1.69). Instead of computing the LL, NLL etc. sums in turn, one can use the RG equation to show that all logarithmic contributions to R can be expressed in terms of the log-independent contributions. By using this summation, the explicit dependence of $R_{e^+e^-}$ on μ occurring in eq. (1.69) through L is exactly cancelled by the implicit dependence on μ through the running coupling $a(\mu)$ [49].

Instead of using

$$R = R_\Sigma = \sum_{n=0}^{\infty} a^{n+1} S_n(aL) \tag{1.78}$$

by directly substituting eq. (1.69) into eq. (1.74) one find that

$$T_{ii} = T_{i-1,i-1} \tag{1.79a}$$

$$T_{21} = (c + 2T_{10}) \tag{1.79b}$$

$$2T_{32} = (2cT_{11} + 3T_{21}) \tag{1.79c}$$

and

$$T_{31} = c_2 + 3T_{20} + 2cT_{10} \tag{1.79d}$$

etc.

Instead of the grouping of eq. (1.73) we can introduce

$$A_n = \sum_{k=0}^{\infty} T_{n+k,n} a^{n+k+1} \tag{1.80}$$

so R can be rearranged as

$$R = R_A = \sum_{n=0}^{\infty} A_n(a) L^n. \tag{1.81}$$

Substituting eq. (1.81) into the RG equation (1.71), we get the recursive relation for $A_n(a)$

$$A_n(a) = -\frac{\beta(a)}{bn}\frac{d}{da}A_{n-1}(a). \tag{1.82}$$

One now can introduce $\eta = \ln\frac{\mu}{\Lambda}$ where Λ is a universal scale. Definition of η [37,50] is associated with the boundary condition on eq. (1.69) so that

$$\eta = \int_{a_I}^{a(\eta)} \frac{dx}{\beta(x)} \quad (a_I = a(\eta = 0) = \text{const.}). \tag{1.83}$$

By eqs. (1.71,1.82) we find that

$$A_n(a(\eta)) = \frac{-1}{bn}\frac{d}{d\eta}A_{n-1}(a(\eta)) = \frac{1}{n!}\left(-\frac{1}{b}\frac{d}{d\eta}\right)^n A_0(a(\eta)). \tag{1.84}$$

Together, eqs. (1.81,1.84) lead to

$$R_A = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{L}{b}\right)^n \frac{d^n}{d\eta^n} A_0(a(\eta)) \tag{1.85}$$

$$= A_0\left(a\left(\eta - \frac{1}{b}L\right)\right). \tag{1.86}$$

With the definitions of η and L, eq. (1.86) becomes

$$R_A = A_0\left(a\left(\ln\frac{Q}{\Lambda}\right)\right). \tag{1.87}$$

Eq. (1.87) is an exact equation that expresses R in terms of its log independent contributions and the running coupling a evaluated at $\ln\frac{Q}{\Lambda}$ with all dependence of R on μ, both implicit and explicit, removed. This disappearance of dependence on μ is to be expected as μ is unphysical. In Chapter 4 the removal of μ dependence in eq. (1.87) will play an important role in a new characterization method to be introduced.

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Chapter 2

Light Front Quantization with the Light Cone Gauge

2.1 Introduction

The idea of light front quantization was originally introduced by Dirac [1]. This idea is based on the introduction of a set of light front coordinates, and plays a practical role as an alternative to ordinary equal-time quantization. Light front quantization has received consistent attention since its invention. It is applied to a theory in the reference frame with infinite momentum [2], and has played an important role in a wide range of areas such as gauge theories [3-11], supersymmetry [12], general relativity [13-16] and superstrings [17].

In the standard quantization procedure, a particular gauge is introduced to eliminate variables occurring in the original gauge invariant action before the resulting reduced action is quantized under the specific gauge. However, by following the Dirac constraint formalism to quantize gauge systems [18-19], one should first identify and classify all constraints in a system, then introduce a gauge condition to accompany each of the first class constraints. These gauge conditions are not used to eliminate degrees of freedom from the action prior to applying the Dirac constraint procedure.

We apply this procedure to light front quantization using light-front variables. It is worth noticing that, applying Dirac constraint formalism does not necessarily result in the same quantized theory that arises if the light cone gauge is used at the outset to eliminate "superfluous" degrees of freedom before applying the Dirac procedure. We illustrate this by considering Yang-Mills theory and the superparticle in a $2+1$ dimensional target space.

2.2 Yang-Mills Theory and the Light-Cone

The light front coordinates we use for a covariant vector $a^\mu(\mu = 0, 1, \ldots, D-1)$ with $g_{\mu\nu} =$ diag$(+, -\ldots)$ in Yang-Mills theory are

$$\begin{aligned} a^\pm &= \frac{1}{\sqrt{2}}(a^0 \pm a^{D-1}) \\ a^i &= a^\mu(\mu = 1\ldots D-2). \end{aligned} \tag{2.1}$$

After applying the light front coordinates, we have

$$a \cdot b = a^+ b^- + a^- b^+ - a^i b^i. \tag{2.2}$$

Adopting these notations, the well-known Yang-Mills (YM) action under light front coordinates now becomes

$$
\begin{aligned}
S_{YM} &= \int d^d x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \right) \\
&= \int d^d x \left(\frac{1}{2} F^{a+-} F^{a+-} + F^{a+i} F^{a-i} - \frac{1}{4} F^{aij} F^{aij} \right)
\end{aligned} \tag{2.3}
$$

where

$$
F^{a\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + \epsilon^{abc} A^{b\mu} A^{c\nu}. \tag{2.4}
$$

This action, along with actions in which $A^{a\mu}$ is coupled with spinor and/or scalar fields, has been studied in a number of papers [3-11], mostly by imposing the following gauge condition to reduce the number of independent fields in the initial action

$$
A^{a+} = 0 \tag{2.5}
$$

and making use of any resulting equation of motion that does not contain the "time" derivative

$$
\partial^+ f \equiv \dot{f}. \tag{2.6}
$$

In my thesis, I will instead apply the Dirac constraint formalism [18-19] to the Yang-Mills action of eq. (2.3), with imposition of gauge conditions in conjunction with first class constraints that arise during Dirac procedure. This has been considered when applying path integral quantization to the action of eq. (2.3) [11]. This approach has been previously used to analyze the spin-two action (i.e., linearized gravity) in ref. [15]. It is also worth mentioning that following the methodology in Appendix the first class constraints arising from the action of eq. (2.3) lead to a generator of the usual gauge transformation

$$
\delta A^a_\mu = D^{ab}_\mu \theta^b \equiv \left(\partial_\mu \delta^{ab} + \epsilon^{apb} A^p_\mu \right) \theta^b \tag{2.7}
$$

despite the presence of second class constraints.

From the action of eq. (2.3), we compute the canonical momenta in our system

$$\pi_i^a = \partial \mathcal{L}_{YM}/\partial \dot{A}^{ai} = F^{a-i} \tag{2.8a}$$

$$\pi_+^a = \partial \mathcal{L}_{YM}/\partial \dot{A}^{a+} = 0 \tag{2.8b}$$

$$\pi_-^a = \partial \mathcal{L}_{YM}/\partial \dot{A}^{a-} = F^{a+-}. \tag{2.8c}$$

Following standard Hamiltonian mechanics, these canonical momenta result in the canonical Hamiltonian

$$\mathcal{H}_c = \frac{1}{2}\pi_-^a \pi_-^a + \frac{1}{4}F^{aij}F^{aij} - A^{a+}\left(D^{abi}\pi_i^b + D^{ab-}\pi_i^b\right). \tag{2.9}$$

Next we shall identify and classify constraints from the canonical momenta. Eq. (2.8a) is a second class primary constraint

$$\theta_i^a = \pi_i^a - F^{a-i}. \tag{2.10}$$

From the primary constraint of eq. (2.8b)

$$\phi_1^a = \pi_+^a \tag{2.11a}$$

and the canonical Hamiltonian of eq. (2.9) we obtain the secondary constraint

$$\phi_2^a = D^{abi}\pi_i^b + D^{ab-}\pi_-^b; \tag{2.11b}$$

ϕ_1^a and ϕ_2^a are both first class therefore no further constraints arise.

The constraints of eqs. (2.10,2.11) have the Poisson bracket (PB) algebra

$$\left\{\phi_2^a, \phi_2^b\right\} = \epsilon^{abc}\phi_2^c \tag{2.12a}$$

$$\left\{\phi_2^a, \theta_i^b\right\} = \epsilon^{abc}\theta_i^c \tag{2.12b}$$

$$\left\{\theta_i^a(x), \theta_j^b(y)\right\} = -2\delta_{ij}D^{ab-}\delta(x-y), \tag{2.12c}$$

Constraints are classified into first class and second class also via their Poisson brackets. Eq. (2.12c) is the only non-vanishing Poisson bracket on the constraint plane, (i.e., when all the constraints themselves equal to zero), therefore identifies the only second class constraint. Upon having these Poisson brackets between constraints, by eq. (2.12c), we can eliminate the second class constraint θ_i^a by defining the Dirac bracket (DB)

$$\{M, N\}^* = \{M, N\} - \{M, \theta_i^a(z)\} \frac{-1}{2D_z^{ab-}} \delta(z-w) \left\{\theta_i^b(w), N\right\}. \tag{2.13}$$

As in eq. (A.7), we define the generator of the gauge transformation that leaves S_{YM} of eq. (2.3) invariant to be

$$G = \mu_1^a \phi_1^a + \mu_2^a \phi_2^a \tag{2.14}$$

with μ_1^a determined in terms of μ_2^a by those terms in eq. (A.11) at least linear in ϕ_A,

$$(\dot{\mu}_1^a \phi_1^a + \dot{\mu}_2 \phi_2^a) + \{\mu_1^a \phi_1^a + \mu_2^a \phi_2^a, \mathcal{H}_c\} - \delta\mu_1^a \phi_1^a = 0 \tag{2.15}$$

which by eqs. (2.9, 2.12) leaves us with

$$G = \left(\dot{\mu}_2^a + \epsilon^{abc} A^{b+} \mu_2^c\right) \phi_1^a + \mu_2^a \phi_2^a. \tag{2.16}$$

From eq. (2.16) we find the gauge transformation of eq. (2.7) with $\theta^a = \mu_2^a$, as expected.

The first class constraints ϕ_I^a of eqs. (2.11a,b) are accompanied by gauge conditions γ_I^a so that together ϕ_I^a and γ_I^a form a set of second class constraints. Here we will use the same gauge conditions that were suggested in ref. [11], and will proceed to find the resulting DB.

The constraint of eq. (2.11a) suggests the gauge condition

$$\gamma_1^a = A^{a+} \tag{2.17}$$

while that of eq. (2.11b) suggests either

$$\gamma^a_{2I} = A^{a-} \tag{2.18a}$$

or

$$\gamma^a_{2II} = \partial^i A^{ai}. \tag{2.18b}$$

Having already eliminated θ^a_i of eq. (2.10) by defining the DB of eq. (2.13), we can now eliminate ϕ^a_1 and γ^a_1 by the "second stage" DB

$$\{M, N\}^{**} = \{M, N\}^{*} - \left[\{M, \pi^a_+(z)\}^{*}\,\delta(z-w)\,\{A^{a+}(w), N\}^{*} - (M \leftrightarrow N)\right]. \tag{2.19}$$

In the same way ϕ^a_2 and γ^a_{2I} give rise to a "third stage" DB. This involves using

$$\left\{\gamma^a_{2I}, \phi^b_2\right\}^{**} = -D^{ab-}\delta(x-y) \tag{2.20a}$$

$$\left\{\phi^a_2, \phi^b_2\right\}^{**} = \epsilon^{abc}\phi^c_2 - \left[\epsilon^{apm}\theta^m_i(x)\right]\frac{-1}{2D^{pq}}\delta(x-y)\left[-\epsilon^{bqn}\theta^n_i(y)\right]. \tag{2.20b}$$

When forming the DB to eliminate γ^a_{2I} and ϕ^a_2, we set ϕ^a_2 and θ^a_i to zero in eq. (2.20b) and so our third stage DB is

$$\{M, N\}^{***} = \{M, N\}^{**} - \left[\{M, \phi^a_2(z)\}^{**}\frac{-1}{D^{ab-}_z}\delta(z-w)\{\gamma^a_{2I}(w), N\}^{**} - (M \rightleftharpoons N)\right]. \tag{2.21}$$

Computing the third stage DB when using the gauge condition γ^a_{2II} of eq. (2.18b) in conjunction with the first class constraint ϕ^a_2 of eq. (2.11b) is more involved. Eq. (2.20b) still holds, but now we also have

$$\left\{\gamma^a_{2II}, \gamma^b_{2II}\right\}^{**} = \frac{1}{2}\partial^k\frac{1}{D^{ab-}}\partial^k\delta(x-y) \tag{2.22}$$

as well as

$$\left\{\gamma^a_{2II}, \phi^b_2\right\}^{**} = -\partial^i D^{abi}\delta(x-y) - \frac{1}{2}\partial^i\frac{1}{D^{aq-}}\delta(x-y)\epsilon^{bqr}\theta^r_i(y). \tag{2.23}$$

Again, in eqs. (2.20b, 2.22, 2.23) we can set $\phi_2^a = \theta_i^a = 0$ when forming the DB to eliminate γ_{2II}^a and ϕ_2^a.

Since

$$\begin{pmatrix} \frac{1}{2}\partial^k \frac{1}{D^{ab-}}\partial^k & -\partial^i D^{abi} \\ -D^{abi}\partial^i & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\frac{1}{D^{ab-}\partial^j} \\ -\frac{1}{\partial^i D^{abi}} & -\frac{1}{2}\frac{1}{\partial^i D^{api}}\partial^k \frac{1}{D^{pq-}}\partial^k \frac{1}{D^{abj}\partial^j} \end{pmatrix} \tag{2.24}$$

we find that

$$\begin{aligned} \{M,N\}^{***} = \{M,N\}^{**} - &\Big[\{M, \gamma_{2II}^a(z)\}^{**} \frac{-1}{\partial^j D^{abj}} \delta(z-w) \\ &\left\{\phi_2^b(w), N\right\}^{**} - (M \rightleftharpoons) N \Big] \\ &- \Big[\{M, \phi_2^a(z)\}^{**} \left(\frac{-1}{2}\right) \frac{1}{\partial^i D^{abi}} \partial^k \frac{1}{D^{pq-}} \partial^k \frac{1}{D^{abj}\partial^j} \delta(z-w) \\ &\quad \{\phi_2^a(w), N\}^{**} \Big]. \end{aligned} \tag{2.25}$$

One example from eq. (2.25) is that

$$\begin{aligned} &\left\{A^{ai}(x), A^{bj}(y)\right\}^{***} \\ &= \frac{1}{2}\Big[-\delta^{ij}\frac{1}{D^{ab-}} + \frac{1}{D^{ap-}}\partial^i \frac{1}{D^{pqk}\partial^k} D^{qbj} + D^{api}\frac{1}{\partial^k D^{pqk}}\partial^j \frac{1}{D^{qb-}} \\ &- D^{api}\frac{1}{\partial^k D^{pqk}}\partial^m \frac{1}{D^{qr-}}\partial^m \frac{1}{D^{rs\ell}\partial^\ell} D^{sbj}\Big]\delta(x-y). \end{aligned} \tag{2.26}$$

We can also derive

$$\left\{\partial^i A^{ai}, A^{bj}\right\}^{***} = 0 \tag{2.27}$$

which is consistent with the gauge condition of eq. (2.18b). In the $U(1)$ limit, eq. (2.26) reduces to

$$\left\{A^i(x), A^j(y)\right\}^{***} = \frac{1}{2}\left(-\delta^{ij} + \frac{\partial^i\partial^j}{\partial^k\partial^k}\right)\frac{1}{\partial^-}\delta(x-y). \tag{2.28}$$

We thus see that applying the Dirac canonical analysis to YM theory right from the outset (i.e., only introducing constraints after the first class constraints which follow from the initial YM action when written in light front coordinates) yields different DB than what arises when the light cone gauge is used to eliminate degrees of freedom from the YM action before employing the Dirac formalism.

There is a similar treatment of the spin two action in a manner consistent with the approach used here with YM theory in ref. [15].

We now turn to examining the superparticle in the light cone gauge.

2.3 The Superparticle and the Light Cone

The superparticle [20] has Bosonic variables $x^\mu(\tau)$ and Fermionic variables $\theta(\tau)$; its action is written as

$$S = \int d\tau \frac{1}{2e}\left(\dot{x}^\mu + i\dot{\bar{\theta}}\gamma^\mu\theta\right)\left(\dot{x}_\mu + i\dot{\bar{\theta}}\gamma_\mu\theta\right). \tag{2.29}$$

A discussion of its constraint structure appears in ref. [21] (see also ref. [22]). Quite often, the light cone gauge conditions

$$x^+ = p_+\tau \tag{2.30a}$$

$$\gamma^+\theta = 0 \tag{2.30b}$$

are used [17] to eliminate degrees of freedom from the action of eq. (2.29) prior to applying Dirac's formalism; here we will instead use the gauge conditions of eq. (2.30) in conjunction with the first class constraints arising from eq. (2.29).

The spinor θ has different properties in every dimension of the target space; we restrict our attention to $2+1$ dimensions to specify our discussion. We keep our conventions consistent with ref. [21], so that

$$\gamma^0 = \sigma_2 \quad \gamma^1 = i\sigma_3 \quad \gamma^2 = i\sigma_1 \tag{2.31}$$

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + i\epsilon^{\mu\nu\lambda}\gamma_\lambda$$

$$C = -\gamma^0 \tag{2.32}$$

$$\theta = C\overline{\theta}^T = (-\gamma^0)(\theta^+\gamma^0)^T$$

so that

$$\theta = \begin{pmatrix} U \\ d \end{pmatrix} = \begin{pmatrix} U^* \\ d^* \end{pmatrix}. \tag{2.33}$$

With these light cone coordinates, we find our action eq. (2.29) becomes

$$S = \int \frac{d\tau}{2e} \left[\left(\dot{x}^0 + i(\dot{U}U + \dot{d}d) \right)^2 - \left(\dot{x}^1 - i(\dot{U}d + \dot{d}U) \right)^2 - \left(\dot{x}^2 + i(\dot{U}U + \dot{d}d) \right)^2 \right] \tag{2.34}$$

so that the momenta conjugate to e, x^μ, U and d are

$$P_e = 0 \tag{2.35a}$$

$$p_\mu = \frac{1}{e} \left(\dot{x}^0 + i(\dot{U}U + \dot{d}d), -\dot{x}^1 + i(\dot{U}d + \dot{d}U), -\dot{x}^2 - i(\dot{U}U + \dot{d}d) \right) \tag{2.35b}$$

$$\pi_U = -idp_1 + iUp_+ \tag{2.35c}$$

$$\pi_d = i(dp_- - Up_1) \tag{2.35d}$$

where $p_\pm \equiv p_0 \pm p_2$. We can see that eqs. (2.35a,c,d) are primary constraints. Following ref. [21], we treat $\sigma_1 = \pi_U + idp_1 - iUp_+$ as a second class constraint and eliminate it by defining the DB

$$\{M, N\}^* = \{M, N\} - \{M, \sigma_1\} \frac{1}{2ip_+} \{\sigma_1, N\}. \tag{2.36}$$

With this DB, the constraint $\sigma_2 = \pi_d - idp_- + iUp_1$ satisfies

$$\{\sigma_2, \sigma_2\}^* = 2ip^2/p_+. \tag{2.37}$$

Since the canonical Hamiltonian is

$$H_c = \frac{e}{2}p^2, \tag{2.38}$$

we see that the primary constraint of eq. (2.35a) leads to the secondary first class constraint

$$p^2 = 0, \tag{2.39}$$

and hence by eq. (2.37), we see that once σ_1 has been taken to be second class, σ_2 becomes first class. (The roles of σ_1 and σ_2 can be reversed.)

It is at this stage we introduce gauge conditions to accompany the first class constraints that have been derived. In conjunction with

$$\phi_1 = p_e, \quad \phi_2 = p^2, \quad \phi_3 = \sigma_2 \tag{2.40a,b,c}$$

we introduce the respective gauges

$$\gamma_1 = e - 1, \quad \gamma_2 = x^+ - p_+\tau, \quad \gamma_3 = \gamma^+\theta = U. \tag{2.41a,b,c}$$

From the first class constraints of eq. (2.40), one can use the approach of ref. [23] to derive a generator of a set of Bosonic and Fermionic gauge transformations. the Fermionic ones being half of the so-called κ-symmetry transformations of ref. [24]. (The other half can be generated by reversing the rules of σ_1 and σ_2.)

Together, ϕ_I and γ_I in eqs. (2.40,2.41) constitute a set of second class constraints that can be eliminated by forming a "second stage" DB. This involves inverting the matrix

$$M = \left\{(\gamma_1, \phi_1, \gamma_2, \phi_2, \gamma_3, \phi_3)^T, (\gamma_1, \phi_1, \gamma_2, \phi_2, \gamma_3, \phi_3)\right\}^* \tag{2.42}$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p_- & -U/p_+ & -2iUp_1/p_+ \\ 0 & 0 & -2p_- & 0 & 0 & 0 \\ 0 & 0 & U/p_+ & 0 & i/2p_+ & -p_1/p_+ \\ 0 & 0 & 2iUp_1/p_+ & 0 & -p_1/p_+ & 2ip^2/p_+ \end{pmatrix}.$$

To find M^{-1}, we use the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} \Delta^{-1} & -\Delta^{-1}BD^{-1} \\ -D^{-1}C\Delta^{-1} & D^{-1} + D^{-1}C\Delta^{-1}BD^{-1} \end{pmatrix} (\Delta = A - BD^{-1}C)$$

and $U^2 = 0$ (since U is Grassmann); we arrive at

$$M^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2p_- & 0 & 0 \\ 0 & 0 & 1/2p_- & 0 & -iU/p_- & 0 \\ 0 & 0 & 0 & -iU/p_+ & -2ip^2/p_- & -p_1/p_- \\ 0 & 0 & 0 & 0 & -p_1/p_- & 1/2ip_- \end{pmatrix}. \tag{2.43}$$

From the resulting DB, it follows, for example that

$$\left\{x^1, x^2\right\}^{**} = \left\{x^1, x^2\right\}^* - \left\{x^1, \Phi^T\right\}^* M^{-1} \left\{\Phi, x^2\right\}^*$$
$$= \frac{p_1\tau}{p_-} + \frac{iUdp_0}{2p_+p_-}. \tag{2.44}$$

where $\Phi^T = (\gamma_1, \phi_1, \gamma_2, \phi_2, \gamma_3, \phi_3)^T$. This result serves to illustrate how using the light cone gauge conditions of eq. (2.41) in conjunction with the first class constraints of eq. (2.42) (arrived at by applying Dirac's canonical procedure to the initial action of eq. (2.29)) leads to results different from those obtained by using eq. (2.41) to eliminate fields from eq. (2.29) and only then applying the Dirac procedure (as is normally done).

These considerations can also be applied to string theories. For the Bosonic string, the action is [25]

$$S = \int d\tau d\sigma \left(\frac{1}{2} \sqrt{-g}\; g^{ab} x^A_{,a} x_{A,b} \right). \tag{2.45}$$

The canonical momenta associated with g^{ab} and x_A are

$$\boldsymbol{P}_{ab} = 0 \tag{2.46a}$$

$$p_A = \sqrt{-g} \left(g^{00} x_{A,0} + g^{01} x_{A,1} \right) \tag{2.46b}$$

which lead to the secondary first class constraints

$$\Sigma_S = \frac{1}{2} \left(p_A^2 + x_{A,1}^2 \right) \tag{2.47a}$$

$$\Sigma_p = p_A x^A_{,1} \tag{2.47b}$$

both of these in principle should be accompanied by a suitable gauge condition. However, the usual practice is to use a single gauge condition (the "light cone gauge") and then using this to simplify the initial action of eq. (2.45). Only at this stage is the Dirac procedure invoked. A similar approach is generally used with the superstring. (A discussion of the canonical structure of the superstring appears in ref. [26].)

2.4 Discussion

The Dirac procedure for treating the canonical structure of dynamical systems which have a local gauge invariance is well defined; all constraints are first obtained and then classified, and those which are first class are then paired with suitable gauge conditions. All superfluous degrees of freedom arising on account of there being a local gauge symmetry are then eliminated by replacing the PB by a DB defined using both the first and second class constraints and the gauge conditions. This procedure can be tedious especially for such common theories, as YM theory on the light front and the superparticle (as was done above). Both of these systems are commonly simplified by using a "light cone" gauge condition to eliminate superfluous degrees of freedom at the outset from the classical action, and then using Dirac's procedure. However, we derive the DB in a way that is to fully consistent with the Dirac procedure; superfluous degrees of freedom are not eliminated at the outset. The two procedures lead to different quantum theories from what is obtained if one were to use the DB to define a quantum mechanical commutator.

2.5 Appendix

In refs.[19,23] it is shown how to obtain the generator of a gauge transformation for systems involving exclusively first class constraints. In fact, a really interesting aspect of Dirac constraint formalism is that it allows us to derive a complete gauge generator of any theory, through deriving all first class constraints. Here we will extend this discussion to include the situation in which there are also primary second class constraints so that one can consider the light front formulation of Yang-Mills theory.

In the presence of primary second class constraints θ_α and first class constraints ϕ_{A_i} (where i denotes the generation of the constraint-primary is $i = 1$, secondary is $i = 2$ etc.), then suppose we have the PB algebra

$$\left\{\theta_\alpha, \theta_\beta\right\} = \Delta_{\alpha\beta}, \tag{A.1}$$

as well as

$$\{\phi_A, \phi_B\} = C^C_{AB}\phi_C + C^\alpha_{AB}\theta_\alpha \tag{A.2}$$

and

$$\{\phi_A, \theta_\alpha\} = C^\beta_{A\alpha}\theta_\beta + +C^B_{A\alpha}\phi_B. \tag{A.3}$$

We then can define the DB

$$\{M, N\}^* = \{M, N\} - \{M, \theta_\alpha\}\,\Delta^{-1}_{\alpha\beta}\left\{\theta_\beta, N\right\}. \tag{A.4}$$

Upon using the constraints θ_α and ϕ_{A_i}, the canonical Hamiltonian H_C can be defined

$$H_C = p_i\dot{q}^i - L(q^i, \dot{q}^i); \tag{A.5}$$

this leads to the extended Hamiltonian

$$H_E = H_C + \sum_\alpha U_\alpha\theta_\alpha + \sum_{A_i} V_{A_i}\phi_{A_i}. \tag{A.6}$$

If the sum over A_i in eq. (A.6) is restricted to having $i = 1$ (i.e., just the primary constraints) then H_E reduces to H_T, the total Hamiltonian.

We now can consider the generator

$$G = \sum_{A_i} \mu_{A_i}\phi_{A_i} \tag{A.7}$$

of "gauge" transformations that leave the extended action S_E invariant, that is the change induced by G on a dynamical quantity f is given by

$$\delta f = \{f, G\}\,. \tag{A.8}$$

The change in the extended action is given by

$$\begin{aligned}\delta S_E &= \int dt\; \delta\left(p_i \dot{q}^i - H_E\right) \\ &\int dt\Big[\delta p_i \dot{q}^i + p_i \delta \dot{q}_i - \{H_C, G\} \\ &\quad - \sum_\alpha \left(\delta U_\alpha \theta_\alpha + U_\alpha \{\theta_\alpha, G\}\right) \\ &\quad - \sum_{A_i} \left(\delta V_{A_i} \phi_{A_i} + V_{A_i} \{\phi_{A_i}, G\}\right).\end{aligned} \tag{A.9}$$

But now into eq. (A.9) we can substitute

$$\begin{aligned}\delta p_i \dot{q}^i + p_i \delta \dot{q}^i &= -\frac{\partial G}{\partial q^i}\dot{q}^i + \frac{d}{dt}\left(p_i \frac{\partial G}{\partial p_i}\right) - \dot{p}^i \frac{\partial G}{\partial p_i} \\ &= \frac{d}{dt}\left(p_i \frac{\partial G}{\partial p_i} - G\right) + \left[\left(\frac{\partial}{\partial t} + \dot{U}_\alpha \frac{\partial}{\partial U_\alpha} + \dot{V}_{A_i}\frac{\partial}{\partial V_{A_i}}\right)\mu_{B_j}\right]\phi_{B_j}\end{aligned} \tag{A.10}$$

yielding

$$\begin{aligned}\delta S_E = \int dt\Big[&\left(\frac{D}{Dt}\mu_{B_i}\right)\phi_{B_i} + U_\alpha \mu_{A_i}\left(D^{B_j}_{A_i}\phi_{B_j} + D^\gamma_{A_i}\theta_\gamma\right) \\ &- \sum_\alpha \left(\delta U_\alpha \theta_\alpha - U_\alpha \mu_{B_j}\left(C^\gamma_{B_{j\alpha}}\theta_\gamma + C^C_{B_{j\alpha}}\phi_C\right)\right) \\ &- \sum_{A_i}\left(\delta V_{A_i}\phi_{A_i} - V_{A_i}\mu_{B_j}\left(C^{C_k}_{B_j A_i}\phi_{C_k} + C^\gamma_{B_j A_i}\theta_\gamma\right)\right)\Big].\end{aligned} \tag{A.11}$$

In eq. (A.11), we have dropped all surface terms, defined

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{U}_\alpha \frac{\partial}{\partial U_\alpha} + \dot{V}_{A_i}\frac{\partial}{\partial V_{A_i}} \tag{A.12}$$

and have used the fact the ϕ_{A_i} are all first class so that

$$\{\phi_{A_i}, H_C\} = D^{B_j}_{A_i}\phi_{B_j} + D^{j}_{A_i}\theta_j. \tag{A.13}$$

In eq. (11), we can arrange for $\delta S_E = 0$ by choosing δU_α so that all coefficients of θ_α vanish, and by having the μ_{B_i} satisfy a differential equation that answers that the coefficients of ϕ_{B_i} sum to zero. Upon having [19,23] $\delta V_{A_i} = V_{A_i} = 0 (i \geq 2)$, S_E reduces to S_T, the total action, and G becomes the generator of gauge transformations that leave

$$S_C = \int dt L(q^i, \dot{q}^i) \tag{A.14}$$

invariant, as S_T and S_C have the same dynamical content.

We can replace eq. (A.8) with

$$\delta f = \{f, G\}^* \tag{A.15}$$

as by eq. (A.3), $\{f, G\}^*$ and $\{f, G\}$ differ by an expression that is at least linear in θ_α; in eq. (A.11) this term can be absorbed into δU_α. The advantage of using the DB over the PB in finding δf is that we can set $\theta_\alpha = 0$ at the outset of any calculation.

It would be interesting to see how the approach of ref. [27] to finding gauge symmetries could be adapted to the case in which second class constraints are present.

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Chapter 3

Quantizing the Palatini Action using a Transverse Traceless Propagator

3.1 Introduction

It has been shown with both Yang-Mills (YM) action and the Einstein-Hilbert (EH) action for gravity, that by using the first order form of the action, there is only a single vertex arising from the classical action and this is independent of momentum [1, 2, 3, 4, 5]. This simplifies the computation of loop diagrams, even though the number of propagating fields is increased.

It has also been shown that imposing both the conditions of tracelessness and transversality on the spin two propagator associated with the EH action requires use of a non-quadratic gauge fixing Lagrangian [6, 7, 8, 9, 10]. Such gauge fixing results in the need to consider the contributions of two complex Fermionic ghosts and one real Bosonic ghost analogous to the usual complex "Faddeev-Popov" ghosts.

In this chapter of my thesis we consider how the full first order Einstein-Hilbert (1EH) action can be used in conjunction with the transverse-traceless (TT) gauge. We will show that the spin two propagator is TT only if the gauge fixing parameter α is allowed to vanish. This limit for α results in a well defined set of Feynman rules with two propagating Bosonic fields, two complex Fermionic ghost fields, one real Bosonic ghost, three three-point vertices for the Bosonic fields and four ghost vertices.

3.2 The TT gauge for the 1EH Action

The Einstein-Hilbert action in first order (Palatini) form

$$S = \int \mathrm{d}^d x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma) \tag{3.1}$$

when written in terms of the variables

$$h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \tag{3.2a}$$

and

$$G^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \frac{1}{2}\left(\delta^{\lambda}_{\mu}\Gamma^{\sigma}_{\nu\sigma} + \delta^{\lambda}_{\nu}\Gamma^{\sigma}_{\mu\sigma}\right) \tag{3.2b}$$

becomes

$$S = \int \mathrm{d}^d x h^{\mu\nu}\left(G^{\lambda}_{\mu\nu,\lambda} + \frac{1}{d-1} G^{\lambda}_{\lambda\mu} G^{\sigma}_{\sigma\nu} - G^{\lambda}_{\mu\sigma} G^{\sigma}_{\nu\lambda}\right). \tag{3.3}$$

This "Palatini" form of the action facilitates a canonical analysis of S [11]. The diffeomorphism invariance of S in Eq. (3.1) leads to the local gauge transformations

$$\delta h^{\mu\nu} = h^{\mu\lambda}\partial_{\lambda}\theta^{\nu} + h^{\nu\lambda}\partial_{\lambda}\theta^{\mu} - \partial_{\lambda}(h^{\mu\nu}\theta^{\lambda}) \tag{3.4a}$$

$$\begin{aligned}\delta G^{\lambda}_{\mu\nu} &= -\partial^2_{\mu\nu}\theta^{\lambda} + \frac{1}{2}\left(\delta^{\lambda}_{\mu}\partial_{\nu} + \delta^{\lambda}_{\nu}\partial_{\mu}\right)\partial_{\rho}\theta^{\rho} - \theta^{\rho}\partial_{\rho}G^{\lambda}_{\mu\nu} \\ &+ G^{\rho}_{\mu\nu}\partial_{\rho}\theta^{\lambda} - \left(G^{\lambda}_{\mu\rho}\partial_{\nu} + G^{\lambda}_{\nu\rho}\partial_{\mu}\right)\theta^{\rho} \end{aligned} \tag{3.4b}$$

The term bilinear in h and G in Eq. (3.3) does not lead to a well defined propagator, irrespective of the choice of gauge fixing. However, upon making an expansion of $h^{\mu\nu}$ about a flat background

$$h^{\mu\nu} = \eta^{\mu\nu} + \phi^{\mu\nu}(x) \quad (\eta^{\mu\nu} = \mathrm{diag}(+---\dots)) \tag{3.5}$$

the term bilinear in ϕ and G arising from Eq. (3.3) does have a well defined propagator once an appropriate gauge fixing is chosen. These bilinear terms are the first order form of the action for a spin two field [11].

In order to have a TT propagator for the spin two field we must consider a general gauge fixing Lagrangian that is not quadratic [6]. If the classical Lagrange density appearing in Eq. (3.3) is $\mathcal{L}(h^{\mu\nu}, G^{\lambda}_{\mu\nu})$, then this entails inserting into the generating functional

$$Z[j_{\mu\nu}, J^{\mu\nu}_{\lambda}] = \int \mathcal{D}\phi^{\mu\nu}\mathcal{D}G^{\lambda}_{\mu\nu} \exp i \int \mathrm{d}^d x \left(\mathcal{L}(\eta+\phi, G) + j_{\mu\nu}\phi^{\mu\nu} + J^{\mu\nu}_{\lambda}G^{\lambda}_{\mu\nu}\right) \tag{3.6}$$

two factors of "1"

$$1 = \int \mathcal{D}\boldsymbol{\theta}_i\, \delta\left(\underset{\sim}{F}_i(\boldsymbol{\phi} + \underset{\sim}{A}\boldsymbol{\theta}_i) - \boldsymbol{p}_i\right)\det(\underset{\sim}{F}_i\underset{\sim}{A}); \quad (i = 1, 2) \tag{3.7}$$

where $\boldsymbol{\phi} = (\phi^{\mu\nu}, G^{\lambda}_{\mu\nu})$. The gauge transformations of Eq. (3.4) are of the form

$$\delta_i\boldsymbol{\phi} = \underset{\sim}{A}\boldsymbol{\theta}_i \tag{3.8}$$

and the gauge fixing conditions are

$$\underset{\sim}{F}_i\boldsymbol{\phi} = 0. \tag{3.9}$$

Insertion of a third factor of "1" that is of the form

$$1 = \frac{1}{(\pi\alpha)^d}\int \mathcal{D}\boldsymbol{p}_1\mathcal{D}\boldsymbol{p}_2 \exp\frac{-i}{\alpha}\int \mathrm{d}^d x(\boldsymbol{p}_1^{\mathrm{T}}\,\underset{\sim}{N}\boldsymbol{p}_2)\det(\,\underset{\sim}{N}) \tag{3.10}$$

into Eq. (3.6) leads to

$$\begin{aligned} Z[\boldsymbol{j}] = \textstyle\int \mathcal{D}\boldsymbol{\phi}\det(\,\underset{\sim}{F}_1\underset{\sim}{A})\det(\,\underset{\sim}{F}_2\underset{\sim}{A})\det(\,\underset{\sim}{N}/\pi\alpha)\int\mathcal{D}\boldsymbol{\theta}_1\mathcal{D}\boldsymbol{\theta}_2 \\ \exp i\textstyle\int \mathrm{d}^d x\left\{\mathcal{L}(\boldsymbol{\phi}) - \frac{1}{\alpha}\left[\,\underset{\sim}{F}_1(\boldsymbol{\phi}+\underset{\sim}{A}\boldsymbol{\theta}_1)\right]^{\mathrm{T}}\,\underset{\sim}{N}\left[\,\underset{\sim}{F}_2(\boldsymbol{\phi}+\underset{\sim}{A}\boldsymbol{\theta}_2)\right] + \boldsymbol{j}^{\mathrm{T}}\cdot\boldsymbol{\phi}\right\}; \end{aligned} \tag{3.11}$$

where $\boldsymbol{j} \equiv (j_{\mu\nu}, J_\lambda^{\mu\nu})$.

Since the gauge transformation of Eq. (3.8) leaves $\mathcal{L}(\boldsymbol{\phi})$, $\mathcal{D}\boldsymbol{\phi}$ and $\det(\,\underset{\sim}{F}_i\underset{\sim}{A})$ invariant [12, 13], we can make the shift

$$\boldsymbol{\phi} \to \boldsymbol{\phi} - \underset{\sim}{A}(\boldsymbol{\theta}_+ + \epsilon\boldsymbol{\theta}_-) \tag{3.12}$$

in Eq. (3.11) ($\boldsymbol{\theta}_\pm \equiv (\boldsymbol{\theta}_1 \pm \boldsymbol{\theta}_2)/2$) leaving us with

$$\begin{aligned} Z[\boldsymbol{j}] = \int \mathcal{D}\boldsymbol{\phi}\mathcal{D}\boldsymbol{\theta}_-\det(\,\underset{\sim}{F}_1\underset{\sim}{A})\det(\,\underset{\sim}{F}_2\underset{\sim}{A})\det(\,\underset{\sim}{N}) \\ \exp i\int \mathrm{d}^d x\left\{\mathcal{L}(\boldsymbol{\phi}) - \frac{1}{\alpha}\left[\,\underset{\sim}{F}_1(\boldsymbol{\phi}+\underset{\sim}{A}(1-\epsilon)\boldsymbol{\theta}_-)\right]^{\mathrm{T}}\right. \\ \left.\underset{\sim}{N}\left[\,\underset{\sim}{F}_2(\boldsymbol{\phi}-\underset{\sim}{A}(1+\epsilon)\boldsymbol{\theta}_-)\right] + \boldsymbol{j}^{\mathrm{T}}\cdot\boldsymbol{\phi}\right\}. \end{aligned} \tag{3.13}$$

A factor $1/(\pi\alpha)^{d/2}\int\mathcal{D}\boldsymbol{\theta}_+$ has been absorbed into the normalization of Z. We now choose the gauge fixing to be

$$\underset{\sim}{F}_i\boldsymbol{\phi} = g_i\partial_\rho\phi_\mu^\mu + \partial_\mu\phi_\rho^\mu \tag{3.14a}$$

and

$$\underset{\sim}{N} = \eta^{\mu\nu}/2. \tag{3.14b}$$

The gauge fixing contribution of Eq. (3.13) becomes

$$\begin{aligned}
& \left[\underset{\sim}{F}_1(\boldsymbol{\phi} + \underset{\sim}{A}(1-\epsilon)\boldsymbol{\theta}_-)\right]^{\mathrm{T}} \underset{\sim}{N}\left[\underset{\sim}{F}_2(\boldsymbol{\phi} - \underset{\sim}{A}(1+\epsilon)\boldsymbol{\theta}_-)\right] \\
= \; & (\underset{\sim}{F}_1\boldsymbol{\phi})^{\mathrm{T}} \underset{\sim}{N}(\underset{\sim}{F}_2\boldsymbol{\phi}) + (\epsilon^2-1)\left\{\left[\boldsymbol{\theta}_-^{\mathrm{T}} + \frac{1}{2}\boldsymbol{\phi}^{\mathrm{T}}\left(-(1+\epsilon)\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2 + (1-\epsilon)\,\underset{\sim}{F}_2^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_1\right)\underset{\sim}{A}\right.\right. \\
& \left.\left((\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1}/(\epsilon^2-1)\right)\right]\left[\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A}\right] \\
& \left.\left[\boldsymbol{\theta}_- + \frac{1}{2}\left((\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1}/(\epsilon^2-1)\right)\underset{\sim}{A}^{\mathrm{T}}\left(-(1+\epsilon)\,\underset{\sim}{F}_2^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_1 + (1-\epsilon)\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\right)\boldsymbol{\phi}\right]\right\} \\
& -\frac{1}{4(\epsilon^2-1)}\boldsymbol{\phi}^{\mathrm{T}}\left(-(1+\epsilon)\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2 + (1-\epsilon)\,\underset{\sim}{F}_2^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_1\right)\underset{\sim}{A}(\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1}\underset{\sim}{A}^{\mathrm{T}} \\
& \left(-(1+\epsilon)\,\underset{\sim}{F}_2^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_1 + (1-\epsilon)\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\right)\boldsymbol{\phi}
\end{aligned} \tag{3.15}$$

(In Eq. (3.15) we use the convention $\partial^{\mathrm{T}} = -\partial$.)

Provided $\epsilon \neq \pm 1$, the shift in $\boldsymbol{\theta}_-$

$$\boldsymbol{\theta}_- \to \boldsymbol{\theta}_- - \frac{1}{2}\left((\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1}/(\epsilon^2-1)\right)\underset{\sim}{A}^{\mathrm{T}}\left(-(1+\epsilon)\,\underset{\sim}{F}_2^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_1 + (1-\epsilon)\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\right)\boldsymbol{\phi} \tag{3.16}$$

can be made to diagonalize Eq. (3.15) in $\boldsymbol{\theta}_-$ and $\boldsymbol{\phi}$. In Refs. [6, 7, 8] and eq. (1.53) above, $\epsilon = \pm 1$ and a shift in $\boldsymbol{\phi}$ was used to diagonalize the gauge fixing, but as such a shift is not a gauge transformation, $\mathcal{L}(\boldsymbol{\phi})$ is not invariant under this transformation and new vertices involving $\boldsymbol{\phi}$ and $\boldsymbol{\theta}_-$ must be introduced. We take $\epsilon \neq \pm 1$ in order to be able to make a shift in $\boldsymbol{\theta}_-$ that eliminates mixed propagators for these fields without introducing extra vertices.

Together Eqs. (3.15) and (3.16) result in

$$\begin{aligned}
Z[\boldsymbol{j}] \;=\; & \int \mathcal{D}\boldsymbol{\phi}\mathcal{D}\boldsymbol{\theta}_- \det(\underset{\sim}{F}_1\underset{\sim}{A})\det(\underset{\sim}{F}_2\underset{\sim}{A})\det(\underset{\sim}{N}) \\
& \exp i\int \mathrm{d}^d x\left\{\mathcal{L}(\boldsymbol{\phi}) - \frac{1}{\alpha}(\underset{\sim}{F}_1\boldsymbol{\phi})^{\mathrm{T}}\underset{\sim}{N}(\underset{\sim}{F}_2\boldsymbol{\phi}) - \frac{1}{\alpha(\epsilon^2-1)}\boldsymbol{\theta}_-^{\mathrm{T}}(\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})\boldsymbol{\theta}_-\right. \\
+ \; & \frac{1}{4\alpha(\epsilon^2-1)}\boldsymbol{\phi}^{\mathrm{T}}\left(-(1+\epsilon)\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2 + (1-\epsilon)\,\underset{\sim}{F}_2^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_1\right)\underset{\sim}{A}(\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1} \\
& \left.\underset{\sim}{A}^{\mathrm{T}}\left(-(1+\epsilon)\,\underset{\sim}{F}_2^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_1 + (1-\epsilon)\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\right)\boldsymbol{\phi} + \boldsymbol{j}^{\mathrm{T}}\cdot\boldsymbol{\phi}\right\}.
\end{aligned} \tag{3.17}$$

The integral over $\boldsymbol{\theta}_-$ can now be evaluated in Eq. (3.17); it results in a contribution

$$\det{}^{-1/2}(\underset{\sim}{F}_1\underset{\sim}{A})\det{}^{-1/2}(\underset{\sim}{N})\det{}^{-1/2}(\underset{\sim}{F}_2\underset{\sim}{A}). \tag{3.18}$$

We now treat the last term in Eq. (3.17) as an interaction term. Due to its structure, the two fields $\boldsymbol{\phi}$ that occur explicitly ($\underset{\sim}{A}$ also is $\boldsymbol{\phi}$ dependent on account of Eq. (3.4)) are contracted with a propagator for $\phi_{\mu\nu}$ and a factor of $\underset{\sim}{X}$ where

$$\begin{aligned}
X_{\mu\nu,\lambda\sigma} &\equiv \left(-(1+\epsilon)\,\underset{\sim}{F}_2^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_1 + (1-\epsilon)\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\right)_{\mu\nu,\lambda\sigma} \\
&= \frac{1}{2}(g_1-g_2)\left(\partial_\mu\partial_\nu\eta_{\lambda\sigma} - \eta_{\mu\nu}\partial_\lambda\partial_\sigma\right) \\
&+ \epsilon\left[g_1g_2\eta_{\mu\nu}\eta_{\lambda\sigma}\partial^2 + \frac{g_1+g_2}{2}\left(\partial_\mu\partial_\nu\eta_{\lambda\sigma} + \eta_{\mu\nu}\partial_\lambda\partial_\sigma\right)\right. \\
&+ \left.\frac{1}{4}\left(\partial_\mu\partial_\lambda\eta_{\nu\sigma} + \partial_\nu\partial_\lambda\eta_{\mu\sigma} + \partial_\mu\partial_\sigma\eta_{\nu\lambda} + \partial_\nu\partial_\sigma\eta_{\mu\lambda}\right)\right]
\end{aligned} \tag{3.19}$$

by Eq. (3.14).

We know from Refs. [6, 7, 8] and eq. (1.59) above that as $\alpha \to 0$, the propagator for the field $\phi_{\mu\nu}$ that comes from $\mathcal{L}(\boldsymbol{\phi}) - \frac{1}{\alpha}(\underset{\sim}{F}_1\boldsymbol{\phi})^{\mathrm{T}}\,\underset{\sim}{N}(\underset{\sim}{F}_2\boldsymbol{\phi})$ is transverse and traceless in the limit $\alpha \to 0$ provided $g_1 \neq g_2$. Only terms of order α are not transverse and traceless. Thus, on account of the structure of Eq. (3.19), the contribution of the vertex coming from the last term in Eq. (3.17) vanishes as $\alpha \to 0$, even though this vertex is proportional to $1/\alpha$. There is one exception to this; when a sequence of these vertices lies in a ring, then a finite contribution arises in the limit $\alpha \to 0$. To see this in more detail, write this last term in Eq. (3.17) as

$$\frac{1}{\alpha}\boldsymbol{\phi}^{\mathrm{T}}\underset{\sim}{V}\boldsymbol{\phi} = \frac{1}{\alpha}\boldsymbol{\phi}^{\mathrm{T}}(\underset{\sim}{X}^{\mathrm{T}}\underset{\sim}{A})\frac{(\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1}}{4(\epsilon^2-1)}\underset{\sim}{A}^{\mathrm{T}}\underset{\sim}{X}\boldsymbol{\phi}. \tag{3.20}$$

A ring in which a sequence of these vertices occurs results in a contribution proportional to

$$\begin{aligned}
\mathrm{Tr}\Bigg\{&\left[\tfrac{1}{\alpha}\underset{\sim}{X}^{\mathrm{T}}\underset{\sim}{A}(\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1}\underset{\sim}{A}^{\mathrm{T}}\underset{\sim}{X}\right]\underset{\sim}{D}\left[\tfrac{1}{\alpha}\underset{\sim}{X}^{\mathrm{T}}\underset{\sim}{A}(\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1}\underset{\sim}{A}^{\mathrm{T}}\underset{\sim}{X}\right]\underset{\sim}{D} \\
&\ldots\left[\tfrac{1}{\alpha}\underset{\sim}{X}^{\mathrm{T}}\underset{\sim}{A}(\underset{\sim}{A}^{\mathrm{T}}\,\underset{\sim}{F}_1^{\mathrm{T}}\,\underset{\sim}{N}\,\underset{\sim}{F}_2\underset{\sim}{A})^{-1}\underset{\sim}{A}^{\mathrm{T}}\underset{\sim}{X}\right]\underset{\sim}{D}\Bigg\},
\end{aligned} \tag{3.21}$$

where $\underset{\sim}{D}$ is the propagator of $\boldsymbol{\phi}$. From Eq. (3.19) it is apparent that since when $\alpha \to 0$ $\underset{\sim}{D}$ is transverse and traceless, then $\underset{\sim}{X}\underset{\sim}{D}$ is of order α; since we have a factor of $1/\alpha$ for each factor of $\underset{\sim}{X}\underset{\sim}{D}$ on account of these vertices occurring in a ring, we can let

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\underset{\sim}{X}\underset{\sim}{D} = \underset{\sim}{X}\underset{\sim}{D}^{(0)}. \tag{3.22}$$

Furthermore, a contribution of a closed loop of these vertices can be written as

$$\begin{aligned} &\det{}^{-1/2}\left[\underset{\sim}{X}^{\mathrm{T}}\underset{\sim}{A}(\underset{\sim}{A}^{\mathrm{T}}\ \underset{\sim}{F}_1^{\mathrm{T}}\ \underset{\sim}{N}\ \underset{\sim}{F}_2\underset{\sim}{A})^{-1}\underset{\sim}{A}^{\mathrm{T}}\underset{\sim}{X}\underset{\sim}{D}^{(0)}\right] \\ = \;&\det{}^{1/2}(\ \underset{\sim}{F}_1\underset{\sim}{A})\det{}^{1/2}(\ \underset{\sim}{N})\det{}^{1/2}(\ \underset{\sim}{F}_2\underset{\sim}{A})\det{}^{-1/2}(\underset{\sim}{A}^{\mathrm{T}}\underset{\sim}{X}\underset{\sim}{D}^{(0)}\underset{\sim}{X}^{\mathrm{T}}\underset{\sim}{A}). \end{aligned} \tag{3.23}$$

Together Eqs. (3.18) and (3.23) reduce Eq. (3.17) to

$$\begin{aligned} Z[\boldsymbol{j}] \;=\; &\lim_{\alpha\to 0}\int \mathcal{D}\boldsymbol{\phi}\det(\ \underset{\sim}{F}_1\underset{\sim}{A})\det(\ \underset{\sim}{N})\det(\ \underset{\sim}{F}_2\underset{\sim}{A})\det{}^{-1/2}(\underset{\sim}{A}^{\mathrm{T}}\underset{\sim}{X}\underset{\sim}{D}^{(0)}\underset{\sim}{X}^{\mathrm{T}}\underset{\sim}{A}) \\ &\exp i\int \mathrm{d}^d x\left\{\mathcal{L}(\boldsymbol{\phi}) - \frac{1}{\alpha}(\ \underset{\sim}{F}_1\boldsymbol{\phi})^{\mathrm{T}}\ \underset{\sim}{N}(\ \underset{\sim}{F}_2\boldsymbol{\phi}) + \boldsymbol{j}^{\mathrm{T}}\cdot\boldsymbol{\phi}\right\} \end{aligned} \tag{3.24}$$

provided $g_1 \neq g_2$. The functional determinants in Eq. (3.24) can be exponentiated using "ghost" fields; $\det(\ \underset{\sim}{F}_i\underset{\sim}{A})$ $(i = 1, 2)$ using complex Fermionic "Faddeev-Popov" ghosts $\mathbf{c}_i$ [14, 15, 16, 17], $\det(\ \underset{\sim}{N})$ by a complex Fermionic Nielsen-Kalosh ghost [18, 19] and $\det{}^{-1/2}(\underset{\sim}{A}^{\mathrm{T}}\underset{\sim}{X}\underset{\sim}{D}^{(0)}\underset{\sim}{A})$ by a real Bosonic ghost ζ. By Eq. (3.4a), it follows that

$$(\underset{\sim}{A}\boldsymbol{\theta})_{\mu\nu} = \left[\partial_\mu\eta_{\nu\rho} + \partial_\nu\eta_{\mu\rho} - \partial_\rho\eta_{\mu\nu} + \left(\phi_\mu{}^\sigma\partial_\sigma\eta_{\nu\rho} + \phi_\nu{}^\sigma\partial_\sigma\eta_{\mu\rho} + \partial_\rho\phi_{\mu\nu}\right)\right]\theta^\rho. \tag{3.25}$$

Using Eqs. (3.19) and (3.25) and the propagator for $\boldsymbol{\phi}$ given in Ref [6] we find that the contribution that is bilinear in the ghost $\zeta$is given by

$$4p^2\zeta_\mu\left\{\epsilon^2 p^2\eta^{\mu\nu} + \left[\left(g_1g_2(d-2)^2 - (g_1+g_2)(d-2)\right)(\epsilon^2-1) - 1\right]p^\mu p^\nu\right\}\zeta_\nu \tag{3.26}$$

which becomes

$$4p^4\epsilon^2\zeta_\mu\eta^{\mu\nu}\zeta_\nu. \tag{3.27}$$

when

$$g_1 = -g_2 = \frac{1}{(d-2)\sqrt{1-\epsilon^2}}. \tag{3.28}$$

Similarly, the vertex for $\phi_{\mu\nu}(p) - \zeta_\alpha(q) - \zeta_\beta(r)$ comes from

$$\begin{aligned}
&\frac{1}{2}\Big\{\Big[(d-2)g_1(\epsilon-1)^2 + (d-2)g_2((\epsilon+1)^2-2\Big] q^\mu q^\alpha \left(p^\beta q^\nu + r^\beta q^\nu - 2q^\beta r^\nu\right) \\
+ \quad & \epsilon^2 q^2 q^\mu \left[2r^\nu \eta^{\alpha\beta} - p^\beta \eta^{\alpha\nu} + r^\beta \eta^{\alpha\nu}\right] \\
+ \quad & q^2 \left(2r^\nu q^\alpha \eta^{\mu\beta} - p^\beta q^\alpha \eta^{\mu\nu} - r^\beta q^\alpha \eta^{\mu\nu}\right)\left[g_1(\epsilon+1)^2 + g_2(\epsilon-1)^2 - 2g_1 g_2 (d-1)(\epsilon+1)^2\right] \\
+ \quad & \epsilon^2 q^2 \eta^{\mu\nu}\left(2r^\nu q^\beta - p^\beta q^\nu - r^\beta q^\nu\right)\Big\} + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta;\ q \leftrightarrow r).
\end{aligned} \tag{3.29}$$

Finally, a vertex for $\phi_{\mu_1\nu_1}(p) - \phi_{\mu_2\nu_2}(q) - \zeta_\alpha(r) - \zeta_\beta(s)$ can also be worked out. The vertices $\phi - \phi - \zeta - \zeta$ and $\phi - \zeta - \zeta$ are both quartic in the external momenta.

The two complex "Faddeev-Popov" ghosts $\mathbf{c}_1$ and $\mathbf{c}_2$ and the real Bosonic ghost ζ reduce to a single complex Fermionic Faddeev-Popov ghost $\mathbf{c} = \mathbf{c}_1 + i\mathbf{c}_2$ if we deal with a quadratic gauge fixing Lagrangian when $\underset{\sim}{F}_1 = \underset{\sim}{F}_2$.

If we now define $M_{\lambda\ \sigma}^{\mu\nu\pi\tau}(h)$ by the equation

$$h^{\mu\nu}\left(\frac{1}{d-1}G^\lambda_{\lambda\mu}G^\sigma_{\sigma\nu} - G^\lambda_{\sigma\mu}G^\sigma_{\lambda\nu}\right) = \frac{1}{2}M_{\lambda\ \sigma}^{\mu\nu\pi\tau}(h) G^\lambda_{\mu\nu}G^\sigma_{\pi\tau} \tag{3.30}$$

then the shift

$$G^\lambda_{\mu\nu} \to G^\lambda_{\mu\nu} + M^{-1\lambda\ \sigma}_{\ \ \mu\nu\pi\tau}(\eta)\,\phi^{\pi\tau}_{,\sigma} \tag{3.31}$$

in $\mathcal{L}(\boldsymbol{\phi})$ in Eq. (3.17) leads to

$$\begin{aligned}
\mathcal{L}(\boldsymbol{\phi}) = -\frac{1}{2}\phi^{\mu\nu}_{,\lambda} M^{-1\lambda\ \sigma}_{\ \ \mu\nu\pi\tau}(\eta)\,\phi^{\pi\tau}_{,\sigma} + \frac{1}{2}G^\lambda_{\mu\nu}M_{\lambda\ \sigma}^{\mu\nu\pi\tau}(\eta)G^\sigma_{\pi\tau} \\
+\frac{1}{2}\left(G^\lambda_{\mu\nu} + \phi^{\alpha\beta}_{,\xi}M^{-1\xi\ \lambda}_{\ \ \alpha\beta\mu\nu}(\eta)\right) M_{\lambda\ \sigma}^{\mu\nu\pi\tau}(\phi)\left(G^\sigma_{\pi\tau} + M^{-1\sigma\ \zeta}_{\ \ \pi\tau\gamma\delta}(\eta)\phi^{\gamma\delta}_{,\zeta}\right)
\end{aligned} \tag{3.32}$$

so that off diagonal propagators $\phi - G$ are eliminated. However, two new momentum dependent vertices now arise. They are $\phi - \phi - \phi$ and $\phi - \phi - G$.

With the gauge fixing of Eq. (3.14) we find from Ref. [6] that the propagator for the field $G^{\lambda}_{\mu\nu}$ is

$$
\begin{aligned}
{}^{\lambda}_{\mu\nu} \rule[0.5ex]{3em}{0.4pt} {}^{\rho}_{\pi\tau} \quad = \quad & \frac{1}{4}\eta^{\lambda\rho}\left(\eta_{\mu\tau}\eta_{\nu\pi} + \eta_{\mu\pi}\eta_{\nu\tau} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\pi\tau}\right) \\
& -\frac{1}{4}\left(\delta^{\lambda}_{\tau}\delta^{\rho}_{\mu}\eta_{\nu\pi} + \delta^{\lambda}_{\tau}\delta^{\rho}_{\nu}\eta_{\mu\pi} + \delta^{\lambda}_{\pi}\delta^{\rho}_{\nu}\eta_{\mu\tau} + \delta^{\lambda}_{\pi}\delta^{\rho}_{\mu}\eta_{\nu\tau}\right) \qquad (3.33a)
\end{aligned}
$$

The propagator for $\phi_{\mu\nu}$ is [6]

$$
\begin{aligned}
\mu\nu \underset{k}{\sim\!\sim\!\sim} \lambda\sigma \quad = \quad & \frac{1}{k^2}\Bigg\{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - 2\frac{(g_1-g_2)^2 + 2(g_1+1)(g_2+1)\alpha}{\Delta}\eta_{\mu\nu}\eta_{\lambda\sigma} \\
+ \quad & (\alpha-1)\frac{1}{k^2}\left[k_\mu k_\lambda \eta_{\nu\sigma} + (\mu \leftrightarrow \nu) + (\lambda \leftrightarrow \sigma)\right] \\
+ \quad & 2\frac{(g_2-g_1)^2 + [4(g_1+1)(g_2+1) - g_2 - g_1 - 2]\alpha}{\Delta}\frac{1}{k^2}\left[k_\mu k_\nu \eta_{\lambda\sigma} + k_\lambda k_\sigma \eta_{\mu\nu}\right] \\
+ \quad & \frac{1}{\Delta}\Big[4\alpha\Big[(g_1+g_2)(d-4) + (2g_1g_2+1)(d-3) - \left(g_1^2+g_2^2\right)(d-1)\Big] \\
+ \quad & 2(d-2)\left[(g_1-g_2)^2 - \alpha^2(4(g_1+1)(g_2+1)-1)\right]\Big]\frac{1}{k^4}k_\mu k_\nu k_\lambda k_\sigma\Bigg\}, \qquad (3.33b)
\end{aligned}
$$

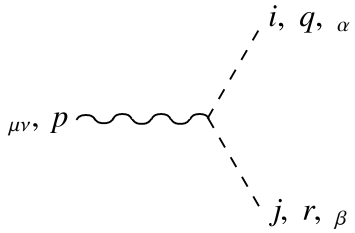
where $\Delta = (d-1)(g_1-g_2)^2 + 2(d-2)(g_1+1)(g_2+1)\alpha$.

When $\alpha \to 0$ $(g_1 \neq g_2)$ this becomes the transverse-traceless propagator.

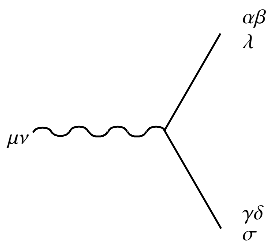
For the real fields c_i we have

$$
\mu \underset{k}{- - - - -} \nu \quad = \quad D^{(i)}_{\mu\nu} \quad = \frac{\dfrac{(d-2)g_i k_\mu k_\nu}{k^2[(d-2)g_i - 1]} - \eta^{\mu\nu}}{k^2}. \qquad (3.33c)
$$

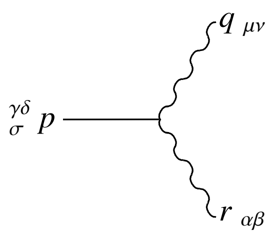
The vertices are given by



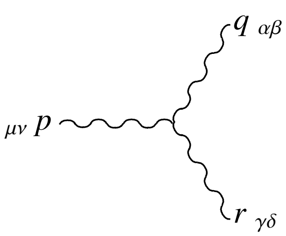
$$= \frac{\delta^{ij}}{4}\Big[-p^{\beta}q^{\nu}\eta^{\mu\alpha} - r^{\beta}q^{\nu}\eta^{\mu\alpha} - g_i p^{\beta}q^{\alpha}\eta^{\mu\nu} - g_i r^{\beta}q^{\alpha}\eta^{\mu\nu} + 2g_i r^{\nu}q^{\alpha}\eta^{\mu\beta} + (q,\alpha)\leftrightarrow(r,\beta)\Big] + \mu\leftrightarrow\nu \tag{3.34a}$$



$$= \frac{1}{8}\left\{\left[\left(\frac{\delta^{\beta}_{\mu}\delta^{\delta}_{\nu}\delta^{\alpha}_{\lambda}\delta^{\gamma}_{\sigma}}{d-1} - \delta^{\beta}_{\mu}\delta^{\delta}_{\nu}\delta^{\alpha}_{\sigma}\delta^{\gamma}_{\lambda} + \mu\leftrightarrow\nu\right) + \alpha\leftrightarrow\beta\right] + \gamma\leftrightarrow\delta\right\} + (\lambda,\alpha,\beta)\longleftrightarrow(\sigma,\gamma,\delta) \tag{3.34b}$$



$$= \frac{ir_{\theta}}{4}\left\{\left[\left(\frac{1}{d-1}\delta^{\gamma}_{\mu}\delta^{\delta}_{\sigma}\mathcal{D}^{\theta\;\;\;\;\rho}_{\alpha\beta\nu\rho} - \delta^{\gamma}_{\mu}\mathcal{D}^{\theta\;\;\;\;\delta}_{\alpha\beta\nu\sigma} + \mu\leftrightarrow\nu\right) + \alpha\leftrightarrow\beta\right] + \gamma\leftrightarrow\delta\right\} + (q,\alpha,\beta)\longleftrightarrow(r,\mu,\nu) \tag{3.34c}$$



$$= \frac{q_{\kappa}r_{\theta}}{8}\left\{\left[\left(\mathcal{D}^{\kappa\;\;\;\;\pi}_{\alpha\beta\mu\sigma}\mathcal{D}^{\theta\;\;\;\;\sigma}_{\gamma\delta\nu\pi} - \frac{1}{d-1}\mathcal{D}^{\kappa\;\;\;\;\sigma}_{\alpha\beta\mu\sigma}\mathcal{D}^{\theta\;\;\;\;\pi}_{\gamma\delta\nu\pi} + \mu\leftrightarrow\nu\right) + \alpha\leftrightarrow\beta\right] + \gamma\leftrightarrow\delta\right\} + \text{six permutations of } (p,\mu,\nu)\ (q,\alpha,\beta)\ (r,\gamma,\delta) \tag{3.34d}$$

If $g_1 = g_2$, we cannot recover the TT propagator from Eq. (3.33b) even if $\alpha \to 0$ [6].

For the Bosonic ghost ζ^{μ} we have a propagator and vertices that follow from Eqs. (3.27) and (3.28).

The arguments used in ref. [12, 13] can be used to show that when using a non-quadratic gauge fixing Lagrangian, physical results are independent of the gauge choice.

Beginning with the insertion of Eq. (3.7) into Eq. (3.6), we have

$$\begin{aligned} Z[\boldsymbol{j}] &= \int \mathcal{D}\boldsymbol{\phi} \int \mathcal{D}\boldsymbol{\theta}_1 \mathcal{D}\boldsymbol{\theta}_2 \exp i \int \mathrm{d}^d x \, [\mathcal{L}(\boldsymbol{\phi}) + \boldsymbol{j} \cdot \boldsymbol{\phi}] \\ & \delta(\underset{\sim}{F}_1(\boldsymbol{\phi} + \underset{\sim}{A}\boldsymbol{\theta}_1) - \boldsymbol{p}_1)\delta(\underset{\sim}{F}_2(\boldsymbol{\phi} + \underset{\sim}{A}\boldsymbol{\theta}_2) - \boldsymbol{p}_2) \\ & \det(\underset{\sim}{F}_1\underset{\sim}{A}) \det(\underset{\sim}{F}_2\underset{\sim}{A}). \end{aligned} \tag{3.35}$$

We can now insert into this equation a further factor of "1"

$$1 = \int \mathcal{D}\vec{\omega}\delta(\underset{\sim}{F}_3(\boldsymbol{\phi} + \underset{\sim}{A}\omega) - \vec{q}) \det(\underset{\sim}{F}_3\underset{\sim}{A}) \tag{3.36}$$

and then by interchanging ω and $\boldsymbol{\theta}_1$, and $\boldsymbol{p}_1$ and q we see that $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_3$ are interchanged without altering $Z[\boldsymbol{j}]$, demonstrating that Z is independent of the gauge fixing condition.

3.3 Discussion

In this Chapter we considered how the transverse-traceless(TT) gauge could be applied on the first order Einstein-Hilbert(1EH) action. We modified the ordinary Faddeev-Popov gauge-fixing procedure and introduced two gauge-fixing conditions at same time to allow gravition propagator to be transverse and traceless at same time. We derived the resulting action and associated Feynman rules under the transverse-traceless condition. There are now two Fermionic and one Bosonic ghost fields.

It would be interesting to derive a set of WTST and BRST identities associated with the gauge transformation of Eq. (3.4) and the gauge choices of Eq. (3.14).

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Chapter 4

Renormalization Scheme Ambiguities and Multiple Couplings

4.1 Introduction

In order to excise divergences arising in the perturbative evaluation of physical quantities using quantum field theory, it is necessary to perform a subtraction to "renormalize" the parameters that characterize the theory[1]. Ambiguities in perturbative results arise both from the introduction of an unphysical scale parameter μ and from the possibility of performing a finite renormalization in addition to what is required to eliminate the divergence. The requirement that the exact expression for physical quantities be unambiguous leads to the renormalization group (RG) equations [2-4].

The renormalization scheme (RS) ambiguities when one uses a mass independent RS [5,6] in

[1] Analytic continuation can be used to avoid explicit occurrence of divergences [1].

theories with a single coupling constant a can be parameterized by the coefficients $c_i (i \geq 2)$ of the RG β-function that arise beyond two loop order, with the one and two loop coefficients being RS invariant [7]. It is possible to find a function $B_i(a, c_k)$ that shows how this coupling a depends on these coefficients c_i [8]. Furthermore, it is possible to use the RG equation associated with μ to sum these terms which in perturbation theory explicitly depend on μ through $\ln\mu$ so that this explicit dependence of a physical quantity R on μ cancels against implicit dependence on μ through $a(\mu)$ [9-11].

In this chapter, I extend these considerations to deal with the situation in which there are two coupling constants in a massless theory. It turns out there are significant differences when one goes from one to two couplings. I first review how when using mass independent renormalization the β-functions associated with these couplings are RS dependent at two loop order and beyond. This is unlike the situation in which there is only one coupling where at two loop order the β-function is RS independent. (This has been noted in ref. [12] and again in ref. [19].)

A second feature of a theory in which there are two couplings is that, unlike the situation in which there is but one coupling, there is no RS in which the β-functions can be terminated beyond two loop order. When there is only one coupling, the β-function receives only one and two loop contributions when the 't Hooft [13] RS is used.

At N loop order the β-functions in a model with two couplings involve $2(N + 2)$ parameters. We show how the RS used can be characterized by $2(N + 1)$ of these parameters; in general the two other parameters are dependent on these $2(N + 1)$ parameters. This motivates developing a way of characterizing a RS by use of parameters that arise in the expansion of the coupling in one RS in terms of the coupling in another RS.

The RS dependence of perturbative expressions for a physical quantity R is considered when there are two couplings. It is demonstrated how R is independent of μ when RG summation is performed and once this is done, how R depends on parameters that characterize the change in RS.

It is shown that when there are either one or two coupling constants, a RS can be chosen so that the perturbative expansion for R terminates and the effect of all higher loop effects is absorbed into the behavior of the running coupling.

In the next section we study some features of RS dependence when there are five couplings. By way of contrast, the analogous results when there are two couplings and five couplings are presented. There are qualitative differences in the RS dependence of models with one and models with two or five couplings.

We wish to emphasize that we are exclusively using mass independent renormalization schemes. When using a mass dependent RS, there are non-trivial differences in the RS ambiguities in the theory [26,27].

4.2 Renormalization Scheme Dependence With One Coupling

Quantum chromodynamics (QCD) is characterized by a single couplant a. When using the notation of ref. [8], the dependency of a on the renormalization scale parameter μ is given by

$$\begin{aligned} \mu\frac{da}{d\mu} &= \beta(a) \\ &= -ba^2\left(1 + ca + c_2a^2 + \dots\right) \end{aligned} \tag{4.1}$$

when using a mass independent renormalization scheme [5,6].

If a finite renormalization is performed [12], then

$$\overline{a} = a + x_2 a^2 + x_3 a^3 + \ldots \tag{4.2}$$

obeys an equation like (4.1). We find that since

$$\mu \frac{d\overline{a}}{d\mu} = \beta(a) \left(1 + 2x_2 a + 3x_3 a^2 + \ldots\right) \tag{4.3a}$$

as well as

$$\begin{aligned} &= -\overline{b}\left(a + x_2 a^2 + x_3 a^3 + \ldots\right)^2 \left[1 + \overline{c}\left(a + x_2 a^2 + \ldots\right)\right. \\ &\qquad \left. + \overline{c}_2 \left(a + x_2 a^2 + \ldots\right)^2 + \ldots\right] \end{aligned} \tag{4.3b}$$

then by eqs. (4.3a, 4.3b) we find that [23]

$$\overline{b} = b \tag{4.4a}$$

$$\overline{c} = c \tag{4.4b}$$

$$\overline{c}_2 = c_2 - c x_2 + x_3 - x_2^2 \tag{4.4c}$$

$$\overline{c}_3 = c_3 - 3c x_2^2 + 2\left(c_2 - 2\overline{c}_2\right) x_2 + 2x_4 - 2x_2 x_3 \tag{4.4d}$$

$$\begin{aligned} \overline{c}_4 = c_4 - 2x_4 x_2 - x_2^3 + c\left(x_4 - x_2^3 - 6x_2 x_3\right) + 3x_3 c_2 - 4x_3 \overline{c}_2 \\ - 6x_2^2 \overline{c}_2 + 2x_2 c_3 - 5x_2 \overline{c}_3 + 3x_5 \end{aligned} \tag{4.4e}$$

etc.

From eqs. (4.4a-4.4e) we find

$$x_3 = \bar{c}_2 - c_2 + cx_2 + x_2^2 \tag{4.5a}$$

$$x_4 = \frac{1}{2}\left[\bar{c}_3 - c_3 + (6\bar{c}_2 - 4c_2)\,x_2 + 5cx_2^2 + 2x_2^3\right] \tag{4.5b}$$

$$\begin{aligned} x_5 = \frac{1}{3}\Big\{&\bar{c}_4 - c_4 + (5x_2\bar{c}_3 - 2x_2c_3) + (4\bar{c}_2 - 3c_2 + 6x_2c) \\ &\left(\bar{c}_2 - c_2 + cx_2 + x_2^2\right) + \left(\bar{c}_2 - c_2 + cx_2 + x_2^2\right)^2 + 6x_2^2\bar{c}_2 \\ &+ x_2^3c + (2x_2 - c)\Big[\frac{1}{2}\,(\bar{c}_3 - c_3) + x_2\,(3\bar{c}_2 - 2c_2) \\ &+ \frac{5}{2}cx_2^2 + x_2^3\Big]\Big\} \end{aligned} \tag{4.5c}$$

etc.

We see that the renormalization of a in eq. (4.2) leads to a change in $c_i (i \geq 2)$ that fix $x_i (i \geq 3)$ with x_2 not determined. In ref. [23,24], some restrictions on the transformation of eq. (4.2) are considered.

The fact that a RS is characterized by c_i means that a itself is dependent on c_i. If

$$\frac{da}{dc_i} = B_i(a, c_k) \tag{4.6}$$

then the function B_i can be determined by the consistency condition

$$\left[\mu\frac{\partial}{\partial\mu}, \frac{\partial}{\partial c_i}\right] a = 0 \tag{4.7}$$

which leads to [8]

$$\begin{aligned} B_i(a, c_k) &= -b\beta(a)\int_0^a dx\frac{x^{i+2}}{\beta^2(x)} \\ &\approx \frac{a^{i+1}}{i-1}\left[1 + \left(\frac{(-i+2)c}{i}\right)a + \left(\frac{(i^2 - 3i + 2)c^2(-i^2 + 3i)c_2}{(i+1)i}\right)a^2 + \ldots\right]. \end{aligned} \tag{4.8}$$

If now

$$\mu\frac{d}{d\mu}a\,(\bar{\mu}, c_i) = 0 = \left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a}\right)\left(a\,(\mu, c_i) + (\sigma_{21}\ell)\,a^2\,(\mu, c_i) + \ldots\right) \quad \left(\ell \equiv b\ln\left(\frac{\mu}{\bar{\mu}}\right)\right) \tag{4.9}$$

then we have

$$\sigma_{21}=1,\quad \sigma_{31}=c,\quad \sigma_{32}=1,\quad \sigma_{41}=c_2,\quad \sigma_{42}=\frac{5}{2}c,\quad \sigma_{43}=1 \tag{4.10}$$
$$\sigma_{51}=c_3,\quad \sigma_{52}=3c_2+\frac{3}{2}c^2,\quad \sigma_{53}=\frac{13}{3}c,\quad \sigma_{54}=1$$
$$\sigma_{61}=c_4,\quad \sigma_{62}=\frac{7}{2}\left(cc_2+c_3\right),\quad \sigma_{63}=\frac{1}{6}\left(35c^2+36c_2\right),\quad \sigma_{64}=\frac{77}{12}c,\quad \sigma_{65}=1$$
$$\sigma_{71}=c_5,\quad \sigma_{72}=2\left(c_2^2+2cc_3+2c_4\right),\quad \sigma_{73}=\frac{1}{6}\left(15c^2+92cc_2+48c_3\right),$$
$$\sigma_{74}=\frac{5}{6}\left(17c^2+12c_2\right),\quad \sigma_{75}=\frac{87}{10}c,\quad \sigma_{76}=1.$$

Knowing these coefficients σ_{mn} gives $a(\bar{\mu},c_i)$ in terms of $a(\mu,c_i)$; this amounts to having a perturbative solution of eq. (4.1) [28]. If one defines $S_n(a)=\sum_{k=0}^{\infty}\sigma_{k+n+1,k}a^{k+n+1}$ (n=0,1,2...), one can solve sequentially for S_n using eq. (4.9).

Similarly, if

$$a\left(\mu,\overline{c}_k\right)=a\left(\mu,c_k\right)+\lambda_2\left(c_k,\overline{c}_k\right)a^2\left(\mu,c_k\right)+\lambda_3\left(c_k,\overline{c}_k\right)a^3\left(\mu,c_k\right)+\dots \tag{4.11}$$

with $\lambda_i(c_k,c_k)=0$, then the equation

$$\frac{d}{dc_i}a\left(\mu,\overline{c}_k\right)=0=\left(\frac{\partial}{\partial c_i}+B_i\left(a,c_k\right)\frac{\partial}{\partial a}\right)\left(a(\mu,c_k)+\lambda_2\left(c_k,\overline{c}_k\right)a^2\left(\mu,c_k\right)+\dots\right) \tag{4.12}$$

results in

$$\lambda_2=(\overline{c}_2-c_2),\quad \lambda_3=\frac{1}{2}(\overline{c}_3-c_3),\quad \lambda_4=\frac{1}{6}\left(\overline{c}_2^2-c_2^2\right)+\frac{3}{2}(\overline{c}_2-c_2)-\frac{c}{6}\left(\overline{c}_3-c_3\right)+\frac{1}{3}(\overline{c}_4-c_4) \tag{4.13}$$

etc.

Eqs. (4.11-4.13) is essentially a series solution of eq. (4.6) [28].

If in eq. (4.2) we eliminate $x_n (n\geq 3)$ in favour of x_2, c_i, $\overline{c}_i (i\geq 2)$ using eq. (4.5) and then set $c_i=\overline{c}_i$, we end up with the series of eq. (4.9) for $a(\overline{\mu},c_i)$ provided $x_2=b\ln\left(\frac{\mu}{\overline{\mu}}\right)$ [11]. This shows that x_2 can be identified with $b\ln\left(\frac{\mu}{\overline{\mu}}\right)$ as postulated in ref. [8].

If now a physical quantity, such as the cross section $R_{e^+e^-}$ for $e^+e^- \longrightarrow$ (hadrons), is expanded in the form

$$R = \sum_{n=0}^{\infty} A_n(a) L^n \tag{4.14}$$

where $L = b \ln \frac{\mu}{Q}$ and [9, 10]

$$A_n(a) = \sum_{k=0}^{\infty} T_{n+k,n} a^{n+k+1}, \tag{4.15}$$

then from the RG equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(a) \frac{\partial}{\partial a}\right) R = 0$$

it follows that

$$A_n(a) = -\frac{\beta(a)}{bn} \frac{d}{da} A_{n-1}(a) \tag{4.16}$$

so that since by eq. (4.1) [8]

$$\ln\left(\frac{\mu}{\Lambda}\right) = \int_0^{a\left(\ln \frac{\mu}{\Lambda}\right)} \frac{dx}{\beta(x)} + \int_0^{\infty} \frac{dx}{bx^2(1+cx)} \tag{4.17}$$

we find from eqs. (4.14-4.17)

$$R = A_0\left(a\left(\ln \frac{Q}{\Lambda}\right)\right) \tag{4.18}$$

and the explicit and implicit dependence of R on the unphysical scale parameter μ has cancelled [10].

By eqs. (4.15, 4.18) we see that

$$R = \sum_{n=0}^{\infty} T_n \left(a\left(\ln \frac{Q}{\Lambda}\right)\right)^{n+1} \quad (T_n \equiv T_{n,0}). \tag{4.19}$$

so that from the requirement that

$$\left(\frac{\partial}{\partial c_i} + B_i(a) \frac{\partial}{\partial a}\right) R = 0 \tag{4.20}$$

we find that

$$T_0 = \tau_0, \quad T_1 = \tau_1, \quad T_2 = -c_2 + \tau_2, \quad T_3 = -2c_2\tau_1 - \frac{1}{2}c_3 + \tau_3 \tag{4.21}$$
$$T_4 = -\frac{1}{3}c_4 - \frac{c_3}{2}\left(-\frac{1}{3}c + 2\tau_i\right) + \frac{4}{3}c_2^2 - 3c_2\tau_2 + \tau_4$$

etc.

where the τ_i are constants of integration and hence are RS invariants [9, 10]. One RS of particular interest is the one in which $T_i = 0 (i \geq 2)$ so that R is represented by a perturbative series that terminates. A second interesting RS due to 't Hooft has $c_i = 0 (i \geq 2)$ [13, 14], so that the β function is a finite series in the coupling.

We will now see how the results obtained in this section are modified when one considers models in which there are two coupling constants. Again, we will deal with massless theories and employ mass independent renormalization schemes.

4.3 Renormalization Scheme Dependence With Two Couplings

We now will consider the consequences of having two couplings $g_a (a = 1, 2)$ in a model with the β-functions

$$\mu \frac{dg_a}{d\mu} = \beta_a(g_1, g_2) = \sum_{i=2}^{\infty} \sum_{j=0}^{i} c^a_{ij} (g_1)^{i-j} (g_2)^j \tag{4.22}$$

in place of eq. (4.1). In order to compute c^a_{ij}, a calculation of diagrams involving i-1 loops is required. For example, in the limit of the Standard Model in which there is only the $SU(2)$ gauge field and the Higgs doublet, with the gauge coupling g and the Higgs self coupling λ, the coefficient c^a_{ij} are to two loop order [15] in the $\overline{MS}$ RS if $g^2 = 16\pi^2 g_1$ and $\lambda = 16\pi^2 g_2$,

$$\begin{aligned} c^1_{20} &= -\frac{19}{3}, \quad c^1_{30} = \frac{35}{3}, \quad c^2_{20} = \frac{27}{4}, \quad c^2_{21} = -9, \quad c^2_{22} = 4 \\ c^2_{30} &= \frac{915}{8}, \quad c^2_{31} = -\frac{73}{8}, \quad c^2_{32} = 18, \quad c^2_{33} = -\frac{26}{3}. \end{aligned} \tag{4.23}$$

The analogue to eq. (4.2) for a finite renormalization of g_a is

$$\overline{g}_a = g_a + \sum_{i=2}^{\infty} \sum_{j=0}^{i} x^a_{ij} (g_1)^{i-j} (g_2)^j. \tag{4.24}$$

In analogy with eqs. (4.3a) and (4.3b) we then see that

$$\mu \frac{d\overline{g}_a}{d\mu} = \beta_a(g_b) + \sum_{i=2}^{\infty} \sum_{j=0}^{i} x^a_{ij} \left[(i-j) g_1^{i-j-1} g_2^j \beta_1(g_b) + j g_1^{i-j} g_2^{j-1} \beta_2(g_b) \right] \tag{4.25a}$$

and

$$\mu \frac{d\overline{g}_a}{d\mu} = \sum_{i=2}^{\infty} \sum_{j=0}^{i} \overline{c}^a_{ij} \left[g_1 + \sum_{k=2}^{\infty} \sum_{\ell=0}^{k} x^1_{k\ell} g_1^{k-\ell} g_2^{\ell} \right]^{i-j} \left[g_2 + \sum_{m=2}^{\infty} \sum_{n=0}^{m} x^2_{mn} g_1^{m-n} g_2^n \right]^j. \tag{4.25b}$$

Upon comparing terms in eqs. (4.25a) and (4.25b) that are quadratic in the couplings (i.e., that are of one loop order) we find that much like eq. (4.4a)

$$\overline{c}^a_{2j} = c^a_{2j} \quad (j = 0, 1, 2) \tag{4.26}$$

and so one loop contributions to $\beta_a(g_b)$ are RS independent. However, terms in eqs. (4.25a) and (4.25b) that are cubic in the couplings (i.e., that are at two loop order) show that at order g_2^3, g_2^3, $g_1^2 g_2$ and $g_1 g_2^2$ respectively [19]

$$\overline{c}^a_{30} = c^a_{30} + 2x^a_{20} c^1_{20} - 2c^a_{20} x^1_{20} + x^a_{21} c^2_{20} - c^a_{21} x^2_{20} \tag{4.27a}$$

$$\overline{c}^a_{33} = c^a_{33} + 2x^a_{22} c^2_{22} - 2c^a_{22} x^2_{22} + x^a_{21} c^1_{22} - c^a_{21} x^1_{22} \tag{4.27b}$$

$$\overline{c}^a_{31} = c^a_{31} + 2x^a_{20} c^1_{21} - 2c^a_{20} x^1_{21} + x^a_{21} \left(c^1_{20} + c^2_{21} \right) - c^a_{21} \left(x^1_{20} + x^2_{21} \right) + 2x^a_{22} c^2_{20} - 2c^a_{22} x^2_{20} \tag{4.27c}$$

$$\overline{c}^a_{32} = c^a_{32} + 2x^a_{20} c^1_{22} - 2c^a_{20} x^1_{22} + x^a_{21} \left(c^1_{21} + c^2_{22} \right) - c^a_{21} \left(x^1_{21} + x^2_{22} \right) + 2x^a_{22} c^2_{21} - 2c^a_{22} x^2_{21} \tag{4.27d}$$

with $a = 1, 2$. From eq. (4.27) it is immediately apparent that the two loop contributions to $\beta_a(g_1, g_2)$ are RS dependent, unlike what happens when there is one coupling (see. eq. (4.4b)) [12, 19]. However, as there are now eight equations fixing changes in the eight quantities $c^a_{3i}(a = 1, 2; i = 0, 1, 2, 3)$ in terms of just the six independent coefficients $x^a_{2i}(a = 1, 2; i = 0, 1, 2)$, it is evident that it is in general not possible to vary each of the quantities c^a_{3i} independently. Only if the coefficients $c^a_{2i}(a = 1, 2; i = 0, 1, 2)$ were to have special values would it be possible to find values of x^a_{2i} so that each of the c^a_{3i} equals zero, which would be the analogue of the 't Hooft RS when there is one coupling [13,14].

When one goes beyond two loop order, equations much like eq. (4.27) can be found. At N loop order, $\overline{c}^a_{N+1,i} - c^a_{N+1,i}(a = 1, 2; i = 0 \ldots N + 1)$ is related to $x^a_{N,i}(a = 1, 2; i = 0 \ldots N)$ through $2(N+2)$ equations. Consequently, in general, 2 of the $2(N+2)$ quantities $c^a_{N+1,i}$ cannot be varied independently by altering the RS by adjusting only the $2(N + 1)$ independent parameters $x^a_{N,i}$. However, there is the intriguing possibility that for some choice of $x^a_{i,j}$ that either $\beta_1(g_1, g_2)$ or $\beta_2(g_1, g_2)$ vanishes beyond one loop order.

Since not all of the coefficients c^a_{mn} can be varied independently by a change of RS, it is apparent that these coefficients are no longer suitable for characterizing a RS where there is more than one coupling. In the next section we show how the coefficients x_i in eq. (4.2) (when there is one coupling) or x^a_{ij} in eq. (4.24) (when there are two couplings), all of which are independent, can be used to characterize a RS.

RS ambiguities are of practical importance, as is illustrated by the discrepancy between the calculations presented in refs. [20] and [21]. This is discussed in ref. [22].

4.4 An Alternate Way to Characterize a Renormalization Scheme

We begin by considering the case of one coupling a and showing how the parameters x_i in eq. (4.2) can be used to characterize a RS. Suppose that a refers to the coupling in some "base scheme" such as $\overline{MS}$, and the $\overline{a}$ is the coupling in some other scheme with a and $\overline{a}$ related by eq. (4.2). If now

$$a = \overline{a} + y_2\overline{a}^2 + y_3\overline{a}^3 + \dots \tag{4.28}$$

then eqs. (4.2, 4.28) are consistent provided

$$a = \overline{a} - x_2\overline{a}^2 + \left(2x_2^2 - x_3\right)\overline{a}^3 + \left(5x_2x_3 - 5x_2^3 - x_4\right)\overline{a}^4 + \dots. \tag{4.29}$$

It is clear that $\overline{a}$ depends on x_i; from eq. (4.2) we see that

$$\frac{d\overline{a}}{dx_n} = a^n \qquad (\overline{a}(x_n = 0) = a) \tag{4.30a}$$

which by eq. (4.29) becomes

$$\frac{d\overline{a}}{dx_n} \equiv \overline{B}_n(\overline{a}, x_m) = \left(\overline{a} - x_2\overline{a}^2 + \left(2x_2^2 - x_3\right)\overline{a}^3 + \dots\right)^n. \tag{4.30b}$$

There are two consistency checks on eq. (4.30b). First of all, we have

$$\frac{da}{dx_n} = 0 \tag{4.31a}$$

which by (29) and (30b) leads to

$$\left[\frac{\partial}{\partial x_n} + \left(\overline{a} - x_2\overline{a}^2 + \left(2x_2^2 - x_3\right)\overline{a}^3 + \dots\right)^n \frac{\partial}{\partial\overline{a}}\right]\left(\overline{a} - x_2\overline{a}^2 + \left(2x_2^2 - x_3\right)\overline{a}^3 + \dots\right) = 0 \tag{4.31b}$$

which can be verified. A second test follows from eq. (4.1)

$$\mu\frac{da}{d\mu} = -ba^2(1 + ca + c_2a^2 + \ldots); \tag{4.32a}$$

if we eliminate a in eq. (4.32a) by eq. (4.29) and use

$$\mu\frac{d\overline{a}}{d\mu} = -\overline{b}\overline{a}^2(1 + \overline{c}\ \overline{a} + \overline{c}_2\overline{a}^2 + \ldots) \tag{4.32b}$$

we recover eq. (4.4).

We can now employ this approach to characterizing a RS to the situation in which there are two couplings. In this case, a RS is defined in terms of a "base scheme" in which the couplings are given by (g_1, g_2) and the coefficients x^a_{mn} appearing in eq. (4.24). The advantage of this approach is that all of the x^a_{mn} can be independently varied. We have shown that it is not possible to independently vary the coefficients c^a_{ij} appearing in the functions β_a in eq. (4.22) by use of eq. (4.24).

We begin by noting that from eq. (4.24), it follows that if

$$g_a = \overline{g}_a + \sum_{m=2}^{\infty}\sum_{n=0}^{m} Y^a_{mn}\overline{g}_1^{\,m-n}\ \overline{g}_2^n \tag{4.33}$$

then

$$Y^a_{2k} + x^a_{2k} = 0 \quad (a = 1, 2; k = 0, 1, 2) \tag{4.34}$$

and

$$Y^1_{30} = 2(x^1_{20})^2 + x^1_{21}x^2_{20} - x^1_{30}; \quad Y^2_{33} = 2(x^2_{22})^2 + 2x^2_{21}x^1_{22} - x^2_{33} \tag{4.35a,b}$$

$$Y^1_{33} = 2x^1_{22}x^2_{22} + x^1_{21}x^1_{22} - x^1_{33}; \quad Y^2_{30} = 2x^2_{20}x^1_{20} + x^2_{21}x^2_{20} - x^2_{30} \tag{4.35c,d}$$

$$Y^1_{31} = 2x^1_{20}x^1_{21} + x^1_{21}(x^1_{20} + x^2_{21}) + 2x^1_{22}x^2_{20} - x^1_{31}; \quad Y^2_{32} = 2x^2_{22}x^2_{21} + x^2_{21}(x^2_{22} + x^1_{21}) + 2x^2_{20}x^1_{22} - x^2_{32} \tag{4.35e,f}$$

$$Y^1_{32} = 2x^1_{22}x^2_{21} + x^1_{21}(x^2_{22} + x^1_{21}) + 2x^1_{20}x^1_{22} - x^1_{32}; \quad Y^2_{31} = 2x^2_{20}x^1_{21} + x^2_{21}(x^1_{20} + x^2_{21}) + 2x^2_{22}x^2_{20} - x^2_{31}. \tag{4.35g,h}$$

etc.

The inversion of series with several variables is discussed in, for example, ref. [18].

It also follows from eq. (4.24) that

$$\frac{d\overline{g}_a}{dx^b_{mn}} \equiv \overline{B}^a_{b;m,n}(\overline{g}_a) = \delta^a_b g_1^{m-n} g_2^n \tag{4.36}$$

so that, for example

$$\frac{d\overline{g}_1}{dx^1_{21}} = g_1 g_2 = \overline{g}_1\overline{g}_2 - x^2_{20}\overline{g}_1^3 - \left(x^1_{20} + x^2_{21}\right)\overline{g}_1^2\overline{g}_2 \\ - \left(x^1_{21} + x^2_{22}\right)\overline{g}_1\overline{g}_2^2 - x^1_{22}\overline{g}_2^3 \dots \tag{4.37}$$

We now can consider the RS dependence of a physical quantity using the parameters x_n when there is one coupling a and x^a_{mn} when there are two couplings g_1, g_2.

Again considering R given by eq. (4.19), we take $a\left(\ln \frac{Q}{\Lambda}\right)$ to be the coupling in a "base scheme" (such as $\overline{MS}$). Under a renormalization such as in eq. (4.2) we must have

$$\frac{d}{dx_n}R = 0 = \left(\frac{\partial}{\partial x_n} + \overline{B}_n(\overline{a}, x_m)\frac{\partial}{\partial \overline{a}}\right)\sum_{n=0}^{\infty}\overline{T}_n\left(\overline{a}\left(\ln\frac{Q}{\Lambda}\right)\right)^{n+1}. \tag{4.38}$$

In eq. (4.38), $\overline{T}_n \equiv \overline{T}_{n,0}$ are the coefficients of an expansion of R in powers of $\overline{a}$, a coupling related to the coupling a through the renormalization of eq. (4.2). Using eq. (4.30b), we find that for $k = 2, 3\dots$

$$\sum_{n=0}^{\infty}\left[\frac{\partial\overline{T}_n}{\partial x_k}\overline{a}^{n+1} + (n+1)\overline{a}^n\left(\overline{a} - x_2\overline{a}^2 + \left(2x_2^2 - x_3\right)\overline{a}^3 + \dots\right)^k\overline{T}_n\right] = 0. \tag{4.39}$$

From eq. (4.39) it follows that

$$\frac{\partial\overline{T}_0}{\partial x_2} = \frac{\partial\overline{T}_0}{\partial x_3} = \frac{\partial\overline{T}_0}{\partial x_4} = 0 \tag{4.40a-c}$$

$$\frac{\partial \overline{T}_1}{\partial x_2} + \overline{T}_0 = \frac{\partial \overline{T}_1}{\partial x_3} = \frac{\partial \overline{T}_1}{\partial x_4} = 0 \tag{4.41a-c}$$

$$\frac{\partial \overline{T}_2}{\partial x_2} + 2\overline{T}_1 - 2x_2\overline{T}_0 = \frac{\partial \overline{T}_2}{\partial x_3} + \overline{T}_0 = \frac{\partial \overline{T}_2}{\partial x_4} = 0 \tag{4.42a-c}$$

$$\frac{\partial \overline{T}_3}{\partial x_2} + 3\overline{T}_2 - 4x_2\overline{T}_1 + \overline{T}_0\left(5x_2^2 - 2x_3\right) = \frac{\partial \overline{T}_3}{\partial x_3} + 2\overline{T}_1 - 3x_2\overline{T}_0 = \frac{\partial \overline{T}_3}{\partial x_4} + \overline{T}_0 = 0. \tag{4.43a-c}$$

Since when $x_i = 0$, $\overline{a} = a$ and $\overline{T}_n = T_n$, we see that from eqs. (4.40-4.43) that

$$\overline{T}_0 = T_0 \tag{4.44a}$$

$$\overline{T}_1 = T_1 - x_2T_0 \tag{4.44b}$$

$$\overline{T}_2 = T_2 + \left(-x_3 + 2x_2^2\right)T_0 + (-2x_2)T_1 \tag{4.44c}$$

$$\overline{T}_3 = T_3 + \left(-x_4 + 5x_2x_3 - 5x_2^3\right)T_0 + \left(-2x_3 + 5x_2^2\right)T_1 - 3x_2T_2 \tag{4.44d}$$

etc.

One interesting feature of eq. (4.44) is that $x_2, x_3 \ldots$ can all be selected so that $\overline{T}_1 = \overline{T}_2 = \overline{T}_3 \cdots = 0$, leaving R given by the single term

$$R = T_0\overline{a}\left(\ln\frac{Q}{\Lambda}\right). \tag{4.45}$$

In eq. (4.45), $\overline{a}$ runs according to eq. (4.32b) with $\overline{b}, \overline{c}, \overline{c}_k$ given by eq. (4.4) once x_k is computed in terms of T_n from eq. (4.44). As is apparent upon comparing eqs. (4.19,4.45), the solution for x_k is $x_k = \frac{T_{k-1}}{T_0}$.

If there are two couplings g_1, g_2 then the general form of R is

$$R = \sum_{m=1}^{\infty}\sum_{n=0}^{m}\sum_{k=0}^{m-1} T_{m,n;k}\, g_1^{m-n} g_2^{n} L^k \tag{4.46}$$

where $L = \ln\left(\frac{\mu}{Q}\right)$ and g_1, g_1 satisfy eq. (4.22) so that $g_a = g_a\left(ln\frac{\mu}{\Lambda}\right)$. Since R is independent of the unphysical renormalization mass scale μ, then

$$\mu\frac{d}{d\mu}R = \left(\mu\frac{\partial}{\partial\mu} + \beta_a\frac{\partial}{\partial g_a}\right)\sum_{k=0}^{\infty} A_k(g_1, g_2)L^k = 0 \tag{4.47}$$

where

$$A_k = \sum_{m=0}^{\infty} \sum_{n=0}^{m+k+1} T_{m+k+1,n;k} g_1^{m+k+1-n} g_2^n. \tag{4.48}$$

By eqs. (4.47) and (4.22), we find that

$$A_{k+1}\left(g_a\left(\ln \frac{\mu}{\Lambda}\right)\right) = \frac{-1}{k+1} \frac{d}{d\left(\ln \frac{\mu}{\Lambda}\right)} A_k\left(g_a\left(\ln \frac{\mu}{\Lambda}\right)\right) \tag{4.49}$$

so that R in eq. (4.46) becomes

$$R = \sum_{k=0}^{\infty} \frac{(-L)^k}{k!} \left(\frac{d}{d\left(\ln \frac{\mu}{\Lambda}\right)}\right)^k A_0\left(g_a\left(\frac{\mu}{\Lambda}\right)\right) = A_0\left(g_a\left(\frac{Q}{\Lambda}\right)\right). \tag{4.50}$$

As in eq. (4.18), all implicit and explicit dependence on μ has cancelled once the RG has been used to sum the logarithmic contributions to R.

In analogy with eq. (4.38) we now have

$$\frac{dR}{dx^a_{mn}} = 0 = \left(\frac{\partial}{\partial x^a_{mn}} + \overline{B}^b_{a;m,n}(\overline{g}_b)\frac{\partial}{\partial \overline{g}^b}\right) \sum_{k=0}^{\infty} \sum_{\ell=0}^{k+1} \overline{T}_{k+1,\ell;0} (\overline{g}_1)^{k+1-\ell} (\overline{g}_2)^{\ell}. \tag{4.51}$$

Using eq. (4.36) for $\overline{B}^b_{a;m,n}$, eq. (4.51) becomes (with $\bar{T}_{m,n} \equiv \bar{T}_{m,n;0}$)

$$\sum_{k=0}^{\infty} \sum_{\ell=0}^{k+1} \left\{ \frac{\partial \overline{T}_{k+1,\ell}}{\partial x^a_{mn}} (\overline{g}_1)^{k+1-\ell} (\overline{g}_2)^{\ell} + \overline{T}_{k+1,\ell}\Big[\overline{B}^1_{a;m,n}(k+1-\ell)(\overline{g}_1)^{k-\ell}(\overline{g}_2)^{\ell} \right. \tag{4.52}$$

$$\left. + \overline{B}^2_{a;m,n}(\ell)(\overline{g}_1)^{k+1-\ell}(\overline{g}_2)^{\ell-1}\Big] \right\} = 0.$$

From eq. (4.52) it follows

$$\frac{\partial \overline{T}_{1\ell}}{\partial x^a_{mn}} = 0 \quad (\ell = 0, 1) \tag{4.53}$$

$$\frac{\partial \overline{T}_{20}}{\partial x^a_{mn}} + \overline{T}_{10}\,(\delta^a_1 \delta_{m2} \delta_{n0}) + \overline{T}_{11}\,(\delta^a_2 \delta_{m2} \delta_{n0}) = 0 \tag{4.54a}$$

$$\frac{\partial \overline{T}_{22}}{\partial x^a_{mn}} + \overline{T}_{11}\,(\delta^a_2 \delta_{m2} \delta_{n2}) + \overline{T}_{10}\,(\delta^a_1 \delta_{m2} \delta_{n2}) = 0 \tag{4.54b}$$

$$\frac{\partial \overline{T}_{21}}{\partial x^a_{mn}} + \overline{T}_{10}\,(\delta^a_1 \delta_{m2} \delta_{n1}) + \overline{T}_{11}\,(\delta^a_2 \delta_{m2} \delta_{n1}) = 0 \tag{4.54c}$$

etc.

with $\overline{T}_{k\ell} = T_{k\ell}$ when $x^a_{mn} = 0$. Eqs. (53, 54) lead to

$$\overline{T}_{1\ell} = T_{1\ell} \tag{4.55}$$

$$\overline{T}_{20} = T_{20} - x^1_{20}T_{10} - x^2_{20}T_{11} \tag{4.56a}$$

$$\overline{T}_{22} = T_{22} - x^2_{22}T_{11} - x^1_{22}T_{10} \tag{4.56b}$$

$$\overline{T}_{21} = T_{21} - x^1_{21}T_{10} - x^2_{21}T_{11} \tag{4.56c}$$

etc.

It is evident that x^a_{mn} can be selected so that $\overline{T}_{mn}(m \geq 2)$ are all zero so that R is given by just two terms

$$R = T_{10}\overline{g}_1\left(\ln\frac{Q}{\Lambda}\right) + T_{11}\overline{g}_2\left(\ln\frac{Q}{\Lambda}\right) \tag{4.57}$$

and no higher powers of $\overline{g}_a$ contribute to R. The functions $\overline{\beta}_a(\overline{g}_b)$ that govern the evolution of $\overline{g}_a$ with $\ln\frac{Q}{\Lambda}$ can be found using eq. (4.27) once x^a_{ij} has been determined.

4.5 Renormalization Scheme Ambiguities in the Standard Model

In its simplest form, the Standard Model of particle physics involves five coupling constants g_a, the SU(3), SU(2) and U(1) gauge couplings as well as the quartic SU(2) scalar self coupling and the Yukawa coupling of the top quark. As with any renormalizable theory, renormalization introduces a mass scale μ and these couplings all vary as μ varies in a way dictated by the renormalizaion group (RG) β-functions.

$$\mu \frac{dg_a}{d\mu} = \beta_a(g_b) = \sum_{k=2}^{\infty} \sum_{i_1=0}^{k} \sum_{i_2=0}^{k} \sum_{i_3=0}^{k} \sum_{i_4=0}^{k} \sum_{i_5=0}^{k} c^a_{k;i_1i_2i_3i_4i_5} (g_1)^{i_1} (g_2)^{i_2} (g_3)^{i_3} (g_4)^{i_4} (g_5)^{i_5} \delta_{k-(i_1+i_2+\ldots+i_5)}. \tag{4.58}$$

However, the value of any physical quantity R when computed to finite order in perturbative theory has explicit dependence on μ. This explicit dependence must be conpensated for by the implicit dependence through $g_a(\mu)$; this leads to the RG equation [2-4]

$$\mu \frac{d}{d\mu} R = (\mu \frac{\partial}{\partial \mu} + \beta_a(g_b) \frac{\partial}{\partial g_a}) R = 0. \tag{4.59}$$

In addition to the ambiguity in the perturbative value of R resulting from the necessity of introducing the renormalization mass scale μ, it is possible to make finite renormalizations of the couplings g_a, even when using a mass-dependent renormalization scheme (RS) [5-6], so that g_a is replaced by $\overline{g}_a$ where

$$\overline{g}_a = g_a + \sum_{k=2}^{\infty} \sum_{i_1=0}^{k} \sum_{i_2=0}^{k} \sum_{i_3=0}^{k} \sum_{i_4=0}^{k} \sum_{i_5=0}^{k} x^a_{k;i_1i_2i_3i_4i_5} (g_1)^{i_1} (g_2)^{i_2} (g_3)^{i_3} (g_4)^{i_4} (g_5)^{i_5} \delta_{k-(i_1+i_2+\ldots+i_5)}. \tag{4.60}$$

There is an extensive literature dealing with the RS ambiguities (for example refs. [8,10]). We have extended these considerations to the case of two couplings in the preceeding sections. We have seen that there are qualitative differences between the RS ambiguities occurring when there are one and two couplings. When there is one coupling a, the RS ambiguities can be characterized by the coefficients of the β-function $\beta(a)$ [8] and a RS can be chosen so that $\beta(a)$ receives no contribution beyond two loop order. Furthermore, it is possible to have a RS so that R(a) vanishes beyond one-loop order and all higher loop effects serve only to affect the β-function [16]. This can be implemented after the RG equation (4.59) is used to sum all logarithmic contribution to R which results in a cancellation between the implicit and explicit dependence on μ [10].

When there are two couplings, the number of coefficients arising in the perturbative expansion of $\beta_a(g_b)$ is inadequate to fully characterize a RS. It also becomes impossible to choose a RS when using mass independent renormalization to choose a RS that eliminates all higher loop contributions to $\beta_a(g_b)$, although only the one loop contribution to $\beta_a(g_b)$ is RS invariants [12,19]. However, as in the one coupling case, upon using the RG equation to sum logarithmic effects, the implicit and explicit dependence on μ cancels in R and it becomes possible to choose a RS in which higher loop contributions to R vanish with all of the higher loop effects contributing to β_a.

Here we examine the effects of RS ambiguities on the couplings in the Standard Model. We note that when using modified minimal substraction $\overline{MS}$ as a RS, then all $\beta_a(g_b)$ have been computed to two loop order [15] while the β-function for the gauge couplings are known to three loop order [29].

If a β-function $\overline{\beta}(\overline{g}_a)$ dictates how $\overline{g}_a$ evolves under change of μ and $\overline{\beta}(\overline{g}_a)$ has the same form as eq. (4.58) with $\overline{c}^a_{k;i_1i_2i_3i_4i_5}$ replacing $c^a_{k;i_1i_2i_3i_4i_5}$, then since both

$$\mu\frac{d\overline{g}_a}{d\mu} = \overline{\beta}_a(\overline{g}_b(g_c)) \tag{4.61}$$

and

$$\mu \frac{d\overline{g}_a}{d\mu} = \sum_{c=1}^{5} \frac{\partial \overline{g}_a(g_c)}{\partial g_c} \beta_c(g_b) \tag{4.62}$$

where $\overline{g}_b(g_c)$ is given by eq. (4.60), we find from eqs. (4.61,4.62) that upon looking at terms quadratic and cubic in the couplings

$$\overline{c}^a_{2;i_1i_2i_3i_4i_5} = c^a_{2;i_1i_2i_3i_4i_5} \tag{4.63a}$$

$$\begin{aligned}\overline{c}^1_{3;30000} &= c^1_{3;30000} + c^5_{2;20000}x^1_{2;10001} + c^4_{2;20000}x^1_{2;10010} + c^3_{2;20000}x^1_{2;10100} + c^2_{2;20000}x^1_{2;11000} \\ &\quad - c^1_{2;11000}x^2_{2;20000} - c^1_{2;10100}x^3_{2;20000} - c^1_{2;10010}x^4_{2;20000} - c^1_{2;10001}x^5_{2;20000}\end{aligned} \tag{4.63b}$$

and

$$\begin{aligned}\overline{c}^1_{3;21000} &= c^1_{3;21000} + c^5_{2;20000}x^1_{2;01001} + c^4_{2;20000}x^1_{2;01010} + c^3_{2;20000}x^1_{2;01100} + 2c^2_{2;20000}x^1_{2;02000} \\ &\quad + c^5_{2;11000}x^1_{2;10001} + c^4_{2;11000}x^1_{2;10010} + c^3_{2;11000}x^1_{2;10100} + c^2_{2;11000}x^1_{2;11000} + c^1_{2;11000}x^1_{2;20000} \\ &\quad - 2c^1_{2;02000}x^2_{2;20000} - c^1_{2;01100}x^3_{2;20000} - c^1_{2;01010}x^4_{2;20000} - c^1_{2;01001}x^5_{2;20000} \\ &\quad - c^1_{2;20000}x^1_{2;11000} - c^1_{2;11000}x^2_{2;11000} - c^1_{2;10100}x^3_{2;11000} - c^1_{2;10010}x^4_{2;11000} - c^1_{2;10001}x^5_{2;11000}.\end{aligned} \tag{4.63c}$$

etc.

As we showed above, if there were but one coupling, eq. (4.63) shows that c_2 and c_3 are unaltered by a change of RS of the form of eq. (4.60) [7]; $c_n (n > 3)$ which arise from an (n-1) loop calculation can all be altered. In fact, $x_n (n > 2)$ can be chosen so that $c_n (n > 3)$ vanishes [15]. A RS can be characterized either by $c_n (n > 3)$ [8] with μ being identified with x_2 [16], or by the parameters $x_n (n \geq 2)$ themselves.

It is possible to see that with five coupling constants, as with two coupling constants considered above, there simply are not enough constants appearing in the expansion of $\overline{g}_a$ given in eq. (4.60) to independently vary the constants in the expansion of $\beta_a(g_b)$ in eq. (4.58). (In particular, at N-loop order, there are more constants $c^a_{N+1;i_1 i_2 i_3 i_4 i_5}$ than constants $x^a_{N;i_1 i_2 i_3 i_4 i_5}$.) Thus, unlike what happens when there is one coupling, the coefficients of the expansion of $\beta_a(g_b)$ are not suitable for characterizing a RS and as in the case of two couplings, we will employ directly the coefficients $x^a_{N;i_1 i_2 i_3 i_4 i_5}$ of eq. (4.60) to relate the parameters that occur when using a particular RS to that of a "base scheme".

In particular, since

$$\frac{\partial \overline{g}_a}{\partial x^b_{k;i_1 i_2 i_3 i_4 i_5}} = \overline{B}^a_{b;k;i_1 i_2 i_3 i_4 i_5}(\overline{g}_c) = \delta^a_b \delta_{k-(i_1+i_2+...+i_5)} g_1^{i_1} g_2^{i_2} g_3^{i_3} g_4^{i_4} g_5^{i_5}. \tag{4.64}$$

and as eq. (4.60) can be inverted to give

$$g_a = \overline{g}_a + \sum_{k=2}^{\infty} \sum_{i_1=0}^{k} \sum_{i_2=0}^{k} \sum_{i_3=0}^{k} \sum_{i_4=0}^{k} \sum_{i_5=0}^{k} y^a_{k;i_1 i_2 i_3 i_4 i_5} (\overline{g}_1)^{i_1} (\overline{g}_2)^{i_2} (\overline{g}_3)^{i_3} (\overline{g}_4)^{i_4} (\overline{g}_5)^{i_5} \delta_{k-(i_1+i_2+...+i_5)} \tag{4.65}$$

where

$$y^a_{2;i_1 i_2 i_3 i_4 i_5} + x^a_{2;i_1 i_2 i_3 i_4 i_5} = 0 \; (a = 1, 2, ...5; i_1 + i_2 + i_3 + i_4 + i_5 = 2) \tag{4.66}$$

$$y^1_{3;30000} = -x^1_{3;30000} + 2(x^1_{2;20000})^2 + x^1_{2;11000}x^2_{2;20000} + x^1_{2;10100}x^3_{2;20000} + x^1_{2;10010}x^4_{2;20000} + x^1_{2;10001}x^5_{2;20000} \tag{4.67a}$$

$$y^1_{3;21000} = -x^1_{3;21000} + 3(x^1_{2;20000})^2 + x^1_{2;11000}x^2_{2;20000} + x^1_{2;10100}x^3_{2;20000} + x^1_{2;10010}x^4_{2;20000} + x^1_{2;10001}x^5_{2;20000} \tag{4.67b}$$

$$\begin{aligned} y^1_{3;11100} = {} & -x^1_{3;11100} + 2x^1_{2;10100}x^1_{2;11000} + 2x^1_{2;01100}x^1_{2;20000} \\ & + x^1_{2;11000}x^2_{2;01100} + 2x^1_{2;02000}x^2_{2;10100} + x^1_{2;01100}x^2_{2;11000} \\ & + x^1_{2;10100}x^3_{2;01100} + x^1_{2;01100}x^3_{2;10100} + 2x^1_{2;00200}x^3_{2;11000} \\ & + x^1_{2;10010}x^4_{2;01100} + x^1_{2;01010}x^4_{2;10100} + x^1_{2;00110}x^4_{2;11000} \\ & + x^1_{2;10001}x^5_{2;01100} + x^1_{2;01001}x^5_{2;10100} + x^1_{2;00101}x^5_{2;11000} \end{aligned} \tag{4.67c}$$

etc.

we find that eq. (4.64) leads to, for example

$$\begin{aligned} \frac{d\bar{g}_1}{dx^1_{2;02000}} = \bar{B}^1_{1;2;02000}(\bar{g}_c) = g_2^2 = \bar{g}_2^2 - x^2_{2;20000}\bar{g}_1^2\bar{g}_2 - x^2_{2;02000}\bar{g}_2^3 - x^2_{2;00200}\bar{g}_3^2\bar{g}_2 - x^2_{2;00020}\bar{g}_4^2\bar{g}_2 \\ - x^2_{2;00002}\bar{g}_5^2\bar{g}_2 - x^2_{2;11000}\bar{g}_1\bar{g}_2^2 - x^2_{2;01100}\bar{g}_2\bar{g}_3^2 - x^2_{2;00110}\bar{g}_2\bar{g}_3\bar{g}_4 \ldots \end{aligned} \tag{4.68}$$

As noted above, in ref [16] it is shown that if there is one coupling, there exists a RS in which $c_n = 0$ beyond two loop order. In contrast, by eq. (4.63) we cannot find a scheme when there are five couplings such that $c^a_{k;i_1i_2i_3i_4i_5}$ all vanish beyond a certain order in the loop expansion. However, it is possible to find a RS in which at least one of the couplings has a β-function that receives no contribution beyond one loop order. For example, if $x^a_{k;i_1i_2i_3i_4i_5} = 0 (a \neq 1)$ then eqs. (4.63a-c) simplify and we obtain the relations between $\overline{c}^a_{3;i_1i_2i_3i_4i_5}$ and $c^a_{3;i_1i_2i_3i_4i_5}$

$$\overline{c}^1_{3;30000} = c^1_{3;30000} + c^5_{2;20000}x^1_{2;10001} + c^4_{2;20000}x^1_{2;10010} + c^3_{2;20000}x^1_{2;10100} + c^2_{2;20000}x^1_{2;11000} \tag{4.69a}$$

$$\begin{aligned}\overline{c}^1_{3;21000} = {} & c^1_{3;21000} + c^5_{2;20000}x^1_{2;01001} + c^4_{2;20000}x^1_{2;01010} + c^3_{2;20000}x^1_{2;01100} + 2c^2_{2;20000}x^1_{2;02000} \\ & + c^5_{2;11000}x^1_{2;10001} + c^4_{2;11000}x^1_{2;10010} + c^3_{2;11000}x^1_{2;10100} + c^2_{2;11000}x^1_{2;11000} \\ & + c^1_{2;11000}x^1_{2;20000} - c^1_{2;20000}x^1_{2;11000}\end{aligned} \tag{4.69b}$$

etc. and

$$\overline{c}^2_{3;30000} = c^2_{3;30000} - 2c^2_{2;20000}x^1_{2;20000} \tag{4.70a}$$

$$\overline{c}^2_{3;21000} = c^2_{3;21000} - 2c^2_{2;20000}x^1_{2;11000} - c^2_{2;11000}x^1_{2;20000} \tag{4.70b}$$

$$\overline{c}^2_{3;11100} = c^2_{3;21000} - 2c^2_{2;20000}x^1_{2;01100} - c^2_{2;11000}x^1_{2;10100} - c^2_{2;10100}x^1_{2;11000}. \tag{4.70c}$$

etc.

with all other $\overline{c}^a_{k;i_1i_2i_3i_4i_5}$ similarly computed. We see that it is possible to choose $x^1_{k;i_1i_2i_3i_4i_5}$ so that $c^1_{k;i_1i_2i_3i_4i_5} = 0$ for all $k > 2$. We could, for example, identify g_1 with the strong SU(3) coupling $16\pi^2 a$ in which case $\overline{a}$ would by ref. [15] satisfy simply

$$\mu\frac{d\overline{a}}{d\mu} = -14\overline{a}^2 \tag{4.71}$$

with no higher loop corrections. Of course, in this scheme, $\overline{g}_2 ... \overline{g}_5$ would all satisfy eq. (4.58) with coefficients $\overline{c}^a_{k;i_1i_2i_3i_4i_5}(a = 2, 3, 4, 5)$ that depend on the values of $x^a_{k;i_1i_2i_3i_4i_5}$ chosen to give rise to eq. (4.71).

We now will consider RS dependence for a physical quantity R expanded as

$$R = \sum_{k=0}^{\infty} A_k(a)L^k \tag{4.72}$$

where $L = \ln(\frac{\mu}{Q})$ and

$$A_k(a) = \sum_{m=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}...\sum_{i_5=0}^{\infty} T_{m;i_1i_2...i_5;k}\delta_{m+k+1-i_1-i_2...-i_5}(g_1)^{g_1}(g_2)^{g_2}...(g_5)^{g_5}. \tag{4.73}$$

With g_2 satisfying eq. (4.58), substitution of eq. (4.72) into eq.(4.59) leads to

$$A_{k+1}(g_a(\ln(\frac{\mu}{\Lambda}))) = \frac{-1}{k+1}\frac{d}{d(\ln\frac{\mu}{\Lambda})}A_k(g_a(\ln\frac{\mu}{\Lambda})) \tag{4.74}$$

where Λ is a mass scale associated with the boundary conditions on eq. (4.58). As a result [10]

$$R = \sum_{k=0}^{\infty}\frac{(-L)^k}{k!}(\frac{d}{d(\ln\frac{\mu}{\Lambda})})^k A_0(g_a(\ln\frac{\mu}{\Lambda})) = A_0(g_a(\ln\frac{Q}{\Lambda})), \tag{4.75}$$

just like eq (4.50).

All explicit dependence of R on μ through L in eq. (4.72) has been canceled with the implicit dependence on μ through $g_a(\ln(\frac{\mu}{\Lambda}))$ upon summing the logarithmic terms in eq. (4.72), which is possible on account of the RG equation (4.59). The apparent ambiguity in the perturbative expansion for R due to μ has disappeared.

Together eqs. (4.73) and (4.75) lead to

$$R = \sum_{m=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\ldots\sum_{i_5=0}^{\infty} T_{m;i_1 i_2 \ldots i_5}\delta_{m+1-i_1-i_2\ldots-i_5}(g_1)^{g_1}(g_2)^{g_2}\ldots(g_5)^{g_5} \tag{4.76}$$

where

$$T_{m;i_1 i_2 \ldots i_5} = T_{m;i_1 i_2 \ldots i_5;0}. \tag{4.77}$$

Under the change in RS in eq. (4.60), we have T and g_a in eq. (4.75) replaced by $\overline{T}$ and $\overline{g}_a$. However, as R is RS independent, we must have by eq. (4.64)

$$\begin{aligned}\frac{dR}{dx^a_{k;i_1 i_2 i_3 i_4 i_5}} = 0 = &\left(\frac{\partial}{\partial x^a_{k;i_1 i_2 i_3 i_4 i_5}} + \overline{B}^b_{a;k;i_1 i_2 i_3 i_4 i_5}(\overline{g}_b)\frac{\partial}{\partial \overline{g}^b}\right)\\ &\sum_{h=1}^{\infty}\sum_{j_1=0}^{h}\sum_{j_2=0}^{h}\sum_{j_3=0}^{h}\sum_{j_4=0}^{h}\sum_{j_5=0}^{h}\delta_{h-(j_1+j_2+\ldots+j_5)}\overline{T}_{h;j_1 j_2 j_3 j_4 j_5}(\overline{g}_1)^{j_1}(\overline{g}_2)^{j_2}(\overline{g}_3)^{j_3}(\overline{g}_4)^{j_4}(\overline{g}_5)^{j_5}.\end{aligned} \tag{4.78}$$

Upon using eq. (4.64) for $\overline{B}^b_{a;k;i_1 i_2 i_3 i_4 i_5}$, eq. (4.78) becomes

$$\begin{aligned}= \sum_{h=1}^{\infty}\sum_{j_1=0}^{h}\sum_{j_2=0}^{h}\sum_{j_3=0}^{h}\sum_{j_4=0}^{h}\sum_{j_5=0}^{h}\delta_{h-(j_1+j_2+\ldots+j_5)}&\left\{\frac{\partial \overline{T}_{h;j_1 j_2 j_3 j_4 j_5}}{\partial x^a_{k;i_1 i_2 i_3 i_4 i_5}}(\overline{g}_1)^{j_1}(\overline{g}_2)^{j_2}(\overline{g}_3)^{j_3}(\overline{g}_4)^{j_4}(\overline{g}_5)^{j_5}\right.\\ &+ \overline{T}_{h;j_1 j_2 j_3 j_4 j_5}\Big[\overline{B}^1_{a;k;i_1 i_2 i_3 i_4 i_5} j_1(\overline{g}_1)^{j_1-1}(\overline{g}_2)^{j_2}(\overline{g}_3)^{j_3}(\overline{g}_4)^{j_4}(\overline{g}_5)^{j_5}\\ &\left.+ \overline{B}^2_{a;k;i_1 i_2 i_3 i_4 i_5} j_2(\overline{g}_1)^{j_1}(\overline{g}_2)^{j_2-1}(\overline{g}_3)^{j_3}(\overline{g}_4)^{j_4}(\overline{g}_5)^{j_5} + \ldots\Big]\right\} = 0.\end{aligned} \tag{4.79}$$

Terms of a given order in $\overline{g}_a$ lead to, for example

$$\frac{\partial \overline{T}_{1;j_1j_2j_3j_4j_5}}{\partial x^a_{k;i_1i_2i_3i_4i_5}} = 0 \tag{4.80}$$

$$\frac{\partial \overline{T}_{2;20000}}{\partial x^a_{k;i_1i_2i_3i_4i_5}} + \left(\overline{T}_{1;10000}\delta^a_1 + \overline{T}_{1;01000}\delta^a_2 + \overline{T}_{1;00100}\delta^a_3 + \overline{T}_{1;00010}\delta^a_4 + \overline{T}_{1;00001}\delta^a_5\right)\delta_{j_1 2}\delta_{j_2 0}\delta_{j_3 0}\delta_{j_4 0}\delta_{j_5 0} = 0 \tag{4.81a}$$

$$\frac{\partial \overline{T}_{2;11000}}{\partial x^a_{k;i_1i_2i_3i_4i_5}} + \left(\overline{T}_{1;10000}\delta^a_1 + \overline{T}_{1;01000}\delta^a_2 + \overline{T}_{1;00100}\delta^a_3 + \overline{T}_{1;00010}\delta^a_4 + \overline{T}_{1;00001}\delta^a_5\right)\delta_{j_1 1}\delta_{j_2 1}\delta_{j_3 0}\delta_{j_4 0}\delta_{j_5 0} = 0 \tag{4.81b}$$

$$\frac{\partial \overline{T}_{2;02000}}{\partial x^a_{k;i_1i_2i_3i_4i_5}} + \left(\overline{T}_{1;10000}\delta^a_1 + \overline{T}_{1;01000}\delta^a_2 + \overline{T}_{1;00100}\delta^a_3 + \overline{T}_{1;00010}\delta^a_4 + \overline{T}_{1;00001}\delta^a_5\right)\delta_{j_1 0}\delta_{j_2 2}\delta_{j_3 0}\delta_{j_4 0}\delta_{j_5 0} = 0 \tag{4.81c}$$

etc.

These equations have the boundary conditions that $\overline{T} = T$ when $x^a_{k;i_1i_2i_3i_4i_5} = 0$ and so we have the solutions

$$\overline{T}_{1;j_1j_2j_3j_4j_5} = T_{1;j_1j_2j_3j_4j_5} \tag{4.82}$$

$$\overline{T}_{2;20000} = T_{2;20000} - x^1_{2;20000}T_{1;10000} - x^2_{2;20000}T_{1;01000} - x^3_{2;20000}T_{1;00100} - x^4_{2;20000}T_{1;00010} - x^5_{2;20000}T_{1;00001} \tag{4.83a}$$

$$\overline{T}_{2;11000} = T_{2;11000} - x^1_{2;11000}T_{1;10000} - x^2_{2;11000}T_{1;01000} - x^3_{2;11000}T_{1;00100} - x^4_{2;11000}T_{1;00010} - x^5_{2;11000}T_{1;00001} \tag{4.83b}$$

$$\overline{T}_{2;02000} = T_{2;02000} - x^1_{2;02000}T_{1;10000} - x^2_{2;02000}T_{1;01000} - x^3_{2;02000}T_{1;00100} - x^4_{2;02000}T_{1;00010} - x^5_{2;02000}T_{1;00001} \tag{4.83c}$$

etc.

It is evident that these equations can be used to find values of $x^a_{k;i_1i_2i_3i_4i_5}$ that lead to $\overline{T}_{m;j_1j_2j_3j_4j_5} = 0$ with $m \geq 2$. In this case we have

$$R = T_{1;10000}\overline{g}_1\left(\ln\frac{Q}{\Lambda}\right)+T_{1;01000}\overline{g}_2\left(\ln\frac{Q}{\Lambda}\right)+T_{1;00100}\overline{g}_3\left(\ln\frac{Q}{\Lambda}\right)+T_{1;00010}\overline{g}_4\left(\ln\frac{Q}{\Lambda}\right)+T_{1;00001}\overline{g}_5\left(\ln\frac{Q}{\Lambda}\right) \tag{4.84}$$

and no higher powers of $\overline{g}_a$ contribute to R. For example, if we choose to have $x^a_{h;i_1i_2i_3i_4i_5} = 0$ for $a \neq 1$, then $\overline{T}_{m;j_1j_2j_3j_4j_5} = 0(m > 1)$ results in

$$x^1_{2;i_1i_2i_3i_4i_5} = \frac{T_{2;i_1i_2i_3i_4i_5}}{T_{1;10000}} \tag{4.85}$$

$$x^1_{3;30000} = \frac{2(T_{2;20000})^2}{(T_{1;10000})^2} + \frac{T_{3;30000}}{T_{1;10000}} \tag{4.86a}$$

$$x^1_{3;12000} = \frac{(T_{2;11000})^2 + T_{2;02000}T_{2;20000}}{(T_{1;10000})^2} + \frac{T_{3;12000}}{T_{1;10000}} \tag{4.86b}$$

$$x^1_{3;11100} = \frac{T_{2;10100}T_{2;11000} + T_{2;01100}T_{2;20000}}{(T_{1;10000})^2} + \frac{T_{3;11100}}{T_{1;10000}} \tag{4.86c}$$

etc.

The β-functions associated with $\overline{g}_a$ are now given by eqs. (4.69,4.70) with $x^a_{m;i_1i_2i_3i_4i_5} = 0(m = 2,3)$ given by eq. (4.85,4.86).

4.6 Discussion

In this chapter we have considered some aspects of RS ambiguities when using mass independent renormalization in a theory in which there are no physical mass scales and two coupling constants. Unlike what happens when there is but one coupling, the β_a-functions that dictate how the couplings vary with the renormalization mass scale μ when there are two couplings are ambiguous at two loop order (and beyond). Furthermore, these ambiguities do not permit one to vary the coefficients of the expansions of these functions independently, making these coefficients unsuitable for characterizing a RS when using mass-independent renormalization. Instead, it is convenient to parameterize a RS by directly using the coefficients of an expansion of the couplings used in a "new" RS in terms of the couplings used in a base RS.

A change in RS can affect the perturbative expansion for a physical quantity R in powers of the coupling. When there is a single coupling a, one can change the coefficients c_i ($i \geq 2$) in eq. (4.1) by a renormalization of the form of eq. (4.2), as is apparent from eq. (4.4). This means that one can characterize a RS by the values of c_i ($i \geq 2$). If one chooses a RS in which $c_i = 0$ ($i \geq 2$) then the power series for $\beta(a)$ in eq. (4.1) terminates (the 't Hooft scheme [13,14]) and the behavior of the running coupling found exactly in terms of a Lambert function. A second choice of c_i ($i \geq 2$) can be made using eq. (4.21) so that only T_0 and T_1 in the expansion of eq. (4.19) is non-zero, which means that the perturbative expansion for R in powers of a terminates.

A different situation arises when there are two couplings, g_1 and g_2. In this case, the expansion coefficients c^a_{ij} in eq. (4.22) cannot be used to characterize a RS as a renormalization like that of eq. (4.24) does not allow all of the c^a_{ij} to independently vary, as can be seen by eq (27).

It is, however, possible to use the coefficients x_i of eq. (4.2) (when there is one coupling) and the coefficients x^a_{ij} of eq. (4.24) (when there are two couplings) to characterize how a change of RS from some "base scheme" (such as minimal subtraction) can be affected. In the former case, a choice of the x_i so that $c_i = 0$ $(i \geq 2)$ can be made, while in the latter case it is not in general possible to choose the x^a_{ij} so that the expansion of eq. (4.22) is finite. However, in both the cases of one and two couplings, the x_i and x^a_{ij} respectively can be chosen so that the perturbative expansion for a physical quantity R in powers of the coupling terminates, as can be seen from eqs. (4.45,4.57). With such a choice of renormalization, the expansion coefficients of the β function (c_i and c^a_{ij}) are now dependent on the physical quantity being considered and all higher order loop effects are absorbed into the behavior of the running coupling.

The fixed point in such a RS is clearly important. In ref. [16] the behavior of the running couping a when the quantity R in eq. (4.19) is the total cross section ($e^+e^- \rightarrow$hardrons) is discussed. There it is shown that in a RS in which $T_n = 0$ $(n \geq 2)$, the four-loop contribution to $\beta(a)$ results in an infrared fixed point and a well defined low energy limit for R. Since the perturbative series for R terminates, its convergence need not be considered. It would be quite interesting to see if fixed points arise in models with more than one coupling when a finite series is used to compute particular physical quantities.

We then demonstrated that the possibility of making a finite renormalization of the five couplings provides a great deal of flexibility in the way perturbative results can be presented. It is possible to reduce the β-function for one of the couplings to the one loop result. It is also possible to sum all logarithmic contributions to a physical quantity R, thereby eliminating dependence on the renormalization mass scale μ and to make it possible to eliminate all higher order contributions to R. In this scheme, any higher loop calculation only serves to affect the contributions to the β-functions beyond one loop order. We plan to examine how finite renormalization of mass parameters can affect a theory.

4.7 Appendix - Evolution of two running couplings

When there is one coupling $a(\mu)$ whose evolution under changes in the renormalization mass scale μ is given by eq. (4.1), the relationship between $a(\mu)$ and $a(\overline{\mu})$ can be found using eqs. (4.9-4.11). In this appendix we consider the same problem when there are two couplings $g_a(\mu)(a = 1, 2)$ that satisfy eq. (4.24). We begin by making the expansion

$$g_a(\overline{\mu}) = g_a(\mu) + \sum_{i=2}^{\infty} \sum_{j=0}^{i} \sum_{k=1}^{i-1} \sigma^a_{i,j;k} g_1^{i-j}(\mu) g_2^j(\mu) \ell^k. \quad (l \equiv ln(\frac{\mu}{\overline{\mu}})) \tag{A.1}$$

It follows from the condition

$$\begin{aligned} \mu \frac{dg^a(\overline{\mu})}{d\mu} &= 0 \\ &= \left(\mu \frac{\partial}{\partial \mu} + \beta_b(g_1, g_2) \frac{\partial}{\partial g_b} \right) g^a(\overline{\mu}) \end{aligned} \tag{A.2}$$

where β_b is given by eq. (4.24). Substitution of eq. (A.1) into eq. (A.2) results in

$$\sigma^1_{20,1} = -c^1_{20}, \qquad \sigma^1_{21,1} = -c^1_{21}, \qquad \sigma^1_{22,1} = -c^1_{22} \tag{A.3a-c}$$

$$\sigma^1_{30,1} = -c^1_{30}, \qquad \sigma^1_{31,1} = -c^1_{31}, \qquad \sigma^1_{32,1} = -c^1_{32}, \qquad \sigma^1_{33,1} = -c^1_{33} \tag{A.4a-d}$$

$$\sigma^1_{30,2} = \frac{1}{2}\left[2(c^1_{20})^2 + c^1_{21}c^2_{20}\right], \quad \sigma^1_{31,2} = \frac{1}{2}\left[3c^1_{21}c^1_{20} + c^1_{21}c^2_{21} + 2c^1_{22}c^2_{20}\right] \tag{A.5a-d}$$

$$\sigma^1_{32,2} = \frac{1}{2}\left[(c^1_{21})^2 + 2c^1_{22}c^1_{20} + c^1_{21}c^2_{22} + 2c^1_{22}c^2_{11}\right]$$

$$\sigma^1_{33,2} = \frac{1}{2}\left[c^1_{22}c^1_{21} + 2c^1_{22}c^2_{22}\right].$$

The values of $\sigma^a_{4j,k}$ can similarly be computed in terms of $c^a_{ij,k}$. We note that since eqs. (4.21, A.1) are symmetric between g_1 and g_2, we have symmetry in $(c^1_{ij}, c^2_{i,i-j})$ and $(\sigma^1_{i,j;k}, \sigma^2_{i,i-j;k})$. Computing all of the coefficients $\sigma^a_{i,j;k}$ amounts to integrating eq. (4.24) with a fixed boundary value for $g^a(\mu)$.

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Chapter 5

Conclusions

In this thesis we study various topics in the quantization and renormalization of gauge fields.

Chapter 1 gives a review of context for all three topics covered in the following chapters. We introduce Dirac Constraint Formalism which is later used in Chapter 2 to quantize both Yang-Mills field and 2+1 dimensional superparticle in light cone coordinate prior to the process of gauge fixing. After that we also review the transverse-traceless gauge, which uses a non-quadratic gauge fixing procedure that is necessary in order to have a transverse-traceless propagator. In Chapter 3 we apply this non-quadratic gauge fixing procedure to first order gravity. Finally we give an introduction to the renormalization group equation that follows from renormalization scheme ambiguity. I also introduce renormalization group summation which leads to a new characterization of remormalization schemes in Chapter 4.

In Chapter 2 we consider both the Yang-Mills field and 2+1 dimensional superparticle. For both theories, we formulate its Langrangian in light cone coordinate, apply constraint formalism to identify and classify first class and second class constraints, pairing up each first class constraint with a gauge and then compute Dirac brackets to exhaust all second class constraints as well as pairs of first class constraints and their associated gauges. The most promising application of Dirac's constraint formalism is that it allows us to find all symmetries of a specific theory. However for complicated theories like gravity, the process of applying the constraint formalism is really involved. The symmetries revealed by constraint formalism may not be of the form that is desired. It would be interesting to extend our study on these aspects.

In Chapter 3 we apply non-quadratic gauge fixing procedure to first order gravity. With non-quadratic double gauge fixing, we can have our graviton propagator being transverse and traceless at same time. We have also derived the entire set of Feynman rules under such gauge fixing. Having graviton propagator transverse and traceless could lead to cancellation in perturbative computation, which could contribute to the renormalizability of underlying gravity theory.

In Chapter 4 we consider parameterization of renormalization scheme ambiguities. We discover that when there are two or more couplings, there is no scheme in which the β-functions can be terminated beyond one loop order, and ambiguity in the β-functions occurs beyond one loop order. We propose a new characterization method using parameters that arise in the expansion of coupling in one scheme in terms of couplings in another scheme. We tested our new characterization method in theories with one, two and five couplings and discover that with our new characterization method, we can choose a scheme so that a physical quantity R can be perturbatively terminated with higher loop effects absorbed into the behavior of running coupling.

Appendix A

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The following paper on which this thesis is partially based was published in *Canadian Journal of Physics*, a journal of Canadian Science Publishing:
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Appendix B

Code for symbolic computations

In this appendix, I present major components of my code for symbolic computations in Section 4.5. The complete file can be found in my github:
https://github.com/CGZGit.

```
(** RG beta functions **)
gbar[1] = g[1] + Sum[Sum[Sum[Sum[Sum[x[1, i1, i2, i3, i4, itotal - i1 - i2 -
i3 - i4]*g[1]^i1*g[2]^i2*g[3]^i3*g[4]^i4*g[5]^(itotal - i1 - i2 - i3 - i4),
{i1, 0, itotal - i4 - i3 - i2}], {i2, 0, itotal - i4 - i3}], {i3, 0,
itotal - i4}], {i4, 0, itotal}], {itotal, 2, k}].

Do[gbar[ii] = g[ii] + Sum[Sum[Sum[Sum[Sum[
x[ii, i1, i2, i3, i4, itotal - i1 - i2 - i3 - i4]*g[1]^i1*
g[2]^i2*g[3]^i3*g[4]^i4*g[5]^(itotal - i1 - i2 - i3 - i4), {i1, 0,
itotal - i4 - i3 - i2}], {i2, 0, itotal - i4 - i3}], {i3, 0,
itotal - i4}], {i4, 0, itotal}], {itotal, 2, k}], {ii, 1, 5}].

(** Construct polynomials containing coefficient relations between x and
y and then extract coefficient relation equations **)
Do[pol1[ii] = gbar[ii] - g[ii] + Sum[Sum[Sum[Sum[Sum[
y[ii, i1, i2, i3, i4, itotal - i1 - i2 - i3 - i4]*
```

```
gbar[1]^i1*gbar[2]^i2*gbar[3]^i3*gbar[4]^i4*
gbar[5]^(itotal - i1 - i2 - i3 - i4), {i1, 0,
itotal - i4 - i3 - i2}], {i2, 0, itotal - i4 - i3}], {i3, 0,
itotal - i4}], {i4, 0, itotal}], {itotal, 2, k}], {ii, 1, 5}].

Do[pol2[ii] = pol1[ii]*g[1] g[2] g[3] g[4] g[5], {ii, 1, 5}].

Do[Do[ce[ii, i1, i2, i3, i4, i5] =
Coefficient[Coefficient[Coefficient[Coefficient[Coefficient
[pol2[ii], g[1], i1], g[2], i2], g[3],
i3], g[4], i4], g[5], i5], {i1, 1, k}, {i2, 1, k}, {i3, 1,
k}, {i4, 1, k}, {i5, 1, k}], {ii, 1, 5}]

(** Solve for coefficient relations **)
Do[Do[Do[Do[Do[y1[ii, i1, i2, i3, i4, 2 - i1 - i2 - i3 - i4] =
y[ii, i1, i2, i3, i4, 2 - i1 - i2 - i3 - i4] /.
Solve[ce[ii, i1 + 1, i2 + 1, i3 + 1, i4 + 1, 3 - i1 - i2 - i3 - i4] == 0,
y[ii, i1, i2, i3, i4, 2 - i1 - i2 - i3 - i4]][[1]], {i1, 0, 2 - i4 - i3 - i2}],
{i2, 0, 2 - i4 - i3}], {i3, 0, 2 - i4}], {i4, 0, 2}], {ii, 1, 5}]

(** Reconstruct RG beta functions for new characterization method **)
Do[gg[ii] = ggbar[ii] + Sum[Sum[Sum[Sum[Sum[
y[ii, i1, i2, i3, i4, itotal - i1 - i2 - i3 - i4]*ggbar[1]^i1*ggbar[2]^i2*
ggbar[3]^i3*ggbar[4]^i4*ggbar[5]^(itotal - i1 - i2 - i3 - i4), {i1, 0,
itotal - i4 - i3 - i2}], {i2, 0, itotal - i4 - i3}], {i3, 0,
itotal - i4}], {i4, 0, itotal}], {itotal, 2, k}], {ii, 1, 5}]

(** Dependence of g on x **)
Do[Do[Do[Do[Do[Do[
Bbar[ii, i1, i2, i3, i4, itotal - i1 - i2 - i3 - i4] =
ggbar[1]^i1*ggbar[2]^i2*ggbar[3]^i3*ggbar[4]^i4*ggbar[5]
```

```
^(itotal - i1 - i2 - i3 - i4), {i1, 0, itotal - i4 - i3 - i2}],
{i2, 0, itotal - i4 - i3}], {i3, 0, itotal - i4}], {i4, 0, itotal}],
{itotal, 2, 3}], {ii, 1, 5}]

(** The ultimate consistency condition (4.79) **)
Ult[2, 2, 0, 0, 0, 0] = Sum[Sum[Sum[Sum[Sum[
Tx[2, 2, 0, 0, 0, 0, itotal, i1, i2, i3, i4, itotal - i1 - i2 - i3 - i4]
*ggbar[1]^i1*ggbar[2]^i2*ggbar[3]^i3*ggbar[4]^i4*ggbar[5]
^(itotal - i1 - i2 - i3 - i4), {i1, 0, itotal - i4 - i3 - i2}],
{i4, 0, itotal - i2 - i3}], {i3, 0, itotal - i2}], {i2, 0, itotal}],
{itotal, 1, k}] + Sum[Sum[Sum[Sum[Sum[
T[itotal, i1, i2, i3, i4, itotal - i1 - i2 - i3 - i4]*Bbar[2, 2, 0, 0, 0, 0]
*i2*ggbar[1]^i1*ggbar[2]^(i2 - 1)*ggbar[3]^i3*ggbar[4]^i4*ggbar[5]
^(itotal - i1 - i2 - i3 - i4), {i1, 0, itotal - i4 - i3 - i2}],
{i4, 0, itotal - i2 - i3}], {i3, 0, itotal - i2}], {i2, 1, itotal}],
{itotal, 1, k}]

(** Deriving diferential Equations of T-bar **)
Do[Do[Do[Do[Do[dTdx[2, m1, m2, m3, m4, mtotal - m1 - m2 - m3 - m4]
= -Sum[Sum[Sum[Sum[Sum[Sum[T[itotal, i1, i2, i3, i4,
itotal - i1 - i2 - i3 - i4]*
Coefficient[Coefficient[Coefficient[Coefficient[Coefficient[
Bbar[ii, m1, m2, m3, m4, mtotal - m1 - m2 - m3 - m4], ggbar[1], m1],
ggbar[2], m2], ggbar[3], m3], ggbar[4], m4], ggbar[5],
mtotal - m1 - m2 - m3 - m4]*x[ii, m1, m2, m3, m4, mtotal - m1 - m2 - m3 - m4]
D[ggbar[1]^i1*ggbar[2]^i2*ggbar[3]^i3*ggbar[4]^i4*ggbar[5]
^(itotal - i1 - i2 - i3 - i4), ggbar[ii]]
, {i1, 0, itotal - i4 - i3 - i2}], {i4, 0, itotal - i2 - i3}],
{i3, 0, itotal - i2}], {i2, 0, itotal}], {itotal, 1, 1}], {ii, 1, 5}],
{m1, 0, mtotal - m4 - m3 - m2}], {m4, 0, mtotal - m2 - m3}],
{m3, 0, mtotal - m2}], {m2, 0, mtotal}], {mtotal, 2, 2}]
```

```
Do[Do[Do[Do[Do[dTdx1[3, t1, t2, t3, t4, ttotal - t1 - t2 - t3 - t4]
= -Sum[Sum[Sum[Sum[Sum[Sum[Sum[Sum[Sum[Sum[Sum[
T[ktotal, k1, k2, k3, k4, ktotal - k1 - k2 - k3 - k4]*
Coefficient[Coefficient[Coefficient[Coefficient[Coefficient[
Bbar[aa, m1, m2, m3, m4, mtotal - m1 - m2 - m3 - m4], ggbar[1], t1],
ggbar[2], t2], ggbar[3], t3], ggbar[4], t4], ggbar[5],
ttotal - t1 - t2 - t3 - t4]*delta[aa, m1, m2, m3, m4,
mtotal - m1 - m2 - m3 - m4]D[ggbar[1]^k1*ggbar[2]^k2*ggbar[3]^k3
*ggbar[4]^k4*ggbar[5]^(ktotal - k1 - k2 - k3 - k4), ggbar[aa]]
, {k1, 0, ktotal - k4 - k3 - k2}], {k4, 0, ktotal - k2 - k3}],
{k3, 0, ktotal - k2}], {k2, 0, ktotal}], {ktotal, 1, 1}], {aa, 1, 5}]
, {m1, 0, mtotal - m4 - m3 - m2}], {m4, 0, mtotal - m2 - m3}],
{m3, 0, mtotal - m2}], {m2, 0, mtotal}], {mtotal, 2, 3}],
{t1, 0, ttotal - t4 - t3 - t2}], {t4, 0, ttotal - t2 - t3}],
{t3, 0, ttotal - t2}], {t2, 0, ttotal}], {ttotal, 3, 3}]

Do[Do[Do[Do[Do[dTdx2[3, t1, t2, t3, t4, ttotal - t1 - t2 - t3 - t4] =
-Sum[Sum[Sum[Sum[Sum[Sum[Sum[Sum[Sum[Sum[Sum[
T[ktotal, k1, k2, k3, k4, ktotal - k1 - k2 - k3 - k4]*
delta[aa, m1, m2, m3, m4, mtotal - m1 - m2 - m3 - m4]*
Coefficient[Coefficient[Coefficient[Coefficient[Coefficient[
D[ggbar[1]^k1*ggbar[2]^k2*ggbar[3]^k3*ggbar[4]^k4*ggbar[5]
^(ktotal - k1 - k2 - k3 - k4), ggbar[aa]], ggbar[1], t1 - m1],
ggbar[2], t2 - m2], ggbar[3], t3 - m3], ggbar[4], t4 - m4],
ggbar[5], ttotal - t1 - t2 - t3 - t4 - mtotal + m1 + m2 + m3 + m4],
{k1, 0, ktotal - k4 - k3 - k2}], {k4, 0, ktotal - k2 - k3}],
{k3, 0, ktotal - k2}], {k2, 0, ktotal}], {ktotal, 2, 2}], {aa, 1, 5}],
{m1, 0, mtotal - m2 - m3 - m4}], {m4, 0, mtotal - m2 - m3}],
{m3, 0, mtotal - m2}], {m2, 0, mtotal}], {mtotal, 2, 2}],
```

```
{t1, 0, ttotal - t4 - t3 - t2}], {t4, 0, ttotal - t2 - t3}],
{t3, 0, ttotal - t2}], {t2, 0, ttotal}], {ttotal, 3, 3}]

Do[Do[Do[Do[Do[
dTdx[3, t1, t2, t3, t4, ttotal - t1 - t2 - t3 - t4] =
dTdx1[3, t1, t2, t3, t4, ttotal - t1 - t2 - t3 - t4] +
dTdx2[3, t1, t2, t3, t4, ttotal - t1 - t2 - t3 - t4]
, {t1, 0, ttotal - t4 - t3 - t2}], {t4, 0, ttotal - t2 - t3}],
{t3, 0, ttotal - t2}], {t2, 0, ttotal}], {ttotal, 3, 3}]

(** Set x=0 for a>1 **)
Do[Do[Do[Do[Do[Do[x[kzeros, t1, t2, t3, t4, ttotal - t1 - t2 - t3 - t4]
= 0, {t1, 0, ttotal - t4 - t3 - t2}], {t4, 0, ttotal - t2 - t3}],
{t3, 0, ttotal - t2}], {t2, 0, ttotal}], {ttotal, 2, 2}], {kzeros, 2, 5}]

(** Reconstruct beta-functions to solve for c-bar **)
beta3[ii_] = Sum[Sum[Sum[Sum[Sum[
c[ii, i1, i2, i3, i4, itotal - i1 - i2 - i3 - i4]*g[1]^i1
*g[2]^i2*g[3]^i3*g[4]^i4*g[5]^(itotal - i1 - i2 - i3 - i4),
{i1,0, itotal - i4 - i3 - i2}], {i2, 0, itotal - i4 - i3}],
{i3, 0, itotal - i4}], {i4, 0, itotal}], {itotal, 2, 3}]

(** Construct polymonials to equate (4.61) with (4.62) **)
Do[polc[ii] = beta3[ii] + Sum[Sum[Sum[Sum[Sum[
xxx[ii, i1, i2, i3, i4, itotal - i1 - i2 - i3 - i4]*
Sum[D[g[1]^i1*g[2]^i2*g[3]^i3*g[4]^i4*
g[5]^(2 - i1 - i2 - i3 - i4), g[id]]*beta2[id], {id, 1,
5}], {i1, 0, itotal - i4 - i3 - i2}], {i2, 0, itotal - i4 - i3}],
{i3, 0, itotal - i4}], {i4, 0, itotal}], {itotal, 2, 2}] -
Sum[Sum[Sum[Sum[cbar[ii, i1, i2, i3, i4, 2 - i1 - i2 - i3 - i4]
*(g[1] + Sum[Sum[Sum[Sum[xxx[1, j1, j2, j3, j4, 2 - j1 - j2 - j3 - j4]
```

```
*g[1]^j1*g[2]^j2*g[3]^j3*g[4]^j4*g[5]^(2 - j1 - j2 - j3 - j4),
{j1, 0, 2 - j4 - j3 - j2}], {j2, 0, 2 - j4 - j3}], {j3, 0, 2 - j4}],
{j4, 0, 2}])^i1*(g[2] + Sum[Sum[Sum[Sum[
xxx[2, j1, j2, j3, j4, 2 - j1 - j2 - j3 - j4]*g[1]^j1*g[2]^j2*g[3]^j3
*g[4]^j4*g[5]^(2 - j1 - j2 - j3 - j4), {j1, 0, 2 - j4 - j3 - j2}],
{j2, 0, 2 - j4 - j3}], {j3, 0, 2 - j4}], {j4, 0, 2}])^i2*(g[3] +
Sum[Sum[Sum[Sum[xxx[3, j1, j2, j3, j4, 2 - j1 - j2 - j3 - j4]*g[1]^j1
*g[2]^j2*g[3]^j3*g[4]^j4*g[5]^(2 - j1 - j2 - j3 - j4),
{j1, 0, 2 - j4 - j3 - j2}], {j2, 0, 2 - j4 - j3}], {j3, 0, 2 - j4}],
{j4, 0, 2}])^i3*(g[4] + Sum[Sum[Sum[Sum[
xxx[4, j1, j2, j3, j4, 2 - j1 - j2 - j3 - j4]*g[1]^j1*g[2]^j2*g[3]^j3
*g[4]^j4*g[5]^(2 - j1 - j2 - j3 - j4), {j1, 0, 2 - j4 - j3 - j2}],
{j2, 0, 2 - j4 - j3}], {j3, 0, 2 - j4}], {j4, 0, 2}])^i4*(g[5] +
Sum[Sum[Sum[Sum[xxx[5, j1, j2, j3, j4, 2 - j1 - j2 - j3 - j4]
*g[1]^j1*g[2]^j2*g[3]^j3*g[4]^j4*g[5]^(2 - j1 - j2 - j3 - j4),
{j1, 0, 2 - j4 - j3 - j2}], {j2, 0, 2 - j4 - j3}], {j3, 0, 2 - j4}],
{j4, 0, 2}])^(2 - i1 - i2 - i3 - i4), {i1, 0, 2 - i4 - i3 - i2}],
{i2, 0, 2 - i4 - i3}], {i3, 0, 2 - i4}], {i4, 0, 2}]
- Sum[Sum[Sum[Sum[cbar[ii, i1, i2, i3, i4, 3 - i1 - i2 - i3 - i4]
*g[1]^i1*g[2]^i2*g[3]^i3*g[4]^i4*g[5]^(3 - i1 - i2 - i3 - i4),
{i1, 0, 3 - i4 - i3 - i2}], {i2, 0, 3 - i4 - i3}], {i3, 0, 3 - i4}],
{i4, 0, 3}], {ii, 1, 5}]

Do[polc2[ii] = polc[ii]*g[1] g[2] g[3] g[4] g[5], {ii, 1, 5}]


(** Extract coefficient equation relations **)
Do[Do[Do[Do[Do[Do[ce2[ii, i1, i2, i3, i4, i5] =
Coefficient[Coefficient[Coefficient[Coefficient[Coefficient
[polc2[ii], g[1], i1], g[2], i2], g[3], i3], g[4], i4], g[5], i5],
{i1, 1, k + 5 - i5 - i4 - i3 - i2}], {i2, 1, k + 4 - i5 - i4 - i3}],
```

```
{i3, 1, k + 3 - i5 - i4}], {i4, 1, k + 2 - i5}], {i5, 1, k + 1}], {ii, 1, 5}]

(** Solve for symbolic values of c-bar **)
Do[Do[Do[Do[Do[cbarSymb[ii, i1, i2, i3, i4, 3 - i1 - i2 - i3 - i4] =
cbar[ii, i1, i2, i3, i4, 3 - i1 - i2 - i3 - i4] /.
Solve[ce2[ii, i1 + 1, i2 + 1, i3 + 1, i4 + 1, 4 - i1 - i2 - i3 - i4]
== 0, cbar[ii, i1, i2, i3, i4, 3 - i1 - i2 - i3 - i4]][[1]],
{i1, 0, 3 - i4 - i3 - i2}], {i2, 0, 3 - i4 - i3}],
{i3, 0, 3 - i4}], {i4, 0, 3}], {ii, 1, 5}]
```

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