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Teaching through Mathematics Problems: Re-designed for a Focus on Mathematics

Immaculate Kizito Namukasa
The University of Western Ontario, inamukas@uwo.ca

Elena Polotskaia
Polotski Consultancy Inc.

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Teaching through Mathematics Problems: Re-designed for a Focus on Mathematics

Immaculate K. Namukasa[#1], with Elena Polotskaia[*2]

[*2]Polotski Consultancy Inc.

[#1]*University of Western Ontario, Canada*

Abstract

Recent research on problem solving explores its potential as a pedagogical practice. This emphasis rejuvenates the interest in problem solving as a learning activity. This paper presents the practice of using a selected problem together with its variants in a single lesson. The practice was implemented in middle school classroom settings with gifted students and with mixed ability students as well as in teacher education classrooms. Experiences from practice are used to illustrate that the use of a set of closely related problems is likely to make students more eager to share their solutions, to generate several solution strategies, and to show various connections among the ideas involved. The shift toward exploring multiple math strategies and representations, and big ideas is at the center of innovative and successful approaches to teaching mathematics. The paper is guided by an evaluation of literature that considers teaching through problem solving and of literature on complex professional tasks.

Keywords—Problem solving, pedagogy, middle school, gifted students, strategies, innovative approaches, complex professional tasks.

I. PROBLEM SOLVING AS A PEDAGOGICAL TASK

Several jurisdictions including Ontario focus on mathematics problem solving. A review of two special journal issues on problem solving around the world traces this focus in many countries [1]. Schoenfeld's[2] review of problem solving in the United States, and Clarke's [3] review of problem solving in Australia interpret the current widespread emphasis on problem solving as a sign of a mature reform practice. In North America, the focus has shifted from problem solving as one of the several recommended classroom activities to teaching most mathematics through problem solving [4]. Several innovative approaches such as use of a variety of tools, promotion of reasoning and communication among students are all components of well designed and well orchestrated problem solving lessons. Problem solving is not only a learning approach but also as a pedagogical practice [5], which is also referred to as teaching through problem solving.

Reference [6] warns that explicitly teaching problem solving by explicating key aspects of mathematical problem solving might not be appropriate as it burdens learners and may reduce sophisticated processes to routine, and direct step- by-step exercises. More sophisticated practices such as modeling and situation problems are promising. Presenting well designed problem solving tasks together with their variations is an instructional practice of teaching through problem solving that I have experimented with. This approach is more likely, as I shall illustrate, to encourage students to explore the mathematics involved in the problems.

The paper begins off by considering teaching through problem solving as a complex pedagogical practice. It then makes an assessment of research on selecting rich mathematics tasks before introducing a rich problem together with its variations. The paper finally shares how the use of a problem together with its variants facilitates exploration of mathematics ideas involved in a problem.

II. PROBLEM SOLVING AS A COMPLEX PEDAGOGICAL PRACTICE

Problem solving understood as a pedagogical practice is a more complex task than teaching a mathematics procedure or teaching through random sets of worked out examples. I read complexity in terms of composition, interactions and relations among components. Complexity here does not refer to the level of difficulty. That complex tasks are composed of several components, which have many interwoven relationships, makes their implementation sophisticated. Even when a rich learning task is selected, the possibility of offering it or of students taking it on as a routine problem is still high. A problem solving lesson might turn out to center only on the teaching of a strategy. For this reason, after [7], I take the practice of teaching through problem solving to be a complex professional practice. For an example of a complex professional practice, [7] takes the healthcare procedure of prescribing and administering medical treatment. Both prescribing and administering medication, and planning and implementing a problem solving lesson involve many stages and several actors, although at different scales of expertise and sites: [7] deconstructs complex tasks by considering the number of possible options at every stage, the space of possibilities. He summarizes that for complex tasks the complexity of the systems performing and supporting the task must at every scale of action—individual or collective, human or material actors, and orchestration or follow-up match the complexity of the task. Further, for such complex tasks, achieving desired outcomes is much about reducing the possibility of invalid actions as it is about increasing the possibility of valid actions. Witness all the planning details about students' and teacher's activities that are recommended at the beginning, during and to conclude a problem solving lesson. Bar-Yam warns that reducing the scale of complexity involved in complex tasks, say by reducing the number of actors and details is not a good way to mitigate invalid outcomes, especially because the complexity generated by the variety of possible actions at every stage is the basis for *rich* action. Practitioners “must identify where the complexity arises and create a system that has adequately complex capabilities” (p. 158) to support this complexity, he suggests. This suggestion is very relevant for teaching with rich learning tasks.

Several mathematics educators agree that the practice of teaching through problem solving is sophisticated and demanding. Reference [3] asserts that research is needed that connects problem solving practices to problem solving lessons taught by teachers. Reference [2] maintains that problem solving research has to move to broader theoretical descriptions than, say, studying methods used by expert problems solvers. Still, [3] identifies problem solving learning and teaching as complex performance tasks that require aligning of curriculum, instruction and assessment. Put differently, the complexity of teaching through problem solving needs to be matched at several scales — say, planning, carrying out, orchestrating and reflecting on students’ activities — involved in teaching. This understanding raises questions: In what ways do teachers, who are innovative and successful at teaching through problem solving, design classroom environments, cultures and dispositions that promote this complex practice? Taking the level of planning, how do they design, select or generate the tasks? How do they create instructional materials and classroom environments that have adequately sophisticated capabilities to support rich problem solving activities?

III. RICH MATHEMATICS TASKS: WHAT RESEARCH SAYS ABOUT THEIR SELECTION, DESIGN OR GENERATION

A rich mathematics task is one that is interesting for teachers, students, and even the public. In the research literature, several characteristics have been identified in the design, selection or generation of rich problems. These characteristics are:

a) Variable entry problems [8] also referred to as low floor and high ceiling math [9] that encourage participation of all;

b) Non-routine, original (unfamiliar) problems [10] non-standard, or rich learning tasks [11]that go beyond mere practice of procedures;

c) Insightful tasks [12] that focus on central mathematical structures [13] that involve mathematizing activities [14] and dynamically stretch how students think about the mathematics involved [15];

d) Tasks that potentially lead to challenging and sophisticated [16] yet interesting and beautiful mathematics, especially for the teachers [5];

e) Real-life, practice, investigation, and story contexts that focus on understanding rather than fluency [17];

f) Reflection problems [18] that go further beyond building connections among practiced knowledge towards more planning and implementing of solutions. Elsewhere, this feature is referred to as mathematics modelling [19] or non-goal specific [3] but worthwhile mathematics tasks. In Quebec, the term situational problem is preferred in a manner that draws from the French didactic theory [6].

IV. HANDSHAKE MATHEMATICS

Several textbooks and web resources offer the timeless handshake problem to illustrate problem solving strategies such as use of a simpler case or use of a variety of tools. Cases in point are two Ontario textbooks — [20], [21] — and one, Ontario Ministry of Education teachers' support resource — [22]. This classical problem is about the number of handshakes that occur at a party if people decide to shake each other's hands only once. The problem highlights the possibility of many solutions strategies and of generalizing from specific cases [23]. The handshake problem is a closed problem with one answer but it leads to a variety of solution strategies, some of which are likely to be invented by students. It is usually engaging and not limited to practicing procedures. The distinction open-middle (Kotsopoulos, personal communication), adaptive problem (after, adaptive, flexible problem solving) is appropriate for the handshake problem.

Figures 1 to 3 present the handshake problem with two related problems — the real championship problem and the stair shapes problem. With Polotskaia, I designed and tested several variants that complement classical mathematics tasks such as the handshake, the consecutive number sums and the chessboard square problems. The Chessboard problem involves figuring out the number of squares on a chessboard. (See Figure 4 for a description of the consecutive number sums task.) The above characteristics on rich tasks guide my selection of the problems as well as the generation of variants to the problems. The two problems — the real championship problem and the stair shapes problem were designed to share some underlying concepts, procedures, reasoning, thinking and representations with the handshake problem. Unique contexts were sought. To generate these variants Polotskaia and I studied the solutions of the handshake problem which involved unique representations and models. The stair shape problem arose from the grid, cube-a-link solution in which students add consecutive numbers 1, 2, 3, 4 and 5 to find out the total number of handshakes that occur. Students notice that the first person shakes 5 hands and the next 4, then the next 3, then 2 and the second last shakes 1 hand and the last person finally shakes no new hand. The championship version arose from the table of league solution in which students cross tabulated persons shaking hands. Several other variants that share underlying mathematics structures, models and ideas with the handshake problem, if necessary, could be generated. One other variant uses the context of high ways directly connecting a set of cities. This context would map onto the polygon solution in which students use sides and diagonals to mark handshakes among, say, six people standing to form a hexagon.

- There are 6 people at a party, To become acquainted with one another, each person shakes hands just once with everyone else. How many handshakes occur?
- If there were more people at the party, perhaps as many as we are here, how many handshakes would occur?

Fig. 1 The handshake problem

- You need to organize in your group a “real” chess championship. “Real” means that on the first round every student will play every other only once. How many parties will be played on the first round? How much time would take if you organize this round the best way you can?
- (You can take an hour as an average party time, in order to estimate the total time for the first round).

Fig. 2 The real championship problem

- The factory produces stair- shaped pieces of plastic of different dimensions and numbers of “steps.” An n-step piece is n inches high and n inches long. What is the weight of a 6-steps piece, if 1 square inch of this plastic weights 1gramm?

Fig. 3 The star shapes problem

Some numbers can be expressed as the sum of a string of consecutive whole numbers. For example,

9 = 2 + 3 + 4
11 = 5 + 6
18 = 3 + 4 + 5 + 6

Let us investigate numbers that have this property.

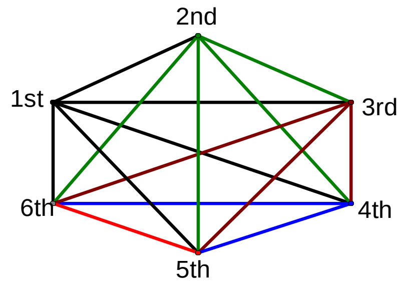
Fig. 4 The consecutive number sums

I have used these problems in several settings including pre-service teacher education, in design experiments with grades 7 to 10 students, and in classroom settings with middle school students. At the beginning of the problem solving part of the lesson in which the handshake problem was offered to teacher candidates, as an instructor, I could hear remarks such as, “There are 6 X 6, 36 handshakes” for a party of 6 people.” Other students first considered that the answer could be, 6!, 6/2, 6-1, 6^2. Soon after such initial conjectures about the structure of mathematics involved, learners begin to explore strategies that can be used to systematically solve the problem. Circulating around the classroom, a teacher might hear students making the following remarks: “Wait, some handshakes are repeated — so we have to subtract those.” “Do you notice that we are adding consecutive numbers …”, or “It is doubled but what is it when not doubled...” Students use several strategies ranging from concrete, pictorial, graphical, numerical through narrated algebra and logical reasoning to analytical representations. Figures 5-12 are a copy of some of the slides used when conferencing about solution strategies in a teacher education setting.

More often than not, learners do not come up with some key handshake solution strategies. For instance, whereas acting the problem out is a very concrete strategy, it might not emerge in some classes. Concrete strategies show relations to geometry and to discrete mathematics of counting and combining. Pictorial and geometrical solutions of stacking, grids and networks are helpful at visualizing the mathematics involved in a problem. In one

Grade 7 lesson, the geometrical solution did not spontaneously emerge, leaving the teacher with the option of introducing this strategy to the classroom. When students work only with pictorial solutions (shown in figures 5, 6&12) and counting strategies (shown in Figure 10) there is a greater chance that their thinking is restricted to arithmetic skills as opposed to when they work with charts, lists, sequences and tables to seek for patterns and algebraic relations shown in figures 7, 10 and 11. Students also miss the opportunity to make connections among the arithmetic, visual- graphic and algebraic solutions when they work with only a couple of solution models. Reference [24] found multiple models to encourage flexibility among learners, to challenge several students to modify and extend their thinking, and to facilitate linking of representations.

The TIMSS video study [25] shows a Japanese teacher using pre-planned hint cards to facilitate students who are stuck say because they are not familiar with the context of the problem. A warm up activity that foresees several of the desired mathematics could be offered to students at the beginning of a lesson to pre-empt students to recall models needed to solve the problem. A teacher could moderate the difficulty of a task for selected students by reducing the challenge [26] created by the big numerical values involved. Reference [27] suggests the use of parallel tasks, which provide easier entry points for weaker students when potential sources of difficulty are removed. Parallel tasks focus on the same learning concept but differ by skill level. An alternative that focuses more on exploring a variety of varied solutions needs to be created.



We thought that Geometry – Took each person to be standing in the corner of a polygon. We then drew the handshakes.

Fig. 5 Polygon diagonals and vertices strategy

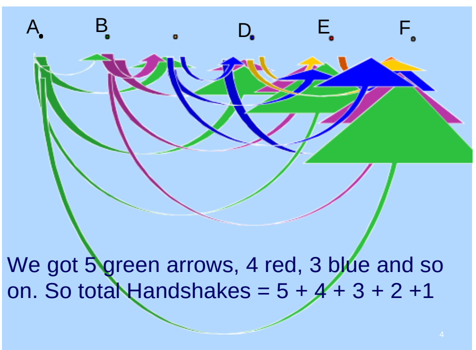


Fig. 6 Pictorial solution: Ordered Pairs

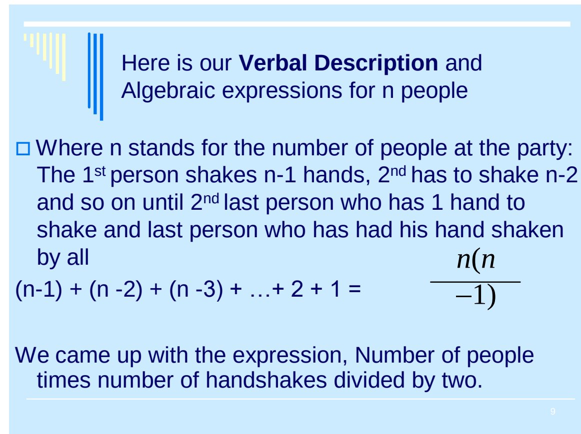


Fig. 9 Narrated algebra and logical reasoning

We used a T-Table

Person at Party	Handshakes
1	0
2	1
3	3
4	6
5	

We found the next term; then the pattern rule and also graphed the relationship using ordered papers (1,0), (2,1), (3,3) and so on. We got a curve. Two of us used the spreadsheet rule that keeps adding a varying difference of 1, then 2, then 3, and so on

Fig. 7 T-tables and patterning

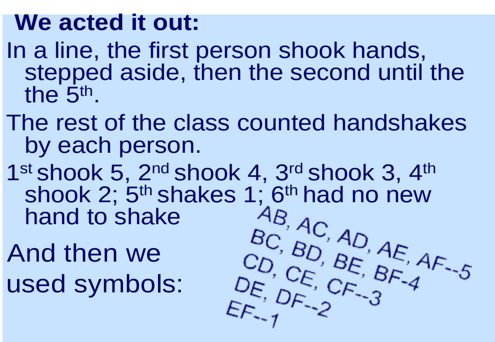


Fig. 8 Acting it out and lists

Counting Strategies 1 + 2 + 3 + 4 + 5 =
5 + 4 + 3 + 4 + 1 =

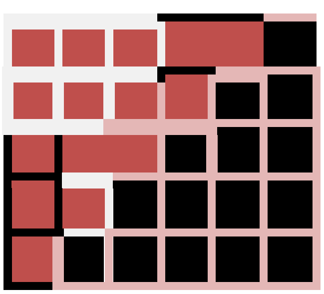
One may reverse the sequence, sum both the original and reversed sequence to get 5+1, 4+2 and so on. This would result in five sixes, 5X6. So the sum of the original sequence is (5X6)/2. In general terms, this is the sum of the first and last term, (n -1 + 1) times number of terms, (n-1) divided by two.(Or, it is the sum of the first and last, 2nd & 2nd last and so on to count how many fives there are, 2 ½ .)
This strategy is attributed to Carl Friedrich Gauss (1777-1855) who used it to mentally compute 1+2+3+4….+96+97+98+99 as (99X100)/2

Fig. 10 Number sequences sums

	1	2	3	4	5	6
1		X	X	X	X	X
2			X	X	X	X
3				X	X	X
4					X	X
5						X
6						

The total is 15 handshakes, the X's above the diagonal line, for a party of 6 people. We used this to get the expressions: 6^2 minus 6^{th} triangular number represented by the shaded cells. In general terms this is n^2 minus n^{th} triangular number.

Fig. 11 Table of leagues or math in genetics



The shape of 1+2+3+4+5 resembles a triangular figure when we used linking cubes. So that stacking two similar such shapes forms a rectangle with dimensions 6 by 5. The area of the original stair shape of 5 by 5 is ½ the area of the rectangle 6 by 5. In general terms this is ½ n(n-1).

Fig. 12 Figurate numbers: Number shapes and area formulae

V. VARIATIONS: FOCUSING ON THE CONTENT, PROCESSES AND MODELING IDEAS INVOLVED

With an enrichment of Grade 7 and 8 combined groups of students, pairs of students were assigned to solve one of the three problems — the stair shapes problem, the handshake problem and real championship problems. For a sample work sheet please refer to Figure 13. The magnitudes of the numbers were similarly small for the first prompts. Follow-up prompts utilized bigger numerical values to encourage generalization of rules and refining of strategies. Students shared the solutions in a whole group in the last quarter of the lesson. The whole activity took one 50-minute lesson.

In another class, a mixed ability class with a good number of students identified to be on modified education plans (i.e., Individual Education Plans) the teacher paired students in homogeneous groups according to their abilities. Taking championship task followed by the handshake task to be more complicated, she offered tasks by level of difficulty. The activity took two 50-minute lessons. This teacher assigned specific writing ink colors at different stages — understanding the problem, solving the problem, during interaction with teacher and during whole group conferencing on solutions. Other such planning details to facilitate the success of the lesson include use of mathematical organizers and highly organized resources (see for example notes on the teacher's role in teaching through problem solving, [28]). These details are as much about reducing the possibility of invalid actions as they are about increasing the possibility of successful student engagement with the problem.

In a preservice education class, teacher candidates worked individually before working with an elbow partner. They then shared solutions in the whole group. One hour was sufficient time to share solution strategies and to address related curriculum, pedagogical and assessment issues.

Variations of a problem are conventionally offered to students in follow-up, consolidation exercises, or for enrichment purposes. Variations based on the handshake problem, as shown in textbooks and online lesson plans such as those at LessonPlanet.com, are thought of in terms of changing the context say to contexts of social networking — friendship requests or phone calls among class members, of spread of seasonal infections, or of physical networks — roads directly connecting cities in a region. In [29] the illuminations lesson utilizes the Supreme Court context. A teacher, studied in [30], asked her students to explore number of gifts received by one's *true love*, when on each day to Christmas the true love receives an accumulation of gifts. Reference [31] sets the problem in the context of counting the number of streamers required to connect each child to each other in a group of 6 children.

Modifications of using lower magnitudes of numbers, say for a party of four people, work well with students who need such accommodation or in the lower grades where the intent of the lesson is not to reach a general solution. In [29] a follow-up question centered on finding the number of people at a party at which there were 45 handshakes and to show how students knew the answer. Consolidation exercises in McGraw-Hill Ryerson Grade 9 involve a set of problems to be solved using a similar strategy by a visual representation. Enrichment activities could involve changing the nature of shaking hands to, say, families present not shaking hands among themselves. The [29] extension involves examining Gauss's method for finding sums of numbers, which relates well to an early grades activity making tens or hundreds by paring single or double digit numbers respectively. A connection could be made to the math story of successive doubling say of one grain of rice or of a one-dollar payment. I have asked teacher candidates to make connections to triangular (and square) numbers and to the consecutive sums task, a task we would have worked on months earlier. Some teacher candidates also make connections to rows in Pascal's triangle.

Working on a set of variants of a rich problem in the same lesson is one way of matching the pedagogical requirements of teaching through problem solving at three levels, that of planning, of offering the task and of sharing the solution strategies. Shaking hands, stacking stairs, and graphically organizing tournaments each lend themselves to specific solution strategies. For example, in one classroom, the table of leagues shown in Figure 11 was more common among students that solved the championship problem and less common with those who solved the stair shapes problem. The polygon diagonal approach in Figure 5 was common among both the handshake and championship problem. Figurate numbers such as triangular and rectangular numbers in figures 11 and 12 were common among students who solved the stair shape and handshake problem. The connection to Gauss summation of a sequence although rarely re-invented by learners makes much sense in the context of stair shapes stacking.

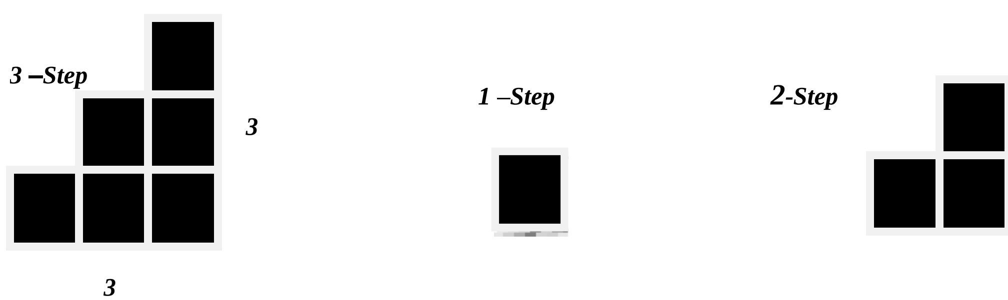
During whole group conferencing about the problems and their solutions, students make connections among the solutions, contexts and representations. One teacher candidate was very excited to relate the sum of the first five natural numbers to the area of the rectangle of dimensions 5 by 6 and, hence, to area of the stair shape of dimension 5 by 5. Classroom experiences with this approach of offering a problem together with its variants suggest that the mathematics ideas involved in terms of the variety of models, representations and tools, and the depth of connections and extensions made is raised when students share solution strategies. Students experience the lesson as an exploration of the mathematics involved in the problems rather than as a lesson in which problems are solved. They focus more on the mathematics involved.

You are invited to participate in the Mr. HS Mathematics Problem Solving Conference. As a member of Mr. HS 7-8 team, you need to prepare your answers to the following problem. Based on your problem-solving experience, you will be expected to share your problem solution with other members of the 7-8 teams. Be sure to keep all copies of your problem-solving attempts to share with the conference chair, Dr. SSP.

Problem A: Stair Shapes Problem

A factory produces stair-shaped pieces of plastic with different dimensions and number of steps.

For example, it produces a three-step stair-shaped piece. A one-step stair-shaped piece is one piece and has dimension 1 by 1, a two-step stair-shaped piece is 2 pieces and has dimension 2 by 2. An n-step piece is n units long and n units wide as shown in the figure below:



1) What is the dimension of a piece that is 5-steps? And how many unit squares does it contain?

2) If you needed to deliver many pieces of the same dimension, how would you pack them?

3) Would your way of packing them work for pieces that are 24 steps? And how many unit squares would such a piece contain? How about for 99 steps pieces?

4) See if you can find at least one alternative way to solve the problem of finding the total number pieces in a stair shaped piece. (You may use words, symbols and pictures to explain your way)

5) Bonus Question: If one 1 square unit of plastic weighs 5 grams. What would be the weight for a 6-steps piece? 24- Steps?

Be sure to show all of your work and keep all rough copies to show the conference chair.

Note: The worksheets for the Handshake and Real Championship problems mirror the instructions in this worksheet.

That is to say both worksheets contain questions on: solving the problem for a party of or tournament of 6 (or, that the number 6 here mirrors the solution for a 5-step stair shape); solving the problem one other way, and a prompt to check the method to see if it works for a bigger number say, 24, or, even a larger number, 100. Finally, a bonus question in the form of application or extension, say to Gauss's summation or to another problem that students would have solved previously that involves related mathematics.

Fig. 13 An activity sheet for the stair shapes problem

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