1-11-2019

The Duals of *-Operator Frames for End*A(H)

Abdelkrim Bourouihiya
Nova Southeastern University, ab1221@nova.edu

M. Rossafi
University of Ibn Tofail - Kenitra, Morocco

H. Labrigui
University of Ibn Tofail - Kenitra, Morocco

A. Touri
University of Ibn Tofail - Kenitra, Morocco

THE DUALS OF $*$-OPERATOR FRAMES FOR $End_{\mathcal{A}}^{*}(H)$

A. BOUROUIHIYA[2], M. ROSSAFI[1*], H. LABRIGUI [1] and A. TOURI[1]

Abstract. Frames play significant role in signal and image processing, which leads to many applications in differents fields. In this paper we define the dual of $*$-operator frames and we show their propreties obtained in Hilbert $\mathcal{A}$-modules and we establish some results.

Frame theory is recently an active research area in mathematics, computer science, and engineering with many exciting applications in a variety of different fields. They are generalizations of bases in Hilbert spaces. Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [5] for study of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [4], and popularized from then on. Hilbert C^*-modules is a generalization of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the eld of complex numbers. The aim of this papers is to study the dual of $*$-operator frames.

The paper is organized as follows:

In section 2, we briefly recall the definitions and basic properties of operator frame and $*$-operator frame in Hilbert C^*-modules.

In section 3, we introduce the dual $*$-operator frame, the $*$-operator frame transform and the $*$-frame operator.

In section 4, we investigate tensor product of Hilbert C^*-modules, we show that tensor product of dual $*$-operator frames for Hilbert C^*-modules $\mathcal{H}$ and $\mathcal{K}$, present a dual $*$-operator frames for $\mathcal{H} \otimes \mathcal{K}$.

1. Preliminaries

Let I be a countable index set. In this section we briefly recall the definitions and basic properties of C^*-algebra, Hilbert C^*-modules, frame, $*$-frame in Hilbert C^*-modules. For information about frames in Hilbert spaces we refer to [1]. Our reference for C^*-algebras is [3, 2]. For a C^*-algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is positive ($a \geq 0$) if $a = a^*$ and $sp(a) \subset \mathbf{R}^+$. $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. [6]

A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, if there exist positive

Date: Received: 15/09/2018;

*Corresponding author.

2010 *Mathematics Subject Classification.* Primary 42C15; Secondary 46L06.

Key words and phrases. $*$-frame, $*$-operator frame, C^*-algebra, Hilbert $\mathcal{A}$-modules.

constants $A, B > 0$ such that

$$A\langle x,x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_ix, T_ix\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, \forall x \in \mathcal{H}. \tag{1.1}$$

The numbers A and B are called lower and upper bound of the operator frame, respectively. If $A = B = \lambda$, the operator frame is λ-tight. If $A = B = 1$, it is called a normalized tight operator frame or a Parseval operator frame.

Definition 1.2. [6]

A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an $*$-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$, if there exists two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle x,x\rangle_{\mathcal{A}}A^* \leq \sum_{i\in I}\langle T_ix, T_ix\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}B^*, \forall x \in \mathcal{H}. \tag{1.2}$$

The elements A and B are called lower and upper bounds of the $*$-operator frame, respectively. If $A = B = \lambda$, the $*$-operator frame is λ-tight. If $A = B = 1_{\mathcal{A}}$, it is called a normalized tight $*$-operator frame or a Parseval $*$-operator frame. If only upper inequality of hold, then $\{T_i\}_{i\in i}$ is called an $*$-operator Bessel sequence for $End^*_{\mathcal{A}}(\mathcal{H})$.

If the sum in the middle of (2.1) is convergent in norm, the operator frame is called standard. If only upper inequality of (2.1) hold, then $\{T_i\}_{i\in I}$ is called an operator Bessel sequence for $End^*_{\mathcal{A}}(\mathcal{H})$.

2. Dual of $*$-operator Frame for $End^*_{\mathcal{A}}(\mathcal{H})$

We begin this section with the following definition.

Definition 2.1.

Let $\{T_i\}_{i\in I} \subset End^*_{\mathcal{A}}(\mathcal{H})$ be an $*$-operator frame for $\mathcal{H}$. If there exists an $*$-operator frame $\{\Lambda_i\}_{i\in I}$ such that $x = \sum_{i\in I} T_i^*\Lambda_i x$ for all $x \in \mathcal{H}$. then the $*$-operator frames $\{\Lambda_i\}_{i\in I}$ is called the duals $*$-operator frames of $\{T_i\}_{i\in I}$.

Example 2.2.

Let $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module over itself, let $\{f_j\}_{j\in J}$ be an $*$-frame for $\mathcal{A}$.

We define the adjointable $\mathcal{A}$-module map $\Lambda_{f_j} : \mathcal{A} \to \mathcal{A}$ by $\Lambda_{f_j} f = \langle f, f_j\rangle$. Clearly, that $\{\Lambda_{f_j}\}_{j\in J}$ is an $*$-operator frame for $\mathcal{A}$.

Theorem 2.3.

Every $*$-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ has a dual $*$-operator frame.

Proof.

Let $\{T_i\}_{i\in I} \subset End^*_{\mathcal{A}}(\mathcal{H})$ be an $*$-operator for $End^*_{\mathcal{A}}(\mathcal{H})$, with $*$-operator S.

We see that $\{T_iS^{-1}\}_{i\in I}$ is an $*$-operator frame.

Or, $\forall x \in \mathcal{H}$ we have :

$$Sx = \sum_{i\in I} T_i^*T_ix$$

then

$$x = \sum_{i\in I} T_i^* T_i S^{-1} x$$

hence $\{T_i S^{-1}\}_{i\in I}$ is a dual $*$-operator frame of $\{T_i\}_{i\in I}$.
It is called the canonique dual $*$-operator frame of $\{T_i\}_{i\in I}$. □

Remark 2.4.
Assume that $T = \{T_i\}_{i\in I}$ is an $*$-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with analytic operator R_T and $\tilde{T} = \{\tilde{T}_i\}_{i\in I}$ is a dual $*$-operator frame of T with analytic operator R_T, then for any $x \in \mathcal{H}$ we have:

$$x = \sum_{i\in I} T_i^* \tilde{T}_i x = R_T^* R_{\tilde{T}} x$$

this show that every element of H can be reconstructed with a $*$-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ and its dual.

Theorem 2.5.
Let $\{\Lambda_i\}_{i\in I}$ be an $*$-operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ with $*$-operator frame transform θ, the $*$-operator frame S and the canonical dual $*$-operator frames $\{\tilde{\Lambda}_i\}_{i\in\mathbb{J}}$.
Let $\{\Omega_i\}_{i\in I}$ be an arbitrary dual $*$-operator frame of $\{\Lambda_i\}_{i\in I}$ with the $*$-operator frame transform η; then the folowing statements are true:

(1) *$\theta^*\eta = I$.*
(2) *$\Omega_i = \Pi_i \eta$ for all $i \in I$.*
(3) *If $\eta' : \mathcal{H} \longmapsto l^2(\mathcal{H})$ is any adjointable right inverse of θ^* then $\{\Pi_i \eta'\}_{i\in I}$ is a dual $*$-operator frame of $\{\Lambda\}_{i\in I}$ with the operator frames transform η'.*
(4) *The $*$-operator frame S_Ω of $\{\Omega_i\}_{i\in I}$ is equal to $S^{-1} + \eta^*(I - \theta S^{-1}\theta^*)\eta$.*
(5) *Every adjointable right inverse η' of θ^* is the forme :*
 $\eta' = \theta S^{-1} + (I - \theta S^{-1}\theta^*)\psi$ for some adjointable map $\psi : \mathcal{H} \longmapsto l^2(\mathcal{H})$ and vice versa.
(6) *There exist a $*$-bessel operator $\{\Delta_j\}_{j\in J} \in End_{\mathcal{A}}^{*}(\mathcal{H})$ $\{\Delta\}_{i\in\mathbb{I}}$ whose $*$-operator frame transform is η and yields is η and yields*
 $\Omega_j = \tilde{\Lambda}_j + \Delta_j - \sum_{k\in J} \tilde{\Lambda}_j \Lambda_k^* \Delta_k, \forall j \in J$

Proof.
(1) For $f, g \in \mathcal{H}$ we have :

$$\begin{aligned}
\langle \theta^*\eta f, g\rangle &= \langle \eta f, \theta g\rangle \\
&= \langle \sum_{i\in I} \Omega_i f, \sum_{i\in I} \Lambda_i g\rangle \\
&= \sum_{i\in I} \langle \Omega_i f, \Lambda_i g\rangle = \sum_{i\in I} \langle \Lambda^* \Omega_i f, g\rangle \\
&= \langle \sum_{i\in I} \Lambda^* \Omega_i f, g\rangle = \langle f, g\rangle
\end{aligned}$$

then $\theta^*\eta = I$.
(2) The proof is clear from the definition

(3) Since $\eta^{'}$ is adjointable, it follows from prop3.1 that $\{\Pi_i\eta^{'}\}_{i\in I}$ is a $*$-bessel sequence in $\mathcal{H}$.

Also, since $(\eta^{'})^*\theta = I$; $(\eta^{'})^*$ is surjective, by lemme 2.7, for $f \in \mathcal{H}$ we have:

$$||(\eta^{'})^*\eta)^{-1}||^{-1}\langle f, f\rangle \leq \langle \eta^{'} f, \eta^{'} f\rangle = \sum_{i\in I}\langle \pi_i\eta' f, \pi_i\eta' f\rangle$$

clearly, $\eta^{'}$ is the pre-frame $*$-operator frame transform $\{\Pi_i\eta^{'}\}_{i\in I}$

(4)

$$\begin{aligned} S_\Omega &= \eta^*\eta \\ &= \eta^*\theta S^{-1} + \eta^*\eta - \eta^*\theta S^{-1} \\ &= \eta^*\theta S^{-1} + \eta^*\eta - \eta^*\theta S^{-1}\theta^*\eta \\ &= \eta^*\theta S^{-1} + \eta^*(I - \theta S^{-1}\theta^*)\eta \end{aligned}$$

(5) If $\eta^{'}$ is such a right inverse of θ, then

$$\theta S^{-1} + (I - \theta S^{-1}\theta^*)\eta^{'} = \theta S^{-1} + \eta' - \theta S^{-1}\theta^*\eta^{'} = \theta S^{-1} + \eta' - \theta S^{-1} I = \eta'$$

(6) Let $\{\Delta_i\}_{i\in I}$ be an $*$-operator bessel sequence for $End_{\mathcal{A}}^*(\mathcal{H})$ with the preframe operator η. For $i \in I$, let $\Omega_i = \tilde{\Lambda_i} + \Delta_i - \sum_{k\in I}\tilde{\Lambda_i}\Lambda_k^*\Delta_k$ Let S and θ be the $*$-frame operator and the preframe operator of $\{\Delta_i\}_{i\in I}$, resp. we define the linear operator $\psi : \mathcal{H} \longmapsto l^2(\mathcal{H})$ by $\psi f = (\Omega_i f)_{i\in I}$. clearly, ψ is adjointable, for every $i \in I$, we have

$$\begin{aligned} \pi_i\psi &= \Omega_i \\ &= \Lambda_i S^{-1} + \Delta_i - \Lambda_i S^{-1}\sum_{k\in I}\Lambda_k^*\Delta_k \\ &= \Lambda_i S^{-1} + \pi_i\eta - \sum_{k\in I}\Lambda_k^*\Delta_k \\ &= \pi_i\theta S^{-1} + \pi_i\eta - \pi_i\theta S^{-1}\theta^*\eta \\ &= \pi_i(\theta S^{-1} + \eta - \theta S^{-1}\theta^*\eta) \end{aligned}$$

then

$$\psi = \theta S^{-1} + \eta - \theta S^{-1}\theta^*\eta$$

by parts (3) and (5) of the theorem; $\{\Omega_i\}_{i\in I}$ becomes a dual of $*$-operator $\{\Lambda_i\}_{i\in I}$

□

Example 2.6.

Let $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module over itself, let $\{f_j\}_{j\in J} \subset \mathcal{A}$.

We define the adjointable $\mathcal{A}$-module map $\Lambda_{f_j} : \mathcal{A} \to \mathcal{A}$ with $\Lambda_{f_j}.f = \langle f, f_j\rangle$, clearly $\{f_j\}_{j\in J}$ is a $*$-frame in $\mathcal{A}$ if and only if $\{\Lambda_{f_j}\}_{j\in J}$ is a $*$-operator frame in $\mathcal{A}$.

In the folowing, we study the duals of such $*$-operator frame.

(a) Let $\{g_j\}_{j\in J} \subset \mathcal{A}$ for all $f \in \mathcal{A}$:

$$\sum_{j\in J}\Lambda_{g_j}^{*}\Lambda_{f_j}f=\sum_{j\in J}\langle f,f_j\rangle g_j=\sum_{j\in J}\langle f,g_j\rangle f_j=\sum_{j\in J}\Lambda_{f_j}^{*}\Lambda_{g_j}f.$$

Therefore, $\{g_j\}_{j\in J}$ is a dual $*$-frame of $\{f_j\}_{j\in J}$ if and only if $\{\Lambda_{g_j}\}_{j\in J}$; is a dual $*$-operator of $\{\Lambda_{f_j}\}_{j\in J}$

(b) Let S and S_Λ be the $*$-frame operators of $\{f_j\}_{j\in J}$ and $\{\Lambda_{f_j},\mathcal{A}\}_{j\in J}$ respectively.

For all $f\in\mathcal{A}$ we have:

$$\sum_{j\in J}\langle f,f_j\rangle f_j=\sum_{j\in J}ff_j^{*}f_j=\sum_{j\in J}\langle\langle f,f_j\rangle,f_j^{*}\rangle=\sum_{j\in J}\Lambda_{f_j}^{*}\Lambda_{f_j}f.$$

It follows that $S=S_\Lambda$

(c) It is clearly to see that $\{h_j\}_{j\in J}\subset\mathcal{A}$ is an $*$-bessel sequence if and only if $\{\Lambda_{h_j},\mathcal{A}\}_{j\in J}$ is an $*$-bessel operator.

(d) for a $*$-bessel sequence $\{h_j\}_{j\in J}$ we define

$$g_j=S^{-1}f_j+h_j-\sum_{k\in J}\langle S^{-1}f_j,f_k\rangle h_k$$

then the sequence $\{g_j\}_{j\in J}$ is a dual $*$-frame of $\{f_j\}_{j\in J}$.

By the last theorem, the sequence $\{\Gamma_j\}_{j\in J}$ is a dual $*$-operator frame of $\{\Lambda_{f_j}\}_{j\in J}$, where

$$\Gamma_j=\tilde{\Lambda}_{f_j}+\Lambda_{h_j}+\sum_{k\in J}\tilde{\Lambda}_{f_j}\Lambda_{f_k}^{*}\Lambda_{h_k},\forall j\in J$$

now we clain that $\Gamma_j=\Lambda_{g_j}$

In fact, $\forall f \in \mathcal{A}$ we have

$$\begin{aligned}
\Gamma_j f &= \tilde{\Lambda}_{f_j} f + \Lambda_{h_j} f - \sum_{k\in J} \tilde{\Lambda}_{f_j} \Lambda^*_{f_k} \Lambda_{h_k} f \\
&= \Lambda_{f_j} S^{-1} f + \Lambda_{h_j} f - \sum_{k\in J} \Lambda_{f_j} S^{-1} \Lambda^*_{f_k} \langle f, h_k\rangle \\
&= \langle S^{-1} f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J} \langle S^{-1} \Lambda^*_{f_k} \langle f, h_k\rangle, f_j\rangle \\
&= \langle S^{-1} f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J} \langle S^{-1} \Lambda^*_{f_k} \Lambda_{h_k} f, f_j\rangle \\
&= \langle S^{-1} f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J} \langle S^{-1} \Lambda_{h_k} f f_k, f_j\rangle \\
&= \langle S^{-1} f, f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J} \langle S^{-1} f h^*_k f_k, f_j\rangle \\
&= \langle f, S^{-1} f_j\rangle + \langle f, h_j\rangle - \sum_{k\in J} \langle f h^*_k f_k, S^{-1} f_j\rangle \\
&= \langle f, S^{-1} f_j + h_j\rangle - \sum_{k\in J} \langle f, f^*_k h_k S^{-1} f_j\rangle \\
&= \langle f, S^{-1} f_j + h_j - \sum_{k\in J} \langle S^{-1} f_j, f_k\rangle h_k\rangle \\
&= \langle f, g_j\rangle = \Lambda_{g_j} f.
\end{aligned}$$

therefore, every $*$-operator frame of $\{\Lambda_{f_j}\}_{j\in J}$ has the form :

$$\tilde{\Lambda}_{f_j} + \Lambda_{h_j} - \sum_{k\in J} \tilde{\Lambda}_{f_j} \Lambda^*_{f_k} \Lambda_{h_k}$$

where $\{h_j\}_{j\in J}$ is a $*$-bessel sequence in $\mathcal{A}$.

3. **Tensor product**

In this section, we study the tensor product of the duals $*$-operator frames.

Theorem 3.1.
Let $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert C^*-modules over unitary C^*-Algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, let $\{\Lambda_i\}_{i\in I} \subset End^*_{\mathcal{A}}(\mathcal{H})$ and $\{\Gamma_j\}_{j\in J} \subset End^*_{\mathcal{B}}(\mathcal{K})$ are an $*$-operators frames.
If $\{\tilde{\Lambda}_i\}_{i\in I}$ is a dual of $\{\Lambda_i\}_{i\in I}$ and $\{\tilde{\Gamma}_j\}_{j\in J}$ is a dual of $\{\Gamma_j\}_{j\in J}$ then $\{\tilde{\Lambda}_i \otimes \tilde{\Gamma}_j\}_{i\in I, j\in J}$ is a dual $*$-operator frame of $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$.

Proof.

Let $x \in \mathcal{H}$ and $y \in \mathcal{K}$, we have :

$$\begin{aligned}\sum_{i\in I,j\in J}(\Lambda_i\otimes\Gamma_j)^*(\tilde{\Lambda_i}\otimes\tilde{\Gamma_j})(x\otimes y) &= \sum_{i\in I,j\in J}(\Lambda_i^*\otimes\Gamma_j^*)(\tilde{\Lambda_i}x\otimes\tilde{\Gamma_j}y)\\ &= \sum_{i\in I,j\in J}(\Lambda_i^*\tilde{\Lambda_i}x\otimes\Gamma_j^*\tilde{\Gamma_j}y)\\ &= \sum_{i\in I}\Lambda_i^*\tilde{\Lambda_i}x\otimes\sum_{j\in J}\Gamma_j^*\tilde{\Gamma_j}y\\ &= x\otimes y\end{aligned}$$

then

$$\sum_{i\in I,j\in J}(\Lambda_i\otimes\Gamma_j)^*(\tilde{\Lambda_i}\otimes\tilde{\Gamma_j}) = I$$

hence $\{\tilde{\Lambda_i}\otimes\tilde{\Gamma_j}\}_{i\in I,j\in J}$ is a dual $*$-operator frames of $\{\Lambda_i\otimes\Gamma_j\}_{i\in I,j\in J}$. □

Corollary 3.2.

Let $(\Lambda_{ij})_{0\leq i\leq n;j\in J}$ be a family of $*$-operator and $(\tilde{\Lambda}_{ij})_{0\leq i\leq n;j\in J}$ its their dual, then $(\tilde{\Lambda}_{0j}\otimes\tilde{\Lambda}_{1j}\otimes ...\otimes\tilde{\Lambda}_{nj})_{j\in J}$ is a dual of $(\Lambda_{0j}\otimes\Lambda_{1j}\otimes ...\otimes\Lambda_{nj})_{j\in J}$.

Acknowledgment. The authors would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help to improve the paper.

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[1]Department of Mathematics, University of Ibn Tofail, B.P. 133, Kenitra, Morocco
E-mail address: rossafimohamed@gmail.com; hlabrigui75@gmail; touri.abdo68@gmail.com

[2]Department of Mathematics, Nova Southeastern University, 3301 College Avenue, Fort Lauderdale, Florida, USA
E-mail address: ab1221@nova.edu