

2016

# Decomposition of Finite Schmidt Rank Bounded Operators on the Tensor Product of Separable Hilbert Spaces

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## NSUWorks Citation

Bourouihiya, Abdelkrim, "Decomposition of Finite Schmidt Rank Bounded Operators on the Tensor Product of Separable Hilbert Spaces" (2016). *Mathematics Faculty Articles*. 209.  
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## DECOMPOSITION OF FINITE SCHMIDT RANK BOUNDED OPERATORS ON THE TENSOR PRODUCT OF SEPARABLE HILBERT SPACES

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ABSTRACT. Inverse formulas for the tensor product are used to develop an algorithm to compute Schmidt decompositions of Finite Schmidt Rank (FSR) bounded operators on the tensor product of separable Hilbert spaces. The algorithm is then applied to solve inverse problems related to the tensor product of bounded operators. In particular, we show how properties of a FSR bounded operator are reflected by the operators involved in its Schmidt decomposition. These properties include compactness of FSR bounded operators and convergence of sequences whose terms are FSR bounded operators.

### 1. INTRODUCTION

For this paper, all Hilbert spaces are assumed to be separable and denoted by  $\mathcal{H}$  or  $\mathcal{K}$ . The norm is the usual norm that is induced by the inner product of the Hilbert space [5]. The space of bounded operators:  $\mathcal{H} \rightarrow \mathcal{K}$  is denoted  $\mathcal{B}(\mathcal{H}, \mathcal{K})$ .

**Definition 1.1.** We say that  $F \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$  is a *finite Schmidt rank* (FSR) if it can be written in the form

$$(1.1) \quad F = \sum_{k=1}^r F_{1,k} \otimes F_{2,k},$$

where  $\{F_{1,k}\}_{k=1}^r \subset \mathcal{B}(\mathcal{H}_1, \mathcal{K}_1)$  and  $\{F_{2,k}\}_{k=1}^r \subset \mathcal{B}(\mathcal{H}_2, \mathcal{K}_2)$ . If  $r$  is the minimum number such that  $F$  can be written in form (1.1), we say  $r$  is the *Schmidt rank* of  $F$ , we denote  $\text{rank}_{\otimes}(F) = r$ , and call equality (1.1) a *Schmidt decomposition* of  $F$ .

Singular value decomposition (SVD) is often used to compute Schmidt decompositions of matrices [10]. For the infinite dimensional case, the SVD of a bounded operator exists if and only if the operator is compact [9]. However, to the best of our knowledge, there is no published algorithm to compute a Schmidt decomposition in form (1.1) of a FSR bounded operator  $A$  in the infinite dimensional case, even if  $A$  is assumed to be compact.

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Received December 25, 2016.

2010 *Mathematics Subject Classification.* Primary 47A80, 46C05, 47B40, 47L05, 47B10.

*Key words and phrases.* Hilbert spaces; Tensor product of operators; Schmidt decomposition; Compact Operators; Inverse problems.

Section 2 uses inverse formulas of the tensor product to develop a new algorithm to compute Schmidt decompositions of FSR bounded operators on the tensor product of separable Hilbert spaces with finite or infinite dimensions. The need for infinite dimensional Schmidt decompositions is not only theoretical, but it is also for applications. For example, there are states in quantum physics acting on infinite dimensional Hilbert spaces. Meanwhile, Schmidt decomposition in the finite case is a tool to study some physics phenomena such as quantum entanglement [4]. Other applications involve integral operators acting on infinite dimensional Hilbert spaces. These operators are traditionally discretized before their decompositions. The process of discretization eventually induces numerical imprecisions [3].

In [6], C.S. Kubrusly and P. Vieira, proved that the tensor product of two uniformly (strongly) convergent sequences of bounded operators is a uniformly (strongly) convergent sequence. They also proved that the converse holds in case of convergence to zero under the semigroup assumption. The following theorem that is proved in Section 3, states the converse in case of convergence to a nonzero operator.

**Theorem 1.2.** *Let  $\{F^n\}$  be a sequence of bounded operators that is uniformly convergent to  $F \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2) \setminus \{0\}$ .*

- (i) *If  $\text{rank}_{\otimes}(F^n) \leq M$  for each  $n > 0$ , then  $\text{rank}_{\otimes}(F) \leq M$ .*
- (ii) *Assume that  $F$  has a Schmidt decomposition in form (1.1) and*

$$(1.2) \quad F^n = \sum_{m=1}^M F_{1,m}^n \otimes F_{2,m}^n,$$

*where the sequences  $\{F_{1,m}^n\}_{n>0}$  and  $\{F_{2,m}^n\}_{n>0}$  are bounded for each  $m = 1, \dots, M$ . Then there are  $c_{(i,k)}^1, \dots, c_{(i,k)}^M \in \mathbb{C}$  such that the sequence*

$$\sum_{m=1}^M c_{(i,k)}^m F_{i,m}^n$$

*has a subsequence that converges to  $F_{i,k}$  for each  $(i, k) \in \{1, 2\} \times \{1, \dots, r\}$ .*

- (iii) *Statements (i) and (ii) hold if uniform convergence is replaced with strong convergence.*

It is known that the tensor product of two compact operators is compact. J. Zanni and C. S. Kubrusly [11] recently proved that the converse of this property also holds. The main theorem of Section 4 proves that the Schmidt decomposition of a FSR compact operator has each of its term as a tensor product of compact operators. The same result is also shown for the class of Hilbert-Schmidt operators, the class of nuclear operators, and all Schatten classes of operators.

## 2. SCHMIDT DECOMPOSITIONS OF FSR BOUNDED OPERATORS

### 2.1. Preliminaries and notations

Most definitions and results in this subsection can be found in [5]. We denote by  $\mathcal{H}'$  the dual of  $\mathcal{H}$ . If  $x \in \mathcal{H}$ , we denote by  $x^*$  the linear form  $x^*(y) = \langle y, x \rangle$  for all

$y \in \mathcal{H}$ . The transpose of the linear mapping  $F: \mathcal{H} \rightarrow \mathcal{K}$  is  $F^t: \mathcal{K}' \rightarrow \mathcal{H}'$ , where  $F^t(y^*) = y^* \circ F$  for all  $y^* \in \mathcal{K}'$ .

The Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_2$  can be interpreted as the Hilbert space  $\mathcal{L}_2(\mathcal{H}'_1, \mathcal{H}_2)$ , the space of Hilbert-Schmidt operators. The tensor product of  $F_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{K}_1)$  and  $F_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{K}_2)$  can be defined by  $F_1 \otimes F_2(H) = F_2 H F_1^t$ , for each  $H \in \mathcal{L}_2(\mathcal{H}'_1, \mathcal{H}_2)$ .

For all  $h, k \in \mathcal{H}$ , we define the one-rank operator  $E_{(h,k)}(x) = \langle x, k \rangle h$  for all  $x \in \mathcal{H}$ .

## 2.2. An algorithm to compute Schmidt decompositions

The following Proposition collects results from [2].

**Proposition 2.1.** *Let  $u = u_1 \otimes u_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$  and let  $v = v_1 \otimes v_2 \in \mathcal{K}_1 \otimes \mathcal{K}_2$ . We define the bilinear operator*

$$\begin{aligned} \mathcal{P}_{u,v}: \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)^2 &\longrightarrow \mathcal{B}(\mathcal{H}_1, \mathcal{K}_1) \otimes \mathcal{B}(\mathcal{H}_2, \mathcal{K}_2) \\ (F, G) &\longrightarrow V^{v_2} F U^{u_2} \otimes V_{v_1} G U_{u_1}, \end{aligned}$$

where

$$\begin{aligned} U_{u_1}: \mathcal{H}_2 &\rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2, & \text{given by } U_{u_1}(x_2) &= u_1 \otimes x_2, \\ U^{u_2}: \mathcal{H}_1 &\rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2, & \text{given by } U^{u_2}(x_1) &= x_1 \otimes u_2, \\ V_{v_1}: \mathcal{K}_1 \otimes \mathcal{K}_2 &\rightarrow \mathcal{K}_2, & \text{given by } V_{v_1}(H) &= H(v_1^*), \\ V^{v_2}: \mathcal{K}_1 \otimes \mathcal{K}_2 &\rightarrow \mathcal{K}_1, & \text{given by } V^{v_2}(H) &= H^t(v_2^*). \end{aligned}$$

(i) *The operators  $V_{v_1}, V^{v_2}, U_{u_1}, U^{u_2}, \mathcal{P}_{u,v}$ , are bounded and we have*

$$\|V_{v_1}\| = \|v_1\|, \quad \|V^{v_2}\| = \|v_2\|, \quad \|\mathcal{P}_{u,v}\| \leq \|u\| \|v\|.$$

(ii) *The mapping  $\mathcal{D}_{u,v}(F) = \mathcal{P}_{u,v}(F, F)$  is continuous and we have*

$$\|\mathcal{D}_{u,v}(F) - \mathcal{D}_{u,v}(G)\| \leq \|u\| \|v\| (\|F\| + \|G\|) \|F - G\|.$$

(iii)  *$\text{rank}_{\otimes}(F) = 1$  if and only if  $\mathcal{D}_{u,v}(F) = \langle F(u), v \rangle F$ .*

**Definition 2.2.** We say that  $\{F_{1,k} \otimes F_{2,k}\}_{k=1}^m$  is a *finite minimal system* (FMS) in  $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$  if the system  $\{F_{i,k}\}_{k=1}^m$  is independent for each  $i \in \{1, 2\}$ .

**Theorem 2.3.** *Let  $F \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$ . Equality (1.1) is a Schmidt decomposition of  $F$  if and only if  $\{F_{1,k} \otimes F_{2,k}\}_{k=1}^r$  is a FMS.*

*Proof.* If  $\{F_{1,k} \otimes F_{2,k}\}_{k=1}^r$  is not a FMS, it is easy to deduce that  $\text{rank}_{\otimes}(F) < r$ . For the converse, it suffices to prove by induction on  $n$  the following claim.

$C(n)$ : For each FMS  $\{F_{1,k} \otimes F_{2,k}\}_{k=1}^r$  for which  $r > n$ , we have

$$(2.1) \quad \text{rank}_{\otimes} \left( \sum_{k=1}^r F_{1,k} \otimes F_{2,k} \right) > n.$$

$C(0)$  is obvious. Now, assume that  $C(n-1)$  holds for some  $n > 0$  and (2.1) does not hold for some FMS  $\{F_{1,k} \otimes F_{2,k}\}_{k=1}^r$  for which  $r > n$ . Therefore,

$$(2.2) \quad \sum_{k=1}^{n-1} G_{1,k} \otimes G_{2,k} = \sum_{k=1}^r F_{1,k} \otimes F_{2,k} - G_{1,r} \otimes G_{2,r},$$

where  $\{G_{1,k} \otimes G_{2,k}\}_{k=1}^n$  is a FMS, and so  $\{F_{1,1} \otimes F_{2,1}, \dots, F_{1,r} \otimes F_{2,r}, G_{1,r} \otimes G_{2,r}\}$  is not a FMS by using  $C(n-1)$ . Consequently, we may assume WLG that  $G_{1,r} \in \text{span}\{F_{1,1}, \dots, F_{1,r}\}$ , and so we can find  $c_1, \dots, c_r \in \mathbb{C}$  to rewrite (2.2) as

$$(2.3) \quad \sum_{k=1}^{n-1} G_{1,k} \otimes G_{2,k} = \sum_{k=1}^r F_{1,k} \otimes (F_{2,k} - c_k G_{2,r}).$$

On the one hand,  $C(n-1)$  implies  $\{F_{1,k} \otimes (F_{2,k} - c_k G_{2,r})\}_{k=1}^r$  is not a FMS, and so the system  $\{H_{2,k} = F_{2,k} - c_k G_{2,r}\}_{k=1}^r$  is dependent. On the other hand, the dimension of  $\text{span}\{H_{2,k}\}_{k=1}^r \geq r-1$ . Hence, we may assume WLG that  $H_{2,r} \in \text{span}\{H_{2,k}\}_{k=1}^{r-1}$  and  $\{H_{2,k}\}_{k=1}^{r-1}$  is an independent system. Using the last fact and (2.3), we can find  $a_1, \dots, a_{r-1} \in \mathbb{C}$  such that

$$\sum_{k=1}^{n-1} G_{1,k} \otimes G_{2,k} = \sum_{k=1}^{r-1} (F_{1,k} + a_k F_{1,r}) \otimes H_{2,k}.$$

The last equality contradicts  $C(n-1)$  since  $\{(F_{1,k} + a_k F_{1,r}) \otimes H_{2,k}\}_{k=1}^{r-1}$  is a FMS.  $\square$

A finite dimensional version of Theorem 2.3 is proved in [7].

**Theorem 2.4.** *Let  $F \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2) \setminus \{0\}$  be a FSR. Let  $(u, v) = (u_1 \otimes u_2, v_1 \otimes v_2) \in \mathcal{H}_1 \otimes \mathcal{H}_2 \times \mathcal{K}_1 \otimes \mathcal{K}_2$ . If  $\langle F(u), v \rangle = 1$ , then*

$$\text{rank}_{\otimes}(F - \mathcal{D}_{u,v}(F)) = \text{rank}_{\otimes}(F) - 1.$$

*Proof.* Assume that  $F$  has a Schmidt decomposition in form (1.1). Using Proposition 2.1, we have

$$(2.4) \quad \mathcal{D}_{u,v}(F) = \sum_{k,l=1}^r a_{1,k} a_{2,l} F_{1,k} \otimes F_{2,l},$$

where  $a_{1,k} = \langle F_{1,k}(u_1), v_1 \rangle$  and  $a_{2,k} = \langle F_{2,k}(u_2), v_2 \rangle$  for each  $k \in \{1, \dots, r\}$ . Thus, (2.4) and the fact that  $\langle F(u), v \rangle = 1$  imply

$$(2.5) \quad \sum_{l=1}^r a_{1,l} a_{2,l} = 1,$$

and so

$$(2.6) \quad F - \mathcal{D}_{u,v}(F) = \sum_{l=2}^r \Delta_{1,l}^1 \otimes \Delta_{1,l}^2 + \sum_{k=2}^r \sum_{l=k+1}^r \Delta_{k,l}^1 \otimes \Delta_{k,l}^2,$$

where for each  $k \in \{1, \dots, r\}$  and for each  $l \in \{k+1, \dots, r\}$ ,

$$\Delta_{k,l}^1 = a_{1,l} F_{1,k} - a_{1,k} F_{1,l} \quad \text{and} \quad \Delta_{k,l}^2 = a_{2,l} F_{2,k} - a_{2,k} F_{2,l}.$$

Owing (2.5), we may assume WLOG that  $a_{1,1} \neq 0$ . If  $l > k$ , we then have  $a_{1,k}\Delta_{1,l}^1 - a_{1,1}\Delta_{k,l}^1 = a_{1,l}\Delta_{1,k}^1$ , and so

$$(2.7) \quad \begin{aligned} F - \mathcal{D}_{u,v}(F) &= \sum_{k=3}^r \Delta_{1,k}^1 \otimes \left( \Delta_{1,k}^2 - \sum_{l=k+1}^r \frac{a_{1,l}}{a_{1,1}} \Delta_{k,l}^2 + \sum_{l=2}^{k-1} \frac{a_{1,l}}{a_{1,1}} \Delta_{l,k}^2 \right) \\ &\quad + \Delta_{1,2}^1 \otimes \left( \Delta_{1,2}^2 - \sum_{l=3}^r \frac{a_{1,l}}{a_{1,1}} \Delta_{2,l}^2 \right). \end{aligned}$$

Using Theorem 2.4,  $\text{rank}_{\otimes}(F - \mathcal{D}_{u,v}(F)) \geq r - 1$  while the Schmidt rank of the right hand side of (2.7) is less or equal to  $r - 1$  and this finishes the proof.  $\square$

Based on Theorem 2.4, one can develop an algorithm to compute the Schmidt decomposition of any FSR  $F \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2) \setminus \{0\}$ . The algorithm leads to a decomposition of  $F$  after  $r$  steps if and only if  $\text{rank}_{\otimes}(F) = r$ .

*Example 2.5.* Let the matrix

$$M = \begin{pmatrix} -4 & 13 & -2 & 24 \\ -11 & 0 & -13 & -5 \\ 6 & 5 & -12 & 11 \\ 6 & -7 & -21 & 8 \end{pmatrix}$$

be identified with an operator on the Hilbert space  $\mathbb{C}^2 \otimes \mathbb{C}^2$  endowed with the basis  $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ , where  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$ .

We have  $\langle M(e_1 \otimes e_1), e_1 \otimes e_1 \rangle = -4$ . Therefore, a first term in the decomposition of  $M$  is  $-1/4M_{1,1} \otimes M_{1,2}$ , where

$$M_{1,1} = V^{e_1} M U^{e_1} = \begin{pmatrix} -4 & -2 \\ 6 & -12 \end{pmatrix}, \quad M_{1,2} = V_{e_1} M U_{e_1} = \begin{pmatrix} -4 & 13 \\ -11 & 0 \end{pmatrix}.$$

Let  $M_1 = M + 1/4M_{1,1} \otimes M_{1,2}$ . We have  $\langle M_1(e_1 \otimes e_2), e_1 \otimes e_2 \rangle = -21/2$ . Therefore, a second term in the decomposition of  $M$  is  $-2/21M_{1,2} \otimes M_{2,2}$ , where

$$\begin{aligned} M_{2,1} &= V^{e_2} M_1 U^{e_1} = \begin{pmatrix} 0 & -15/2 \\ -21/2 & 12 \end{pmatrix}, \\ M_{2,2} &= V_{e_2} M_1 U_{e_1} = \begin{pmatrix} 0 & 49/2 \\ -21/2 & -7 \end{pmatrix}. \end{aligned}$$

We have  $M_1 + 2/21M_{1,2} \otimes M_{2,2} = 0$ . Consequently,  $\text{rank}_{\otimes}(M) = 2$  and

$$M = -\frac{1}{4}M_{1,1} \otimes M_{1,2} - \frac{2}{21}M_{2,1} \otimes M_{2,2}.$$

### 3. CONVERGENT SEQUENCES OF FSR BOUNDED OPERATORS

We write  $F^n \xrightarrow{u} F$  to mean the sequence  $\{F^n\}_{n>0}$  uniformly converges to  $F$ , i.e.,  $\|F^n - F\|$  converges to zero. We write  $F^n \xrightarrow{s} F$  to mean the sequence  $\{F^n\}_{n>0}$  strongly converges to  $F$ , i.e.,  $\|F^n(x) - F(x)\|$  converges to zero for all  $x \in \mathcal{H}$ .

If two sequences converge uniformly (strongly), then the tensor product of two sequences converges uniformly (strongly) [6]. The following theorem states the converse under boundness conditions and the fact that the limit is nonzero.

**Theorem 3.1.** *Let  $\{F_1^n \otimes F_2^n\}_{n>0}$  be a sequence that is uniformly convergent to  $F \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2) \setminus \{0\}$ . Then the following statements hold:*

- (i)  $\text{rank}_\otimes(F) = 1$ .
- (ii) *If  $\{F_1^n\}_{n>0}$  and  $\{F_2^n\}_{n>0}$  are bounded and  $F = F_1 \otimes F_2$ , then there is a constant  $c \neq 0$  such that  $\{F_1^n\}_{n>0}$  has a subsequence that uniformly converges to  $F_1/c$  and  $\{F_2^n\}_{n>0}$  has a subsequence that uniformly converges to  $cF_2$ .*
- (iii) *Statements (i) and (ii) hold if uniform convergence is replaced with strong convergence.*

*Proof.* (i) Let  $u = u_1 \otimes u_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$  and  $v = v_1 \otimes v_2 \in \mathcal{K}_1 \otimes \mathcal{K}_2$  for which we have  $\langle F(u), v \rangle = 1$ . Let  $F_1 \otimes F_2 = \mathcal{D}_{u,v}(F)$ . Using Proposition 2.1, we obtain

$$\begin{aligned} \|F_1^n \otimes F_2^n - F_1 \otimes F_2\| &\leq \|F - F_1^n \otimes F_2^n\| \|u\| \|v\| \|F_1^n \otimes F_2^n\| \\ &\quad + \|u\| \|v\| (\|F\| + \|F_1^n \otimes F_2^n\|) \|F - F_1^n \otimes F_2^n\|. \end{aligned}$$

Consequently,  $F_1^n \otimes F_2^n \xrightarrow{u} F_1 \otimes F_2$ , and so  $F = F_1 \otimes F_2$ .

(ii) Let  $u = u_1 \otimes u_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$  and  $v = v_1 \otimes v_2 \in \mathcal{K}_1 \otimes \mathcal{K}_2$  for which we have  $\langle F_1(u_1), v_1 \rangle = \langle F_2(u_2), v_2 \rangle = 1$ . Since  $\{F_2^n\}_{n>0}$  is bounded, then so is the sequence  $\{c_n = \langle F_2^n(u_2), v_2 \rangle\}_{n>0}$ . Therefore, by Borel Theorem, there is a subsequence  $\{c_{i(n)}\}_{n>0}$  that converges to a constant  $c$ . Using Proposition 2.1, we have

$$\begin{aligned} (3.1) \quad \|cF_1^{i(n)} - F_1\| &= \|(c - c_{i(n)})F_1^{i(n)} + V^{v_2}[F_1^{i(n)} \otimes F_2^{i(n)} - F_1 \otimes F_2]U^{u_2}\| \\ &\leq |c - c_{i(n)}| \|F_1^{i(n)}\| + \|u_2\| \|v_2\| \|F_1^{i(n)} \otimes F_2^{i(n)} - F_1 \otimes F_2\|. \end{aligned}$$

The last inequality and the fact that the sequence  $\{F_1^n\}_{n>0}$  is bounded imply  $F_1^{i(n)} \xrightarrow{u} F_1/c$ . Similarly, there is a constant  $d \neq 0$  such that  $\{F_2^{i(n)}\}_{n>0}$  has a subsequence  $\{F_2^{j(n)}\}_{n>0}$  that uniformly converges to  $F_2/d$ . Finally, the fact that the sequence  $\{F_1^{j(n)} \otimes F_2^{j(n)}\}_{n>0}$  uniformly converges to  $F_1 \otimes F_2$  leads to  $d = 1/c$ .

(iii) Assume  $F_1^n \otimes F_2^n \xrightarrow{s} F$ . Therefore, for all  $x = x_1 \otimes x_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$ ,  $F^n E_{x,x} \xrightarrow{u} F E_{x,x}$ , and so using (i),  $\text{rank}_\otimes(F E_{x,x}) = 1$ . Hence,  $\text{rank}_\otimes(F) = 1$ .

Following the same steps to prove (ii) and replacing (3.1) with the inequality

$$\begin{aligned} (3.2) \quad \|cF_1^{i(n)}(x_1) - F_1(x_1)\| &\leq |\langle F_2^{i(n)}(u_2), v_2 \rangle| \|F_1^{i(n)}(x_1) - F_1(x_1)\| \\ &\quad + |c - c_{i(n)}| \|F_1^{i(n)}(x_1)\| \\ &\quad + |\langle F_2^{i(n)}(u_2) - F_2(u_2), v_2 \rangle| \|F_1(x_1)\|, \end{aligned}$$

where  $x_1 \in \mathcal{H}_1$ , we can obtain the desired result.  $\square$

*Remark 3.2.*

1. For every type of convergence, each one of the sequences  $\{1/nF_1 \otimes nF_2\}_{n>0}$  and  $\{(-1)^n F_1 \otimes (-1)^n F_2\}_{n>0}$  is obviously convergent to  $F_1 \otimes F_2$ . Therefore, in general, one cannot state stronger than the last two statements of Theorem 3.1.

2. It is not clear to us how to prove the equivalent to Statement (i) in the case of weak convergence. However, in that case, Inequality 3.2 can be used to prove the equivalent to Statement (ii) provided the equivalent to Statement (i) is assumed.

**Lemma 3.3.** *Let  $\{A, B\}$  be an independent set in  $\mathcal{B}(\mathcal{H}, \mathcal{K})$ . There is  $u \in \mathcal{H}$  and  $v \in \mathcal{K}$  for which we have  $\langle A(u), v \rangle = 0$  and  $\langle B(u), v \rangle \neq 0$ .*

*Proof.* Let  $\{e_n\}_{n>0}$  be an ONB of  $\mathcal{K}$  and assume the following.

If  $u, v \in \mathcal{H}$  and  $\langle A(u), v \rangle = 0$ , then  $\langle B(u), v \rangle = 0$ .

(i) If  $A(u) = 0$ , then for all  $n > 0$ ,  $\langle A(u), e_n \rangle = 0$ , and so  $\langle B(u), e_n \rangle = 0$ . Hence,  $B(u) = 0$ .

(ii) Now assume that  $A(u) \neq 0$ . For each  $n > 0$ , let  $f_n = \langle A(u), e_n \rangle$  and let  $g_n = \langle B(u), e_n \rangle$ . WLG, we can assume that  $f_1 \neq 0$ . For all  $n > 0$ , we have  $\langle A(u), f_1 e_n - f_n e_1 \rangle = 0$ , and so  $\langle B(u), f_1 e_n - f_n e_1 \rangle = 0$ , i.e.,  $f_1 g_n = f_n g_1$ . Therefore,

$$\sum_{n>0} g_n e_n = \lambda(u) \sum_{n>0} f_n e_n,$$

where  $\lambda(u) = g_1/f_1$ . This with (i) imply that  $B(u) = \lambda(u)A(u)$  for all  $u \in \mathcal{H}$ .

(iii) Let's fix  $u \in \mathcal{H}$  for which we have  $B(u) \neq 0$  and let  $\lambda = \lambda(u)$ . Let  $v \in \mathcal{H}$  for which we have  $B(v) \neq 0$  and let  $w \in \mathcal{H}$  for which we have  $\langle B(u), w \rangle \neq 0$  and  $\langle B(v), w \rangle \neq 0$ . We then have  $\lambda(v) = \lambda$  since

$$\lambda \langle B(u), w \rangle \langle B(v), w \rangle = \lambda(v) \langle B(u), w \rangle \langle B(v), w \rangle.$$

iv) Let  $v \in \mathcal{H}$  for which we have  $B(v) = 0$  and  $A(v) \neq 0$ . Therefore, we have  $B(u+v) \neq 0$ , and so by (iii), we have  $\lambda(u+v) = \lambda$ . The equalities

$$B(u+v) = \lambda(u+v)A(u+v) = \lambda A(u) + \lambda A(v),$$

$$B(u+v) = B(u) + B(v) = \lambda A(u) + \lambda(v)A(v).$$

then imply that  $\lambda(v) = \lambda$ , and so  $B = \lambda A$ , which is a contradiction.  $\square$

Now, we are ready to prove Theorem 1.2 stated at the introduction.

*Proof.* Theorem 3.1 states (i) for  $M = 1$ .

Assume that (i) holds for  $M - 1 \geq 1$ . Let  $F^n \xrightarrow{u} F$  and for each  $n > 0$ , that  $\text{rank}_{\otimes}(F^n) \leq M$ . Let  $u = u_1 \otimes u_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$  and  $v = v_1 \otimes v_2 \in \mathcal{K}_1 \otimes \mathcal{K}_2$ , for which we have  $\langle F(u), v \rangle = 1$ . Let  $G^n = \langle F^n(u), v \rangle F^n - \mathcal{D}_{u,v}(F^n)$  and let  $G = F - \mathcal{D}_{u,v}(F)$ .

Using Theorem 2.4,  $\text{rank}_{\otimes}(G^n) \leq M - 1$ , and using Proposition 2.1, we obtain

$$\begin{aligned} \|G^n - G\| &\leq \|F^n - F\| \|u\| \|v\| \|F^n\| + \|F^n - F\| \\ &\quad + \|u\| \|v\| (\|F^n\| + \|F\|) \|F - F^n\|. \end{aligned}$$

Therefore,  $G^n \xrightarrow{u} G$  and by induction,  $\text{rank}_{\otimes}(G) \leq M - 1$ , and so  $\text{rank}_{\otimes}(F) \leq M$ .

We will prove Statement (ii) by induction on  $r$ . Assume that  $r = 1$ , i.e.,  $F = F_1 \otimes F_2$ . Let  $u = u_1 \otimes u_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$  and  $v = v_1 \otimes v_2 \in \mathcal{K}_1 \otimes \mathcal{K}_2$  for which we have  $\langle F_1(u_1), v_1 \rangle = \langle F_2(u_2), v_2 \rangle = 1$ . The sequence

$$\{(\langle F_{2,1}^n(u_2), v_2 \rangle, \dots, \langle F_{2,M}^n(u_2), v_2 \rangle, \langle F_{1,1}^n(u_1), v_1 \rangle, \dots, \langle F_{1,M}^n(u_1), v_1 \rangle)\}_{n>0}$$



is bounded in  $\mathbb{C}^M \times \mathbb{C}^M$ , and so by Borel Theorem, it has a subsequence

$$\{(\langle F_{2,1}^{i(n)}(u_2), v_2 \rangle, \dots, \langle F_{2,M}^{i(n)}(u_2), v_2 \rangle, \langle F_{1,1}^{i(n)}(u_1), v_1 \rangle, \dots, \langle F_{1,M}^{i(n)}(u_1), v_1 \rangle)\}_{n>0}$$

that converges to  $(c_1^1, \dots, c_1^M, c_2^1, \dots, c_2^M) \in \mathbb{C}^M \times \mathbb{C}^M$ . Therefore,

$$\begin{aligned} \left\| \sum_{m=1}^M c_1^m F_{1,m}^{i(n)} - F_1 \right\| &\leq \left\| \sum_{m=1}^M (c_1^m - \langle F_{2,m}^{i(n)}(u_2), v_2 \rangle) F_{1,m}^{i(n)} + V v_2 (F^{i(n)} - F) U u_2 \right\| \\ &\leq \sum_{m=1}^M |c_1^m - \langle F_{2,m}^{i(n)}(u_2), v_2 \rangle| \|F_{1,m}^{i(n)}\| + \|F^{i(n)} - F\| \|u_2\| \|v_2\|. \end{aligned}$$

This with the fact that  $\{F_{1,m}^{i(n)}\}_{n>0}$  is bounded imply

$$\sum_{m=1}^M c_1^m F_{1,m}^{i(n)} \xrightarrow{u} F_1.$$

Similarly, we can prove that

$$\sum_{m=1}^M c_2^m F_{2,m}^{j(n)} \xrightarrow{u} F_2,$$

where  $j(n)$  is a subsequence of  $i(n)$ . WLG, we can assume that  $i(n) = j(n)$ .

Now, assume that (ii) holds for  $r-1 > 0$  and  $F$  has a Schmidt decomposition in form (1.1). Let  $u = u_1 \otimes u_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$  and  $v = v_1 \otimes v_2 \in \mathcal{K}_1 \otimes \mathcal{K}_2$ . For each  $k \in \{1, \dots, r\}$ , we set  $a_{1,k} = \langle F_{1,k}(u_1), v_1 \rangle$  and  $a_{2,k} = \langle F_{2,k}(u_2), v_2 \rangle$ .

Using Lemma 3.3, we can choose  $u_1, v_1, u_2,$  and  $v_2$  so that

$$\begin{aligned} a_{1,1} &= \langle F_{1,1}(u_1), v_1 \rangle = 1, & a_{1,2} &= \langle F_{1,2}(u_1), v_1 \rangle = 0, \\ a_{2,1} &= \langle F_{2,1}(u_2), v_2 \rangle = 1, & \left\langle \sum_{k=2}^r a_{1,k} F_{2,k}(u_2), v_2 \right\rangle &= 0. \end{aligned}$$

Therefore,  $\langle F(u), v \rangle = 1$ , and so we can apply Theorem 2.4 from which we borrow the notations. Since  $\Delta_{1,2}^1 = -F_{1,2}$ , (2.7) then implies the equality

$$\begin{aligned} (3.3) \quad F - \mathcal{D}_{u,v}(F) &= \sum_{k=3}^r \Delta_{1,k}^1 \otimes \left( \Delta_{1,k}^2 - \sum_{l=k+1}^r \frac{a_{1,l}}{a_{1,1}} \Delta_{k,l}^2 + \sum_{l=2}^{k-1} \frac{a_{1,l}}{a_{1,1}} \Delta_{l,k}^2 \right) \\ &\quad - F_{1,2} \otimes \left( \Delta_{1,2}^2 - \sum_{l=3}^r \frac{a_{1,l}}{a_{1,1}} \Delta_{2,l}^2 \right). \end{aligned}$$

Assume that  $F^n$  has the form (1.2). Therefore,

$$F^n - \mathcal{D}_{u,v}(F^n) = \sum_{m=1}^M F_{1,m}^n \otimes [F_{2,m}^n - b_{2,m}^n \sum_{j=1}^M b_{1,j}^n F_{2,j}^n],$$

where,  $b_{1,m}^n = \langle F_{1,k}^n(u_1), v_1 \rangle$  and  $b_{2,m}^n = \langle F_{2,m}^n(u_2), v_2 \rangle$  for all  $m \in \{1, \dots, M\}$ .

The continuity of  $\mathcal{D}_{u,v}$  implies  $(F^n - \mathcal{D}_{u,v}(F^n)) \xrightarrow{u} (F - \mathcal{D}_{u,v}(F))$ ; by Theorem 2.4, we have  $\text{rank}_{\otimes}(F - \mathcal{D}_{u,v}(F)) = r-1$ ; and the second hand of (3.3)

shows  $F_{1,2}$ . By induction, there are then  $c_{1,2}^1, \dots, c_{1,2}^M \in \mathbb{C}$  for which there is a subsequence of

$$\sum_{k=1}^M c_{1,2}^k F_{1,m}^n$$

that uniformly converges to  $F_{1,2}$ . Similarly, we can obtain the desired conclusion for each  $F_{i,k}$ , where  $(i,k) \in \{1,2\} \times \{1, \dots, M\}$ .

(iii) As we did in the proof of Theorem 3.1, we use (i) to prove  $\text{rank}_{\otimes}(F) \leq M$ .

Following similar steps as in the proof of (ii), we use the following inequality

$$\begin{aligned} & \|\mathcal{D}_{u,v}(F^n)(x_1 \otimes x_2) - \mathcal{D}_{u,v}(F)(x_1 \otimes x_2)\| \\ & \leq \|v\| \|F^n(x_1 \otimes u_2) - F^n(u_1 \otimes x_2) - F(u_1 \otimes x_2)\| \\ & \quad + \|v\| \|F^n(x_1 \otimes u_2) - F(x_1 \otimes u_2)\| \|F(u_1 \otimes x_2)\| \text{ for all } x_1 \otimes x_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2, \end{aligned}$$

to obtain  $\mathcal{D}_{u,v}(F^n)(x) \rightarrow \mathcal{D}_{u,v}(F)(x)$  for each  $x$ , in a dense subspace of  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . The uniform boundedness principle implies that the sequence  $\{F^n\}$  is bounded, which implies  $(F^n - \mathcal{D}_{u,v}(F^n)) \xrightarrow{s} (F - \mathcal{D}_{u,v}(F))$  and allows us to finish the proof.  $\square$

#### 4. SCHMIDT DECOMPOSITION OF BOUNDED OPERATORS WITH IDEAL PROPERTIES

We say that  $\mathcal{C}$  is an ideal class of bounded operators if for all  $F, G \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ , we have  $FG, GF \in \mathcal{C}$  whenever  $F \in \mathcal{C}$ . Ideal classes that are closed under the tensor product include the class of compact operators, the class of Hilbert-Schmidt operators, the class of nuclear operators, and all Schatten classes of operators [8].

**Theorem 4.1.** *Let  $\mathcal{C}$  be an ideal class of bounded operators that is closed under the tensor product. Let  $\mathcal{H}_1, \dots, \mathcal{H}_m$  be  $m$  Hilbert spaces. Let*

$$(4.1) \quad F = \sum_{k=1}^N F_{1,k} \otimes F_{2,k} \otimes \cdots \otimes F_{m,k},$$

where for each  $i \in \{1, \dots, m\}$ ,  $\{F_{i,k}\}_{k=1}^N$  is independent in  $\mathcal{B}(\mathcal{H}_i)$ . The operator  $F \in \mathcal{C}$  if and only if the operators  $F_{1,k}, \dots, F_{m,k} \in \mathcal{C}$  for each  $k \in \{1, \dots, N\}$ .

*Proof.* By Theorem 2.3, every FMS is independent. Using this fact and the associativity of the tensor product, it suffices to prove Theorem 4.1 for  $m = 2$ . The necessary condition is obvious. We will prove the sufficient by induction on  $N$ .

If  $N = 1$ , then  $F = F_{1,1} \otimes F_{2,1}$ . Let  $u = u_1 \otimes u_2, v = v_1 \otimes v_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$  for which we have  $\langle F_{1,1}(u_1), v_1 \rangle = \langle F_{2,1}(u_2), v_2 \rangle = 1$ . Using Proposition 2.1, we have

$$F_{1,1} = V^{v_2} F U^{u_2} \quad \text{and} \quad F_{2,1} = V_{v_1} F U_{u_1}.$$

The fact that  $\mathcal{C}$  is an ideal class implies  $F_{1,1}, F_{2,1} \in \mathcal{C}$ .

Now, assume that the induction hypothesis holds for  $N - 1 \geq 0$  and let  $F$  be in the form (4.1) for  $m = 2$ . Following the same steps to prove Theorem 1.2(ii), we can obtain equality (3.1) for which the right hand side shows  $F_{1,2}$ . By Theorem

2.4, we have  $\text{rank}_{\otimes}(F - \mathcal{D}_{u,v}(F)) = N - 1$ , and so by induction, the operator  $F_{1,2} \in \mathcal{C}$ . Similarly, we can prove that  $F_{1,k}, F_{2,k} \in \mathcal{C}$  for each  $k \in \{1, \dots, N\}$ .  $\square$

The case  $N = 1$  for compact operators was recently obtained by J. Zanni and C. S. Kubrusly using different techniques [11].

Theorem 2.4, our main theorem, leads to an algorithm to compute Schmidt decompositions of bounded operators on the tensor product of separable Hilbert spaces. This paper includes some applications in operator theory of that algorithm. Other applications include a compression method published in [1].

**Acknowledgment.** The author wants to thank the anonymous referee for his/her authoritative and thorough review. The author also wants to acknowledge that because of comments by the referee, results about convergence in the strong topology of operators were added to Theorem 1.2 and Theorem 3.1.

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