LITERATURE SURVEY ON THE DYNAMICS
OF PLATE AND SHELL STRUCTURES

By

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Scope of Study: The aim of this report is to outline and summarize the study that has been done in the area of vibration of plate and shell structures. A brief description of vibration of grids is also made.

Findings and Conclusions: The Rayleigh-Ritz method appears to be the most useful method for finding a reasonable approximate solution for natural frequencies of vibration of thin elastic plates and shells. This literature survey will serve the first step toward the complete comprehension of the vibration problems in plate and shell structures; it will be very beneficial in future investigations of this problem.

ADVISER'S APPROVAL ____________________

LITERATURE SURVEY ON THE DYNAMICS

OF PLATE AND SHELL STRUCTURES

Report Approved:

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Report Adviser

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NOMENCLATURE

D	$= \dfrac{Eh^3}{12(1 - \nu^2)}$, flexural rigidity
E	= modulus of elasticity
T	= kinetic energy
V	= potential energy
X_m	= function of x
Y_n	= function of y
a, b	= length in x and y directions
h	= thickness
l	= length of beam
l_0	= z coordinate to small edge of conical shell
l_1	= z coordinate to large edge of conical shell
t	= time
u_1	= meridianal displacement at position z, θ at time t of conical shell
u_2	= tangential displacement at position z, θ at time t of conical shell
w	= displacement in z direction, inward displacement of conical shell
α_r	= parameter in expressions for φ_r
ε_r	= parameter in expressions for φ_r
φ_r	= characteristic function of a vibrating beam

ρ = mass density of plate per unit area, mass density of shell per unit volume

ν = Poisson's ratio

ω = natural angular frequency

PART I

INTRODUCTION

It is the purpose of this report to outline and summarize the study that has been done in the area of vibration of plate and shell structures. This literature survey should be very beneficial in future investigations of this problem.

The contents of this report are divided into two areas: vibration of plates, and vibration of shells; also, a brief description of vibration of grids is included in the plate section.

The survey on vibration of plates includes rectangular plates, circular plates, triangular plates, and skew plates, with various edge conditions, and a simply supported isosceles trapezoidal plate. The shell section includes vibrations of cylindrical shells, shallow spherical shells, conical shells and paraboloidal shells of revolution.

This literature survey concentrates on the field of free vibration of plate and shell structures. The ordinary assumptions of elastic analysis are made in the reviewed literature.

PART II

VIBRATIONS OF PLATES

1. General.

Thin plates which consist of elastic, homogeneous isotropic material will be taken into account in this study. The well-known plate equation[1] is obtained as follows:

$$\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z}{D} \qquad (a)$$

where p_z is the intensity of load.

The equation of vibration is obtained from equation (a) by substituting for p_z the expression[2], $-\rho\,\dfrac{\partial^2 w}{\partial t^2}$, thus,

$$\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{\rho}{D}\,\frac{\partial^2 w}{\partial t^2} \qquad (1.1)$$

or

$$\left(\nabla^4 + \frac{\rho}{D}\,\frac{\partial^2}{\partial t^2}\right) w = 0.$$

1-1. Rayleigh-Ritz Method[3].

The potential energy accumulated in the plate element during the deformation is:

$$V = \frac{D}{2} \iint \left\{ \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\,\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}\right.$$

$$+ 2(1 - \nu) \left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2 \Bigg\} dx\ dy. \qquad (1.2)$$

The kinetic energy of a vibrating plate is:

$$T = \frac{\rho}{2} \iint \dot{w}^2\ dx\ dy. \qquad (1.3)$$

Expressing the deflection as:

$$w = W \cos \omega t$$

and substituting in equations (1.2) and (1.3) and equating them

$$\omega^2 = \frac{2}{\rho} \ \frac{V}{\iint W\, dx\, dy} \ . \qquad (1.4)$$

Let

$$W = A_1 g_1 + A_2 g_2 + A_3 g_3 + \dots\dots + A_n g_n \qquad (1.5)$$

Equation (1.5) is minimized to obtain

$$\frac{\frac{\partial}{\partial A_i} \left(\frac{2}{\rho} V - \omega^2 \iint W^2\ dx\ dy \right)}{\iint W\ dx\ dy} = 0 \qquad (1.6)$$

from which

$$\frac{\partial}{\partial A_i} \iint \Bigg\{ \left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2 \ \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2}$$

$$+ 2(1 - \nu) \left(\frac{\partial^2 W}{\partial x \ \partial y}\right)^2 - \omega^2 \frac{\rho}{D} W^2 \Bigg\} dx\ dy = 0. \qquad (1.7)$$

Equation (1.7) represents a set of n linear homogeneous equations; for nontrivial solution, the determinant of the coefficients must be zero. This yields the approximate values of the natural frequencies in the problem being considered.

2. Vibration of Rectangular Plates.

2-1. Rectangular Plate Simply Supported on All Four Edges[3].

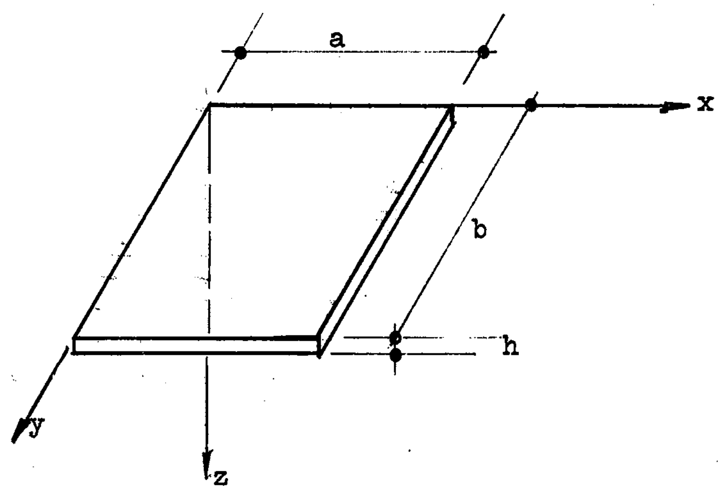


Fig. 1

Rectangular Plate

Let

$$w = \sum_{m=1}^{p} \sum_{n=1}^{q} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where q_{mn} is a time function.

Substituting into equation (1.2),

$$V = \frac{\pi^4 ab}{8} D \sum_{m=1}^{p} \sum_{n=1}^{q} q_{mn}{}^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 .$$

The kinetic energy is

$$T = \frac{\rho}{2} \frac{ab}{4} \Sigma\Sigma \, \dot{q}_{mn}{}^2 .$$

Consider a virtual displacement

$$\delta q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b} .$$

Thus, the differential equation of normal vibration is

$$\rho\ddot{q}_{mn} + \pi^4 D \, q_{mn} \left(\frac{m^2}{q^2} + \frac{n^2}{b^2}\right)^2 = 0$$

from which

$$q_{mn} = C_1 \cos \omega_{mn} t + C_2 \sin \omega_{mn} t$$

where

$$\omega_{mn} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \sqrt{\frac{D}{\rho}} \ .$$

2-2. Vibration of a Rectangular Plate With Various Edge Conditions.

Rayleigh-Ritz method is employed to solve these problems. Characteristic beam functions appropriate to the boundary conditions are used for deriving closed formulas for the frequencies of vibration of plates.

The series approximation for W is taken in the form

$$W(x,y) = \sum_{m=1}^{p} \sum_{n=1}^{q} A_{mn} X_m(x) \, Y_n(y) \ . \tag{2.1}$$

From equation (1.6)

$$\frac{\partial V}{\partial A_{ik}} - \frac{\rho \omega^2}{2} \frac{\partial}{\partial A_{ik}} \iint W^2 dx \, dy = 0. \tag{2.2}$$

Following are characteristic functions for vibrating beams:

(A) Clamped - Clamped Beam(5)

$$\varphi_r = \cosh \frac{\varepsilon_r x}{l} - \cos \frac{\varepsilon_r x}{l} - \alpha_r \left(\sinh \frac{\varepsilon_r x}{l} - \sin \frac{\varepsilon_r x}{l}\right). \tag{2.3}$$

(B) Clamped - Free Beam(5)

$$\varphi_r = \cosh \frac{\varepsilon_r x}{l} - \cos \frac{\varepsilon_r x}{l} - \alpha_r \left(\sinh \frac{\varepsilon_r x}{l} - \sin \frac{\varepsilon_r x}{l}\right) \tag{2.4}$$

(C) Free - Free Beam(5)

$$\varphi_1 = 1 \tag{2.5a}$$

$$\varphi_2 = \sqrt{3} \left(1 - 2 \frac{x}{l}\right) \tag{2.5b}$$

$$\varphi_r = \cosh \frac{\varepsilon_r x}{l} + \cos \frac{\varepsilon_r x}{l} - \alpha_r \left(\sin \frac{\varepsilon_r x}{l} + \sin \frac{\varepsilon_r x}{l}\right) \qquad (2.5c)$$

(D) Simply Supported[6]

$$\varphi_r = \sin \frac{\varepsilon_r x}{l} \qquad (2.6)$$

(E) Clamped - Simply Supported[6]

$$\varphi_r = \cosh \frac{\varepsilon_r x}{l} - \cos \frac{\varepsilon_r x}{l} - \alpha_r \left(\sinh \frac{\varepsilon_r x}{l} - \sin \frac{\varepsilon_r x}{l}\right) \qquad (2.7)$$

$$(r = 1,\ 2,\ 3,\ 4,\ \ldots)$$

The numerical values of α_r and ε_r can be tabulated[5][6].

The characteristic functions listed are used for X_m and Y_n in equation (2.1). The particular sets to be used in any problem will depend upon the boundary conditions of the plate.

The available numerical results are summarized in Tables I to IV[7], using the abbreviations F = free, S = simply supported, and C = clamped. The quantity entered in Tables I, III, and IV is $k = \omega a^2 \sqrt{\frac{\rho}{D}}$ and in Table II is either k, or $k' = \omega b^2 \sqrt{\frac{\rho}{D}}$.

3. Vibration of Circular Plates[3].

Rayleigh-Ritz method will be used for the approximate solution of the vibration of a circular plate. Transforming the equations (1.2) and (1.3)

$$V = \frac{D}{2}\int_0^{2\pi}\int_0^{a} \left\{ \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + 2(1-\nu)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right]^2 \right\} r\,d\theta\,dr \qquad (3.1)$$

TABLE I

k FOR MODES OF A SQUARE PLATE[7]

Edge Conditions	Authority	Mode Number 1	2	3	4	5	6
F / F F / C	Young	03.494	08.547	021.44	027.46	031.17	
F / C F / C	Young	06.958	24.080	026.80	048.05	063.14	
F / F F / F	Ritz	14.100	20.550	023.91	035.96	061.60	065.24
S / S S / S	Eqn(3)	19.740	49.340	078.96	098.69	128.30	167.80
S / S S / C	Iguchi	23.650	51.680	058.65	086.13	100.30	113.20
S / C C / S	Iguchi	28.950	54.750	069.32	094.59	102.2	129.10
C / C C / C	Iguchi	35.980	73.400	108.20		132.20	165.00
	Young	35.990	73.410	108.30	131.60	132.30	165.10

TABLE II

k AND k' FOR FUNDAMENTAL MODES OF RECTANGULAR PLATES

(IGUCHI)[7]

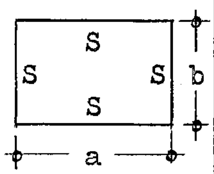
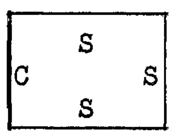
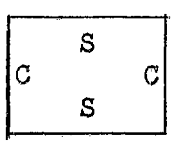
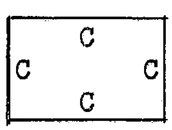
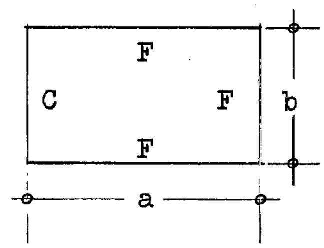
	b/a	01.00	01.50	02.00	02.50	03.00	∞
	k	19.74	14.26	12.34	11.45	10.97	09.87
	b/a	01.00	01.50	02.00	02.50	03.00	∞
	k	23.65	18.90	17.33	16.63	16.26	15.43
	a/b	01.00	01.50	02.00	02.50	03.00	∞
	k'	23.65	15.57	12.92	11.75	11.14	09.87
	b/a	01.00	01.50	02.00	02.50	03.00	∞
	k	28.95	25.05	23.82	23.27	22.99	22.37
	a/b	01.00	01.50	02.00	02.50	03.00	∞
	k'	28.95	17.37	13.69	12.13	11.36	09.87
	b/a	01.00	01.50	02.00	02.50	03.00	∞
	k	35.98	27.00	24.57	23.77	23.19	22.37

TABLE III

k FOR MODES OF RECTANGULAR CANTILEVER PLATES

(BARTON)[7]

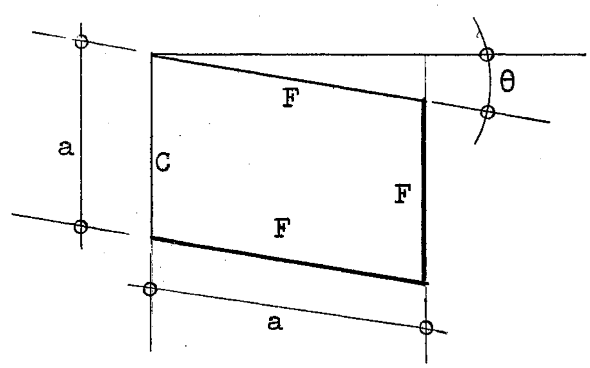


a/b	Mode Number				
	1	2	3	4	5
1/2	3.508	05.372	21.96	010.26	024.85
1	3.494	08.547	21.44	027.46	031.17
2	3.472	14.930	21.61	094.49	048.71
3	3.450	34.730	21.52	563.90	105.90

TABLE IV

k FOR MODES OF SKEW CANTILEVER PLATES

(BARTON)[7]



Mode Number	θ		
	15°	30°	45°
1	3.60	03.96	04.82
2	8.87	10.19	13.75

$$T = \frac{\rho}{2}\int_0^{2\pi}\int_0^a \dot{w}^2 r d\theta dr \qquad (3.2)$$

where a is the radius of the plate.

3.1. Vibration of Circular Plate Clamped at the Boundary.

For the case of the lowest mode of vibration, equations (3.1) and (3.2) reduce to

$$V = \pi D\int_0^a \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)^2 r dr \qquad (3.3)$$

$$T = \pi\rho\int_0^a \dot{w}^2 \, r dr \qquad (3.4)$$

Assuming

$$w = W \cos \omega t \qquad (3.5)$$

and substituting equation (3.5) into equations (3.3) and (3.4) and equating them

$$\omega^2 = \frac{D}{\rho}\,\frac{\int_0^a \left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right)^2 r dr}{\int_0^a W^2 \, r dr} \qquad (3.6)$$

The function W is taken in the form of the series

$$W = A_1\left(1 - \frac{r^2}{a^2}\right) + A_2\left(1 - \frac{r^2}{a^3}\right)^3 + \cdots \qquad (3.7)$$

using equation (1.7)

$$\frac{\partial}{\partial A_i}\int_0^a \left\{\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right)^2 - \frac{\omega^2\rho}{D} W^2\right\} r dr = 0 \qquad (3.8)$$

Substituting equation (3.7) into equation (3.8), and setting its determinant to zero, the frequencies of successive modes can be obtained. In all cases the frequency of vibration has the pattern

$$\omega = \frac{\alpha}{a^2}\sqrt{\frac{D}{\rho}} \, . \qquad (3.9)$$

The constant α for a given number s, of nodal circles, and for a given number n, of nodal diameters, is given in Table V.

TABLE V

THE VALUES OF α OF CIRCULAR PLATE CLAMPED AT BOUNDARY(3)

s	n = 0	n = 1	n = 2
0	10.21	21.22	34.84
1	39.78		
2	88.90		

3.2. Vibration of Circular Plate With Other Kinds of Boundary Conditions.

(A) For a Free Circular Plate ($\nu = 1/3$)

TABLE VI

THE VALUES OF α OF FREE CIRCULAR PLATE(3)

s	n = 0	n = 1	n = 2	n = 3
0			05.251	12.23
1	09.076	20.52	35.240	52.91
2	38.520	59.86		

(B) For a Circular Plate With Its Center Fixed

TABLE VII

THE VALUES OF α OF CIRCULAR PLATE WITH ITS CENTER FIXED[3]

s	0	1	2	3
α	3.75	20.91	60.68	119.7

4. Vibration of Triangular Plates.

4-1. Vibration of Triangular Cantilever Plate[8].

Taking the coordinates as shown in Fig. 2, the following coordinate transformation is made:

$$u = \frac{x}{a} \quad , \quad v = k\,\frac{y}{x} \tag{4.1}$$

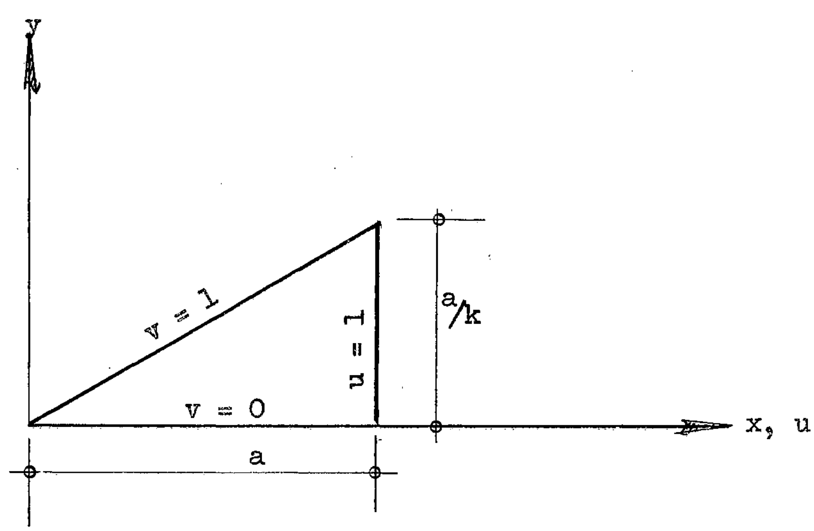


Fig. 2

Illustration of Coordinate u and v

In the coordinates u and v, equation (1.6) becomes:

$$\frac{\partial}{\partial A_i}\iint\Bigg[u\left(\frac{\partial^2 W}{\partial u^2}\right)^2 - 4v\,\frac{\partial^2 W}{\partial u^2}\,\frac{\partial^2 W}{\partial u\partial v}$$

$$+\frac{2}{u}\left\{\left[2v^2 + k^2(1-\nu)\right]\left(\frac{\partial^2 W}{\partial u\partial v}\right)^2\right.$$

$$\left.+(v^2 + k^2\nu)\,\frac{\partial^2 W}{\partial u^2}\,\frac{\partial^2 W}{\partial v^2} + 2v\,\frac{\partial W}{\partial v}\,\frac{\partial^2 W}{\partial u^2}\right\}$$

$$-\frac{4}{u^2}\left\{\left[2v^2 + k^2(1-\nu)\right]\frac{\partial^2 W}{\partial u\partial v}\,\frac{dW}{dv}\right. \qquad (4.2)$$

$$\left.+(v^3 + k^2 v)\,\frac{\partial^2 W}{\partial u\partial v}\,\frac{\partial^2 W}{\partial v^2}\right\}$$

$$+\frac{1}{u^3}\left\{2\,[2v^2 + k^2(1-\nu)\left(\frac{\partial^2 W}{\partial v^2}\right)^2\right.$$

$$\left.+4(v^3 + k^2 v)\,\frac{\partial W}{\partial v}\,\frac{\partial^2 W}{\partial v^2} + (v^2 + k^2)^2\left(\frac{\partial^2 W}{\partial v^2}\right)^2\right\}$$

$$-\gamma^2 u W^2\Bigg]\,du\,dv = 0$$

in which W is a function of u and v and

$$\gamma = \omega\sqrt{\frac{\rho a^4}{D}}\,.$$

(A) First Case-Symmetrical Triangle

A symmetric triangle with apex at the origin and length a and base 2a/k is obtained by taking the limits

$$0 \le u \le 1, \qquad -1 \le v \le +1\,.$$

For symmetric modes, let

$$W = [A_{11} + A_{31}\,u^2\,\phi_3(v)]\,\varphi_1(u) + [A_{12} + A_{32}\,u^2\phi_3(v)]\,\varphi_2(u) \qquad (4.3)$$

for antisymmetric modes,

$$W = [A_{21}\, v + A_{41}\, \phi_4(v)]\; u^2\varphi_1(u) + [A_{22}\, v + A_{42}\, \phi_4(v)]\; u^2\varphi_2(u). \tag{4.4}$$

φ_1 and φ_2 represent the first two modes of a cantilever beam free at $u = 0$ and clamped at $u = 1$. ϕ_3 and ϕ_4 represent the first symmetric and antisymmetric modes of a beam free at $v = \pm 1$. The values of γ are shown in Table VIII.

(B) Second Case - Unsymmetrical Triangle

An unsymmetric triangle with apex at origin and of length a, and base a/k is obtained by taking the limits

$$0 \leq u \leq 1, \qquad\qquad 0 \leq v \leq 1 .$$

Let

$$W = [A_{11} + A_{21}u^2v + A_{31}u^2\phi_3(v)]\, \varphi_1(u)$$

$$+ [A_{12} + A_{22}u^2v + A_{32}u^2\phi_3(v)]\, \varphi_2(u) . \tag{4.5}$$

The values of γ are shown in Table IX.

4-2. Vibration of Clamped Triangular Plate[9].

The method of collocation[18, 19, 20] is employed to obtain reasonable approximate solutions. The method of collocation consists essentially in satisfying a given differential equation, or set of equations, at a finite number of points.

Skew coordinate axes x and y are taken in the middle surface of the plate as shown in Fig. 3.

TABLE VIII

THE VALUES OF γ OF CANTILEVER, SYMMETRICAL TRIANGULAR PLATE

$$\gamma = \omega\sqrt{\frac{\rho a^4}{D}}$$

mode \ k	2	4	8	14
1st	007.149	007.122	007.080	007.068
2nd	030.803	030.718	030.654	030.638
3rd	061.131	090.105	157.700	265.980
4th	148.800	259.400	493.400	853.600

TABLE IX

THE VALUES OF γ OF CANTILEVER, UNSYMMETRICAL TRIANGULAR PLATE

$$\gamma = \omega\sqrt{\frac{\rho a^4}{D}}$$

mode \ k	2	4	7
1st	05.887	06.617	06.897
2nd	25.400	28.800	30.280

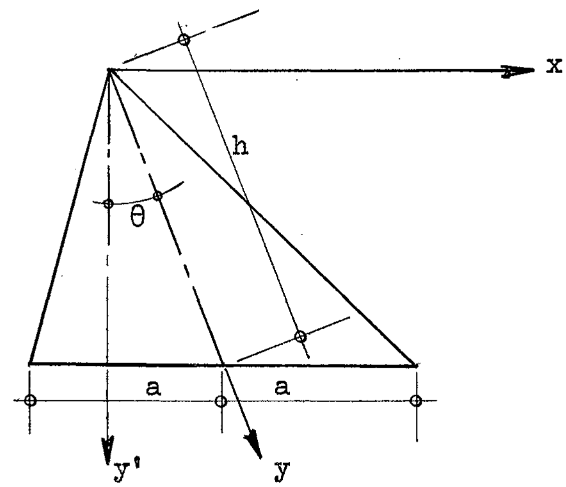


Fig. 3

Clamped Triangular Plate

The differential equation of free vibration is

$$\frac{\partial^4 w}{\partial x^4} + 2(1 + 2\sin^2\theta)\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}$$

$$- 4\sin\theta\left(\frac{\partial^4 w}{\partial x^3 \partial y} + \frac{\partial^4 w}{\partial x \partial y^3}\right) = \frac{\rho\omega^2}{D} w \qquad (4.6)$$

where

θ = skew angle.

Boundary conditions are

$$(w)_{y = h} = (w)_{x = (\pm a/h)y} = 0$$

$$\left(\frac{\partial w}{\partial y}\right)_{y = h} = \left(\frac{\partial w}{\partial n}\right)_{x = (\pm a/h)y} = 0$$

where

h = median distance from the origin

n = normal direction to a boundary.

The deflection function is

$$w = \left(\alpha_1 y^2 \sin^2 \frac{\pi y}{h} + \alpha_2 y^2 \sin \frac{\pi y}{h} \sin \frac{2\pi y}{h}\right)$$

$$\left[1 - \left(\frac{h}{a}\frac{x}{y}\right)^2\right] \cos\left(k \frac{\pi}{2} \frac{h}{a} \frac{x}{y}\right) \qquad (4.7)$$

where

α = generalized coefficient.

Differentiating equation (4.7) substituting into equation (4.6),

$$P\alpha_1 + Q\alpha_2 = 0 \qquad \text{at} \qquad y = h/2$$

$$R\alpha_1 + S\alpha_2 = 0 \qquad \text{at} \qquad y = 2h/3$$

where

P, Q, R, and S are in terms of β and θ

$$\beta = \frac{\rho \omega^2 h^4}{D} .$$

For various ratios of h/a and θ, values of β may be determined from the condition

$$\begin{vmatrix} P & Q \\ R & S \end{vmatrix} = 0$$

Fig. 4 gives value of γ, $\gamma = \sqrt{\beta}$, where

$$0 \leq \theta \leq 25°$$

$$\omega = \frac{\gamma}{h^2}\sqrt{\frac{D}{\rho}}$$

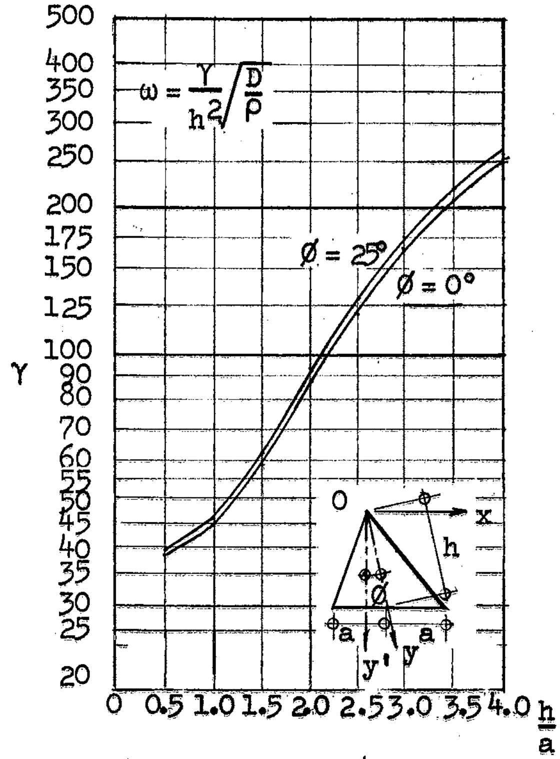


Fig. 4

Vibration Coefficients for Clamped Triangular Plates[9]

4-3. Vibration of Isosceles Triangular Plate Having the Base Clamped and the Other Edges Simply Supported[10].

The method of collocation is employed to solve this problem. Let x and y be coordinates in the middle surface of the uniform elastic plate as shown in Fig. 5.

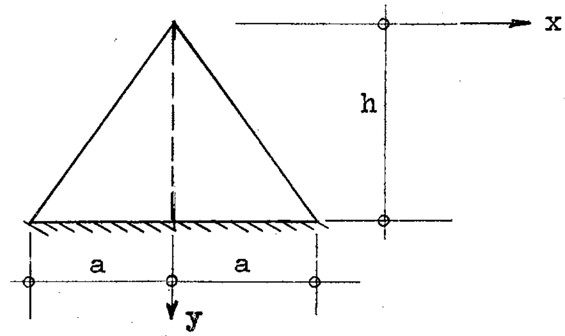


Fig. 5

Isosceles Triangular Plate

The governing differential equation is written as

$$\nabla^4 w - \frac{\rho \omega_n^2}{D} w = 0. \tag{4.8}$$

Boundary conditions are

$$(w)_{y = h} = (w)_{x = \pm (a/h)y} = 0$$

$$\left(\frac{\partial w}{\partial y}\right)_{y = h} = 0$$

$$\left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2}\right)_{x = \pm (a/h)y} = 0$$

where

n = normal direction to the lines $x = \pm (a/h)y$

t = tangential direction of any line along a rectilinear edge

$\frac{\partial^2 w}{\partial t^2} = 0$, $\frac{\partial^2 w}{\partial n^2} = 0$, on the boundary.

The deflection function is

$$W = \left\{ \alpha_1 y^2 \sin^2 \frac{\pi y}{h} + \alpha_2 y^2 \sin \frac{\pi y}{h} \sin \frac{2\pi y}{h} + \alpha_3 \frac{y^2}{h^4} [y^2 (y - h)^2] \right\} \cos \left(\frac{\pi}{2} \frac{h}{a} \frac{x}{y} \right) \tag{4.9}$$

Differentiating equation (4.9) and substituting the proper derivative into equation (4.8),

$$\left.\begin{aligned} A\,\alpha_1 + B\,\alpha_2 + C\,\alpha_3 &= 0 \qquad \text{at} \quad y = h/2 \\ J\,\alpha_1 + E\,\alpha_2 + F\,\alpha_3 &= 0 \qquad \text{at} \quad y = 2h/3 \\ G\,\alpha_1 + H\,\alpha_2 + I\,\alpha_3 &= 0 \qquad \text{at} \quad y = 3h/3 \end{aligned}\right\} \tag{4.10}$$

where A, B, C, D, E, F, G, H, and I are interms of h/a and β,

$$\beta = \frac{\rho \omega_1^2 h^4}{D}, \quad \gamma = \beta^{1/2}.$$

Values of γ for various ratio of h/a may be determined from the condition

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix} = 0$$

The relationship between the vibration coefficient γ, and h/a is shown in Fig. 6.

5. Vibration of Simply Supported Isosceles Trapezoidal Plates[11].

The approximate solutions are obtained by using the method of collocation. Let x and y be rectangular coordinates in the middle surface of the plate as shown in Fig. 7.

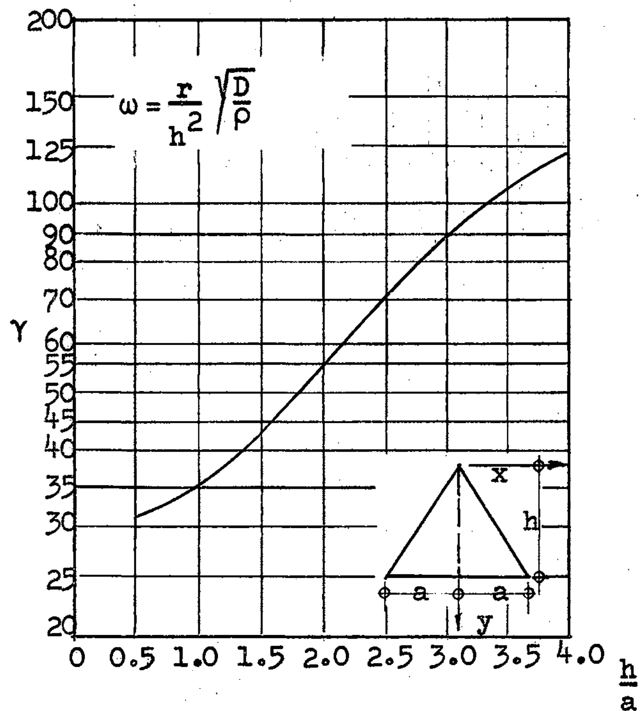


Fig. 6

Vibration Coefficients for Triangular Plates Having the Base Clamped and Equal Sides Simply - Supported

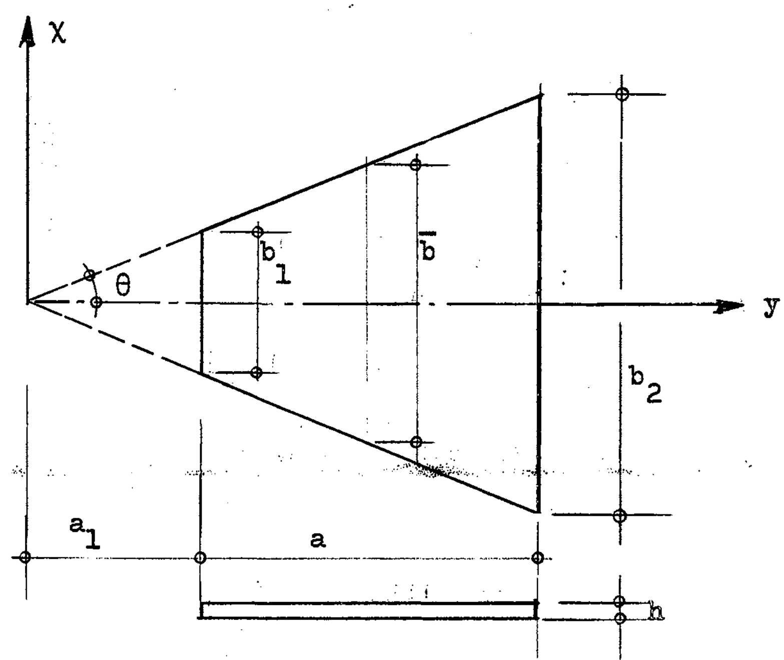


Fig. 7

Isosceles Trapezoidal Plate

The governing differential equation is written:

$$\nabla^4 w - \frac{\rho \omega_n^2}{D} w = 0 . \tag{5.1}$$

Boundary conditions are

$$(w)_{y = a_1} = (w)_{y = a_1 + a} = (w)_{x = \pm y \tan\theta} = 0 \tag{5.2}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y = a_1} = \left(\frac{\partial^2 w}{\partial y^2}\right)_{y = a_1 + a} = 0 \tag{5.3}$$

$$\left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2}\right)_{x = \pm y \tan\theta} = 0 \tag{5.4}$$

where

n = normal direction to lines $x = \pm y \tan\theta$

t = tangential direction of the lines

$\frac{\partial^2 w}{\partial t^2} = 0$, on the boundary.

The deflection function is

$$w = \left[\alpha_1 \sin \frac{\pi(y - a_1)}{a} + \alpha_2 \sin \frac{2\pi(y - a_1)}{a} + \alpha_3 \sin \frac{3\pi(y - a_1)}{a}\right] \cos\left(\frac{\pi}{2} \frac{x}{y} \cot\theta\right) . \tag{5.5}$$

Differentiating equation (5.5) and substituting the proper derivatives into equation (5.1),

$$\left.\begin{aligned} A\alpha_1 + B\alpha_2 + C\alpha_3 &= 0 \quad \text{at } \frac{y - a_1}{a} = 1/3 \\ T\alpha_1 + E\alpha_2 + F\alpha_3 &= 0 \quad \text{at } \frac{y - a_1}{a} = 1/2 \\ G\alpha_1 + H\alpha_2 + I\alpha_3 &= 0 \quad \text{at } \frac{y - a_1}{a} = 2/3 \end{aligned}\right\} \tag{5.6}$$

where A, B, C, J, E, F, G, H, and I are interms of β, θ and $\frac{b_1}{a}$

$$\beta = \frac{\rho \omega_1^2 h^4}{D}.$$

Values of β for various values of b_1/a and θ may be determined from the condition

$$\begin{vmatrix} A & B & C \\ J & E & F \\ G & H & I \end{vmatrix} = 0$$

Fig. 8 shows the relationship between b_1/h and the values of β.

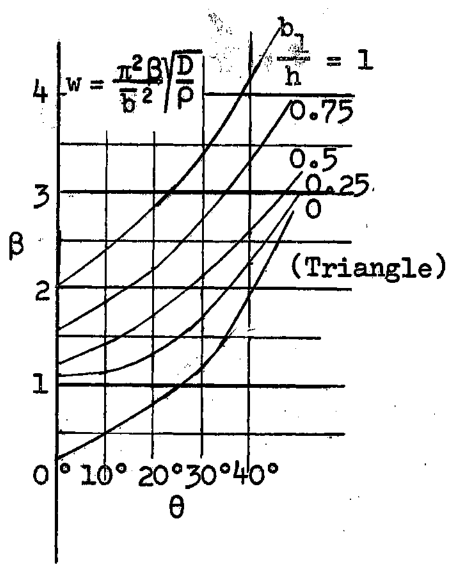


Fig. 8

Fundamental Frequency of Isosceles Trapezoidal Plate vs θ for Various Values of b_1/h (11)

6. Vibration of Thin Skew Plates[17].

Rayleigh's method will be employed to determine the upper bound to the natural frequency and Kato's theorem is used for determining a closer lower bound.

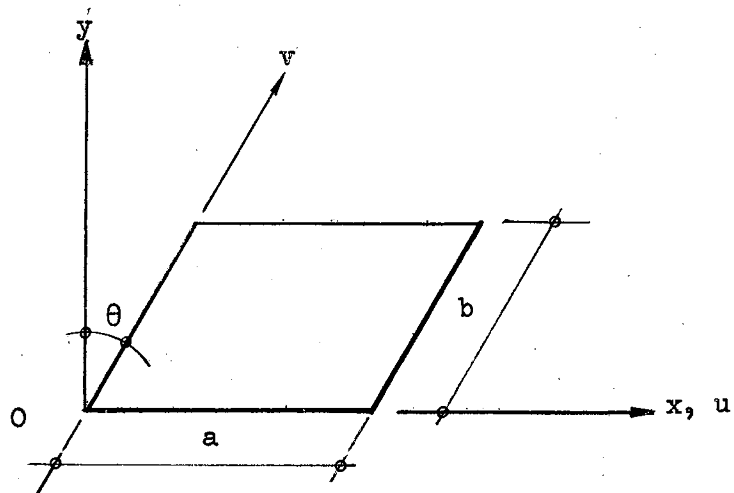


Fig. 9

Skew Coordinates

6-1. Rayleigh Method.

The frequency equations in terms of the skew coordinate system (u, v), as shown in Fig. 9, are

$$\frac{\partial}{\partial A_i} \iint \left[\left(\frac{\partial^2 W}{\partial u^2} + 2 \frac{\partial^2 W}{\partial u \partial v} \sin \theta + \frac{\partial^2 W}{\partial v^2} \right)^2 + \rho_R^{\ 2}\, W \right] dudv = 0 \qquad (6.1)$$

where

$\rho_R^{\ 2}$ = Rayleigh's ratio .

Taking the deflection W in the form

$$W = \sum_{m=1}^{p} \sum_{n=1}^{q} A_{mn} \varphi_m(u) \phi_n(v)$$

The normal orthogonal bar eigenfunctions are:

(A) Clamped - Clamped bar

$$\varphi_m = \frac{1}{\sqrt{a}} \left\{ \frac{\sin [K_m(u - a/2)]}{\sin (K_m a/2)} - \frac{\sinh [K_m(u - a/2)]}{\sinh (K_m a/2)} \right\} \cos^2 \frac{m\pi}{2}$$

$$+ \frac{1}{\sqrt{a}} \left\{ \frac{\cos [K_m(u - a/2)]}{\cos (K_m a/2)} - \frac{\cosh [K_m(u - a/2)]}{\cosh (K_m a/2)} \right\} \sin^2 \frac{m\pi}{2}$$

where $K_m a$ is the mth positive root of the transcendental equation

$$\tan (K_m a/2) = (-1)^m \tanh (K_m a/2)$$

$$m = 1, 2, 3, \ldots .$$

(B) Clamped-Simply Supported bar

$$\varphi_m(u) = \frac{1}{\sqrt{a}} \left\{ \frac{\sin [K_m(u - a)]}{\cos K_m a} - \frac{\sinh [K_m(u - a)]}{\cosh K_m a} \right\}$$

where $K_m a$ is the mth root of the transcendental equation

$$\tan K_m a = \tanh K_m a .$$

The values of ρ_R for various edge conditions of a rombic skew plate with different skew angle is shown in Table VII.

6-2. Kato's Method.

The equation of motion of a thin plate, in the skew coordinate system, is

$$\nabla^2 \left\{ \nabla^2 w - 4 \sin \theta \frac{\partial^2 w}{\partial u \partial v} \right\} + 4 \sin^2 \theta \frac{\partial^4 w}{\partial u^2 \partial v^2} - \lambda_r^2 w = 0$$

where

λ_r = eigenvalue.

The measure of accuracy ε_0^2 is

$$\varepsilon_o^2 = \frac{\int_0^a\int_0^a \left\{ \nabla^2 \left[\nabla^2 W - 4 \sin\theta \frac{\partial^2 W}{\partial u \partial v}\right] + 4 \sin^2\theta \frac{\partial^4 W}{\partial u^2 \partial v^2} - \rho_R^2 W \right\}^2 du dv}{\int_0^a \int_0^b W^2 \, du \, dv}$$

In applying Kato's theory for determining the lower bound to an eigenvalue λ_1^2, for which the closer upper bound is $\rho_{R1}^2 \geq \lambda_1^2$, $\beta^2 = \mu^2$ is taken, where μ^2 is the smallest eigenvalue greater than λ_1^2 and a lower estimate to λ_2^2,

$$\left(\rho_R^2 - \frac{\varepsilon_0^2}{\beta^2 - \rho_R^2}\right) \leq \lambda_1^2 \leq \rho_R^2$$

$$\rho_K = \left(\rho_R^2 - \frac{\varepsilon_0^2}{\beta^2 - \rho_R^2}\right)^{1/2} \text{. Kato's lower bound.}$$

The values of ρ_K for a rombic skew plate with various edge conditions is shown in Table X.

7. Free Vibration of a Gridwork(4).

A gridwork of beams extending in the x and y directions as shown in Fig. 10 is considered. The portion of the total load p(x, y) carried by the beams in the x direction and the y direction is given by

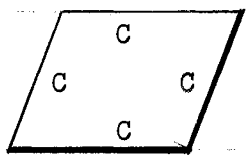
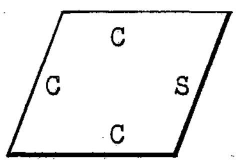
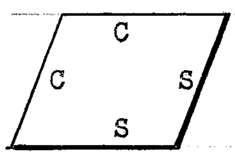
$$D \frac{\partial^4 w}{\partial x^4} = p(x) \quad ; \quad D \frac{\partial^4 w}{\partial y^4} = p(y) \text{ .} \tag{7.1}$$

For a gridwork of beams, the torsional resistance is small in comparison with the bending resistance; thus, the deflection equation can be

TABLE X

LIMITING BOUNDS FOR ROMBIC SKEW PLATES

(m = 1 , n = 1)

edge conditions	θ	ρ_K	ρ_R
	0°	35.33322	36.10868
	15°	34.69011	36.66593
	30°	32.95941	38.14697
	45°	30.63837	40.08173
	0°	31.46043	31.95364
	15°	31.46798	32.54105
	30°	30.35069	34.09421
	45°	29.46388	36.10806
	0°	26.22513	27.19478
	15°	24.91261	27.83775
	30°	21.45018	29.52310

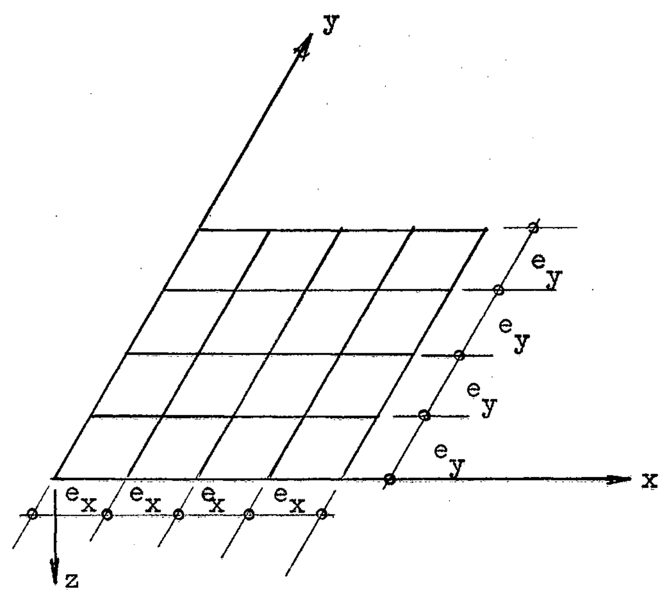


Fig. 10

Gridwork of Beams

written as follows:

$$D\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4}\right) = p(x,\ y). \tag{7.2}$$

Taking $\nu = 0$, and assuming the moment of inertia per unit length of the gridwork is not the same in the two principal directions,

$$\frac{E_x I_x}{e_x}\frac{\partial^4 w}{\partial x^4} + \frac{E_y I_y}{e_y}\frac{\partial^4 w}{\partial y^4} = p(x,\ y) \tag{7.3}$$

where $E_x I_x$ and $E_y I_y$ represent the flexural rigidity of an individual beam in the x and y directions, respectively; e_x and e_y are the spacings between two adjacent beams in the x and y directions, respectively.

The equation of free vibration is

$$\frac{E_x I_x}{\rho_x e_x}\frac{\partial^4 w}{\partial x^4} + \frac{E_y I_y}{\rho_y e_y}\frac{\partial^4 w}{\partial y^4} + \frac{\partial^2 w}{\partial t^2} = 0 \tag{7.4}$$

Let

$$\frac{E_x I_x}{e_x} = D_x \qquad ; \qquad \frac{E_y I_y}{e_y} = D_y \tag{7.5}$$

Solutions of the form

$$w = X(x)\ Y(y)\ q(t) \tag{7.6}$$

are investigated.

Substitution of equations (7.5) and (7.6) into equation (7.4) yields

$$\frac{D_x X^{iv}}{\rho_x X} + \frac{D_y Y^{iv}}{\rho_y Y} = -\frac{\ddot{q}}{q} \tag{7.7}$$

Let equation (7.7) equal to a constant p^2, thus

$$\ddot{q} + p^2 q = 0 \tag{7.8}$$

$$\frac{D_x X^{iv}}{\rho_x X} = -\frac{D_y Y^{iv}}{\rho_y Y} + p^2 \tag{7.9}$$

Let equation (7.9) be equal to a new constant k^2, thus

$$D_x X^{iv} - \rho_x k^2 X = 0 \tag{7.10}$$

$$D_y Y^{iv} - \rho_y (p^2 - k^2) Y = 0\ . \tag{7.11}$$

The solutions of equations (7.8), (7.10), and (7.11) are

$$q(t) = A \sin p\,t + B \cos p\,t \tag{7.12}$$

$$X = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x \tag{7.13}$$

$$Y = G_1 \sin \lambda' x + G_2 \cos \lambda' y + G_3 \sinh \lambda' y + G_4 \cosh \lambda' y \tag{7.14}$$

where

$$\lambda^4 = \rho_x K^2 / D_x$$

$$\lambda'^4 = \rho_y (p^2 - K^2) D_y\ .$$

PART III

VIBRATION OF THIN SHELLS

1. General[2].

Consider a shell element bounded by curves of the curvilinear rectangle α, $\alpha + \delta\alpha$, β and $\beta + \delta\beta$ as shown in Fig. 11.

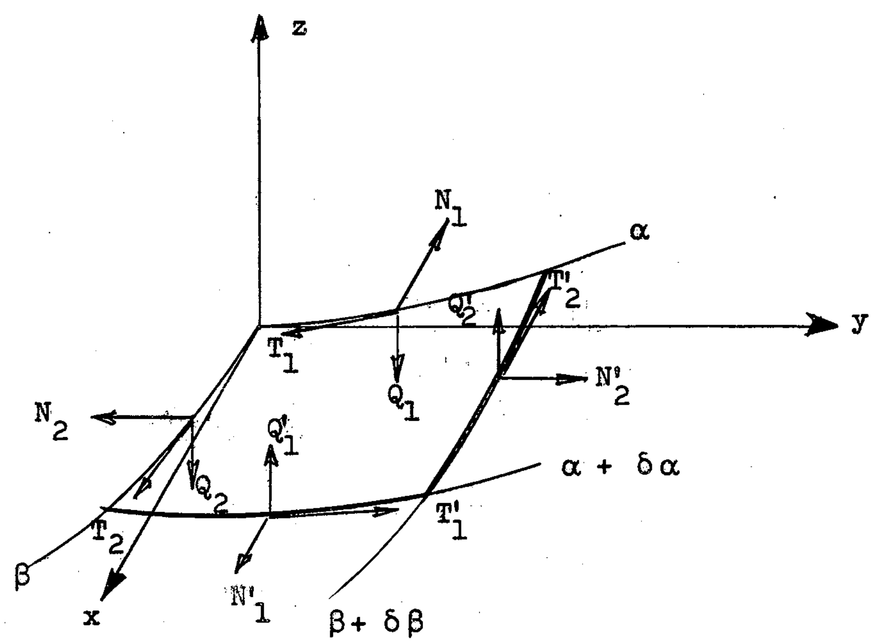


Fig. 11

Element of Shell

The equations of vibration can be written as follow

$$\frac{1}{AB}\left[\frac{\partial(N_1 B)}{\partial\alpha} - \frac{\partial(T_2 A)}{\partial\beta} + T_1\frac{\partial A}{\partial\beta} - N_2\frac{\partial\beta}{\partial\alpha}\right] - \frac{Q_1}{R_1} = 2\rho h\frac{\partial^2 u}{\partial t^2}$$

$$\frac{1}{AB}\left[\frac{\partial(T_1B)}{\partial\alpha}+\frac{\partial(N_2A)}{\partial\beta}-N_1\frac{\partial A}{\partial\beta}-T_2\frac{\partial\beta}{\partial\alpha}\right]-\frac{Q_2}{R_2}=2\rho h\frac{\partial^2 v}{\partial t^2} \qquad (8.1)$$

$$\frac{1}{AB}\left[\frac{\partial(Q_1B)}{\partial\alpha}+\frac{\partial(Q_2A)}{\partial\beta}\right]+\frac{N_1}{R_1}+\frac{N_2}{R_2}=2\rho h\frac{\partial^2 w}{\partial t^2}$$

where

N_1 , N_2 = normal stress

T_1 , T_2 = tangential stress

Q_1 , Q_2 = transverse shearing stress

H_1 , G_1 = stress couple in the same directions as N_1 , T_1

H_2 , G_2 = stress couple in the same directions as N_2 , T_2

u, v, w = deflections in the x, y and z directions, respectively

A , B = function of α, β.

$\frac{1}{R_1}$, $\frac{1}{R_2}$ = curvatures in the x and y directions, respectively

2. Free Vibration of Thin Cylindrical Shells(12).

Neglecting the rotatory inertia, the equations of vibration for an element of a cylindrical shell can be written as

$$\nabla^4 u-\frac{\nu}{R}\frac{\partial^3 w}{\partial x^3}+\frac{1}{R}\frac{\partial^3 w}{\partial x\,\partial s^2}$$

$$=-\frac{2(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\left(\frac{1-\nu^2}{E}\rho\frac{\partial^2 u}{\partial t^2}-\frac{3-\nu}{2}\nabla^2 u+\frac{\nu}{R}\frac{\partial w}{\partial x}\right) \qquad (9.1)$$

$$\nabla^4 v-\frac{2+\nu}{R}\frac{\partial^3 w}{\partial x^2\,\partial s}-\frac{1}{R}\frac{\partial^3 w}{\partial s^3}$$

$$=-\frac{2(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\left(\frac{1-\nu^2}{E}\rho\frac{\partial^2 v}{\partial t^2}-\frac{3-\nu}{2}\nabla^2 v+\frac{1}{R}\frac{\partial w}{\partial s}\right) \qquad (9.2)$$

$$\frac{h^2}{12} \nabla^8 w + \frac{1 - \nu^2}{R^2} \frac{\partial^4 w}{\partial x^4}$$

$$= - \frac{2(1 + \nu)}{E} \rho \frac{\partial^2}{\partial t^2} \left[\left(\frac{1 - \nu^2}{E} \rho \frac{\partial^2}{\partial t^2} - \frac{3 - \nu}{2} \nabla^2 \right) \right.$$

$$\left(\frac{1 - \nu^2}{E} \rho \frac{\partial^2 w}{\partial t^2} + \frac{w}{R^2} + \frac{h^2}{12} \nabla^4 w \right) + \frac{1 - \nu}{2} \nabla^4 w + \frac{\nu^2}{R^2} \frac{\partial^2 w}{\partial x^2}$$

$$\left. + \frac{1}{R^2} \frac{\partial^2 w}{\partial s^2} \right] \qquad (9.3)$$

where

$$s = R \phi$$

The displacement components are assumed in the form

$$\left. \begin{aligned} u &= \sum_i A_i e^{\lambda_i \frac{x}{l}} \cos m \phi \sin \omega t \\ v &= \sum_i B_i e^{\lambda_i \frac{x}{l}} \sin m \phi \sin \omega t \\ w &= \sum_i C_i e^{\lambda_i \frac{x}{l}} \cos m \phi \sin \omega t \end{aligned} \right\} \qquad (9.4)$$

Substituting equation (8.4) into equations (9.1), (9.2) and (9.3), and assuming

$$\frac{\left| \lambda_i^2 \right| R^2}{m^2 l^2} << 1 \qquad (9.5)$$

the following expressions are obtained

$$A_i = C_i \lambda_i M \frac{R}{l} \qquad (9.6)$$

$$B_i = C_i N \qquad (i = 1, 2, 3, \cdot \cdot \cdot) \qquad (9.7)$$

$$F = (1 - \nu)(1 - \nu^2)\left(\frac{\lambda_i R}{l}\right)^4 \qquad (9.8)$$

in which

$$M = \frac{2\nu\Omega + (1-\nu)m^2}{2\Omega^2 - (3-\nu)m^2\Omega + (1-\nu)m^4}$$

$$N = \frac{-2m\Omega + (1-\nu)m^3}{2\Omega^2 - (3-\nu)m^2\Omega + (1-\nu)m^4}$$

$$F = 2\Omega^3 - \Omega^2\left[2 + (3-\nu)m^2 + 2km^4\right] + \Omega\left[(1-\nu)m^2(m^2+1) + (3-\nu)km^6\right] - (1-\nu)km^8$$

where

l = length of shell

m = positive integer equal to the number of circumferential waves

A_i , B_i , C_i = constant coefficients

$$\Omega = \frac{1-\nu^2}{E}\rho R^2\omega^2 \quad , \quad k = \frac{h^2}{12R^2}$$

The roots of λ_i of equation (9.8) are of the form

$$\lambda_1 = K \;,\; \lambda_2 = -K \;,\; \lambda_3 = iK \;,\; \lambda_4 = -iK \tag{9.9}$$

where K is a real number.

By application of equations (9.6), (9.7), (9.8) and (9.9), the frequency equations and displacement components have been obtained for the following two cases.

(A) Shell with Both Edges Freely Supported

The frequency equation is

$$2\Omega^3 - \Omega^2\left[2 + (3-\nu)m^2 + 2km^4\right] + \Omega\left[(1-\nu)m^2(m^2+1) + (3-\nu)km^6\right] - (1-\nu)km^8 - (1-\nu)(1-\nu^2)\left(\frac{n\pi R}{l}\right)^4 = 0 \tag{9.10}$$

The displacement components are

$$\left.\begin{aligned} u &= MC \frac{n\pi R}{l} \cos \frac{n\pi x}{l} \cos m\phi \sin \omega t \\ v &= NC \sin \frac{n\pi x}{l} \sin m\phi \sin \omega t \\ \omega &= C \sin \frac{n\pi x}{l} \cos m\phi \sin \omega t \end{aligned}\right\} \tag{9.11}$$

where $n = 1, 2, 3, 4, \ldots$

(B) Shells With Both Edges Clamped

The frequency equation is

$$\begin{aligned} 2\Omega^3 - \Omega^2 & \left[2 + (3 - \nu)m^2 + 2km^4\right] \\ & + \Omega\left[(1-\nu)m^2(m^2+1) + (3 + \nu)km^6\right] \\ & - (1-\nu)km^8 - (1-\nu)(1-\nu^2)\left(\frac{n\pi R}{l}\right)^4 = 0 \end{aligned} \tag{9.12}$$

The displacement components are

$$\left.\begin{aligned} w = 2C & \Big[(\sinh n\pi - \sin n\pi) - \\ & (\cosh n\pi - \cos n\pi)\Big]^{-1}\Big[(\sinh n\pi - \sin n\pi) \\ & \left(\cosh \frac{n\pi x}{l} - \cos \frac{n\pi x}{l}\right) - (\cosh n\pi - \cos n\pi) \\ & \left(\sinh \frac{n\pi x}{l} - \sin \frac{n\pi x}{l}\right)\Big] \cos m\phi \sin \omega t \\ u = & MR \frac{\partial w}{\partial x} \\ v = & -\frac{NR}{m} \frac{\partial w}{\partial s} \end{aligned}\right\} \tag{9.13}$$

$n = 1.506, 2.500, 3.500, 4.500$

3. Vibration of Shallow Spherical Shells.

The equations of vibration for a shallow spherical shell can be written as(13)

$$r \frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} - \frac{v}{r} + (1 + \nu)\frac{r}{R} \frac{\partial w}{\partial r} + \frac{h \rho \omega^2}{N'} rv = 0 \tag{10.1}$$

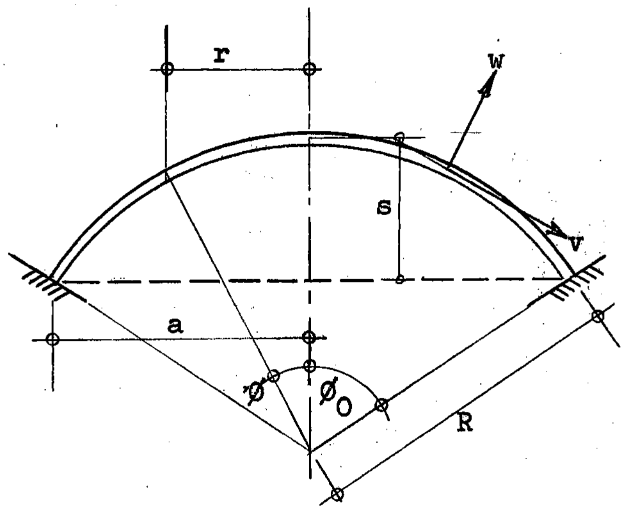


Fig. 12

Section of Spherical Shell

$$\frac{\partial}{\partial r}\left[r\,\frac{\partial^3 w}{\partial r^3} + \frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\,\frac{\partial w}{\partial r}\right]$$

$$+\left[(1+\nu)\,\frac{N'}{RD}\right]\left[r\,\frac{\partial v}{\partial r} + v + \frac{2rw}{R}\right] - \left(\frac{h\rho\,\omega^2}{D}\right) rw = 0 \qquad (10.2)$$

where

$$N' = \frac{Eh}{1-\nu^2}$$

Expressing equations (10.1) and (10.2) in terms of Bessel functions, the solutions of which turn out to be [14]

$$v = -m_1^2\left\{\frac{B_1 J_1(\mu_1 r)}{\alpha^2-\mu_1^2} + \frac{B_2 J_1(\mu_2 r)}{\alpha^2-\mu_2^2} + \frac{B_3 J_1(\mu_3 r)}{\alpha^2-\mu_3^2}\right\}$$

$$w = -\left\{\frac{1}{\mu_1}J_0(\mu_1 r) + \frac{2}{\mu_2}J_0(\mu_2 r) + \frac{B_3}{\mu_3}J_0(\mu_3 r)\right\}$$

For a given frequency ω_n

$$x_i = (\mu_i a)^2$$

which are the roots of the following cubic equation:

$$\left[\frac{(1-\nu^2)\rho a^2}{E}\omega_n^2 - x\right]\left[x^2 - 12(1-\nu^2)\rho\frac{\omega_n^2 a^4}{Eh^2} + 96\frac{s}{h^2}(1+\nu)\right] + 48(1+\nu)^2\frac{s^2}{h^2}x = 0$$

where

a = half the base chord

s = rise of arc

r = radial distance from point on sphere to axis of symmetry

$$\alpha^2 = \frac{\rho h \omega_n^2}{N'}$$

$$m_1^2 = \frac{1+\nu}{R} = \frac{2(1+\nu)s}{R}$$

$J_0(x)$ = Bessel function of order zero

$J_1(x)$ = Bessel function of order one

Boundary conditions are:

Case A. Clamped Edge

$$w_n(a) = w_n'(a) = v_n(a) = 0$$

The possible frequencies follow from the determinantal equation

$$\begin{vmatrix} \frac{J_0(x_1)}{x_1} & \frac{J_0(x_2)}{x_2} & \frac{J_0(x_3)}{x_3} \\ J_1(x_1) & J_1(x_2) & J_1(x_3) \\ \frac{J_1(x_1)}{(\alpha a)^2 - x_1^2} & \frac{J_1(x_2)}{(\alpha a)^2 - x_2^2} & \frac{J_1(x_3)}{(\alpha a)^2 - x_3^2} \end{vmatrix} = 0$$

and also

$$B_2 = \frac{x_3^2 - x_1^2}{x_2^2 - x_3^2} \cdot \frac{(\alpha\, a)^2 - x_2^2}{(\alpha\, a)^2 - x_1^2} \; \frac{J_1(x_1)}{J_1(x_2)} \; B_1$$

$$B_3 = - \left[1 + \frac{x_3^2 - x_1^2}{x_2^2 - x_3^2} \cdot \frac{(\alpha a)^2 - x_2^2}{(\alpha a)^2 - x_1^2} \right] \frac{J_1(x_1)}{J_1(x_3)} \, B_1$$

Case B. Simply Supported Edge

$$w_n(a) = v_n(a) = M\, \emptyset(a) = 0$$

and so

$$\begin{vmatrix} \dfrac{J_0(x_1)}{x_1} & \dfrac{J_0(x_2)}{x_2} & \dfrac{J_0(x_3)}{x_3} \\ \begin{bmatrix} J_1(x_1)(1 - \nu) \\ - x_1 J_0(x_1) \end{bmatrix} & \begin{bmatrix} J_1(x_2)(1 - \nu) \\ - x_2 J_0(x_2) \end{bmatrix} & \begin{bmatrix} J_1(x_3)(1 - \nu) \\ - x_3 J_0(x_3) \end{bmatrix} \\ \dfrac{J_1(x_1)}{(\alpha a)^2 - x_1^2} & \dfrac{J_1(x_2)}{(\alpha a)^2 - x_2^2} & \dfrac{J_1(x_3)}{(\alpha a)^2 - x_3^2} \end{vmatrix} = 0$$

and

$$B_3 = - \frac{\dfrac{(\alpha\, a)^2 - x_1^2}{(\alpha\, a)^2 - x_2^2} \cdot \dfrac{J_1(x_2)}{J_1(x_1)} - \dfrac{x_1 J_0(x_2)}{x_2 J_0(x_1)}}{\dfrac{(\alpha\, a)^2 - x_1^2}{(\alpha\, a)^2 - x_3^2} \cdot \dfrac{J_1(x_3)}{J_1(x_1)} - \dfrac{x_1 J_0(x_3)}{x_3 J_0(x_1)}} \; B_2 = \lambda B_2$$

$$B_1 = \left[\frac{- x_1 J_0(x_2)}{x_2 J_0(x_1)} + \lambda \, \frac{x_1 J_0(x_3)}{x_3 J_0(x_1)} \right] B_2$$

The functions $J_0(x)$ and $J_1(x)$ are evaluated with the aid of standard tables.

The frequencies of vibration for both two cases are evaluated (Table 11) by assuming

$$\nu = \frac{1}{3} \;, \quad \frac{h}{R} = \frac{1}{60} \;, \quad \frac{E}{\rho} = \frac{30 \times 10^6}{0.2836}$$

TABLE XI

FREQUENCIES OF VIBRATION

Frequencies for Clamped Edge in rps

Mode \ s/a	0	$\frac{0.5}{6}$	$\frac{1.0}{6}$	$\frac{1.6}{6}$
1st	4,400	9,000	16,000	22,000
2nd	17,160	19,000	22,000	29,000
3rd	38,390	39,000	40,000	43,000

Frequencies for Simply Supported Edge in rps

Mode \ s/a	0	$\frac{0.5}{6}$	$\frac{1.0}{6}$	$\frac{1.6}{6}$
1st	2,100	9,000	16,000	21,000
2nd	12,760	15,000	20,000	29,000
3rd	31,850	32,000	34,000	38,000

4. Vibration of Conical Shells[15].

The Rayleigh-Ritz method is used to determine the natural frequency of the conical shell.

Each displacement is assumed in the form

$$w(z, \theta, t) = w(z, \theta) \sin \omega t$$

$$u_1(z, \theta, t) = u_1(z, \theta) \sin \omega t \qquad (11.1)$$

$$u_2(z, \theta, t) = u_2(z, \theta) \sin \omega t \,.$$

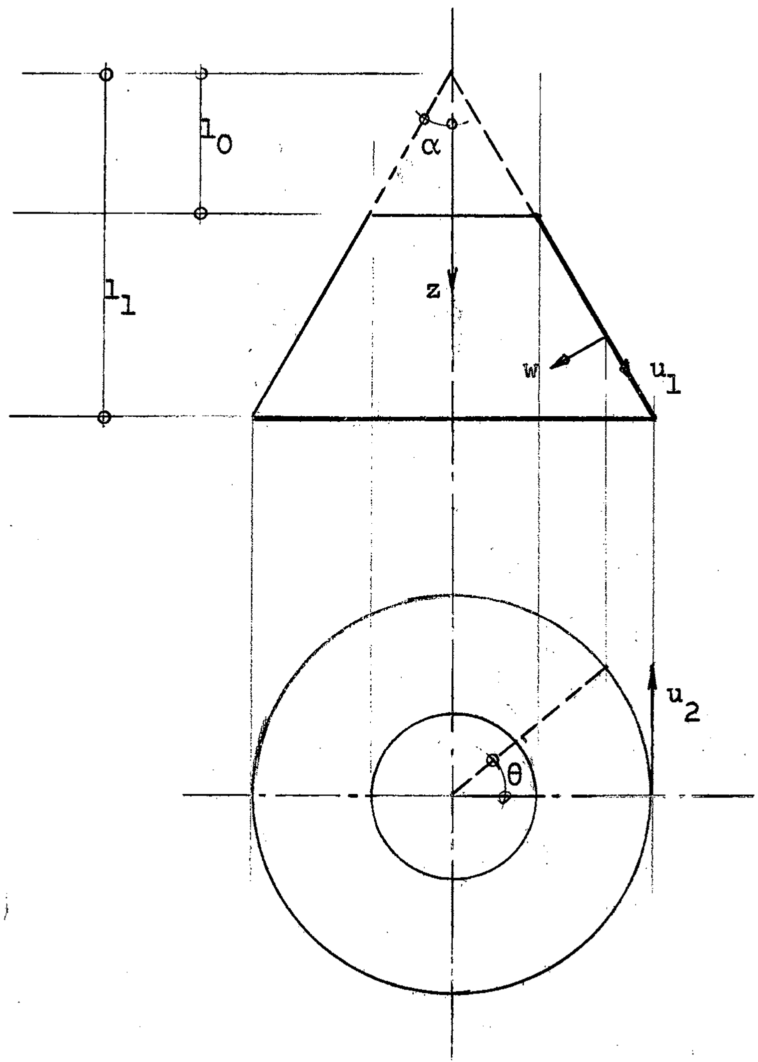


Fig. 13

Section of Conical Shell

The middle surface strains ε_1 , ε_2 and ε_3 , and the changes of curvature k_1 , k_2 and k_{12} are given by Love[21].

$$\varepsilon_1 = \frac{\partial u_1}{\partial z} \cos \alpha$$

$$\varepsilon_2 = \frac{\partial u_2}{\partial \theta} \frac{\cos \alpha}{z \sin \alpha} + \frac{u_1}{z} \cos \alpha - \frac{w}{z} \frac{\cos^2 \alpha}{\sin \alpha}$$

$$\varepsilon_{12} = z \cos\alpha \quad \frac{\partial}{\partial z}\left(\frac{u_2}{z}\right) + \frac{\cos\alpha}{z \sin\alpha} \frac{\cos^2\alpha}{\sin\alpha}$$

$$K_1 = \frac{\partial^2 w}{\partial z^2} \cos^2\alpha$$

$$K_2 = \frac{\partial u_2}{\partial\theta} \frac{\cos^3\alpha}{z^2 \sin^2\alpha} + \frac{\cos^2\alpha}{z^2 \sin^2\alpha} \frac{\partial^2 w}{\partial\theta^2} + \frac{\cos^2\alpha}{z} \frac{\partial w}{\partial z}$$

$$K_{12} = \frac{\partial u_2}{\partial z} \frac{\cos^3\alpha}{z \sin\alpha} - \frac{\cos^3\alpha}{z^2 \sin\alpha} u_2 + \frac{\cos^2\alpha}{z \sin\alpha} \frac{\partial^2 w}{\partial z \partial\theta}$$

$$- \frac{\cos^2\alpha}{z^2 \sin\alpha} \frac{\partial w}{\partial\theta}$$

The potential energy and kinetic energy are

$$V = \frac{1}{2} \frac{Eh}{(1 - \nu^2)} \sin^2 \omega t \int_{\theta=0}^{2\pi} \int_{z=l_0}^{l_1} \left\{ \frac{h^2}{12} \left[(K_1 + K_2)^2 - 2(1 - \nu)(K_1 K_2 - K_{12}^2) \right] + (\varepsilon_1 + \varepsilon_2)^2 - 2(1 - \nu)(\varepsilon_1 \varepsilon_2 - \varepsilon_{12}^2) \right\} \frac{z \sin\alpha}{\cos^2\alpha} \, dz \, d\theta$$

$$= V_{max} \sin^2 \omega t \tag{11.2}$$

and

$$T = \frac{1}{2} \rho h \omega^2 \cos^2 \omega t \int_{\theta=0}^{2\pi} \int_{z=l_0}^{l} (w^2 + u_1^2 + u_2^2) \frac{z \sin\alpha}{\cos^2\alpha} \, dz \, d\theta$$

$$= T_{max} \cos^2 \omega t \tag{11.3}$$

The Rayleigh-Ritz procedure applied to Hamilton's principle leads to

$$\frac{\partial}{\partial A_i} (T_{max} - V_{max}) = 0 , \tag{11.4}$$

The values of ω^2 can be obtained from equation (11.4).

5. Vibration of Thin Paraboloidal Shells of Revolution[16].

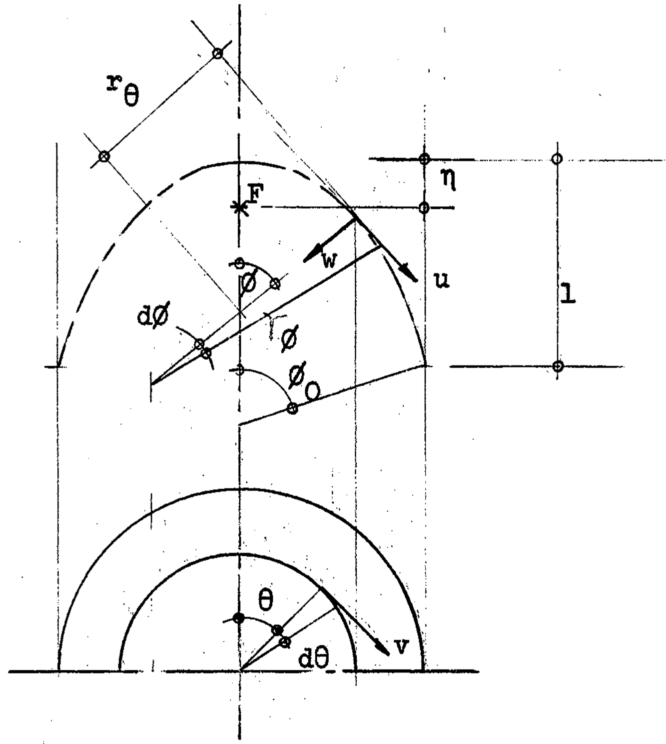


Fig. 14

Paraboloidal Shell

The governing equations for normal modes of vibration are

$$\frac{\partial u}{\partial \phi} - w = 0$$

$$\frac{\partial v}{\partial \theta} + \frac{u}{\sin \phi} - w \sin \phi = 0 \qquad (12.1)$$

$$\tan \phi \frac{\partial v}{\partial \phi} + \sec^3 \phi \ \frac{\partial u}{\partial \theta} - v \sec^2 \phi = 0$$

The solutions of equations (11.1) are the following

$$u_n = a_n \sin \phi \tan^n \phi \cos n\theta$$

$$v_n = a_n \tan^{n+1} \phi \sin n\theta$$

$$w_n = a_n \tan^n \phi (\cos \phi + n \sec \phi) \cos n\theta$$

where n is an integer representing the number of circumferential waves for the corresponding mode shape.

By equating the maximum kinetic and potential energies of the vibrating system, the natural frequencies of vibration can be obtained as follows:

$$\omega_n = \left\{ \frac{n^2(n^2-1)^2 E}{12(1-\nu^2)(2\eta)^4 \rho} \cdot \frac{\int_0^{\phi_0} h^3 \tan^{2n-3}\phi \sec^3\phi \,(\cos^2\phi + \sec^2\phi + 2 - 4\nu)\, d\phi}{\int_0^{\phi_0} h \tan^{2n+1}\phi \sec^3\phi \left[2n + (n^2+1)\sec^2\phi\right] d\phi} \right\}^{\frac{1}{2}} \qquad (12.2)$$

where

η = focal length of shell .

Fig. 15 shows the relationship between the frequency parameter

$$\Delta_n = \frac{\omega_n^2 \eta^4 h \rho}{D} ,$$

and the limit angle ϕ_0 (or $1/\eta$ ratio) at the boundary for uniform paraboloidal shells of revolution made of aluminum or steel ($\nu = 0.3$).

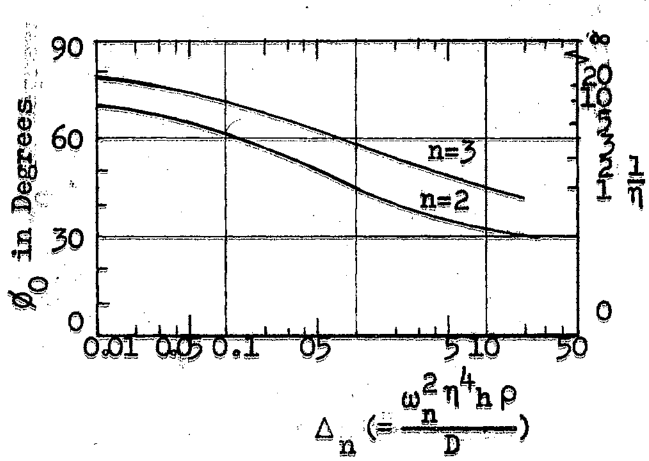


Fig. 15

Relation Between the Frequency Parameter and the Boundary Coordinate ϕ_0

PART IV

SUMMARY AND CONCLUSIONS

1. Summary.

In this report, a literature survey was made in the area of vibration of plate and shell structures. This will be of considerable value in future investigations in this area.

An exact solution for the natural frequencies of a simply supported rectangular plate has been obtained. The Rayleigh-Ritz method is employed to determine the approximate solution for the rectangular plate with other kinds of edge conditions. Characteristic functions of a vibrating beam are used for representing the deformations which lead to the solution.

In circular plates, the Rayleigh-Ritz method is also employed; Timoshenko[3] found that in all cases the frequencies of vibration of circular plates has the pattern

$$\omega = (\alpha/a^2)\sqrt{D/\rho} .$$

For the vibration of triangular plates, investigations have been conducted for the three kinds of boundary conditions: cantilever, all edges clamped, and the triangular plate with the base clamped and other edges simply supported. The method of collocation is employed for the latter two cases. This method is also extended to solving the simply supported isosceles trapezoidal plate.

The Rayleigh-Ritz method, with the aid of characteristic bar functions, is employed to solve for the natural frequencies of a skew plate with various edge conditions. Kato's method is also used for determining closer lower bounds for which upper bounds are provided by the Rayleigh-Ritz method.

In the shell section, four types of shells: cylindrical, spherical, conical, and paraboloidal shells of revolution, are observed. The vibration of a cylindrical shell has been investigated on the basis of a set of three different equations. Direct solutions of determinantal frequency equations for shallow spherical shells with clamped and simply supported edges are given. For the conical shell, a Rayleigh-Ritz procedure is used for determining the natural frequencies. The same method is also employed to obtain the approximate solution for frequencies of vibration of paraboloidal shells.

In this report, many numerical results are drawn from many investigators. They will be useful for further investigations.

2. Conclusions.

Extremely accurate solutions for the natural frequencies of vibration of thin elastic plates and shells may be difficult and laborious to obtain. Usually the Rayleigh-Ritz method is considered to be the most useful method for finding a reasonable approximate solution. But the results and the practicability of the computation depend to a great extent upon the set of functions that are chosen to represent the deformation. It is generally known that the Rayleigh-Ritz method yields frequencies that are higher than the actual frequencies, however, it is considered to be of sufficient accuracy for most design purposes.

In addition to the Rayleigh-Ritz method, the method of collocation is also one of the several possible procedures for obtaining approximate solutions for vibrating plates, especially for triangular plates and trapezoidal plates.

For determining a closer lower bound to the natural frequencies of thin skew plates for which an upper bound is provided by the Rayleigh-Ritz principle, Kato's method has been employed. The mean value of these two bounds give more reasonable results.

For shell structures, the differential equations of vibration are complicated; Bessel functions are introduced to simplify the evaluation.

In this report, the literature survey is conducted in the area of free vibrations. This will be the first step toward the complete comprehension of the vibration problems in shell and plate structures. Also more literature survey on the free and forced vibrations of plate and shell structures is needed.

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