COVERAGE DISTRIBUTIONS ASSOCIATED WITH THREE RECTANGULAR REGIONS

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CHAPTER I

INTRODUCTION

The study of the occurrence of numerous physical phenomena is currently being aided through the use of the technique of Monte Carlo simulation. In such problems, the basis for such a study is the development of a probabilistic model describing the phenomena. The present study is concerned with some of the probabilistic aspects of the Runway Cutter Program currently in use by the Oklahoma State University Field Office Computations Group at Eglin AFB, Florida.

The Runway Cutter Program was initially provided by the Operations Evaluation Group of the Center for Naval Analysis, (Lakin, 1966). The current program used is a modification of the above, (Jackett, 1970).

A Description of the Runway Cutter Program

The Runway Cutter Program uses simulation techniques to determine the number of passes necessary to cut the runway when various stick bombing methods are used. According to the present definition, a runway is considered to be cut when no longitudinal segment of the runway remains intact with width greater than a previously specified width. The cutting is usually done with a stick of weapons whose ideal impact points can be assumed to form a rectangular shaped pattern.

The size of the rectangular shaped pattern is dependent on a variety of input conditions. Some of these are: type of aircraft, aircraft airspeed, dive angle, intervalometer setting, type and number of weapons, just to mention some of the conditions. The various patterns for the above conditions are filed on tape and are entered into the program at the appropriate times.

The program uses a coordinate system corresponding to the line of flight of the aircraft. These coordinates are called the range and deflection coordinates where range refers to the axis in the direction of the line of flight and deflection to the axis perpendicular to the range axis. This coordinate system is a rotation through the approach angle θ, $0 \leq \theta \leq \pi/2$, of the coordinate system corresponding to the length and the width of the target with the origin at the center of the target. See Figure 1.

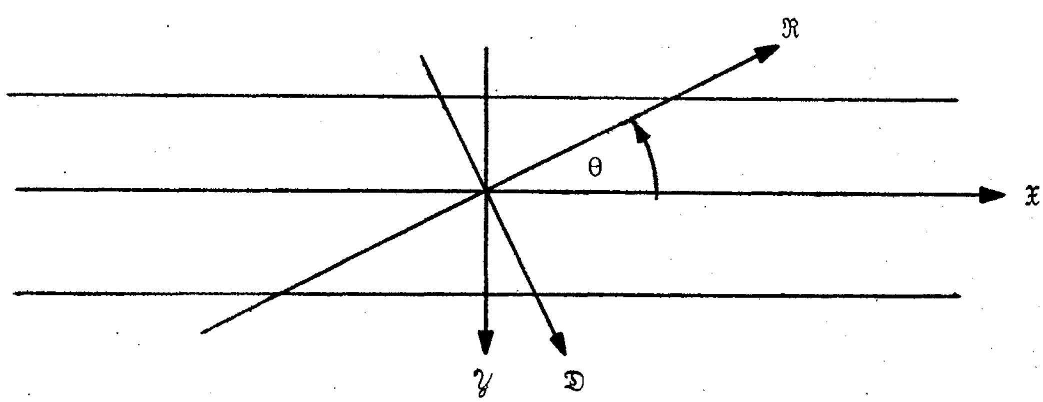

Figure 1. Runway Cutter Program Coordinate System

The (x, y) coordinates of a point are given in terms of the (r, d) coordinates by the orthogonal transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix} . \tag{1.1}$$

The aiming error of the center of the rectangular pattern containing the weapons is assumed to follow a bivariate normal distribution. That is, if (r_0, d_0) is the center of the pattern, then

$$\begin{pmatrix} r_0 \\ d_0 \end{pmatrix} \sim N\left[\begin{pmatrix} A_R \\ A_D \end{pmatrix} , \begin{pmatrix} \sigma_{AR} & 0 \\ 0 & \sigma_{AD} \end{pmatrix} \right] ; \tag{1.2}$$

where (A_R, A_D) are the coordinates of the aim point, and where σ_{AR} and σ_{AD} are the standard deviations of the aiming error in the range and deflection directions respectively.

The weapons within a pattern have a displacement (Δ_R, Δ_D) from the pattern center point (PCP). The displacements are determined relative to the previously stated input conditions. However, since some of the conditions influence the size of the pattern, the expansion or shrinkage of the pattern due to this influence is attained by using the scale factors (S_R, S_D). When the scale factors are used, the particular weapons retain the same relative position with respect to the PCP; however, the displacements will be changed. Consequently, the displacements will now be denoted by (δ_R, δ_D) where $\delta_R = S_R \Delta_R$ and $\delta_D = S_D \Delta_D$. Thus, if no ballistic errors were involved with the individual weapons, each weapon would impact at a point having the displacement (δ_R, δ_D) from the PCP's point of impact.

The ballistic error of each weapon in the pattern is assumed to follow a bivariate normal distribution about its ideal impact point within the pattern. This distribution is assumed to be the same for each weapon in the pattern. That is, if (r_i, d_i) denote the coordinates of the impact of the i^{th} weapon in the pattern, then

$$\begin{pmatrix} r_i \\ d_i \end{pmatrix} \sim N\left[\begin{pmatrix} \delta_{R_i} \\ \delta_{D_i} \end{pmatrix}, \begin{pmatrix} \sigma_{BR} & 0 \\ 0 & \sigma_{BD} \end{pmatrix}\right]; \tag{1.3}$$

where $(\delta_{R_i}, \delta_{D_i})$ is the aimpoint of the i^{th} weapon with respect to the PCP, and where σ_{BR} and σ_{BD} are the standard deviations of the ballistic error in the range and deflection directions respectively.

The program employs normally distributed random numbers from the above distributions $(A_R, A_D, \delta_{R_i}, \delta_{D_i}, \sigma_{AR}, \sigma_{BR}, \sigma_{AD}, \sigma_{BD}$ have specified values as determined from the input conditions) to locate the impact points of a pattern on the target. These points are then transformed onto the runway coordinate system by equation (1.1). The crater diameters of the impact points can either be fixed or can be randomly assigned according to a uniform distribution. Each crater is then tested to see if it is within the target hit zone. If so, that portion of the crater on the runway is projected onto two vectors representing the runway length and width. After each hit is thusly recorded, the contents of the width vector are examined to determine if the cut criterion is satisfied. If the criterion is satisfied, the cut is recorded and the program goes to another iteration.

The program output includes the frequency distribution of the situations where cuts occur on the first pass, second pass, etc. The cumulative distribution of the above frequency distribution, the quartile probabilities of the frequency distribution and the average number of passes required to achieve a cut are also part of the output.

The objective of the Runway Cutter Program is to determine some of the characteristics of the distribution of passes necessary to cut a runway. Since large amounts of computer time are used in the generation of data for the distribution of passes, the determination of an analytic solution to the prediction problem would be valuable. Using the information in the description of the program given above, the probability of achieving a cut in a prescribed number of passes will be discussed.

An integral part of the above discussion hinges on the single pass hit probabilities; that is, the probability of obtaining $0, 1, 2, \ldots, b$ hits in a single pass. These probabilities are dependent upon several factors, for example, the distribution of the pattern center point, the number of bombs per stick, the size of the pattern, the size of the runway, and the approach angle, to mention a few factors. Since the runway and pattern are rectangular in shape, the probability distribution of the proportion of the pattern covering the runway might prove to be useful in approximating the single pass hit probabilities. Due to computational difficulties involved when the distribution of the pattern center point is assumed to follow the normal distribution, several alternative distributions are proposed, evaluated, and compared with the program output.

Statement of the Problem

The purpose of this investigation is to obtain the probability distribution for the proportion of a rectangle covering another rectangle. This distribution is obtained for several different "aiming" distributions.

Chapter II pertains to the discussion of the probability of cutting the runway in k passes. Chapters III and IV relate to the development of the probability distribution of rectangles which cover other rectangles. An application of the results of Chapter IV to the probabilities discussed in Chapter II is presented in Chapter V.

CHAPTER II

PROBABILITY OF CUTTING A RUNWAY IN K PASSES

The principal use of the output of the Runway Cutter Program is the determination of the number of aircraft passes required to interdict a runway. One would like to know the number of aircraft bombing passes needed to render a target unusable, but this is impossible to know with certainty. However, a realistic approach to the problem indicates that one would be interested in determining the probability of rendering a runway unusable in a certain number of passes.

The purpose of this chapter is to develop a general form for the probability of cutting a runway in a prescribed number of passes for any cut criterion. After the form is developed, it is illustrated with several examples.

A General Form for the Probability of a Cut in k Passes

When bombing a target many factors have an influence on the outcome. The output of the Runway Cutter Program is a function of these factors because they are used in the determination of the size of the pattern rectangle and the error distributions involved in dropping the weapons. The set of factors that will be considered in determining the probability of cutting the runway are:

(1) the cut criterion denoted by C,

(2) the set of coordinates of the bomb impact points with respect to the range and deflection axes denoted by χ,

(3) the number of passes denoted by k,

(4) the number of hits denoted by j, and

(5) the assumed probability density used to drop the b bombs per pass denoted by δ_b.

In some instances, certain of the above factors may be related. For example, including one of the above factors may necessitate deletion of another. In particular, if one uses a cut criterion that considers only the number of hits, then the coordinates of the impact point are unimportant. In considering the probability of satisfying C for the first time on the k^{th} pass, factors (1) and (3) are related in that C has not been satisfied on the previous passes. Since only those bombs which damage the target are of interest to us, let χ' denote the set of impact points with respect to the range and deflection axes of those bombs damaging the target.

Now let us incorporate the factors given above into a function that generically describes the probability of cutting the runway. Let $f(\cdot\,; C, \chi', k, j, \delta)$ denote the probability sought. This function can be written as

$$f(C, \chi', k, \delta_b) = \sum_{j=1}^{bk} \Pr\{C \text{ is satisfied} \mid j \text{ hits in } k \text{ passes}\} \Pr\{j \text{ hits in } k \text{ passes}\}, \tag{2.1}$$

since $\Pr\{j \text{ hits in } k \text{ passes}\}$ is a function of δ_b and since C can depend on χ'. Therefore, equation (2.1) indicates a need for determining $\Pr\{j \text{ hits in } k \text{ passes}\}$ which is a function of the single pass hit probabilities. This determination will be discussed later.

Examples Using Formula (2.1)

Two examples now will be given to illustrate the usage of the above formula. Although the conditions will be simplified in order to facilitate easier manipulations, the simplified conditions will not represent a realistic situation. In these two examples, the cut criterion C is given as the maximum spacing between the projected impact points on the axis representing the runway width. Let w^* represent the width which satisfies C. For these examples it will be assumed that only one bomb per pass is used and that all k bombs hit the target. Consequently, χ' and χ are the same sets; however, only the projected coordinates on the width axis are of interest in these examples.

Example 2.1. The probability density of the projected ordinates on the width axis is assumed to be the uniform distribution with density function

$$f(y) = \frac{1}{2w_r}, \quad -w_r \le y \le w_r,$$

$$= 0, \text{ otherwise};$$

where w_r represents the half-width of the runway. The actual derivation of the probability in this example is given in Appendix A. Thus, Pr{C is satisfied in k independent passes with one bomb per pass}

$$= \Pr\{C \leq w^*\}$$

$$= 1 - \binom{k+1}{1}\left(1 - \frac{w^*}{2w_r}\right)^k$$

$$+ \binom{k+1}{2}\left(1 - \frac{2w^*}{2w_r}\right)^k - \dots \pm (-1)^i \binom{k+1}{i}\left(1 - \frac{iw^*}{2w_r}\right)^k , \qquad (2.2)$$

where

$$\frac{2w_r}{i+1} < w^* \leq \frac{2w_r}{i} \; ; \; i = k, k-1, \dots, 2, 1 \; .$$

The following example is given to illustrate a density function for which it is relatively easy to find the cumulative distribution for the given cut criterion.

Example 2.2. The context of this example is the same as that of the previous one with the exceptions that the probability density function of the projected impact points on the width axis is different and the spacings involved vary slightly from the previous example. The density function is given by

$$f(y) = \lambda e^{-\lambda y}, \; y > 0 , \; \lambda > 0 ,$$
$$= 0 , \text{ otherwise} .$$

Note that in the following, the large case letters are used to denote random variables while the small case letters represent realizations of the random variables.

Let $Y_1, Y_2, \dots, Y_k$ be independent random variables each distributed according to the above density function and let

$X_1, X_2, \ldots, X_k$ be the order statistics of $Y_1, Y_2, \ldots, Y_k$. The joint density function of $Y_1, Y_2, \ldots, Y_k$ is given by

$$f(y_1, y_2, \ldots, y_k) = \prod_{i=1}^{k} (\lambda e^{-\lambda y_i}), \quad y_1, y_2, \ldots, y_k > 0 \tag{2.3}$$

$$= 0, \text{ otherwise}.$$

The joint density function of $X_1, X_2, \ldots, X_k$ is given by

$$g(x_1, x_2, \ldots, x_k) = k! \prod_{i=1}^{k} (\lambda e^{-\lambda x_i}), \quad 0 \leq x_i \leq x_2 \leq \ldots \leq x_k < \infty, \tag{2.4}$$

$$= 0, \text{ otherwise}.$$

Define the spacings S_i as:

$$S_1 = X_1, S_2 = X_2 - X_1, \ldots, S_k = X_k - X_{k-1}. \tag{2.5}$$

Since $X_1 = S_1$, $X_2 = S_1 + S_2, \ldots, X_k = S_1 + S_2 + \ldots + S_k$, the Jacobian has a value of one. Thus, the joint density function of the S_i, $i = 1, 2, \ldots, k$ is given by

$$h(s_1, s_2, \ldots, s_k) = g(s_1, s_1+s_2, \ldots, s_1+s_2 + \ldots + s_k) \cdot 1$$

$$= k! \, \lambda^k e^{-\lambda s_1} e^{-\lambda(s_1+s_2)} \ldots e^{-\lambda(s_1+s_2+\ldots+s_k)}$$

$$= k! \prod_{i=1}^{k} [\lambda e^{-\lambda(k+1-i)s_i}] \tag{2.6}$$

$$= \prod_{k=1}^{k} [\lambda(k+1-i) e^{-\lambda(k+1-i)s_i}], \; 0 < s_i < \infty, \; i = 1, 2, \ldots, k.$$

From (2.6) it is seen that the S_i are independent since their joint density function can be factored into a product of marginal density functions. Also, each S_i is exponentially distributed with a parameter of $\lambda(k+1-i)$.

In order to determine the probability distribution of the maximum spacing, let $S = \max\{S_1, S_2, \ldots, S_k\}$. Then

$$\begin{aligned} \Pr\{S \le w^*\} &= \Pr\{S_1 \le w^*, S_2 \le w^*, \ldots, S_k \le w^*\} \\ &= \prod_{i=1}^{k} \Pr\{S_i \le w^*\} \\ &= \prod_{i=1}^{k} [1 - e^{-\lambda(k+1-i)w^*}] , \end{aligned} \qquad (2.7)$$

since

$$\begin{aligned} \Pr\{S_i \le w^*\} &= \int_0^{w^*} \lambda(k+1-i)\, e^{-\lambda(k+1-i)s_i}\, d\, s_i \\ &= 1 - e^{-\lambda(k+1-i)w^*} . \end{aligned}$$

Thus, the probability distribution of the maximum spacing is not difficult to determine for this example. However, the exponential distribution is not a realistic aiming or ballistic error distribution. Also the spacings as defined do not consider the spacing from the edge of the runway to the last impact point, $w_r - X_k$.

Application to the Runway Cutter Program

As mentioned in Chapter I, the Runway Cutter Program projects those craters on the runway onto the width axis. In order to determine the probability of cutting the runway in k passes, one needs to know the probability distribution governing the projected hits on the width axis.

Let us assume that (r', d') are the true impact coordinates of a bomb in a stick. Then the independence of the aiming and ballistic errors implies that (r', d') has a bivariate normal distribution given by:

$$\begin{pmatrix} r' \\ d' \end{pmatrix} \sim N\left[\begin{pmatrix} A_R + \delta_R \\ A_D + \delta_D \end{pmatrix}, \begin{pmatrix} \sigma_{AR}^2 + \sigma_{BR}^2 & 0 \\ 0 & \sigma_{AD}^2 + \sigma_{BD}^2 \end{pmatrix}\right]. \qquad (2.8)$$

For the transformation given in (1.1), the distribution of the (x, y) coordinates of (r', d') is given as

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}\right], \qquad (2.9)$$

where

$$\begin{aligned}
\mu_x &= (A_R + \delta_R)\cos\theta + (A_D + \delta_D)\sin\theta \ , \\
\mu_y &= -(A_R + \delta_R)\sin\theta + (A_D + \delta_D)\cos\theta \ , \\
\sigma_x^2 &= (\sigma_{AR}^2 + \sigma_{BR}^2)\cos^2\theta + (\sigma_{AD}^2 + \sigma_{BD}^2)\sin^2\theta \ , \qquad (2.10) \\
\sigma_y^2 &= (\sigma_{AR}^2 + \sigma_{BR}^2)\sin^2\theta + (\sigma_{AD}^2 + \sigma_{BD}^2)\cos^2\theta \ , \quad \text{and} \\
\sigma_{xy} &= \sin\theta\cos\theta\,[\sigma_{AD}^2 + \sigma_{BD}^2 - \sigma_{AR}^2 - \sigma_{BR}^2] .
\end{aligned}$$

Consequently, the density function of the impact points along the Y-axis is given as

$$f(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right], \quad -\infty < y < \infty. \qquad (2.11)$$

Normalization of the distribution in (2. 11) indicates that one can consider the projected impact points as coming from a $N(0, 1)$ distribution. Using this distribution and the assumptions of one bomb per pass and all k bombs hitting the runway, let us proceed with the determination of the probability of cutting the runway in k passes.

Let $Y_1, Y_2, \ldots, Y_k$ be independent random variables, each with a density function given by

$$f(y) = \frac{1}{\sqrt{2\pi}} \exp(-y^2/2), \quad -\infty < y < \infty .$$

Let $X_1, X_2, \ldots, X_k$ denote the order statistics of $Y_1, Y_2, \ldots, Y_k$. Then the joint density function of $Y_1, Y_2, \ldots, Y_k$ is given as

$$f(y_1, y_2, \ldots, y_k) = \frac{1}{(2\pi)^{k/2}} \exp\left(-1/2 \sum_{i=1}^{k} y_i^2\right), \quad -\infty < y_i < \infty, \qquad (2.12)$$

and that of $X_1, X_2, \ldots, X_k$ is given as

$$g(x_1, x_2, \ldots, x_k) = \frac{k!}{(2\pi)^{k/2}} \exp\left(-1/2 \sum_{i=1}^{k} x_i^2\right),$$

$$-\infty < x_1 \leq x_2 \leq \ldots \leq x_k < \infty . \qquad (2.13)$$

Define the spacings S_i as $S_i = X_i - X_{i-1}$, $i = 2, 3, \ldots, k$. Note that the intervals $(-\infty, X_1)$ and (X_k, ∞) are not considered as spacings.

In order to determine the joint density function of the S_i , let $S_1 = X_1$. Then $X_1 = S_1, X_2 = S_1 + S_2, \ldots, X_k = S_1 + S_2 + \ldots + S_k$. The Jacobian of this transformation has a value of 1 , therefore, the joint density function of $S_1, S_2, \ldots, S_k$ is given by

$$h(s_1, s_2, \ldots, s_k)$$

$$= g(s_1, s_1 + s_2, \ldots, s_1 + s_2 + \ldots + s_k)|J|$$

$$= \left\{ \frac{k!}{(2\pi)^{k/2}} \exp\left(-\frac{s_1^2}{2}\right) \exp\left(-\frac{(s_1+s_2)^2}{2}\right) \ldots \exp\left(-\frac{(s_1+s_2+\ldots+s_k)^2}{2}\right) \right\} \cdot 1 ,$$

$$-\infty < s_1 < \infty , \ 0 < s_i < \infty , \ i = 2, 3, \ldots, k , \qquad (2.14)$$

However, since one is interested in the joint distribution of S_i, $i = 2, 3, \ldots, k$, the s_1 variable needs to be integrated out of (2. 14). That is,

$$h_1(s_2, s_3, \ldots, s_k) = \int_{-\infty}^{\infty} h(s_1, s_2, \ldots, s_k)\, ds_1$$

$$= \frac{k!}{(2\pi)^{k/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{s_1^2}{2}\right) \prod_{i=2}^{k} \exp\left(-\frac{(s_1+s_2+\ldots+s_i)^2}{2}\right) ds_1 .$$

(2. 15)

Equation (2. 15) indicates the complication involved in evaluating such an integral, since no closed form exists for this integral. Even if it were possible to integrate (2. 15) successfully, the resulting joint density function $h_1(s_2, s_3, \ldots, s_k)$ presents a similar problem in the determination of the density function for the maximum spacing.

As a consequence of the preceding study, the derivation of the probability distribution the maximum spacing between impact points will be abandoned at this stage. Although this cut interion provides working solutions for the Runway Cutter Program, the foregoing discussion indicates the difficulty in using this criterion for making probability statements about the number of passes required to cut the runway.

A Different Cut Criterion

A pertinent question to be considered is "Are there any other feasible cut criterion for which it would be easier to determine the probability of a cut?" Consider the cut criterion of "t or more hits in k passes." This criterion is easier to evaluate in that it does not depend on the location of the hits.

If we adopt this as the cut criterion, the expansion of the function $f(C; \chi'; k, \delta_b)$ in (2.1) reduces to

$$\sum_{j=t}^{bk} \Pr\{j \text{ hits in } k \text{ passes}\},$$

since $\Pr\{C \text{ is satisfied} | j \text{ hits in } k \text{ passes}\}$ is either 0 or 1, depending on whether or not t or more hits have occurred. Thus, one only needs to be concerned about $\Pr\{j \text{ hits in } k \text{ passes}\}$ or essentially the single pass hit probabilities.

For this situation one will assume that a stick of b bombs is dropped on each pass. Let X_i be a discrete random variable that records the number of hits per pass. The X_i are identically and independently distributed with a probability distribution given by $\Pr\{X_i = j\} = p_j$, $j = 0, 1, 2, \ldots, b$. The p_j are the single pass hit probabilities. Let S_n represent the partial sums of the X_i, that is, $S_1 = X_1$, $S_2 = X_1 + X_2, \ldots, S_n = X_1 + X_2 + \ldots + X_n$. Then S_n gives the total number of hits at the end of n passes.

The probability distribution of S_n can be determined from that of the X_i. According to Feller (1957), if one lets $F(s) = \sum_{m=0}^{b} p_m s^m$, $|s| < 1$, be the probability generating function (p. g. f.) associated with X_i, then the p. g. f. of S_n, say $G(s)$, is given by

$$G(s) = [F(s)]^n . \qquad (2.16)$$

Thus, the probability, $Pr\{S_n = i\}$, is the coefficient of s^i in the expansion of (2. 16).

For the cut criterion of t or more hits in k passes, one is interested in the probability of satisfying the cut criterion for the first time in k passes. Let us define the event C_k as

C_k = the event of obtaining t or more hits for the first time in k passes.

The event C_k has a chance of occurring only if the number of hits recorded at the end of the first $(k-1)$ passes is between $t-b$ and $t-1$, that is $t-b \leq S_{k-1} \leq t-1$. For C_k to occur, then the number of hits at the termination of the k^{th} pass will be $t, t+1, \ldots, t+b-1$. Thus,

$$Pr\{C_k\} = Pr\{C_k | t-b \leq S_{k-1} \leq t-1\} \, Pr\{t-b \leq S_{k-1} \leq t-1\} . \qquad (2.17)$$

Since the event $\{t-b \leq S_{k-1} \leq t-1\}$ is the union of the mutually exclusive events $\{S_{k-1} = i\}$, $i = t-b, t-b+1, \ldots, t-1$, equation (2. 17) can be written as

$$P(C_k) = \sum_{i=1}^{b} Pr\{C_k | S_{k-1} = t-i\} \, Pr\{S_{k-1} = t-i\} . \qquad (2.18)$$

Now the conditional event $\{C_k | S_{k-1} = t-i\}$ occurs if one obtains $i, i+1, \ldots, b$ hits on the k^{th} pass. Consequently,

$$\Pr\{C_k | S_{k-1} = t-i\} = \sum_{j=i}^{b} p_j . \tag{2.19}$$

Therefore,

$$\begin{aligned} \Pr\{C_k\} &= \sum_{i=1}^{b} \Pr\{C_k | S_{k-1} = t-i\} \Pr\{S_{k-1} = t-i\} \\ &= \sum_{i=1}^{b} \sum_{j=i}^{b} p_j \Pr\{S_{k-1} = t-i\} , \end{aligned} \tag{2.20}$$

where $\Pr\{S_{k-1} = t-i\}$ is obtained from the expansion of (2.16).

Example 2.3. Let us suppose that $b=3$, $k=3$, and $t=4$. Then the p.g.f. of S_2 is given by

$$\begin{aligned} G(s) &= [p_0 + p_1 s + p_2 s^2 + p_3 s^3]^2 \\ &= p_0^2 + 2p_0p_1 s + (2p_0p_2 + p_1^2)s^2 + 2(p_0p_3 + p_1p_2)s^3 + (2p_1p_3 + p_2^2)s^4 \\ &\quad + 2p_2p_3 s^5 + p_3^2 s^6 . \end{aligned}$$

If one evaluates the possible outcomes that satisfy the cut criterion of four or more hits for the first time in three passes, then a list of the possible outcomes and their associated probabilities are given in the following table. The numbers in parentheses represent the number of hits recorded in the first, second, and third passes, respectively.

TABLE I

POSSIBLE OUTCOMES SATISFYING THE CUT CRITERION AND THEIR ASSOCIATED PROBABILITIES

t = 4, b = 3, k = 3		
4 Hits	5 Hits	6 Hits
(0, 1, 3) - $p_0p_1p_3$	(0, 2, 3) - $p_0p_2p_3$	(0, 3, 3) - $p_0p_3^2$
(0, 2, 2) - $p_0p_2^2$	(0, 3, 2) - $p_0p_2p_3$	(3, 0, 3) - $p_0p_3^2$
(0, 3, 1) - $p_0p_1p_3$	(1, 1, 3) - $p_1^2p_3$	(1, 2, 3) - $p_1p_2p_3$
(1, 0, 3) - $p_0p_1p_3$	(1, 2, 2) - $p_1p_2^2$	(2, 1, 3) - $p_1p_2p_3$
(1, 1, 2) - $p_1^2p_2$	(2, 0, 3) - $p_0p_2p_3$	
(1, 2, 1) - $p_1^2p_2$	(2, 1, 2) - $p_1p_2^2$	
(2, 0, 2) - $p_0p_2^2$	(3, 0, 2) - $p_0p_2p_3$	
(2, 1, 1) - $p_1^2p_2$		
(3, 0, 1) - $p_0p_1p_3$		

Thus, the probability of obtaining 4 or more hits for the first time on the third pass is the sum of the probabilities in Table I, namely,

$$4\,p_0p_1p_3 + 3\,p_1^2p_2 + 2\,p_0p_2^2 + 4\,p_0p_2p_3 + 2\,p_1p_2^2 + 2\,p_1p_2p_3 + p_1^2p_3 + 2\,p_0p_3^2 \,.$$

Using the results of (2.20) and (2.21), the probability 4 or more hits for the first time on the third pass is given by

$$\sum_{i=1}^{3}\sum_{j=i}^{3} p_j \Pr\{S_2 = 4-i\}$$

$$= (p_1+p_2+p_3)\Pr\{S_2 = 3\} + (p_2+p_3)\Pr\{S_2 = 2\}$$

$$+ p_3 \Pr\{S_2 = 1\} ,$$

$$= (p_1+p_2+p_3)(2p_0p_3+2p_1p_2) + (p_2+p_3)(2p_0p_2+p_1^2)$$

$$+ p_3(2p_0p_1)$$

$$= 4p_0p_1p_3 + 3p_1^2p_2 + 2p_0p_2^2 + 4p_0p_2p_3 + 2p_1p_2^2 + p_1^2p_3$$

$$+ 2p_0p_3^2 + 2p_1p_2p_3$$

From Example 2.3, the preceding discussion, and the form of the probability of a cut in (2.1), the single pass hit probabilities, that is, the p_j, $j = 0, 1, 2, \ldots, b$, are seen to be quite important. Thus, in order to make use of the cut criteria considered so far, one needs to determine these probabilities. The material set forth in Chapters III and IV will enable us to determine them.

CHAPTER III

DERIVATION OF THE PROPORTION OF PATTERN AREA ON THE RUNWAY

The primary purpose of this chapter is to determine the regions that generate the various configurations of the pattern upon the runway. After some general notation and general assumptions pertaining to formulation of the problem are defined, the region of possible pattern center impact points is examined for sub-regions which generate the pattern configurations on the runway.

In the following discussion, let R denote the runway rectangle with dimensions $2L_r$ by $2W_r$, D the distribution rectangle with dimensions $2L_d$ by $2W_d$, and P the pattern rectangle with dimensions $2L_p$ by $2W_p$. The length of each rectangle is assumed to be greater than its width, and it is assumed that $L_r > L_d + L_p$. Let us assume that the centers of D and R are coincident such that D can be rotated about its center point through an angle θ measured in the clockwise direction, $0 < \theta < \pi/2$. In this discussion, θ can also take on an "optimum angle" value. The criterion for the optimum angle is discussed in Appendix B. The cases for $\theta = 0^{\circ}$ and $\theta = \pi/2$ are considered in Appendix C. The coordinate system to be used is a rotation, through θ, of the usual $(\mathfrak{X}, \mathfrak{Y})$ coordinate system and is denoted as the $(\mathfrak{D}, \mathfrak{R})$ coordinate system. See Figure 2.

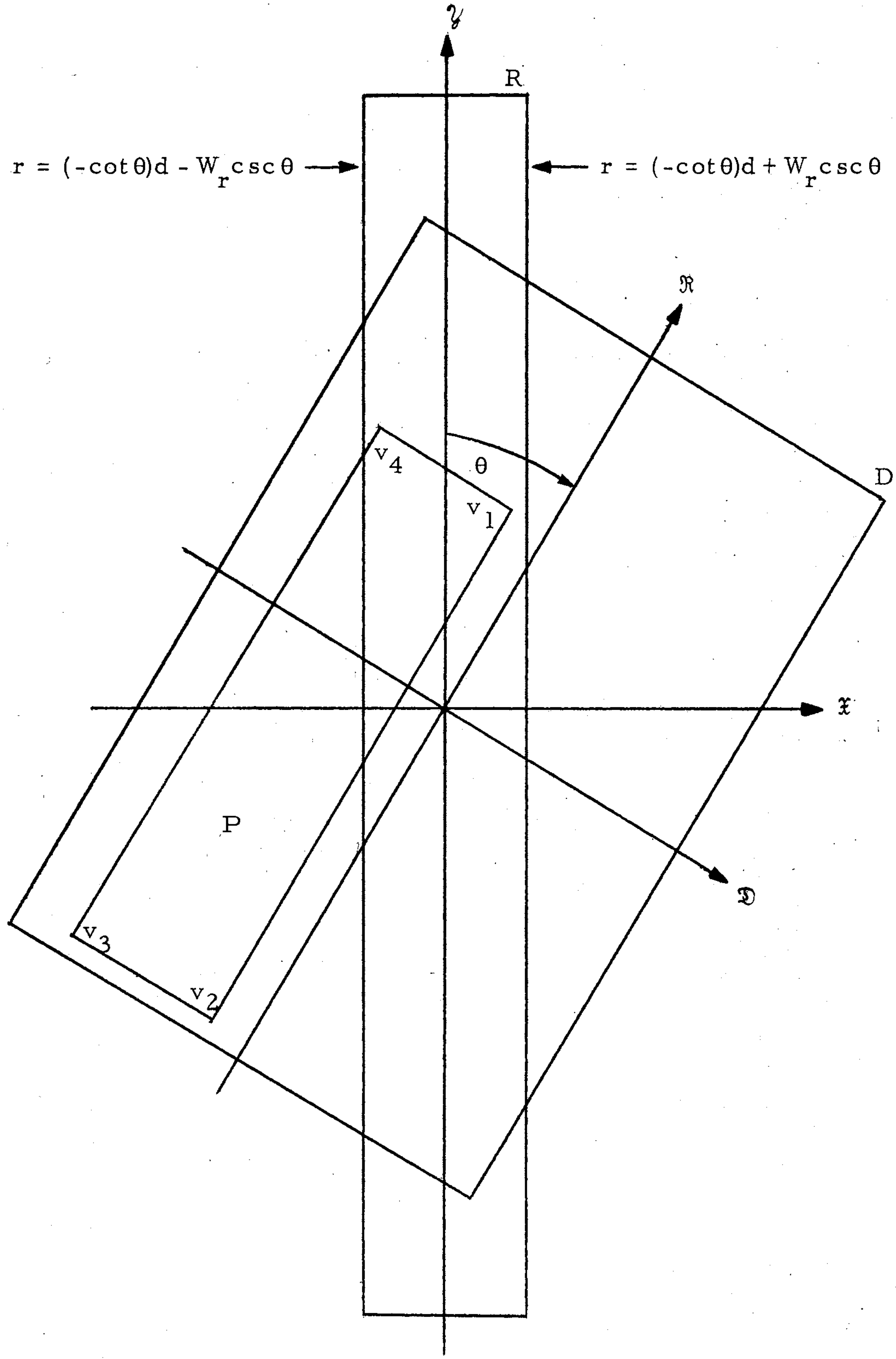

Figure 2. (𝔇, ℜ) Coordinate System

The rectangle P is assumed to have the same orientation as D. It is also assumed that the center point of P (hereafter denoted by PCP) is always contained within D. According to Figure 2, the vertices of P are labeled v_1, v_2, v_3, v_4.

Determination of the Sub-regions

The configurations of P on R change from one form to another when the respective vertices of P meet the sides of R. The sides of R are denoted by the lines $r = (-\cot\theta)d \pm W_r \csc\theta$. The set of PCPs in D at which a given vertex of P meets a given side of R forms a straight line ℓ parallel to the line given by $r = (-\cot\theta)d$. Since the line ℓ intersects the perimeter of D in two places, one can consider the PCPs on the perimeter of D at which the configurations change. Thus, the method of determining the regions of D that generate the different configurations of P on R consists of moving the PCP around the perimeter of D and noting the coordinates of the PCP whenever the configurations change. The point $(-W_d, -L_d)$ will be considered as the starting point as PCP moves in a clockwise direction around D. However, due to symmetry with respect to the origin, one needs only to consider the movement of the PCP from $(-W_d, -L_d)$ to (W_d, L_d).

Three possible situations can occur as the PCP proceeds along its designated path. They are:

(1) P forms its first configuration on R when the PCP moves along the line $r = L_d$,

(2) P forms its first configuration on R when the PCP moves along the line $d = -W_d$, and

(3) P forms a configuration on R when the PCP is located at $(-W_d, -L_d)$.

Determination of Sub-regions in D
for Situation (1)

Let us examine situation (1) more closely. In this case, P fails to make contact with R as the PCP moves along $d = -W_d$. The first contact that P makes with R, and the subsequent configurations that P forms on R, occur when the PCP is located on the line $r = L_d$. Note, in situation (1), each of the four vertices of P meet both sides of R as the PCP moves along the line $r = L_d$. Accordingly, there are eight PCP abscissas on $r = L_d$ at which the configurations change. These abscissas can be determined in the following manner.

Since P has the same orientation as D, the first contact occurs when the vertex v_1 meets the left side of R. Let (d', L_d) be the coordinates of the PCP when the vertex v_1 meets the left side of R which is given by the line $r = (-\cot\theta)d - W_r\csc\theta$. Consequently, the coordinates of the point of contact are $(d' + W_p, L_d + L_p)$. Since this point is on the previously stated line, it can be determined that $d' = -(L_d + L_p)\tan\theta - W_r\sec\theta - W_p$. Thus $(-(L_d + L_p)\tan\theta - W_r\sec\theta - W_p, L_d)$ are the PCP coordinates when v_1 makes contact with the left side of R in situation (1).

The remaining PCP abscissas on $r = L_d$ at which the respective vertices of P meet the sides of R can be determined by the same method used above. A list of the abscissas is given as follows:

$$c_1 = -(L_d+L_p)\tan\theta - W_r\sec\theta - W_p \qquad c_5 = -(L_d+L_p)\tan\theta + W_r\sec\theta + W_p$$

$$c_2 = -(L_d+L_p)\tan\theta - W_r\sec\theta + W_p \qquad c_6 = -(L_d-L_p)\tan\theta - W_r\sec\theta + W_p$$

$$c_3 = -(L_d+L_p)\tan\theta + W_r\sec\theta - W_p \qquad c_7 = -(L_d-L_p)\tan\theta + W_r\sec\theta - W_p$$

$$c_4 = -(L_d-L_p)\tan\theta - W_r\sec\theta - W_p \qquad c_8 = -(L_d-L_p)\tan\theta + W_r\sec\theta + W_p. \tag{3.1}$$

Let d_i, $i = 1, 2, \ldots, 8$, be the corresponding abscissas on $r = -L_d$, where $d_i = -c_i$, $i = 1, 2, \ldots, 8$. The subscripts do not denote the order in which abscissas occur.

Since the PCP abscissas at which the configurations change are dependent upon which of the vertices of P meets the side of R, one needs to know what factors influence the order in which the vertices of P meet the sides of R. Figure 3 given below illustrates the fact that as the PCP moves right on $r = L_d$ from the point (c_1, L_d), the order in which the vertices v_1, v_2, v_3, v_4 meet both sides of R depends on the relative magnitudes of $L_p\tan\theta$, $W_r\sec\theta$, and W_p. Thus, the order of the PCP abscissas on $r = L_d$ also depends on the relative magnitude of $L_p\tan\theta$, $W_r\sec\theta$, and W_p. Consequently, a specified ordering of the terms $L_p\tan\theta$, $W_r\sec\theta$, and W_p, determines an arrangement of PCP abscissas given in (3.1).

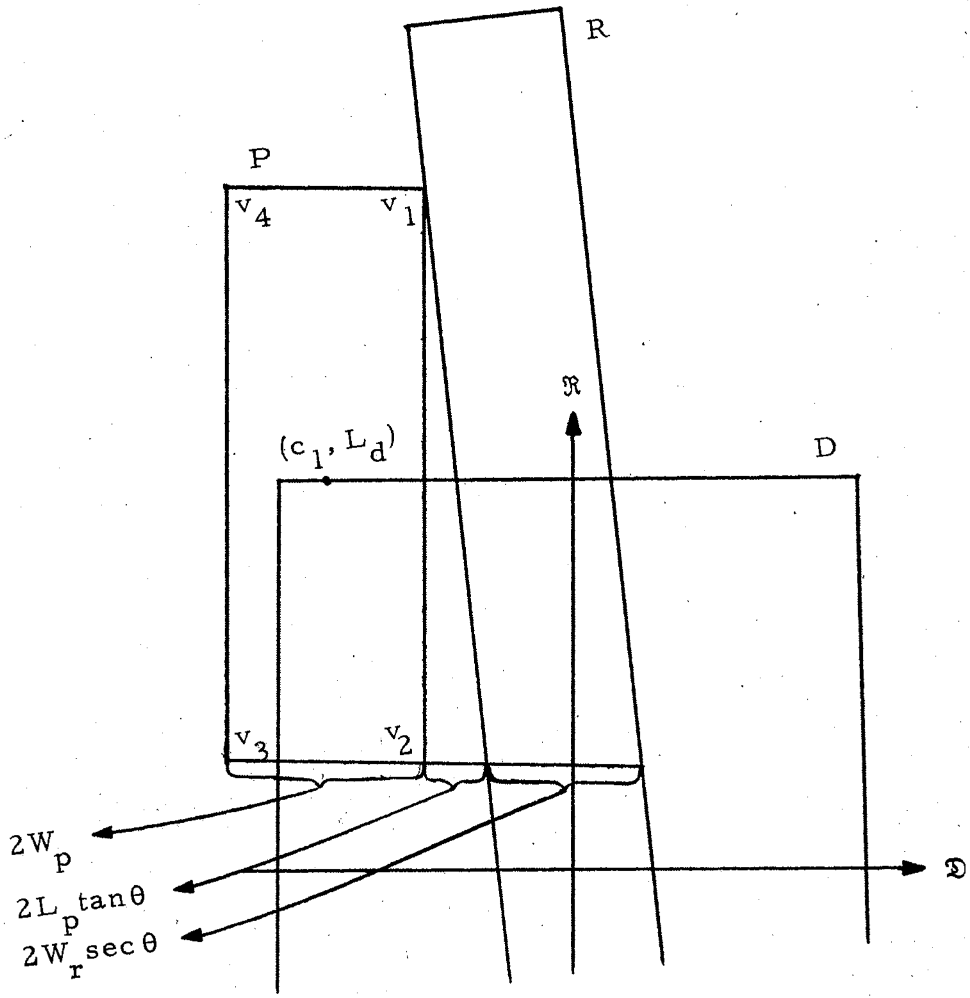

Figure 3. Geometrical Interpretation of $2L_p \tan\theta$, $2W_r \sec\theta$ and $2W_p$

Table II below gives the distances that the PCP must move along $r = L_d$ from (c_1, L_d) for the respective vertices of P to meet the sides of R. The distances can be determined by referring to Figure 3.

TABLE II

PCP DISTANCE FROM (c_1, L_d) FOR A VERTEX OF P TO MEET A SIDE OF R

Vertex of P	Side of R	Distance
v_1	right side	$2W_r \sec\theta$
v_2	left side	$2L_p \tan\theta$
v_2	right side	$2(L_p \tan\theta + W_r \sec\theta)$
v_3	left side	$2(W_p + L_p \tan\theta)$
v_3	right side	$2(W_p + L_p \tan\theta + W_r \sec\theta)$
v_4	left side	$2W_p$
v_4	right side	$2(W_p + W_r \sec\theta)$

Examples 3.1, 3.2, and 3.3 will be used to illustrate the determination of the appropriate sub-regions of D that generate the configurations of P on R that arise in situations (1), (2), and (3), respectively. For each of the examples, it will be assumed that
$W_p > W_r \sec\theta > L_p \tan\theta$.

Example 3.1. Using Table II and assuming that $W_p > W_r \sec\theta > L_p \tan\theta$, one can determine the sequence in which the respective vertices of P meet the sides of R . For these conditions, the sequence can be determined as $\{(v_1, \text{left}), (v_2, \text{right}), (v_1, \text{right}), (v_4, \text{left}), (v_2, \text{right}),$

(v_3, left), (v_4, right), $(v_3, \text{right})\}$. However, if $W_p > W_r \sec\theta + L_p \tan\theta$, then using Table II, one sees that vertex v_2 will meet the right side of R before vertex v_4 meets the left side of R. Thus, if $W_p > W_r \sec\theta + L_p \tan\theta$, (v_4, left) and (v_2, right) interchange positions in the above sequence.

The sequence of PCP abscissas on $r = L_d$ at which the configurations change is determined by noting the order of the elements in the above sequence and adding the appropriate distance determined from Table II to the abscissa c_1. That is, the PCP abscissa when vertex v_2 meets the left side of R is determined by adding $2L_p \tan\theta$ to c_1, the PCP abscissa when vertex v_1 meets the right side of R is obtained by adding $2W_r \sec\theta$ to c_1, etc. Thus, the sequence of PCP abscissas on $r = L_d$ at which the configurations change is given by $\{c_1, c_4, c_3, c_2, c_7, c_6, c_5, c_8\}$. If $W_p > W_r \sec\theta + L_p \tan\theta$, then c_2 and c_7 interchange positions in the above sequence.

Table III below gives the sequence of configurations of P that develop on R as the PCP moves along $r = L_d$. The quantities in parentheses replace the given quantities when $W_p > W_r \sec\theta + L_p \tan\theta$. Note, the set of configurations that P forms on R is restricted by the assumption that $L_r > L_d + L_p$.

TABLE III

CONFIGURATIONS OF P ON R WHEN THE PCP IS ON $r = L_d$ AND $W_p > W_r \mathrm{SEC}\,\theta > L_p \mathrm{TAN}\,\theta$

< PCP $\leq$	Configuration
c_1, c_4	triangle
c_4, c_3	trapezoid
c_3, c_2 (c_3, c_7)	pentagon
c_2, c_7 (c_7, c_2)	hexagon (parallelogram)
c_7, c_6 (c_2, c_6)	pentagon
c_6, c_5	trapezoid
c_5, c_8	triangle

As the PCP moves from $(-W_d, -L_d)$ to $(W_d, -L_d)$ the corresponding sequence of PCP abscissas on $r = -L_d$ can be determined from the above sequence of abscissas by means of symmetry. This sequence is given by $\{d_8, d_5, d_6, d_7, d_2, d_3, d_4, d_1\}$ with d_2 and d_7 interchanging positions if $W_p > W_r \sec\theta + L_p \tan\theta$. The appropriate sub-regions of D generating the configurations are determined by connecting the respective abscissas on the lines $r = L_d$ and $r = -L_d$ by lines parallel to the line given by $r = (-\cot\theta)d$, see Figure 4. A tabular arrangement of the corresponding sequence of abscissas at which the configurations change for other relationships between $L_p \tan\theta$, $W_r \sec\theta$ and W_p will be given later in this chapter.

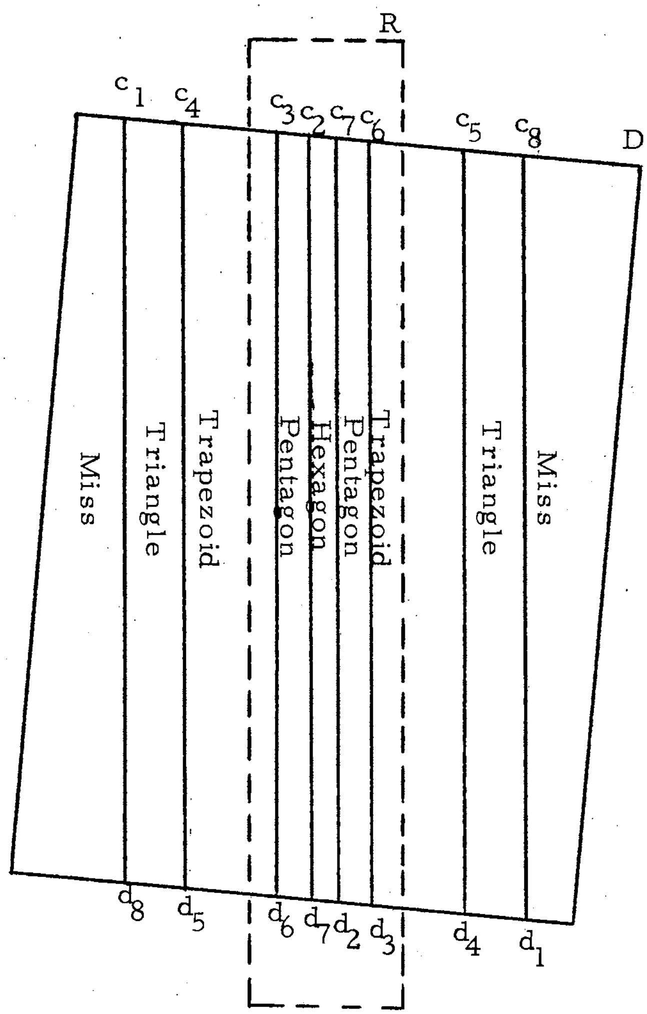

Figure 4. Generating Sub-regions for Example 3.1

Determination of Sub-regions in D for Situation (2)

In this situation, P makes its first contact with R when the PCP is on the line $d = -W_d$. Again, since P has the same orientation as D, the first contact occurs when the vertex v_1 meets the left side of R. Let $(-W_d, r')$ be the coordinates of the PCP when v_1 meets the left side of R. Accordingly, v_1 has coordinates $(-W_d+W_p, r'+L_p)$ and is a point on the line $r = (-\cot\theta)d - W_r\csc\theta$. Thus, $r' + L_p = (-\cot\theta)(-W_d + W_p) - W_r\csc\theta$ and the PCP coordinates are given by $(-W_d, (W_d - W_p)\cot\theta - W_r\csc\theta - L_p)$. The remaining PCP ordinates on $d = -W_d$ at which the respective vertices of P meet the sides of R can be determined by the method used above. A list of all the possible ordinates which may arise is given as follows:

$$
\begin{aligned}
a_1 &= (W_d - W_p)\cot\theta - W_r\csc\theta - L_p & a_5 &= (W_d - W_p)\cot\theta + W_r\csc\theta + L_p \\
a_2 &= (W_d - W_p)\cot\theta - W_r\csc\theta + L_p & a_6 &= (W_d + W_p)\cot\theta - W_r\csc\theta + L_p \\
a_3 &= (W_d - W_p)\cot\theta + W_r\csc\theta - L_p & a_7 &= (W_d + W_p)\cot\theta + W_r\csc\theta - L_p \\
a_4 &= (W_d + W_p)\cot\theta - W_r\csc\theta - L_p & a_8 &= (W_d + W_p)\cot\theta + W_r\csc\theta + L_p .
\end{aligned}
\tag{3.2}
$$

Let b_i, $i = 1, 2, \ldots, 8$, be the corresponding ordinates on $d = W_d$, determined by $b_i = -a_i$, $i = 1, 2, \ldots, 8$.

When the PCP is located on the line $d = -W_d$, Figure 5 indicates that the sequence in which the vertices of P meet the sides of R and the corresponding sequence of possible PCP ordinates are dependent upon the magnitudes of L_p, $W_r\csc\theta$, and $W_p\cot\theta$. Note that these terms are equal to $\cot\theta$ times the similar terms given in situation (1).

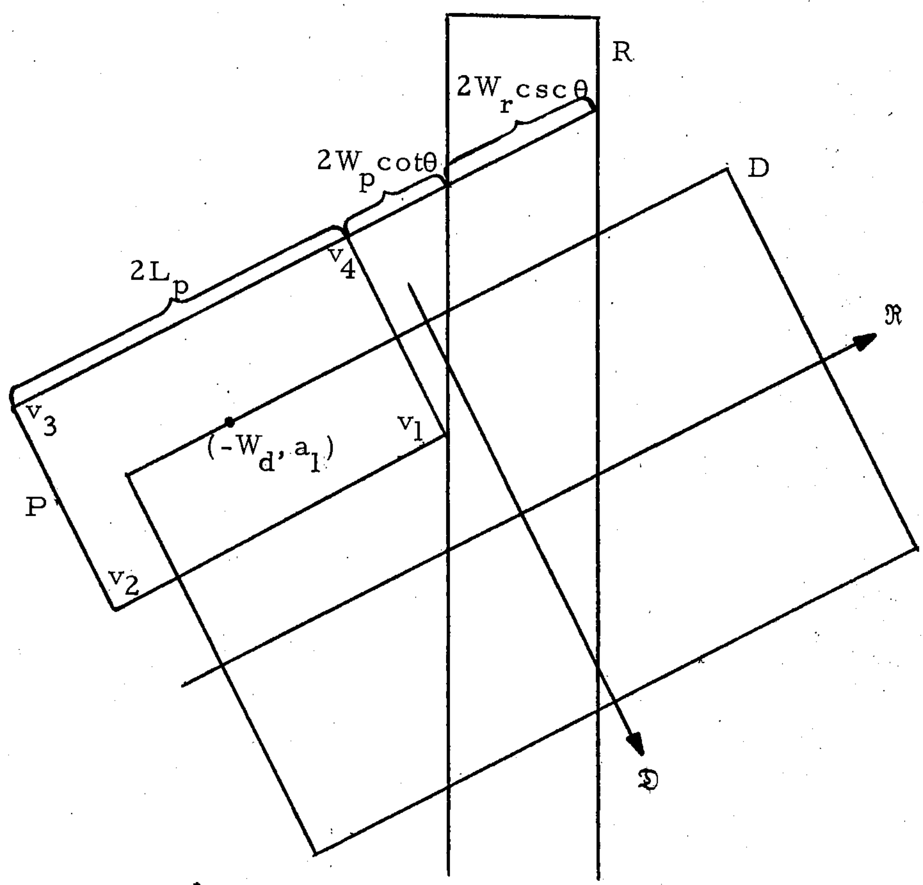

Figure 5. Geometrical Interpretation of $2L_p$, $2W_p \cot\theta$ and $2W_r \csc\theta$

The distances that the PCP must move along $d = -W_d$ from $(-W_d, a_1)$ for the respective vertices of P to meet the sides of R are given in Table IV. Note that the PCP may meet $(-W_d, L_d)$ before it moves some of the distances given in the table.

TABLE IV

DISTANCE FROM $(-W_d, a_1)$ FOR THE PCP TO MOVE FOR A VERTEX OF P TO MEET A SIDE OF R

Vertex of P	Side of R	Distance
v_1	right side	$2W_r \csc\theta$
v_2	left side	$2L_p$
v_2	right side	$2(L_p + W_r \csc\theta)$
v_3	left side	$2(W_p \cot\theta + L_p)$
v_3	right side	$2(W_p \cot\theta + L_p + W_r \csc\theta)$
v_4	left side	$2W_p \cot\theta$
v_4	right side	$2(W_p \cot\theta + W_r \csc\theta)$

For situation (2), P will miss R when the PCP is located at $(-W_d, -L_d)$. Then, due to symmetry with respect to the origin, P will also miss R when the PCP is located at (W_d, L_d). Consequently, as the PCP moves from $(-W_d, -L_d)$ to (W_d, L_d), each of the four vertices of P will meet both sides of R. Thus, for situation (2), the set of possible P on R configurations will be the same set of configurations that arise in situation (1). This result occurs because of the relationship between $L_p \tan\theta$, $W_r \sec\theta$, and W_p and L_p, $W_r \csc\theta$, and $W_p \cot\theta$.

In situation (2), when the PCP is located at $(-W_d, L_d)$, a configuration of P on R results. Now, if the PCP could continue along $d = -W_d$, this configuration and succeeding configurations would terminate when the vertices of P meet appropriate sides of R as determined by the conditions on the relative sizes of L_p, $W_r \csc\theta$, and $W_p \cot\theta$. However, due to the nature of the problem, the PCP must move from $(-W_d, L_d)$ to (W_d, L_d) and cannot continue along $d = -W_d$. When the PCP moves from $(-W_d, L_d)$ to (W_d, L_d), does the configuration of P on R terminate in the same manner as when the PCP continues along $d = -W_d$? Since the configurations terminate when the vertices of P meet the sides of R, one could ask the above question in the following way. When the PCP moves from $(-W_d, L_d)$ to (W_d, L_d), do the vertices of P meet the sides of R in the same manner as when the PCP continues along $d = -W_d$? The answer to this question is given in the following theorem.

Theorem 3.1. For a specified relationship between $L_p \tan\theta$, $W_r \sec\theta$, and W_p, the sequence in which the vertices of P meet the sides of R is invariant to whether the PCP moves along

(i) the line $d = -W_d$, or

(ii) the line $r = L_d$, or

(iii) the line segments $\ell_1 = \{(d, r) : d = -W_d, -L_d \le r \le L_d\}$ and $\ell_2 = \{(d, r) : -W_d \le d \le W_d, r = L_d\}$.

Proof: The specified relationship between $L_p \tan\theta$, $W_r \sec\theta$, and W_p determines the sequence in which the vertices of P meet the sides of R as the PCP moves along the line $r = L_d$. Since the product of $\cot\theta$ and the terms in the above relationship produce the same relationship between L_p, $W_r \csc\theta$, and $W_p \cot\theta$, the same sequence results when the PCP moves along $d = -W_d$.

For the specified relationship between $L_p \tan\theta$, $W_r \sec\theta$, and W_p, the a_i, $i = 1, 2, \ldots, 8$, have a specific ordering. Let us assume that $a_j < L_d < a_k$. When the PCP is located at $(-W_d, a_j)$ let the ordered distances in the positive r direction of each vertex of P from the appropriate side of R be denoted by $\delta_1, \delta_2, \delta_3, \delta_4$. Let $\delta = L_d - a_j$. Then, the ordered distances in the positive r direction of the vertices from the appropriate sides of R when the PCP is at $(-W_d, L_d)$ are given by $\delta_n - \delta$, $n = 1, 2, 3, 4$. When the PCP is at $(-W_d, L_d)$, the distance in the positive d direction of each vertex from the appropriate side of R is given by $(\delta_n - \delta)\tan\theta$, $n = 1, 2, 3, 4$. Thus, the distances $(\delta_n - \delta)\tan\theta$, $n = 1, 2, 3, 4$, retain the same order as $\delta_1, \delta_2, \delta_3, \delta_4$. Consequently, the sequence in which the vertices of P meet the sides of R is the same as those sequences for (i) and (ii).

Example 3.2. For the condition $W_p > W_r \sec\theta > L_p \tan\theta$, the sequence of possible PCP ordinates along $d = -W_d$ at which the configurations change is given as $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$. For illustrative purposes, let us assume that $a_3 < L_d < a_4$. Since $a_3 < L_d < a_4$, the configuration of P on R forms a pentagon when the PCP is located at $(-W_d, L_d)$. Theorem 3.1 implies that the sequence in which the respective vertices meet the sides of R for this example is the same as in Example 3.1. Since the PCP moves right along $r = L_d$ after it meets $(-W_d, L_d)$, the sequence of PCPs at which the configuations change is now given by $\{a_1, a_2, a_3, c_2, c_7, c_6, c_5, c_8\}$ with c_2 and c_7 interchanging positions if $W_p > W_r \sec\theta + L_p \tan\theta$. The corresponding sequence on the perimeter of D along the lines $r = -L_d$ and $d = W_d$ is given by $\{d_8, d_5, d_6, d_7, d_2, b_3, b_2, b_1\}$, with

d_2 and d_7 changing positions if $W_p > L_p \tan\theta + W_r \sec\theta$. The sub-regions in D generating the configurations for this example are found by connecting the respective PCPs in the two preceding sequences by lines parallel to the lines given by $r = (-\cot\theta)d$. The sequences of PCP ordinates and abscissas at which the configurations change in situation (2) for the other relationships between $L_p \tan\theta$, $W_r \sec\theta$ and W_p are given later in tabular form.

Determination of Sub-regions in D for Situation (3)

In order to determine the sub-regions for situation (3), one needs to know the configuration of P on R when the PCP is located at $(-W_d, -L_d)$. When the PCP is located on the line $d = -W_d$, the configurations of P which develop on R are related to the a_i, $i = 1, 2, \ldots, 8$. Recall that a_1 is the PCP ordinate on $d = -W_d$ at which P first makes contact with R. Thus, in situation (3), $a_1 < -L_d$. The relative magnitude of the a_i to $-L_d$ indicates the configuration of P on R when the PCP is at $(-W_d, -L_d)$.

Let us suppose, for example, that a_j is the PCP ordinate at which the triangular configuration terminates for some specified condition on $L_p \tan\theta$, $W_r \sec\theta$, and W_p. If $-L_d < a_j$, then $a_1 < -L_d < a_j$. Thus, the configuration of P on R is triangular when the PCP is at $(-W_d, -L_d)$. Extensions to this relationship can be given for the other configurations.

The knowledge of the relative magnitude of the a_i to $-L_d$ and L_d indicates the configurations of P on R when the PCP is located at $(-W_d, -L_d)$ and $(-W_d, L_d)$, respectively. Note that due to symmetry, the configurations of P on R when the PCP is located at $(-W_d, -L_d)$ and (W_d, L_d) are the same.

<u>Example 3.3.</u> From Examples 3.1 and 3.2, the sequences of possible PCP ordinates and abscissas are $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ and $\{c_1, c_4, c_3, c_2, c_7, c_6, c_5, c_8\}$, respectively. For the sake of illustration, let us assume that $a_1 < -L_d < a_2$ and $a_4 < L_d < a_5$. Thus, when the PCP is located at $(-W_d, -L_d)$, the configuration of P on R forms a triangle and when the PCP is located at $(-W_d, L_d)$, the resulting configuration is a hexagon. Since $a_1 < -L_d$, a_1 cannot appear in a sequence of PCPs at which the configurations change for this example. Due to symmetry, neither can c_8 appear in such a sequence. Since $-L_d < a_2 < a_3 < a_4 < L_d$, the sequence of PCP change points for this example is given as $\{a_2, a_3, a_4, c_7, c_6, c_5\}$. The corresponding sequence on the lines $r = -L_d$ and $d = W_d$ is $\{d_5, d_6, d_7, b_4, b_3, b_2\}$. If $W_p > W_r \sec\theta + L_p \tan\theta$, then a_4 and c_7 are replaced by a_5 and a_4 in the first sequence and d_7 and b_4 are replaced by b_4 and b_5 in the second sequence. The appropriate sub-regions of D are then determined by connecting the successive elements in the two sequences by lines parallel to the line $r = (-\cot\theta)d$ as the elements are arranged on the perimeter of D. The PCPs at which the configurations change in situation (3) for other conditions on $L_p \tan\theta$, $W_r \sec\theta$ and W_p are given later.

If values are assigned to the three sets of length and width parameters and θ, then how does one determine which of the three situations occur? The answer to this question is given in the following theorem.

Theorem 3.2. (1) Situation (1) occurs if and only if $L_d < a_1$,

(2) Situation (2) occurs if and only if $-L_d \leq a_1 \leq L_d$,

(3) Situation (3) occurs if and only if $a_1 < -L_d$.

The proof of this theorem is obtained by recalling that a_1 is the PCP ordinate on $d = -W_d$ at which P first makes contact with R and by recalling the physical interpretations of the three different situations.

A Tabular Representation of the PCP Change Points

The regions in D that generate the configurations of P on R depend on θ, the angle of rotation, since the regions are determined from the relationships between $L_p \tan\theta$, $W_r \sec\theta$ and W_p. In order to obtain the sub-regions in D, the PCP abscissas and ordinates on the perimeter of D at which the configurations change (or terminate) have been tabulated in Tables V - X. These tables are indexed by the different relationships between $L_p \tan\theta$, $W_r \sec\theta$ and W_p. The appropriate sub-regions result when the points obtained from the table are connected by lines parallel to the line $r = (-\cot\theta)d$.

An explanation is now given on the usage and format of Tables V - X. The configurations of P on R are labeled as the geometric figures in the top row of the table. If the relationship between $L_p \tan\theta$, $W_r \sec\theta$, and W_p which indexes the table is such that the largest term is greater than the sum of the smaller two terms, then note the following remarks. In such cases, the right hand column under the leftmost "Pentagon" heading is used instead of the left hand column under that heading. Also, in this case, the column headed by "Parallelogram" in Tables V, VI, VII, and IX or the column headed by "Rectangle" in Tables VIII and X are used instead of the column headed by "Hexagon."

The leftmost column headed by "$< -L_d, L_d \leq$" is used to determine the P on R configurations when the PCP is located at $(-W_d, -L_d)$ and $(-W_d, L_d)$. The use of this column is dependent upon the ranking of the a_i, $-L_d$, and L_d. The ranking of the a_i and L_d determines which row of the table to use. The ranking of $-L_d$ and the a_i determines which columns of the previously chosen row to use. If a configuration of P on R results when the PCP is located at $(-W_d, -L_d)$, that is, $a_1 < -L_d$, the offset "a" and "c" are used to indicate that the appropriate region generating that configuration is initiated at $(-W_d, -L_d)$ or that a similar region terminates at (W_d, L_d). The "a" is used to denote $-L_d$ in such cases and the "c" is used to denote W_d for notational convenience and consistency in Tables V - X.

If specific values are assigned to the three sets of length and width parameters and θ, the following set of instructions indicates how to use Tables V - X.

Step 1. Evaluate and rank $L_p \tan\theta$, $W_r \sec\theta$, and W_p. This indicates which table to use. Note if the largest term is greater than the sum of the smaller two terms.

Step 2. Evaluate and rank the a_i, $-L_d$, and L_d. The relationship of L_d to the a_i determines which row of the table to use. The ranking of the a_i and $-L_d$ determines which columns of the row to use. If $-L_d < a_1$, all of the appropriate columns are used. Since the relationship between the a_i and $-L_d$ determines the configuration of P on R when the PCP is located at $(-W_d, -L_d)$, use the column on the left headed by this configuration to initiate the sequence of PCP change points. The sequence of PCP change points terminates in the right hand column headed by that configuration.

Step 3. Once the appropriate row and columns have been determined, the upper elements in the row are the PCP change point ordinates and/or abscissas on the lines $d = -W_d$ and $r = L_d$. The lower elements in the row are the PCP change point ordinates and/or abscissas on the lines $d = W_d$ and $r = -L_d$.

Three examples illustrating the use of these instructions for Tables V - X follow Table X.

TABLE V

PCP CHANGE POINTS ON THE PERIMETER OF D

	Configuration of P on R ($W_p > W_r \sec\theta > L_p \tan\theta$)							
$< -L_d, L_d \leq$	Triangle	Trapezoid	Pentagon	Hexagon	Parallelogram	Pentagon	Trapezoid	Triangle
a, a_1	c_1 c_4 d_8 d_5	c_3 d_6	c_2 c_7 d_7 d_2	c_7 d_2	c_2 d_7	c_6 d_3	c_5 d_4	c_8 d_1
a_1, a_2	a a_1 c_4 d_8 d_5	c_3 d_6	c_2 c_7 d_7 d_2	c_7 d_2	c_2 d_7	c_6 d_3	c_5 d_4	c c_8 b_1
a_2, a_3	a a_1 a_2 d_8 d_5	a c_3 d_6	c_2 c_7 d_7 d_2	c_7 d_2	c_2 d_7	c_6 d_3	c c_5 b_2	c c_8 b_1
a_3, a_4 (a_3, a_5)	a a_1 a_2 d_8 d_5	a a_3 d_6	a c_2 c_7 d_7 d_2	c_7 d_2	c_2 d_7	c c_6 b_3	c c_5 b_2	c c_8 b_1
a_4, a_5 (a_5, a_4)	a a_1 a_2 d_8 d_5	a a_3 d_6	a a_4 a_5 d_7 d_2	a c_7 c b_4	a c_2 c b_5	c c_6 b_3	c c_5 b_2	c c_8 b_1
a_5, a_6 (a_4, a_6)	a_1 a_2 d_8 d_5	a_3 d_6	a_4 a_5 b_5 b_4	a_5 b_4	a_4 b_5	c_6 b_3	c_5 b_2	c_8 b_1
a_6, a_7	a_1 a_2 d_8 d_5	a_3 b_6	a_4 a_5 b_5 b_4	a_5 b_4	a_4 b_5	a_6 b_3	c_5 b_2	c_8 b_1
a_7, a_8	a_1 a_2 d_8 b_7	a_3 b_6	a_4 a_5 b_5 b_4	a_5 b_4	a_4 b_5	a_6 b_3	a_7 b_2	c_8 b_1

TABLE VI

PCP CHANGE POINTS ON THE PERIMETER OF D

	Configuration of P on R ($L_p \tan\theta > W_r \sec\theta > W_p$)							
$< -L_d, L_d \le$	Triangle	Trapezoid	Pentagon	Hexagon	Parallelogram	Pentagon	Trapezoid	Triangle
a, a_1	c_1 c_2 d_8 d_7	c_3 d_6	c_4 c_5 d_5 d_4	c_5 d_4	c_4 d_5	c_6 d_3	c_7 d_2	c_8 d_1
a_1, a_4	a a_1 c_2 d_8 d_7	c_3 d_6	c_4 c_5 d_5 d_4	c_5 d_4	c_4 d_5	c_6 d_3	c_7 d_2	c c_8 b_1
a_4, a_3	a a_1 a_4 d_8 d_7	a c_3 d_6	c_4 c_5 d_5 d_4	c_5 d_4	c_4 d_5	c_6 d_3	c c_7 d_4	c c_8 b_1
a_3, a_2 (a_3, a_7)	a a_1 a_4 d_8 d_7	a a_3 d_6	a c_4 c_5 d_5 d_4	c_5 d_4	c_4 d_5	c c_6 b_3	c c_7 b_4	c c_8 b_1
a_2, a_7 (a_7, a_2)	a a_1 a_4 d_8 d_7	a a_3 d_6	a a_2 a_7 d_5 d_4	a c_5 c b_2	a c_4 c b_7	c c_6 b_3	c c_7 b_4	c c_8 b_1
a_7, a_6 (a_2, a_6)	a_1 a_4 d_8 d_7	a_3 d_6	a_2 a_7 b_7 b_2	a_7 b_2	a_2 b_7	c_6 b_3	c_7 b_4	c_8 b_1
a_6, a_5	a_1 a_4 d_8 d_7	a_3 b_6	a_2 a_7 b_7 b_2	a_7 b_2	a_2 b_7	a_6 b_3	c_7 b_4	c_8 b_1
a_5, a_8	a_1 a_4 d_8 b_5	a_3 b_6	a_2 a_7 b_7 b_2	a_7 b_2	a_2 b_7	a_6 b_3	a_5 b_4	c_8 b_1

TABLE VII

PCP CHANGE POINTS ON THE PERIMETER OF D

	Configuration of P on R ($W_p > L_p \tan\theta > W_r \sec\theta$)							
$< -L_d, L_d \leq$	Triangle	Trapezoid	Pentagon	Hexagon	Parallelogram	Pentagon	Trapezoid	Triangle
a, a_1	c_1 c_3 d_8 d_6	c_4 d_5	c_2 c_7 d_7 d_2	c_7 d_2	c_2 d_7	c_5 d_4	c_6 d_3	c_8 d_1
a_1, a_3	a a_1 c_3 d_8 d_6	c_4 d_5	c_2 c_7 d_7 d_2	c_7 d_2	c_2 d_7	c_5 d_4	c_6 d_3	c c_8 b_1
a_3, a_2	a a_1 a_3 d_8 d_6	a c_4 d_5	c_2 c_7 d_7 d_2	c_7 d_2	c_2 d_7	c_5 d_4	c c_6 b_3	c c_8 b_1
a_2, a_4 (a_2, a_5)	a a_1 a_3 d_8 d_6	a a_2 d_5	a c_2 c_7 d_7 d_2	c_7 d_2	c_2 d_7	c c_5 b_2	c c_6 b_3	c c_8 b_1
a_4, a_5 (a_5, a_4)	a a_1 a_3 d_8 d_6	a a_2 d_5	a a_4 a_5 d_7 d_2	a c_7 c b_4	a c_2 c b_5	c c_5 b_2	c c_6 b_3	c c_8 b_1
a_5, a_7 (a_4, a_7)	a_1 a_3 d_8 d_6	a_2 d_5	a_4 a_5 b_5 b_4	a_5 b_4	a_4 b_5	c_5 b_2	c_6 b_3	c_8 b_1
a_7, a_6	a_1 a_3 d_8 d_6	a_2 b_7	a_4 a_5 b_5 b_4	a_5 b_4	a_4 b_5	a_7 b_2	c_6 b_3	c_8 b_1
a_6, a_8	a_1 a_3 d_8 b_6	a_2 b_7	a_4 a_5 b_5 b_4	a_5 b_4	a_4 b_5	a_7 b_2	a_6 b_3	c_8 b_1

TABLE VIII

PCP CHANGE POINTS ON THE PERIMETER OF D

	Configuration of P on R ($W_r \sec\theta > W_p > L_p \tan\theta$)							
$< -L_d, L_d \leq$	Triangle	Trapezoid	Pentagon	Hexagon	Rectangle	Pentagon	Trapezoid	Triangle
a, a_1	c_1 c_4 d_8 d_5	c_2 d_7	c_3 c_6 d_6 d_3	c_6 d_3	c_3 d_6	c_7 d_2	c_5 d_4	c_8 d_1
a_1, a_2	a a_1 c_4 d_8 d_5	c_2 d_7	c_3 c_6 d_6 d_3	c_6 d_3	c_3 d_6	c_7 d_2	c_5 d_4	c c_8 b_1
a_2, a_4	a a_1 a_2 d_8 d_5	a c_2 d_7	c_3 c_6 d_6 d_3	c_6 d_3	c_3 d_6	c_7 d_2	c c_5 b_2	c c_8 b_1
a_4, a_3 (a_4, a_6)	a a_1 a_2 d_8 d_5	a a_4 d_7	a c_3 c_6 d_6 d_3	c_6 d_3	c_3 d_6	c c_7 b_4	c c_5 b_2	c c_8 b_1
a_3, a_6 (a_6, a_3)	a a_1 a_2 d_8 d_5	a a_4 d_7	a a_3 a_6 d_6 d_3	a c_6 c b_3	a c_3 c b_6	c c_7 b_4	c c_5 b_2	c c_8 b_1
a_6, a_5 (a_3, a_5)	a_1 a_2 d_8 d_5	a_4 d_7	a_3 a_6 b_6 b_3	a_6 b_3	a_3 b_6	c_7 b_4	c_5 b_2	c_8 b_1
a_5, a_7	a_1 a_2 d_8 d_5	a_4 b_5	a_3 a_6 b_6 b_3	a_6 b_3	a_3 b_6	a_5 b_4	c_5 b_2	c_8 b_1
a_7, a_8	a_1 a_2 d_8 b_7	a_4 b_5	a_3 a_6 b_6 b_3	a_6 b_3	a_3 b_6	a_5 b_4	a_7 b_2	c_8 b_1

TABLE IX

PCP CHANGE POINTS ON THE PERIMETER OF D

	Configuration of P on R ($L_p \tan\theta > W_p > W_r \sec\theta$)							
$< -L_d, L_d \leq$	Triangle	Trapezoid	Pentagon	Hexagon	Parallelogram	Pentagon	Trapezoid	Triangle
a, a_1	c_1 c_3 d_8 d_6	c_2 d_7	c_4 c_5 d_5 d_4	c_5 d_4	c_4 d_5	c_7 d_2	c_6 d_3	c_8 d_1
a_1, a_3	a a_1 c_3 d_8 d_6	c_2 d_7	c_4 c_5 d_5 d_4	c_5 d_4	c_4 d_5	c_7 d_2	c_6 d_3	c c_8 b_1
a_3, a_4	a a_1 a_3 d_8 d_6	a c_2 d_7	c_4 c_5 d_5 d_4	c_5 d_4	c_4 d_5	c_7 d_2	c c_6 b_3	c c_8 b_1
a_4, a_2 (a_4, a_7)	a a_1 a_3 d_8 d_6	a a_4 d_7	a c_4 c_5 d_5 d_4	c_5 d_4	c_4 d_5	c c_7 b_4	c c_6 b_3	c c_8 b_1
a_2, a_7 (a_7, a_2)	a a_1 a_3 d_8 d_6	a a_4 d_7	a a_2 a_7 d_5 d_4	a c_5 c b_2	a c_4 c b_7	c c_7 b_4	c c_6 b_3	c c_8 b_1
a_7, a_5 (a_2, a_5)	a_1 a_3 d_8 d_6	a_4 d_7	a_2 a_7 b_7 b_2	a_7 b_2	a_2 b_7	c_7 b_4	c_6 b_3	c_8 b_1
a_5, a_6	a_1 a_3 d_8 d_6	a_4 b_5	a_2 a_7 b_7 b_2	a_7 b_2	a_2 b_7	a_5 b_4	c_6 b_3	c_8 b_1
a_6, a_8	a_1 a_3 d_8 b_6	a_4 b_5	a_2 a_7 b_7 b_2	a_7 b_2	a_2 b_7	a_5 b_4	a_6 b_3	c_8 b_1

TABLE X

PCP CHANGE POINTS ON THE PERIMETER OF D

	Configuration of P on R ($W_r \sec\theta > L_p \tan\theta > W_p$)							
$< -L_d, L_d \le$	Triangle	Trapezoid	Pentagon	Hexagon	Rectangle	Pentagon	Trapezoid	Triangle
a, a_1	c_1 c_2 d_8 d_7	c_4 d_5	c_3 c_6 d_6 d_3	c_6 d_5	c_3 d_6	c_5 d_4	c_7 d_2	c_8 d_1
a_1, a_4	a a_1 c_2 d_8 d_7	c_4 d_5	c_3 c_6 d_6 d_3	c_6 d_3	c_3 d_6	c_5 d_4	c_7 d_2	c c_8 b_1
a_4, a_2	a a_1 a_4 d_8 d_7	a c_4 d_5	c_3 c_6 d_6 d_3	c_6 d_3	c_3 d_6	c_5 d_4	c c_7 b_4	c c_8 b_1
a_2, a_3 (a_2, a_6)	a a_1 a_4 d_8 d_7	a a_2 d_5	a c_3 c_6 d_6 d_3	c_6 d_3	c_3 d_6	c c_5 b_2	c c_7 b_4	c c_8 b_1
a_3, a_6 (a_6, a_3)	a a_1 a_4 d_8 d_7	a a_2 d_5	a a_3 a_6 d_6 d_3	a c_6 c b_3	a c_3 c b_6	c c_5 b_2	c c_7 b_4	c c_8 b_1
a_6, a_7 (a_3, a_7)	a_1 a_4 d_8 d_7	a_2 d_5	a_3 a_6 b_6 b_3	a_6 b_3	a_3 b_6	c_5 b_2	c_7 b_4	c_8 b_1
a_7, a_5	a_1 a_4 d_8 d_7	a_2 b_7	a_3 a_6 b_6 b_3	a_6 b_3	a_3 b_6	a_7 b_2	c_7 b_4	c_8 b_1
a_5, a_8	a_1 a_4 d_8 b_5	a_2 b_7	a_3 a_6 b_6 b_3	a_6 b_3	a_3 b_6	a_7 b_2	a_5 b_4	c_8 b_1

Example 3.4. Let $\theta = 5^{\circ}$, $W_r = 75'$, $W_p = 100'$, $W_d = 250'$, $L_p = 250'$, and $L_d = 500'$.

Step 1. $L_p \tan\theta = 21.8725$, $W_r \sec\theta = 75.2868$, $W_p = 100.0$. Then $W_p > W_r \sec\theta > L_p \tan\theta$ and $W_p > W_r \sec\theta + L_p \tan\theta$. Thus, one uses Table V for this example.

Step 2. When the a_i, $i = 1, 2, \ldots, 8$ are evaluated, the following values result:

$$\begin{array}{ll} a_1 = 603.95 & a_5 = 2825.05 \\ a_2 = 1103.95 & a_6 = 3389.95 \\ a_3 = 2325.05 & a_7 = 4611.05 \\ a_4 = 2889.95 & a_8 = 5111.05 \,. \end{array}$$

The ranked values of the a_i are given by $a_1, a_2, a_3, a_5, a_4, a_6, a_7, a_8$.

Since $L_d < a_1$, the first row of Table V is the appropriate row to use. Also, since $-L_d < a_1$, all the appropriate columns are used. Note that $W_p > W_r \sec\theta + L_p \tan\theta$. Accordingly, the second column under the heading "Pentagon" is used instead of the first column and the "Parallelogram" column is used instead of the "Hexagon" column. Since $-L_d < a_1$, no "a" or "c" need to be considered.

Step 3. The appropriate PCP abscissas on $r = L_d$ and $r = -L_d$ at which the configurations change are given as

$$\left\{ \begin{pmatrix} c_1 \\ d_8 \end{pmatrix}, \begin{pmatrix} c_4 \\ d_5 \end{pmatrix}, \begin{pmatrix} c_3 \\ d_6 \end{pmatrix}, \begin{pmatrix} c_7 \\ d_2 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_7 \end{pmatrix}, \begin{pmatrix} c_6 \\ d_3 \end{pmatrix}, \begin{pmatrix} c_5 \\ d_4 \end{pmatrix}, \begin{pmatrix} c_8 \\ d_1 \end{pmatrix} \right\} .$$

The appropriate sub-regions in D generating the configurations are then determined by connecting the elements in the above parentheses by lines parallel to the line $r = (-\cot\theta)d$.

Example 3.5. Let $\theta = 20^{\circ}$ and suppose that W_r, W_p, W_d, L_p, and L_d assume the same values as in Example 3.4.

Step 1. $L_p \tan\theta = 90.9925$, $W_r \sec\theta = 79.815$, $W_p = 100.0$. Then $W_p > L_p \tan\theta > W_r \sec\theta$, but $W_p < L_p \tan\theta + W_r \sec\theta$. Thus, one uses Table VII for this example.

Step 2. The values of the a_i, $i = 1, 2, \ldots, 8$, are as follows:

$$\begin{array}{llll} a_1 = & -57.16 & a_5 = & 881.41 \\ a_2 = & 442.84 & a_6 = & 992.34 \\ a_3 = & 381.91 & a_7 = & 930.91 \\ a_4 = & 492.34 & a_8 = & 1430.91 \end{array}$$

The ranked values of the a_i are given by $a_1, a_3, a_2, a_4, a_5, a_7, a_6, a_8$.

Since $a_4 < L_d < a_5$, one uses the fifth row of Table VII. Again, since $-L_d < a_1$, all appropriate columns are used and "a" and "c" are not considered. Because $W_p < L_p \tan\theta + W_r \sec\theta$, one uses the first column under the "Pentagon" heading and uses the "Hexagon" column instead of the "Parallelogram" column.

Step 3. The sequence of ordinates and abscissas on the perimeter of D at which the configurations change is given by

$$\left\{ \begin{pmatrix} a_1 \\ d_8 \end{pmatrix}, \begin{pmatrix} a_3 \\ d_6 \end{pmatrix}, \begin{pmatrix} a_2 \\ d_5 \end{pmatrix}, \begin{pmatrix} a_4 \\ d_7 \end{pmatrix}, \begin{pmatrix} c_7 \\ b_4 \end{pmatrix}, \begin{pmatrix} c_5 \\ b_2 \end{pmatrix}, \begin{pmatrix} c_6 \\ b_3 \end{pmatrix}, \begin{pmatrix} c_8 \\ b_1 \end{pmatrix} \right\}.$$

Connecting the points in parentheses by lines parallel to the line

$r = (-\cot\theta)d$ determines the appropriate sub-regions of D that generate the given configurations.

Example 3.6. Let $\theta = 30^{\circ}$, $W_r = 100'$, $W_p = 150'$, $W_d = 200'$, $L_p = 600'$, and $L_d = 300'$.

Step 1. $L_p \tan\theta = 346.41$, $W_r \sec\theta = 115.47$, and $W_p = 150.0$. Thus, $L_p \tan\theta > W_p > W_r \sec\theta$ and $L_p \tan\theta > W_p + W_r \sec\theta$. Consequently, Table IX is the appropriate table to use.

Step 2. The values of the a_i, $i = 1, 2, \ldots, 8$ are as follows:

$$\begin{aligned} a_1 &= -713.395 & a_5 &= 886.605 \\ a_2 &= 486.605 & a_6 &= 1006.235 \\ a_3 &= -313.395 & a_7 &= 206.235 \\ a_4 &= -193.765 & a_8 &= 1406.235 \end{aligned}$$

The ranking of the a_i is given as $a_1, a_3, a_4, a_7, a_2, a_5, a_6, a_8$.

Since $a_7 < L_d < a_2$, the fifth row of the table is the appropriate row to use. $L_p \tan\theta > W_p + W_r \sec\theta$ implies that the second column under the "Pentagon" heading is used and the "Parallelogram" column is used instead of the "Hexagon" column.

In this example $a_1 < a_3 < -L_d$. This result indicates that when the PCP is located at $(-W_d, -L_d)$ and (W_d, L_d) the P on R configuration forms a trapezoid. Accordingly, the appropriate columns to use in Table IX are headed by "Trapezoid," "Pentagon," "Parallelogram," "Pentagon," and "Trapezoid." In this case, "a" is used to indicate that the sub-region in D generating trapezoids originates at $(-W_d, -L_d)$. Likewise, "c" is used to indicate that sub-region in the upper right hand corner of D which generates trapezoids terminates at (W_d, L_d).

Step 3. The sequence of abscissas and ordinates on the perimeter of D at which the configurations change or terminate is given as

$$\left\{ \begin{pmatrix} a \\ \ \end{pmatrix}, \begin{pmatrix} a_4 \\ d_7 \end{pmatrix}, \begin{pmatrix} a_7 \\ d_4 \end{pmatrix}, \begin{pmatrix} c_4 \\ b_7 \end{pmatrix}, \begin{pmatrix} c_7 \\ b_4 \end{pmatrix}, \begin{pmatrix} c \\ \ \end{pmatrix} \right\}.$$

Again connecting the points in parentheses by lines parallel to the line $r = (-\cot\theta)d$ determines the sub-regions in D which generate the P on R configurations.

CHAPTER IV

THE PROBABILITY DISTRIBUTION OF THE PROPORTION OF PATTERN AREA ON THE RUNWAY

The purpose of this chapter is to develop a representation for the cumulative distribution function (c.d.f.) of the proportion of the area of P on R to the total area of P. First the formulas for the areas of the various P on R configurations are developed when the PCP is located in the regions generating those configurations. The form for the desired c.d.f. is discussed for a general PCP probability density restricted to D. The desired c.d.f. is then attained in graphical form for several specific PCP probability densities.

In the discussion, the proportion of the area of P on R to the total area of P will be denoted by "A_p." The range of values for A_p is $0 \leq A_p \leq 1$. The desired c.d.f. will be denoted by $P(\alpha) = \Pr[A_p \leq \alpha],\ 0 \leq \alpha \leq 1$.

Since the range of values of A_p changes for the various P on R configurations, one needs to know the possible values of A_p for these configurations. However, in order to obtain these values, it is necessary to know the areas of the configurations. Consequently, formulas for the area of the different P on R configurations relative to the center of P are needed. These formulas are developed in the following section of this chapter.

Formulation of the Proportion of Area for P on R Configurations

In order to develop the probability distribution of the proportion of area of P on R to the total area of P, one needs to obtain a representation of the area for the various P on R configurations. For each of the P on R configurations in the following discussion, the PCP is located at (d, r), an arbitrary point in the sub-region of D generating that P on R configuration. A formula for the area of each of the P on R configurations is obtained by using (d, r) as the PCP.

Let us suppose that the PCP is in a sub-region of D that generates a specific P on R configuration. When the PCP is located on a line parallel to the line given by $r = (-\cot\theta)d$, the area of the P on R configuration remains constant. As such, the proportion of the area of the P on R configuration to the total area of P also remains constant when the PCP remains on the stated line. Let α represent the desired proportion. Then by setting $4L_p W_p \alpha$ equal to the formula for the area of the configuration, one can obtain the equation of a line in terms involving d and r. This line is parallel to the line whose equation is $r = (-\cot\theta)d$. When the PCP is located on the derived line, the proportion of the area of the P on R configuration to the total area of P is given by α. Hereafter, the derived lines shall be referred to as "α lines."

Due to symmetry with respect to the origin, the following discussion considers only those generating sub-regions to the left of the line $r = (-\cot\theta)d$. The illustrations given for each of the configurations consider the point (d, r) as being in a sub-region of D generating that configuration. However, the rectangle D is not illustrated for convenience purposes.

Triangle: From Figure 6, one obtains the following representation for the area of a triangular P on R configuration:

$$A(d, r) = \frac{1}{2}\{(r+L_p) + (d+W_p)\cot\theta + W_r\csc\theta\}\{(d+W_p) + (r+L_p)\tan\theta + W_r\sec\theta\} . \tag{4.1}$$

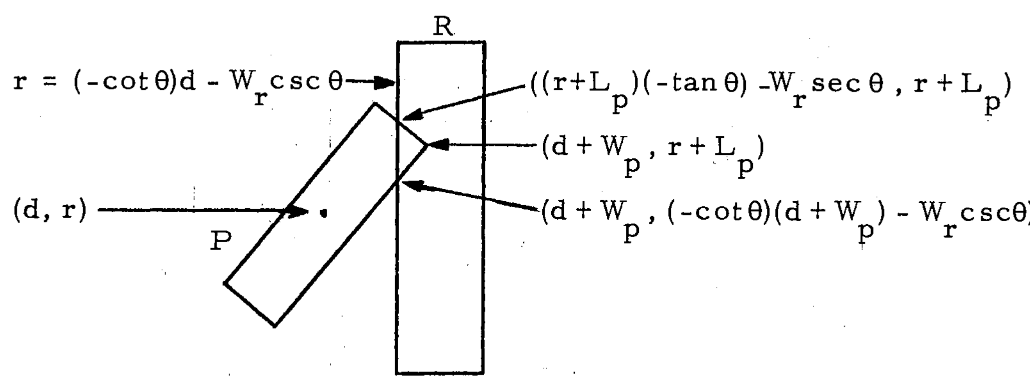

Figure 6. Triangular P on R Configuration

After some trigonometric substitutions are employed in (4. 1), one obtains the following result:

$$A(d, r) = \frac{\{(r+L_p)\sin\theta + (d+W_p)\cos\theta + W_r\}^2}{2\sin\theta\cos\theta} \tag{4.2}$$

For a triangular P on R configuration, α ranges from zero to α', where α' is the ratio of the maximum area of a triangular P on

R configuration to the total area of P. Then for $0 \leq \alpha \leq \alpha'$, the α line is determined from $4L_pW_p\alpha = A(d, r)$, where $A(d, r)$ is given in (4.2). The equation of the resulting α line is given by

$$r = (-\cot\theta)(d+W_p) - L_p - W_r\csc\theta + (\alpha 8L_pW_p\cot\theta)^{1/2}. \qquad (4.3)$$

In the remainder of this discussion, the range of α for a specific configuration is not considered.

Rectangle: If the P on R configuration forms a rectangle, then the context of the problem implies that rectangle P is entirely contained with rectangle R. Consequently, $\alpha = 1$. In such cases, any line within the sub-region of D generating rectangles that is parallel to the line $r = (-\cot\theta)d$ can serve as an α line.

Trapezoid: When the P on R configuration forms a trapezoid, three possible trapezoids may occur. Figure 7 indicates their forms.

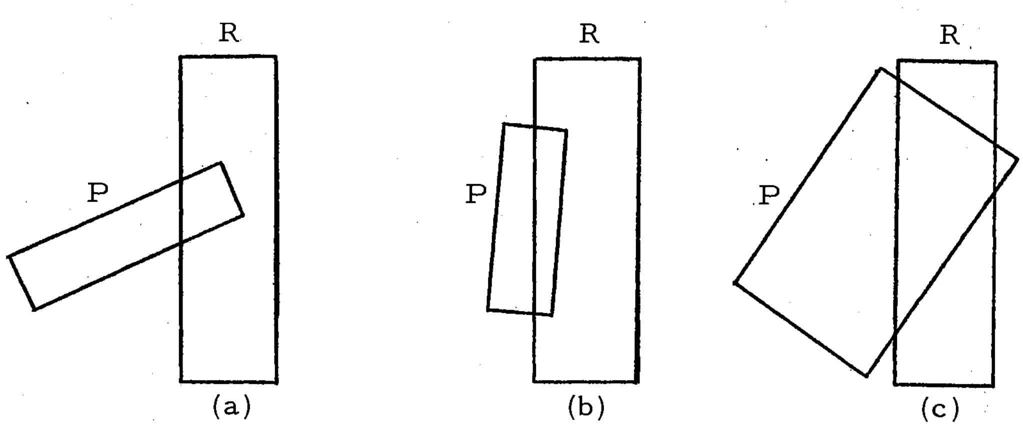

Figure 7. Possible Trapezoidal P on R Configurations

The area of the trapezoid illustrated in Figure 7(a) is given by

$$A(d, r) = 2W_p\{(r+L_p) + d\cot\theta + W_r\csc\theta\} . \qquad (4.4)$$

The resulting α line is given by

$$r = (-\cot\theta)d - W_r\csc\theta + (2\alpha - 1)L_p . \qquad (4.5)$$

The area of the trapezoid in Figure 7(b) is given by

$$A(d, r) = 2L_p\{(d + W_p) + r\tan\theta + W_r\sec\theta\} , \qquad (4.6)$$

with the resulting α line given by

$$r = (-\cot\theta)d - W_r\csc\theta + (2\alpha - 1)W_p\cot\theta . \qquad (4.7)$$

The area of the trapezoidal P on R configuration in Figure 7(c) is more difficult to obtain than the area for the preceding trapezoids. For this case, one needs to obtain the lengths of the two parallel sides of the trapezoid. Combination of terms in the representations of the lengths of the two parallel sides, enables one to write the area in the following form:

$$A(d, r) = \frac{2W_r\{(r+L_p)\sin\theta + (d+W_p)\cos\theta\}}{\sin\theta\cos\theta} . \qquad (4.8)$$

The α line for the trapezoid illustrated in Figure 7(c) is given by

$$r = (-\cot\theta)(d + W_p) - L_p + \frac{2\alpha L_p W_p \cos\theta}{W_r} . \qquad (4.9)$$

<u>Parallelograms</u>: The two forms of parallelograms resulting from P on R configurations are illustrated in Figure 8.

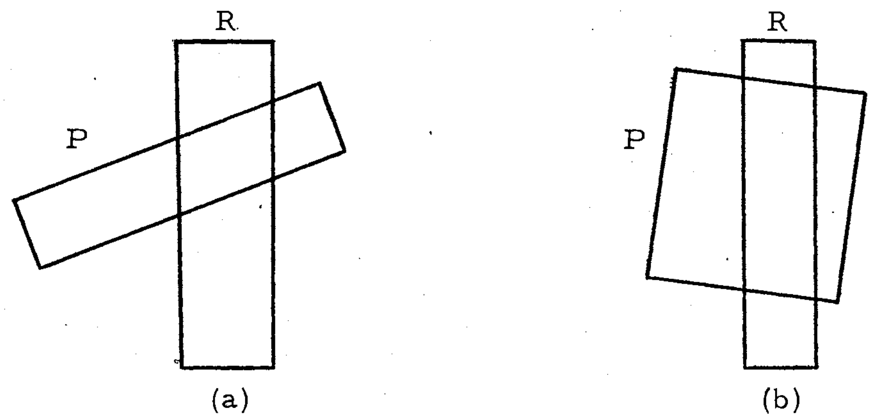

Figure 8. Parallelogram Configurations of P on R

The area of a parallelogram remains constant when the point (d, r) is located in the sub-regions that generate parallelograms. The area of the parallelograms given in Figure 8(a) and Figure 8(b) are given by

$$A(d, r) = 4W_pW_r\csc\theta \qquad (4.10)$$

and

$$A(d, r) = 4L_pW_r\sec\theta\,, \qquad (4.11)$$

respectively. Since the area of the parallelogram remains constant, the proportion α also remains constant. The α lines for parallelogram configurations correspond to any line that is parallel to the line $r = (-\cot\theta)d$ and that is located within the region generating parallelograms.

Pentagon: Three different pentagon configurations of P on R can occur. They are illustrated in Figure 9.

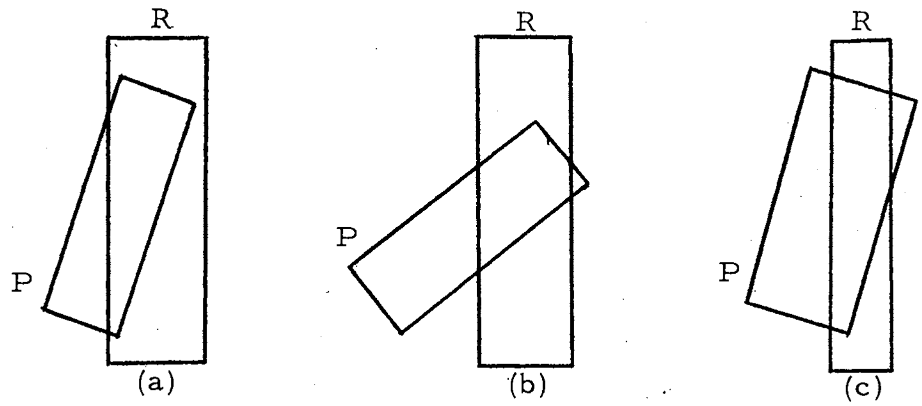

Figure 9. Pentagon P on R Configurations

The area of the pentagon in Figure 9(a) is determined by subtracting the area of the triangle exterior from the total area of P. The area of this pentagon is given as

$$A(d, r) = 4L_pW_p - \frac{\{(d-W_p)\cos\theta + (r-L_p)\sin\theta + W_r\}^2}{2\sin\theta\cos\theta} . \qquad (4.12)$$

The resulting α line is given by

$$r = (-\cot\theta)(d - W_p) - W_r\csc\theta + L_p - (8(1-\alpha)L_pW_p\cot\theta)^{1/2} . \qquad (4.13)$$

To determine the area of the pentagon in Figure 9(b), one subtracts the area of P exterior to R from the total area of P. The area of the pentagon is given as

$$A(d, r) = 4L_pW_p + 2W_p\{(r - L_p) + d\cot\theta + W_r\csc\theta\}$$

$$- \frac{\{(r + L_p)\sin\theta + (d + W_p)\cos\theta - W_r\}^2}{2\sin\theta\cos\theta}$$

$$= \frac{\{4W_rW_p\cos\theta - 1/2\,[(r + L_p)\sin\theta + (d - W_p)\cos\theta - W_r]^2\}}{\sin\theta\cos\theta} \quad (4.14)$$

The corresponding α line is determined as

$$r = (-\cot\theta)(d - W_p) + W_r\csc\theta - L_p - [8W_p\cot\theta\,(W_r\csc\theta - \alpha L_p)]^{1/2} . \quad (4.15)$$

The area of the pentagon in Figure 9(c) is determined in the same manner as the area of the preceding pentagon. This area is given by

$$A(d, r) = 4L_pW_p + 2L_p\{(d - W_p) + r\tan\theta + W_r\sec\theta\}$$

$$- \frac{\{(r + L_p)\sin\theta + (d + W_p)\cos\theta - W_r\}^2}{2\sin\theta\cos\theta}$$

$$= \frac{\{4L_pW_r\sin\theta - 1/2\,[(r - L_p)\sin\theta + (d + W_p)\cos\theta - W_r]^2\}}{\sin\theta\cos\theta} \quad (4.16)$$

The α line is determined as

$$r = (-\cot\theta)(d + W_p) + W_r\csc\theta + L_p + [8L_p(W_r\csc\theta - \alpha W_p\cot\theta)]^{1/2} \quad (4.17)$$

Hexagon: The area of the hexagon illustrated in Figure 10 is determined by subtracting the area of the two triangular regions exterior to R from the total area of P.

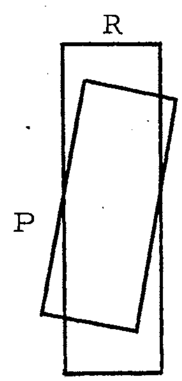

Figure 10. Hexagon P on R Configuration

The area of the hexagon is determined as

$$A(d, r) = 4L_pW_p - \frac{\{[(r+L_p)\sin\theta+(d+W_p)\cos\theta - W_r]^2 + [(r-L_p)\sin\theta+(d-W_p)\cos\theta+W_r]^2\}}{2\sin\theta\cos\theta}. \tag{4.18}$$

The equation of the resulting α line is given by

$$r = (-\cot\theta)d - \{4L_pW_p(1-\alpha)\cot\theta - (L_p + W_p\cot\theta - W_r\csc\theta)^2\}^{1/2}. \tag{4.19}$$

When values are assigned to the set of length and width parameters and θ, the following table, Table XI, is helpful in determining the general shape of the P on R configuration and consequently, which of the area and α line formulas to use. The table entries refer to the preceding figures in this chapter.

TABLE XI

SPECIFIC P ON R CONFIGURATIONS
FOR TABLES V - X

Table Number	Triangle	Trapezoid	Pentagon	Hexagon	Parallelogram
V	Figure (6)	Figure 7(b)	Figure 9(c)	Figure 10	Figure 8(b)
VI	Figure (6)	Figure 7(a)	Figure 9(b)	Figure 10	Figure 8(a)
VII	Figure (6)	Figure 7(c)	Figure 9(c)	Figure 10	Figure 8(b)
VIII	Figure (6)	Figure 7(b)	Figure 9(a)	Figure 10	
IX	Figure (6)	Figure 7(c)	Figure 9(b)	Figure 10	Figure 8(a)
X	Figure (6)	Figure 7(a)	Figure 9(a)	Figure 10	

Rectangular P on R configurations occur for the situations where the entries in the parallelogram column are blank.

The Probability Distribution $P(\alpha) = \Pr[A_p \leq \alpha]$

In the following discussion let us denote the probability density function for the PCP by $f(d, r)$. It will be assumed that $f(d, r)$ is continuous and that $f(d, r)$ equals zero for points (d, r) exterior to the rectangle D. It will also be assumed that $f(d, r)$ is symmetric.

In order to obtain the desired c.d.f., $P(\alpha)$, one needs to integrate $f(d, r)$ over the regions in D that produce values of A_p for which $A_p \leq \alpha$. Now, since the sub-regions in D that generate the P on R configurations are symmetric with respect to the origin, the integration process needs only to be considered in sub-regions of D to the left of the line $r = (-\cot\theta)d$. Once this integral is obtained, twice its value gives the desired result.

A formulation of the appropriate integral would be in the form

$$P(\alpha) = 2\int_{c}^{d}\int_{a}^{b} f(d,r)\,d\,d\,d\,r\ . \qquad (4.20)$$

The limits of integration in (4.20) are as follows: $a = -W_d$, $c = -L_d$, b is either an equation of an appropriate α line or the value W_d, depending on the situation; and, d is either the point at which the α line and the line $d = -W_d$ intersect or the value L_d.

In an evaluation of $P(\alpha)$ for a specific situation, it is necessary to know the range of the values for A_p for the various P on R configurations. Table XII given below gives the maximum value of A_p for each configuration listed in Tables V - X. Also, the table entries give the α lines on which the PCP would be located to give the configuration that value. For the sake of notational convenience, the maximum A_p values and the corresponding α lines are listed below. Note that the α lines are the lines at which the configurations change form. Let us now define the maximum A_p values as α_i, $i = 1, 2, \ldots, 8$, and the α lines as λ_i, $i = 1, 2, \ldots, 8$. The α_i are given as follows:

$$\alpha_1 = \frac{L_p \tan\theta}{2W_p} \qquad \alpha_5 = \frac{(W_p \cos\theta - L_p \sin\theta - W_r)^2}{2L_pW_p \sin\theta \cos\theta}$$

$$\alpha_2 = \frac{W_r \sec\theta}{W_p} \qquad \alpha_6 = \frac{(L_p \sin\theta + W_p \cos\theta - W_r)^2}{4L_pW_p \sin\theta \cos\theta}$$

$$\alpha_3 = \frac{W_p \cot\theta}{2L_p} \qquad \alpha_7 = \frac{(L_p \sin\theta - W_p \cos\theta - W_r)^2}{2L_pW_p \sin\theta \cos\theta}$$

$$\alpha_4 = \frac{W_r \csc\theta}{L_p} \qquad \alpha_8 = \frac{W_r^2}{2L_pW_p \sin\theta \cos\theta} \; . \tag{4.21}$$

The λ_i are given as follows:

$$\lambda_1 : r = (-\cot\theta)(d + W_p) - W_r \csc\theta - L_p$$

$$\lambda_2 : r = (-\cot\theta)(d + W_p) - W_r \csc\theta + L_p$$

$$\lambda_3 : r = (-\cot\theta)(d + W_p) + W_r \csc\theta - L_p$$

$$\lambda_4 : r = (-\cot\theta)(d - W_p) - W_r \csc\theta - L_p$$

$$\lambda_5 : r = (-\cot\theta)(d + W_p) + W_r \csc\theta + L_p$$

$$\lambda_6 : r = (-\cot\theta)(d - W_p) - W_r \csc\theta + L_p$$

$$\lambda_7 : r = (-\cot\theta)(d - W_p) + W_r \csc\theta - L_p$$

$$\lambda_8 : r = (-\cot\theta)d \; . \tag{4.22}$$

The lower row in the "pentagon" row in Table XII is used when the relationship between $L_p \tan\theta$, $W_r \sec\theta$ and W_p for Tables V - X is such that the largest term is greater than the sum of the other two terms.

TABLE XII

MAXIMUM VALUES OF A_p AND THE ASSOCIATED α LINES

	Table V		Table VI		Table VII		Table VIII		Table IX		Table X	
Miss	0	λ_1	0	λ_1	0	λ_1	0	λ_1	0	λ_1	0	λ_1
Triangle	α_1	λ_2	α_3	λ_4	α_8	λ_3	α_1	λ_2	α_8	λ_3	α_3	λ_4
Trapezoid	$\alpha_2-\alpha_1$	λ_3	$\alpha_4-\alpha_3$	λ_3	$\alpha_2-\alpha_8$	λ_2	$1-\alpha_1$	λ_4	$\alpha_4-\alpha_8$	λ_4	$1-\alpha_3$	λ_2
Pentagon	$\alpha_2-\alpha_5$	λ_4	$\alpha_4-\alpha_7$	λ_2	$\alpha_2-\alpha_5$	λ_4	$1-2\alpha_6$	λ_3	$\alpha_4-\alpha_7$	λ_2	$1-2\alpha_6$	λ_3
	α_2	λ_5	α_4	λ_7	α_2	λ_5	1	λ_6	α_4	λ_7	1	λ_6
Hexagon	$1-\alpha_6$	λ_8	$1-\alpha_6$	λ_8	$1-\alpha_6$	λ_8	$1-\alpha_6$	λ_8	$1-\alpha_6$	λ_8	$1-\alpha_6$	λ_8
Parallelogram	α_2	λ_5	α_4	λ_7	α_2	λ_5			α_4	λ_7		
Rectangle							1	λ_8			1	λ_8

Examples of $P(\alpha)$

$P(\alpha)$ is dependent upon $f(d, r)$, the probability density function associated with the PCP. Several examples of $P(\alpha)$ will be given for different density functions. The two families of PCP density functions that will be considered in the examples are given as

$$
\begin{aligned}
f(d, r) &= \frac{h_2 - h_1}{L_d} r + h_2 , \quad -W_d \le d \le W_d , \quad -L_d \le r < 0 \\
&= -\frac{h_2 - h_1}{L_d} r + h_2 , \quad -W_d \le d \le W_d , \quad 0 \le r \le L_d \\
&= 0 , \text{ elsewhere} , \quad 0 \le h_1 \le \frac{1}{4L_d W_d} , \quad h_1 + h_2 = \frac{1}{2L_d W_d} . \qquad (4.23)
\end{aligned}
$$

and

$$
\begin{aligned}
f(d, r) &= -\frac{3}{8L_d^3 W_d} (r^2 - L_d^2) , \quad -W_d \le d \le W_d , \quad -L_d \le r \le L_d , \\
&= 0 , \text{ elsewhere} . \qquad (4.24)
\end{aligned}
$$

Although the bivariate normal probability density function would have been an appropriate PCP density function to use, the difficulty of the computation of $P(\alpha)$ for these examples prohibited its usage. Consequently, the above density functions were developed as approximations to the bivariate normal probability density function.

When $h_1 = 0$ in (4.23) above, $f(d, r)$ forms a triangular density function in the r direction. A bivariate uniform density results when $h_1 = h_2 = \frac{1}{4L_d W_d}$. The three density functions to be used in the examples of $P(\alpha)$ are (4.23) with $h_1 = h_2 = \frac{1}{4L_d W_d}$; (4.23) with $h_2 = \frac{1}{2L_d W_d}$; and (4.24). These PCP probability density functions will be referred to as the uniform, triangular, and parabolic densities, respectively.

In the following examples, the P on R configurations and the subsequent sub-regions of D which generate them are determined according to the methods described in Chapter III. The value of the maximum A_p for each configuration is then determined as an aid in the evaluation of $P(\alpha)$. $P(\alpha)$ is calculated by evaluating (4.20) for the given density functions and the values of α in the range of A_p for the various configurations.

Example 4.1. In this example let $\theta = 10^{o}$, $W_r = 100'$, $W_d = 600'$, $W_p = 100'$, $L_r = 5000'$, $L_d = 1200'$, and $L_p = 400'$. This case corresponds to situation (1) and Table VIII mentioned in Chapter III. When the three PCP density functions are integrated over the appropriate sub-regions of D for this example, the resulting form of $P(\alpha)$ is given by

$$P(\alpha) = \frac{L_d \tan\theta + W_d + b}{W_d} , \qquad (4.25)$$

where b is the abscissa of the point of intersection of an α line and the line $r = L_d$. The α lines used in this example are determined from Table XI and are given by equations (4.3), (4.7), (4.13), and (4.19). The range of A_p for each of the resulting configurations is determined from Table XII and are given as

$$\begin{aligned} &\text{triangle: } 0 \le A_p \le .353 && \text{pentagon: } .647 \le A_p \le .663 \\ &\text{trapezoid: } .353 \le A_p \le .647 && \text{hexagon: } .663 \le A_p \le .831 . \end{aligned} \qquad (4.26)$$

If L_p is changed to 800', the resulting case corresponds to situation (1) and Table VI. Consequently (4.25) is used to evaluate $P(\alpha)$. However, Table XI indicates that the α lines are now given by equations (4.3), (4.5), (4.15), and (4.19). Table XII indicates that the range of A_p values for each configuration is given by

$$\begin{aligned} &\text{triangle: } 0 \le A_p \le .354 && \text{pentagon: } .365 \le A_p \le .591 \\ &\text{trapezoid: } .354 \le A_p \le .365 && \text{hexagon: } .591 \le A_p \le .655\,. \end{aligned} \tag{4.27}$$

The graphs of the c.d.f., $P(\alpha)$, for these two cases are given in Figure 11.

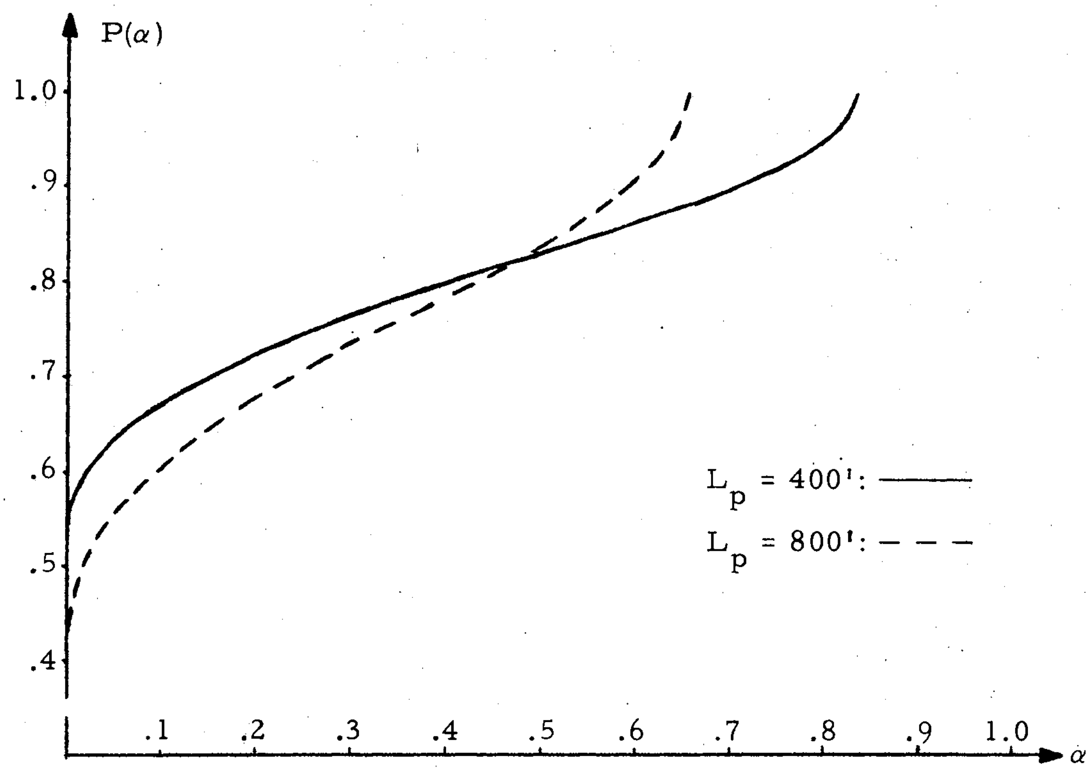

Figure 11. $P(\alpha)$ for Example 4.1

In the following examples, the values of the set of length and width parameters and θ indicate that one is in situation (2) mentioned in Chapter III. For this situation let us define b as the ordinate of the point of intersection of an α line and the line $d = -W_d$. When the integral given in (4.20) is integrated for this situation for the three given PCP probability density functions, the following formulas result:

$$\text{uniform PCP density: } \frac{(b + L_d)^2}{4L_p W_p}$$

$$\text{triangular PCP density: } \frac{[(b + L_d)^3 - 2b^3]\tan\theta}{6L_d^2 W_d}$$

$$\text{parabolic PCP density: } \frac{[8bL_d^3 + 3L_d^4 - b^4 + 6b^2 L_d^2]\tan\theta}{16L_d^3 W_d} \quad . \qquad (4.28)$$

These formulas are used to calculate $P(\alpha)$ as long as $b \leq L_d$. If $b > L_d$, then equation (4.24) is used to evaluate $P(\alpha)$.

<u>Example 4.2.</u> Let us assume that $\theta = 10^o$, $W_r = 100'$, $W_p = 100'$, $W_d = 600'$, $L_p = 800'$ and $L_d = 2400'$. This example corresponds to situation (2) and Table VI in Chapter III. The appropriate α lines as indicated by Table XI are given by equation (4.3), (4.5), (4.15) and (4.19). The range of values of A_p are the same as those given in Example 4.1, equation (4.27). For this example, the formulas in (4.28) are used for $0 \leq \alpha \leq .244$, with (4.24) being used when $.244 \leq \alpha \leq .655$. This procedure is used since $.244$ is the value of A_p when the PCP is located at $(-W_d, L_d)$. The graph of $P(\alpha)$ for this example is given in Figure 12.

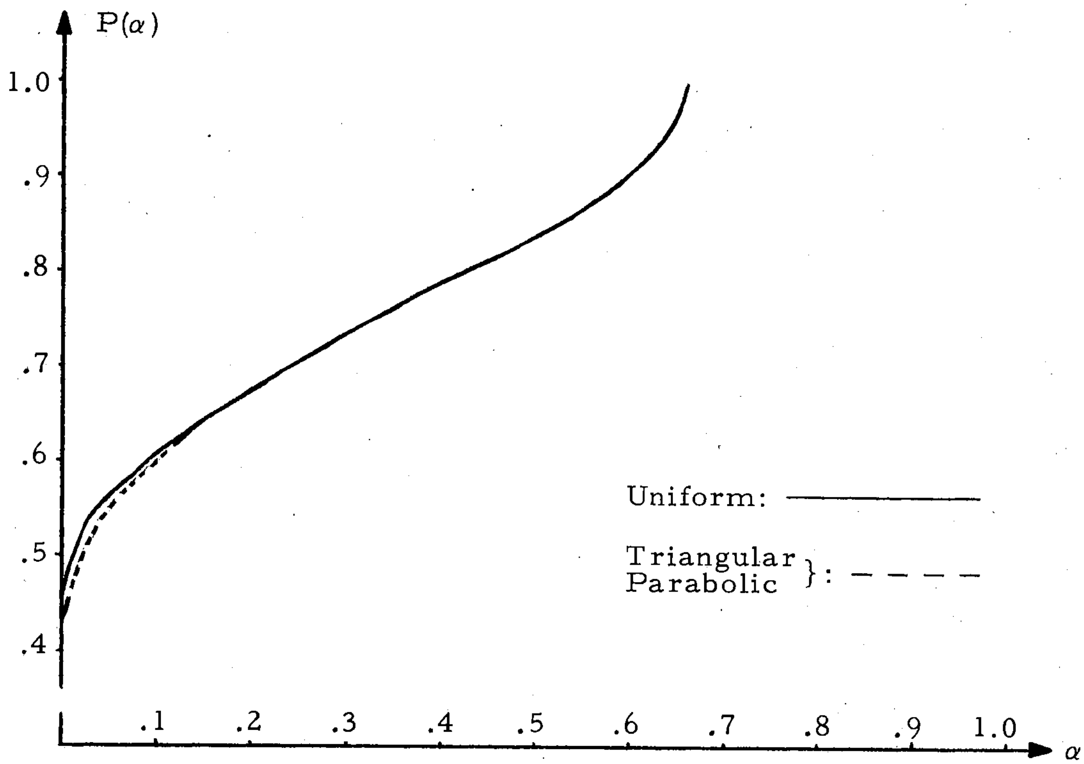

Figure 12. $P(\alpha)$ for Example 4.2

Example 4.3. Let us assume that $\theta = 20^{\circ}$, $W_r = 100'$, $W_p = 100'$, $W_d = 600'$, $L_p = 400'$ and $L_d = 1200'$. This example again corresponds to situation (2) and Table VI in Chapter III. Consequently the same formulas are used for α lines as were used in Example 4.2. The range of values of A_p as indicated in Table XII are given by

$$\text{triangle: } 0 \leq A_p \leq .343 \qquad \text{pentagon: } .388 \leq A_p \leq .604$$
$$\text{trapezoid: } .343 \leq A_p \leq .388 \qquad \text{hexagon: } .604 \leq A_p \leq .667 \tag{4.29}$$

The formulas for calculating $P(\alpha)$ in (4.28) are used for $0 \leq \alpha \leq .306$, with formula (4.24) being used when $.306 \leq \alpha \leq .667$, since .306 is the value of A_p when the PCP is located at $(-W_d, L_d)$. The graph of $P(\alpha)$ for this example is given in Figure 13.

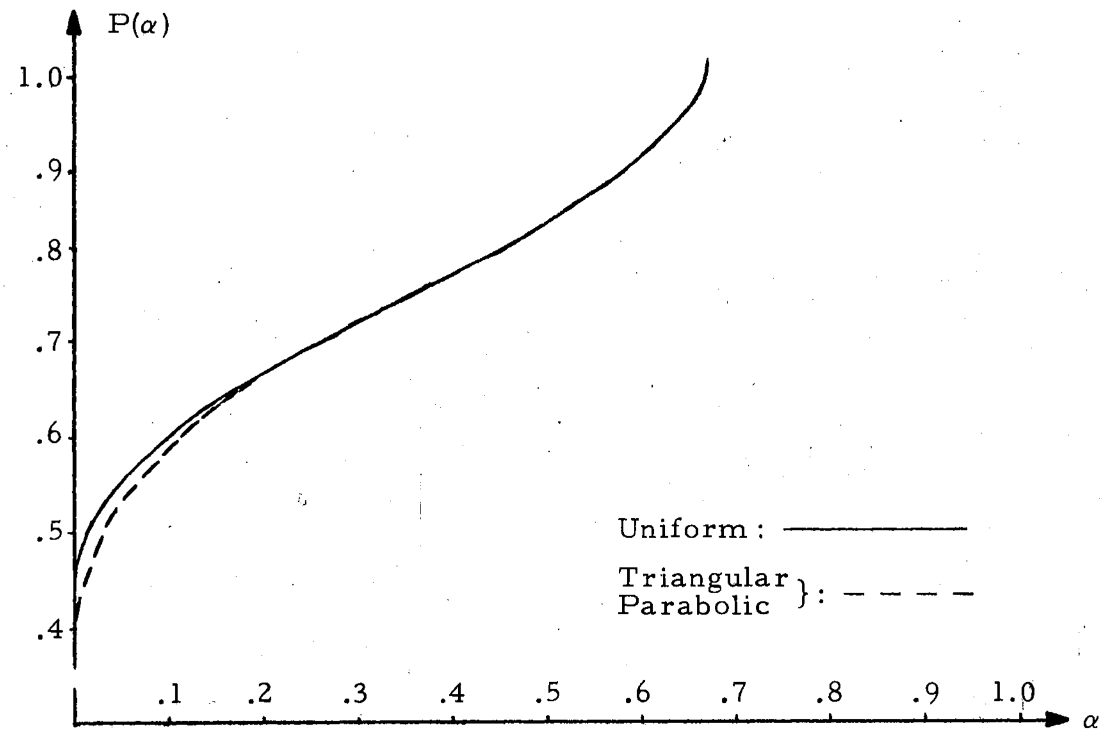

Figure 13. $P(\alpha)$ for Example 4.3

Example 4.4. The conditions on the parameter values are the same as those in Example 4.3, with the exception that $L_p = 800'$. Again, this example corresponds to situation (2) and Table VI in Chapter III. According to Table IX, the formulas to be used for the α lines are given in equation (4.3), (4.5), (4.15), and (4.19). The range of values of A_p as determined from Table XII are given as:

$$\text{triangle: } 0 \leq \alpha \leq .172 \qquad \text{pentagon: } .194 \leq \alpha \leq .365$$
$$\text{trapezoid: } .172 \leq \alpha \leq .194 \qquad \text{parallelogram: } \alpha = .365 \tag{4.30}$$

When the PCP is located at $(-W_d, L_d)$, the corresponding value of A_p is given as $A_p = .339$. Consequently, the formulas in (4.28) are used for $0 \leq \alpha \leq .339$ and (4.24) is used for $.339 \leq \alpha \leq .365$. The graph of $P(\alpha)$ for this example is given in Figure 14. Note the jump in $P(\alpha)$ in Figure 14 as α approaches the value of $.365$. This jump occurs because the P on R configuration for this value of α is a parallelogram. As such, the area of the parallelogram remains constant. Consequently, no other values of A_p are produced. Thus, the integral given in (4.20) is integrated over the portion of D to the left of the line $r = (-\cot\theta)d$. Since $r = (-\cot\theta)d$ divides D into two symmetrical regions each with equal area and since the PCP densities used are symmetric with respect to the origin, the value of (4.20) in this case is equal to one.

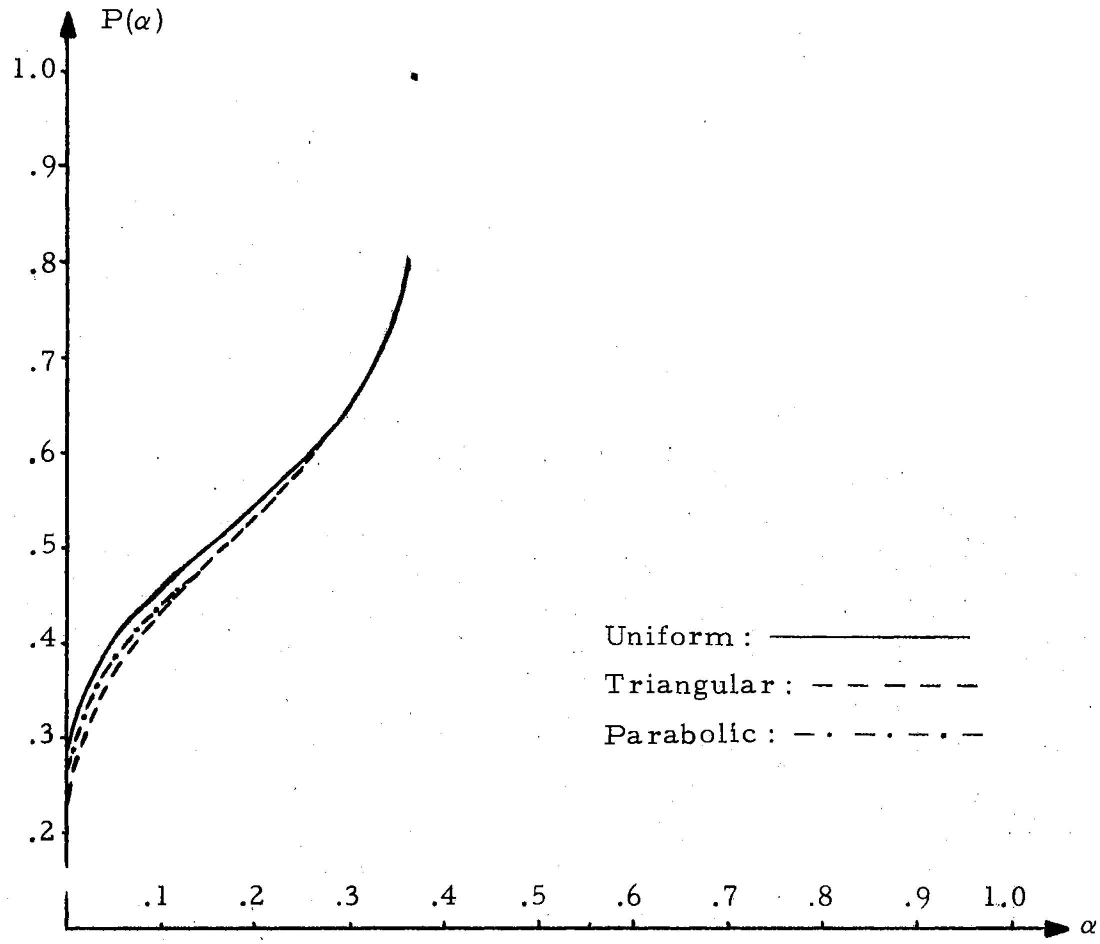

Figure 14. $P(\alpha)$ for Example 4.4

<u>Example 4.5.</u> Let us assume that $\theta = 20^{\circ}$, $W_r = 100'$, $W_r = 100'$, $W_d = 600'$, $L_r = 5000'$, $L_d = 2400'$ and $L_p = 800'$. Again, this example corresponds to situation (2) and Table VI in Chapter III. The α lines and the ranges for the values of A_p are the same as those in Example 4.4., equation (4.30). When the PCP is located at $(W_d, -L_d)$, the corresponding value of A_p is given as $A_p = .212$. The formulas of (4.28) are used when $0 \leq \alpha \leq .212$. For the cases when $.212 \leq \alpha \leq .365$, the following formulas are used to calculate $P(\alpha)$:

uniform PCP density: $\dfrac{b + L_d - W_d \cot\theta}{L_d}$,

triangular PCP density:

$$\frac{\{6W_d K_1^2 + 3b^2 L_d \tan\theta - b^3 \tan\theta + 2K_2^3 \tan\theta - 3(b - L_d)K_2^2 \tan\theta - 6b\, L_d \tan\theta K_2\}}{6L_d^2\, W_d}, \tag{4.31}$$

where $K_1 = b + L_d - 2W_d \cot\theta$ and $K_2 = b - 2W_d \cot\theta$,

parabolic PCP density: $\dfrac{K_2^4 - b^4}{16L_d^3\, W_d \cot\theta} - \dfrac{3(K_1^2 - (b + L_d)^2)}{8L_d W_d \cot\theta} - 1/2$.

The graph of $P(\alpha)$ is given in Figure 15. Note the jump in $P(\alpha)$ in Figure 15.

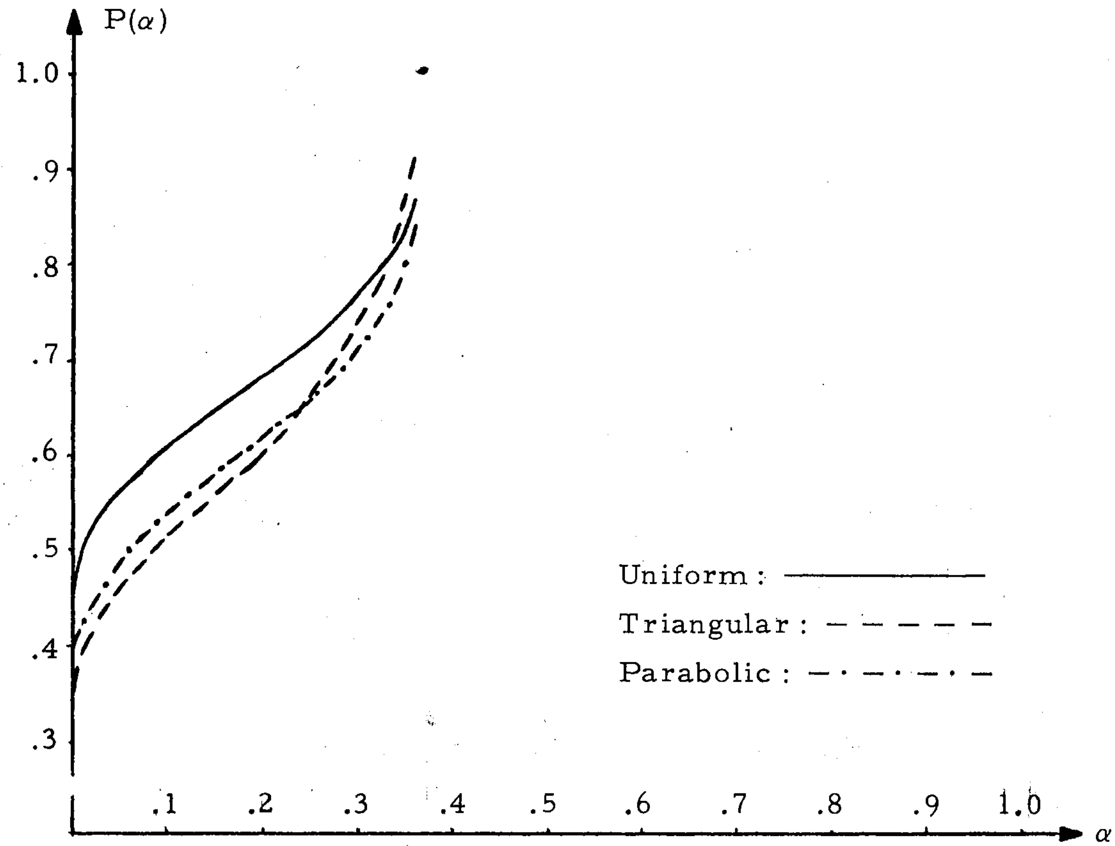

Figure 15. $P(\alpha)$ for Example 4.5

The above example concludes Chapter IV. The next chapter will give a method to determine the single pass hit probabilities from the graph of $P(\alpha)$.

CHAPTER V

A METHOD TO DETERMINE THE SINGLE PASS HIT PROBABILITIES

The discussion in Chapter II indicated that the probability of satisfying any cut criterion is dependent upon the single pass hit probabilities. The intent of this chapter is to present one method of determining the single pass hit probabilities from the c.d.f., $P(\alpha)$.

It will be assumed that the rectangular pattern P contains b weapons. Since the b weapons in a stick are usually released sequentially, it will be assumed that each weapon impact point is contained within a specific segment of P. That is, P can be considered as being divided into b sub-rectangles each with width $2W_p$ and length $2L_p/b$. Accordingly, each weapon is assumed to impact within a specific sub-rectangle of P according to its release time in the release sequence. Furthermore, it is assumed that the density of each weapon impact point is distributed uniformly within its specific rectangle.

Since each weapon falls within a sub-rectangle of P, the method of determining the single pass hit probabilities, p_i, $i = 1, 2, \ldots, b$, assumes that one hit occurs when $100(1/b)\%$ of the area of P is located on R. In a similar fashion, it will be assumed that two hits occur when $100(2/b)\%$ of the area of P is located on R, etc. However, according to this method, in most situations it is not possible for all b weapons in a stick to impact on the target. This fact is illustrated by the examples in Chapter IV.

The c.d.f., $P(\alpha)$, discussed in Chapter IV, determines the probability of obtaining at most a $100\alpha\%$ coverage of P on R. When $\alpha = 0$, then $P(0)$ is the probability of obtaining no coverage of P on R or equivalently, the probability of no hits occurring. Therefore, let us denote by p_0, the probability of obtaining no hits in a single pass, which is determined by $p_0 = P(0)$.

Let H_0 be the event of obtaining no hits in one pass, H_1, the event of obtaining exactly one hit in one pass, etc. Let H_j' be the event of obtaining at most j hits in one pass. Since the H_i are mutually exclusive events, the probability of the occurrence of H_j' is the sum of the probabilities of the events $H_0, H_1, \ldots, H_j$. The method for determining the single pass hit probabilities is given as follows:

Step 1. Determine the maximum value of A_p and denote this by α^*. That is, find α^* such that $P(\alpha^*) = 1$.

Step 2. Now determine the next integer $\geq$ $b\alpha^*$, where b is the number of weapons in the stick. Denote this integer as $[b\alpha^*]$. Thus, $[b\alpha^*] - 1 < b\alpha^* \leq [b\alpha^*]$, $[b\alpha^*]$ indicates the maximum number of hits that can occur when dropping a stick of b weapons.

Step 3. Divide α^* by $[b\alpha^*]$. This essentially divides the interval $[0, \alpha^*]$ into $[b\alpha^*]$ equal segments. Then calculate the end-points of the respective segments, namely, $\alpha^*/[b\alpha^*] = \alpha_1^*$,

$$2\alpha^*/[b\alpha^*] = \alpha_2^*, \ldots, ([b\alpha^*]-1)\alpha^*/[b\alpha^*] = \alpha^*_{[b\alpha^*]-1}.$$

Step 4. From the graph of $P(\alpha)$, determine the following:

$$P(0),\ P(\alpha_1^*),\ \ldots,\ P(\alpha^*_{[b\alpha^*]-1}),\ P(\alpha^*),$$

where $P(0)$ is defined as the probability of no hits, $P(\alpha_1^*)$ is defined as the probability of zero or one hits, ..., $P(\alpha^*)$ is defined as the probability of zero, or one, or two, ..., or $[b\alpha^*]$ hits.

Step 5. Since $P(\alpha_1^*)$ is the probability of zero or one hit, determine the probability of exactly one hit by $P(\alpha_1^*) - P(0) = P(\alpha_1^*) - p_0$. Denote this difference by p_1, the probability of obtaining exactly one hit when dropping a stick of b weapons. The remaining p_i, $i = 2, 3, \ldots, [b\alpha^*]$, are determined in a similar manner. That is,

$$p_2 = P(\alpha_2^*) - p_1$$

$$p_3 = P(\alpha_3^*) - p_2$$

$$p_{[b\alpha^*]} = 1 - P([b\alpha^*] - 1).$$

For $j > [b\alpha^*]$, p_j is defined to be zero.

The following table is an array of the single pass hit probabilities for the examples given in Chapter IV. The "U", "T", "P" columns refer to the uniform, triangular, and parabolic PCP density functions used in Chapter IV. The entries in the table were obtained by the method given above. The table extends no further than p_{11} since in the examples, a maximum of 11 hits were obtainable according to the stated method of determining the single pass hit probabilities.

TABLE XIII

SINGLE PASS HIT PROBABILITIES FOR THE EXAMPLES IN CHAPTER IV

	$\theta = 10^o$									$\theta = 20^o$								
	$L_p = 400$ 8 bombs			$L_p = 800$ 16 bombs						$L_p = 400$ 8 bombs			$L_p = 800$ 16 bombs					
	$L_d = 1200$						$L_d = 2400$			$L_d = 1200$						$L_d = 2400$		
	U	T	P	U	T	P	U	T	P	U	T	P	U	T	P	U	T	P
p_0	.55	.55	.55	.43	.43	.43	.46	.44	.44	.45	.42	.43	.28	.22	.24	.45	.34	.37
p_1	.13	.13	.13	.14	.14	.14	.11	.13	.13	.16	.19	.18	.14	.17	.16	.13	.14	.13
p_2	.06	.06	.06	.05	.05	.05	.05	.05	.05	.1	.1	.1	.07	.09	.08	.08	.07	.06
p_3	.04	.04	.04	.05	.05	.05	.03	.03	.03	.04	.04	.04	.06	.06	.06	.05	.07	.06
p_4	.04	.04	.04	.03	.03	.03	.04	.04	.04	.07	.07	.07	.08	.08	.08	.05	.09	.06
p_5	.04	.04	.04	.04	.04	.04	.03	.03	.03	.08	.08	.08	.13	.14	.14	.10	.15	.12
p_6	.05	.05	.05	.03	.03	.03	.03	.03	.03	.1	.1	.1	.24	.24	.24	.16	.14	.20
p_7	.09	.09	.09	.02	.02	.02	.04	.04	.04	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
p_8	0.0	0.0	0.0	.04	.04	.04	.04	.04	.04	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
p_9				.03	.03	.03	.04	.04	.04				0.0	0.0	0.0	0.0	0.0	0.0
p_{10}				.05	.05	.05	.03	.03	.03				0.0	0.0	0.0	0.0	0.0	0.0
p_{11}				.09	.09	.09	.09	.09	.09				0.0	0.0	0.0	0.0	0.0	0.0

CHAPTER VI

SUMMARY AND EXTENSIONS

This investigation dealt with the derivation of the cumulative distribution function (c.d.f.), $P(\alpha)$, for the percentage of coverage of a rectangular region upon another rectangular region. The c.d.f. was derived when the region of possible impact points was also considered to be rectangular. An application of $P(\alpha)$ was made to the determination of the single pass hit probabilities in the context of the Runway Cutter Program. In Chapter I, a description of the input, output, and functions of the Runway Cutter Program were described.

Chapter II was concerned with determining the probability of cutting the runway in k passes. This probability was expressed in terms of several factors influencing the probability of a cut. The single pass hit probabilities are a basic part of this function. Several examples of the probability of a cut were given for different cut criteria, one of which was solely dependent upon the single pass hit probabilities.

In Chapter III, the basic work was done for obtaining the c.d.f., $P(\alpha)$, for the percentage of pattern area on the runway. The region of possible impact points of the center of the pattern was examined for the different configurations that the pattern forms on the target. An extensive table which can be used to determine the regions generating the configurations was developed. Several examples to illustrate the use of these tables were given.

Chapter IV contains the actual development of $P(\alpha)$. Since $P(\alpha)$ is dependent upon the probability density associated with the pattern centroid, several examples were given illustrating $P(\alpha)$ for different pattern centroid probability densities.

A method for obtaining the single pass hit probabilities from $P(\alpha)$ was presented in Chapter V. This method was illustrated by using the results of the examples in Chapter IV.

A possible area of further study would be an analogous development of the material in Chapters III and IV when other assumptions are placed on the set of length and width parameters. Of particular interest would be the case where $L_r < L_d + L_p$. Since the angle of approach was considered fixed in this investigation, another area of possible study would be in the development of $P(\alpha)$ when the angle of approach is considered as a random variable.

Other areas of possible investigation related to this problem include, a study of the $P(\alpha)$'s for various length and width parameters in order to determine how these parameters affect $P(\alpha)$ and the single pass hit probabilities, a search for other methods of determining the single pass hit probabilities from $P(\alpha)$, and obtaining $P(\alpha)$ by means of numerical integration processes when the pattern centroid density is assumed to be a bivariate normal density.

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APPENDIX A

PROBABILITY DISTRIBUTION OF THE LARGEST SUB-INTERVAL

The problem to be considered is the determination of the probability distribution of the largest of $(n+1)$ intervals created by n random points in an interval not necessarily of unit length. The desired probability distribution for the case of the unit interval has been solved by various methods. See Darling (1952) and (1953), Fisher (1929), Flatto and Konheim (1962), Garwood (1940), Irwin (1955) and Mauldon (1951). Portions of the development below are analogous to Garwood's (1940) solution of the problem.

Let us consider n points in the unit interval $[0, 1]$ following the uniform distribution. These n points generate $(n+1)$ intervals. Let E be the event that the length of the largest interval denoted by I, is $\geq \alpha$. Then $\overline{E}$ is the event that the length of the largest interval, I, is $\leq \alpha$. Also, $\overline{E}$ is equivalent to the event F, that no interval has length $> \alpha$. Hence, $\overline{F}$ is the event that at least one interval is $> \alpha$. Thus, one needs to determine the probability of the occurrence of the event $\overline{F}$, $Pr(\overline{F})$.

In order to determine $Pr(\overline{F})$, let us use the following theorem according to Feller (1957).

<u>Theorem A. 1.</u> (Feller) Let $P_{1,N}$ = probability of the occurrence of at least one of the N events $A_1, A_2, \ldots, A_N$; that is

$$P_{1,N} = \Pr\{A_1 \cup A_2 \cup \ldots \cup A_N\} .$$

Then

$$P_{1,N} = S_1 - S_2 + S_3 - S_4 + \ldots \pm S_N ,$$

where

$$S_1 = \sum_i \Pr(A_i) , \quad S_2 = \sum_{i_1 < i_2} \Pr(A_{i_1} \cap A_{i_2}) ,$$

$$S_3 = \sum_{i_1 < i_2 < i_3} \Pr(A_i \cap A_{i_2} \cap A_{i_3}) , \ldots , S_N = \Pr(A_1 \cap A_2 \cap \ldots \cap A_N).$$

The proof of this theorem can be obtained by the use of mathematical induction.

Let $I_1, I_2, \ldots, I_{n+1}$ denote the $(n+1)$ intervals formed by dropping n points at random on the unit interval. Let A_1 be the event that the length of I_1 is $> \alpha$, A_2, the event that the length of I_2 is $> \alpha, \ldots, A_{n+1}$, the event that the length of interval I_{n+1} is $> \alpha$. Thus, $\Pr(\overline{F}) = \Pr(A_1 \cup A_2 \cup \ldots \cup A_{n+1})$.

Let P_k denote the probability of the simultaneous occurrence of any k of the above $(n+1)$ events. That is, $P_k = \Pr(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k})$. Let us assume that k of the events $A_1, A_2, \ldots, A_{n+1}$ have occurred. Since the resulting probability is multiplied by the factor $\binom{n+1}{k}$, the combination of $(n+1)$ things taken k at a time, one can assume that the first k of the events $A_1, A_2, \ldots, A_{n+1}$ have occurred. Thus, one is interested in determining the probability that the first k intervals each have length $> \alpha$.

Let us suppose that an amount α is cut off from each of these intervals, the shortened intervals transformed to the right until the gaps are filled, and a length of $k\alpha$ added to the beginning of the first interval. By this process, any division of the unit interval such that the first k intervals each have length $> \alpha$ corresponds with one in which all the dividing points lie to the right of $k\alpha$ and vice versa. Because of this correspondence, the probabilities of the two types of division are equal. Since the probability of the latter situation is given by $(1-k\alpha)^n$, one has that $P_k = (1-k\alpha)^n$. Consequently, $S_k = \binom{n+1}{k}(1-k\alpha)^n$.

In order to determine $\Pr(\overline{F})$ = probability of the occurrence of at least one of the events $A_1, A_2, \ldots, A_{n+1}$, Theorem A. 1 implies that

$$\Pr(\overline{F}) = S_1 - S_2 + S_3 - S_4 + \ldots \pm S_{n+1} .$$

From the above discussion, $S_1 = \binom{n+1}{1}(1-\alpha)^n$, $S_2 = \binom{n+1}{2}(1-2\alpha)^n$, etc. Thus,

$$\Pr(\overline{F}) = \binom{n+1}{1}(1-\alpha)^n - \binom{n+1}{2}(1-2\alpha)^n + \ldots + (-1)^j\binom{n+1}{j}(1-j\alpha)^n , \qquad \text{(A. 1)}$$

where $j = [1/\alpha]$, $0 < \alpha \leq 1$. [ ] denotes the greatest-integer function. That is, $\frac{1}{j+1} < \alpha \leq \frac{1}{j}$, $j = n, n-1, \ldots, 2, 1$. Therefore,

$$\begin{aligned} \Pr(I \leq \alpha) &= \Pr(\overline{E}) = 1 - \Pr(\overline{F}) \\ &= 1 - \binom{n+1}{1}(1-\alpha)^n + \binom{n+1}{2}(1-2\alpha)^n - \ldots + (-1)^j\binom{n+1}{j}(1-j\alpha)^n , \qquad \text{(A. 2)} \end{aligned}$$

where $j = [1/\alpha]$, $0 < \alpha \leq 1$.

To extend the result of (A. 2) to the uniform density on any interval, let us assume that $X_1, X_2, \ldots, X_n$ are independently and identically distributed random variables with density function given as

$$f(x\,;\mu,\theta) = \frac{1}{2\theta}\,,\ \mu-\theta \le x \le \mu+\theta\,,\ -\infty < \mu < \infty\,,\ 0 < \theta < \infty$$

$$= 0\,,\ \text{elsewhere}\,. \qquad \text{(A. 3)}$$

Consider taking a random sample of size n from this distribution and determining the probability distribution of the interval having maximum length of the $(n+1)$ intervals formed. If one follows the same method used above, and assumes that the first k intervals have length $> \alpha$, then these assumptions imply that the n points must fall in the interval $[\mu-\theta+k\alpha\,,\mu+\theta]$. The probability that the n points fall in the interval $[\mu-\theta+k\alpha\,,\mu+\theta]$ is

$$\left(\frac{2\theta - k\alpha}{2\theta}\right)^n = \left(1 - \frac{k\alpha}{2\theta}\right)^n .$$

This probability results because the probability of a single point falling in that interval is equal to the length of the interval divided by the length of the interval $[\mu-\theta\,,\mu+\theta]$. The single point probability is then raised to the n^{th} power because of the independence of $X_1, X_2, \ldots, X_n$. Therefore,

Pr (largest interval $\le \alpha$)

$$= 1 - \binom{n+1}{1}\left(1-\frac{\alpha}{2\theta}\right)^n + \binom{n+1}{1}\left(1-\frac{2\alpha}{2\theta}\right)^n + \ldots + (-1)^j\binom{n+1}{j}\left(1-\frac{j\alpha}{2\theta}\right)^n \qquad \text{(A. 4)}$$

where $j = [\frac{2\theta}{\alpha}]$, $0 < \alpha \le 2\theta$. That is $\frac{2\theta}{j+1} < \alpha \le \frac{2\theta}{j}$, $j = n,\ n-1, \ldots, 2, 1$.

APPENDIX B

A CRITERION FOR AN OPTIMUM APPROACH ANGLE

As mentioned in the text, the Runway Cutter Program considers approach angles in the range of 0 to $\pi/2$. The nature of the program output indicates that there may be certain approach angles which generate higher hit probabilities. The appendix pertains to a criterion for the determination of an optimum approach angle for specific realistic conditions on the three sets of length and the width parameters.

Let us assume the following conditions on the length and the width parameters:

$$L_r > W_r \; , \; L_d > W_d \; , \; L_p > W_p \; , \; L_r > L_d > L_p \; , \; W_d > W_p > W_r$$

$$W_d > W_p + W_r \; , \; L_r > L_p + W_r \; . \qquad \text{(B. 1)}$$

The optimum angle criterion is defined as that angle of rotation of D about its center point that minimizes the area of the region of D generating a complete miss of P on R.

For the case $\theta = 0$, the "miss area" of D forms two rectangular regions, each with dimensions given as $2L_d$ by $W_d - (W_p + W_r)$. The dimensions of the rectangular regions when $\theta = \pi/2$ are $2W_d$ by $L_d - (L_p + W_r)$.

If we consider rotating D in the clockwise direction from 0 to $\pi/2$, three different miss regions result. The first region generating the "miss area" is trapezoidal in shape. Followed by a triangular shaped "miss area," and finally another trapezoidal shaped "miss area." The angles at which the above regions change from one shape to another are given respectively as

$$\theta_1 = \tan^{-1}\left(\frac{W_d - W_p}{L_d + L_p}\right) - \sin^{-1}\left(\frac{W_r}{\sqrt{(L_d + L_p)^2 + (W_d - W_p)^2}}\right), \qquad (B.2)$$

$$\theta_2 = \tan^{-1}\left(\frac{W_d + W_p}{L_d - L_p}\right) + \sin^{-1}\left(\frac{W_r}{\sqrt{(L_d - L_p)^2 + (W_d + W_p)^2}}\right). \qquad (B.3)$$

See Figures 16 and 17.

$$\alpha_1 = \tan^{-1}\left(\frac{W_d - W_p}{L_d + L_p}\right)$$

$$\beta_1 = \sin^{-1}\left[\frac{W_r}{[(L_d+L_p)^2 + (W_d - W_p)^2]^{1/2}}\right]$$

$$\theta_1 = \alpha_1 - \beta_1$$

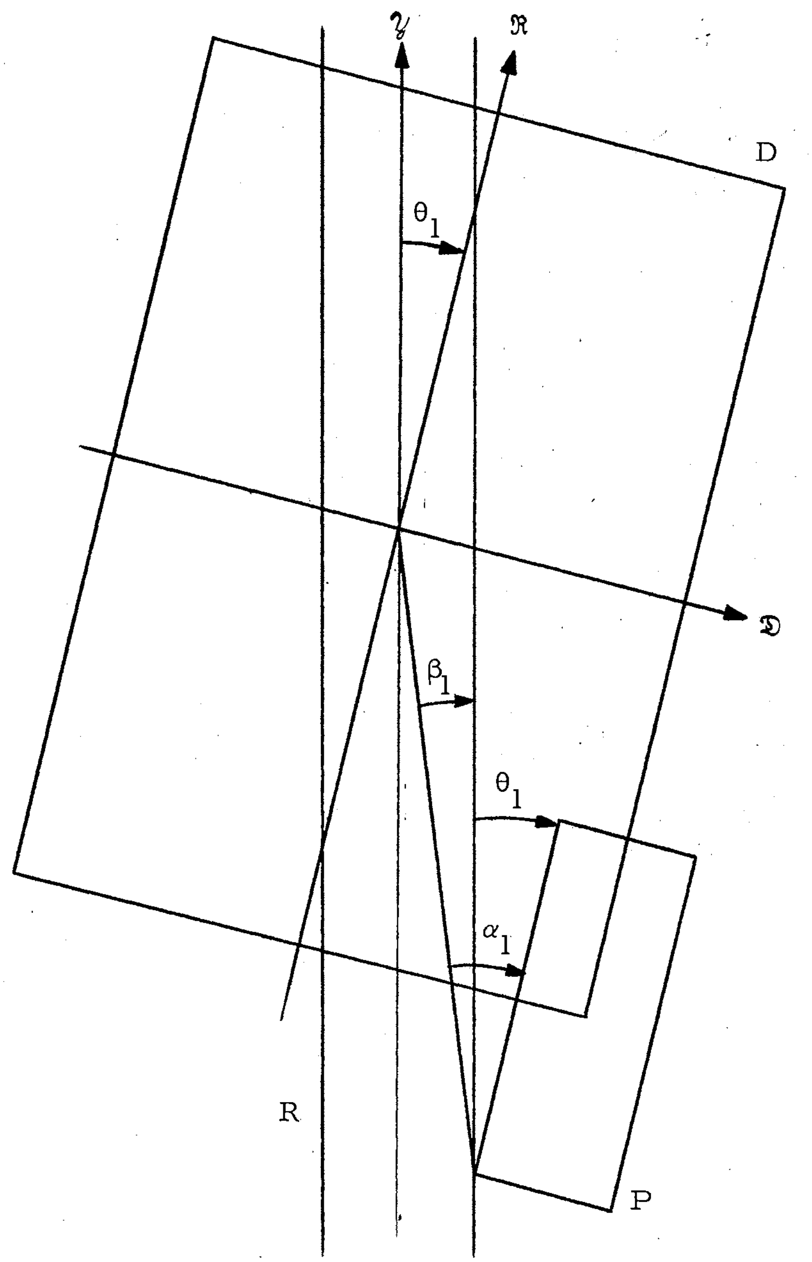
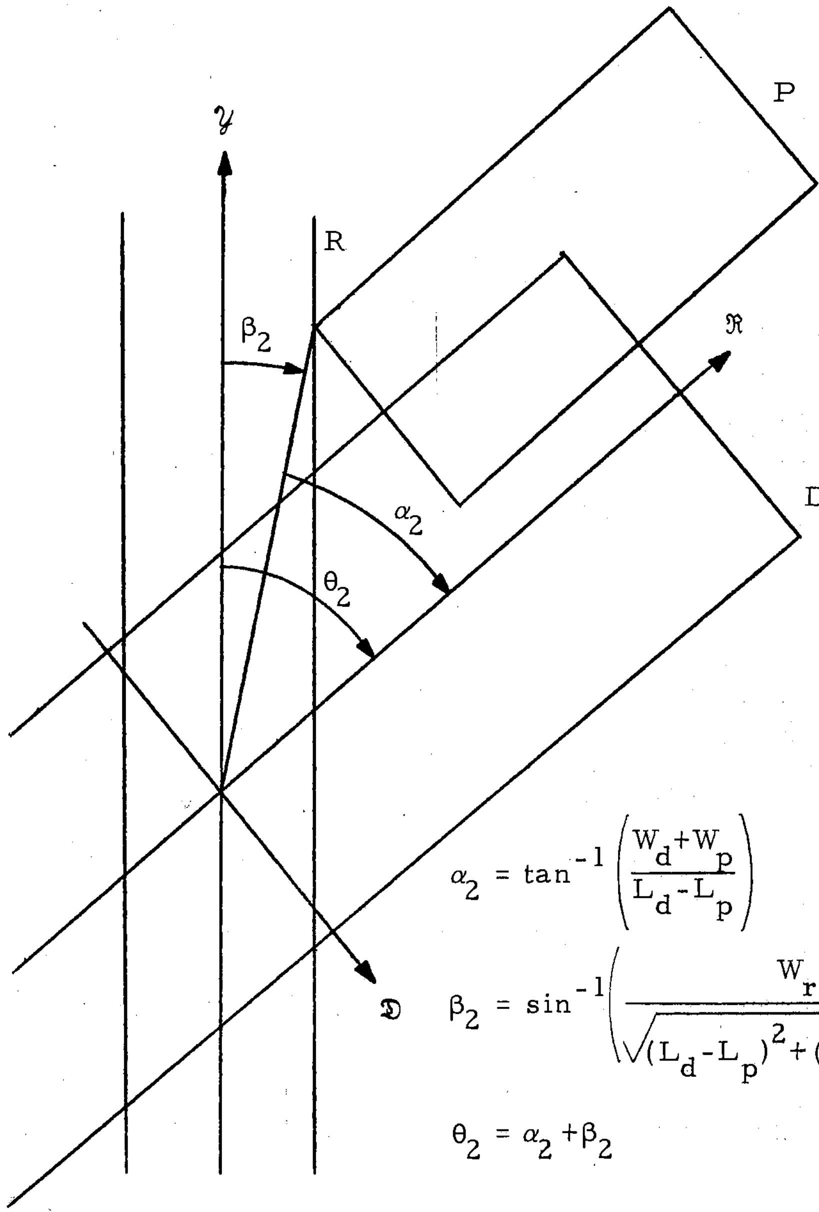


Figure 16. θ_1 - Change Angle



$$\alpha_2 = \tan^{-1}\left(\frac{W_d + W_p}{L_d - L_p}\right)$$

$$\beta_2 = \sin^{-1}\left(\frac{W_r}{\sqrt{(L_d - L_p)^2 + (W_d + W_p)^2}}\right)$$

$$\theta_2 = \alpha_2 + \beta_2$$

Figure 17. θ_2 - Change Angle

For $0 \leq \theta < \theta_1$, the trapezoidal "miss area" has an area given by $A_1(\theta) = 2L_d(W_d - W_p - W_r \sec\theta - L_p \tan\theta)$. Since both $\sec\theta$ and $\tan\theta$ are increasing functions of θ as θ increases from 0 to θ_1, $A_1(\theta)$ attains a minimum at $\theta = \theta_1$. For $\theta_2 < \theta \leq \pi/2$, the trapezoidal "miss area" has an area given by $A_2(\theta) = 2W_d(L_d - L_p - W_r \csc\theta - W_p \cot\theta)$. Because $\csc\theta$ and $\cot\theta$ are decreasing functions of θ as θ increases from θ_2 to $\pi/2$, $A_2(\theta)$ attains a minimum at θ_2. The triangular region describing the "miss area" for $\theta_1 \leq \theta \leq \theta_2$, has an area given by $A_3(\theta) = [(L_d - L_p)\sin\theta + (W_d - W_p)\cos\theta - W_r]^2/\sin(2\theta)$.

The usual methods for determining the extrema of functions can be applied to $A_3(\theta)$, $\theta_1 \leq \theta \leq \theta_2$. The first derivative of $A_3(\theta)$ can be factored into

$$\frac{2[(L_d - L_p)\sin\theta + (W_d - W_p)\cos\theta - W_r]}{\sin^2(2\theta)} \qquad (B.4)$$

times

$$2W_r\cos^2\theta + (L_d - L_p)\sin\theta - (W_d - W_p)\cos\theta - W_r . \qquad (B.5)$$

The solution, θ, obtained in equation (B.4) is inappropriate since θ is not in the interval $[\theta_1, \theta_2]$. To solve (B.5) for θ necessitates solving a quartic equation, for which it is unfeasible to write an exact solution. However, since a rearrangement of terms in $A_3'(\theta_1)$ and $A_3'(\theta_2)$ indicates that $A_3'(\theta_1) < 0$ and $A_3'(\theta_2) > 0$, we know that $A_3(\theta)$ attains a local minimum for some θ in the interval $[\theta_1, \theta_2]$.

The following procedure was used to obtain an approximation to within 1^o for the value of θ which minimizes $A_3(\theta)$ for θ in the interval $[\theta_1, \theta_2]$:

(1) Specific values were assigned to the length and width parameters

(2) For the values in (1), θ_1 and θ_2 were determined

(3) The values of $A_3(\theta)$ and equation (B. 5) were computed for values of θ between θ_1 and θ_2 in increments of 1^o.

The results of this procedure appear in Table XIV.

TABLE XIV

OPTIMUM APPROACH ANGLE FOR SPECIFIC CONDITIONS ON THE LENGTH AND WIDTH PARAMETERS*

L_d	W_d	L_p	W_p	W_r	opt. angle
2500	1000	1200	400	75	23^o
2500	1000	500	200	75	20^o
1500	600	1200	400	75	26^o
1500	600	500	200	75	19^o
2500	1000	1200	400	150	20^o
2500	1000	500	200	150	19^o
1500	600	1200	400	150	11^o
1500	600	500	200	150	15^o

*Parameter values are given in feet

APPENDIX C

DERIVATION OF $P(\alpha)$ FOR $\theta = 0^{\circ}$ AND $\theta = \pi/2$

This appendix deals with the derivation of $P(\alpha)$ for the angles of rotation $\theta = 0^{\circ}$ and $\theta = \pi/2$. In this derivation, the set of conditions on the length and width parameters for the three rectangles R, D, and P, will be made more restrictive than those in Chapter III.

Determination of $P(\alpha)$ When $\theta = 0^{\circ}$

Let us first consider $\theta = 0^{\circ}$. As in Chapter III, let us assume that $L_r > L_d + L_p$. The assumptions on the width parameters of the three rectangles will be considered in the following cases:

$$\text{(i)}\quad W_r < W_p < W_d\,, \qquad \text{and} \qquad \text{(ii)}\quad W_p < W_r < W_d\,.$$

Case (i). For this case let us consider the situations where $W_r + W_p < W_d$ and $W_r + W_p > W_d$. If $W_d > W_p + W_r$, then P will miss R when the PCP is located in the region given by the set of points

$$R_1 = \{(d, r) : -L_d \le r \le L_d\,, -W_d \le d \le -(W_p + W_r) \text{ or } W_p + W_r \le d \le W_d\}\,.$$

However, if $W_d < W_p + W_r$, then P cannot miss R, even when the PCP is located on the perimeter of D.

The P on R configuration is rectangular for these cases. If $W_d > W_p + W_r$, the minimum value for A_p is zero. Since $W_p > W_r$, the maximum value of A_p occurs when the PCP is located in the region given by

$$R_2 = \{(d, r) : -L_d \leq r \leq L_d , -(W_p - W_r) \leq d \leq W_p - W_r\} .$$

When the PCP is located in this region, the value of A_p is given by

$$A_p(\max) = \frac{(2L_p)(2W_r)}{4L_pW_p} = \frac{W_r}{W_p} . \qquad (C.1)$$

If $A_p = \alpha_0$, then the PCP is located on the lines $d = W_r + (1-2\alpha_0)W_p$ or $d = -W_r - (1-2\alpha_0)W_p$ to give a $100\alpha_0\%$ coverage. Note that A_p remains constant at W_r/W_p when the PCP is located in R_2.

The c.d.f. $P(\alpha)$ is obtained from the following integral:

$$P(\alpha_0) = 4\int_0^{L_d}\int_{W_r+(1-2\alpha_0)W_p}^{W_d} f(d, r)\,d\,d\,d\,r , \quad 0 \leq \alpha_0 \leq \alpha_{max} = \frac{W_r}{W_p} . \qquad (C.2)$$

Note that $f(d, r)$ is assumed to be symmetric with respect to the origin. Also note that $P(\alpha)$ will have a jump at α_{max}. This jump results because A_p attains its maximum when the PCP is in the region R_2 and because

$$\begin{aligned} P(\alpha_{max}) &= \Pr\,[A_p \leq \alpha_{max}] \\ &= 4\int_0^{L_d}\int_0^{W_d} f(d, r)\,d\,d\,d\,r . \end{aligned}$$

If $W_d < W_p + W_r$, then P never misses R. Consequently, the minimum value of A_p is given by

$$A_p(\text{min}) = \frac{W_p + W_r - W_d}{2W_p} .$$

The same integral as in equation (C. 2) is used to determine $P(\alpha)$ with the exception that α_0 is now restricted to $\alpha_{min} \le \alpha_0 \le \alpha_{max}$.

<u>Case (ii).</u> When $W_p < W_r < W_d$, the situation where $W_d > W_p + W_r$ corresponds to the similar situation in case (i). That is, when the PCP is located in the region R_1, P misses R. Similarly, if $W_d < W_p + W_r$, then P cannot miss R when the PCP is located in D.

The minimum value of A_p is given by zero or $(W_p+W_r-W_d)/2W_p$ depending on whether $W_d > W_p + W_r$ or $W_d < W_r + W_r$. Since $W_p < W_r$, the maximum value of A_p is one which occurs when the PCP is located in the region given by

$$R_3 = \{(d, r) : -L_d \le r \le L_d , -(W_r - W_p) \le d \le W_r - W_p\} .$$

If $A_p = \alpha_0$, then a $100\alpha_0\%$ P on R coverage is obtained when the PCP is located on the lines $d = -W_r - (1-2\alpha_0)W_p$ or $d = W_r + (1-2\alpha_0)W_p$.

When $W_d > W_p + W_r$, the c. d. f. $P(\alpha)$ is obtained from the following integral

$$P(\alpha_0) = 4\int_0^{L_d}\int_{W_r+(1-2\alpha_0)W_p}^{W_d} f(d, r)\,d d\, d r \qquad 0 \le \alpha_0 \le 1 . \qquad \text{(C. 3)}$$

If $W_d < W_p + W_r$, $P(\alpha)$ is obtained by the integral in equation (C. 3) with the exception that α is restricted to $\alpha_{min} \le \alpha \le 1$. $P(\alpha)$ will again have a jump due to A_p attaining its maximum in the region R_3 .

Determination of $P(\alpha)$ When $\theta = \pi/2$

For the angle of rotation $\theta = \pi/2$, let us make the following assumptions on the set of length and width parameters:

$$L_d > L_p > W_r \quad \text{such that} \quad L_d > L_p + W_r , \quad \text{and} \quad L_r > W_d + W_p .$$

Since $L_d > L_p + W_r$, P will miss R when the PCP is located in the region given by

$$R_4 = \{(d, r): -W_d \leq d \leq W_d , -L_d \leq r \leq -(L_p + W_r) \text{ or } L_p + W_r \leq r \leq L_d\} .$$

In this situation the maximum value of A_p is given by

$$A_p(\max) = \frac{(2W_r)(2W_p)}{4L_pW_p} = \frac{W_r}{L_p} .$$

For a $100\alpha_0\%$ coverage of P on R, the PCP must be located on the lines $r = -W_r - (1-2\alpha_0)L_p$ or $r = W_r + (1-2\alpha_0)L_p$.

The c.d.f $P(\alpha)$ is obtained by integrating the following integral:

$$P(\alpha_0) = 4\int_0^{W_d}\int_{W_r+(1-2\alpha_0)L_p}^{L_d} f(d, r)\,d r\,d d , \quad 0 \leq \alpha_0 \leq \alpha_{\max} = \frac{W_r}{L_p} . \qquad (C.4)$$

$P(\alpha)$ also has a jump when A_p attains its maximum value.

VITA[2]

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