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NECESSARY AND SUFFICIENT CONDITIONS FOR F RATIOS IN REPEATED MEASURES DESIGNS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

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degree of

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By

JORGE L. MENDOZA

Norman, Oklahoma

1974

NECESSARY AND SUFFICIENT CONDITIONS FOR F RATIOS

IN REPEATED MEASURES DESIGNS

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF PSYCHOLOGY

By

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NECESSARY AND SUFFICIENT CONDITIONS FOR F RATIOS IN REPEATED MEASURES DESIGNS

Jorge L. Mendoza

University of Oklahoma

Abstract

The relationship between circularity and type H matrices is explored. Some theorems proved by Rouanet and Lepine (1970) are generalized to cover cases where the number of subjects is less than the number of repeated measures. In addition, the necessary and sufficient condition is derived for the validity of each F ratio in a L X J X K factorial design with two repeated factors. This is shown to be

$$C' \Sigma_1 C = C' \Sigma_2 C = . . . = C' \Sigma_L C = \sigma^2 I,$$

where C' is the contrast matrix representing the comparison of interest. Tests for this condition are outlined, and some of the problems involved in testing it are discussed.

NECESSARY AND SUFFICIENT CONDITIONS FOR F RATIOS IN REPEATED MEASURES DESIGNS

Jorge L. Mendoza

University of Oklahoma

Introduction

In recent articles Rouanet and Lepine (1970) and Huynh and Feldt (1970) have discussed the necessary and sufficient conditions for repeated measures designs to have appropriate univariate models. Rouanet and Lepine (1970), working with a repeated measures design with two repeated factors (see Table 1), showed that in order for the F ratios in that design to be valid, the circularity assumption must be met. On the other hand, Huynh and Feldt (1970), working with a simple repeated measures design with one repeated factor and a groups by trial design (see Table 2), claimed that in order for these designs to have valid F ratios, the variance-covariance matrix had to have special structure, which they called a Type H matrix (definitions of circularity and Type H matrices are given later).

In brief, the method develcped by Rouanet and Lepine (1970) used concepts of linear algebra to represent sum of squares as well as the so-called planned comparisons. Instead of working with the usual sum of squares, they introduced a more general method which one could use to perform either the overall tests or planned comparison(s). The method requires the generation of a contrast matrix, say C, which represents the comparison of interest. The matrix C is an orthogonal matrix satisfying $C'C = I_p$. In addition, each row of C must sum to zero. The null hypothesis is represented as

$$Ho: \quad C'\underline{U} = 0$$

and is tested with a F test which is carried out by dividing tr(C'BC)/p by tr(C'AC)/d, where tr is the trace operator, and p and d are the degrees of freedom for tr(C'BC) and tr(C'AC), respectively. Matrix A is a function of the sample variance-covariance matrix for the repeated factor(s), and matrix B, a function of the sample means. This F test is possible since tr(C'BC) is proportional to a central chi-square under the null hypothesis and proportional to a non-central chi-square under the alternative hypothesis. Under either hypothesis tr(C'AC) is proportional to a central chi-square, and tr(C'AC) is independent of tr(C'BC).

Insert tables 1 & 2 about here

One of the major theoretical contributions of the Rouanet and Lepine (1970) article was to show, assuming a multinormal model, that tr(C'AC) is proportional to $X^2(d)$ and tr(C'BC) is proportional to $X^2(p, \delta)$ if, and only if,

$$C' \Sigma C = I \sigma^2,$$

where Σ is the population variance-covariance matrix. This is called the circularity assumption by Rouanet and Lepine (1970), and was shown to be less restrictive than the assumption of compound symmetry. The assumption of compound symmetry simply means that all pairwise correlations are equal among the levels of the repeated factor. More specifically, they showed that

$$\frac{tr(C'BC)/p}{tr(C'AC)/d}$$

is distributed as F if, and only if, $C' \Sigma C = I \sigma^2$.

In contrast to Rouanet and Lepine (1970), Huynh and Feldt (1970) used a more classical approach to show the necessary and sufficient conditions for validity of the treatments and interaction F ratios in the repeated measures designs. Working with the designs illustrated in Table 2, Huynh and Feldt demonstrated that in order for the treatments and interaction F ratios to be valid, the population variance-covariance matrix, Σ, had to have a special structure that they called a Type H matrix. When the elements of the matrix Σ can be expressed as

$$(1.1) \qquad \sigma_{ij} = \tfrac{1}{2}\sigma_{ii} + \tfrac{1}{2}\sigma_{jj} - \lambda \qquad (i \neq j)$$

for any given λ ($\lambda > 0$), then it is said that Σ is a Type H matrix. Or equivalently, Σ is said to be a Type H matrix when all possible differences, $X_i - X_j$, between levels of the repeated factor are equally variable, i.e.,

$$(1.2) \qquad Var(X_i - X_j) = 2\lambda \qquad (i \neq j).$$

Accordingly, the treatments and interaction F ratios are valid if, and only if, Σ is Type H, that is, if the elements of Σ can be expressed as in (1.1) or if (1.2) holds.

To recapitulate, Rouanet and Lepine (1970) showed that the necessary and sufficient condition for the validity of the F ratios in a J X K factorial design with two repeated factors, is $C' \Sigma C = \sigma^2 I$.

On the other hand, Huynh and Feldt (1970) demonstrated that the necessary and sufficient condition for the validity of the F ratios in the designs schematized in Table 2 is $\mathrm{Var}(X_i - X_j) = 2\lambda$, $(i \neq j)$.

The present paper has several purposes. The first is to integrate and relate the two approaches discussed above. The second is to generalize some of the theorems proved by Rouanet and Lepine (1970). Finally, the necessary and sufficient conditions for the validity of the F ratios in a L X J X K factorial design with two repeated factors are specified.

2. Circularity and Type H Matrices

As the first step in the discussion of the relationship between circularity and Type H matrices, it will be shown that the concept of overall circularity is equivalent to that of a Type H matrix. Consider Theorem 3 of Huynh and Feldt (1970). This theorem states that Σ is a Type H matrix if, and only if, $T' \Sigma T = I\xi$, $(\xi > 0)$ where T has the following properties:

$$\text{(i)} \quad T'T = I_{t-1}$$

$$\text{(ii)} \quad TT' = I_t - W/t.$$

Matrix I is the identity matrix, t is the number of repeated measures, and

$$W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ & & \cdot & \\ & & \cdot & \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

It can be shown for any repeated measures design with variance-covariance matrix, Σ, that the overall contrast matrix,

$$\underset{(t-1,\ t)}{C'_o} = \begin{bmatrix} 1/c_1 & -1/c_1 & 0 & \cdots & 0 \\ 1/c_2 & 1/c_2 & -2/c_2 & \cdots & 0 \\ & & & \cdot & \\ & & & \cdot & \\ & & & \cdot & \\ 1/c_{t-1} & 1/c_{t-1} & 1/c_{t-1} & \cdots & -(t-1)/c_{t-1} \end{bmatrix}, c_i = (\Sigma \text{ weights}^2)^{\frac{1}{2}}$$

and represents the overall comparison for t repeated measures, C_o conforms to properties (i) and (ii) as stated in Theorem 3. Consequently, the overall circularity assumption will be met only when Σ is of Type H, and vice versa. The relationship between circularity and Type H matrices becomes more elusive as we consider contrast matrices representing hypotheses other than the overall comparison. For example, the contrast matrix associated with the comparison for the A effect in the repeated measures design schematized in Table 1 can be written as

$$C'_A = \begin{bmatrix} 1/c_1\sqrt{K}\,\underline{1}' & -1/c_1\sqrt{K}\,\underline{1}' & 0\,\underline{1}' & \cdots & 0\,\underline{1}' \\ & & & \cdot & \\ & & & \cdot & \\ & & & \cdot & \\ 1/c_{J-1}\sqrt{K}\,\underline{1}' & 1/c_{J-1}\sqrt{K}\,\underline{1}' & 1/c_{J-1}\sqrt{K}\,\underline{1}' & \cdots & -(J-1)/c_{J-1}\sqrt{K}\,\underline{1}' \end{bmatrix},$$

where $\underline{1}'$ is a 1 by K vector of ones and $(c_j\sqrt{K})$ is the normalizing constant for the j th row.[1] Following the partitioning of C_A,

$$\not{Z} = \begin{bmatrix} \not{Z}_{11} & \not{Z}_{12} & \cdots & \not{Z}_{1J} \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ \not{Z}_{J1} & \not{Z}_{J2} & \cdots & \not{Z}_{JJ} \end{bmatrix} ,$$

where $\not{Z}_{ij}$ is a K by K matrix. After some algebraic manipulation,

$$C_A'\not{Z}C_A = \frac{1}{K} \left[\begin{bmatrix} 1/c_1 & -1/c_1 & 0 & \cdots & 0 \\ 1/c_2 & 1/c_2 & -2/c_2 & \cdots & 0 \\ 1/c_{J-1} & 1/c_{J-1} & 1/c_{J-1} & \cdots & -(J-1)/c_{J-1} \end{bmatrix} \cdot \begin{bmatrix} \underline{1}'\not{Z}_{11}\underline{1} & \cdots & \underline{1}'\not{Z}_{1J}\underline{1} \\ & \cdot & \\ & & \cdot \\ & & \cdot \\ \underline{1}'\not{Z}_{J1}\underline{1} & \cdots & \underline{1}'\not{Z}_{JJ}\underline{1} \end{bmatrix} \cdot \begin{bmatrix} 1/c_1 & 1/c_2 & \cdots & 1/c_{J-1} \\ -1/c_1 & 1/c_2 & \cdots & 1/c_{J-1} \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ 0 & 0 & \cdots & -(J-1)/c_{J-1} \end{bmatrix} \right]$$

Then, if we let

$$C_a' = \begin{bmatrix} 1/c_1 & -1/c_1 & 0 & \cdots & 0 \\ 1/c_2 & 1/c_2 & -2/c_2 & \cdots & 0 \\ 1/c_{J-1} & 1/c_{J-1} & 1/c_{J-1} & \cdots & -(J-1)/c_{J-1} \end{bmatrix} ,$$

it can be shown that C_a satisfies properties (i) and (ii) of Huynh & Feldt (1970) Theorem 3. Therefore, $C_A' \not{Z} C_A = I\sigma^2$ if, and only if, $\not{Z}^*$

[1]The normalizing constant is expressed in terms of K so that later on K can be factored out.

is a Type H matrix, where $\Sigma^* = 1/K\ \Sigma_o$ and

$$\Sigma_o = \begin{bmatrix} \underline{1}'\ \Sigma_{11}\ \underline{1} & \cdots & \underline{1}'\ \Sigma_{J1}\ \underline{1} \\ & \vdots & \\ \underline{1}'\ \Sigma_{J1}\ \underline{1} & \cdots & \underline{1}'\ \Sigma_{JJ}\ \underline{1} \end{bmatrix}$$

It may be noticed, however, that if $1/K\ \Sigma_o$ is of Type H, then by Theorem 3 of Huynh and Feldt (1970) there exists a matrix C, such that

$$C'\ 1/K\ \Sigma_o\ C = I\ \lambda, \text{ and}$$

$$C'\ \Sigma_o\ C = I\ \lambda\ K.$$

Hence, Σ_o is of Type H. On the other hand, if Σ_o is of Type H then

$$C'\ \Sigma_o\ C = I\ \lambda, \qquad \text{and}$$

$$C'\ 1/K\ \Sigma_o\ C = 1/K\ C'\ \Sigma_o\ C$$

$$= 1/K\ \lambda\ I.$$

These results may be summarized as:

<u>LEMMA 2.1</u>

$\Sigma^* = c\ \Sigma_o$ $(c > 0)$ is of Type H if, and only if, Σ_o is of Type H.

From Lemma 2.1, $C_A\ \Sigma\ C_A = I\sigma^2$ if, and only if, the variance-covariance matrix, Σ_o, for the A treatments collapsed over trials is of Type H. Likewise, $C_T'\ \Sigma\ C_T = I\sigma_T^2$ if, and only if, the variance-covariance matrix for the T treatments collapsed over levels of A is of Type H.

The previous paragraph suggests a way of relating circularity to Type H matrices. The point to bear in mind, however, is that it is possible for $C' \Sigma C$ to be equal to $I\sigma^2$ even though Σ is not of Type H, when C represents other than the overall comparison. On the other hand, if Σ is of Type H, $C' \Sigma C = I\sigma^2$ for any contrast matrix C. Generalizing (1.1), when Σ is of Type H, we can write

$$\Sigma = P + P' + I\lambda,$$

where

$$P = \begin{bmatrix} \tfrac{1}{2}(\sigma_{11} - \lambda) & \cdots & \tfrac{1}{2}(\sigma_{11} - \lambda) \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \tfrac{1}{2}(\sigma_{KK} - \lambda) & \cdots & \tfrac{1}{2}(\sigma_{KK} - \lambda) \end{bmatrix} .$$

Hence, if Σ is of Type H,

$$\begin{aligned} C' \Sigma C &= C'PC + C'P'C + C'\lambda IC \\ &= \lambda C'IC \\ &= I\lambda, \end{aligned}$$

since by the definition of P, it is obvious that $C'P' = 0$ and $PC = 0$ for any contrast matrix, C. To recapitulate, if Σ is of Type H, then $C' \Sigma C = I\lambda$ for any contrast matrix, C, and this implies that

$$\frac{tr(C'BC)/p}{tr(C'AC)/d}$$

will follow the F distribution under normality conditions. Note that the same is true when the overall circularity assumption is met, since if Σ is of Type H then $C_o' \Sigma C_o = I\sigma^2$ and vice versa.

Some of the relationships between circularity and Type H matrices have been shown in this section. More important, it has been demonstrated that Type H matrices are a special case of the circularity concept. Therefore, from this point on, this paper will deal only with the circularity concept.

3. Modification and Extension of Some Theorems by Rouanet & Lepine (1970)

Before proceeding any further with the presentation of this paper, let us state the following theorem:

Theorem 3.1

Let C be a t by p contrast matrix representing a comparison and satisfying

$$C'C = I_p.$$

Then, if $\underline{Y}_1, \ldots, \underline{Y}_n$ (n > 1) are independent and $\underline{Y}_i$ is distributed according to N_t ($\underline{U}$, Σ), where Σ is nonsingular:

(i) $Q/\sigma^2 = tr(C'AC)/\sigma^2$ is distributed according to $X^2(p(n-1))$,

(ii) $Q_h/\sigma^2 = \underline{K}'CC'\underline{K}/\sigma^2$ is distributed according to

$X^2(p,\ n/2\ \underline{U}'CC'\underline{U})$, and

(iii) Q and Q_h are independent, where

$$A = \sum_{i=1}^{n} \underline{Y}_i\, \underline{Y}_i' - n\underline{\bar{Y}}\underline{\bar{Y}}', \text{ and}$$

$$\underline{K} = n^{-\frac{1}{2}} \sum_{i=1}^{n} \underline{Y}_i.$$

Conditions (i) and (ii) are true if, and only if, $C' \Sigma C = \sigma^2 I$ ($\sigma^2 > 0$) and (iii) is an immediate consequence of $\underline{Y}_i \sim N_t(\underline{U}, \Sigma)$. (See Appendix A for the proof.)

The theorem is very similar to one stated by Rouanet and Lepine (1970) and Courrége and Rouanet (1972). The advantage of Theorem 3.1, however, is that its proof does not require that n be greater than t. Accordingly, this theorem, unlike that of Rouanet and Lepine, may be used to analyze a repeated measures design where the number of subjects is less than the total number of treatments. Further, it enables one to use the F distribution regardless of the relationship between number of subjects and total number of treatments.

Provided that $\underline{Y}_i \sim N_t\ (U, \Sigma)$ with a nonsingular variance-covariance matrix, Σ, and that C is a contrast matrix as defined in Theorem 3.1

$$\frac{K'CC'K/p}{tr(C'AC)/p(n-1)} \tag{3.1}$$

is distributed as F with p and p(n-1) degrees of freedom if, and only if, $C' \Sigma C = I\sigma^2$. This follows from the definition of the F distribution and from Theorem 3.1. Also, Theorem 3.1 implies that the non-centrality parameter for this ratio is $n/2\ \underline{U}'CC'\underline{U}$, where $\underline{U} = E(\underline{Y}_i)$. Consequently, if $C'\underline{U} = 0$, then $n/2\ \underline{U}'CC'\underline{U} = 0$. Thus, when $C'\underline{U} = 0$, ratio (3.1) is a central F and is noncentral otherwise. This provides a way to test Ho: $C'\underline{U} = 0$. The design schematized in Table 1 may be used to illustrate the rationale for testing $C'\underline{U} = 0$. An acceptable linear model for this design is

$$\begin{aligned} y_{ijk} &= \mu + \alpha_j + \tau_k + \pi_i + \alpha\tau_{JK} + \alpha\pi_{ji} \\ &\quad + \tau\pi_{ki} + \alpha\tau\pi_{jki} + e_{ijk}, \end{aligned} \tag{3.2}$$

where μ, α_J, and τ_K are the grand mean, treatment effect for the A factor, and repeated factor (T), respectively; π_i is a random effect associated with subject i such that $E(\pi) = 0$, and the remaining terms are interactions and error. The usual restrictions are placed on this linear model, i.e., that the sums of the components representing fixed main effects and interactions must be zero. Let the vector $\underline{Y}_i'$ contain the scores across treatments for subject i, and say that it is reasonable to assume that

$$\underline{Y}_i \sim N(\underline{U}, \Sigma),$$

where Σ is nonsingular. From (3.2) it follows that

$$(3.3)\quad E(\underline{Y}_i) = \begin{bmatrix} \mu + \alpha_1 + \tau_1 + \alpha\tau_{11} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \mu + \alpha_1 + \tau_K + \alpha\tau_{1K} \\ \mu + \alpha_2 + \tau_1 + \alpha\tau_{21} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \mu + \alpha_2 + \tau_K + \alpha\tau_{2K} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \mu + \alpha_J + \tau_1 + \alpha\tau_{J1} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \mu + \alpha_J + \tau_K + \alpha\tau_{JK} \end{bmatrix} = \underline{U}.$$

Identity (3.3), the definition of a contrast matrix, and the restrictions placed on the model enable one to see that it is possible to select matrix C such that $C'\underline{U}$ will isolate any main or interaction effect which may be of interest. Providing that $C' \Sigma C = I\sigma^2$, the F distribution can be used to test the hypothesis that $C'\underline{U} = 0$. For example, suppose that one is interested in a design like the one shown in Table 1 with 3 levels in A, 3 levels in T, and n subjects. The contrast matrix associated with the A treatment effect can be written as

$$C_A' = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 0 & 0 & 0 \\ 1/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} & -2/\sqrt{18} & -2/\sqrt{18} & -2/\sqrt{18} \end{bmatrix},$$

Since

$$(3.4) \qquad \underline{U} = \begin{bmatrix} \mu + \alpha_1 + \tau_1 + \alpha\tau_{11} \\ \mu + \alpha_1 + \tau_2 + \alpha\tau_{12} \\ \mu + \alpha_1 + \tau_3 + \alpha\tau_{13} \\ \mu + \alpha_2 + \tau_1 + \alpha\tau_{21} \\ \mu + \alpha_2 + \tau_2 + \alpha\tau_{22} \\ \mu + \alpha_2 + \tau_3 + \alpha\tau_{23} \\ \mu + \alpha_3 + \tau_1 + \alpha\tau_{31} \\ \mu + \alpha_3 + \tau_2 + \alpha\tau_{32} \\ \mu + \alpha_3 + \tau_3 + \alpha\tau_{33} \end{bmatrix},$$

and, accordingly,

$$C_A'\underline{U} = \begin{bmatrix} 3/\sqrt{6}\ (\alpha_1 - \alpha_2) \\ 3/\sqrt{18}\ (\alpha_1 + \alpha_2) - 6/\sqrt{18}\ \alpha_3 \end{bmatrix}.$$

Note that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ when $C_A'\underline{U} = 0$, since $\alpha_1 + \alpha_2 + \alpha_3 = 0$ by restriction. From (3.4) the contrast matrices associated with the T and AT effects are obtained:

$$C_T' = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{6} & 0 & 1/\sqrt{6} & -1/\sqrt{6} & 0 & 1/\sqrt{6} & -1/\sqrt{6} & 0 \\ 1/\sqrt{18} & 1/\sqrt{18} & -2/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} & -2/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} & -2/\sqrt{18} \end{bmatrix}$$

and

$$C'_{AT} = \begin{bmatrix} 1/18 & 1/18 & -2/18 & 1/18 & 1/18 & -2/18 & -2/18 & -2/18 & 4/18 \\ 1/3\sqrt{12} & -1/3\sqrt{12} & 0 & 1/3\sqrt{12} & -1/3\sqrt{12} & 0 & -2/3\sqrt{12} & 2/3\sqrt{12} & \\ 1/3\sqrt{12} & 1/3\sqrt{12} & -2/3\sqrt{12} & -1/3\sqrt{12} & -1/3\sqrt{12} & 2/3\sqrt{12} & 0 & 0 & 0 \\ 1/6 & -1/6 & 0 & -1/6 & 1/6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A design frequently used in psychology is the L X J X K factorial in which there are repeated observations on the last two factors. This design may be conceptualized as an extension of the design shown in Table 1 with L groups of subjects instead of one group. To specify the necessary and sufficient conditions for the validity of the F ratios in this design, the following theorems are in order.

Theorem 3.2

Let the following be given:

C is a matrix as defined in Theorem 3.1,

$$A_g = \sum_{i=1}^{n} \underline{Y}_{gi}\underline{Y}'_{gi} - n\,\bar{\underline{Y}}_g\bar{\underline{Y}}_g,$$

$$\underline{K}_g = n^{-\frac{1}{2}} \sum_{i=1}^{n} \underline{Y}_{gi}, \text{ and}$$

$$\underline{K}^* = L^{-\frac{1}{2}} \sum_{g=1}^{L} \underline{K}_g .$$

Then, if $\underline{Y}_{11},\ldots,\underline{Y}_{1n}\underline{Y}_{21},\ldots,\underline{Y}_{2n}, \ldots ,\underline{Y}_{L1},\ldots,\underline{Y}_{Ln}$ are independent and $\underline{Y}_{gi}$ is distributed according to $N_t\ (\underline{U}_g, \Sigma_g)$, where Σ_g is nonsingular:

(i) $\quad Q^+/\sigma^2 = \sum_{g=1}^{L} tr(C'A_gC)/\sigma^2$ is $X^2(p(N-L))$,

(ii) $\quad Q_h^+/\sigma^2 = \sum_{g=1}^{L} \underline{K}'_gCC'\underline{K}_g/\sigma^2$ is $X^2(Lp,\ n/2 \sum_{g=1}^{L} \underline{U}'_gCC'\underline{U}_g)$,

(iii) $Q_h^*/\sigma^2 = \underline{K}^{*\prime}CC'\underline{K}^*/\sigma^2$ is $X^2(p, N/2\ \underline{U}^{*\prime}CC'\underline{U}^*)$,

where $N = nL$, $\underline{U}^* = \sum_{g=1}^{L} \underline{U}_g/L$, and

(iv) Q_h^+ and Q_h^* are independent of Q^+.

Conditions (i) and (ii) are true if, and only if, $C' \Sigma_g C = I\sigma^2$ ($\sigma^2 > 0$) for all g. The necessary and sufficient conditions for (iii) to be true is that $C' \Sigma^* C = I\sigma^2$, where $\Sigma^* = 1/L \sum_{g=1}^{L} \Sigma_g$. Condition (iv) is true when $\underline{Y}_{gi} \sim N_t (U_g, \Sigma_g)$, (See Appendix B for the proof of this theorem).

Theorem 3.3

Let C be a matrix as defined in Theorem 3.1, then under the normality conditions specified in Theorem 3.2, and when $C' \Sigma_g C = \sigma^2 I$ ($\sigma^2 > 0$) for all g:

(i) $Q_h^- = Q_h^+ - Q_h^*$ is $\sigma^2 X^2(p(L-1), \sum_{g=1}^{L} (\lambda_g - \lambda^*))$. $\sum_{g=1}^{L} \lambda g$ and λ^* are the noncentrality parameters for Q_h^+/σ^2 and Q_h^+/σ^2, respectively.

(ii) Q_h^-, Q_h^+ and Q_h^* are independent. (See Appendix C for proof.)

Insert tables 3 & 4 about here

Theorem 3.2 and Theorem 3.3, the major theoretical contributions of this paper, may be used to test hypotheses frequently encountered in a L X J X K factorial design with two repeated factors. To examine this let the vector $\underline{Y}_{gi}$ be distributed according to $N_{JK}(U_g, \Sigma_g)$, where Σ_g, for $g=1,2,\ldots,L$, is nonsingular. The vector $\underline{Y}_{gi}$ contains JK ($JK \equiv t$) repeated measures for subject i in group g. Accordingly, Theorems 3.2 and 3.3 may be used to obtain the results illustrated in Table 3, which shows besides the usual ANOVA table, the necessary and sufficient conditions for the validity of each F ratio. Modifying identity (3.1) to allow for the group effect and interactions it can be shown that the F's in Table 3 are central under their respective null hypotheses. One can test numerous hypotheses with this procedure by keeping in mind the linear model and by arranging the contrast matrix such that the non-centrality parameter will be zero under the hypothesis of interest. This is possible, of course, only when the circularity assumption is met for that hypothesis. In Table 4 the flexibility of the procedure is illustrated. Table 4 shows a method for analyzing the L X J X K factorial with two repeated factors. Here, only one error term is necessary, and the validity of the F ratios lies in the assumption of overall circularity i.e., that

$$C_o' \Sigma_1 C_o = C_o' \Sigma_2 C_o = \ldots = C_o' \Sigma_L C_o = I\sigma^2, \qquad (3.5)$$

where

$$C_o' = \begin{bmatrix} 1/c_1 & -1/c_1 & 0 & \cdots & 0 \\ 1/c_2 & 1/c_2 & -2/c_2 & \cdots & 0 \\ & & & \vdots & \\ 1/c_{JK-1} & 1/c_{JK-1} & 1/c_{JK-1} & \cdots & -(JK-1)/c_{JK-1} \end{bmatrix} \text{ and } c_i = (\Sigma \text{ weights}^2)^{\frac{1}{2}}$$

Assumption (3.5) is obviously more general than any one hypothesis specified in Table 3. But if met, then

$$C' \Sigma_g C = I\sigma^2, \; g = 1,2,\ldots,L$$

for any contrast matrix, C. (See Section 2.) In other words, if assumption (3.5) is true, i.e.,

$$C_o' \Sigma_g C_o = I\sigma^2 \text{ for } g = 1,2,\ldots,L,$$

then

$$C_o' \Sigma_g C_o = C' \Sigma_g C = I\sigma^2, \; g = 1,2,\ldots,L$$

for any contrast matrix, C. The F ratios in Table 4 are, then, an immediate consequence of this fact.

The conveniences of the procedure discussed in this section are three-fold. First, this procedure enables one to generate and test a variety of hypotheses. Second, assumptions are hypothesis-specific and not general and global. Finally, one is free to select whatever linear model seems appropriate, and from it, generate a contrast matrix which could be used to test the hypothesis of interest.

Before concluding this section, perhaps it would be wise to note that the techniques discussed here apply to the repeated measures designs schematized in Table 1 & 2.

4. Some Aspects of Testing for Circularity

As mentioned before, the circularity assumption is the necessary and sufficient condition for the validity of the variance ratios discussed here. The circularity assumption assumed two basic forms for the designs examined. First, for the simple designs

$$C' \Sigma C = \sigma^2 I$$

was assumed. Second, for the more complex designs

$$C'\Sigma_1 C = C'\Sigma_2 C = . . . = C'\Sigma_L C = \sigma^2 I$$

was assumed. (C in both cases may be any contrast matrix of interest.) This section will illustrate how to test for circularity in either the two forms.

Following Huynh and Feldt (1970) and Rouanet and Lepine (1970), Mauchly's criterion W may be used to test the hypothesis that

$$Ho: \; C' \Sigma C = \sigma^2 I$$

If S is the unbiased sample variance-covariance matrix, i.e.,

$$S=A/(n-1),$$

then W is given by

$$W=|C'SC|/[tr(C'SC)/p]^p,$$

where p=r(C'), the number of rows in C'. Under the null hypothesis, normality conditions, and when $n > t$, the statistic

$$x=-n_1 d \ln W$$

has a sampling distribution which is approximated by $X^2((p^2+p)/2-1)$, and is exact when p=2. In the computation of x, $n_1=(n-1)$, and

$$d = 1 - (2p^2 + p + 2)/6pn_1.$$

For more complex designs, the circularity hypothesis

$$Ho: \; C' \Sigma_1 C = C' \Sigma_2 C = . . . = C' \Sigma_L C = \sigma^2 I$$

is tested in two stages. In the first stage the hypothesis

$$H_1: \; C' \Sigma_1 C = C' \Sigma_2 C = . . . = C' \Sigma_L C = \Sigma$$

is tested. Then if H_1 is not rejected, the hypothesis

$$H_2\colon\ C' \not\Sigma\, C = \sigma^2 I$$

is tested. If both hypotheses, i.e., H_1 and H_2, are not rejected, the Ho is not rejected. If either H_1 or H_2 is rejected, then Ho is rejected. Box's test (1950) may be used to test H_1 by computing

$$M=(N-L)\ln|C'S_pC| - (n-1)\sum_{g=1}^{L} \ln|C'S_gC|,$$

where S_g is the unbiased variance-covariance matrix corresponding to group g, and

$$S_p = \sum_{g=1}^{L} A_g/(N-L),$$

is the pooled variance-covariance matrix. Under H_1, normality conditions, and when $n > t$, the statistic

$$x_1 = M/b$$

is approximately distributed as F and f_1 and f_2 degrees of freedom. For the F approximation f_1, f_2 and b are calculated as follows;

$$f_1 = .5[(L-1)t(t+1)],$$

$$f_2 = f_1 + 2/(\theta_2 - \theta_1^2),$$

$$b = f_1/(1 - \theta_1 - f_1/f_2),$$

where

$$\theta_1 = \frac{(6t^2 + 3t - 1)(L-1)}{6(t + 1)Ln_1}$$

and

$$\theta_2 = (t-1)(t+2)(L^2+L+1)/6L^2n_1^2.$$

After computing x_1 we refer to the F table, with f_1 degrees of freedom in the numerator and f_2 degrees of freedom in the denominator, to see whether or not x_1 is significant. The F approximation is given rather than the chi-square approximation since the chi-square approximation appears to be good only when $t \leq 5$ and $L \leq 5$. (See Box 1949.) If the hypothesis H_1 is not rejected, the hypothesis H_2: $C' \Sigma C = \sigma^2 I$ may be tested by forming $x_2 = -n_2 d \ln W$ where, in this case, $W = |C'S_pC| / [tr(C'S_pC)/p]^p$, and $n_2 = N-L$.

There are two problems with tests of circularity. The first is that when one is working in a situation where $n < t$, the tests suggested here are not applicable. The second and most important problem is that very little is known about these tests. Specifically, it is not known whether these tests are robust to violations of assumptions, and there are problems in determining their power and probability of Type I error. Therefore one should exercise caution when making inferences from these tests.

5. Discussion

In section 3 it was shown that it is possible to extend the concepts involved in the circularity assumption to complex repeated measures designs. The nature of the assumptions was shown to have an immediate effect on the layout of the ANOVA table. In section 4 procedures were given on how to test for these assumptions. Taking all of this into consideration it is suggested that the overall circularity assumption should ideally be tested first for several reasons. On the basis of the outcome of the test, it is important to consider whether or not the ANOVA table displayed in Table 4 is appropriate. In order to be able to use this ANOVA table the overall circularity assumption must be met. If it is met, the F tests shown in Table 4 are definitely more powerful than those shown in Table 3. Subsequently, if the overall circularity assumption is not met one may individually test the circularity assumptions necessary for the F tests depicted in Table 3. On the basis of the results one should decide whether or not Table 3 is suitable.

In the event that one cannot assume any type of circularity, one may use a multivariate technique to analyze the data since the assumption of equality of variance-covariance matrices, a necessary assumption for multivariate techniques, does not imply circularity and vice versa. Hitherto, this line of reasoning has not been possible because the first step in testing for compound symmetry has been to test the hypothesis that Ho: $\Sigma_1 = \Sigma_2 = \ldots = \Sigma_L$.

Before concluding this part of our discussion, perhaps it would be wise to consider again some of the issues which were discussed in regard to tests of the circularity assumption. If it is recalled that very little is known about circularity tests, the discussion of which technique to use emerges as one which is somewhat complicated, practically speaking. Specifically, if it is suspected that one or more of the assumptions needed for the circularity test are not being met, perhaps, one should use a technique like the one described by Geisser and Greenhouse (1958) rather than testing for circularity in order to decide the technique to use.

One should also note that the relatively simple formation of F ratios is a convenient aspect of the procedure developed in section 3. To illustrate, when the circularity assumption is met, the tr(C'AC) is proportional to a central chi-square and $\underline{K}'CC'\underline{K}$ is proportional to a non-central chi-square in accordance with Theorem 3.1. Furthermore, tr(C'AC) is independent of $\underline{K}'CC'\underline{K}$. Therefore,

$$\frac{\underline{K}'CC'\underline{K}}{tr(C'AC)}$$

must be proportional to F. Note that we did not need the expected value of tr(C'AC), nor the expected value of $\underline{K}'CC'\underline{K}$ to form the F ratio. In general, all that is required is that the non-centrality parameter for the numerator whould be equal to zero under the null. This is true since by theorems 3.1, 3.2, and 3.3, when the circularity assumption is met, the numerator follows a non-central chi-square while the denominator follows one which is central.

6. Summary

A different approach to the analysis of repeated measures designs was demonstrated, and was shown to more flexible. In addition, the necessary and sufficient condition was derived for the validity of the F ratios in a L X J X K factorial design with two repeated factors. It was shown that the circularity assumption

$$C' \Sigma_1 C = C' \Sigma_2 C = \dots = C' \Sigma_L C = \sigma^2 I$$

is the necessary and sufficient condition for the validity of the F ratios. Also, it was shown that Type H matrices are a special case of the circularity assumption. Procedures for testing for circularity in this design were outlined. It was pointed out that these circularity tests are not applicable when the number of subjects per group is less than that of the total number of repeated measures and their use is questionable given lack of knowledge about their robustness.

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TABLE 1

Schematic for the Repeated Measures Design with Two Repeated Factors

	A_1				A_2				. . .	A_J				
	T_1	T_2	...	T_K	T_1	T_2	...	T_K	. . .	T_1	T_2	...	T_K	
S_1	(Y_{111}	Y_{121}	...	Y_{1K1}	Y_{112}	Y_{122}	...	Y_{1K2}	. . .	Y_{11J}	Y_{12J}	...	Y_{LKJ})	$= Y'_1$
.		.				.					.			.
.		.				.					.			.
.		.				.					.			.
S_n	(Y_{n11}	Y_{n21}	...	Y_{nK1}	Y_{n12}	Y_{n22}	...	Y_{nK2}	. . .	Y_{n1J}	Y_{n2J}	...	Y_{nKJ})	$= Y'_n$

n = Number of Subjects

K = Number of Levels in T

J = Number of Levels in A

TABLE 2

Schematic for the Simple Repeated Measures and Groups by Trials Designs

	T_1	T_2	. . .	T_K	
S_1	$(Y_{11}$	Y_{12}	. . .	$Y_{1K})$	$= \underline{Y}'_1$
.			.		.
.			.		.
.			.		.
S_n	$(Y_{n1}$	Y_{n2}	. . .	$Y_{nK})$	$= \underline{Y}'_n$

Simple Repeated Measures Design

			T_1	T_2	. . .	T_K	
1		S_1	$(Y_{111}$	Y_{112}	. . .	$Y_{11K})$	$= \underline{Y}'_{11}$
2		.			.		.
	G_1	.			.		.
		.			.		.
		S_n	$(Y_{n11}$	Y_{n12}	. . .	$Y_{n1K})$	$= \underline{Y}'_{1n}$
		S_1	$(Y_{n121}$	Y_{122}	. . .	$Y_{12K})$	$= \underline{Y}'_{21}$
		.			.		.
.	G_2	.			.		.
.		.			.		.
.		S_n	$(Y_{n21}$	Y_{n22}	. . .	$Y_{n2K})$	$= \underline{Y}'_{2n}$
		.			.		.
		.			.		.
		.			.		.
		S_1	$(Y_{1L1}$	Y_{1L2}	. . .	$Y_{1LK})$	$= \underline{Y}'_{L1}$
		.			.		.
	G_L	.			.		.
		.			.		.
N		S_n	$(Y_{nL1}$	Y_{nL2}	. . .	$Y_{nLK})$	$= \underline{Y}'_{Ln}$

Groups by Trial Repeated Measures Design
L = Number of Groups
N = Total Number of Subjects

TABLE 3

Partial Analysis of Variance Table for the L X J X K Factorial with Two Repeated Factors (A & T) and One Non-Repeated Factor (G).

Source of Variance	SS	df	F	Necessary and Sufficient Condition for the Validity of F
A	$Q_A = \underline{K}^{*\prime} C_A C_A' \underline{K}^*$	$r(C_A C_A') \equiv p$	$\frac{Q_A/p}{Q_1/p(N-L)}$	$C_A \Sigma_g C_A = I\sigma_A^2$ for g=1,...,L
GA	$Q_{GA} = \sum_{g=1}^{L} \underline{K}_g' C_A C_A' \underline{K}_g - Q_A$	Lp − p	$\frac{Q_{GA}/p(L-1)}{Q_1/p(N-L)}$	$C_A' \Sigma_g C_A = I\sigma_A^2$ for g=1,...,L
AS/G	$Q_1 = \sum_{g=1}^{L} tr(C_A' A_g C_A)$	p(N-L)		
T	$Q_T = \underline{K}^{*\prime} C_T C_T' \underline{K}^*$	$r(C_T C_T') \equiv q$	$\frac{Q_T/q}{Q_2/p(N-L)}$	$C_T' \Sigma_g C_T = I\sigma_T^2$ for g=1,...,L
TG	$Q_{TG} = \sum_{g=1}^{L} \underline{K}_g' C_T C_T' \underline{K}_g - Q_T$	Lq-q	$\frac{Q_{GT}/q(L-1)}{Q_2/q(N-L)}$	$C_T' \Sigma_g C_T = I\sigma_T^2$ for g=1,...,L
TS/G	$Q_2 = \sum_{g=1}^{L} tr(C_T' A_g C_T)$	q(N-L)		
AT	$Q_{AT} = \underline{K}^{*\prime} C_{AT} C_{AT}' \underline{K}^*$	$r(C_{AT} C_{AT}') \equiv W$	$\frac{Q_{AT}/W}{Q_3/W(N-L)}$	$C_{AT}' \Sigma C_{AT} = I\sigma_{AT}^2$ for g=1,...,L
ATG	$Q_{ATG} = \sum_{g=1}^{L} \underline{K}_g' C_{AT} C_{AT}' \underline{K}_g - Q_{AT}$	LW-W	$\frac{Q_{ATG}/W(L-1)}{Q_3/W(N-L)}$	$C_{AT}' \Sigma C_{AT} = I\sigma_{AT}^2$ for g=1,...,L
TAS/G	$Q_3 = \sum_{g=1}^{L} tr(C_{AT}' A_g C_{AT})$	W(N-L)		

TABLE 3 (continued)

K = Number of Levels in T

J = Number of Levels in A

L = Number of Levels in G

n = Number of SS per group

N = Total Number of Subjects

Y'_{gi} = Vector of Scores for Subject i in Group g

$K'_g = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y'_{gi}$, $K^* = \frac{1}{\sqrt{L}} \sum_{g=1}^{L} K_g$ and $A_g = \sum_{i=1}^{L} Y_{gi} Y'_{gi} - n \bar{Y}_g \bar{Y}'_g$

r (CC') = Number of Rows in C'

TABLE 4

Partial Analysis of Variance Table for the LxJxK Factorial with Two Repeated Factors, When the Overall Circularity Assumption Holds, i.e., $C_o' \Sigma_g C_o = I\sigma^2$ for g=1,2,...,L.

Source of Variance	SS	df	F
A	*	*	$\frac{Q_A/p}{Q_e/(JK-1)(N-L)}$
GA	*	*	$\frac{Q_{GA}/p(L-1)}{Q_e/(JK-1)(N-L)}$
T	*	*	$\frac{Q_T/q}{Q_e/(JK-1)(N-L)}$
TG	*	*	$\frac{Q_{TG}/q(L-1)}{Q_e/(JK-1)(N-L)}$
AT	*	*	$\frac{Q_{AT}/t}{Q_e/(JK-1)(N-L)}$
ATG	*	*	$\frac{Q_{ATG}/t(L-1)}{Q_e/(JK-1)(N-L)}$
Error	$Q_e = \sum_{g=1}^{L} tr(C_o' A_g C_o)$	(JK-1)(N-L)	

* = Same as in Table 3.

C_o = The Overall Contrast Matrix.

APPENDIX A

PROOF OF THEOREM 3.1

APPENDIX A

Proof of Theorem 3.1

According to Anderson (1958, p. 53) there exists an n by n orthogonal matrix $T = \{t_{\alpha i}\}$, with its last row equal to $(1/\sqrt{n}, \ldots, 1/\sqrt{n})$, such that

(1) $$A = \sum_{\alpha=1}^{n} \underline{Z}_\alpha \underline{Z}'_\alpha - \underline{Z}_n \underline{Z}'_n = \sum_{\alpha=1}^{n-1} \underline{Z}_\alpha \underline{Z}'_\alpha ,$$

where $\underline{Z}_\alpha = \sum_{i=1}^{n} t_{\alpha i} \underline{Y}_i$. Also, $\underline{Z}_1, \underline{Z}_2, \ldots, \underline{Z}_n$ are independent and normally distributed with

(2) $$E(\underline{Z}_n) = \sqrt{n}\underline{U}$$
$$E(\underline{Z}_\alpha) = 0 \text{ for } \alpha = 1, \ldots, n-1$$

and variance-covariance matrix Σ.

Statement (iii) is easily proved by noticing that

$$\underline{K} = \underline{Z}_n .$$

Since $\underline{Z}_n$ is independent of $\underline{Z}_1, \ldots, \underline{Z}_{n-1}$, Q is independent of Q_h. We proceed to prove (i) by rewriting Q as

$$Q = tr(C'AC) = tr\left(C' \sum_{\alpha=1}^{n-1} \underline{Z}_\alpha \underline{Z}'_\alpha C\right)$$

$$= tr\left(CC' \sum_{\alpha=1}^{n-1} \underline{Z}_\alpha \underline{Z}'_\alpha\right)$$

$$= \mathrm{tr}\left(\sum_{\alpha=1}^{n-1} CC'\underline{Z}_\alpha \underline{Z}_\alpha'\right)$$

$$= \sum_{\alpha=1}^{n-1} \mathrm{tr}(CC'\underline{Z}_\alpha \underline{Z}_\alpha')$$

$$= \sum_{\alpha=1}^{n-1} \mathrm{tr}(\underline{Z}_\alpha' CC'\underline{Z}_\alpha)$$

$$= \sum_{\alpha=1}^{n-1} \underline{Z}_\alpha' CC'\underline{Z}_\alpha .$$

It follows from Theorem 2 in Searle (1971, p. 57) that $\underline{Z}_\alpha' CC'\underline{Z}_\alpha =$ $\sigma^2 X^2(p)$ if, and only if, $(\sigma^{-2}CC' \not{Z})^2 = \sigma^{-2}CC' \not{Z}$. Thus, since the $\underline{Z}$'s are independent, $Q/\sigma^2 = X^2(p(n-1))$ when $(\sigma^{-2}CC' \not{Z})^2 = \sigma^{-2} CC' \not{Z}$.
Note that if $C' \not{Z} C = \sigma^2 I$ then

$$
\begin{aligned}
(\sigma^{-2}CC' \not{Z})^2 &= \sigma^{-4} CC' \not{Z} CC' \not{Z} \\
&= \sigma^{-4}CI\sigma^2 C' \not{Z} \qquad (4)\\
&= \sigma^{-2}CC' \not{Z}.
\end{aligned}
$$

This shows that if $C' \not{Z} C = \sigma^2 I$ then $Q/\sigma^2 = X^2 (p(n-1))$. Necessity is proven by letting

$$\underline{Z}^{*\prime} = (\underline{Z}_1', \underline{Z}_2', \ldots, \underline{Z}_{n-1}')$$

and rewriting $\sum_{\alpha=1}^{n-1} \underline{Z}_\alpha \underline{Z}_\alpha'$ as

$$\sum_{\alpha=1}^{n-1} \underline{Z}_\alpha \underline{Z}_\alpha' = \underline{Z}^{*\prime} V \underline{Z}^{*},$$

where

$$V = \frac{1}{\sigma^2} \begin{bmatrix} CC' & & \cdots & 0 & 0 \\ & CC' & & & 0 \\ \cdot & & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & & & \cdot & \\ 0 & 0 & \cdots & & CC' \end{bmatrix} .$$

It is common knowledge that $\underline{Z}^*$ is distributed according to $N(0,\Lambda)$, where

$$\Lambda = \begin{bmatrix} \Sigma & \cdots & 0 & 0 \\ \cdot & \Sigma & & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & & \cdot & \cdot \\ 0 & 0 & \cdots & \Sigma \end{bmatrix} .$$

Consequently, if $\underline{Z}^{*\prime} V \underline{Z}^* = X^2(p(n-1))$, $V\Lambda$ must be idempotent i.e.,

$$(V\Lambda)^2 = V\Lambda.$$

But in order for $V\Lambda$ to be idempotent, $\sigma^{-2}CC'\Sigma$ must be idempotent, since

$$(V\Lambda)^2 = \begin{bmatrix} (\sigma^{-2}CC'\Sigma)^2 & & \cdots & 0 & 0 \\ \cdot & (\sigma^{-2}CC'\Sigma)^2 & & & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & & & \cdot & \cdot \\ 0 & 0 & \cdots & & (\sigma^{-2}CC'\Sigma)^2 \end{bmatrix} .$$

The final step in the proof is to show that $\sigma^{-2}CC'\Sigma$ is idempotent if, and only if, $C' \Sigma C = \sigma^2 I$. Consider the statement

$$\sigma^{-2}CC'\Sigma = (\sigma^{-2}CC'\Sigma)^2$$

$$(5) \qquad = \sigma^{-4}CMC'\Sigma,$$

where $M = C' \Sigma C$. Given that Σ is positive definite, equation (5) implies that

$$(6) \qquad \sigma^2 CC' = CMC'.$$

The, if we premultiply and postmultiply both sides of equation (6) by C' and C, respectively, we find that

$$(7) \qquad M = \sigma^2 I.$$

In light of the results obtained in equations (4) and (7), $(\sigma^{-2}CC'\Sigma)^2 = \sigma^{-2}CC'\Sigma$ if, and only if, $C' \Sigma C = \sigma^2 I$, which concludes the proof of statement (i).

The random vector K is $N_t(\sqrt{n}\underline{U}, \Sigma)$. From Searle's Theorem 2, $\underline{K}'CC'\underline{K}/\sigma^2$ is $X^2(p, n/2\ \underline{U}'CC'\underline{U})$ if, and only if, $(\sigma^{-2}CC'\Sigma)^2 = \sigma^{-2}CC'\Sigma$. By the proof of (i) it is known that $(\sigma^{-2}CC'\Sigma)^2 = \sigma^{-2}CC'\Sigma$ if, and only if, $C' \Sigma C = \sigma^2 I$. Therefore, $K'CC'K/\sigma^2$ is $X^2(p, n/2\ \underline{U}'CC'\underline{U})$ if, and only if, $C' \Sigma C = \sigma^2 I$.

APPENDIX B

PROOF OF THEOREM 3.2

APPENDIX B

Proof of Theorem 3.2

From the proof of Theorem 3.1 it is evident that

$$(1) \qquad Q^{+} = \sum_{g=1}^{L} \left(\sum_{i=1}^{n-1} \underline{Z}'_{gi} CC' \underline{Z}_{gi} \right),$$

and that $\sum_{i=1}^{n-1} \underline{Z}'_{gi} CC' \underline{Z}_{gi} = \sigma_g^2 \chi_g^2 (p(n-1))$ if, and only if, $C' \Sigma_g C = \sigma_g^2 I$ $(\sigma_g^2 > 0)$. Hence, when $C' \Sigma_g C = \sigma_g^2 I$ for all g, identity (1) becomes

$$Q^{+} = \sum_{g=1}^{L} \sigma_g^2 \chi_g^2 (p(n-1)).$$

Furthermore, if $C' \Sigma_g C = \sigma^2 I$ for all g,

$$Q^{+} = \sum_{g=1}^{L} \sigma^2 \chi_g^2 (p(n-1)),$$

and

$$(2) \qquad Q^{+}/\sigma^2 = \chi^2 (p(N-L)),$$

where N=nL. Necessity is proven by assuming that

$$(3) \qquad Q^{+}/\sigma^2 = \chi^2 (p(N-L)).$$

Equation (3) implies that $V^{+}\Lambda^{+}$ is idempotent, where

$$V^{+}\Lambda^{+} = \begin{bmatrix} \sigma^{-2}CC'\Sigma_1 & & & \cdots & & 0 & 0 \\ \cdot & \ddots & \sigma^{-2}CC'\Sigma_1 & & & & 0 \\ \cdot & & & \ddots & & & \cdot \\ \cdot & & & & & & \cdot \\ & & & & \sigma^{-2}CC'\Sigma_L & & \cdot \\ 0 & & & & & \ddots & \\ 0 & 0 & \cdots & & & & \sigma^{-2}CC'\Sigma_L \end{bmatrix} .$$

Thus, from the proof of Theorem 3.1, $Q^{+}/\sigma^2 = X^2(p(N-L))$ if, and only if, $C' \Sigma_g C = \sigma^2 I$, for all g.

Following the same logic as above, it can be shown that part (ii) of the theorem is true. Since the K's are independent and distributed according to $N_t(\sqrt{n}\, \underline{U}_g, \Sigma_g)$, K* is $N_t(\sqrt{n/L} \sum_{g=1}^{L} \underline{U}_g, 1/L \sum_{g=1}^{L} \Sigma_g)$.

Accordingly, $\underline{K}^{*\prime}CC'\underline{K}^*/\sigma^2$ is $X^2(p, N/2\ \underline{U}^*CC'\underline{U}^*)$ if, and only if, $C'\Sigma^*C = \sigma^2 I$, where $U^* = 1/L \sum_{g=1}^{L} \underline{U}_g$, and $\Sigma^* = 1/L \sum_{g=1}^{L} \Sigma_g$. Notice that when $C'\Sigma_g C = \sigma^2 I$ for all g, $C'\Sigma^*C = \sigma^2 I$. Finally, part (iv) follows from condition (iii) of Theorem 3.1.

APPENDIX C

PROOF OF THEOREM 3.3

APPENDIX C

Proof of Theorem 3.3

In order to prove this Theorem we first have to show that $Q_h^- \geq 0$. Consider the linear transformation $\underline{Y}_g = P'\underline{K}_g$ where P is an orthogonal matrix (P'P = PP' = I) such that P'CC'P is a diagonal matrix D. P exists since CC' is real and symmetric. The random vector $\underline{Y}_g$ is distributed according to $N_t\ (P\underline{U}_g,\ P'\ \not{\Sigma}_g\ P)$ and

$$Q^*_h = \underline{K}^{*\prime}CC'\underline{K}^* = 1/L\ (\sum_{g=1}^{L} \underline{K}'_g)CC'\ (\sum_{g=1}^{L} \underline{K}_g)$$

$$= 1/L\ (\sum_{g=1}^{L} \underline{Y}'_g P')CC'\ (\sum_{g=1}^{L} P\underline{Y}_g)$$

(1) $$= 1/L\ (\sum_{g=1}^{L} \underline{Y}'_g)\ D\ (\sum_{g=1}^{L} \underline{Y}_g)$$

$$= 1/L \sum_{\alpha=1}^{t} d_\alpha\ (\sum_{g=1}^{L} y_{g\alpha})^2.$$

$D^2 = P'CC'PP'CC'P = D$, consequently, the elements of D must be zeroes and ones. As $r(D) = r(P'CC'P) = r(CC') = p$ there are p ones and t-p zeroes. Hence,

(2) $$Q^* = 1/L \sum_{\alpha=1}^{P} (\sum_{g=1}^{L} y_{g\alpha})^2.$$

Similarly,

$$Q_h^+ = \sum_{g=1}^{L} \underline{K}_g' CC' \underline{K}_g$$

(3)
$$= \sum_{g=1}^{L} \underline{Y}_g' D \underline{Y}_g$$

$$= \sum_{\alpha=1}^{p} \sum_{g=1}^{L} y_{g\alpha}^2 .$$

It follows from equations (3) and (2) that

$$Q_h^- = \sum_{\alpha=1}^{p} \sum_{g=1}^{L} y_{g\alpha}^2 - 1/L \sum_{\alpha=1}^{p} (\sum_{g=1}^{L} y_{g\alpha})^2$$

(4)
$$= \sum_{\alpha=1}^{p} \sum_{g=1}^{L} (y_{g\alpha} - y_{.\alpha})^2 .$$

Equation (4) shows that $Q_h^- \geq 0$. Next, we must express Q_h^-, Q_h^* and Q_h^+ as quadratic forms of the same random vector, say $\underline{Z}$. Let $\underline{Z}' = (y_{11}, \ldots, y_{1L}, y_{21}, \ldots, y_{2L}, \ldots, y_{p1}, \ldots, y_{pL})$ and W be a Lp by Lp matrix such as

$$W = \begin{bmatrix} J & . & . & . & 0 & 0 \\ . & J & & & & 0 \\ . & & . & & & . \\ . & & & . & & . \\ 0 & & & & . & . \\ 0 & 0 & . & . & . & J \end{bmatrix} ,$$

where J is a L by L matrix written

$$J = \frac{1}{\sqrt{L}} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ 1 & 1 & \dots & 1 \end{bmatrix} .$$

Then, we can express:

$$Q^*_h = \underline{Z}'W\,\underline{Z},$$

$$Q^+_h = \underline{Z}'\underline{Z}, \text{ and}$$

$$Q^-_h = \underline{Z}'\,(I - W)\underline{Z}.$$

Thus, by a Theorem in Hogg and Craig (1958) and by Theorem 3.2, Q^-_h is distributed according to $X^2(p(L-1),\ n/2 \sum_{g=1} \underline{U}'_g CC'\underline{U}_g - N/2\ \underline{U}^*CC'\underline{U}^*)$, and Q^-_h, Q^+_h, and Q^*_h are independent, when $C' \not{\Sigma}_g\ C = \sigma^2$ for all g, where $U^* = \frac{1}{L} \sum_{g=1}^{L} \underline{U}_g$.