 69-8603

FRAWLEY, William James, 1937-
LOCALLY DISCONJUGATE FAMILIES OF CONTINUOUS FUNCTIONS.

The University of Oklahoma, Ph.D., 1968
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

LOCALLY DISCONJUGATE FAMILIES
OF CONTINUOUS FUNCTIONS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

WILLIAM JAMES FRAWLEY

Norman, Oklahoma

1968

LOCALLY DISCONJUGATE FAMILIES

OF CONTINUOUS FUNCTIONS

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

BY

W T Reid

John C. Brixey

George M. Ewing

Gene Levy

T. K. Pan

ACKNOWLEDGEMENT

The author gratefully acknowledges the guidance and inspiration of his major professor, Dr. William T. Reid, during the preparation of this paper.

The research for this paper was done while the author was a National Aeronautics and Space Administration Trainee at The University of Oklahoma.

TABLE OF CONTENTS

LOCALLY DISCONJUGATE FAMILIES OF CONTINUOUS FUNCTIONS

1. Introduction. In the study of the linear homogeneous second-order differential equation

$$(1.1) \qquad L(u) = (r\cdot u'+q\cdot u)'-(q\cdot u'+p\cdot u) = 0$$

on the open interval I of the real line R where p, q, and r are continuous real-valued functions with r positive on I, it is found that given any triple (t_o,u_o,u_o') in $I\times R^2$ there is a unique solution u of (1.1) on I such that $u(t_o) = u_o$ and $u'(t_o) = u_o'$, so that for some neighborhood I_o of any point t_o in I there is a solution u_1 of (1.1) which is never zero in I_o. Moreover, any solution of (1.1) on I_o is a linear combination of u_1 and a solution u_2 given by

$$(1.2) \qquad u_2(t) = \left(\int_{t_o}^{t} \frac{ds}{r(s)\cdot u_1^2(s)}\right)\cdot u_1(t)$$

for t in I_o. Then, given a pair of points (t_1,x_1) and (t_2,x_2) in $I_o\times R$ with $t_1\neq t_2$, the determinant $u_1(t_1)\cdot u_2(t_2) - u_1(t_2)\cdot u_2(t_1)$ is different from zero and there is a unique solution u of (1.1) which satisfies the conditions

$$(1.3) \qquad u(t_i) = x_i, \quad i = 1,2;$$

namely the function $u = A \cdot u_1 + B \cdot u_2$ where A and B are given by

$$(1.4) \qquad \begin{pmatrix} A \\ B \end{pmatrix} = \frac{\begin{pmatrix} u_2(t_2) & -u_2(t_1) \\ -u_1(t_2) & u_1(t_1) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}{u_1(t_1) \cdot u_2(t_2) - u_1(t_2) \cdot u_2(t_1)}$$

If solutions to the differential equation (1.1) connect pairs of points with distinct abscissae in the strip $I_o \times R$ uniquely, (1.1) is said to be disconjugate on I_o. We prefer to say that the set of solutions, $\mathcal{S}$, of (1.1) is disconjugate on I_o and, in view of the remarks above, that $\mathcal{S}$ _is a locally disconjugate family_. It is our purpose to study locally disconjugate families of continuous functions with no assumed differentiability properties.

In Section 2 we formalize the notion of local disconjugacy for subfamilies of C(I,R) and characterize locally disconjugate families as a certain class of homeomorphs of R^2 in C(I,R). The concept of consecutive conjugate points relative to the differential equation (1.1) is generalized to apply to these families and a sufficient condition for continuous functional relationships to exist between a point and its nearest left- and right-conjugates is given. Finally it is shown that the continuous functions obtained by putting together restrictions of members of a locally disconjugate family $\mathfrak{F}$, called piecewise-$\mathfrak{F}$-linear functions, are dense in the real-valued continuous functions on intervals of disconjugacy of $\mathfrak{F}$.

A function f is <u>concave</u> relative to the differential equation $L(u) = 0$ on an interval of disconjugacy, sometimes said "f is super-L", if it dominates those segments of solutions of $L(u) = 0$ whose endpoints lie below the graph of f. (See references [1], [3], [12]). Section 3 concerns $\mathfrak{F}$-concave (super-$\mathfrak{F}$) functions. In Theorem 3.4 a result that pertains to the family $\mathfrak{F}$ is derived from sequences of $\mathfrak{F}$-concave functions. Theorems 3.6 and 3.7 describe the least $\mathfrak{F}$-concave function that dominates a given continuous function on a compact interval.

In discussing disconjugate families, Beckenbach [2] cites as an example "the set of images of all non-vertical straight lines under a 1-to-1 continuous transformation of the domain $a<x<b$ of the plane into itself in such a way that every vertical line is transformed into itself." Theorem 4.1 provides a partial converse to this remark for disconjugate linear subspaces of $C(I,R)$: these are always images of nonvertical straight lines under a topological map of some strip $I'\times R$ onto $I\times R$ which carries vertical lines into vertical lines. Generally, Section 4 provides results for locally disconjugate linear subspaces of $C(I,R)$ which are typical of those for the space of solutions to (1.1). In particular, such a family determines local operators, θ, for which the two-point boundary-value problem

$$\theta(x) = y,\ x(t_i) = y_i,\ i = 1,2,$$

is uniquely solvable and whose inverses have integral formulations. Moreover, a function f in the domain of θ is $\mathfrak{J}$-concave if and only if $\theta(f) \leq 0$. This generalizes a result of Bonsall [3]. Also, following the lead of Ashley [1], we derive from a theorem of Choquet [4] an explicit integral representation of the elements of the cone of nonnegative $\mathfrak{J}_K$- concave functions.

In Section 5, we generalize a result of Reid [12] which characterizes $\mathscr{A}$-concave functions as those which satisfy certain unilateral variational relations stated in terms of the functional J, where

(1.5) $$J_K(u) = \int_K (r\cdot u'^2+2\cdot q\cdot u\cdot u'+p\cdot u^2),$$

by introducing a class of functionals, which includes those of the form (1.5), relative to which a type of unilateral variational problem is solvable. The class of solutions of these unilateral variational problems is precisely the set of generalized concave functions.

In the sequel, I = (a,b), where a and b are extended reals and a<b, represents a fixed open interval in R and I_α represents a generic open subinterval of I. All intervals are assumed to be nondegenerate. The set $C(I_\alpha,R)$ of continuous real-valued functions on I_α is assigned the topology of uniform convergence on compact subsets of I_α, and subsets thereof are assigned the relativized topology. For a set $\mathfrak{J}$ in $C(I_\alpha,R)$ and a subset A of I, $\mathfrak{J}|_A$ is the set

of restrictions of members of $\mathfrak{F}$ to $A \cap I_\alpha$. We use $S(I_\alpha)$ to denote the set $\{(t_1,x_1,t_2,x_2) \in I_\alpha \times R \times I_\alpha \times R : t_1 < t_2\}$ and assign to it the relativized R^4 topology. The subset A of I is said to be <u>bounded</u> in I_α if A is a subset of a compact subset of I_α. When R^n or $C(I,R)$ are considered as ordered spaces they are assigned their respective product orderings. For a natural number n, the symbol $\hat{n}$ denotes the set $\{1,\ldots,n\}$.

2. <u>Locally disconjugate families of continuous functions</u>. Our starting point is the notion of disconjugacy for a family of continuous functions. The family $\mathfrak{F}$ contained in $C(I,R)$ is <u>disconjugate on the subinterval</u> I_o <u>of</u> I if and only if for every pair of points (t_1,x_1), (t_2,x_2) in $I_o \times R$ with distinct abscissae there is one and only one member F of the family which satisfies the conditions:

(2.1) $$F(t_i) = x_i, \quad i = 1,2.$$

In such instances, the interval I_o is called an <u>interval of disconjugacy of the family</u> $\mathfrak{F}$. The behavior of members of the family $\mathfrak{F}$ on intervals of disconjugacy of $\mathfrak{F}$ is restricted, as is shown in the following theorem due to Beckenbach [2].

THEOREM 2.1 <u>Suppose that</u> $\mathfrak{F}$ <u>in</u> $C(I,R)$ <u>is disconjugate on</u> $I_o = (a_o,b_o)$ <u>and that</u> $t_o \in I_o$. <u>If</u> F_1 <u>and</u> F_2 <u>are distinct members of</u> $\mathfrak{F}$ <u>such that</u> $F_1(t_o) = F_2(t_o)$, <u>then either</u> $F_1 > F_2$

<u>on</u> (t_o,b_o) <u>and</u> $F_1 < F_2$ <u>on</u> (a_o,t_o) <u>or</u> $F_1 < F_2$ <u>on</u> (t_o,b_o) <u>and</u> $F_1 > F_2$ <u>on</u> (a_o,t_o).

It is to be noted that it follows from the definition of disconjugacy and from Theorem 2.1 that if t_1, t', and t'' belong to an interval of disconjugacy I_o of $\mathfrak{F}$, and are such that either $t_1 < t' < t''$ or $t'' < t' < t_1$, and if $\{F_\alpha\}$ is a subfamily of $\mathfrak{F}$ such that $\{F_\alpha(t_1)\}$ is bounded below, [above], by the real number r and $\{F_\alpha(t')\}$ is bounded above, [below], by the real number s, then $\{F_\alpha(t'')\}$ is bounded above, [below], by $F(t'')$, where F is the unique member of $\mathfrak{F}$ connecting the points (t_1,r) and (t',s). Using an indirect argument, this implies that if $[t_1,t_2] \subset I_o$ and if the sequence (F_n) in $\mathfrak{F}$ is such that $(F_n(t_i)) \to +\infty$, $[(F_n(t_i)) \to -\infty]$, $i = 1,2$, then $(F_n(t')) \to +\infty$, $[(F_n(t')) \to -\infty]$, for all t' in $[t_1,t_2]$.

Beckenbach has also proved the following theorem and corollary.

THEOREM 2.2. <u>If the subfamily</u> $\mathfrak{F}$ <u>of</u> C(I,R) <u>is disconjugate on</u> I <u>then the map</u> $\Phi: S(I) \to C(I,R)$ <u>such that</u> $\Phi(t_1,x_1,t_2,x_2)$ <u>is the unique member of</u> $\mathfrak{F}$ <u>whose graph contains the points</u> (t_1,x_1) <u>and</u> (t_2,x_2) <u>is continuous</u>.

COROLLARY. <u>If the subfamily</u> $\mathfrak{F}$ <u>of</u> C(I,R) <u>is disconjugate on</u> I, <u>then the subset</u> $\mathfrak{a}$ <u>of</u> $\mathfrak{F}$ <u>is compact if for distinct elements</u> t_1 <u>and</u> t_2 <u>of</u> I <u>the sets</u> $\mathfrak{a}[t_i] = \{F(t_i) : F \in \mathfrak{a}\}$, $i = 1,2$, <u>are compact</u>.

These results can be generalized so as to apply to a larger class of families of functions. The subfamily $\mathfrak{F}$ of C(I,R) is said to be <u>locally disconjugate</u> if and only if every point in I is interior to some interval of disconjugacy of $\mathfrak{F}$.

THEOREM 2.3. *Suppose that $\mathfrak{F}$ is a locally disconjugate subfamily of C(I,R) and that I_o is an interval of disconjugacy of $\mathfrak{F}$. Then the map $\Phi: S(I_o) \to C(I,R)$ such that $\Phi(t_1,x_1,t_2,x_2)$ is the unique member of $\mathfrak{F}$ that passes through (t_1,x_1) and (t_2,x_2) is continuous and its range is $\mathfrak{F}$. Moreover, if t_1 and t_2 are distinct fixed points of I_o, the restriction ϕ of Φ to $\{(t_1,x_1,t_2,x_2) : x_1,x_2 \in R\}$ is a homeomorphism of R^2 onto $\mathfrak{F}$. In particular, if $\mathfrak{F}$ is a linear subspace of C(I,R), then ϕ is a linear homeomorphism of R^2 onto $\mathfrak{F}$.*

Before proving Theorem 2.3 we remark that consideration of the family $\mathfrak{F} = \{F(\alpha,\beta) : \alpha,\beta \in R\}, \mathfrak{F} \subset C(R,R)$, where $[F(\alpha,\beta)](0) = \beta$ and, for $x \neq 0$, $[F(\alpha,\beta)](x) = \alpha \cdot x^2 \cdot \sin(x^{-1}) + \beta$, leads to the conclusion that not all homeomorphs of R^2 contained in C(R,R) are locally disconjugate families.

To prove the theorem, suppose that $[a_o,b_o]$ is a subinterval of I not wholly contained in I_o and choose a subinterval $[a',b']$ of I containing $[a_o,b_o]$ whose intersection with I_o is an interval. Cover $[a',b']$ by open intervals of disconjugacy of $\mathfrak{F}$, choosing I_o as the particular interval of disconjugacy for each point in $I_o \cap [a',b']$. This cover has a minimum finite subcover containing I_o whose elements can be strictly ordered by their left-hand endpoints. Suppose the cover contains an interval which extends to the left of I_o. Then the rightmost of these intervals, J_1, contains a pair of points, t_1 and t_2, of I_o. By Theorem 2.2, a net $(p^{(\alpha)})$ in $S(I_o)$ convergent to p in $S(I_o)$ defines a convergent net $(q^{(\alpha)})$,

$$q^{(\alpha)} = (t_1, [\Phi(p^{(\alpha)})](t_1), t_2, [\Phi(p^{(\alpha)})](t_2)),$$

in $S(J_1)$ with limit $q = (t_1, [\Phi(p)](t_1), t_2, [\Phi(p)](t_2))$. Since $\mathfrak{F}$ is disconjugate on J_1, Theorem 2.2 implies that for any compact subset K of J_1 the net of restrictions of $(\Phi(p^{(\alpha)}))$ to K converges uniformly to the like restriction of $\Phi(p)$, whence the same holds for compact subsets K of $J_1 \cup I_o$. Since the cover is finite, applying the above method to elements of the cover adjacent to the rightmost and leftmost intervals of the cover for which this result has been demonstrated yields the uniform convergence of $\Phi(p^{(\alpha)})$ to $\Phi(p)$ on compact subsets contained in the union of the elements of the cover, hence on $[a_o, b_o]$. Therefore, Φ is continuous.

Then, for t_1, t_2 distinct in I_o, the one-to-one map ϕ such that $\phi(x_1, x_2) = \Phi(t_1, x_1, t_2, x_2)$ is continuous and, since $\phi(F(t_1), F(t_2)) = F$ for F in $\mathfrak{F}$, ϕ is onto $\mathfrak{F}$. The continuity of ϕ implies that $\mathfrak{F}$ is a closed, hence complete, subset of C(I,R). For, if the sequence (F_n) in $\mathfrak{F}$ has limit f in C(I,R), then, for any choice of t_1, t_2 in an interval of disconjugacy I_o of $\mathfrak{F}$, $F_n = \phi(F_n(t_1), F_n(t_2))$ implies that $f = \phi(f(t_1), f(t_2))$. Moreover, $(F_n) \to F$ in $\mathfrak{F}$ implies that

$((F_n(t_1), F_n(t_2))) \longrightarrow (F(t_1), F(t_2))$ in R^2 which is equivalent to $(\phi^{-1}(F_n)) \longrightarrow \phi^{-1}(F)$. Hence the continuity of ϕ^{-1}.

If $\mathfrak{F}$ is a linear subspace, then $\lambda\cdot\phi(x_1,x_2)$ belongs to $\mathfrak{F}$ for real λ, whence $\lambda\cdot\phi(x_1,x_2)$ is the unique member of $\mathfrak{F}$ passing through $(t_1, \lambda\cdot x_1)$ and $(t_2, \lambda\cdot x_2)$. Thus, $\lambda\cdot\phi(x_1,x_2)$ equals $\phi(\lambda\cdot x_1, \lambda\cdot x_2)$. Similarly, $\phi(x_1,x_2) + \phi(x_1',x_2') = \phi(x_1+x_1', x_2+x_2')$.

COROLLARY. _If_ $\mathfrak{A}$ _is a subset of the locally disconjugate family_ $\mathfrak{F}$, _then_ $\mathfrak{A}$ _is compact if and only if for some pair of points_ t_1, t_2 _belonging to a common open interval of disconjugacy of_ $\mathfrak{F}$ _the sets_ $\mathfrak{A}[t_i] = \{F(t_i): F\in\mathfrak{A}\}$, $i = 1,2$, _are compact_.

The corollary follows from the fact that projections are continuous after noting that $\mathfrak{A}[t_i] = p_i\circ\phi^{-1}[\mathfrak{A}]$, $i = 1,2$, where ϕ is defined as in Theorem 2.3 by any pair t_1,t_2 contained in some interval of disconjugacy of $\mathfrak{F}$ and $p_i: R^2 \to R$ is the i^{th} coordinate projection map.

We now characterize those $C(I,R)$-valued maps whose images are locally disconjugate families. We say that the pair of points t_1,t_2 of I _determines_ the homeomorphic imbedding ψ of R^2 into $C(I,R)$ if and only if the map β defined by

$$\beta(y_1,y_2) = ([\psi(y_1,y_2)](t_1), [\psi(y_1,y_2)](t_2)),$$

and denoted by $\beta = (e_{t_1}, e_{t_2})\circ\psi$, is a homeomorphism of R^2. The map $\psi: R^2 \to C(I,R)$ is _pairwise determined_ if and only if every pair of points in I determines ψ; the map ψ is <u>locally pairwise determined</u> if and only if every point of I belongs to some open interval each of whose pairs of points determines ψ.

If $\psi: R^2 \to C(I,R)$ is a homeomorphism whose range is a locally disconjugate family, and if ϕ is the homeomorphism of Theorem 2.3 obtained from $\psi(R^2)$ and from the points t_1 and t_2 belonging to a common interval of disconjugacy of $\psi(R^2)$, then the map $\beta = (e_{t_1}, e_{t_2}) \circ \psi$ equals $\phi^{-1} \circ \psi$ and is a homeomorphism of R^2. Also, if $\psi: R^2 \to C(I,R)$ is a locally pairwise determined homeomorphism, then, for t_o in I, any open interval I_o about t_o each of whose pairs of points determine ψ is an interval of disconjugacy for $\psi(R^2)$. For, with t_1 and t_2 in I_o, $\psi \circ ((e_{t_1}, e_{t_2}) \circ \psi)^{-1}$ at (x_1, x_2) is the unique member of $\psi(R^2)$ whose graph contains (t_1, x_1) and (t_2, x_2). These considerations yield the following theorem.

THEOREM 2.4. <u>The range of a homeomorphic imbedding of R^2 into C(I,R) is disconjugate on I, [locally disconjugate], if and only if the homeomorphism is pairwise determined, [locally pairwise determined].</u>

Therefore, a subfamily of C(I,R) is disconjugate on I, [locally disconjugate], if and only if it is the range of a pairwise determined, [locally pairwise determined], homeomorphic imbedding of R^2 in C(I,R).

In the sequel $\mathfrak{F}$ is a locally disconjugate family in C(I,R) and $F = \mathfrak{F}(t_1, x_1, t_2, x_2)$ indicates that F is the unique member of $\mathfrak{F}$ which satisfies conditions (2.1). Use of this notation presupposes that $t_1 < t_2$ and that $[t_1,t_2]$ is contained in an interval of disconjugacy of $\mathfrak{F}$. Thus $\mathfrak{F}$, as a function, is used to represent any of the continuous functions Φ of Theorem 2.3. Since the topology of uniform convergence on compacta for C(I,R) is jointly continuous [9;p. 224 and Thm. 7.11], the function $\overline{\mathfrak{F}}$ defined by

$$\overline{\mathfrak{F}}(t_1,x_1,t_2,x_2;t) = [\mathfrak{F}(t_1,x_1,t_2,x_2)](t)$$

is a continuous function of five variables on domains of the form $S(I_o)\times I$, I_o being an interval of disconjugacy of $\mathfrak{F}$.

For a pair of points t_1,t_2 in I with $t_1 < t_2$ we say that t_1 _is_ $\mathfrak{F}$-_related_ to t_2, denoted by $t_1 \sim t_2$, if and only if for some (x_1,x_2) in R^2 either two members of $\mathfrak{F}$ satisfy conditions (2.1) or no member of $\mathfrak{F}$ satisfies these conditions. If $\mathfrak{F}$ is also a linear subspace of C(I,R) then the alternative conditions used in defining $\mathfrak{F}$-related pairs are equivalent, [see Section 4] so that if $\mathfrak{L}$ is the space of solutions of the differential equation (1.1), then t_1 is $\mathfrak{L}$-related to t_2 if and only if t_1 and t_2 are conjugate with respect to this differential equation. Even in the general case, the second condition in the definition of $\mathfrak{F}$-relatedness can be expressed in terms of the first condition, as is shown in the theorem below.

THEOREM 2.5. _If no member of a locally disconjugate family_ $\mathfrak{F}$ _joins_ (t_1,x_1) _and_ (t_2,x_2), _then either two members of_ $\mathfrak{F}$ _join_ (t_1,x_1) _and some_ (t_2',x_2') _in_ $(t_1,t_2]\times R$, _or_ t_2

<u>is the infimum of the set of points</u> t_2' <u>for which there is an associated</u> x_2' <u>such that two members of</u> $\mathfrak{F}$ <u>join</u> (t_1,x_1) <u>and</u> (t_2',x_2').

For a choice of t_1' in (t_1,t_2) such that $\mathfrak{F}$ is disconjugate on $[t_1,t_1']$ define F_r to be $\mathfrak{F}(t_1,x_1,t_1',r)$ for real r. Let $\mathfrak{A} = \{F_r : r\epsilon R\}$ and define $\alpha : R \to \mathfrak{A}|_{(t_1,t_2]}$ by $\alpha(r) = F_r|_{(t_1,t_2]}$. The continuity of $\bar{\mathfrak{F}}$ implies that the set of values attained at t_2 by members of $\mathfrak{F}$ passing through (t_1,x_1), namely $\{F_r(t_2) : r\epsilon R\} = \{\bar{\mathfrak{F}}(t_1,x_1,t',r;t_2) : r\epsilon R\}$, is an interval in R. Suppose, for definiteness, that $F_r(t_2) < x_2$ for all real r. It is clear that two members of $\mathfrak{A}$ intersect in $(t_1,t_2]$ if and only if α is not strictly increasing. Suppose that α is strictly increasing. Then $\mathfrak{A}[t_2]$ is an open interval with a supremum $A \le x_2$. If no members of $\mathfrak{A}$ intersect in some closed neighborhood $[t',t'']$ of t_2 on which $\mathfrak{F}$ is disconjugate, then both sequences $(F_n(t'))$ and $(F_n(t''))$ are strictly increasing and, by the remarks following Theorem 2.1, the fact that $(F_n(t_2))$ is bounded above implies that one of the sets $\mathfrak{A}[t']$, $\mathfrak{A}[t'']$ has a finite least upper bound B. Then the sequence (F_n), each term of which satisfies $F_n(t_1) = x_1$, has one of the functions $\mathfrak{F}(t',B,t_2,A)$, $\mathfrak{F}(t_2,A,t'',B)$ as limit and, by Theorem 2.3, this limit function is a member of $\mathfrak{F}$ which passes through (t_1,x_1) and (t_2,A), contradicting the openness of $\mathfrak{A}[t_2]$. Thus, for all $\varepsilon > 0$, $(t_2,t_2+\varepsilon)\times R$ contains a point at which a pair of elements of $\mathfrak{A}$ intersect.

The proof just given makes it clear that Theorem 2.5 has the following dual statement: if no member of $\mathfrak{F}$ joins (t_1,x_1) and (t_2,x_2), then t_1 is less than or equal to the supremum of the values t_1' for which there is an associated x_1' such that two members of $\mathfrak{F}$ join (t_1',x_1') and (t_2,x_2).

THEOREM 2.6. _If_ $\mathfrak{F}$ _is a locally disconjugate family and_ t_1 _is_ $\mathfrak{F}$_-related to_ t_2_, then for every_ t_1' _less than_ t_1, [t_2' _greater than_ t_2], _there is a_ t_2' _less than_ t_2, [t_1' _greater than_ t_1], _such that_ t_1' _is_ $\mathfrak{F}$_-related to_ t_2'.

We consider the case: $t_1' < t_1$. The alternative follows in a similar manner. Note that it is sufficient to prove the result for t_1' such that $\mathfrak{F}$ is disconjugate on $J = [t_1',t_1]$. For if $\mathfrak{F}$ is not disconjugate on J, it is possible to find a mimimal finite open cover of J consisting of intervals of disconjugacy of $\mathfrak{F}$ and, after choosing a point from each of the intersections of pairs of adjacent intervals of the cover, a finite number of applications of the result for intervals of disconjugacy yields the result of the theorem.

Suppose that $\mathfrak{F}$ is disconjugate on $[t_1',t_1]$. Since $t_1 \sim t_2$ there are real numbers x_1 and x_2 such that either two members of $\mathfrak{F}$ join (t_1,x_1) and (t_2,x_2) or no member of $\mathfrak{F}$ joins these points. First, if F and G are distinct members of $\mathfrak{F}$ connecting (t_1,x_1) and (t_2,x_2), we may assume without loss of generality that $F > G$ on (t_1,t_2) and that F and G do not intersect in (t_1',t_1). Moreover, if $F(t_1') = G(t_1')$ the desired result is immediate, so we assume that $F(t_1') < G(t_1')$. For a fixed x_1' in the interval $(F(t_1'), G(t_1'))$, index the members of the set $\mathfrak{A}$ of $\mathfrak{F}$-lines passing through (t_1', x_1') by their values at t_1. That is, $\mathfrak{A} = \{F_r : r \in R\}$ where $F_r \equiv \mathfrak{F}(t_1', x_1', t_1, r)$. We show that some members of $\mathfrak{A}$ intersect in (t_1', t_2), which is equivalent to the statement that the map α which takes r in R to $F_r|(t_1', t_2)$ is not strictly increasing, by deriving a contradiction from the assumption that α is strictly increasing.

Suppose that α is strictly increasing. In view of Theorem 2.5, this implies that every point in $(t_1', t_2) \times R$ is in the graph of some member of $\mathfrak{A}$. Also, a continuous function f is bounded above and below by members of $\mathfrak{A}$ on compact subintervals, K, of (t_1', t_2). For, given f and K, for any t in K, the member F_{r_t} of $\mathfrak{A}$ whose value at t is $2 \cdot f(t)$ strictly dominates f on a neighborhood N_t of t, so that, for any finite cover, $\{N_{t_i} : i \in \hat{n}\}$, of K taken from $\{N_t : t \in K\}$ and with $r \equiv \max \{r_{t_i} : i \in \hat{n}\}$, F_r strictly dominates f on K. There are two cases to consider. If $\alpha(x_1)$ meets F or G in (t_1, t_2), then some member of $\mathfrak{A}$ meets F or G at a point in (t_1, t_2) but does not cross it at this point, a contradiction to the result of Theorem 2.1. For if $\alpha(x_1)$ crosses F in (t_1, t_2) with t_2' the least abscissa of the points of intersection of F and $\alpha(x_1)$ in (t_1, t_2), then, since some member of $\mathfrak{A}$ strictly dominates F on $[t_1, t_2']$ and by the result of Theorem 2.3, there is a nonempty open interval in R with infimum $s > x_1$ consisting of all reals r for which $\alpha(r) > F$ on $[t_1, t_2']$, whence $\alpha(s)$ dominates F on

$[t_1, t_2']$ but meets F somewhere in $[t_1, t_2']$. If $\alpha(x_1)$ meets neither F nor G in (t_1, t_2), let H_t be the unique member of $\mathfrak{a}$ which passes through $(t, F(t))$ for t in $[t_1, t_2)$. If any H_t meets F in (t, t_2), say at some least abscissa t_2', then, by an argument similar to the one given above, there is a member of $\mathfrak{a}$ which dominates F on $[t, t_2']$ but which equals F somewhere in $[t, t_2']$, a contradiction. Otherwise, for every t in $[t_1, t_2)$, $\alpha(x_1) < H_t < F$ on (t, t_2), so that $H_t(t_2) = F(t_2)$ for all t in $[t_1, t_2)$, by virtue of which $\mathfrak{F}$ is not disconjugate on any neighborhood of t_2, also a contradiction.

With $\mathfrak{F}$ still taken to be disconjugate on $[t_1', t_1]$, suppose that no member of $\mathfrak{F}$ connects (t_1, x_1) and (t_2, x_2), and, for definiteness, that $F_r(t_2) < x_2$ for all real r, where $F_r \equiv \mathfrak{F}(t_1', -r, t_1, x_1)$. Since $\mathfrak{F}$ is disconjugate on an open interval containing t_1 and since $\{F_r : r \in R\}$ is the set of all $\mathfrak{F}$-lines through (t_1, x_1), there is an $\varepsilon > 0$ such that, for all t in $(t_1, t_1+\varepsilon)$, the sequence $(F_n(t)) \to +\infty$. Then any member of $\mathfrak{F}$ that passes through (t_2, x_2) meets F_n at least twice in (t_1', t_2) for all sufficiently large n. In view of the results of the preceding paragraphs, this yields the conclusion of the theorem.

If an endpoint t_o of the interval I_o is $\mathfrak{F}$-related to some t' in I_o, then, by Theorem 2.6, for a choice of t" between t' and the other endpoint of I_o, t" is $\mathfrak{F}$-related to some t_o' between t" and t_o, whence $\mathfrak{F}$ is not disconjugate on I_o. Thus we have the following corollary.

COROLLARY. _An endpoint of an open interval of disconjugacy of a locally disconjugate family is not $\mathfrak{F}$-related to any point in the interval._

We say that a locally disconjugate family in C(I,R) is _merely locally disconjugate_ if the family is not disconjugate on I. Clearly, this condition holds if and only if I contains an $\mathfrak{F}$-related pair. If $t_1' \sim t_2'$ in I and if $I_o = (t_1,t_2) \subset I$ is an interval of disconjugacy of $\mathfrak{F}$ containing t_o, then either $t_1 < t_1'$ or $t_2' < t_2$ or $(t_1,t_2) \subset (t_1',t_2')$. By Theorem 2.6, in the first case, t_1 is $\mathfrak{F}$-related to some t_2'' in $[t_2,t_2')$; in the second case, t_2 is $\mathfrak{F}$-related to some t_1'' in $(t_1',t_1]$. Therefore, if $\mathfrak{F}$ is merely locally disconjugate, every point in I belongs to some open interval of disconjugacy of $\mathfrak{F}$ which is contained in a closed subinterval of I whose endpoints are $\mathfrak{F}$-related.

Suppose now that $\mathfrak{F}$ is merely locally disconjugate. By Theorem 2.6, the sets $D_\rho = \{t : \exists t' > t \ni t \sim t'\}$ and $D_\lambda = \{t : \exists t' < t \ni t' \sim t\}$ of left and right members of $\mathfrak{F}$-related pairs in I are of the form (a,t_ρ) and (t_λ,b), respectively, for some t_ρ in $(a,b]$ and t_λ in $[a,b)$. It is also clear that the functions $\lambda : D_\lambda \rightarrow I$ and $\rho : D_\rho \rightarrow I$, called the _left and right conjugate point functions determined by_ $\mathfrak{F}$, defined by

$$\lambda(t) = \sup\{t' < t : t' \sim t\} \quad \text{and} \quad \rho(t) = \inf\{t' > t : t \sim t'\} \tag{2.2}$$

are nondecreasing and that for t in D_ρ, [t in D_λ], the family $\mathfrak{F}$ is disconjugate on the interval $(t,\rho(t))$, $[(\lambda(t),t)]$.

Further, if $t \in D_\rho \cap \rho^{-1}(D_\lambda)$, $[t \in D_\lambda \cap \lambda^{-1}(D_\rho)]$, then $\lambda(\rho(t)) \le t$, $[\rho(\lambda(t)) \ge t]$. For, if $\lambda(\rho(t)) > t$, then, by the definition of λ, $\rho(t)$ is $\mathfrak{J}$-related to a point in $(t, \lambda(\rho(t))]$ which, by the corollary to Theorem 2.6, contradicts the disconjugacy of $\mathfrak{J}$ on the interval $(t, \rho(t))$. Using these alternative results together, if $t \in D_\rho \cap \rho^{-1}(D_\lambda)$, $[t \in D_\lambda \cap \lambda^{-1}(D_\rho)]$, then $\lambda(\rho(t)) \in D_\rho$, $\rho(\lambda \circ \rho(t)) \le \rho(t)$, and $\rho \circ \lambda(\rho(t)) \ge \rho(t)$, whence $\rho(\lambda(\rho(t))) = \rho(t)$, $[\lambda(\rho(\lambda(t))) = \lambda(t)]$. Then, with $R_\lambda = \lambda(D_\lambda)$ and $R_\rho = \rho(D_\rho)$, $\{t : \rho \circ \lambda(t) = t\}$ is precisely $D_\lambda \cap R_\rho$ and $\{t : \lambda \circ \rho(t) = t\}$ equals $D_\rho \cap R_\lambda$, and $\{t : \lambda \circ \rho(t) < t\}$ equals $D_\rho \cap \rho^{-1}(D_\lambda) \setminus R_\lambda$ and $\{t : \rho \circ \lambda(t) > t\}$ is $D_\lambda \cap \lambda^{-1}(D_\rho) \setminus R_\rho$. Suppose that $\lambda(\rho(t)) < t$ and that $t' \in (\lambda(\rho(t)), t)$. Then $\rho(\lambda(\rho(t))) \le \rho(t') \le \rho(t)$, so that $\rho(t') = \rho(t)$. Thus, $D_\rho \cap \rho^{-1}(D_\lambda) \setminus R_\lambda$ consists of intervals of constancy of ρ whose left-hand endpoints are in $D_\rho \cap R_\lambda$. A dual result holds for $D_\lambda \cap \lambda^{-1}(D_\rho) \setminus R_\rho$. Now, if $t \in D_\rho$ and $\rho(t) < t_\lambda$, then t_λ is to the right of an $\mathfrak{J}$-related pair, whence $t_\lambda \in D_\lambda$, a contradiction. Therefore, $R_\rho \subset [t_\lambda, b)$ and, since ρ is nondecreasing, $\{t : \rho(t) = t_\lambda\}$, which is $D_\rho \setminus \rho^{-1}(D_\lambda)$, is an initial interval in I; that is, it has one of the forms (a, t'_ρ), $(a, t'_\rho]$ for some t'_ρ in $[a, t_\rho]$. Similarly, $R_\lambda \subset (a, t_\rho]$ and $D_\lambda \setminus \lambda^{-1}(D_\rho) = \{t : \lambda(t) = t_\rho\}$ has the form (t'_λ, b) or $[t'_\lambda, b)$ for some t'_λ in $[t_\lambda, a]$.

Suppose now that for no t in I is $\lambda(\rho(t)) < t$. Then either $t'_\rho = t_\rho$ and the function ρ is the constant t_λ function, or $t'_\rho = a$ and ρ is strictly increasing, or

$a < t'_\rho < t_\rho$ and ρ is constant on (a,t'_ρ) and strictly increasing on (t'_ρ,t_ρ). If it is further assumed that $\rho(\lambda(t))$ is never greater than t, the functions $\ell: D_\lambda \cap R_\rho \to D_\rho \cap R_\lambda$ and $r: D_\rho \cap R_\lambda \to D_\lambda \cap R_\rho$, defined as restrictions of λ and ρ, respectively, are such that $r \circ \ell$ and $\ell \circ r$ are identity maps, so that λ and ρ are strictly increasing and continuous on (t_λ,t'_λ) and (t'_ρ,t_ρ) respectively. Then ρ is continuous unless $\alpha = \lim_{t \to (t'_\rho)+} \rho(t) > t_\lambda$. But, since $\lambda(\rho(t)) = t$ for t in (t'_ρ,t_ρ), this would imply that $\lambda(\alpha) > t'_\rho$, whence any t' in $(t'_\rho,\lambda(\alpha))$ has $\rho(t') < \alpha$, a contradiction. A dual argument shows the continuity of λ. In this instance, let $\bar{\rho}$ and $\bar{\lambda}$ be the continuous extensions of ρ and λ to $[a,t_\rho]$ and $[t_\lambda,b]$, respectively, with $\bar{r}$ and $\bar{\ell}$ their respective restrictions to $[t'_\rho,t_\rho]$ and $[t_\lambda,t'_\lambda]$.

THEOREM 2.7. _If_ $\mathcal{F}$ _is a merely locally disconjugate family in_ C(I,R), _the conjugate point functions_, ρ _and_ λ, _determined by_ $\mathcal{F}$ _are such that_ $(t,\rho(t))$, $[(\lambda(t),t)]$, _is the largest open interval of disconjugacy with_ t _as endpoint and extending to the right_, [_left_]. _If for no_ t _in_ I _is_ $\lambda(\rho(t)) < t$ _or_ $\rho(\lambda(t)) > t$, ρ _and_ λ _are continuous and define continuous extensions_ $\bar{\rho}: [a,t_\rho] \to [t_\lambda,b]$ _and_ $\bar{\lambda}: [t_\lambda,b] \to [a,t_\rho]$ _in the extended reals such that_ $\bar{\rho}(t) = t_\lambda$ _for_ t _in_ $[a,t'_\rho]$, $\bar{\lambda}(t) = t_\rho$ _for_ t _in_ $[t'_\lambda,b]$, _and the restrictions of_ $\bar{\rho}$ _and_ $\bar{\lambda}$ _to_ $[t'_\rho,t_\rho]$ _and_ $[t_\lambda,t'_\lambda]$, _respectively, are inverse homeomorphisms_.

An open interval of disconjugacy of $\mathcal{F}$ is said to be

<u>maximal</u> if and only if it is not properly contained in any open interval of disconjugacy of $\mathfrak{J}$. The endpoints of a bounded maximal interval of disconjugacy are called $\mathfrak{J}$-<u>conjugates</u> or are said to constitute an $\mathfrak{J}$-conjugate pair. It follows from the fact that the bounded interval (t_1,t_2) of disconjugacy is maximal if and only if $\mathfrak{J}$ is not disconjugate on either $(t_1,t_2+\varepsilon)$ or $(t_1-\varepsilon,t_2)$ for all $\varepsilon > 0$ that, when $a<t_1<t_2<b$, (t_1,t_2) is maximal if and only if $t_2 = \rho(t_1)$ and $t_1 = \lambda(t_2)$. In such an instance, we call t_1 the <u>left-</u>$\mathfrak{J}$-<u>conjugate</u> of t_2 and t_2 the <u>right</u>-$\mathfrak{J}$-<u>conjugate</u> of t_1.

COROLLARY. *If for no t in I is $\lambda(\rho(t)) < t$ or $\rho(\lambda(t)) > t$ and if $t_\rho' < t_\rho$, then the right, [left], $\mathfrak{J}$-conjugate of t is defined, is continuous and strictly increasing, and is given by the right, [left], conjugate point function for t in (t_ρ',t_ρ), $[(t_\lambda,t_\lambda')]$. Moreover, the right-$\mathfrak{J}$-conjugate and left-$\mathfrak{J}$-conjugate functions are inverse homeomorphisms.*

Note that two points in (a,b) are consecutive conjugate points relative to the differential equation (1.1) defined on (a,b) if and only if they are $\mathfrak{L}$-conjugates, where $\mathfrak{L}$ is the family of solutions to (1.1) on (a,b). Thus, our notion of $\mathfrak{J}$-conjugate points is a generalization of the notion of consecutive conjugate points of the differential equation (1.1).

Theorem 2.8. *If $\mathfrak{J}$ is merely locally disconjugate, then every point in I is interior to some maximal open interval of disconjugacy.*

For a fixed t_o and a fixed $\mathfrak{F}$-related pair t_1, t_2 in I such that $t_o \in (t_1, t_2)$, any chain $\mathcal{C} = \{I_\alpha\}_{\alpha \in A}$ in the set $\mathcal{B}$ of open subintervals of $[t_1, t_2]$ which contain t_o and on which $\mathfrak{F}$ is disconjugate, where $\mathcal{B}$ is partially ordered by set inclusion, has $\bigcup_{\alpha \in A} I_\alpha$ as an upper bound. By Zorn's lemma, some member of $\mathcal{B}$ is not properly contained in any other. If some maximal member (t_1', t_2') of $\mathcal{B}$ is such that $t_1' = t_1$ and $t_2' = t_2$ or $t_1 < t_1'$ and $t_2' < t_2$, then it is a bounded maximal interval of disconjugacy containing t_o. Otherwise, maximal members of $\mathcal{B}$ have the form (t_1, t_2') with $t_2' < t_2$ or (t_1', t_2) with $t_1 < t_1'$. In the former case, $t_2' = \rho(t_1)$ and one of the intervals (t_1, t_2'), (a, t_2'), (t_1'', t_2'), where $t_1'' \equiv \inf (\rho^{-1}(\{\rho(t_1)\}))$, is a maximal interval of disconjugacy. In the latter case, either (t_1', t_2), (t_1', b), or (t_1', t_2''), where $t_2'' \equiv \sup(\lambda^{-1}(\{\lambda(t_2)\}))$, is maximal.

The members of $\mathfrak{F}$ are called $\mathfrak{F}$-<u>lines</u> and a function $f: I \to R$ which agrees with some member of $\mathfrak{F}$ on the interval $[t_1, t_2]$ is said to be $\mathfrak{F}$-<u>linear</u> on $[t_1, t_2]$. The function $f: I \to R$ is termed <u>piecewise</u>-$\mathfrak{F}$-<u>linear</u> if and only if there are finite sets $T = \{t_i : i \in \hat{n}\} \subset I$ and $X = \{x_i : i \in \hat{n}\} \subset R$, for some $n \geq 2$, with $t_i < t_{i+1}$ and $\mathfrak{F}$ disconjugate on $[t_i, t_{i+1}]$ for $i = 1, 2, \cdots, n-1$, such that

$$f(t) = \begin{cases} [\mathfrak{F}(t_1, x_1, t_2, x_2)](t), & a < t \leq t_2, \\ [\mathfrak{F}(t_i, x_i, t_{i+1}, x_{i+1})](t), & t_i \leq t \leq t_{i+1}, i = 2, \cdots, n-2, \\ [\mathfrak{F}(t_{n-1}, x_{n-1}, t_n, x_n)](t), & t_{n-1} \leq t < b, \end{cases}$$

denoted by $f = \mathfrak{J}(T,X)$. The collection of piecewise-$\mathfrak{J}$-linear functions is denoted by $\mathfrak{P}(\mathfrak{J})$.

The operations $\wedge$ and $\vee$, defined in $C(I,R)$ such that $f \wedge g$ and $f \vee g$ are the pointwise infimum and pointwise supremum, respectively, of f and g, are continuous and give $C(I,R)$ the structure of a distributive lattice. The following lemma is an immediate result of M. H. Stone's Theorem 1 in [13].

LEMMA 2.9. <u>If</u> $\mathfrak{J}$ <u>is disconjugate on</u> I <u>and if</u> $\mathfrak{P}$ <u>is a sublattice of</u> $C(I,R)$ <u>containing</u> $\mathfrak{J}$, <u>then</u> $\mathfrak{P}$ <u>is dense in</u> $C(I,R)$.

That is, given f in $C(I,R)$, for every $\varepsilon > 0$ and for every compact subinterval K of I, there is a member P of $\mathfrak{P}$ such that $|P(t)-f(t)| < \varepsilon$ for all t in K.

Suppose I_o is an interval of disconjugacy of $\mathfrak{J}$. Since $\mathfrak{J}$-linear segments of pairs of members of $\mathfrak{P}(\mathfrak{J}|_{I_o})$ intersect at most once, if $F' = \mathfrak{J}(T',X')$, $T' \subset I_o$, and $F'' = \mathfrak{J}(T'',X'')$, $T'' \subset I_o$, then $F = F' \vee F''$ has the form $\mathfrak{J}(T,X)$, where T is a strict ordering of the union of T', T", together with the abscissae of the points of intersection of F' and F", and where $x_i = \max\{F'(t_i), F''(t_i)\}$. A similar result holds for $F' \wedge F''$, whence $\mathfrak{P}(\mathfrak{J}|_{I_o})$ is a sublattice of $C(I_o,R)$ containing $\mathfrak{J}|_{I_o}$. This and the result of Lemma 2.9 prove the next theorem.

THEOREM 2.10. <u>Suppose</u> $\mathfrak{J} \subset C(I,R)$. <u>If</u> $\mathfrak{J}$ <u>is disconjugate on</u> I_o, <u>then</u> $\mathfrak{P}(\mathfrak{J}|_{I_o})$ <u>is dense in</u> $C(I_o,R)$.

This means that any continuous function can be uniformly approximated by piecewise-$\mathfrak{J}$-linear functions on any compact interval of disconjugacy of $\mathfrak{J}$. Note that if $\mathfrak{J}$ is locally disconjugate $\mathfrak{P}(\mathfrak{F})$ need not be a sublattice, for the family

$$\mathfrak{F} = \{F(\alpha,\beta) : [F(\alpha,\beta)](t) = \alpha\cdot\cos(t)+\beta\cdot\sin(t), \alpha,\beta,t\in R\}$$

contains the sine and constant zero functions, whence $\mathfrak{P}(\mathfrak{F})$ contains them, but $(\sin) \vee (0)$ is not in $\mathfrak{P}(\mathfrak{F})$.

The theorems to follow will be seen to be of the nature of Theorem 2.10. That is, they pertain to the restrictions of the members of locally disconjugate families to intervals of disconjugacy. Or, said another way, the subsequent theorems concern families which are disconjugate on their common domain and thereby give local results for locally disconjugate families. Maximal intervals of disconjugacy are then the largest open intervals to which these local results apply.

3. *$\mathfrak{F}$-Concave and $\mathfrak{F}$-Convex Functions*. Suppose $\mathfrak{F}$ is a locally disconjugate family in $C(I,R)$. For a function $f:I\rightarrow R$ and a pair of points t_1,t_2 which belong to a common open interval of disconjugacy of $\mathfrak{F}$, let $\mathfrak{F}(f;t_1,t_2)$ represent $\mathfrak{F}(t_1,f(t_1),t_2,f(t_2))$; that is, that member of $\mathfrak{F}$ which agrees with f at t_1 and t_2. We say that $f:I\rightarrow R$ is *$\mathfrak{F}$-concave* (or *super*-$\mathfrak{F}$) if and only if for every pair t_1,t_2 which belongs to a common open interval of disconjugacy of $\mathfrak{F}$ the following holds:

(3.1) $$f\big|_{(t_1,t_2)} \geq \mathfrak{F}(f;t_1,t_2)\big|_{(t_1,t_2)}.$$

The function f is *strictly $\mathfrak{F}$-concave* if the above inequality is strict. Reversal of inequality (3.1) leads to definitions of $\mathfrak{F}$-*convex* (*sub*-$\mathfrak{F}$) and *strictly* $\mathfrak{F}$-*convex* functions. Let $\mathfrak{F}^+$ and $\mathfrak{F}^-$ denote the sets of $\mathfrak{F}$-concave and $\mathfrak{F}$-convex functions, respectively. Note that $\mathfrak{F} = \mathfrak{F}^+ \cap \mathfrak{F}^-$. In this section we will state and prove results for $\mathfrak{F}^+$. It should be kept in mind that dual theorems hold for $\mathfrak{F}^-$.

The function f is convex relative to the locally disconjugate family $\mathfrak{F}$ if and only if, for every open interval of disconjugacy, I_o, of $\mathfrak{F}$, $f|_{I_o}$ is sub-$(\mathfrak{F}|_{I_o})$ in the sense of Beckenbach [2]. Theorem 3.1 restates some of Beckenbach's results in our setting.

THEOREM 3.1. *Suppose that* $\mathfrak{F}$ *is a locally disconjugate family in* $C(I,R)$ *which is disconjugate on* I_o *and that* f *is* $\mathfrak{F}$-*concave. Then, for all* t_1, t_2 *in* I_o, $f|_A \le \mathfrak{F}(f;t_1,t_2)|_A$, *where* $A = I_o \setminus (t_1,t_2)$. *If there is a member* F *of* $\mathfrak{F}$ *and a point* t_o *in* I_o *such that* $f \ge F$ *on* I_o *and* $f(t_o) = F(t_o)$, *then* $f = F$ *on* I_o. *Moreover,* f *is continuous.*

The continuity of an $\mathfrak{F}$-concave function f at any t_o is proved using the inequalities

$$f_{o,h}(t) \le f(t) \le f_{o,-h}(t), \text{ when } t_o \le t \le t_o+h,$$

and

$$f_{o,-h}(t) \le f(t) \le f_{o,h}(t), \text{ when } t_o-h \le t \le t_o,$$

where $h > 0$ and is such that $\mathfrak{F}$ is disconjugate on $[t_o-h, t_o+h]$ and $f_{o,h} \equiv \mathfrak{F}(f;t_o,t_o+h)$ and $f_{o,-h} \equiv \mathfrak{F}(f;t_o-h,t_o)$, and the fact that $f_{o,h}$ and $f_{o,-h}$ are continuous.

A function f *supports* the $\mathfrak{F}$-line F at t_o if and only if $f(t_o) = F(t_o)$ and $F \geq f$ on some maximal interval of disconjugacy containing t_o. Green [7] attributes the following result to Peixoto [11]. Since this latter reference is not available to us we give a proof.

THEOREM 3.2. *The function* f *is* $\mathfrak{F}$-*concave if and only if* f *supports an* $\mathfrak{F}$-*line at each* t_o *in* I.

Suppose f is $\mathfrak{F}$-concave and t_o belongs to the maximal interval of disconjugacy I_o. Choose $h > 0$ such that $[t_o-h, t_o+h] \subset I_o$, and define $G_i:(0,h] \rightarrow R$, $i = 1,2$, as follows:

$$(3.2) \quad G_1(h') = f_{o,h'}(t_o+h) \text{ and } G_2(h') = f_{o,-h'}(t_o+h).$$

G_2 is nondecreasing, G_1 is nonincreasing, and $A \equiv \sup(G_1) \leq B \equiv \inf(G_2)$. Then, for any $c \in [A,B]$, f supports $\mathfrak{F}(t_o, f(t_o), t_o+h, c)$ at t_o. If f is not $\mathfrak{F}$-concave, some maximal interval contains points t_1, t', t_2 with $t_1 < t' < t_2$ for which $f(t') < [\mathfrak{F}(f;t_1,t_2)](t')$. Then any $\mathfrak{F}$-line through $(t', f(t'))$ which is not below f at one of the points t_1, t_2 is strictly below f at the other. That is, f supports no $\mathfrak{F}$-line at t'.

For a compact subinterval $K = [\tau_1, \tau_2]$ of an open interval of disconjugacy of $\mathfrak{F}$, $(\mathfrak{F}|_K)$-concavity is defined by means of (3.1) with $[t_1, t_2] \subset K$.

COROLLARY. *Suppose* $f: K \rightarrow R$. *Then* f *is* $(\mathfrak{F}|_K)$-*concave if and only if* f *supports an* $(\mathfrak{F}|_{(\tau_1, \tau_2)})$-*line at each* t *in* (τ_1, τ_2) *and* $(\liminf f)(\tau_i) \geq f(\tau_i)$, $i = 1,2$.

Since the $\mathfrak{F}|_K$-concavity of f implies that f is $(\mathfrak{F}|_{(\tau_1,\tau_2)})$-concave, we prove the corollary by showing that, under the assumption that f is $(\mathfrak{F}|_{(\tau_1,\tau_2)})$-concave, f is $(\mathfrak{F}|_K)$-concave if and only if the stated limit inferior relations hold. If $\alpha_i \equiv (\lim\inf f)(\tau_i)$ is strictly less than $f(\tau_i)$ in the extended reals for either i = 1 or i = 2, then any $\mathfrak{F}$-line, F, through $(\tau_i, f(\tau_i))$ is bounded away from α_i on some neighborhood of τ_i in K and in this neighborhood F dominates f at the points of a monotonic sequence, $(t^{(n)})$, in (τ_1,τ_2) with limit τ_i chosen such that $(f(t^{(n)}))\to\alpha_i$, whence f is not $(\mathfrak{F}|_K)$-concave. Conversely, if, with f being $(\mathfrak{F}|_{(\tau_1,\tau_2)})$-concave, f is not $(\mathfrak{F}|_K)$-concave, then there is a t' between points t_1,t_2 of K, at least one of which is an endpoint of K, such that $f(t') < F(t')$, where $F \equiv \mathfrak{F}(f;t_1,t_2)$. If only one of the points t_1,t_2, call it τ, is an endpoint of K, then, by Theorem 2.1 and the first result of Theorem 3.1, the $\mathfrak{F}$-line which agrees with f at t' and at the member of $\{t_1,t_2\}$ which is interior to K is strictly dominated by F and dominates f on the closed interval with endpoints τ and t'. Then, since $F(\tau) = f(\tau)$, $(\lim\inf f)(\tau) < f(\tau)$. If $t_1 = \tau_1$ and $t_2 = \tau_2$, suppose that at one of these endpoints, τ, $(\lim\inf f)(\tau) \geq f(\tau)$. This and the continuity result of Theorem 2.3 imply that there is a positive ε such that, for every t in $(\tau-\varepsilon,\tau+\varepsilon)\cap K$, the $\mathfrak{F}$-line which agrees with f at t and at the endpoint of K opposite τ is greater than f at t'. Then, for any such choice of t except τ, the former argument applies, whence f is greater than (lim inf f) at the endpoint of K opposite τ.

REMARK. If f is $\mathfrak{F}_K$-concave ($\mathfrak{F}|_K$-concave) and if

$$\alpha = (\liminf f)(\tau) < (\limsup f)(\tau) = \beta$$

for an endpoint τ of K, there are monotonic sequences (t_n^+) and (t_n^-) in (τ_1, τ_2) whose elements are interlaced, each with limit τ, such that $(f(t_n^+)) \to \beta$, $(f(t_n^-)) \to \alpha$. Then, for sufficiently large n, the $\mathfrak{F}$-line which agrees with f at t_n^+ and t_{n+1}^+ is above f at the element of (t_n^-) between these points, a contradiction. Therefore, $\alpha = \beta$. Thus, _every $\mathfrak{F}_K$-concave function agrees with a unique continuous $\mathfrak{F}_K$-concave function on the interior of_ K.

The following lemma implies that neither $\mathfrak{F}^+ \setminus \mathfrak{F}$ nor $\mathfrak{F}^- \setminus \mathfrak{F}$ is empty. (The function $\mathfrak{F}(T,X)$, used below, was defined in Section 2.)

LEMMA 3.3. _Suppose_ F _is a member of the locally disconjugate family_ $\mathfrak{F}$ _which is disconjugate on_ $[t_1, t_2]$ _and_ $t_o \in (t_1, t_2)$. _Then, if_ k _is positive, [negative], the function_

$$f = \mathfrak{F}(\{t_1, t_o, t_2\}, \{F(t_1), F(t_o)+k, F(t_2)\})$$

_is_ $\mathfrak{F}$_-concave, [_$\mathfrak{F}$_-convex], but not_ $\mathfrak{F}$_-linear_.

Clearly, f is not $\mathfrak{F}$-linear. Now, with $k > 0$, suppose that t_o is interior to an interval of disconjugacy $[t_1', t_2']$ of $\mathfrak{F}$ and define F' to be $\mathfrak{F}(f; t_1', t_2')$. On some interval I' containing t_o, t_1, t_2, t_1', and t_2', $f = F_1 \wedge F_2$ where

$$F_1 = \mathfrak{F}(t_1, F(t_1), t_o, F(t_o)+k) = \mathfrak{F}(f; t_1', t_o)$$

and

$$F_2 = \mathfrak{F}(t_o, F(t_o)+k, t_2, F(t_2)) = \mathfrak{F}(f; t_o, t_2').$$

This and the facts that $F_1(t_1') < F_2(t_1')$ and $F_1(t_2') > F_2(t_2')$ imply that F' is strictly less than both F_1 and F_2 on (t_1', t_2'), whence $F' \leq f$ on $[t_1', t_2']$.

COROLLARY. *If* $[t', t_o]$ *is contained in an open interval of disconjugacy of* $\mathfrak{F}$ *and* α, β, *and* γ *are reals with* $\alpha < \beta$, *then* f *such that* f *equals* $\mathfrak{F}(t', \alpha, t_o, \gamma)$ *on* $(a, t_o]$, f *equals* $\mathfrak{F}(t', \beta, t_o, \gamma)$ *on* $[t_o, b)$, *is* $\mathfrak{F}$-*concave but not* $\mathfrak{F}$-*linear*.

Now if $\mathfrak{F}$ is disconjugate on $I_o = (a_o, b_o)$, $F \in \mathfrak{F}$, and (t_n') and (t_n'') are, respectively, decreasing and increasing sequences in I_o with respective limits a_o and b_o, then, for any t_o in I_o and for sufficiently large n, the function g_n defined as $\mathfrak{F}(\{t_n', t_o, t_n''\}, \{F(t_n'), F(t_o)+1, F(t_n'')\})$ belongs to $\mathfrak{F}^+ \setminus \mathfrak{F}$. For a fixed t' in (a_o, t_o) and again for sufficiently large n, the sequences (c_n) and (d_n), where $c_n \equiv [\mathfrak{F}(t_o, F(t_o)+1, t_n'', F(t_n''))](t')$ and $d_n \equiv g_n(t')$, are, respectively, decreasing and increasing, and $d_n < c_n$. Then (d_n) and (c_n) have limits A and B with $A \leq B$. Define f as in the corollary to Lemma 3.3 taking $\alpha = A$, $\beta = B$, and $\gamma = F(t_o)+1$. Clearly, f is $\mathfrak{F}$-concave and $f > F$ on I_o. Indeed, $f = \lim (g_n)$. Since f supports an $\mathfrak{F}$-line (possibly itself) at t_o, there is an $\mathfrak{F}$-line which is strictly greater than F on I_o.

The results of Section 4 are based on the following statement of this result.

THEOREM 3.4. Every member of a locally disconjugate family $\mathfrak{F}$ is strictly dominated by some member of the family on an open interval of disconjugacy.

The proof given above does not require that a_o or b_o belong to I. But if I_o is bounded in I, then it is clear that F is dominated on I_o by any $\mathfrak{F}$-line supported by $\mathfrak{F}(\{a_o,t_o,b_o\},\{F(a_o),F(t_o)+1,F(b_o)\})$ at t_o. Moreover, given I_o bounded in I, if the construction in the proof yields A = B for any F in $\mathfrak{F}$, then I_o is maximal.

Note that functions are convex in the usual sense if and only if they are $\mathfrak{X}$-convex where $\mathfrak{X}$ is the disconjugate family

$$\{L(\alpha,\beta):[L(\alpha,\beta)](x) = \alpha\cdot x+\beta,\ \alpha,\beta\in R,\ x\in I\}.$$

In view of this, the following theorems are not unexpected.

It suffices to suppose $\mathfrak{F}\subset C(I,R)$ to be disconjugate on I. Let $\mathcal{K}$ represent the collection of compact subintervals of I and, for K in $\mathcal{K}$, let $\mathfrak{F}_K$ stand for $\mathfrak{F}|_K$. A function in C(I,R) is $\mathfrak{F}$-concave on K if and only if its restriction to K is $\mathfrak{F}_K$-concave. The collection of all such functions is denoted by $\mathfrak{F}_K^+$. Note that $K_1\subset K_2$ implies $\mathfrak{F}^+\subset\mathfrak{F}_{K_2}^+\subset\mathfrak{F}_{K_1}^+$.

If $K = [t_1,t_2]\subset I$ and $\alpha\in R$, consideration of the sequence (F_n), where $F_n = \mathfrak{F}(t_1',\alpha,t_2,n)$, t_1' fixed in (a,t_1), leads to the conclusion that some $\mathfrak{F}$-line is never below α in K.

Then every element of $C(I,R)$ is dominated on each K in $\mathcal{K}$ by some $\mathfrak{F}$-line, so that the operator θ^+, which, for K in $\mathcal{K}$, maps $C(I,R)$ into the set of real-valued functions on K according to the rule

$$[\theta_K^+(f)](t) = \inf\{F(t) : F \in \mathfrak{F}, F \geq f \text{ on } K\}$$

is well-defined. Further, if f is dominated on I by an $\mathfrak{F}$-line, we define $[\theta^+(f)](t)$ to be $\inf\{F(t) : F \in \mathfrak{F}, F \geq f\}$. The domain, D_{θ^+}, of θ^+ contains at least $\mathfrak{F}^+$ and the bounded-above members of $C(I,R)$. For any pair K_1, K_2 in $\mathcal{K}$ with $K_1 \subset K_2$, $f \leq \theta_{K_1}^+(f) \leq \theta_{K_2}^+(f)$ on K_1 for f in $C(I,R)$; for f in D_{θ^+}, each of these is dominated by $\theta^+(f)$ on K_1. Also, if $f \leq g$, $f, g \in C(I,R)$, then $\theta_K^+(f) \leq \theta_K^+(g)$ for all K in $\mathcal{K}$ and $\theta^+(f) \leq \theta^+(g)$ if $g \in D_{\theta^+}$. Thus each θ_K^+ and θ^+ are order-preserving. Moreover, $\theta_K^+(f)$ and $\theta^+(f)$, when defined, are, respectively, $\mathfrak{F}_K$-concave and $\mathfrak{F}$-concave. For if, for some $K \in \mathcal{K}$ and some $f \in C(I,R)$, there are points t_1, t', t_2 in K such that $t_1 < t' < t_2$ and for which

$$[\theta_K^+(f)](t') < [\mathfrak{F}(\theta_K^+(f); t_1, t_2)](t'),$$

then any member F of $\mathfrak{F}$ which satisfies

$$[\theta_K^+(f)](t') \leq F(t') < [\mathfrak{F}(\theta_K^+(f); t_1, t_2)](t')$$

and which dominates f on K is not below $\theta_K^+(f)$ at either t_1 or t_2, whence, by Theorem 2.1, F dominates $\mathfrak{F}(\theta_K^+(f); t_1, t_2)$ on $[t_1, t_2]$, a contradiction. This latest result and

Theorem 3.2 and its corollary yield the following lemma.

LEMMA 3.5. <u>The function</u> f <u>in</u> C(I,R) <u>is super</u>-$\mathfrak{F}$, [<u>super</u>-$\mathfrak{F}_K$, <u>where</u> $K\in\mathcal{K}$], <u>if and only if</u> $\theta^+(f) = f$, [$\theta_K^+(f) = f|_K$].

According to Lemma 3.5, θ^+ and θ_K^+, K in $\mathcal{K}$, are idempotent; that is, $\theta^+\circ\theta^+ = \theta^+$ and $\theta_K^+\circ\theta_K^+ = \theta_K^+$. Moreover, $\theta_K^+(f)$, being the pointwise infimum of continuous functions, is upper semicontinuous, and, by Theorem 3.1 and the corollary to Theorem 3.2, it is lower semicontinuous, whence the following corollary is immediate.

COROLLARY. <u>If</u> $K\in\mathcal{K}$ <u>and</u> $f\in C(I,R)$, <u>then</u> $\theta_K^+(f)$ <u>is continuous</u>.

It is clear from Theorem 3.2 and the definition of θ^+ that <u>no</u> $\mathfrak{F}$-<u>concave function that dominates</u> f <u>can be below</u> $\theta^+(f)$ <u>at any point</u>. Thus we have:

THEOREM 3.6. <u>Suppose</u> $\mathfrak{F}\subset C(I,R)$ <u>is disconjugate on</u> I. <u>If</u> f <u>in</u> C(I,R) <u>is dominated by some</u> $\mathfrak{F}$-<u>line</u>, $\theta^+(f)$ <u>is the least</u> $\mathfrak{F}$-<u>concave function dominating</u> f. <u>For</u> g <u>in</u> C(I,R), <u>and</u> $K\in\mathcal{K}$, $\theta_K^+(g)$ <u>is the least</u> $\mathfrak{F}_K$-<u>concave function dominating</u> g <u>on</u> K.

Now if $K = [t_1,t_2]$, $g\in C(I,R)$, and $\varepsilon > 0$, then for all sufficiently large n the function $F_n = \mathfrak{F}(t_1, g(t_1)+\varepsilon, t_2, n)$ dominates g on K, else g is not bounded on K. Thus, $[\theta_K^+(g)](t_1) = g(t_1)$. Similarly, $[\theta_K^+(g)](t_2) = g(t_2)$. Therefore $\theta_K^+(g)$ belongs to the set of members of C(K,R) which agree with g at the endpoints of K and which dominate $g|_K$. We label this set $C_K^+(g)$.

COROLLARY. For K in $\mathcal{K}$ and g in C(I,R), $\theta_K^+(g)$ is the least $\mathfrak{F}_K$-concave member of $C_K^+(g)$.

In Theorem 3.7 we give a different characterization of θ_K^+. For K in $\mathcal{K}$ let $\mathcal{T}_K$ be the set of finite subsets of K which contain the endpoints of K. For $T = \{t_1, \cdots, t_n\}$ in $\mathcal{T}_K$ with $t_1 < \cdots < t_n$ and $f: I \to R$, let $\mathfrak{F}(f;T)$ represent $\mathfrak{F}(\{t_i : i \in \hat{n}\}, \{f(t_i) : i \in \hat{n}\})$. For f in C(I,R) and for t in K,

$$[\psi_K^+(f)](t) \equiv \sup \{[\mathfrak{F}(f;T)](t) : T \in \mathcal{T}_K\}.$$

Then, for any F in $\mathfrak{F}$ such that $F \geq f$ on K and for any T in $\mathcal{T}_K$, $F \geq \mathfrak{F}(f;T)$ on K, whence $\theta_K^+(f) \geq \psi_K^+(f) \geq f|_K$. Also $[\psi_K^+(f)](t_i) = f(t_i)$, i = 1,2. Next we show that for t interior to $K = [t_1, t_2]$

(3.3) $[\psi_K^+(f)](t) = \sup\{[\mathfrak{F}(f;t_1', t_2')](t) : t_1' < t < t_2', t_1', t_2' \in K\}$.

Since $\{t_1, t_1', t_2', t_2\} \in \mathcal{T}_K$, $[\psi_K^+(f)](t)$ is as least as great as the right-hand entity in (3.3). Now if a sequence (T_n) in $\mathcal{T}_K$ such that $[\psi_K^+(f)](t) - 1/n < [\mathfrak{F}(f;T_n)](t)$ is such that no T_n contains t, then for each n there is an integer i_n such that $t_{i_n} < t < t_{i_n+1}$ and $[\psi_K^+(f)](t)$ equals $\lim [\mathfrak{F}(f;t_{i_n}, t_{i_n+1})](t)$. Otherwise, every sequence (T_n) in $\mathcal{T}_K$ for which $x \equiv [\psi_K^+(f)](t)$ equals $\lim([\mathfrak{F}(f;T_n)](t))$ is such that t belongs to each T_n for sufficiently large n. Then, with t_1' chosen less than t in K and with $F_n \equiv \mathfrak{F}(t_1', f(t_1'), t, x-1/n)$, F_n does not intersect f in $(t, t_2]$ for sufficiently large n, whence, for a choice of t_2' in

$(t, t_2]$, the increasing sequence $(F_n(t_2'))$ is bounded above by $f(t_2')$. Then, either $(F_n(t_2'))$ has limit $f(t_2')$, so that $x = [\mathfrak{F}(f;t_1',t_2')](t)$, contradicting the assumption concerning sequences (T_n), or $(F_n(t_2')) \to \alpha < f(t_2')$, whence $x < [\mathfrak{F}(f;t_1',t_2')](t)$, contradicting the definition of x.

As a result of (3.3) there are sequences $(t_{1,n}')$ in $[t_1, t)$ and $(t_{2,n}')$ in $(t, t_2]$ with respective limits L_1 in $[t_1, t]$ and L_2 in $[t, t_2]$ such that

$$(3.4) \qquad [\psi_K^+(f)](t) = \lim([\mathfrak{F}(f;t_{1,n}',t_{2,n}')](t))$$

and (3.4) equals $[\lim(\mathfrak{F}(f;t_{1,n}',t_{2,n}'))](t)$, whenever this indicated limit exists. If $L_1 = t$ or $L_2 = t$ then $[\psi_K^+(f)](t) = f(t)$ and f supports an $\mathfrak{F}$-line at t. If, for some pair of sequences $(t_{1,n}'),(t_{2,n}')$, $L_1 < t < L_2$, then

$$\lim (\mathfrak{F}(f;t_{1,n}',t_{2,n}')) = \mathfrak{F}(f;L_1,L_2),$$

and $\psi_K^+(f)$ supports $F \equiv \mathfrak{F}(f;L_1,L_2)$ at t. For, if there is a t_o in K at which $\psi_K^+(f)$ dominates F, then, by (3.3), there is a t' in K, different from t, L_1, L_2, and t_o, at which f dominates F, which implies that one of the functions $\mathfrak{F}(f;L_1,t')$, $\mathfrak{F}(f;t',L_2)$ is strictly above F at t, contradicting the equality $F(t) = [\psi_K^+(f)](t)$. Also, $\psi_K^+(f)$, being the supremum of a collection of continuous functions, is lower semicontinuous. Then, by the Corollary to Theorem 3.2, $\psi_K^+(f)$ is $\mathfrak{F}_K$-concave and, by Theorem 3.6, $\psi_K^+(f) = \theta_K^+(f)$.

THEOREM 3.7. <u>If</u> $\mathfrak{F}$ <u>is disconjugate on</u> I, $K \in \mathcal{K}$, <u>and</u>

$f \in C(I,R)$, _then_ $[\theta_K^+(f)](t) = \sup\{[\mathfrak{F}(f;T)](t) : T \in \mathcal{T}_K\}$ _for_ t _in_ K.

4. _Locally disconjugate linear subspaces of_ $C(I,R)$. Suppose that $\mathfrak{F}$ is a locally disconjugate linear subspace of $C(I,R)$, that $\mathcal{K}$ is the set of compact subintervals of I, that $\mathfrak{d}(\mathfrak{F})$ is the set of maximal intervals of disconjugacy of $\mathfrak{F}$, and that $\mathcal{K}(\mathfrak{F})$ is the set of compact subintervals of members of $\mathfrak{d}(\mathfrak{F})$. By Theorem 2.3, for a fixed pair t_1, t_2 in some I_o belonging to $\mathfrak{d}(\mathfrak{F})$, the map

$$\phi : R^2 \to C(I,R) :: \phi(x_1,x_2) = \mathfrak{F}(t_1,x_1,t_2,x_2) = x_1 \cdot F + x_2 \cdot G,$$

where $F = \mathfrak{F}(t_1,1,t_2,0)$ and $G = \mathfrak{F}(t_1,0,t_2,1)$, is a topological isomorphism onto $\mathfrak{F}$. When the determinant $D(t_1',t_2') \equiv F(t_1')\cdot G(t_2') - F(t_2')\cdot G(t_1')$ is not zero, $\phi(x_1,x_2)$ is the unique $\mathfrak{F}$-line passing through (t_1',x_1') and (t_2',x_2') if and only if

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\begin{pmatrix} G(t_2') & -G(t_1') \\ -F(t_2') & F(t_1') \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}}{D(t_1',t_2')} .$$

The local disconjugacy of $\mathfrak{F}$ implies that F and G have no zeroes in common, whence $D(t_1',t_2') = 0$ if and only if the matrix

$$\begin{pmatrix} F(t_1') & G(t_1') \\ F(t_2') & G(t_2') \end{pmatrix}$$

has rank one, and this occurs if and only if there are points (x_1', x_2') and (x_1'', x_2'') in R^2 such that two $\mathfrak{F}$-lines pass through (t_1', x_1') and (t_2', x_2') and no $\mathfrak{F}$-lines pass through (t_1', x_1'') and (t_2', x_2''). Thus, $D(t_1', t_2') = 0$ if and only if $t_1' \sim t_2'$. It follows, then, that for every basis $\{F, G\}$ of $\mathfrak{F}$ (as a vector space) and for every point t_o in I, $D(t_1', t_2')$ is of constant sign for all t_1', t_2' with $t_1' < t_2'$ in some neighborhood of t_o.

Also, $t_1' \sim t_2'$ if and only if the one-dimensional subspace of $\mathfrak{F}$ consisting of those $\mathfrak{F}$-lines which pass through the point $(t_1', 0)$ is precisely the one-dimensional subspace of $\mathfrak{F}$ consisting of the $\mathfrak{F}$-lines passing through $(t_2', 0)$. Therefore, when $\mathfrak{F}$ is a linear space the relation ~ satisfies the following transitivity condition: if any two members of a triple of distinct points in I are $\mathfrak{F}$-related to the third, then these two are $\mathfrak{F}$-related to each other. Thus, the set of $\mathfrak{F}$-relatives of t_1' is the set of zeroes of any nonzero member of $\mathfrak{F}$ which passes through $(t_1', 0)$ and each of the zeroes of such a function is $\mathfrak{F}$-related to all the others. If any such set had an accumulation point t_o in I, every neighborhood of t_o would contain $\mathfrak{F}$-related pairs, a contradiction. Therefore, <u>the set of zeroes of any nonzero member of $\mathfrak{F}$ has no accumulation point in</u> I.

It follows, then, that if $\mathfrak{F}$ is not disconjugate on all of I the maps $\bar{\rho}$ and $\bar{\lambda}$ of Theorem 2.7 yield the least right-$\mathfrak{F}$-relatives and the greatest left-$\mathfrak{F}$-relatives, respectively, of their arguments, that $\bar{\rho}: [a, t_\rho] \to [t_\lambda, b]$

and $\bar{\lambda}: [t_\lambda, b] \to [a, t_\rho]$ are inverse homeomorphisms, and that every t_o in I is interior to a bounded maximal interval of disconjugacy whose endpoints are consecutive zeroes of some nonzero $\mathfrak{F}$-line. Further, the iterates of $\bar{\rho}$ and $\bar{\lambda}$ applied to any point in I generate the $\mathfrak{F}$-relatives (now called the $\mathfrak{F}$-<u>conjugates</u>) of that point.

Suppose t_1' and t_2' are zeroes of the nonzero $\mathfrak{F}$-line F and that $\{F,G\}$ is a basis for $\mathfrak{F}$. G is not zero at either t_1' or t_2'. If G has no zero in (t_1', t_2'), then G is strictly positive or strictly negative on $[t_1', t_2']$, whence some multiple of F intersects G at least twice in $(t_1', \bar{\rho}(t_1'))$, a contradiction. Thus, <u>between any pair of zeroes of one member of a basis for $\mathfrak{F}$, there is a zero of the other member of the basis</u>.

Now suppose that $\mathfrak{F}$ is disconjugate on $I_o = (a_o, b_o)$. According to Theorem 3.4 some member, F, of $\mathfrak{F}$ is strictly positive on I_o and, by linearity, F can be chosen to take on the value 1 at any fixed t_1 in I_o. For such a t_1 and F, choose t_2 in (t_1, b_o) and define G to be $\mathfrak{F}(t_1, 0, t_2, 1)$. Then $\{F,G\}$ is a basis for $\mathfrak{F}$, and the constancy of sign of $D(t_1', t_2')$ for $a_o < t_1' < t_2' < b_o$ and the fact that $D(t_1, t_2) = 1$ imply that the continuous function $T_1 \equiv (G/F): I_o \to R$ is strictly increasing. Let I' represent the range of T_1. Note that if I_o is bounded and maximal then $F(a_o) = 0 = F(b_o)$ and $G(a_o) < 0 < G(b_o)$, whence I' is R. For convenience, let t be the <u>map</u> $(T_1)^{-1}$. That is, t is the strictly increasing homeomorphism of I' onto I_o such that $t(s_o) = t_o$ if and only if $s_o \cdot F(t_o) = G(t_o)$. Also, let $\mathfrak{L}$ be the family of straight lines, $\alpha \cdot s' + \beta$, for s' in I'. $\mathfrak{L}$ is a linear subspace of $C(I',R)$ which is disconjugate on I'.

Consider the map $T: I_o \times R \to I' \times R$ defined by

$$T(t_o, x_o) = (T_1(t_o), T_2(t_o, x_o)) = (G(t_o)/F(t_o), x_o/F(t_o)).$$

T is a homeomorphism of $I_o \times R$ onto $I' \times R$ which takes vertical lines into vertical lines. Such a map induces a homeomorphism, T^*, of $C(I_o, R)$ onto $C(I', R)$ in the following way:

$$T^*(f) \text{ at } s_o \text{ in } I' \text{ is } T_2((T_1)^{-1}(s_o), f((T_1)^{-1}(s_o))).$$

In our case, $[T^*(f)](s_o) = f(t(s_o))/F(t(s_o))$, denoted by $T^*(f) = (f/F) \circ (G/F)^{-1}$, and T^* is linear and positive. That is, if f is a nonnegative function on I_o, then $T^*(f)$ is a nonnegative function on I'. Moreover, $[T^*(\alpha \cdot F + \beta \cdot G)](s) = \alpha + \beta \cdot s$. More precisely, the image under T^* of the $\mathfrak{F}$-line determined by the points (t_1, x_1) and (t_2, x_2) in $I_o \times R$ is the $\mathfrak{L}$-line (straight line) through the points $T(t_1, x_1)$ and $T(t_2, x_2)$. This and the positivity of T^* yield the result that f is $\mathfrak{F}$-concave, [$\mathfrak{F}$-convex], on I_o if and only if $T^*(f)$ is concave, [convex], in the usual sense. Note also that s_o is a zero of $T^*(f)$ if and only if $t(s_o)$ is a zero of f. Let $\mathfrak{L}^+$ and $\mathfrak{L}^-$ denote the sets of functions on I' which are concave and convex, respectively.

THEOREM 4.1. <u>Suppose I_o is an interval of disconjugacy</u>

of the locally disconjugate linear subspace $\mathfrak{F}$ *of* $C(I,R)$. *Then there is an interval* I' *in* R *and a homeomorphism* T *of* $I_o \times R$ *onto* $I' \times R$ *which takes vertical lines to vertical lines and which induces a positive, linear homeomorphism* T^* *of* $C(I_o,R)$ *onto* $C(I',R)$ *which maps* $\mathfrak{F}_{I_o}$ *onto* $\mathfrak{L}$, $\mathfrak{F}^+_{I_o}$ *onto* $\mathfrak{L}^+$, *and* $\mathfrak{F}^-_{I_o}$ *onto* $\mathfrak{L}^-$. *Moreover, if* I_o *is bounded and maximal then* I' *equals* R.

In particular, if a linear subspace of $C(I,R)$ is disconjugate on I then the graphs of its members in $I \times R$ are the images of the nonvertical straight line segments in some strip $I' \times R$ under a homeomorphism that preserves vertical lines.

Theorem 4.1 yields a local integral representation for $\mathfrak{F}$-concave functions. First note that, for a fixed choice of I_o, F, and G, $(T^*)^{-1}(z) = F\cdot(z\circ(G/F))$ for any z in $C(I',R)$. It is well known (see Natanson [10;p.230]) that $z: I' \to R$ is concave in the usual sense if and only if it is expressible as an indefinite integral of a nonincreasing function $\phi: I' \to R$ which is bounded on compacta in I'. Then $f: I \to R$ is $\mathfrak{F}$-concave on I_o if and only if $T^*(f)$ has the form

$$[T^*(f)](s) = [T^*(f)](s_o) + \int_{s_o}^{s} \phi \text{ for all } s \text{ in } I',$$

for some real-valued nonincreasing function ϕ. Applying $(T^*)^{-1}$ to the above expression we have

$$f(t) = F(t)\cdot\left(\frac{f(t_o)}{F(t_o)} + \int_{(G/F)(t_o)}^{(G/F)(t)} \phi \right)$$

for all t in I_o. Note that $\phi\circ(G/F)$ is real-valued and nonincreasing on I_o.

THEOREM 4.2. <u>Suppose</u> $\mathfrak{F}$ <u>is a linear subspace of</u> $C(I,R)$ <u>which is disconjugate on</u> I_o. <u>Then</u> f <u>is</u> $\mathfrak{F}$<u>-concave on</u> I_o <u>if and only if for every basis</u> $\{F,G\}$ <u>for</u> $\mathfrak{F}$ <u>with</u> $F > 0$ <u>and</u> (G/F) <u>strictly increasing on</u> I_o <u>and for</u> t_o <u>fixed in</u> I_o, <u>there is a nonincreasing real-valued function</u> $\psi : I_o \to R$ <u>such that</u>

$$f(t) = F(t)\cdot\left(\frac{f(t_o)}{F(t_o)} + \int_{t_o}^{t} \psi\cdot d(G/F)\right)$$

<u>for</u> t <u>in</u> I_o.

Now, assuming the hypotheses of Theorem 4.2 and that F and G are as described there, we construct a linear operator in $C(I_o,R)$ whose solution set is precisely $\mathfrak{F}$ (that is, $\mathfrak{F}_{I_o}$). Any linear operator $\theta:\mathcal{D}\to C(I',R)$ in $C(I',R)$ can be "pulled back" to a linear operator $\theta^*:(T^*)^{-1}\mathcal{D}\to C(I_o,R)$ in $C(I_o,R)$ by defining θ^* as $(T^*)^{-1}\circ\theta\circ T^*$. Then $\theta^*(f) = 0, [\geq 0, \leq 0]$, if and only if $\theta(T^*(f)) = 0, [\geq 0, \leq 0]$, so that $\ker\theta^* = (T^*)^{-1}(\ker\theta)$. We consider the second order linear differential operator

$$D^2 : C^2(I',R)\to C(I',R) \;::\; D^2x = x''$$

because the solution set of $D^2x = 0$, namely $\mathfrak{X}$, is $T^*(\mathfrak{F})$. The operator $(D^2)^* = (T^*)^{-1}\circ D^2\circ T^*$ is called a <u>generalized second-order linear differential operator in</u> $C(I_o,R)$ <u>determined by</u> $\mathfrak{F}$. $\mathfrak{F}$ is the set of solutions of $[(D^2)^*](u) = 0$.

The operator D^2 restricted to the set of all C^2 functions which have fixed values at two fixed points in I' has an inverse; that is, for y continuous on I', y_1 and y_2 arbitrary reals, and s_1 and s_2 in I', there is a unique member z of $C^2(I',R)$ such that

$$D^2(z) = y \text{ and } z(s_i) = y_i, \ i = 1,2,$$

which we denote by $(D^2)^{-1}(y;s_1,y_1,s_2,y_2)$. Explicitly,

$$z(s) = [\mathfrak{L}(s_1,y_1,s_2,y_2)](s) + \int_{s_1}^{s_2} K(s,\sigma)\cdot y(\sigma)\cdot d\sigma$$

where the Green's function K is given by

$$K(s,\sigma) = \frac{(s-s_2)\cdot(\sigma-s_1)}{s_2-s_1}, \ \sigma \le s,$$

$$= \frac{(s-s_1)\cdot(\sigma-s_2)}{s_2-s_1}, \ s \le \sigma.$$

Now $[(D^2)^*](f) = g$ and $f(t_i) = g_i$, $i = 1,2$, if and only if $D^2(T^*(f)) = T^*(g)$ and $[T^*(f)](s_i) = y_i$, $i = 1,2$, with $s_i = (G/F)(t_i)$ and $y_i = g_i/F(t_i)$, $i = 1,2$. This determines $T^*(f)$, hence f, uniquely, and we write $f = ((D^2)^*)^{-1}(g;t_1,g_1,t_2,g_2)$. An explicit representation of f is given below.

THEOREM 4.3. <u>Suppose I_o is an interval of disconjugacy of the linear subspace $\mathfrak{F}$ of $C(I,R)$. A basis $\{F,G\}$ for $\mathfrak{F}$ with $F > 0$ and (G/F) strictly increasing on I_o determines a generalized second-order differential operator $(D^2)^*$ in</u>

$C(I_o,R)$ *whose kernel is* $\mathfrak{F}$ *and for which the two-point boundary-value problem*

$$(D^2)^*(f) = g,\ f(t_i) = g_i,\ i = 1,2,\ t_1 < t_2 \text{ in } I_o,\ g \in C(I_o,R),$$

is uniquely solvable, with solution given by

$$f(t) = [\mathfrak{F}(t_1,g_1,t_2,g_2)]\,(t)$$

$$+$$

$$\int_{(G/F)(t_1)}^{(G/F)(t_2)} \frac{F(t)}{F((G/F)^{-1}(\sigma))} \cdot K\left(\frac{G(t)}{F(t)},\sigma\right)\cdot g((G/F)^{-1}(\sigma))\cdot d\sigma.$$

If (G/F) *is absolutely continuous the integral above can be expressed in terms of a Green's function*

$$K^*(t,\tau) = \frac{F(t)}{F(\tau)} \cdot K\left(\frac{G(t)}{F(t)},\ \frac{G(\tau)}{F(\tau)}\right)\cdot\ [(G/F)']\,(\tau)$$

as

$$f(t) = [\mathfrak{F}(t_1,g_1,t_2,g_2)](t) + \int_{t_1}^{t_2} K(t,\tau)\cdot g(\tau)\cdot d\tau.$$

In view of the result of Lemma 4.4 below, the following corollary includes Bonsall's conclusion [3;Thm.6] that, with $L(u) = u''+p_1\cdot u'+p_2\cdot u$, p_1 and p_2 being continuous, if $f \in C^2(I_o,R)$ and $L(f) \le 0$, $[L(f) < 0]$, then f is concave, [strictly concave], with respect to the set of solutions of $L(u) = 0$ on the interval of disconjugacy I_o. The corollary itself follows from the concavity-preserving property of T* and from the integral representation of concave functions.

COROLLARY. *Suppose* f *belongs to the domain of* $(D^2)^*$. *Then* f *is* $\mathfrak{F}$*-concave on* I_o *if and only if* $(D^2)^*(f) \le 0$.

Moreover, if $(D^2)^*(f) < 0$, *then* f *is strictly* $\mathfrak{F}$*-concave on* I_o.

The following lemma is a result of differentiation.

LEMMA 4.4. *Suppose that* p *is continuous on* I, r *and* q *are continuously differentiable on* I *with* r *positive, and that the differential equation* $L(u) = 0$ *is disconjugate on* I_o, *where*

$$L:C^2(I,R)\to C(I,R) :: L(u) = (r\cdot u'+q\cdot u)'-(q\cdot u'+p\cdot u).$$

If F *solves* $L(u) = 0$ *and is never zero on* I_o, *if* G *is the solution to* $L(u) = 0$ *given by*

$$G(t) = F(t)\cdot\int_{t_o}^{t}\frac{d\tau}{r(\tau)\cdot(F(\tau))^2}$$

and if $T^*(f) \equiv (f/F)\circ(G/F)^{-1}$, *then* $(D^2)^*(f) = r\cdot F^4\cdot L(f)$ *for* f *in* $C^2(I_o,R)$.

The local nature of the results of Theorems 4.1, 4.2, and 4.3 can be summarized by saying that a *locally disconjugate linear subspace of* $C(I,R)$ *gives* $I\times R$ *the structure of a* C° *2-manifold*. (see [8;p.2]). For given such a subspace $\mathfrak{F}$ and a basis $\{F,G\}$ of $\mathfrak{F}$, every t_o in I belongs to at least one of the disjoint intervals which make up the sets $I\setminus F^{-1}(0)$ and $I\setminus G^{-1}(0)$, so that every (t_o,x_o) in $I\times R$ belongs to at least one strip, $I_o\times R$, which is in a one-to-one correspondence with an open set, $I'\times R$, in R^2 via a map having one of the forms

$T'(t,x) = (G(t)/F(t), x/F(t)), T''(t,x) = (F(t)/G(t), x/G(t))$.

Moreover, $T' \circ (T'')^{-1}$ and $T'' \circ (T')^{-1}$ are, when defined, inverse homeomorphisms, since these have the forms
$(s,y) \to ([(G/F) \circ (F/G)^{-1}](s),\ y \cdot ([(G/F) \circ (F/G)^{-1}](s))$ and
$(s,y) \to ([(F/G) \circ (G/F)^{-1}](s),\ y \cdot ([(F/G) \circ (G/F)^{-1}](s))$ respectively.

Ashley [1] has applied a theorem of Choquet [4;p.237], which we state below as Theorem 4.5, to concavity as defined by the family, $\mathcal{S}_K$, of restrictions of solutions to (1.1) to a compact interval, K, of disconjugacy of (1.1), and asserts the existence of certain integral representations for the elements of the cone of nonnegative, continuous $\mathcal{S}_K$-concave functions. We extend his assertion to locally disconjugate linear subspaces, $\mathfrak{F}$, of C(I,R) and make the asserted representations explicit. Lemmas 4.7 and 4.8 and Theorem 4.9 are versions of Ashley's results in our more general setting. Lemma 4.10 is a generalization of a result sought by Ashley.

THEOREM 4.5. <u>Every point in a convex, compact subset</u> X <u>of a Hausdorff locally convex topological vector space is the center of gravity of a probability measure</u> (<u>nonnegative Radon measure of unit mass</u>) <u>on the closure of the set of extreme points of</u> X.

Let V be a real vector space. The <u>segment</u> $\overline{xy}$ with endpoint x and y in V is the set $\{(1-\lambda)\cdot x+\lambda\cdot y : \lambda\in[0,1]\}$. The set $\overline{xy}\setminus\{x,y\}$ is called the <u>interior</u> of $\overline{xy}$. The subset

S of V is <u>convex</u> if and only if, for every pair x,y in S, S contains the segment $\overline{xy}$. The point x of the convex set S is <u>extreme</u>, denoted by $x \in ex(S)$, if x is interior to no segment in S. The subset S of V is a <u>cone</u> (<u>convex cone</u>) if and only if S contains the zero element of V and is closed under addition and nonnegative scalar multiplication. <u>Radon measures</u> are continuous linear functionals on the space of continuous real-valued functions with compact support in a locally compact space given the inductive limit topology. As a general reference we offer Edwards [5;Chap. 4]. If V is a topological vector space, the point x is the <u>center of gravity</u> of the measure μ on $X \subset V$ (sometimes said "μ <u>represents</u> x") if and only if $F(x) = \int_X F \cdot d\mu$ for every continuous linear functional F on V.

Let $K = [t_1, t_2]$ be a fixed but arbitrary member of $\mathfrak{K}(\mathfrak{F})$ with t_o a fixed point interior to K; let C(K) be the set of nonnegative super-$\mathfrak{F}_K$ functions in C(K,R); and let $B(K,t_o)$ be the set of elements f in C(K) such that $f(t_o) = 1$. Where the meaning is clear we use C and B to denote C(K) and $B(K,t_o)$, respectively. Note that the constant zero function belongs to C and that the identities

$$\mathfrak{F}(f;t_1',t_2') + \mathfrak{F}(g;t_1',t_2') = \mathfrak{F}(f+g;t_1',t_2'),$$

$$\mathfrak{F}(\lambda \cdot f;t_1',t_2') = \lambda \cdot \mathfrak{F}(f;t_1',t_2'),$$

for $[t_1',t_2'] \subset K$, which follow from the last sentence in

Theorem 2.3, imply that C is closed under addition and non-negative scalar multiplication, so that C is a convex cone. Then the functions $h_\lambda \equiv (1-\lambda)\cdot f+\lambda\cdot g$ for $\lambda \in [0,1]$, where f and g are in B, belong to C and $h_\lambda(t_o) = 1$. Also, a nonzero element, f, of C is positive at t_o by the second result of Theorem 3.1, whence $g \equiv f/f(t_o)$ belongs to B. Thus we have:

LEMMA 4.6. _The set_ B _is a convex subset of the convex cone_ C _in_ C(K,R) _such that for every_ f _in_ C _there is a unique_ g _in_ B _for which_ $f = f(t_o)\cdot g$.

The functions $F_1 \equiv \mathfrak{F}(t_o,1,t_2,0)$, $F_2 \equiv \mathfrak{F}(t_1,0,t_o,1)$ in B determine compact intervals $\mathcal{B}_t$ for t in K by the rule:

$$\mathcal{B}_t = [F_2(t),F_1(t)],t\in[t_1,t_o];$$

$$\mathcal{B}_t = [F_1(t),F_2(t)],t\in[t_o,t_2].$$

If $f \in B$ and, for some t in K, $f(t) \notin \mathcal{B}_t$, then $t \neq t_o$. Suppose $t \in (t_o,t_2]$. If $f(t) > F_2(t)$, then, since $F' \equiv \mathfrak{F}(f;t_o,t)$ crosses F_2 at t_o, $F'(t_1)$ is negative. But, by the first result of Theorem 3.1, $F'(t_1) > f(t_1)$, whence $f(t_1)$ is negative, a contradiction. A like contradiction follows from the assumption that $f(t) < F_1(t)$ and similar considerations applied to the case $t\in[t_1,t_o)$ yield contradictions. This proves the following lemma.

LEMMA 4.7. _For_ f _in_ B _and_ t _in_ K, $f(t)\in\mathcal{B}_t$.

The following is Ashley's Lemma 2.1.

LEMMA 4.8. _If_ f, f_1, _and_ f_2 _are_ $\mathfrak{F}$_-concave functions_

<u>on</u> K <u>such that</u> $f = f_1+f_2$ <u>on</u> K', <u>where</u> K' <u>is a compact sub-interval of</u> K, <u>and</u> f <u>is</u> $\mathfrak{F}$-<u>linear on</u> K', <u>then</u> f_1 <u>and</u> f_2 <u>are</u> $\mathfrak{F}$-<u>linear on</u> K'.

The functions F_1 and F_2 are extreme elements of B. For if $F_1 = \lambda\cdot f+(1-\lambda)\cdot g$ for f,g in B and λ in (0,1), then, by Lemma 4.8, f and g are $\mathfrak{F}$-linear on K, and since $f(t_2) = 0 = g(t_2)$, $f = g = F_1$. Similarly, $F_2 \in ex(B)$. Moreover, since every $\mathfrak{F}_K$-linear element of B has the form $A\cdot F_1+B\cdot F_2$, where $A \geq 0, B \geq 0$, and $A+B = 1$, F_1 and F_2 are the only $\mathfrak{F}_K$-lines in ex(B). Now for a choice of t in (t_1,t_2), let $F \equiv \mathfrak{F}(\{t_1,t,t_2\},\{0,[F_1 \vee F_2](t),0\})$. F is continuous, non-negative, and, by Lemma 3.3, $\mathfrak{F}_K$-concave. Also $F(t_o) = 1$, so that $F \in B$. Suppose $t \leq t_o$. If $F = \lambda\cdot f+(1-\lambda)\cdot g$ for some f and g in B and λ in (0,1), then, since $F = F_1$ on $[t,t_2]$ and, by Lemma 4.8 and the fact that $f(t_1) = 0 = g(t_1)$, $F = f = g$ on $[t_1,t]$. This and a similar argument for $t > t_o$ imply that $F \in ex(B)$. Now if f in B is not $\mathfrak{F}$-linear and is not zero at both t_1 and t_2, then $F' \equiv \mathfrak{F}(f;t_1,t_2)$ is such that $0 < F'(t_o) < 1$. Then, since $f - F'$ is $\mathfrak{F}_K$-concave, the identity

$$f = (1-F'(t_o))\cdot\frac{f-F'}{1-F'(t_o)} + F'(t_o)\cdot\frac{F'}{F'(t_o)}$$

implies that $f \notin ex(B)$. Finally, suppose that f in B is zero at both t_1 and t_2 but that f is not $\mathfrak{F}$-linear on both $[t_1,t_o]$ and $[t_o,t_2]$. For definiteness, suppose $f(t') > F_1(t')$ for some t' in (t_o,t_2). Let $\lambda_o = \inf\{\lambda:\lambda\cdot F_2 > f-F_1$

on K} and let $\tau = \sup\{t'' : \lambda_o F_2(t'') = f(t'') - F_1(t'')\}$. λ_o exists and is in (0,1) and τ exists and belongs to (t_o, t_2). Define f_1 to equal F_2 on $[t_1, \tau]$ and $\lambda_o^{-1} \cdot (f - F_1)$ on $[\tau, t_2]$. Now if $t_1' \in [t_1, \tau)$ and $t_2' \in (\tau, t_2]$, $\mathfrak{F}(f_1; t_1', t_2')$ lies below $f_1 = F_2$ on (t_1', τ), since $f_1(t_2') < F_2(t_2')$, and intersects f_1 once in $(\tau, t_2]$ since $(f - F_1)$ is $\mathfrak{F}$-concave. Therefore, f_1 belongs to B. Define f_2 to be $(1-\lambda_o)^{-1} \cdot (f - \lambda_o \cdot f_1)$. Now $f - \lambda_o \cdot f_1$ equals $f - \lambda_o \cdot F_2$ on $[t_1, \tau]$ and F_1 on $[\tau, t_2]$, and, for t_1' in $[t_1, \tau]$, $f(t_1') - \lambda_o \cdot F_2(t_1') \le F_1(t_1')$ by the definition of λ_o. Then the $\mathfrak{F}$-concavity of f_2 follows as did the $\mathfrak{F}$-concavity of f_1. Thus, f_2 belongs to B and f is interior to the segment determined by f_1 and f_2, and we have:

THEOREM 4.9. <u>Suppose</u> $\mathfrak{F}$ <u>is a locally disconjugate linear subspace of</u> C(I,R), <u>that</u> $K = [t_1, t_2] \in \mathcal{K}(\mathfrak{F})$, <u>and that</u> t_o <u>is interior to</u> K. <u>The extreme elements of the set of continuous nonnegative</u> $\mathfrak{F}_K$<u>-concave functions whose value at</u> t_o <u>is</u> 1 <u>are</u> $F_1 \equiv \mathfrak{F}(t_o, 1, t_2, 0)$, $F_2 \equiv \mathfrak{F}(t_1, 0, t_o, 1)$, <u>and those functions of the form</u>

$$\mathfrak{F}(\{t_1, t, t_2\}, \{0, [F_1 \vee F_2](t), 0\})$$

<u>for</u> t <u>in</u> (t_1, t_2).

Ashley [1; Lemma 3.2] claims that C is closed in R^K, the set of real-valued functions on K, for the topology of pointwise convergence. This is incorrect since (L_n), with $L_n \equiv \mathcal{L}(\{0, 1/n, 1\}, \{0, 1, 1\})$, is a sequence of nonnegative, continuous $\mathcal{L}_{[0,1]}$-concave functions whose pointwise limit, being discontinuous, is not in C. We now develop a setting in which this result holds.

We will apply Choquet's theorem to the vector space V generated by the cone C; that is, V = C-C. The problem is to topologize V in such a way as to satisfy the hypotheses of Theorem 4.5. Let R^K be assigned the product topology (topology of pointwise convergence), let Q^K be the collection of equivalence classes in R^K modulo the equivalence relation $\approx$ defined by

$$f \approx g \text{ if and only if } f = g \text{ on } (t_1, t_2),$$

assign to Q^K the quotient topology, and let $P: R^K \to Q^K$ be the continuous map which takes the function f in R^K to its $\approx$-equivalence class, [f]. Then $([f_\alpha]) \to [f_0]$ in Q^K if and only if $(f_\alpha(t)) \to f_0(t)$ for all t in (t_1, t_2). Q^K is Hausdorff and, as a topological vector space, locally convex; P is a vector space homomorphism. Since a member of Q^K contains at most one member of C(K,R), the map P restricted to C(K,R) is 1-to-1. In particular, the sets ex(B), B, C, and V can be considered as subsets of Q^K. Assign to V the relativized-Q^K-topology. Then V is a locally convex, Hausdorff topological vector space. Moreover, since the evaluation maps, e_t, $t \in (t_1, t_2)$, defined by

$$e_t : Q^K \to R :: e_t([f]) = f(t)$$

are continuous linear functionals on Q^K, their restrictions being discontinuous, is not in C. We now develop a setting in which this result holds.

We will apply Choquet's theorem to the <u>vector space</u> V <u>generated by the cone</u> C; that is, V = C-C. The problem is to topologize V in such a way as to satisfy the hypotheses of Theorem 4.5. Let R^K be assigned the product topology (topology of pointwise convergence), let Q^K be the collection of equivalence classes in R^K modulo the equivalence relation $\approx$ defined by

$$f \approx g \text{ if and only if } f = g \text{ on } (t_1,t_2),$$

assign to Q^K the quotient topology, and let $P: R^K \to Q^K$ be the continuous map which takes the function f in R^K to its $\approx$-equivalence class, [f]. Then $([f_\alpha]) \to [f_o]$ in Q^K if and only if $(f_\alpha(t)) \to f_o(t)$ for all t in (t_1,t_2). Q^K is Hausdorff and, as a topological vector space, locally convex; P is a vector space homomorphism. Since a member of Q^K contains at most one member of C(K,R), the map P restricted to C(K,R) is 1-to-1. In particular, the sets ex(B), B, C, and V can be considered as subsets of Q^K. <u>Assign to</u> V <u>the relativized-Q^K-topology</u>. Then V is a locally convex, Hausdorff topological vector space. Moreover, since the evaluation maps, e_t, $t \in (t_1,t_2)$, defined by

$$e_t : Q^K \to R :: e_t([f]) = f(t)$$

are continuous linear functionals on Q^K, their restrictions to V are continuous linear functionals on V.

LEMMA 4.10. $C = C(K)$ _is closed in_ Q^K, $B = B(K,t_o)$ _is closed in_ Q^K, _and_ B _is compact in_ V.

Suppose (f_α) is a sequence of nonnegative, continuous, $\mathfrak{F}_K$-concave functions with limit $[g]$ in Q^K. Clearly, $g \geq 0$ on (t_1,t_2). If $t_1 < t_1' < t < t_2' < t_2$ and $[\mathfrak{F}(g;t_1',t_2')](t)-g(t) = \delta > 0$, let N_ε be the neighborhood of $[g]$ consisting of classes $[h]$ determined by functions h which are within ε of g at t_1', t, and t_2', for a choice of ε in $(0,\varepsilon_o)$, where $\varepsilon_o = \delta/(1+M)$ and where M is the supremum of $\mathfrak{F}(t_1',1,t_2',1)$ on $[t_1',t_2']$. Then N_ε contains no classes determined by $\mathfrak{F}_K$-concave functions, a contradiction. Therefore, g is $\mathfrak{F}$-concave on (t_1,t_2) and, by the remark following the corollary to Theorem 3.2, there is a unique continuous $\mathfrak{F}_K$-concave function g' in [g]. Moreover, $g' \geq 0$ on K, so that $(f_\alpha) \to g'$ implies C is closed in Q^K. Since $B = B(K,t_o) = C(K) \cap e_{t_o}^{-1}(\{1\})$, the facts that C is closed and e_{t_o} is continuous imply that B is closed in Q^K. By Lemma 4.7, B, as a subset of R^K, is contained in the Cartesian product $S = \times\{\mathcal{B}_t : t \in K\}$ which, by the Tychonoff theorem, is compact in R^K. Then B, as a subset of Q^K, is a closed subset of the compact set P(S) in Q^K. Therefore B is compact in Q^K, hence in V.

Ashley's Theorem 3.3 states that ex(B) is R^K-closed, which is not true. But ex(B) is Q^K-closed, as is shown in the following lemma.

LEMMA 4.11. *For all t_o interior to K, the set of extreme elements of the set of nonnegative, continuous, $\mathfrak{F}_K$-concave functions whose value at t_o is 1 is homeomorphic to K.*

By Lemma 4.9, the map $\xi:K\to V$ defined by

$$\xi(t_1) = F_1; \ \xi(t_2) = F_2;$$

$$\xi(t) = [F_1 \vee F_2](t)\cdot\mathfrak{F}(\{t_1,t,t_2\},\{0,1,0\}), \ t\epsilon(t_1,t_2),$$

is 1-to-1 and onto $ex(B(K,t_o))$. If $(t^{(n)})$ in K has limit t interior to K, the continuity of $F_1 \vee F_2$ and $\mathfrak{F}$ insure that $\xi(t^{(n)})$ approaches $\xi(t)$ uniformly on K, hence pointwise on the interior of K. If $(t^{(n)})$ approaches an endpoint of K, say $(t^{(n)})\to t_i$, $i = 1$ or 2, then $(\xi(t^{(n)}))\to F_i$ pointwise on $K\setminus\{t_i\}$. Therefore ξ is continuous and, since K is compact and ex(B) Hausdorff, a homeomorphism.

According to Theorem 4.5 applied to V with the relative Q^K topology, every point f in the compact, convex set B determines a nonnegative Radon measure μ_f of unit mass on ex(B) such that for every continuous linear functional L on V

$$L(f) = \int_{ex(B)} L\cdot d\mu_f.$$

Now, since K and ex(B) are homeomorphic and since the nonnegative Radon measures of mass 1 on K are the Stieltjes integrals generated by nondecreasing real-valued functions v on K such that $v(t_2)-v(t_1) = 1$ (see Edwards [5;Sec.4.5-4.7]), every f in B determines a nondecreasing function $v_f: K \to [0,1]$ such that $v_f(t_1) = 0$ and $v_f(t_2) = 1$, for which

$$L(f) = \int_K (L \circ \xi) \cdot dv_f = \int_{t_1}^{t_2} L(\xi(\tau)) \cdot dv_f(\tau)$$

for all continuous linear $L: V \to R$. Then choosing $L = e_t$ for t interior to K we have

$$f(t) = \int_{t_1}^{t_2} [\xi(\tau)](t) \cdot dv_f(\tau).$$

Since v_f is nondecreasing it has left and right one-sided limits throughout K, whence v_f is uniquely determined by requiring it to be left-continuous at its points of discontinuity. Finally, a nonnegative $\mathfrak{F}_K$-concave function whose value at t_o is 1 determines a unique member f' of B which in turn determines a function $v_{f'}$.

THEOREM 4.12. <u>Suppose that</u> $\mathfrak{F}$ <u>is a locally disconjugate linear subspace of</u> C(I,R), <u>that</u> $K = [t_1,t_2] \in \mathcal{K}(\mathfrak{F})$, <u>and that</u> t_o <u>is interior to</u> K. <u>Then for every nonzero, nonnegative</u> $\mathfrak{F}_K$<u>-concave function</u> f <u>there is a unique left-continuous nondecreasing function</u> $v_f: K \to R$ <u>such that for all</u> t <u>interior to</u> K

$$f(t) = f(t_o) \cdot \int_{t_1}^{t_2} H(t,\tau) \cdot dv_f(\tau),$$

where

$$H(t,\tau) = \left.\begin{array}{ll} F_1(t), & \tau\epsilon[t_1,t), \\ = (F_1(\tau)/F_2(\tau))\cdot F_2(t), & \tau\epsilon[t,t_o], \\ = F_2(t), & \tau \epsilon (t_o,t_2], \end{array}\right\} \text{for } t \epsilon (t_1,t_o]$$

and

$$H(t,\tau) = \left.\begin{array}{ll} F_1(t), & \tau \epsilon[t_1,t_o) \\ = (F_2(\tau)/F_1(\tau))\cdot F_1(t), & \tau \epsilon [t_o,t], \\ = F_2(t), & \tau \epsilon(t,t_2], \end{array}\right\} \text{for } t \epsilon (t_o,t_2),$$

and

$$F_1 \equiv \mathfrak{F}(t_o,1,t_2,0), \quad F_2 \equiv \mathfrak{F}(t_1,0,t_o,1).$$

5. Unilateral Extremizing Properties of $\mathfrak{F}$-concave functions. For K a compact interval in I, define the following classes of functions: Γ_K the class of real-valued functions that are absolutely continuous on K and whose derivatives are of integrable square on K; $\Gamma_{K,0}$ the class of functions in Γ_K which vanish at the endpoints of K; $\Gamma_K(f)$ the class of functions in Γ_K which agree with $f:I\to R$ at the endpoints of K; $\Gamma_K^+(f)$ the class of functions in $\Gamma_K(f)$ which dominate f on K.

Suppose that r, q, and p belong to C(I,R) and that r is positive on I. Utilizing a well-known equivalence between the disconjugacy of the differential equation

$$(5.1) \qquad L(u) = (r\cdot u'+q\cdot u)'-(q\cdot u'+p\cdot u) = 0$$

on the open interval I and the positive definiteness of the quadratic functional J_K, where

$$(5.2) \quad J_K(u) = \int_{t_1}^{t_2} (r\cdot u'^2+2\cdot q\cdot u\cdot u'+p\cdot u^2), \ K = [t_1,t_2]\subset I,$$

on $\Gamma_{K,0}$ for all compact subintervals K of I, Reid [12] has proved the following theorem. Our statement of the theorem is based on the fact that the differential equation (5.1) is disconjugate on I if and only if the family $\mathcal{L}$ of solutions to (5.1) on I is disconjugate on I in the sense introduced in Section 2.

THEOREM 5.1. *If* (5.1) *is disconjugate on* I, *then* $f:I\to R$ *is* $\mathcal{L}$-*concave if and only if for each compact subinterval* K *of* I, f *belongs to* Γ_K *and* $J_K(g) \geq J_K(f)$ *for all* g *in* $\Gamma_K^+(f)$.

Loosely speaking, the equivalence mentioned above can be stated: $L(u) = 0$ is the Euler equation associated with the functional J. Or it can be said that the set of solutions of (5.1) on I constitute the set of extremals of the functional J. Now $L(u) = 0$ may be the Euler equation for functionals other than J. For example, $y'' = 0$ gives the extremals of both J and H where $J_K(u) = \int_K u'^2$ and $H_K(u) = \int_K (1+u'^2)^{\frac{1}{2}}$. So it is natural to ask: does the unilateral extremal characterization of $\mathcal{L}$-concave functions given in

Theorem 5.1 for the functional J hold for all functionals whose extremals are given by (5.1)? We describe a class of functionals, which include those of the form (5.2), for which the result of Theorem 5.1 holds. Moreover, for these functionals a certain class of unilateral minimization problems is solvable.

For a subset $\mathcal{D}$ of $C(I,R)$, a compact interval K in I, and a real-valued function f on I, let $\mathcal{D}_K$ be the class of restrictions to K of members of $\mathcal{D}$; $\mathcal{D}_K(f)$ the class of functions in $\mathcal{D}_K$ which agree with f at the endpoints of K; and $\mathcal{D}_K^+(f)$ the class of functions in $\mathcal{D}_K(f)$ which dominate $f|_K$. We say that H is a $\mathcal{D}$-<u>functional on</u> $\mathcal{K}$, $\mathcal{K}$ being the set of compact subintervals of I, if H maps each K in $\mathcal{K}$ into a real-valued map, H_K, on $\mathcal{D}_K$. The $\mathcal{D}$-functional H on $\mathcal{K}$ is <u>sectionally additive</u> if and only if, for every pair K',K" in $\mathcal{K}$ whose union is in $\mathcal{K}$,

$$H_{K'\cup K''} = H_{K'} + H_{K''} - H_{K'\cap K''},$$

and, for functions f' and f" in $\mathcal{D}$ which agree at some t_o in I, $\mathcal{D}$ contains the functions f_1 and f_2, where $f_1 = f'$ on $(a,t_o]$, $f_1 = f''$ on (t_o,b) and $f_2 = f''$ on $(a,t_o]$, $f_2' = f'$ on (t_o,b). H is called <u>lower semicontinuous</u> if each H_K for K in $\mathcal{K}$ is lower semicontinuous; that is, given K in $\mathcal{K}$, the uniform convergence of the sequence (f_n) in $\mathcal{D}_K$ to the limit f_o in $\mathcal{D}_K$ implies that $H_K(f_o)$ is not greater than lim inf $(H_K(f_n))$.

Suppose H is a $\mathcal{D}$-functional on $\mathcal{K}$. Given points (t_1,x_1) and (t_2,x_2) in I×R, with $t_1 < t_2$, the problem of minimizing $H_{[t_1,t_2]}$ on the class of functions g in $\mathcal{D}$ which satisfy $g(t_i) = x_i$, i = 1,2, is called a *fixed endpoint variational problem for* H. A function f $\in$ C(I,R) determines a class of fixed endpoint problems -- those of minimizing H_K on $\mathcal{D}_K(f)$ for K in $\mathcal{K}$ -- called, collectively, the *two-point variational problem for* H *determined by* f. The function g in $\mathcal{D}$ *solves* (*is a solution to*) the two-point variational problem for H determined by f if $g|_K$ minimizes H_K uniquely on $\mathcal{D}_K(f)$ for all K in $\mathcal{K}$. The function f is said to be an *extremal* of the $\mathcal{D}$-functional H on $\mathcal{K}$ if and only if $f \in \mathcal{D}$ and every point in I belongs to an open interval I_o such that, for every member K of the set, $\mathcal{K}_o$, of compact subintervals of I_o, $H_K(f|_K)$ is the uniquely achieved minimum value of H_K on $\mathcal{D}_K(f)$. Thus, the extremals of H solve the two-point variation problems that they determine for H locally. The problem of minimizing H_K on $\mathcal{D}^+_K(f)$ for K in $\mathcal{K}$, where f $\in$ C(I,R), is termed the *upper two-point variational problem for* H *determined by* f. The function f in $\mathcal{D}$ is an *upper extremal* of H if and only if every t_o in I is interior to some I_o such that, for K compact in I_o, $f|_K$ is the unique H_K-minimizing element of $\mathcal{D}^+_K(f)$.

We assume that *the maximal intervals of disconjugacy of a locally disconjugate family of extremals are precisely those which are maximal relative to the property of having*

<u>unique solutions to two-point variational problems</u>. If H has extremals and if the family of extremals, $\mathfrak{F}$, of H is locally disconjugate, then, for t_1 and t_2 sufficiently near to each other, the minimum value of $H_{[t_1,t_2]}$ on the class of functions in $\mathfrak{D}_{[t_1,t_2]}$ having (t_1,x_1) and (t_2,x_2) in their graphs is realized at (and only at) $\mathfrak{F}(t_1,x_1,t_2,x_2)$ restricted to $[t_1,t_2]$. Moreover, if H is also sectionally additive, then $\mathfrak{D}$ contains $\mathfrak{P}(\mathfrak{F})$, the set of piecewise-$\mathfrak{F}$-linear functions, and, by Theorem 2.10, $\mathfrak{D}$ is locally dense in C(I,R); that is, every t_o in I belongs to some I_o such that $\mathfrak{D}|_{I_o}$ is dense in $C(I_o,R)$.

If $\mathfrak{F}\subset C(I,R)$ is locally disconjugate, let $\mathfrak{A}(\mathfrak{F})$ represent the set of maximal intervals of disconjugacy and $\mathfrak{K}(\mathfrak{F})$ represent the set of compact subintervals of members of $\mathfrak{A}(\mathfrak{F})$. For convenience, we will write $H_K(f)$ for $H_K(f|_K)$.

THEOREM 5.2. <u>Suppose, for</u> $\mathfrak{D}\subset C(I,R)$, <u>that</u> H <u>is a lower semicontinuous, sectionally additive</u> $\mathfrak{D}$-<u>functional on</u> $\mathfrak{K}$, <u>which has a locally disconjugate family</u> $\mathfrak{F}$ <u>of extremals and that for all</u> K <u>in</u> $\mathfrak{K}(\mathfrak{F})$ <u>the class</u> $\mathfrak{D}$ <u>contains the functions which are</u> $\mathfrak{F}$-<u>concave on</u> K. <u>Then, for</u> f <u>in</u> C(I,R), <u>the upper two-point variational problem for</u> H <u>determined by</u> f <u>is solvable in this local sense: for</u> K <u>in</u> $\mathfrak{K}(\mathfrak{F})$, $\theta_K^+(f)$ <u>is the unique</u> H_K-<u>minimizing member of</u> $\mathfrak{D}_K^+(f)$. <u>In particular</u>, f <u>is</u> $\mathfrak{F}$-<u>concave if and only if it is an upper extremal of</u> H.

If f in $\mathfrak{D}$ is not $\mathfrak{F}$-concave, there is a $K = [t_1,t_2]\in\mathfrak{K}(\mathfrak{F})$ for which $\mathfrak{F}(f;t_1,t_2)|_K\in\mathfrak{D}_K^+(f)$, whence $H_K(f)$ is strictly greater than the minimum of H_K on $\mathcal{D}_K^+(f)$, so that *if for every* $K \in \mathcal{K}(\mathfrak{F})$ *the minimum of* H_K *on* $\mathcal{D}_K^+(f)$ *is* $H_K(f)$, *then* f *is* $\mathfrak{F}$*-concave*. Moreover, if $K \subset K' \subset I_o \in \mathcal{I}(\mathfrak{F})$, sectional additivity implies that $H_{K'}(f) > H_{K'}(f')$, where $f' = \mathfrak{F}(f;t_1,t_2)$ on $[t_1,t_2]$ with $f' = f$ elsewhere, since $H_{K'}(f) - H_{K'}(f') = H_K(f) - H_K(f')$. This yields the stronger result: *if* f *provides* H_K *with its minimum on* $\mathcal{D}_K^+(f)$, *then* f *is* $\mathfrak{F}$*-concave on* K.

If $K \in \mathcal{K}(\mathfrak{F})$ and $f \in \mathcal{D}$, then, from the result of Theorem 2.10 and the compactness of K, there is a sequence (T_n) in $\mathcal{T}_K$ such that the sequence $(\mathfrak{F}(f;T_n)|_K)$ has $\theta_K^+(f)$ as its uniform limit on K. Moreover, no term of the sequence $(H_K(\mathfrak{F}(f;T_n)))$ is greater than $H_K(f)$. Then, since H_K is lower semicontinuous,

$$H_K(\theta_K^+(f)) \leq \liminf\, (H_K(\mathfrak{F}(f;T_n))) \leq H_K(f).$$

Therefore, for K in $\mathcal{K}(\mathfrak{F})$ and f in $\mathcal{D}$, $H_K(f) \geq H_K(\theta_K^+(f))$.

Now suppose that $K = [t_1,t_2] \in \mathcal{K}(\mathfrak{F})$ and that f and g are distinct $\mathfrak{F}_K$-concave elements of $C(K,R)$ with $g \in C_K^+(f)$ and $g > f$ on (t_1,t_2). Note that if the $\mathfrak{F}$-line F is supported by f at t_o in (t_1,t_2), the function $g \wedge F$ (that is, $g \wedge (F|_K)$) belongs to $C_K^+(f) \cap \mathfrak{F}_K^+$, which is a subset of $\mathcal{D}_K^+(f)$, and $H_K(g \wedge F) < H_K(g)$. Let F be an assignment of $\mathfrak{F}$-lines to the points of (t_1,t_2) such that, for t in (t_1,t_2), f supports $F^{(t)}$ at t. Then, for each $T = \{t_1, t_{(1)}, \cdots, t_{(n)}, t_2\}, n \geq 1$, in $\mathcal{T}_K$, the function

$$g_T^{(F)} \equiv \left(\bigwedge_{i=1}^{n} F^{(t_{(i)})}\right) \wedge g$$

belongs to $C_K^+(f)\cap\mathfrak{F}_K^+$ and $H_K(g_T^{(F)}) < H_K(g)$. Also, with $N_n^{(t_o)} \equiv \{t : F^{(t_o)}(t) - f(t) < n^{-1}\}$, for t_o in (t_1,t_2), and $N_n \equiv \{t : g(t) - f(t) < n^{-1}\}$, the collection $\{N_n^{(t_o)} : t_o \in (t_1,t_2)\} \cup \{N_n\}$ is an open cover of K for each n. For each n, choose a finite set, $\{t_{n,i} : i = 1,\cdots,p_n\}$, in (t_1,t_2) such that $\{N_n, N_n^{(t_{n,1})},\ldots,N_n^{(t_{n,p_n})}\}$ covers K and define T_n in $\mathcal{T}_K$ to consist of t_1,t_2 and all $t_{j,i}$, $j = 1,\cdots,n;\ i = 1,\cdots,p_j$. Then the sequence $(g_{T_n}^{(F)})$ has f as its uniform limit and the sequence $(H_K(g_{T_n}^{(F)}))$, each term of which is less than $H_K(g)$, is nonincreasing. Then the lower semicontinuity of H_K yields: $H_K(f) < H_K(g)$. Thus, *if* $f\in C_K^+(f)\cap\mathfrak{F}_K^+$, *then* $H_K(f)$ *is the uniquely attained minimum value of* H_K *on* $C_K^+(f)\cap\mathfrak{F}_K^+$.

For f in C(I,R) and $K\in\mathcal{K}(\mathfrak{F})$, suppose $g\in\mathcal{D}_K^+(f)$. If $\theta_K^+(g) \neq \theta_K^+(f)$, then $H_K^+(g) \geq H_K^+(\theta_K^+(g)) > H_K(\theta_K^+(f))$. Otherwise, $\theta_K^+(g) = \theta_K^+(f)$ and $H_K(g) \geq H_K(\theta_K^+(f))$. Therefore the infimum of H_K on $\mathcal{D}_K^+(f)$ is attained at $\theta_K^+(f)$. Then, if $H_K(g) = H_K(\theta_K^+(f))$, g is $\mathfrak{F}_K$-concave, that is, $g = \theta_K^+(g)$, so that the minimum of H_K on $\mathcal{D}_K^+(f)$ is assumed uniquely. This completes the proof of the theorem.

That Theorem 5.1 is an instance of Theorem 5.2 follows in all particulars except lower semicontinuity upon defining $\mathcal{D}$ to be $\{f\in C(I,R) : \forall K\in\mathcal{K}, f|_K\in\Gamma_K\}$. For the lower semicontinuity of the functional (5.2) we cite Graves [6;p.164].

REFERENCES

1. F. W. Ashley, A cone of super-(L) functions, Pacific J. Math., Vol. 13 (1963), 1-6.

2. E. F. Beckenbach, Generalized convex functions, Bull. Amer. Math. Soc., Vol. 43 (1937), 363-371.

3. F. F. Bonsall, The characterization of generalized convex functions, Quart. J. Math., Oxford Ser. (2), Vol. 1 (1950), 100-111.

4. Gustave Choquet, Theory of Capacities, Ann. Inst. Fourier (Grenoble), Vol. 5 (1955), 131-295.

5. R. E. Edwards, Functional analysis, Holt, Rinehart, and Winston, New York, 1965.

6. L. M. Graves, On the existence of the absolute minimum in space problems of the calculus of variations, Ann. of Math., Vol. 28, (1926-1927), 153-170.

7. J. W. Green, Support, convergence, and differentiability properties of generalized convex functions, Proc. Amer. Math. Soc., Vol. 4 (1953), 391-396.

8. N. J. Hicks, Notes on differential geometry, Math. Studies #3, Van Nostrand, Princeton, N.J., 1955.

9. J. L. Kelley, General topology, Van Nostrand, Princeton, N.J., 1955.

10. I. P. Natanson, Theory of functions of a real variable, Vol. II, Frederick Ungar, New York, 1960.

11. M. M. Peixoto, On the existence of derivatives of generalized convex functions, Summa Brasil. Math., Vol. 2 (1948), 35-42.

12. W. T. Reid, Variational aspects of generalized convex functions, Pacific J. Math., Vol. 9 (1959), 571-581.

13. M. H. Stone, A generalized Weierstrass approximation theorem, Studies in Modern Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962, 30-87.