EMITTER LOCATION BY A TIME DIFFERENCE

HYPERBOLIC NET

By

MARCUS HAROLD McLEAN

Bachelor of Electrical Engineering
Auburn University
Auburn, Alabama
1951

Master of Science
Ohio State University
Columbus, Ohio
1956

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
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Thesis Approved:

Bennett Basore
Thesis Adviser

Arthur M. Breipohl

James E. Shamblin

Kenneth A. McCollom

D. D. Durham
Dean of the Graduate College

730020

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CHAPTER I

INTRODUCTION

1.1 Background. Electronic systems for location of emitters (radiators of electromagnetic energy) by distant receiving sites have applications in military reconnaissance and other areas of government service and to a limited extent in civilian electronics. Present day operational emitter locators employ the technique of direction finding.

As an effort to improve the accuracy of location, several other techniques have been either experimentally demonstrated or proposed in the literature. One of these, which is the subject of this paper, is emitter location by a time difference hyperbolic net.

1.2 Time Difference Hyperbolic Net Defined. In two dimensions, let an emitter be located at unknown point (X_0, Y_0). Let two receiving stations be located at known points (X_1, Y_1) and (X_2, Y_2) as indicated in Figure 1.1. Let R_1 and R_2 be the undirected distances from emitter to the respective receiving sites. Assume an electromagnetic emission leaves point (X_0, Y_0) at time t_0 and arrives at point (X_1, Y_1) at time t_1 and point (X_2, Y_2) at time t_2. Assuming uniform speed of propagation u,

$$(t_1 - t_0)u = R_1 \qquad (1.1)$$

and

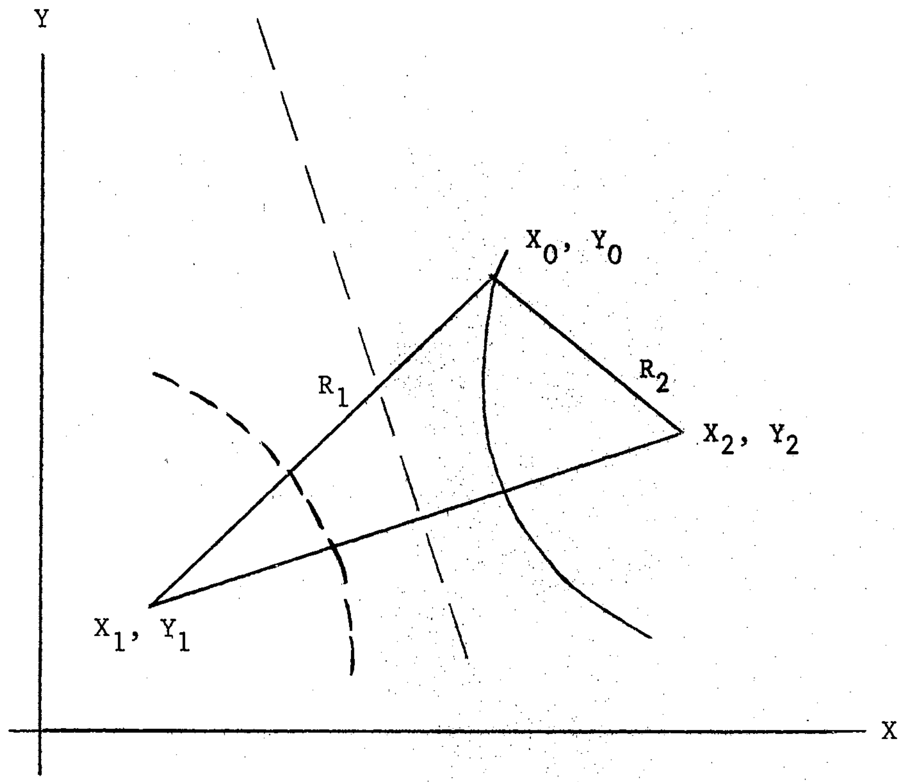

Figure 1.1. A one Baseline Time Difference Hyperbolic net in two Dimensions

$$(t_2 - t_0)u = R_2 \quad . \tag{1.2}$$

One cannot measure $t_1 - t_0$ and $t_2 - t_0$ since he has no access to the emitter. However, the time difference $t_1 - t_2$ can be measured by cooperation of receiving sites at (X_1, Y_1) and (X_2, Y_2). Subtracting Equation 1.2 from Equation 1.1, one obtains the one baseline time difference equation

$$t_1 - t_2 = \frac{1}{u} (R_1 - R_2)$$

which may be written

$$u(t_1 - t_2) = [(X_0 - X_1)^2 + (Y_0 - Y_1)^2]^{\frac{1}{2}} - [(X_0 - X_2)^2 + (Y_0 - Y_2)^2]^{\frac{1}{2}}. \quad (1.3)$$

Observing Equation 1.3, one notes that it is in the form of the equation for a hyperbola. Hence, given $t_1 - t_2$, the equation holds not only for point (X_0, Y_0) but for every point (X, Y) such that

$$u(t_1 - t_2) = [(X - X_1)^2 + (Y - Y_1)^2]^{\frac{1}{2}} - [(X - X_2)^2 + (Y - Y_2)^2]^{\frac{1}{2}} . \quad (1.4)$$

When $t_1 - t_2$ is positive, the branch of the hyperbola defined by Equation 1.4 has its focus at point (X_2, Y_2) as indicated by the solid curve of Figure 1.1. The branch defined when $t_1 - t_2$ is negative has its focus at point (X_1, Y_1) as indicated by the dashed curve of Figure 1.1.

In summary, knowing speed of electromagnetic propagation u, receiver site locations (X_1, Y_1) and (X_2, Y_2) and time difference $t_1 - t_2$ permits one to deduce that emitter location (X_0, Y_0) is on a known branch of a known hyperbola, i.e., on a curved "line of position".

A third receiving site at known location (X_3, Y_3) permits one to generate a second baseline equation

$$u(t_1 - t_3) = [(X - X_1)^2 + (Y - Y_1)^2]^{\frac{1}{2}} - [(X - X_3)^2 + (Y - Y_3)^2]^{\frac{1}{2}} . \quad (1.5)$$

Given $t_1 - t_2$ and $t_1 - t_3$, one may solve Equations 1.4 and 1.5 simultaneously for emitter location $(X = X_0, Y = Y_0)$. Under certain circumstances, the relationship between $(t_1 - t_2, t_1 - t_3)$ and (X_0, Y_0) is one-to-one, but in general, there will be two solutions (X_{01}, Y_{01}) and (X_{02}, Y_{02}). One of these solutions is the actual emitter location, and the other is a "ghost" location. A third baseline formed by a fourth receiving site removes the "ghost" when it exists.

In three dimensions, the basic equation for a one baseline hyperbolic net is

$$u(t_1 - t_2) = [(X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2]^{\frac{1}{2}} - [(X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2]^{\frac{1}{2}} \quad . \tag{1.6}$$

One defines a K - 1 baseline time difference hyperbolic net as an operation of K receiving stations located at points (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), ..., (X_K, Y_K, Z_K) and instrumented with a capability to measure a sufficient set of K - 1 time differences $t_i - t_j$ where t_i and t_j are the arrival times at the i^{th} and j^{th} receivers of a signal emitted from point (X_0, Y_0, Z_0). A set of K - 1 time differences is called sufficient if no member of the set can be expressed as a linear combination of the remaining K - 2 members.

There is no maximum number of baselines for a time difference hyperbolic net, but three is the minimum number permissible (two when emitter and receivers are constrained to the X-Y plane) if a point estimate of emitter location is to be obtained with no a priori information. However, when K is the absolute minimum, point estimation is not always unique. There is an ambiguity between two points as previously mentioned. Additional baselines above the minimum remove this ambiguity and improve the confidence of the estimate.

<u>1.3 Statement of the Problem</u>. The problem considered in this paper is as follows:

1) To develop a general statistical model of a K - 1 baseline time difference hyperbolic net in three dimensions.

2) To develop special models (as special cases of the general model) which describe the several projected modes of operation of a time difference hyperbolic net.

3) To effect a solution of the special models for an estimate of emitter location and variance of the estimated location.

4) To test the sensitivity of estimates of location to errors described by the model.

1.4 Related Previous Work. Marchand (1) found the maximum likelihood estimate of position for a K element time difference hyperbolic net in two dimensions. This study is essentially an extension of Marchand's work. Major extensions not included in his work are:

1) Correlated measurement errors.

2) Three dimensional operation.

3) Receiving site locational errors.

4) Bias errors due to unknown speed of propagation.

5) Bias errors due to multipath propagation.

6) Estimation with a priori information.

CHAPTER II

DETERMINISTIC SOLUTION OF EMITTER LOCATION

2.1 Deterministic Solution of Baseline Equations. In emitter location by a time difference hyperbolic net, one inserts numbers obtained from time difference measurements and receiver location measurements into the time difference equations and then solves for emitter position. If the measurements were without error, the problem essentially reduces to that of finding the intersection of two hyperbolas when the emitter and receiving sites are constrained to the X-Y plane, and finding the intersection of three hyperboloids when otherwise. The solution for these intersections is derived in this chapter.

2.2 A Two Baseline Hyperbolic Net. In a two baseline hyperbolic net, the emitter and three receiving stations are all constrained to the X-Y plane as illustrated in Figure 2.1. An emission from point (X_0, Y_0) travels at speed u and arrives at points (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) at times t_1, t_2 and t_3 respectively. The time difference equations are:

$$u(t_1 - t_2) = [(X_0 - X_1)^2 + (Y_0 - Y_1)^2]^{\frac{1}{2}} - [(X_0 - X_2)^2 + (Y_0 - Y_2)^2]^{\frac{1}{2}} \tag{2.1}$$

$$u(t_1 - t_3) = [(X_0 - X_1)^2 + (Y_0 - Y_1)^2]^{\frac{1}{2}} - [(X_0 - X_3)^2 + (Y_0 - Y_3)^2]^{\frac{1}{2}} \tag{2.2}$$

$$u(t_2 - t_3) = [(X_0 - X_2)^2 + (Y_0 - Y_2)^2]^{\frac{1}{2}} - [(X_0 - X_3)^2 + (Y_0 - Y_3)^2]^{\frac{1}{2}}. \tag{2.3}$$

Note that Equation 2.2 minus Equation 2.1 yields Equation 2.3. Hence, one of the equations is redundant and need not be considered further in solving for (X_0, Y_0). Arbitrarily let the first two equations constitute the solution set. Then the lines connecting point (X_1, Y_1) with (X_2, Y_2) and (X_1, Y_1) with (X_2, Y_2) are the two baselines.

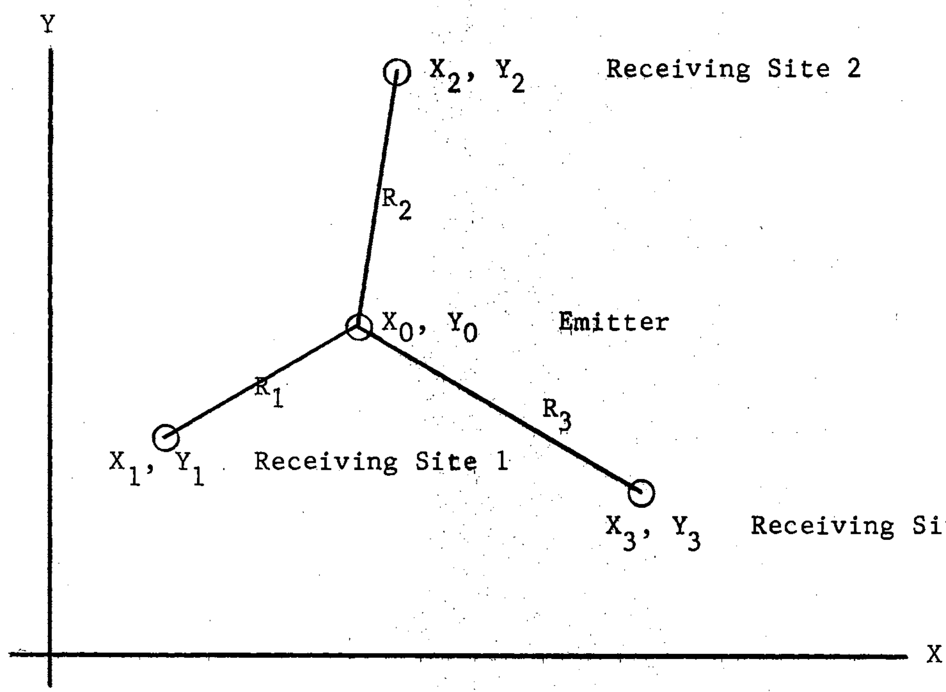

Figure 2.1. A two Baseline Time Difference Hyperbolic net in two Dimensions

The problem may be stated: given measurements for (X_1, Y_1), (X_2, Y_2), (X_3, Y_3), $(t_1 - t_2)$ and $(t_1 - t_3)$, solve Equations 2.1 and 2.2 simultaneously for emitter location (X_0, Y_0).

<u>2.2.1 Intersection of Two Hyperbolas</u>. With X_0 and Y_0 as variables and all other parameters fixed, Equations 2.1 and 2.2 define two

hyperbolas. In general, two hyperbolas may intersect in the X-Y plane as many as four times. However, the situation here is a special case, the unique feature being that point (X_1, Y_1) is a common focal point for the two hyperbolas. It will be shown shortly that the hyperbolas in this speical case intersect at most twice.

Equations 2.1 and 2.2 may be rewritten:

$$u(t_1 - t_2) - [(X_0 - X_1)^2 + (Y_0 - Y_1)^2]^{\frac{1}{2}} = -[(X_0 - X_2)^2 + (Y_0 - Y_2)^2]^{\frac{1}{2}} \tag{2.4}$$

$$u(t_1 - t_3) - [(X_0 - X_1)^2 + (Y_0 - Y_1)^2]^{\frac{1}{2}} = -[(X_0 - X_3)^2 + (Y_0 - Y_3)^2]^{\frac{1}{2}} \tag{2.5}$$

Squaring both equations and rearranging terms,

$$u(t_1 - t_2)[(X_0 - X_1)^2 + (Y_0 - Y_1)^2]^{\frac{1}{2}} = (X_2 - X_1)X_0 + (Y_2 - Y_1)Y_0 + \tfrac{1}{2}[X_1^2 + Y_1^2 - X_2^2 - Y_2^2 + u^2(t_1 - t_2)^2] \quad . \tag{2.6}$$

$$u(t_1 - t_3)[(X_0 - X_1)^2 + (Y_0 - Y_1)^2]^{\frac{1}{2}} = (X_3 - X_1)X_0 + (Y_3 - Y_1)Y_0 + \tfrac{1}{2}[X_1^2 + Y_1^2 - X_3^2 - Y_3^2 + u^2(t_1 - t_3)^2] \quad . \tag{2.7}$$

Squaring the last two equations and summing yields

$$c_1X_0^2 + c_2Y_0^2 + c_3X_0Y_0 + c_4X_0 + c_5Y_0 + c_6 = 0 \quad . \tag{2.8}$$

The coefficients are defined:

$$c_1 = 4[(X_1 - X_2)^2 + (X_1 - X_3)^2 - u^2(t_1 - t_2)^2 - u^2(t_1 - t_3)^2] \; .$$

$$c_2 = 4[(Y_1 - Y_2)^2 + (Y_1 - Y_3)^2 - u^2(t_1 - t_2)^2 - u^2(t_1 - t_3)^2] \; .$$

$$c_3 = 8[(X_1 - X_2)(Y_1 - Y_2) + (X_1 - X_3)(Y_1 - Y_3)] \; .$$

$$c_4 = 4[(X_1 - X_2)(X_2^2 + Y_2^2 - X_1^2 - Y_1^2) + (X_1 - X_3)(X_3^2 + Y_3^2 - X_1^2 - Y_1^2)$$

$$+ u^2(t_1 - t_2)^2(X_1 + X_2) + u^2(t_1 - t_3)^2(X_1 + X_3)] .$$

$$c_5 = 4[(Y_1 - Y_2)(X_2^2 + Y_2^2 - X_1^2 - Y_1^2) + (Y_1 - Y_3)(X_3^2 + Y_3^2 - X_1^2 - Y_1^2)$$

$$+ u^2(t_1 - t_2)^2(Y_1 + Y_2) + u^2(t_1 - t_3)^2(Y_1 + Y_3)] .$$

$$c_6 = [X_2^2 + Y_2^2 - X_1^2 - Y_1^2 - u^2(t_1 - t_2)^2]^2$$

$$+ [X_3^2 + Y_3^2 - X_1^2 - Y_1^2 - u^2(t_1 - t_3)^2]^2$$

$$- 4(X_1^2 + Y_1^2)[u^2(t_1 - t_2)^2 + u^2(t_1 - t_3)^2] .$$

It is now desired to eliminate Y_0 from Equation 2.8. This may be accomplished by solving Equations 2.6 and 2.7 for a linear relation between X_0 and Y_0. Multiplying Equations 2.6 and 2.7 through by $u(t_1 - t_3)$ and $-u(t_1 - t_2)$ respectively and then summing, the term $[(X_0 - X_1)^2 + (Y_0 - Y_1)^2]^{\frac{1}{2}}$ is eliminated, resulting in the equation

$$c_7 Y_0 = c_8 X_0 + c_9 \qquad . \qquad (2.9)$$

The coefficients are defined:

$$c_7 = u(t_1 - t_2)(Y_1 - Y_3) - u(t_1 - t_3)(Y_1 - Y_2) \quad .$$

$$c_8 = u(t_1 - t_3)(X_1 - X_2) - u(t_1 - t_2)(X_1 - X_3) \quad .$$

$$c_9 = \tfrac{1}{2}u(t_1 - t_3)[X_2^2 + Y_2^2 - X_1^2 - Y_1^2 - u^2(t_1 - t_2)^2]$$

$$- \tfrac{1}{2}u(t_1 - t_2)[X_3^2 + Y_3^2 - X_1^2 - Y_1^2 - u^2(t_1 - t_3)^2] .$$

Substituting Equation 2.9 into Equation 2.8, one finally obtains the quadratic equation

$$(c_2c_8^2 + c_3c_7c_8 + c_1c_7^2)X_0^2 + (c_4c_7^2 + c_5c_7c_8 + 2c_2c_8c_9 + c_3c_7c_9)X_0$$
$$+ c_6c_7^2 + c_2c_9^2 + c_5c_7c_9 = 0 \quad . \qquad (2.10)$$

Knowing the parameters X_1, Y_1, X_2, Y_2, X_3, Y_3, $t_1 - t_2$, and $t_1 - t_3$, one may compute c_1, c_2, ..., c_9 and then solve for X_0 by the quadratic formula

$$X_0 = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a} \quad . \qquad (2.11)$$

The parameters a, b, and c are obvious from Equation 2.10. Then Y_0 may be found by use of Equation 2.9.

Example 2.1: Let X_1, Y_1, X_2, Y_2, X_3, and Y_3 equal -30, 10, 50, 10, 10, and 70 kilometers respectively. Assume $t_1 - t_2$ and $t_1 - t_3$ are measured to be 400/3 and 200 microseconds respectively. Assume speed of propagation u = 300,000 kilometers per second. Then:

$c_1 = 11,200$	$c_4 = -1,312,000$	$c_7 = -2,400$
$c_2 = -6,400$	$c_5 = 320,000$	$c_8 = -3,200$
$c_3 = 19,200$	$c_6 = -20,640,000$	$c_9 = -8,000$.

Substituting these numbers into Equation 2.10, one obtains the quadratic equation

$$143\, X_0^2 - 4,940\, X_0 - 110,500 = 0$$

which has the roots: $X_0 = 50$ and $-2,210/143$. The corresponding values for Y_0 are: $Y_0 = 70$ and $-2,470/143$ respectively. The apparent solution $(X_0, Y_0) = (-2,210/143, -2,470/143)$ does not hold in Equations 2.4 and

2.5. Hence, in this example, the solution $(X_0, Y_0) = (50, 70)$ kilometers for emitter location is unique. Had both points (50, 70) and (-2,210/143, -2,470/143) held in the baseline equations, then there would remain an uncertainty as to which were the true emitter location and which were the "ghost" location.

2.3 A Three Baseline Hyperbolic Net in Three Dimensions. Relative to a fixed Cartesian coordinate system with arbitrary choice of origin, let an emitter be located at point (X_0, Y_0, Z_0) and four receiving stations be located at points (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) and (X_4, Y_4, Z_4). A signal is radiated from the emitter at time t_0 and recevied at the four receiving stations at times t_1, t_2, t_3 and t_4 respectively. The emitter and receiving stations may be in motion, but it is assumed that all displacements are negligibly small during the time intervals under consideration.

The time difference equations are:

$$u(t_1 - t_2) = [(X_0 - X_1)^2 + (Y_0 - Y_1)^2 + (Z_0 - Z_1)^2]^{\frac{1}{2}} - [(X_0 - X_2)^2 + (Y_0 - Y_2)^2 + (Z_0 - Z_2)^2]^{\frac{1}{2}} \quad . \quad (2.12)$$

$$u(t_1 - t_3) = [(X_0 - X_1)^2 + (Y_0 - Y_1)^2 + (Z_0 - Z_1)^2]^{\frac{1}{2}} - [(X_0 - X_3)^2 + (Y_0 - Y_3)^2 + (Z_0 - Z_3)^2]^{\frac{1}{2}} \quad . \quad (2.13)$$

$$u(t_1 - t_4) = [(X_0 - X_1)^2 + (Y_0 - Y_1)^2 + (Z_0 - Z_1)^2]^{\frac{1}{2}} - [(X_0 - X_4)^2 + (Y_0 - Y_4)^2 + (Z_0 - Z_4)^2]^{\frac{1}{2}} \quad . \quad (2.14)$$

$$u(t_2 - t_3) = [(X_0 - X_2)^2 + (Y_0 - Y_2)^2 + (Z_0 - Z_2)^2]^{\frac{1}{2}} - [(X_0 - X_3)^2 + (Y_0 - Y_3)^2 + (Z_0 - Z_3)^2]^{\frac{1}{2}} \quad . \quad (2.15)$$

$$u(t_2 - t_4) = [(X_0 - X_2)^2 + (Y_0 - Y_2)^2 + (Z_0 - Z_2)^2]^{\frac{1}{2}} - [(X_0 - X_4)^2 + (Y_0 - Y_4)^2 + (Z_0 - Z_4)^2]^{\frac{1}{2}} \quad . \quad (2.16)$$

$$u(t_3 - t_4) = [(X_0 - X_3)^2 + (Y_0 - Y_3)^2 + (Z_0 - Z_3)^2]^{\frac{1}{2}} - [(X_0 - X_4)^2 + (Y_0 - Y_4)^2 + (Z_0 - Z_4)^2]^{\frac{1}{2}} \quad . \quad (2.17)$$

Note that only three of the above equations are algebraically independent. This is an illustration of the basic fact that from K receiving stations, one may write $\binom{K}{2}$ time difference equations. K - 1 of these are algebraically independent and the remaining $\frac{1}{2}(K - 1)(K - 2)$ are dependent. The convention to be followed throughout this paper is to select the time differences $(t_1 - t_2)$, $(t_1 - t_3)$, ..., $(t_1 - t_K)$ as the K - 1 independent set. Hence, the first three equations are chosen, and the three baselines formed by the four receiving stations are the lines connecting point (X_1, Y_1, Z_1) with point (X_2, Y_2, Z_2), point (X_1, Y_1, Z_1) with point (X_3, Y_3, Z_3) and point (X_1, Y_1, Z_1) with point (X_4, Y_4, Z_4). The problem may be stated: given measurements for (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3), (X_4, Y_4, Z_4), $(t_1 - t_2)$, $(t_1 - t_3)$ and $(t_1 - t_4)$; solve Equations 2.12, 2.13 and 2.14 simultaneously for emitter location (X_0, Y_0, Z_0). The problem just defined may be recognized as the algebraic problem of solving for the point of intersection of three hyperboloids.

2.3.1 Intersection of Three Hyperboloids. In general, three hyperboloids may intersect at eight points. However, it will shortly be seen that the three hyperboloids defined by Equations 2.12, 2.13 and 2.14 present a special case in which there are at most two points of intersection. The procedure to be followed is similar to that of Section

2.2.1, but the algebra here is somewhat more tedious.

Squaring Equations 2.12, 2.13 and 2.14 and rearranging terms, one obtains:

$$u(t_1-t_2)[(X_0-X_1)^2 + (Y_0-Y_1)^2 + (Z_0-Z_1)^2]^{\frac{1}{2}} = (X_2-X_1)X_0 + (Y_2-Y_1)Y_0$$

$$+ (Z_2-Z_1)Z_0 + \tfrac{1}{2}[X_1^2 + Y_1^2 + Z_1^2 - X_2^2 - Y_2^2 - Z_2^2 + u^2(t_1-t_2)^2]. \quad (2.18)$$

$$u(t_1-t_3)[(X_0-X_1)^2 + (Y_0-Y_1)^2 + (Z_0-Z_1)^2]^{\frac{1}{2}} = (X_3-X_1)X_0 + (Y_3-Y_1)Y_0$$

$$+ (Z_3-Z_1)Z_0 + \tfrac{1}{2}[X_1^2 + Y_1^2 + Z_1^2 - X_3^2 - Y_3^2 - Z_3^2 + u^2(t_1-t_3)^2]. \quad (2.19)$$

$$u(t_1-t_4)[(X_0-X_1)^2 + (Y_0-Y_1)^2 + (Z_0-Z_1)^2]^{\frac{1}{2}} = (X_4-X_1)X_0 + (Y_4-Y_1)Y_0$$

$$+ (Z_4-Z_1)Z_0 + \tfrac{1}{2}[X_1^2 + Y_1^2 + Z_1^2 - X_4^2 - Y_4^2 - Z_4^2 + u^2(t_1-t_4)^2]. \quad (2.20)$$

Squaring the last three equations and summing yields

$$c_{11}X_0^2 + c_{12}Y_0^2 + c_{13}Z_0^2 - c_{14}X_0Y_0 - c_{15}X_0Z_0 - c_{16}Y_0Z_0$$

$$- c_{17}X_0 - c_{18}Y_0 - c_{19}Z_0 - c_{10} = 0 \quad . \quad (2.21)$$

The coefficients are defined:

$$c_{11} = u^2[(t_1-t_2)^2 + (t_1-t_3)^2 + (t_1-t_4)^2] - (X_2-X_1)^2 - (X_3-X_1)^2 - (X_4-X_1)^2$$

$$c_{12} = u^2[(t_1-t_2)^2 + (t_1-t_3)^2 + (t_1-t_4)^2] - (Y_2-Y_1)^2 - (Y_3-Y_1)^2 - (Y_4-Y_1)^2$$

$$c_{13} = u^2[(t_1-t_2)^2 + (t_1-t_3)^2 + (t_1-t_4)^2] - (Z_2-Z_1)^2 - (Z_3-Z_1)^2 - (Z_4-Z_1)^2$$

$$c_{14} = 2[(X_2-X_1)(Y_2-Y_1) + (X_3-X_1)(Y_3-Y_1) + (X_4-X_1)(Y_4-Y_1)]$$

$$c_{15} = 2[(X_2-X_1)(Z_2-Z_1) + (X_3-X_1)(Z_3-Z_1) + (X_4-X_1)(Z_4-Z_1)]$$

$$c_{16} = 2[(Y_2-Y_1)(Z_2-Z_1) + (Y_3-Y_1)(Z_3-Z_1) + (Y_4-Y_1)(Z_4-Z_1)]$$

$$c_{17} = (X_2-X_1)(X_1^2+Y_1^2+Z_1^2-X_2^2-Y_2^2-Z_2^2) + (X_3-X_1)(X_1^2+Y_1^2+Z_1^2-X_3^2-Y_3^2-Z_3^2)$$

$$+ (X_4-X_1)(X_1^2+Y_1^2+Z_1^2-X_4^2-Y_4^2-Z_4^2) - u^2[(X_1+X_2)(t_1-t_2)^2 + (X_1+X_3)(t_1-t_3)^2$$

$$+ (X_1+X_4)(t_1-t_4)^2]$$

$$c_{18} = (Y_2-Y_1)(X_1^2+Y_1^2+Z_1^2-X_2^2-Y_2^2-Z_2^2) + (Y_3-Y_1)(X_1^2+Y_1^2+Z_1^2-X_3^2-Y_3^2-Z_3^2)$$

$$+ (Y_4-Y_1)(X_1^2+Y_1^2+Z_1^2-X_4^2-Y_4^2-Z_4^2) - u^2[(Y_1+Y_2)(t_1-t_2)^2 + (Y_1+Y_3)(t_1-t_3)^2$$

$$+ (Y_1+Y_4)(t_1-t_4)^2]$$

$$c_{19} = (Z_2-Z_1)(X_1^2+Y_1^2+Z_1^2-X_2^2-Y_2^2-Z_2^2) + (Z_3-Z_1)(X_1^2+Y_1^2+Z_1^2-X_3^2-Y_3^2-Z_3^2)$$

$$+ (Z_4-Z_1)(X_1^2+Y_1^2+Z_1^2-X_4^2-Y_4^2-Z_4^2) - u^2[(Z_1+Z_2)(t_1-t_2)^2 + (Z_1+Z_3)(t_1-t_3)^2$$

$$+ (Z_1+Z_4)(y_1-y_4)^2]$$

$$c_{10} = \tfrac{1}{4}[X_1^2+Y_1^2+Z_1^2-X_2^2-Y_2^2-Z_2^2+u^2(t_1-t_2)^2]^2 + \tfrac{1}{4}[X_1^2+Y_1^2+Z_1^2-X_3^2-Y_3^2-Z_3^2+u^2(t_1-t_3)^2]^2$$

$$+ \tfrac{1}{4}[X_1^2+Y_1^2+Z_1^2-X_4^2-Y_4^2-Z_4^2+u^2(t_1-t_4)^2]^2 - u^2(X_1^2+Y_1^2+Z_1^2)[(t_1-t_2)^2$$

$$+ (t_1-t_3)^2 + (t_1-t_4)^2]$$

It is now desired to eliminate Y_0 and Z_0 from Equation 2.21. Multiplying Equation 2.18 through by $u(t_1 - t_3)$ and Equation 2.19 through by $-u(t_1 - t_2)$ and summing yields

$$2c_{27}X_0 + 2c_{28}Y_0 + 2c_{29}Z_0 - c_{20} = 0 \qquad (2.22)$$

where

$$c_{27} = u(t_1-t_2)(X_3-X_1) - u(t_1-t_3)(X_2-X_1)$$

$$c_{28} = u(t_1-t_2)(Y_3-Y_1) - u(t_1-t_3)(Y_2-Y_1)$$

$$c_{29} = u(t_1-t_2)(Z_3-Z_1) - u(t_1-t_3)(Z_2-Z_1)$$

$$c_{20} = u(t_1-t_3)[X_1^2+Y_1^2+Z_1^2-X_2^2-Y_2^2-Z_2^2+u^2(t_1-t_2)^2]$$

$$- u(t_1-t_2)[X_1^2+Y_1^2+Z_1^2-X_3^2-Y_3^2-Z_3^2+u^2(t_1-t_3)^2] \quad .$$

To obtain a second linear equation independent of Equation 2.22, multiply Equation 2.18 through by $u(t_1 - t_4)$ and Equation 2.20 through by $-u(t_1 - t_2)$, and then sum the two. The result is

$$2c_{37}X_0 + 2c_{38}Y_0 + 2c_{39}Z_0 - C_{30} = 0 \qquad (2.23)$$

where

$$c_{37} = u(t_1-t_2)(X_4-X_1) - u(t_1-t_4)(X_2-X_1)$$

$$c_{38} = u(t_1-t_2)(Y_4-Y_1) - u(t_1-t_4)(Y_2-Y_1)$$

$$c_{39} = u(t_1-t_2)(Z_4-Z_1) - u(t_1-t_4)(Z_2-Z_1)$$

$$c_{30} = u(t_1-t_4)[X_1^2+Y_1^2+Z_1^2-X_2^2-Y_2^2-Z_2^2+u^2(t_1-t_2)^2]$$

$$- u(t_1-t_2)[X_1^2+Y_1^2+Z_1^2-X_4^2-Y_4^2-Z_4^2+u^2(t_1-t_4)^2] \quad .$$

Note that if $t_1 - t_2 = t_1 - t_3 = t_1 - t_4 = 0$, then the left hand sides of both Equations 2.22 and 2.23 are identically zero for every point (X_0, Y_0, Z_0). Assume this situation does not exist. Solving Equations 2.22 and 2.23 simultaneously, first to eliminate Z_0 and second to eliminate Y_0, one obtains the two equations

$$c_{48}Y_0 = c_{47}X_0 + c_{40} \qquad (2.24)$$

$$-c_{48}Z_0 = c_{57}X_0 + c_{50} \qquad (2.25)$$

where

$$c_{48} = 2(c_{28}c_{39} - c_{29}c_{38})$$

$$c_{47} = 2(c_{29}c_{37} - c_{27}c_{39})$$

$$c_{40} = c_{20}c_{39} - c_{29}c_{30}$$

$$c_{57} = 2(c_{28}c_{37} - c_{27}c_{38})$$

$$c_{50} = c_{20}c_{38} - c_{28}c_{30} \quad .$$

Substituting Equations 2.24 and 2.25 into Equation 2.21 one finally obtains the quadratic equation

$$\begin{aligned}(c_{11}c_{48}^2+c_{12}c_{47}^2+c_{13}c_{57}^2-c_{14}c_{47}c_{48}+c_{15}c_{48}c_{57}+c_{16}c_{47}c_{57})X_0^2 + [2c_{12}c_{40}c_{47}\\ +2c_{13}c_{50}c_{57}-c_{14}c_{40}c_{48}-c_{15}c_{48}c_{57}+c_{16}(c_{40}c_{57}+c_{47}c_{50})-c_{17}c_{48}^2-c_{18}c_{47}c_{48}\\ +c_{19}c_{48}c_{57}]X_0 + c_{12}c_{40}^2+c_{13}c_{50}^2+c_{16}c_{40}c_{50}-c_{18}c_{48}c_{40}+c_{19}c_{48}c_{50}-c_{10}c_{48}=0.\end{aligned} \tag{2.26}$$

After solving Equation 2.26 for X_0 by use of Equation 2.11, one may find Y_0 and Z_0 from Equations 2.24 and 2.25 respectively. In general, Equation 2.26 will yield two solutions for X_0, say X_{01} and X_{02}. One must test each back into Equations 2.12, 2.13 and 2.14. If one of the two points does not hold, it is an ambiguity due to squaring and may be eliminated. If both X_{01} and X_{02} hold, then a fourth independent baseline equation (formed by one more receiving station) is required in order to remove the ambiguity. It is necessary only to insert the X_{01} and X_{02} into the new equation to resolve the uncertainty.

Example 2.2: Let X_1, Y_1, Z_1; X_2, Y_2, Z_2; X_3, Y_3, Z_3; X_4, Y_4, Z_4 be 10, 10, 10; 40, 10, 10; 10, 40, 50; 10, -50, -30 kilometers respectively. Assume $t_1 - t_2$, $t_1 - t_3$, and $t_1 - t_4$ are measured to be -100/3, 100/3 and -200 microseconds respectively. Then Equations 2.24, 2.25 and 2.26 become

$$Y_0 = -5X_0 + 60$$

$$Z_0 = 3X_0 + 20$$

$$247X_0^2 - 380X_0 - 20{,}900 = 0 \quad .$$

These equations have the two solutions $(X_0, Y_0, Z_0) = (10, 10, 50)$ and (-2,090/247, 25,270/247, -1,330/247) kilometers. Testing both solutions as previously discussed, it is determined that the latter does not hold. Hence, in this example, the solution (10, 10, 50) kilometers for emitter location is unique.

2.3.1.1 Special Case, all Receiving Sites Located in X-Y Plane. Suppose all four receiving stations are located in the X-Y plane. Then $Z_1 = Z_2 = Z_3 = Z_4 = 0$, and c_{29} and c_{39} of Equations 2.22 and 2.23 are zero. Hence,

$$2c_{27}X_0 + 2c_{28}Y_0 = c_{20} \tag{2.27}$$

$$2c_{37}X_0 + 2c_{38}Y_0 = c_{30} \quad . \tag{2.28}$$

The unique solution for (X_0, Y_0) may be found by solving these two linear equations simultaneously. The solutions are:

$$X_0 = \frac{1}{2}\,\frac{c_{20}c_{38} - c_{30}c_{28}}{c_{27}c_{38} - c_{28}c_{37}} \tag{2.29}$$

$$Y_0 = \frac{1}{2}\,\frac{c_{30}c_{27} - c_{20}c_{37}}{c_{27}c_{38} - c_{28}c_{37}} \quad . \tag{2.30}$$

Substituting these solutions for X_0 and Y_0 into Equation 2.21, the solution for Z_0 becomes

$$Z_0 = \pm \left[\frac{c_{14}X_0Y_0+c_{17}X_0+c_{18}Y_0+c_{10}-c_{11}X_0^2-c_{12}Y_0^2}{c_{13}} \right]^{\frac{1}{2}} \quad . \tag{2.31}$$

If the emitter is prohibited below the X-Y plane, then the plus or minus ambiguity in Equation 2.31 is eliminated and

$$Z_0 = \left[\frac{c_{14}X_0Y_0+c_{17}X_0+c_{18}Y_0+c_{10}-c_{11}X_0^2-c_{12}Y_0^2}{c_{13}} \right]^{\frac{1}{2}} \quad . \tag{2.32}$$

2.3.1.2 Special Case, a Point With no Unique Solution. As previously mentioned, the solution for (X_0, Y_0, Z_0) given above does not hold at the point $(t_1 - t_2, t_1 - t_3, t_1 - t_4) = (0, 0, 0)$. Let the origin be at (X_1, Y_1, Z_1) for convenience and suppose $t_1 - t_2 = t_1 - t_3 = t_1 - t_4 = 0$. Then Equations 2.18, 2.19 and 2.20 become

$$2X_2X_0 + 2Y_2Y_0 + 2Z_2Z_0 = X_2^2 + Y_2^2 + Z_2^2 \tag{2.33}$$

$$2X_3X_0 + 2Y_3Y_0 + 2Z_3Z_0 = X_3^2 + Y_3^2 + Z_3^2 \tag{2.34}$$

$$2X_4X_0 + 2Y_4Y_0 + 2Z_4Z_0 = X_4^2 + Y_4^2 + Z_4^2 \quad . \tag{2.35}$$

The unique solution for (X_0, Y_0, Z_0), if one exists, is immediately obtainable from the above three equations. Suppose all receiving sites are in the X-Y plane. Then $Z_1 = Z_2 = Z_3 = Z_4 = 0$. The solution for (X_0, Y_0) may be found from the above equations, but not Z_0. Referring back to Equations 2.12, 2.13 and 2.14, it is seen that they hold for every finite Z_0. Therefore, when all time differences are zero and all receiving sites are in the same plane, there is no unique solution.

CHAPTER III

DEVELOPMENT OF THE SYSTEM MODEL

3.1 Introduction. The system model for estimation of emitter location from measurements acquired by operation of a time difference hyperbolic net of K receiving stations is developed in this chapter. Aside from the extensions listed in Section 1.4, the approach here is completely different from that of Marchand (1). Here, the linear model is formulated so as to have the form

$$\underline{\lambda} = \underline{B}\,\underline{\gamma} + \underline{e} \tag{3.1}$$

where

$\underline{\lambda}$ is a column vector of observations (measurements).

$\underline{B}$ is a matrix of known constants.

$\underline{\gamma}$ is a vector of constants to be estimated (coordinates of the emitter position).

$\underline{e}$ is a random vector of errors.

The advantage of this approach is that Equation 3.1 is the standard linear model which has received extensive theoretical treatment in the statistical literature. The reader may want to refer to one of the many texts on the subject (2,3,4).

3.2 The Arrival Time Vector. Relative to a fixed Cartesian coordinate system with arbitrary choice of origin, let a signal be radiated from point (X_0, Y_0, Z_0) at time t_0 and be received at each of K receiving stations at times $t_1, t_2, \ldots, t_K$. The arrival time at the i^{th} receiver may be expressed

$$t_i = t_0 + \frac{1}{u} R_i \qquad (3.2)$$

where

t_i is the time of reception at the i^{th} receiver.

t_0 is the time of radiation by the emitter.

u is the scalar velocity of propagation.

R_i is the distance between emitter and receiver i.

Considering all K receiving stations simultaneously, one has a column vector of K arrival times defined as follows:

$$\underline{Ta} = t_0 J_1^K + \frac{1}{u} \underline{R} \qquad (3.3)$$

where

$\underline{Ta}$ is the arrival time vector $(t_1, t_2, \ldots, t_K)'$.

J_1^K is a column vector of K ones.

$\underline{R}$ is the range vector $(R_1, R_2, \ldots, R_K)'$.

The symbol ' denotes the transpose of a vector.

3.3 <u>The Time Difference Vector</u>. One cannot observe t_0 in Equation 3.3. Measurements are either in the form of time differences $t_1 - t_i$ or arrival times t_i. If the latter, one may convert the data to time differences by taking the difference $t_1 - t_i$ for $i = 2, 3, \ldots, K$. Hence, it is necessary to form a time difference vector from the arrival time vector. To do this, it is convenient to use a time difference generating matrix. The appropriate generating matrix is a K - 1 by K matrix of zeros and ones defined as follows: $C = J_1^{K-1}, -I_{K-1})$ where J_1^{K-1} is a K-1 column vector of ones, and I_{K-1} is the identity matrix of order K-1.

Example 3.3.1: When K = 4

$$C = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} .$$

Multiplying Equation 3.3 through by C, the time difference vector becomes

$$\underline{Td} = \frac{1}{u} C \underline{R} \qquad (3.4)$$

where

$$\underline{Td} = (t_1 - t_2, t_1 - t_3, \ldots, t_1 - t_K)'.$$

<u>3.4 Bounds on the Time Difference Vector</u>. Consider the i^{th} equation of the time difference vector,

$$u(t_1 - t_i) = R_1 - R_i \quad .$$

Let the undirected distance between receiving stations one and i be designated d_{1i}. Reference Figure 3.1 and note R_1, R_i and d_{1i} form a triangle. The magnitude of $R_1 - R_i$ cannot exceed the magnitude of d_{1i} because the difference between two sides of a triangle is never greater than the third. Hence,

$$-\frac{1}{u} d_{1i} \leq t_1 - t_i \leq \frac{1}{u} d_{1i} \quad .$$

The vector relation is

$$-\frac{1}{u} \underline{d} \leq \underline{Td} \leq \frac{1}{u} \underline{d} \qquad (3.5)$$

where

$\underline{d} = (d_{12}, d_{13}, \ldots, d_{1K})'$.

Hence, the time difference vector is bounded between $\pm \frac{1}{u} \underline{d}$ regardless of how distant the emitter. This fact could be of practical significance in the problem of designing instrumentation to measure time difference vector $\underline{T}d$. As mentioned previously, the choice of $\underline{T}d = (t_1 - t_2, t_1 - t_3, \ldots, t_1 - t_K)'$ was arbitrary. For fixed receiving sites, one might want to choose an independent set of time differences so as to minimize the maximum d_{ij} of the set, where d_{ij} is the distance between receiving sites i and j.

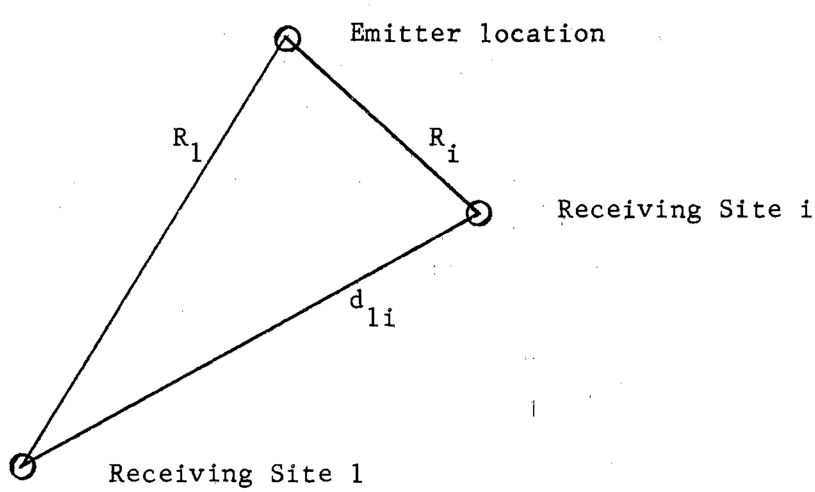

Figure 3.1. Bounds on Time Difference $t_1 - t_i$

3.5 The General Model. Observation of the time difference vector is subject to measurement error. Furthermore, Equation 3.4 is based upon the ideal model of electromagnetic waves propagating in straight-line paths at constant speed. Let the total error due to time difference measurement inaccuracies and propagation anomalies be designated $\underline{e}_{Td}$. Then the time difference observations may be written

$$\underline{\theta} = \frac{1}{u_0} C \underline{R} \underline{e}_{Td} \tag{3.6}$$

where

$\underline{\theta} = (t_1 - t_2, t_1 - t_3, \ldots, t_1 - t_K)'$ plus error is the vector of time difference measurements.

Note that we have replaced u (possibly unknown) by u_0 (known). The difference is accounted for in $\underline{e}_{Td}$.

As an introduction to what follows, let us look at the estimation problem associated with Equation 3.6. Recall $\underline{R} = (R_1, R_2, \ldots, R_K)'$ where R_i is the true distance between emitter and receiver i. R = $[(X_0 - X_i)^2 + (Y_0 - Y_i)^2 + (Z_0 - Z_i)^2]^{\frac{1}{2}}$ where (X_0, Y_0, Z_0) is the true emitter location, and (X_i, Y_i, Z_i) is the true site location of the i^{th} receiver. While (X_0, Y_0, Z_0) and (X_i, Y_i, Z_i) may vary with time, we assume the variation is slow enough that they may be considered as fixed during the time interval under consideration. There are 3K + 3 constants contained in $\underline{R}$. The basic objective is to estimate three of these, the emitter location (X_0, Y_0, Z_0). The remaining 3K are already known; however, in general, these 3K constants $(X_1, Y_1, Z_1; X_2, Y_2, Z_2; \ldots; X_K, Y_K, Z_K)$ are known subject to error. This error will be propagated into our final estimate of emitter location.

Let the uncertainty associated with receiving site locations be expressed

$$\underline{\lambda}_1 = \underline{\alpha} + \underline{e}_s \tag{3.7}$$

where

$\underline{\lambda}_1$ is a random vector of recorded site locations.

$\underline{\alpha}$ is the true location of receiving sites.

$\underline{e}_s$ is the position measurement error.

$\underline{\alpha}$ and $\underline{\lambda}_1$ will be defined more explicitly in Section 3.6.

Let Equations 3.6 and 3.7 be augmented

$$\begin{bmatrix} \underline{\lambda}_1 \\ \underline{\theta} \end{bmatrix} = \begin{bmatrix} & \underline{\alpha} \\ \frac{1}{u_0} & C\,\underline{R} \end{bmatrix} + \begin{bmatrix} \underline{e}_s \\ \underline{e}_{Td} \end{bmatrix} \quad . \tag{3.8}$$

Equation 3.8 will be called the general system model. We have a random vector of 4K - 1 elements (data) equal to a function of 3K + 3 constants (X_0, Y_0, Z_0, X_1, Y_1, Z_1, X_2, Y_2, Z_2, ..., X_K, Y_K, Z_K) plus error. The general model is nonlinear due to the form of $\underline{R}$. In the next section, the case is considered when the errors are small enough that $\underline{R}$ may be approximated by a first order Taylor series.

3.6 <u>The General Linear Model</u>. Consider the range equation $R_i = [(X_0 - X_i)^2 + (Y_0 - Y_i)^2 + (Z_0 - Z_i)^2]^{\frac{1}{2}}$ where R_i is the distance between emitter and receiving site i. Let (X_i^*, Y_i^*, Z_i^*) be chosen as a known point near the true site location (X_i, Y_i, Z_i). One may select the recorded estimate of (X_i, Y_i, Z_i) as the point (X_i^*, Y_i^*, Z_i^*) if desired. Let (X_0^*, Y_0^*, Z_0^*) be chosen as a known point near the true emitter location (X_0, Y_0, Z_0). One can find point (X_0^*, Y_0^*, Z_0^*) by the deterministic solution derived in Chapter II.

As the two unknown points $[(X_0, Y_0, Z_0);(X_i, Y_i, Z_i)]$ vary around the two fixed points $[(X_0^*, Y_0^*, Z_0^*);(X_i^*, Y_i^*, Z_i^*)]$ respectively, the first order Taylor series of R_i may be written

$$R_i = R_i^* + [(X_0^*-X_i^*)(X_0-X_0^*) + (Y_0^*-Y_i^*)(Y_0-Y_i^*) + (Z_0^*-Z_i^*)(Z_0-Z_0^*)$$
$$- (X_0^*-X_i^*)(X_i-X_i^*) - (Y_0^*-Y_i^*)(Y_i-Y_i^*) - (Z_0^*-Z_i^*)(Z_i-Z_i^*)] \qquad (3.9)$$

where

$$R_i^* = [(X_0^*-X_i^*)^2 + (Y_0^*-Y_i^*)^2 + (Z_0^*-Z_i^*)^2]^{\frac{1}{2}} \quad .$$

Letting i range from one to K, the vector $\underline{R}$ becomes

$$\underline{R} = \begin{bmatrix} R_1^* + a_{11}(\beta_1-\alpha_{11}) + a_{12}(\beta_2-\alpha_{12}) + a_{13}(\beta_3-\alpha_{13}) \\ R_2^* + a_{21}(\beta_1-\alpha_{21}) + a_{22}(\beta_2-\alpha_{22}) + a_{23}(\beta_3-\alpha_{23}) \\ \vdots \\ R_K^* + a_{K1}(\beta_1-\alpha_{K1}) + a_{K2}(\beta_2-\alpha_{K2}) + a_{K3}(\beta_3-\alpha_{K3}) \end{bmatrix}$$

where

$$\left.\begin{aligned} a_{i1} &= \frac{(X_0^*-X_i^*)}{R_i^*} \\ a_{i2} &= \frac{(Y_0^*-Y_i^*)}{R_i^*} \\ a_{i3} &= \frac{(Z_0^*-Z_i^*)}{R_i^*} \end{aligned}\right\} \quad \text{i.e. the direction cosines.}$$

$$\left.\begin{aligned} \beta_1 &= X_0-X_0^* \\ \beta_2 &= Y_0-Y_0^* \\ \beta_3 &= Z_0-Z_0^* \end{aligned}\right\} \quad \text{i.e. displacement of the emitter from assumed location.}$$

$$\left.\begin{aligned}\alpha_{i1} &= X_i - X_i^* \\ \alpha_{i2} &= Y_i - Y_i^* \\ \alpha_{i3} &= Z_i - Z_i^*\end{aligned}\right\}$$ i.e. displacement of receiving site i from assumed location.

In order to express $\underline{R}$ in more compact notation, the following vectors and matrices are defined:

$$\underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$\underline{\alpha}_i = \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \alpha_{i3} \end{bmatrix}$$

$$\underline{A}_i = (a_{i1}, a_{i2}, a_{i3})$$

$$\underline{A} = \begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{A}_K \end{bmatrix}$$

$$\underline{G} = \begin{bmatrix} \underline{A}_1 & & & & & \\ & \underline{A}_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \underline{A}_K \end{bmatrix}$$

$$\underline{R}^* = (R_1^*, R_2^*, \ldots, R_K^*)'$$

$$\underline{\alpha} = \begin{bmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{\alpha}_K \end{bmatrix} .$$

Then

$$\underline{R} = \begin{bmatrix} R_1^* + \underline{A}_1(\underline{\beta}-\underline{\alpha}_1) \\ R_2^* + \underline{A}_2(\underline{\beta}-\underline{\alpha}_2) \\ \cdot \\ \cdot \\ \cdot \\ R_K^* + \underline{A}_K(\underline{\beta}-\underline{\alpha}_K) \end{bmatrix} , \tag{3.10}$$

and finally

$$\underline{R} = \underline{R}^* + \underline{A}\,\underline{\beta} - \underline{G}\,\underline{\alpha} . \tag{3.11}$$

Let Equation 3.11 be substituted into Equation 3.6. Then the linear approximation to the time difference model becomes

$$\underline{\theta} = \frac{1}{u_0} C(\underline{R}^* + \underline{A}\,\underline{\beta} - \underline{G}\,\underline{\alpha}) + \underline{e}_{Td} \quad . \qquad (3.12)$$

The receiving site model was defined by Equation 3.7 as

$$\underline{\lambda}_1 = \underline{\alpha} + \underline{e}_s \quad . \qquad (3.13)$$

Let $\underline{\alpha}$ be as defined above, that is $\underline{\alpha} = (X_1-X_1^*, Y_1-Y_1^*, Z_1-Z_1^*; X_2-X_2^*, Y_2-Y_2^*, Z_2-Z_2^*; \ldots; X_K-X_K^*, Y_K-Y_K^*, Z_K-Z_K^*)'$. Then $\underline{\lambda}_1$ approximates the true site locations minus bias vector $(X_1^*, Y_1^*, Z_1^*; X_2^*, Y_2^*, Z_2^*; \ldots; X_K^*, Y_K^*, Z_K^*)'$. We require this arrangement in order to approximate R_i by a linear function of X_0, Y_0, Z_0, X_i, Y_i and Z_i. In the linear model, (X_i^*, Y_i^*, Z_i^*) must be near (X_i, Y_i, Z_i), but in the general model (X_i^*, Y_i^*, Z_i^*) may be arbitrarily chosen. $\underline{\lambda}_1$ of Equation 3.13 is the output of the sensor which measures site coordinates if the net of receiving stations are in motion. If the receiving stations are fixed ground sites, then $\underline{\lambda}_1$ is the vector of recorded site coordinates.

Let Equations 3.12 and 3.13 be augmented

$$\begin{bmatrix} \underline{\lambda}_1 \\ \underline{\theta} \end{bmatrix} = \begin{bmatrix} \underline{\alpha} \\ C(\underline{R}^* + \underline{A}\,\underline{\beta} - \underline{G}\,\underline{\alpha}) \end{bmatrix} + \begin{bmatrix} \underline{e}_s \\ \underline{e}_{Td} \end{bmatrix} \qquad (3.14)$$

Equation 3.14 will be called the general linear system model.

CHAPTER IV

SYSTEM ERRORS

4.1 Introduction. The general system model of a time difference hyperbolic net has error components due to: (a) inaccurate receiving site locations, (b) time difference measurement inaccuracies, (c) instrumental errors associated with the measurement process, and (d) errors due to propagation anomalies. For our purpose, an error component is characterized by its mean and dispersion matrix.

Definition 4.1: Let $\underline{\lambda}$ be an n by 1 random vector. The symbol E is defined as an operator such that $E\underline{\lambda}$ is the mean value of $\underline{\lambda}$. The symbol D is defined as an operator such that $D\underline{\lambda}$ is the dispersion matrix of $\underline{\lambda}$. That is, $D\underline{\lambda} = E[(\underline{\lambda} - E\underline{\lambda})(\underline{\lambda} - E\underline{\lambda})']$.

The form of the dispersion matrix for each error component and for the total system error is derived in this chapter.

4.2 Theory of Error Combination. Before proceeding further, it is necessary to develop some theoretical results. Proofs for the stated theorems are given in Appendix B.

Let Σ be an n by n matrix.

Definition 4.2: Σ is said to be positive definite if and only if $\underline{a}'\Sigma\,\underline{a} > 0$ for every n by 1 non null vector $\underline{a}$.

Definition 4.3: Σ is said to be positive semidefinite if and only if $\underline{a}' \Sigma \underline{a} \geq 0$ for every vector $\underline{a}$, and $\underline{a}' \Sigma \underline{a} = 0$ for some non-null $\underline{a}$.

Theorem 4.4: Let Σ be positive definite, and let $\underline{B}$ be an p by n matrix. Then $\underline{B} \Sigma \underline{B}'$ is positive definite when $\underline{B}$ is of rank p and positive semidefinite when $\underline{B}$ is of rank less than p.

Theorem 4.5: Let Σ and $\underline{B}$ be as defined in Theorem 4.4. Then every diagonal element of $\underline{B} \Sigma \underline{B}'$ is zero only if $\underline{B} = \emptyset$ where $\emptyset$ is a null matrix. Also, $\underline{B} \Sigma \underline{B}' = \emptyset$ only if $\underline{B} = \emptyset$.

Theorem 4.6: Let Σ be positive definite and symmetric. Then Σ^{-1} exists and is positive definite and symmetric.

Theorem 4.7: Let Σ_1 be positive definite and Σ_2 be positive definite or positive semidefinite. Then $\Sigma_1 + \Sigma_2$ is positive definite.

In the following theorems, the symbols $\underline{Y}$ and $\underline{X}$ are used to designate random column vectors and constant matrices respectively. $\underline{Y}$ and $\underline{X}$ should not be confused with location coordinates Y and X used elsewhere in this paper. When the statement is made that a vector or matrix is arbitrary, it is to be understood that the order of the arbitrary quantity must be compatible with the indicated algebraic operation.

Theorem 4.8: Let $\underline{Y}_1 = \underline{X}\,\underline{Y} + \underline{b}$ where $\underline{Y}_1$ and $\underline{Y}$ are random vectors, $\underline{X}$ is an arbitrary constant matrix, and $\underline{b}$ is an arbitrary constant vector. Then $D\underline{Y}_1 = \underline{X}(D\underline{Y})\underline{X}'$.

Definition 4.9: Let $\underline{Y}_1$ and $\underline{Y}_2$ be two random vectors, not necessarily of the same dimensions. The covariance of $(\underline{Y}_1, \underline{Y}_2)$ is defined: $\mathrm{Cov}(\underline{Y}_1, \underline{Y}_2) = E[(\underline{Y}_1 - E\underline{Y}_1)(\underline{Y}_2 - E\underline{Y}_2)']$. Note that $\mathrm{Cov}(\underline{Y}_1, \underline{Y}_2) = [\mathrm{Cov}(\underline{Y}_2, \underline{Y}_1)]'$, and $\mathrm{Cov}(\underline{Y}_1, \underline{Y}_1) = D\underline{Y}_1$.

Theorem 4.10: Let $\underline{Y}_1$ and $\underline{Y}_2$ be two random vectors, and let $\underline{X}_1$ and $\underline{X}_2$ be arbitrary constant matrices. Then $\mathrm{Cov}(\underline{X}_1\underline{Y}_1, \underline{X}_2\underline{Y}_2) = \underline{X}_1[\mathrm{Cov}(\underline{Y}_1, \underline{Y}_2)]\underline{X}_2'$.

Theorem 4.11: Let $\underline{Y}_1$ and $\underline{Y}_2$ be two random vectors, and let $\underline{X}_1$ and $\underline{X}_2$ be two constant matrices. Let the dimensions of $\underline{X}_1$, $\underline{X}_2$, $\underline{Y}_1$ and $\underline{Y}_2$ be such that products $\underline{X}_1\underline{Y}_1$ and $\underline{X}_2\underline{Y}_2$ are each n by one where n is arbitrary. Then $D(\underline{X}_1\underline{Y}_1 \pm \underline{X}_2\underline{Y}_2) = \underline{X}_1(D\underline{Y}_1)\underline{X}_1' + \underline{X}_2(D\underline{Y}_2)\underline{X}_2' \pm \underline{X}_1[\mathrm{Cov}(\underline{Y}_1, \underline{Y}_2)]\underline{X}_2' \pm \underline{X}_2[\mathrm{Cov}(\underline{Y}_2, \underline{Y}_1)]\underline{X}_1'$.

Theorem 4.12: Let $\underline{Y}_1$, $\underline{Y}_2$, $\underline{X}_1$ and $\underline{X}_2$ be as specified in Theorem 4.11. Then if $\underline{Y}_1$ and $\underline{Y}_2$ are uncorrelated, $D(\underline{X}_1\underline{Y}_1 \pm \underline{X}_2\underline{Y}_2) = \underline{X}_1(D\underline{Y}_1)\underline{X}_1' + \underline{X}_2(D\underline{Y}_2)\underline{X}_2'$.

4.3 Receiving Site Errors. The receiving site error model was defined by Equation 3.7 as

$$\underline{\lambda}_1 = \underline{\alpha} + \underline{e}_s \quad . \tag{4.1}$$

where $\underline{e}_s$ is a 3K by 1 random vector of site errors with assumed zero mean. Let the dispersion matrix of $\underline{e}_s$ be designated Σ_s. That is, recorded site coordinate vector $\underline{\lambda}_1$ has mean $\underline{\alpha}$ and dispersion matrix Σ_s. We require Σ_s to be positive definite in the general system model, but this restriction will be removed later in a special model yet to be defined. In general, we require Σ_s to be known. However, if it is not known, one may compute an estimated $\tilde{\Sigma}_s$ from a given uncertainty in $\underline{\lambda}_1$.

Example 4.1: Let the j^{th} element of $\underline{\lambda}_1$ be the one dimensional random variable $\lambda_{1j} = X_i - X_i^* + e_{sj}$. Suppose the uncertainty associated with a given measurement of X_i is $\pm$ one kilometer. Then assuming an equally likely (uniform) distribution, $\sigma^2_{e_{sj}} = 1/3$ kilometer^2. Proceeding in this manner with each of the 3K elements of $\underline{\lambda}_1$, one obtains a diagonal matrix Σ_s which expresses the uncertainty associated with $\underline{\lambda}_1$.

4.4 Time Difference Measurement Errors. The time difference measurement model was defined by Equation 3.6 as

$$\underline{\theta} = \frac{1}{u_0} C \underline{R} + \underline{e}_{Td} \tag{4.2}$$

where $\underline{e}_{Td}$ is a K-1 by 1 random vector of errors. Let

$$\underline{e}_{Td} = \underline{e}_M + \underline{e}_\delta$$

where $\underline{e}_M$ is an error component due to time difference measurement inaccuracies and $\underline{e}_\delta$ is an error component due to propagation anomalities. We require the dispersion matrix $\Sigma_{Td} = D\underline{e}_{Td}$ to be positive definite. By Theorem 4.7, $D\underline{e}_{Td}$ will be positive definite if $D\underline{e}_M$ is. The dispersion matrix Σ_{Td} is required in order to compute the best estimate of emitter location. However, if Σ_{Td} is not known, and if no reasonable estimate is otherwise available, one can find an estimated $\tilde{\Sigma}_{Td}$ as discussed in Section 4.3 above for Σ_s. This procedure allows one to compute the uncertainty of estimated emitter location based upon uncertainty associated with an observation of $\underline{\theta}$.

In order to illustrate the effects of measurement methods upon the form of Σ_M, where $\Sigma_M = D\underline{e}_M$, we describe two procedural arrangements for measuring time differences.

4.4.1 Arrival Time Measurements. Suppose K receiving stations are time synchronized such that each can measure real time events relative to a common time base. Consider the case when an emitted signal has some known set of characteristics such that detection of these characteristics at a receiving site determines (subject to error) arrival time at that site. Hence, at station i, (i = 1, 2, ..., K) an observation b is measured where b_i is equal to the arrival time t_i plus error. Specifically

$$b_i = t_i + e_{t_i} + e_{N_i} \tag{4.3}$$

where

t_i is the true arrival time.

e_{t_i} is time synchronization error at station i.

e_{N_i} is the measurement error at station i due to noise on the signal and "noise" in the measurement process.

By nature of the operation, e_{t_i} and e_{N_i} may be safely assumed to be uncorrelated.

Assume that through prior calibration, one knows $Ee_{t_i} = Ee_{N_i} = 0$, and one has an estimate for the variance of e_{t_i}. Further assume that through prior simulation and analysis, one has an "estimator" which assigns a variance to e_{N_i} based upon signal-to-noise ratio and modulation characteristics of the received signal. Hence,

$$Eb_i = t_i \tag{4.4}$$

and

$$\sigma^2_{b_i} = \sigma^2_{e_{t_i}} + \sigma^2_{e_{N_i}} \tag{4.5}$$

where $\sigma^2_{e_{t_i}}$ and $\sigma^2_{e_{N_i}}$ are assumed known, or at least bonafide estimates are available

Let $\underline{b} = (b_1, b_2, \ldots, b_K)'$. Then

$$\underline{b} = \underline{T}a + \underline{e}_t + \underline{e}_N \tag{4.6}$$

where $\underline{T}a$ is the K by 1 arrival time vector defined by Equation 3.3, and $\underline{e}_t$ and $\underline{e}_N$ are arrival time measurement errors, $\underline{e}_t$ being uncorrelated with $\underline{e}_N$. By the argument above pertaining to the i^{th} element of $\Sigma_t = D\underline{e}_t$ and $\Sigma_N = D\underline{e}_N$, we assume Σ_t and Σ_N are known diagonal matrices.

The time difference measurement is obtained by taking the difference $(b_1 - b_2, b_1 - b_3, \ldots, b_1 - b_K)$. As described in Section 3.3, this is accomplished by multiplying through by the matrix C.

$$C\,\underline{b} = C\,\underline{T}a + C\,\underline{e}_t + C\,\underline{e}_N \quad . \tag{4.7}$$

Recall $\underline{\theta} = C\,\underline{T}a + \underline{e}_M + \underline{e}_\delta$ where $\underline{e}_M$ is the measurement error and $\underline{e}_\delta$ is the error due to propagation anomalies. Hence,

$$C\,\underline{b} = \underline{\theta} - \underline{e}_\delta = C\,\underline{T}a + \underline{e}_M \tag{4.8}$$

where $\underline{e}_M = C\,\underline{e}_t + C\,\underline{e}_N$. By Theorem 4.12, $\Sigma_M = D\underline{e}_M = C(\Sigma_t + \Sigma_N)C'$.

Example 4.2: For $K = 5$, let $\sigma_{e_{t_i}} = 10$ and $\sigma_{e_{N_i}} = 30$ nanoseconds^2 respectively for all i. Find Σ_M.

Solution: $\Sigma_t = 100 \times 10^{-18}\, I_5$ seconds^2.

$\Sigma_N = 900 \times 10^{-18}\, I_5$ seconds^2.

Then $\Sigma_M = C(\Sigma_t + \Sigma_N)C' = 10^{-15}\, CC'$ seconds^2.

From the above example, one may generalize as follows: In the case of arrival time measurements when e_{t_i} has variance $\sigma^2_{e_t}$ at all stations and when e_{N_i} has variance $\sigma^2_{e_N}$ at all stations, then

$$\Sigma_M = (\sigma^2_{e_t} + \sigma^2_{e_N})\, CC' \quad .$$

The inverse of C C' is $I_{K-1} - \frac{1}{K} J_{K-1}^{K-1}$. Therefore Σ_M^{-1} is readily available for the above special case.

Marchand (1) has proposed forming a random vector of $\binom{K}{2} = \frac{1}{2} K(K-1)$ time differences from the K arrival time measurements and then assume the resulting $\binom{K}{2}$ by $\binom{K}{2}$ dispersion matrix has an inverse. Let us show that this cannot be. Let $\underline{e}_N$ be the K by 1 random vector of arrival time measurement errors with dispersion matrix Σ_N. Let C_1 be the time difference generating matrix which blows $\underline{e}_N$ up into a vector of $\binom{K}{2}$ errors designated $\underline{e}_K$. (We define C_1 in Appendix A.) Then $\underline{e}_K = C_1 \underline{e}_N$ and by Theorem 4.8, $D\underline{e}_K = C_1 \Sigma_N C_1'$. Since $\underline{e}_K$ has $\binom{K}{2}$ elements, $D\underline{e}_K$ is $\binom{K}{2}$ by $\binom{K}{2}$ while Σ_N is K by K. This implies the rank of $D\underline{e}_K$ is at most K. Therefore, $D\underline{e}_K$ cannot have an inverse.

Marchand also proposed that $D\underline{e}_K$ be assumed diagonal. Let the first K-1 elements of $\underline{e}_K$ be ordered $(e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_K)$. Then if $D\underline{e}_K$ is partitioned forming a K-1 by K-1 submatrix in the upper left hand corner, the result will be $D\underline{e}_M$ defined above in conjunction with Equation 4.8. We have shown by the special case above that $D\underline{e}_M = \sigma^2 C C' = \sigma^2 (I_{K-1} + J_{K-1}^{K-1})$ when the variance at all sites equals σ^2. Therefore, Marchand's $D\underline{e}_K$ has no inverse and is in general not diagonal.

<u>4.4.2 Time Difference Measurements</u>. Suppose the signal (after detection) is relayed from each of K receiving stations to a central processing station as illustrated in Figure 4.1 below. In this case, receiver site time synchronization is no longer required. Instead it is necessary to know the time delay of the relay link for relay channels 1, 2, ..., K. The total propagation time (neglecting for the moment $\underline{e}_\delta$) from emitter to processing station via the i^{th} receiver is $t_i - t_0 + t_{r_i} + e_{r_i}$, where t_{r_i} is the mean time delay over the relay link from

receiver i to the processing station, and e_{r_i} is an additive random fluctuation to t_{r_i}. Assume that e_{r_i} has the following properties: $Ee_{r_i} = 0$, $\sigma^2_{e_{r_i}}$ is known, and $Cov(e_{r_i}, e_{r_j})$ is zero for every $i \neq j$. Also assume change in e_{r_i} is negligible over time interval under consideration.

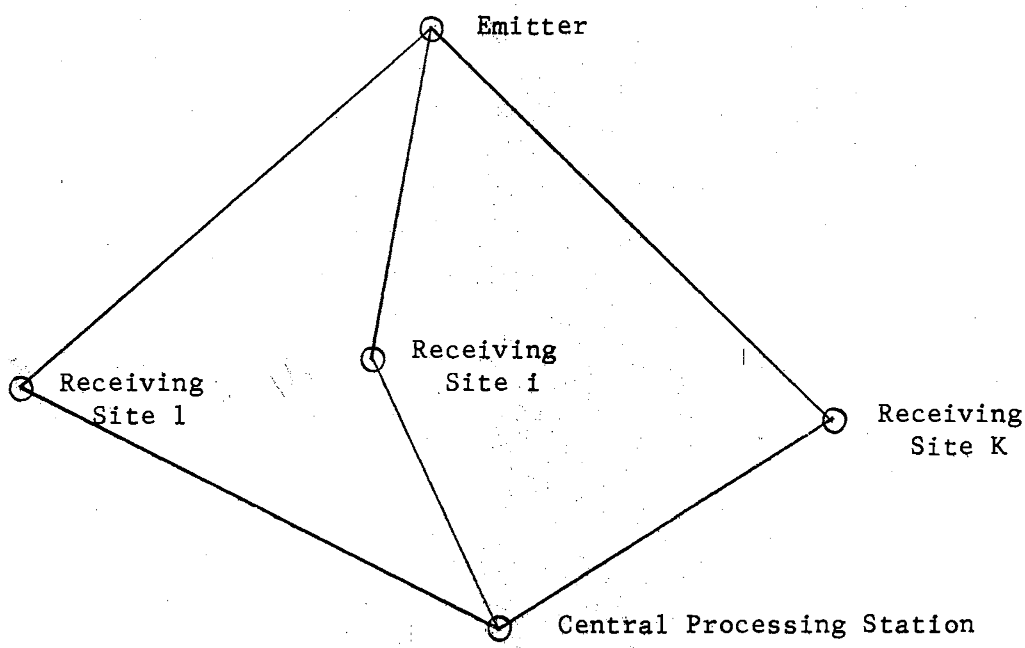

Figure 4.1. Net for Relaying Signal From Each of K Receiving Stations to a Remote Processing Station

At the processing station where all K signals (the emitted signal propagated over K paths) are available at a common terminal, let the time difference measurements b_{1i} for $i = 2, 3, \ldots, K$ be expressed

$$b_{12} = (t_1 - t_0 + t_{r_1} + e_{r_1}) - (t_2 - t_0 + t_{r_2} + e_{r_2}) + e_{N_1} - e_{N_2}$$

$$b_{13} = (t_1 - t_0 + t_{r_1} + e_{r_1}) - (t_3 - t_0 + t_{r_3} + e_{r_3}) + e_{N_1} - e_{N_3}$$

$$\cdots \qquad (4.9)$$

$$\cdots$$

$$\cdots$$

$$b_{1K} = (t_1 - t_0 + t_{r_1} + e_{r_1}) - (t_K - t_0 + t_{r_K} + e_{r_K}) + e_{N_1} - e_{N_K}$$

where e_{N_i} is the error introduced into the measurement due to noise on signal i. Let us consider the case when the noise on signal i is uncorrelated with that of signal j for $i \neq j$. Assume that through prior simulation and analysis, one can estimate $\sigma^2_{N_i}$ based upon signal-to-noise ratio and modulation characteristics of the received signal. Further assume $Ee_{N_i} = 0$ for all i.

Equation 4.9 may be written in vector notation

$$\underline{b} = C(\underline{T}a + \underline{T}r + \underline{e}_r + \underline{e}_N) \qquad (4.10)$$

where

$\underline{b} = (b_{12}, b_{13}, \ldots, b_{1K})' = \underline{\theta} - \underline{e}_\delta$.

$\underline{e}_r = (e_{r_1}, e_{r_2}, \ldots, e_{r_K})'$ is the relay error vector.

$\underline{e}_N = (e_{N_1}, e_{N_2}, \ldots, e_{N_K})'$ is the error vector due to noise.

From the argument above pertaining to e_{r_i} and e_{N_i}, assume $E\underline{e}_r = E\underline{e}_N = \emptyset$, and $\Sigma_r = D\underline{e}_r$ and $\Sigma_N = D\underline{e}_N$ are known diagonal matrices.

Recall that $\underline{\theta} - \underline{e}_\delta = C\ \underline{T}a + \underline{e}_M$ which implies that

$$\underline{e}_M = C(\underline{T}_r + \underline{e}_r + \underline{e}_N) \qquad (4.11)$$

$$E\underline{e}_M = C\ \underline{T}_r \qquad (4.12)$$

and by Theorem 4.12,

$$\Sigma_M = D\underline{e}_M = C(\Sigma_r + \Sigma_N)C' \quad . \qquad (4.13)$$

4.5 Propagation Error. To account for the uncertainty due to propagation anomalies, the error component $\underline{e}_\delta$ was added to the time difference measurement model

$$\underline{\theta} = \underline{T}d + \underline{e}_{Td} = \underline{T}d + \underline{e}_M + \underline{e}_\delta \quad . \qquad (4.14)$$

The error $\underline{e}_\delta$ may be any general function of parameters which causes

$$\underline{T}d = \frac{1}{u} C \underline{R} \qquad (4.15)$$

to not hold when the true value of $\underline{R}$ is known and when $\underline{T}d = (t_1 - t_2, t_1 - t_3, \ldots, t_1 - t_K)$ is measured without error. Since $\underline{e}_\delta$ is a K-1 element vector formed by differencing a K element vector, $\underline{e}_\delta$ may in general be expressed

$$\underline{e}_\delta = C(\underline{T}p + \underline{e}_p) \qquad (4.16)$$

where $\underline{T}p$ and $\underline{e}_p$ are K element vectors which in general are functions of time. Assume the time variation is negligible during the time interval of measurement. Then $\underline{T}p$ is a constant vector and $\underline{e}_p$ is a random vector with zero mean value.

In general, one may not be able to estimate $E\underline{e}_\delta$ and $D\underline{e}_\delta$ individually. In this case, he might want to generate a dispersion matrix $\tilde{\Sigma}_p$ based upon the total uncertainty in $\underline{T}p + \underline{e}_p$, and then compute the effect of this uncertainty in the estimate of emitter location.

As a special case, assume $\underline{e}_\delta$ may be approximated by a linear function of distance. Let 1/u be written $1/u = 1/u_0 + (u_0-u)/u_0^2 + \ldots$ where u is the true speed of propagation, assumed constant throughout the net, and u_0 is the assumed estimate of u. Neglecting all terms above first order, Equation 4.15 becomes

$$\underline{T}d = \frac{1}{u_0} C \underline{R} + \frac{u_0 - u}{u_0^2} C \underline{R}^* \quad . \tag{4.17}$$

Let $\delta_0 = (u_0 - u)/u_0$, and let $\tilde{\delta}_0$ be a random variable which estimates δ . Then

$$\underline{T}p = \frac{\delta_0}{u_0} \underline{R}^* \tag{4.18}$$

and

$$\underline{e}_p = \frac{\tilde{\delta}_0}{u_0} \underline{R}^* \quad . \tag{4.19}$$

That is, random vector $\underline{e}_p$ is an estimator which estimates unknown constant vector $\underline{T}p$. The best available estimate of u is u_0 which implies $E\tilde{\delta}_0$ is zero. Therefore, the estimate of $\underline{T}p$ is $\emptyset$. From a given uncertainty in an estimate of u, say $\pm$ w parts per 10^6, one may estimate $\sigma_{\tilde{\delta}_0}$ and then estimate $D\underline{e}_p$ and $D\underline{e}_\delta$.

Example 4.3: Let u_0 have an uncertainty of $\pm$ 10 parts per million. Estimate Σ_δ.

Solution: Assume the uncertainty in u_0 implies $\tilde{\delta}_0$ (a dimensionless random variable) is uniformly distributed over the interval $\pm 10^{-5}$. This implies $\sigma^2_{\tilde{\delta}_0} = \frac{1}{3} \text{ X } 10^{-10}$. Applying Theorem 4.8 to Equations 4.19 and 4.16 in turn, $\Sigma_p = \frac{1}{3u_0^2} \text{ X } 10^{-10} \underline{R}^*(\underline{R}^*)'$ seconds^2 and $\Sigma_\delta = \frac{1}{3u_0^2}$ X 10^{-10} $(C\,\underline{R}^*)(C\,\underline{R}^*)'$ seconds^2.

<u>4.6 Total Error Dispersion Matrix.</u> Let Equation 3.12 be rewritten

$$u_0\underline{\theta} - C\,\underline{R}^* - u_0 E\underline{e}_{Td} = C(\underline{A}\,\underline{\beta} - \underline{G}\,\underline{\alpha}) + u_0(\underline{e}_{Td} - E\underline{e}_{Td}) \tag{4.20}$$

and let

$$\underline{\lambda}_2 = u_0\underline{\theta} - C\,\underline{R}^* - u_0 E\underline{e}_{Td} \quad . \tag{4.21}$$

Then the general linear system model defined by Equation 3.14 becomes

$$\begin{bmatrix} \underline{\lambda}_1 \\ \underline{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \underline{\alpha} \\ C(\underline{A}\,\underline{\beta} - \underline{G}\,\underline{\alpha}) \end{bmatrix} + \begin{bmatrix} \underline{e}_s \\ u_0(\underline{e}_{Td} - E\underline{e}_{Td}) \end{bmatrix} \tag{4.22}$$

which for convenience is written in the condensed notation

$$\underline{\lambda} = \underline{B}\,\underline{\gamma} + \underline{e} \tag{4.23}$$

where $\underline{\lambda}$ is the vector of total measurements, receiving site locations and coded time difference measurements, coded by Equation 4.21.

$$\underline{\gamma} = \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \end{bmatrix} \quad . \tag{4.24}$$

$\underline{B}$ is a known constant matrix which does not depend upon $\underline{\gamma}$ so long as errors are small enough for linearity to hold.

$$\underline{B} = \begin{bmatrix} I_{3K} & \emptyset_3^{3K} \\ -C\,\underline{G} & C\,\underline{A} \end{bmatrix} \quad . \tag{4.25}$$

$\underline{e}$ is a random vector of total system errors.

$$\underline{e} = \begin{bmatrix} \underline{e}_s \\ u_0(\underline{e}_{Td} - E\underline{e}_{Td}) \end{bmatrix} \tag{4.26}$$

$$E\underline{e} = \emptyset \quad .$$

In general, we make no restrictions on the form of $D\underline{e}$ except that it be positive definite. That is, the receiving site locational errors could be correlated with time difference errors. However, let $D\underline{e} = \Sigma$, and suppose the site errors are uncorrelated with time difference errors. Then

$$\Sigma = \begin{bmatrix} \Sigma_s & \emptyset^{3K}_{K-1} \\ \emptyset^{K-1}_{3K} & \Sigma_2 \end{bmatrix} \tag{4.27}$$

where $\Sigma_s = D\underline{e}_s$, and $\Sigma_2 = u_0^2 D\underline{e}_{Td}$. We further assume estimates of Σ_s and Σ_2 are available. (This problem has been discussed in previous sections.)

CHAPTER V

LINEAR ESTIMATION OF EMITTER POSITION AND VARIANCE OF THE ESTIMATE

5.1 Introduction. The general linear system model was defined by Equation 4.23 as

$$\underline{\lambda} = \underline{B}\,\underline{\gamma} + \underline{e} \tag{5.1}$$

where

$\underline{\lambda}$ is a column vector of measurements.

$\underline{B}$ is a known constant matrix.

$\underline{\gamma}$ is an unknown constant vector (position coordinates).

$\underline{e}$ is the total system error.

It is assumed that $E\underline{e} = \emptyset$ and that we know, or at least have an estimate for, $D\underline{e} = \Sigma$.

The matrix $\underline{B}$ is 4K-1 by 3K+3 and is of rank 3K+3 when $K \geq 4$. To show $\underline{B}$ has rank 3K+3, recall from Equation 4.25

$$\underline{B} = \begin{bmatrix} I_{3K} & \emptyset_3^{3K} \\ -C\,\underline{G} & C\,\underline{A} \end{bmatrix} .$$

$C\,\underline{G}$ and $C\,\underline{A}$ have K-1 rows. Let the first three rows of each be designated $\underline{a}_1$ and $\underline{a}_2$ respectively and consider the matrix

$$\underline{b} = \begin{bmatrix} I_{3K} & \emptyset_3^{3K} \\ -\underline{a}_1 & \underline{a}_2 \end{bmatrix} .$$

Now $\underline{a}_2$ has an inverse because any three rows of C $\underline{A}$ form a square full rank matrix. $\underline{b}$ is 3K+3 by 3K+3 and has inverse

$$\underline{b}^{-1} = \begin{bmatrix} I_{3K} & \emptyset_3^{3K} \\ \underline{a}_2^{-1}\underline{a}_1 & \underline{a}_2^{-1} \end{bmatrix} .$$

Therefore, $\underline{b}$ is of rank 3K+3 which implies $\underline{B}$ is of full rank.

In this chapter, we shall find a linear estimator $\hat{\underline{\gamma}}$ which is unbiased and has minimum variance. Let $\hat{\underline{\gamma}} = \underline{F}_1\underline{\lambda}$ where $\underline{F}_1$ is a given 3K+3 by 4K-1 constant matrix. Then random vector $\hat{\underline{\gamma}}$ is a linear estimator of the constant vector $\underline{\gamma}$. If $E\hat{\underline{\gamma}} = \underline{\gamma}$, then $\hat{\underline{\gamma}}$ is a linear unbiased estimator of $\underline{\gamma}$. Suppose

$$\hat{\underline{\gamma}} = \underline{F}_1 \, \underline{Y}$$

and

$$\tilde{\underline{\gamma}} = \underline{F} \, \underline{Y}$$

where $\underline{F}_1$ is a given constant matrix and $\underline{F}$ is an arbitrary constant matrix. If $E\,\hat{\underline{\gamma}} = E\,\tilde{\underline{\gamma}} = \underline{\gamma}$, and if each diagonal element of $D\,\hat{\underline{\gamma}}$ is less than or equal to the corresponding diagonal element of $D\,\tilde{\underline{\gamma}}$ for every constant matrix $\underline{F}$, then $\hat{\underline{\gamma}}$ is the minimum variance unbiased linear estimator of $\underline{\gamma}$.

5.2 A Linear Unbiased Estimator of $\underline{\gamma}$. Let Equation 5.1 be rewritten

$$\underline{e} = \underline{\lambda} - \underline{B}\,\underline{\gamma} \tag{5.2}$$

and let

$$q = \underline{e}'\, V\, \underline{e} \tag{5.3}$$

where q is a one dimensional random variable and V is an arbitrary 4K-1 by 4K-1 positive definite symmetric matrix. Substituting Equation 5.2 into Equation 5.3 and expanding, one obtains

$$q = \underline{\lambda}'\, V\, \underline{\lambda} - \underline{\gamma}'\, \underline{B}'\, V\, \underline{\lambda} - \underline{\lambda}'\, V\, \underline{B}\, \underline{\gamma} + \underline{\gamma}'\, \underline{B}'\, V\, \underline{B}\, \underline{\gamma} \quad . \tag{5.4}$$

The partial derivative of q with respect to $\underline{\gamma}$ is

$$\frac{\partial q}{\partial \underline{\gamma}} = 2\, \underline{\gamma}'\, \underline{B}'\, V\, \underline{B} - 2\, \underline{\lambda}'\, V\, \underline{B} \quad .$$

Setting $\partial q / \partial \underline{\gamma} = \emptyset$, and replacing $\underline{\gamma}$ by $\tilde{\underline{\gamma}}$,

$$\tilde{\underline{\gamma}}'\, \underline{B}'\, V\, \underline{B} = \underline{\lambda}'\, V\, \underline{B} \quad .$$

Transposing both sides,

$$\underline{B}'\, V\, \underline{B}\, \tilde{\underline{\gamma}} = \underline{B}'\, V\, \underline{\lambda} \quad . \tag{5.5}$$

Recall that $\underline{B}$ is full rank, i.e., the rank of $\underline{B}$ is equal to the number of columns in $\underline{B}$. Therefore, $\underline{B}'\, V\, \underline{B}$ has an inverse. Premultiplying both sides of Equation 5.5 by $(\underline{B}'\, V\, \underline{B})^{-1}$, one obtains

$$\tilde{\underline{\gamma}} = (\underline{B}'\, V\, \underline{B})^{-1}\, \underline{B}'\, V\, \underline{\lambda} \quad . \tag{5.6}$$

We will now show that $\tilde{\underline{\gamma}}$ minimizes q defined by Equation 5.3. Let

$$\tilde{q} = (\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}})'\ V\ (\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}}) \tag{5.7}$$

and let

$$q^* = (\underline{\lambda} - \underline{B}\,\underline{\gamma}^*)'\ V\ (\underline{\lambda} - \underline{B}\,\underline{\gamma}^*) \tag{5.8}$$

where $\underline{\gamma}^*$ is any estimator of $\underline{\gamma}$. Expanding Equation 5.8

$$q^* = (\underline{\lambda} - \underline{B}\,\underline{\gamma}^* + \underline{B}\,\tilde{\underline{\gamma}} - \underline{B}\,\tilde{\underline{\gamma}})'\ V\ (\underline{\lambda} - \underline{B}\,\underline{\gamma}^* + \underline{B}\,\tilde{\underline{\gamma}} - \underline{B}\,\tilde{\underline{\gamma}})$$

$$q^* = [(\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}})' + (\tilde{\underline{\gamma}} - \underline{\gamma}^*)'\underline{B}']\ V\ [(\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}}) + \underline{B}(\tilde{\underline{\gamma}} - \underline{\gamma}^*)]$$

$$q^* = (\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}})'\ V\ (\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}}) + (\tilde{\underline{\gamma}} - \underline{\gamma}^*)'\ \underline{B}'\ V\ \underline{B}\ (\tilde{\underline{\gamma}} - \underline{\gamma}^*)$$

$$+ (\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}})'\ V\ \underline{B}(\tilde{\underline{\gamma}} - \underline{\gamma}^*) + (\tilde{\underline{\gamma}} - \underline{\gamma}^*)'\ \underline{B}'\ V\ (\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}}) \quad .$$

By Equation 5.5,

$$\underline{B}'V(\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}}) = \emptyset \quad .$$

Hence

$$(\underline{\lambda} - \underline{B}\,\tilde{\underline{\gamma}})'\ V\ \underline{B} = \emptyset \quad ,$$

and

$$q^* = \tilde{q} + (\tilde{\underline{\gamma}} - \underline{\gamma}^*)'\ \underline{B}'\ V\ \underline{B}\ (\tilde{\underline{\gamma}} - \underline{\gamma}^*) \quad .$$

Since $\underline{B}$ has rank 3K+3, $\underline{B}'\ V\ \underline{B}$ is positive definite by Theorem 4.4, which implies

$$(\tilde{\underline{\gamma}} - \underline{\gamma}^*)'\ \underline{B}'\ V\ \underline{B}\ (\tilde{\underline{\gamma}} - \underline{\gamma}^*) > 0 \tag{5.9}$$

for every $\underline{\gamma}^* \neq \tilde{\underline{\gamma}}$. Therefore, $\tilde{\underline{\gamma}}$ is the unique linear function of $\underline{\lambda}$ that minimizes the weighted sum of squares q.

As mentioned earlier, the estimator $\tilde{\underline{\gamma}}$ is a random vector since it is a function of random vector $\underline{\lambda}$. Let us find the mean and dispersion matrix for $\tilde{\underline{\gamma}}$. Taking the expectation of both sides of Equation 5.6

$$E\,\tilde{\underline{\gamma}} = E[(\underline{B}'\,V\,\underline{B})^{-1}\,\underline{B}'\,V\,\underline{\lambda}] = [(\underline{B}'\,V\,\underline{B})^{-1}\,\underline{B}'\,V]E\,\underline{\lambda} \quad .$$

Recall $E\,\underline{\lambda} = E(\underline{B}\,\underline{\gamma}) + E\underline{e} = \underline{B}\,\underline{\gamma}$. Therefore,

$$E\,\tilde{\underline{\gamma}} = \underline{\gamma} \tag{5.10}$$

which proves that $\tilde{\underline{\gamma}}$ is an unbiased estimator for $\underline{\gamma}$.

Applying Theorem 4.8 to Equation 5.6,

$$D\,\tilde{\underline{\gamma}} = [(\underline{B}'\,V\,\underline{B})^{-1}\,\underline{B}'\,V]\,D\,\underline{\lambda}\,[(\underline{B}'\,V\,\underline{B})^{-1}\,\underline{B}'\,V]'$$

$$= (\underline{B}'\,V\,\underline{B})^{-1}\,\underline{B}'\,(V\,\Sigma\,V)\,\underline{B}\,(\underline{B}'\,V\,\underline{B})^{-1} \quad . \tag{5.11}$$

Note that if Σ^{-1} is chosen as the weighting matrix V, then

$$D\,\tilde{\underline{\gamma}} = (\underline{B}'\,\Sigma^{-1}\,\underline{B})^{-1} \quad . \tag{5.12}$$

It was shown that $\tilde{\underline{\gamma}}$ minimizes the sum of squares q for arbitrary weighting matrix V. When V = I, $\tilde{\underline{\gamma}}$ is called an unweighted estimator. We will find the "best" weight V in the next section.

5.3 <u>The Minimum Variance Unbiased Linear Estimator for $\underline{\gamma}$</u>. In a one dimensional regression model, Papoulis (5) develops an appealing solution to the problem by use of a technique which he calls the orthogonality principle. We shall extend this idea to the problem of minimum variance unbiased estimation of $\underline{\gamma}$.

In order to establish a more general theory, consider the arbitrary linear model

$$\underline{Y} = \underline{X}\ \underline{a} + \underline{e} \tag{5.13}$$

where

$\underline{Y}$ is an n by 1 random vector (the data).

$\underline{e}$ is an n by 1 random vector (the error).

$\underline{X}$ is an n by p matrix of known constants.

$\underline{a}$ is an p by 1 vector of unknown constants.

Assume $\underline{X}$ is of rank p, $E\underline{e} = \emptyset$, and Σ is known, where $\Sigma = D\ \underline{Y} = D\ \underline{e}$.

We wish to find a matrix $\underline{F}$ such that the random vector

$$\underline{\hat{a}} = \underline{F}\ \underline{Y} \tag{5.14}$$

is the minimum variance linear unbiased estimator for $\underline{a}$. The following theorem applies.

Theorem 5.1: The estimator $\underline{\hat{a}} = \underline{F}\ \underline{Y}$ is the minimum variance linear unbiased estimator for $\underline{a}$ of Equation 5.13 if and only if $\underline{F}$ is chosen such that $\mathrm{Cov}[(\underline{Y} - \underline{X}\ \underline{\hat{a}}),\ \underline{\hat{a}}] = \emptyset$.

To prove the theorem, let us first find $\mathrm{Cov}[(\underline{Y} - \underline{X}\ \underline{\hat{a}}),\ \underline{\hat{a}}]$ for arbitrary $\underline{F}$.

$$\mathrm{Cov}[(\underline{Y} - \underline{X}\ \underline{\hat{a}}),\ \underline{\hat{a}}] = \mathrm{Cov}[(\underline{Y} - \underline{X}\ \underline{F}\ \underline{Y}),\ (\underline{F}\ \underline{Y})]$$

$$= \mathrm{Cov}[(I - \underline{X}\ \underline{F})\underline{Y},\ \underline{F}\ \underline{Y}] \quad .$$

Applying Theorem 4.10,

$$\mathrm{Cov}[(\underline{Y} - \underline{X}\ \underline{\hat{a}}),\hat{a}] = (I - \underline{X}\ \underline{F})\Sigma\ \underline{F}' \quad . \tag{5.15}$$

To find a candidate for $\underline{F}$, initially assume $\underline{F} = \underline{F}_1$ where $\underline{F}_1$ has rank p, and $\mathrm{Cov}[(\underline{Y} - \underline{X}\ \underline{\hat{a}}] = \emptyset$. Then Equation 5.15 may be written

$$\underline{X}\ \underline{F}_1\ \Sigma\ \underline{F}_1' = \Sigma\ \underline{F}_1' \quad . \tag{5.16}$$

Multiplying through by $\underline{F}_1$,

$$\underline{F}_1\ \underline{X}\ \underline{F}_1\ \Sigma\ \underline{F}_1' = \underline{F}_1\ \Sigma\ \underline{F}_1' \quad .$$

Since $\underline{F}_1$ is full rank, $\underline{F}_1\ \Sigma\ F_1'$ is positive definite and $(\underline{F}_1\ \Sigma\ \underline{F}_1')^{-1}$ exists. Therefore

$$\underline{F}_1\ \underline{X} = I_p \quad . \tag{5.17}$$

Premultiplying both sides of Equation 5.16 by $\underline{X}'\ \Sigma^{-1}$,

$$\underline{X}\ \Sigma^{-1}\ \underline{X}\ \underline{F}_1\ \Sigma\ \underline{F}_1' = \underline{X}'\ \Sigma^{-1}\ \Sigma\ \underline{F}_1' = \underline{X}'\ \underline{F}_1'$$

which implies $\underline{X}'\ \Sigma^{-1}\ \underline{X} = (\underline{F}_1\ \Sigma\ \underline{F}_1)^{-1}$ since $\underline{X}'\ \underline{F}_1' = I$. Substituting this result into Equation 5.16 and transposing

$$\underline{F}_1 = (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1}\ \underline{X}'\ \Sigma^{-1} \quad . \tag{5.18}$$

Let us test $\underline{F}_1$ for bias.

$$\hat{\underline{a}} = \underline{F}_1\ \underline{Y} \quad .$$

$$E\ \hat{\underline{a}} = E\ \underline{F}_1\ \underline{Y} = \underline{F}_1\ E\ \underline{Y} = \underline{F}_1\ \underline{X}\ \underline{a} = I\ \underline{a} = \underline{a} \quad .$$

Therefore, the estimator $\underline{a} = \underline{F}_1\ \underline{Y}$ is unbiased.

Recall it was specified in advance that $\underline{F}_1$ be restricted to the class of full rank p by n matrices. To prove the theorem, we must show $\underline{F}_1$ is the unique solution to Equation 5.15 when the left-hand side is null, and we must show that $\hat{\underline{a}}$ has minimum variance when $\underline{F} = \underline{F}_1$. Let

$$\underline{F} = \underline{F}_1 + \underline{F}_2 \tag{5.19}$$

where $\underline{F}_1$ is defined by Equation 5.18 and $\underline{F}_2$ is unspecified.

Since $\underline{F}_2$ is completely arbitrary, $\underline{F}$ is a general p by n matrix with no restrictions. Equation 5.14 becomes

$$\hat{\underline{a}} = (\underline{F}_1 + \underline{F}_2)\underline{Y} \quad . \qquad (5.20)$$

For unbiasedness, E $\hat{\underline{a}}$ must equal $\underline{a}$. Hence,

$$(\underline{F}_1 + \underline{F}_2)\ \underline{X}\ \underline{a} = \underline{a} \quad .$$

Since $\underline{F}_1\ \underline{X} = I$,

$$(I + \underline{F}_2\ \underline{X})\ \underline{a} = \underline{a} \quad . \qquad (5.21)$$

Equation 5.21 is an identity which must hold for every $\underline{a}$ in order for $\hat{\underline{a}}$ to be unbiased. Therefore $\hat{\underline{a}} = \underline{F}\ \underline{Y}$ is unbiased only if

$$\underline{F}_2\ \underline{X} = \emptyset \quad . \qquad (5.22)$$

Letting Cov[$(\underline{Y} - \underline{X}\ \hat{\underline{a}}), \hat{\underline{a}}$] = $\emptyset$ and substituting Equation 5.19 into Equation 5.15,

$$\underline{X}\ (\underline{F}_1 + \underline{F}_2)\ \Sigma\ (\underline{F}_1' + \underline{F}_2') = \Sigma\ (\underline{F}_1' + \underline{F}_2') \quad . \qquad (5.23)$$

Does there exist a non null $\underline{F}_2$ such that Equations 5.22 and 5.23 hold? Upon expansion, Equation 5.23 becomes:

$$\underline{X}(\underline{F}_1\ \Sigma\ \underline{F}_1' + \underline{F}_2\ \Sigma\ \underline{F}_2' + \underline{F}_2\ \Sigma\ \underline{F}_1' + \underline{F}_1\ \Sigma\ \underline{F}_2') = \Sigma(F_1' + F_2')$$

$$\underline{X}[(\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1} + \underline{F}_2\ \Sigma\ \underline{F}_2' + \underline{F}_2\ \underline{X}\ (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1} + (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1}\ \underline{X}'\ \underline{F}_2']$$

$$= \underline{X}\ (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1} + \Sigma\ \underline{F}_2' \quad .$$

Applying the unbiasedness condition specified by Equation 5.22 and clearing,

$$\underline{X}\ \underline{F}_2\ \Sigma\ \underline{F}_2' = \Sigma\ \underline{F}_2' \quad . \tag{5.24}$$

Multiplying through by $\underline{F}_2$,

$$\underline{F}_2\ \underline{X}\ \underline{F}_2\ \Sigma\ \underline{F}_2' = \underline{F}_2\ \Sigma\ \underline{F}_2' \quad .$$

The left-hand side equals zero by Equation 5.22; however, by Theorem 4.5, the right-hand side can be zero only if $\underline{F}_2 = \emptyset$. Therefore, when estimator $\hat{\underline{a}} = \underline{F}\ \underline{Y}$ is unbiased, $\mathrm{Cov}[(\underline{Y} - \underline{X}\ \hat{\underline{a}}),\ \hat{\underline{a}}] = \emptyset$ if and only if $\underline{F} = \underline{F}_1 = (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1}\ \underline{X}'\ \Sigma^{-1}$.

We will now show $\hat{\underline{a}}$ has minimum variance when $\underline{F} = \underline{F}_1$. Let $\underline{F} = \underline{F}_1 + \underline{F}_2$ and let $\mathrm{Cov}[(\underline{Y} - \underline{X}\ \hat{\underline{a}}),\ \hat{\underline{a}}]$ be arbitrary. $\underline{F}$ is a general matrix subject only to Equation 5.22.

$$\begin{aligned} D\ \hat{\underline{a}} &= (\underline{F}_1 + \underline{F}_2)\ \Sigma\ (\underline{F}_1' + \underline{F}_2') \\ &= (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1} + \underline{F}_2\ \Sigma\ \underline{F}_2' + \underline{F}_2\ \underline{X}\ (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1} + (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1}\ \underline{X}'\ \underline{F}_2' \ . \end{aligned}$$

The last two terms on the right-hand side are zero because of the biasedness restriction of Equation 5.22. Consider the term $\underline{F}_2\ \Sigma\ \underline{F}_2'$. Since Σ is positive definite, each diagonal element of $\underline{F}_2\ \Sigma\ \underline{F}_2'$ is equal to or greater than zero for every $\underline{F}_2$. Each diagonal element of $D\ \hat{\underline{a}}$ will be minimized only when $\underline{F}_2\ \Sigma\ \underline{F}_2'$ has all zero diagonal elements. By Theorem 4.5, this can happen only when $\underline{F}_2 = \emptyset$. Therefore

$$\hat{\underline{a}} = (\underline{X}'\ \Sigma^{-1}\ \underline{X})^{-1}\ \underline{X}'\ \Sigma^{-1}\ \underline{Y} \tag{5.25}$$

is the minimum variance unbiased linear estimator in the model defined by Equation 5.13. This completes the proof of Theorem 5.1.

In our time difference hyperbolic net model defined by Equation 5.1, the minimum variance unbiased linear estimator for $\underline{\gamma}$ is

$$\hat{\underline{\gamma}} = (\underline{B}' \Sigma^{-1} \underline{B})^{-1} \underline{B}' \Sigma^{-1} \underline{\lambda} \tag{5.26}$$

and

$$D \hat{\underline{\gamma}} = (\underline{B}' \Sigma^{-1} \underline{B})^{-1} \quad . \tag{5.27}$$

These correspond to the solutions found in Section 5.2 by least squares if Σ^{-1} is chosen for the weighting matrix V.

5.4 A Maximum Likelihood Estimator for $\underline{\gamma}$. Assume $\underline{\lambda}$ has a multivariate normal distribution. Its density function is

$$f(\underline{\lambda}) = \frac{1}{(2\pi)^{\frac{4K-1}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}q} \tag{5.28}$$

where

$$q = (\underline{\lambda} - \underline{B}\,\underline{\gamma})' \Sigma^{-1} (\underline{\lambda} - \underline{B}\,\underline{\gamma}). \tag{5.29}$$

The maximum likelihood estimator of $\underline{\gamma}$ is the particular $\underline{\gamma}$ that maximizes $f(\underline{\lambda})$. Since $f(\underline{\lambda})$ is a decreasing function of q, the $\underline{\gamma}$ that minimizes q will maximize $f(\underline{\lambda})$. We have already found this $\underline{\gamma}$ in Section 5.2. Therefore

$$\hat{\underline{\gamma}} = (\underline{B}' \Sigma^{-1} \underline{B})^{-1} \underline{B}' \Sigma^{-1} \underline{\lambda} \tag{5.30}$$

is the maximum likelihood estimator for $\underline{\gamma}$ under the assumption of normality. Applying Theorem 4.8,

$$D \hat{\underline{\gamma}} = (\underline{B}' \Sigma^{-1} \underline{B})^{-1} \quad . \tag{5.31}$$

Summarizing the results of the last three sections, $\hat{\underline{\gamma}}$ defined by

Equation 5.30 is the weighted least squares, minimum variance, and maximum likelihood estimator for $\underline{\gamma}$. The dispersion matrix for $\hat{\underline{\gamma}}$ is defined by Equation 5.31.

Recall $\hat{\underline{\gamma}}$ estimates both receiving site locations and the emitter location. The implication of this is that we allow the time difference measurements and the total error dispersion matrix to influence the final estimate of receiving site locations. In turn, the variance of the emitter location estimator is minimized.

As a final result, we are primarily interested in $\hat{\underline{\beta}}$ and D $\hat{\underline{\beta}}$. $\hat{\underline{\beta}}$ is the last three elements in $\hat{\underline{\gamma}}$, and D $\hat{\underline{\beta}}$ is the three by three submatrix in the lower right-hand corner of D $\hat{\underline{\gamma}}$.

5.5 Site Errors Uncorrelated With Time Difference Errors. When the receiving site locational errors are uncorrelated with the time difference measurement errors, the total error dispersion matrix may be written

$$\Sigma = \begin{bmatrix} \Sigma_s & \emptyset^{3K}_{K-1} \\ \emptyset^{K-1}_{3K} & \Sigma_2 \end{bmatrix} . \tag{5.32}$$

Then

$$\underline{B}' \Sigma^{-1} \underline{B} = \begin{bmatrix} \Sigma_s^{-1} - \underline{G}' C' \Sigma_2^{-1} C \underline{G} & - \underline{G}' C' \Sigma_2^{-1} C \underline{A} \\ - \underline{A}' C' \Sigma_2^{-1} C \underline{G} & \underline{A}' C' \Sigma_2^{-1} C \underline{A} \end{bmatrix} . \tag{5.33}$$

The positive definite symmetric matrix

$$\underline{c} = \begin{bmatrix} \underline{a}_1 & \underline{b} \\ \underline{b}' & \underline{a}_2 \end{bmatrix}$$

has inverse

$$\underline{c}^{-1} \quad \begin{bmatrix} \underline{a}_1^{-1} + \underline{a}_1^{-1}\ \underline{b}\ \underline{a}_3\ \underline{b}'\ \underline{a}_1^{-1} & -\ \underline{a}_1^{-1}\ \underline{b}\ \underline{a}_3 \\ -\ \underline{a}_3\ \underline{b}'\ \underline{a}_1^{-1} & \underline{a}_3 \end{bmatrix}$$

where

$$\underline{a}_3 = (\underline{a}_2 - \underline{b}'\ \underline{a}_1^{-1}\ \underline{b})^{-1}.$$

Applying this result to Equation 5.33,

$$D\ \hat{\underline{\beta}} = [\underline{A}'C'\Sigma_2^{-1}C\underline{A} - \underline{A}'C'\Sigma_2^{-1}C\underline{G}(\Sigma_s^{-1} - \underline{G}'C'\Sigma_2^{-1}C\underline{G})\ \ \underline{G}'C'\Sigma_2^{-1}C\underline{A}]^{-1}. \quad (5.34)$$

Let

$$V_{12} = (\Sigma_s^{-1} - \underline{G}'\ C'\ \Sigma_2^{-1}\ C\ \underline{G})^{-1}\ (\underline{G}'\ C'\ \Sigma_2^{-1}\ C\ \underline{A}) \quad .$$

Then

$$D\ \hat{\underline{\alpha}} = (\Sigma_s^{-1} - \underline{G}'\ C'\ \Sigma_2^{-1}\ C\ \underline{G})^{-1} + V\ \ (D\ \hat{\underline{\beta}})\ V_{12}' \quad (5.35)$$

and

$$D\ \hat{\underline{\gamma}} = \begin{bmatrix} D\ \hat{\underline{\alpha}} & -\ V_{12}(D\ \hat{\underline{\beta}}) \\ -\ (D\ \hat{\underline{\beta}})V_{12}' & D\ \hat{\underline{\beta}} \end{bmatrix} \quad (5.36)$$

where

$$\underline{\hat{\gamma}} = \begin{bmatrix} \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \end{bmatrix} .$$

The product $\underline{B}' \Sigma^{-1} \underline{\lambda}$ upon expansion becomes

$$\underline{B}' \Sigma^{-1} \underline{\lambda} = \begin{bmatrix} \Sigma_s^{-1} \underline{\lambda}_1 - \underline{G}' C' \Sigma_2^{-1} \underline{\lambda}_2 \\ \underline{A}' C' \Sigma_2^{-1} \underline{\lambda}_2 \end{bmatrix} .$$

Substituting these results into Equation 5.30,

$$\hat{\underline{\alpha}} = (D\ \hat{\underline{\alpha}})(\Sigma_s^{-1} - \underline{G}' C' \Sigma_2^{-1} \underline{\lambda}_2) - V_{12}(D\ \hat{\underline{\beta}})(\underline{A}' C' \Sigma_2^{-1} \underline{\lambda}_2) \qquad (5.37)$$

$$\hat{\underline{\beta}} = (D\ \hat{\underline{\beta}})[\underline{A}' C' \Sigma_2^{-1} \underline{\lambda}_2 - V_{12}' \Sigma_s^{-1} \underline{\lambda}_1 + V_{12}' \underline{G}' C' \Sigma_2^{-1} \underline{\lambda}_2] . \qquad (5.38)$$

Let us summarize. In general, $\hat{\underline{\beta}}$ (the minimum variance linear unbiased estimator for $\underline{\beta}$) and D $\hat{\underline{\beta}}$ (the dispersion matrix of the estimator) are defined implicitly by Equations 5.30 and 5.31 as previously described. When site errors are uncorrelated with time difference measurement errors, then $\hat{\underline{\beta}}$ and D $\hat{\underline{\beta}}$ are given by Equations 5.38 and 5.34 respectively. The utility of the latter two equations is that the matrices to be inverted are of lower order than in the former equations.

CHAPTER VI

SPECIAL MODELS

6.1 Introduction. In the general linear system model, the receiving site error dispersion matrix Σ_s must be inverted. If location of one of the three coordinates X_i, Y_i, Z_i for any i is known with negligible uncertainty, then Σ_s has a zero element on the diagonal, and Σ_s has no inverse. We will redesign the model in this chapter so as to handle the situation when Σ_s is positive semidefinite. Recall that a null matrix is positive semidefinite.

Also in this chapter, we redesign the model so as to offer an alternate method of accounting for the propagation error component $\underline{e}_\delta$.

6.2 Model 3. In Equation 5.30, the minimum variance unbiased estimator for emitter location contains the term Σ_s^{-1}. If one coordinate of one receiving site is known with negligible uncertainty, then Σ_s^{-1} does not exist. Suppose this is the case, and $\underline{e}_s$ has n elements whose variance is other than zero, where $n \leq 3K$. Let us:

(1) Redefine Equation 3.7 by

$$\underline{\lambda}_n = \underline{\alpha}_n + \underline{e}_{sn} \quad . \tag{6.1}$$

(2) Redefine Equation 3.11 by

$$\underline{R} = \underline{R}^* + \underline{A}\,\underline{\beta} - \underline{G}_n\,\underline{\alpha}_n \quad , \tag{6.2}$$

where $\underline{G}_n$ and $\underline{\alpha}_n$ are chosen to fit the new situation, and $\underline{e}_{sn}$ is an n element error vector such that $D\underline{e}_{sn}$ is positive definite. With Equations 6.1 and 6.2 we could proceed as before and develop a general linear system model with n+K-1 observations and n+3 constants to be estimated.

Rather than proceed as outlined above, we will now develop an alternate general solution which always holds, even when $D\underline{e}_s = \emptyset$. Let Equation 6.1 be rewritten

$$\underline{\lambda}_n = I\ \underline{\alpha}_n + \underline{e}_{sn} \quad . \tag{6.3}$$

By the theory of Section 5.3, the minimum variance unbiased prior estimator for $\underline{\alpha}_n$ is

$$\tilde{\underline{\alpha}}_n = (I'\ \Sigma_{sn}^{-1}\ I)^{-1}\ I'\ \Sigma_{sn}^{-1}\ \underline{\lambda}_n = \underline{\lambda}_n \quad . \tag{6.4}$$

The remaining 3K-n elements of $\underline{\lambda}_1$ are constants. Hence, the minimum variance unbiased prior estimator for $\underline{\alpha}$ is $\tilde{\underline{\alpha}} = \underline{\lambda}_1$ regardless of whether $D\ \underline{\lambda}_1$ is positive definite or positive semidefinite. Let $\tilde{\underline{\alpha}} = \underline{\lambda}_1$ be substituted for $\underline{\alpha}$ in Equation 3.7. Then after rearranging terms, Equation 3.12 becomes

$$u_0\underline{\theta} - u_0 E\underline{e}_{Td} + C\ \underline{G}(E\ \underline{\lambda}_1) - C\ \underline{R}^* = C\ \underline{A}\ \underline{\beta} - C\ \underline{G}(\underline{\lambda}_1 - E\ \underline{\lambda}_1) + u_0(\underline{e}_{Td} - E\underline{e}_{Td}) . \tag{6.5}$$

Let

$$\underline{\lambda}_3 = u_0\underline{\theta} - u_0 E\underline{e}_{Td} + C\ \underline{G}(E\ \underline{\lambda}_1) - C\ \underline{R}^* \quad . \tag{6.6}$$

Then Equation 6.5 may be written

$$\underline{\lambda}_3 = C\ \underline{A}\ \underline{\beta} + \underline{e}_3 \tag{6.7}$$

where

$$\underline{e}_3 = \underline{e}_2 - C\,\underline{G}\,\underline{e}_s \quad . \qquad (6.8)$$

The usage of all terms in this chapter is compatible with their previous definitions. Recall that the total system error $\underline{e}$ was defined

$$\underline{e} = \begin{bmatrix} \underline{e}_s \\ \underline{e}_2 \end{bmatrix}$$

and it was assumed that $\underline{e}$ had zero mean and that Σ was known, where $\Sigma = D\underline{e}$. This implies that $E\underline{e}_3 = \emptyset$ and that Σ_2, Σ_s, and $\mathrm{Cov}(\underline{e}_2, \underline{e}_s)$ are known. By Theorem 4.11,

$$\Sigma_3 = \Sigma_2 + C\,\underline{G}\,\Sigma_s\,\underline{G}'\,C' - [\mathrm{Cov}(\underline{e}_2, \underline{e}_s)]\underline{G}'\,C' - C\,\underline{G}[\mathrm{Cov}(\underline{e}_s, \underline{e}_2)] \quad .$$

Equations 6.6, 6.7 and 6.8 will be called Model 3. By the theory of Chapter V,

$$\hat{\underline{\beta}} = (\underline{A}'\,C'\,\Sigma_3^{-1}\,C\,\underline{A})^{-1}\,\underline{A}'\,C'\,\Sigma_3^{-1}\,\underline{\lambda}_3 \qquad (6.9)$$

and

$$D\,\hat{\underline{\beta}} = (\underline{A}'\,C'\,\Sigma_3^{-1}\,C\,\underline{A})^{-1} \quad . \qquad (6.10)$$

Let us examine Equation 6.10. It will be shown in Section 6.5 that under conditions of general regularity, Σ_3 is of the form such that its inverse is defined by a simple formula. The total expression $(\underline{A}'\,C'\,\Sigma_3^{-1}\,C\,\underline{A})$ is three by three. Hence, by Model 3, we have drastically reduced the arithmetic problem of computing $\hat{\underline{\beta}}$ and $D\,\hat{\underline{\beta}}$. A second desirable feature is that it permits dispersion matrix Σ_s to be positive semidefinite. That is, Model 3 holds if some (or even all) of the elements of $\underline{e}_s$ have zero variance.

In order to compute $\hat{\underline{\beta}}$, Equation 6.9 implicitly requires the true site locations. Since we do not know the true site locations, the recorded location vector $\underline{\lambda}_1$ is substituted as an estimate for E $\underline{\lambda}_1$ in Equation 6.6. Due to this approximation, $\hat{\underline{\beta}}$ and D $\hat{\underline{\beta}}$ defined by Equations 6.9 and 6.10 must be considered as sub-optimal solutions when error $\underline{e}_s$ is significant.

6.3 Model 4. The error due to propagation anomalies was expressed by Equation 4.16 as

$$\underline{e}_\delta = C(\underline{T}p + \underline{e}_p) \qquad (6.11)$$

where $\underline{T}p$ is a K element constant vector and $\underline{e}_p$ is a random vector with zero mean. In general, $\underline{e}_\delta$ is not observable. That is, we cannot measure $\underline{T}p$.

Recall that $\underline{e}_\delta$ is a component of $\underline{e}_{Td}$, and we have assumed up to now that E$\underline{e}_{Td}$ and D$\underline{e}_{Td}$ were known. In the solutions previously given for $\hat{\underline{\beta}}$ and D $\hat{\underline{\beta}}$, one can do either of the following:

(1) Empirically estimate $\underline{T}p$ and D$\underline{e}_p$ from experience (the estimate may be $\emptyset$) and proceed to compute $\hat{\underline{\beta}}$ and D $\hat{\underline{\beta}}$ based on these estimates.

(2) Assume $\underline{T}p$ and D$\underline{e}_p$ may have an assumed worst case value and then compute the impact on $\hat{\underline{\beta}}$ and D $\hat{\underline{\beta}}$.

The approach outlined above was proposed in Section 4.5. We can offer an alternate approach when the error $\underline{e}_\delta$ may be assumed to be due entirely to an unknown uniform speed of propagation. Under this assumption, it was shown in Section 4.5 that $\underline{T}p$ could be approximated by

$$\underline{T}p = \frac{\delta_0}{u_0} \underline{R}^* \qquad (6.12)$$

where

$$\delta_0 = \frac{u_0 - u}{u_0} \quad . \tag{6.13}$$

For the remainder of this section, assume in Equation 6.11 that $\underline{e}_p = \emptyset$. This implies that $\underline{e}_\delta$ is now equal to the unknown constant vector C $\underline{T}$p and $D\underline{e}_\delta = \emptyset$. We will account for the propagation error by allowing δ_0 to be one more constant to be estimated.

We will now redesign the general linear model and Model 3 so that δ_0, and hence u, can be estimated from the data. In practice, one may not have any utility for an estimate of u, especially if the variance of the estimate were higher than an estimate already at hand. However, by allowing u to be unknown, we hope to create a model more descriptive of nature.

Let $\underline{\lambda}_1$, $\underline{\lambda}_2$, $\underline{\lambda}_3$, $\underline{e}$, $\underline{e}_2$ and $\underline{e}_3$ be as previously defined except that $D\underline{e}_\delta = \emptyset$. Let $K \geq 5$ and let the following matrices and vectors be defined:

$$\underline{H} = \begin{bmatrix} \underline{A}_1, & R_1^* \\ \underline{A}_2, & R_2^* \\ & \cdot \\ & \cdot \\ & \cdot \\ \underline{A}_K, & R_K^* \end{bmatrix} \tag{6.14}$$

$$\underline{M} = \begin{bmatrix} I_{3K} & \emptyset_4^{3K} \\ -C\,\underline{G} & C\,\underline{H} \end{bmatrix} \tag{6.15}$$

$$\underline{\gamma}_{4A} = \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \\ \delta_0 \end{bmatrix} \tag{6.16}$$

$$\underline{\gamma}_{4B} = \begin{bmatrix} \underline{\beta} \\ \delta_0 \end{bmatrix} \quad . \tag{6.17}$$

Then the system models may be written

$$\underline{\lambda} = \underline{M}\ \underline{\gamma}_{4A} + \underline{e} \tag{6.18}$$

$$\underline{\lambda}_3 = \underline{H}\ \underline{\gamma}_{4B} + \underline{e}_3 \quad . \tag{6.19}$$

Equations 6.18 and 6.19 will be called Models 4A and 4B respectively. The solutions are

$$\hat{\underline{\gamma}}_{4A} = (\underline{M}'\ \Sigma^{-1}\ \underline{M})^{-1}\ \underline{M}'\ \Sigma^{-1}\ \underline{\lambda} \tag{6.20}$$

$$D\ \hat{\underline{\gamma}}_{4A} = (\underline{M}'\ \Sigma^{-1}\ \underline{M})^{-1} \tag{6.21}$$

$$\hat{\underline{\gamma}}_{4B} = (C'\ \underline{H}'\ \Sigma_3^{-1}\ \underline{H}\ C)^{-1}\ C'\ \underline{H}'\ \Sigma_3^{-1}\ \underline{\lambda}_3 \tag{6.22}$$

$$D\ \hat{\underline{\gamma}}_{4B} = (C'\ \underline{H}'\ \Sigma_3^{-1}\ \underline{H}\ C)^{-1} \quad . \tag{6.23}$$

The comments at the end of Section 6.2 regarding solutions to Model 3 being sub-optimal also apply to solutions of Model 4B.

We will partition $\hat{\underline{\beta}}$ and D $\hat{\underline{\beta}}$ from the solutions of Model 4B. Expanding Equation 6.23,

$$D\,\hat{\underline{Y}}_{4B} = \begin{bmatrix} \underline{A}'\,C'\,\Sigma_3^{-1}\,C\,\underline{A} & \underline{A}'\,C'\,\Sigma_3^{-1}\,C\,\underline{R}^* \\ (\underline{R}^*)'\,C'\,\Sigma_3^{-1}\,C\,\underline{A} & (\underline{R}^*)'\,C'\,\Sigma_3^{-1}\,C\,\underline{R}^* \end{bmatrix}^{-1} .$$

Using the algorithm mentioned in Section 5.5 for inverting a partitioned positive definite symmetric matrix, one obtains

$$D\hat{\delta}_0 = \sigma^2_{\hat{\delta}_0} = [(\underline{R}^*)'C'\Sigma_3^{-1}C\underline{R}^* - (\underline{R}^*)'C'\Sigma_3^{-1}C\underline{A}(\underline{A}'C'\Sigma_3^{-1}C\underline{A})^{-1}\underline{A}'C'\Sigma_3^{-1}C\underline{R}^*]^{-1} \quad (6.24)$$

$$D\hat{\underline{\beta}} = (\underline{A}'C'\Sigma_3^{-1}C\underline{A})^{-1}[I + (D\hat{\delta}_0)\underline{A}'C'\Sigma_3^{-1}C\underline{R}^*(\underline{R}^*)'C'\Sigma_3^{-1}C\underline{A}(\underline{A}'C'\Sigma_3^{-1}C\underline{A})^{-1}] \quad .(6.25)$$

Let

$$\underline{b}_{12} = D\,\hat{\delta}_0\,(\underline{A}'\,C'\,\Sigma_3^{-1}\,C\,\underline{A})^{-1}\,\underline{A}'\,C'\,\Sigma_3^{-1}\,C\,\underline{R}^* \quad .$$

Then

$$\hat{\underline{\beta}} = [(D\,\hat{\underline{\beta}})\underline{A}' - \underline{b}_{12}(\underline{R}^*)']C'\,\Sigma_3^{-1}\,\underline{\lambda}_3 \quad (6.26)$$

$$\hat{\delta}_0 = [(D\,\hat{\delta}_0)(\underline{R}^*)' - \underline{b}'_{12}\,\underline{A}']C'\,\Sigma_3^{-1}\,\underline{\lambda} \quad . \quad (6.27)$$

The estimate of the assumed uniform speed of electromagnetic propagation is

$$\hat{u} = u_0(1 - \hat{\delta}_0) \quad (6.28)$$

and

$$D\,\hat{u} = u_0^2\,D\,\hat{\delta}_0 \quad .$$

Equation 6.25 may be rewritten

$$D\,\hat{\underline{\beta}} = (\underline{A}'\,C'\,\Sigma_3^{-1}\,C\,\underline{A})^{-1} + \frac{\underline{b}_{12}\,\underline{b}'_{12}}{D\,\hat{\delta}_0} \quad . \quad (6.29)$$

Note that $(D\ \hat{\delta}_0)^{-1}\ \underline{b}_{12}\ \underline{b}_{12}'$ is positive semidefinite and recall from Model 3,

$$D\ \hat{\underline{\beta}} = (\underline{A}'\ C'\ \Sigma_3^{-1}\ C\ \underline{A})^{-1} \quad . \tag{6.30}$$

We can now compare the effects of propagation error as computed by Models 3 and 4B. Recall in Model 3,

$$\underline{e}_3 = u_0(\underline{e}_M - E\underline{e}_M + \underline{e}_\delta - E\underline{e}_\delta) - C\ \underline{G}\ \underline{e}_s \tag{6.31}$$

while in Model 4B

$$\underline{e}_3 = u_0(\underline{e}_M - E\underline{e}_M) - C\ \underline{G}\ \underline{e}_s \quad . \tag{6.32}$$

First assume that $\Sigma_3 = D\underline{e}_3$ where $\underline{e}_3$ is defined by Equation 6.32, and assume u is known (no propagation error). Then $D\ \hat{\underline{\beta}}$ is given by the first term on the right-hand side of Equation 6.29. Next let Σ_3 be defined as before but let u be unknown. Then $D\ \hat{\underline{\beta}}$ is degraded by the second term on the right-hand side of Equation 6.29. Finally, let $\Sigma_3 = D\underline{e}_3$ where $\underline{e}_3$ is defined by Equation 6.31. Then we are defining the propagation error by a random error vector $\underline{e}_\delta$, and solving the problem after specifying $E\underline{e}_\delta$ and $D\underline{e}_\delta$. As compared to the no propagation error case, the degradation in $D\ \hat{\underline{\beta}}$ results from an enlarged Σ_3 in Equation 6.30.

Thus we have two methods for considering propagation error. In Model 3 and the general linear model, one must define this error component by specifying the mean and dispersion matrix for a random vector. In Models 4A and 4B, the problem takes care of itself.

Recall that Models 4A and 4B are defined only when the propagation error is assumed to be due entirely to a small unknown error in the uniform speed of electromagnetic propagation and when $K \geq 5$. The question now arises as to which is better, Models 4A and 4B, or the previously defined linear models. Obviously, if one is willing to assume that $\underline{e}_\delta$ is equal to the unknown constant vector $\underline{T}p$, then Models 4A and 4B more accurately describe the real world situation. However, if one's experience dictates $\underline{e}_\delta$ should be estimated by a non linear function of distance, and this function has been empirically learned, then the general linear system model or Model 3 is best. Such a situation is encountered when the sky wave correction is made to time difference measurements in Loran (6).

6.4 Two Dimensional Models. Suppose the emitter and all receiving sites are constrained to the X-Y plane. If $Z_0 - Z_0^*$ is deleted from $\underline{\beta}$, the terms $(Z_0^* - Z_i^*)$ deleted from all constant matrices and $(Z_i - Z_i^*)$ deleted from $\underline{\alpha}$ for i = 1, 2, ..., K, then all models previously defined hold for estimating X_0 and Y_0.

6.5 The Total Error Dispersion Matrix Revisited. In all models of this paper, the treatment allows for the possibility that any of the 4K-1 elements of total system error $\underline{e}$ may be correlated with any other element of $\underline{e}$. With this fact reiterated, let us investigate a special case of the total error dispersion matrix Σ_3 of Models 3 and 4B.

Recall $\underline{e}_3 = \underline{e}_2 - C\,\underline{G}\,\underline{e}_s$ where $\underline{e}_2 = u_0(\underline{e}_{Td} - E\underline{e}_{Td})$ and $\underline{e}_{Td}$ is the total time difference error. It was postulated in Sections 4.4 and 4.5 that $\underline{e}_{Td}$ be approximated by

$$\underline{e}_{Td} = C(\underline{e}_N + \underline{e}_r + \underline{e}_t + \underline{e}_p) + C(\underline{T}r + \underline{T}p) \tag{6.33}$$

where each error vector inside the parenthesis is uncorrelated with the other three, and some of the four may be $\emptyset$ either because of measurement method or by assumption. If Equation 6.33 holds, then

$$\Sigma_2 = u_0^2\, C(\underline{De}_N + \underline{De}_r + \underline{De}_t + \underline{De}_p)C' \qquad (6.34)$$

and $\underline{De}_N$, $\underline{De}_r$, $\underline{De}_t$ and $\underline{De}_p$ are each diagonal.

Now consider C $\underline{G}$ $\underline{e}_s$, the second component of $\underline{e}_3$. Assume $\underline{e}_s$ is uncorrelated with $\underline{e}_2$. Let $(\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i)$ be the prior estimator of receiving site location i, and as a special case, permit $\tilde{X}_i$, $\tilde{Y}_i$ and $\tilde{Z}_i$ to be correlated with each other, but assume for every i, the random vector $(\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i)$ is uncorrelated with all other 3K-1 site estimators $(\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i)$. Then $\underline{De}_s$ is a diagonal matrix of matrices. The i^{th} submatrix (not necessarily diagonal itself) is the dispersion matrix of $(\tilde{X}_j, \tilde{Y}_j, \tilde{Z}_j)$. Let this submatrix be called $\underline{De}_{si}$. Then $\underline{G}$ $\underline{e}_s$ $\underline{G}'$ is a K by K diagonal matrix. The i^{th} diagonal element is the real number $\underline{A}_i(\underline{De}_{si})\underline{A}_i'$.

Under these assumptions, the dispersion matrix for Σ_3 may be written

$$\Sigma_3 = u_0^2 C[\Sigma_N + \Sigma_r + \Sigma_t + \Sigma_p + u_0^{-2}\, \underline{G}\, \Sigma_s\, \underline{G}']C' \quad . \qquad (6.35)$$

The total expression inside the brackets is a K by K diagonal matrix. Let this matrix be designated $\underline{a}$ with the i^{th} diagonal element designated a_i. Let $\underline{b}$ be the K-1 by K-1 diagonal matrix with diagonal element $b_i = a_{i+1}/a_i$. Then

$$\Sigma_3 = a_1\, u_0^2\, [\underline{b} + J_{K-1}^{K-1}] \quad . \qquad (6.36)$$

Let $\underline{c}$ be the K-1 by K-1 diagonal matrix with i^{th} diagonal element $c_i = a_i/a_{i+1}$. That is, $\underline{c}$ is the inverse of $\underline{b}$. Then

$$\Sigma_3^{-1} = \frac{1}{a_1 u_0^2}\left[\underline{c} - \frac{\underline{c}\, J_{K-1}^{K-1}\, \underline{c}}{1 + \sum_{i=1}^{K-1} c_i^2}\right] \quad . \qquad (6.37)$$

Thus under assumptions which are intuitively acceptable as reasonable for a time difference net, the inverse of Σ_3 can be computed by the simple formula of Equation 6.37. Should the total expression inside the brackets of Equation 6.35 be equal to $a_1 I_K$, then $\underline{b} = \underline{c} = I_{K-1}$, and Equation 6.37 becomes

$$\Sigma_3^{-1} = \frac{1}{a_1 u_0^2} [I_{K-1} - \frac{1}{K} J_{K-1}^{K-1}] \quad . \qquad (6.38)$$

Note that under the most simplifying assumptions Σ_3 is not diagonal. This conflicts with the work of Marchand (1) and Dutko (7) in which they began with assumptions equivalent to saying Σ_3 is diagonal.

CHAPTER VII

ESTIMATION WITH A PRIORI INFORMATION

7.1 Introduction. In the case of the linear model, we were able to obtain an analytical solution for estimated emitter location and variance of the estimate. This we cannot do in general with the non-linear model. However, the non-linear model is readily suitable for estimation with a priori information. In this chapter, we set up the equation which must be minimized in order to obtain a non-linear estimate of emitter location (X_0, Y_0, Z_0), and solve the problem of estimation with a priori information.

7.2 The Non-Linear Estimation Problem. The general system model was defined in Chapter III by the equation

$$\underline{\lambda} = \begin{bmatrix} \underline{\alpha} \\ C\,\underline{R} \end{bmatrix} + \underline{e} \quad . \tag{7.1}$$

For convenience, let

$$\underline{X} = \begin{bmatrix} \underline{\alpha} \\ C\,\underline{R} \end{bmatrix} \quad .$$

Assuming that $\underline{\lambda}$ is distributed according to the multivariate normal density function,

$$f(\underline{\lambda}) = \frac{1}{(2\pi)^{\frac{4K-1}{2}} \left|\Sigma\right|^{\frac{1}{2}}} e^{-\frac{1}{2}Q}$$

where

$$Q = (\underline{\lambda} - \underline{X})' \Sigma^{-1} (\underline{\lambda} - \underline{X}) \quad .$$

When Q is minimized, $f(\underline{\lambda})$ will be maximized. The maximum likelihood estimate of the 3K+3 unknowns is the particular $(\tilde{X}_0, \tilde{Y}_0, \tilde{Z}_0; \tilde{X}_1, \tilde{Y}_1, \tilde{Z}_1; \tilde{X}_2, \tilde{Y}_2, \tilde{Z}_2; \ldots; \tilde{X}_K, \tilde{Y}_K, \tilde{Z}_K)$ which upon being inserted into $\underline{X}$, minimizes Q for a given observation $\underline{\lambda}$.

Next assume site errors are negligible. Let

$$\underline{\lambda}_4 = u_0(\underline{\theta} - E\underline{e}_{Td}) \quad . \tag{7.2}$$

Then the non-linear model may be written

$$\underline{\lambda}_4 = C\,\underline{R} + \underline{e}_2 \quad . \tag{7.3}$$

Under the assumption of normality,

$$f(\underline{\lambda}_4) = \frac{1}{(2\pi)^{\frac{K-1}{2}} \left|\Sigma_2\right|^{\frac{1}{2}}} e^{-\frac{1}{2}Q_4} \tag{7.4}$$

where

$$Q_4 = (\underline{\lambda}_4 - C\,\underline{R})' \Sigma_2^{-1} (\underline{\lambda}_4 - C\,\underline{R}) \quad . \tag{7.5}$$

The maximum likelihood estimate of emitter location is the particular $(\tilde{X}_0, \tilde{Y}_0, \tilde{Z}_0)$ which upon being inserted into $\underline{R}$, minimizes Q_4.

7.3 The Clustered Emitter Problem. We now consider the clustered emitter problem, a problem which often arises in emitter locational applications. Suppose one measures a vector of time difference measurements $\underline{\theta}$ and is willing to assume with probability one that the unknown emitter position is at one of M known locations. That is, one knows the coordinates (X_{L1}, Y_{L1}, Z_{L1}; X_{L2}, Y_{L2}, Z_{L2}; ...; X_{LM}, Y_{LM}, Z_{LM}) of emitter locations L_1, L_2, ..., L_M and he wishes to assign the emitter position to one of these known locations based on the measured data. We consider two cases below.

7.3.1 Maximum Likelihood Estimation. Let it be assumed that receiving site locations are known with negligible uncertainty. We will use the non-linear model defined by Equation 7.3.

For this problem, $K \geq 2$. That is, we can make a decision based on a net with two or more receiving stations.

Suppose one has a vector of time difference measurements $\underline{\theta}$, but has no information to cause him to favor one of the possible M locations over another. First compute $\underline{\lambda}_4$ from $\underline{\theta}$ by Equation 7.2. By assuming that (X_0, Y_0, Z_0) must be one of M known points, it is a simple matter to sequentially insert $L_i = (X_{Li}, Y_{Li}, Z_{Li})$ for i = 1, 2, ..., M into $\underline{R}$ of Equation 7.5, and choose the one, say $\hat{L}$, which gives the lowest Q_4. Stated mathematically, the maximum likelihood estimate of position under these assumptions is

$$\hat{L} = \underset{L_i}{\text{Min}} \ (\underline{\lambda}_4 - C\ \underline{R})' \ \Sigma_2^{-1} \ (\underline{\lambda}_4 - C\ \underline{R}) \qquad (7.6)$$

where L_i is contained implicitely in $\underline{R}$.

7.3.2 Bayesian Estimate of Emitter Location. Independent of $\underline{\theta}$ and based upon characteristics of the received signal and other information, suppose one is willing to assign an initial probability $p(L_i) = p_i$ for each of the M locations. We now show how to blend this a priori estimate with the data $\underline{\theta}$ in order to arrive at an a posterior estimate.

From Equations 7.4 and 7.5, the density of $\underline{\lambda}_4$ given $L = L_i$ may be written

$$f(\underline{\lambda}_4 \mid L = L_i) = \frac{1}{(2\pi)^{\frac{K-1}{2}} \left|\Sigma_2\right|^{\frac{1}{2}}} e^{-\frac{1}{2}Q_{L_i}} \tag{7.7}$$

where

$$Q_{L_i} = (\underline{\lambda}_4 - C\,\underline{R}_{L_i})'\,\Sigma_2^{-1}\,(\underline{\lambda}_4 - C\,\underline{R}_{L_i}) \tag{7.8}$$

and $\underline{R}_{L_i}$ is the range vector with (X_0, Y_0, Z_0) being set equal to $(X_{L_i}, Y_{L_i}, Z_{L_i})$. By Bayes' Rule,

$$p(L = L_i \mid \underline{\lambda}_4) = \frac{p(L = L_i) f(\underline{\lambda}_4 \mid L = L_i)}{\sum_{i=1}^{M} p(L = L_i) f(\underline{\lambda}_4 \mid L = L_i)} \; . \tag{7.9}$$

Inserting Equations 7.7 and 7.8 into Equation 7.9

$$p(L = L_i \mid \underline{\lambda}_4) = \frac{p_i\, e^{-\frac{1}{2}Q_{L_i}}}{\sum_{i=1}^{M} p_i\, e^{-\frac{1}{2}Q_{L_i}}} \tag{7.10}$$

which is the a posterior probability that the emitter is at location L_i, given the data $\underline{\theta}$. To obtain the Bayesian estimate of emitter location, one sequentially inserts p_i and L_i for $i = 1, 2, \ldots, M$ into Equation 7.10 and chooses the largest. Stated mathematically

$$\hat{L} = \underset{L_i}{\text{Max}}\ p_i\ e^{-\frac{1}{2}(\underline{\lambda}_4 - C\underline{R}_{L_i})'\Sigma_2^{-1}(\underline{\lambda}_4 - C\underline{R}_{L_i})} \quad . \qquad (7.11)$$

Note that the denominator of Equation 7.10 is a positive constant. Hence, we disregarded it when writing Equation 7.11.

To show the maximum likelihood estimate of Section 7.3.1 is equivalent to assuming $p_i = \frac{1}{M}$ for all i, let $p_i = \frac{1}{M}$ be inserted into Equation 7.11. Then

$$\hat{L} = \underset{L_i}{\text{Max}}\ \frac{1}{M}\ e^{-\frac{1}{2}Q_{L_i}}$$

$$= \underset{L_i}{\text{Max}}\ e^{-\frac{1}{2}Q_{L_i}}$$

$$= \underset{L_i}{\text{Min}}\ Q_{L_i}$$

which is equivalent to Equation 7.6.

CHAPTER VIII

FURTHER CONSIDERATIONS OF THE LINEAR MODEL

8.1 Utility of the Time Difference Hyperbolic Net Linear Model. The dispersion matrix of the emitter locational estimator, defined by the solutions for $D\ \hat{\underline{\beta}}$, does not depend upon the data. Rather it is a function of expected measurement accuracy Σ and the geometry of the net. From the solutions for $D\ \hat{\underline{\beta}}$ given in this paper, one may predict (before design) the locational accuracy of an assumed time difference hyperbolic net. It is merely a matter of specifying site coordinates and then computing the three diagonal elements of $D\ \hat{\underline{\beta}}$ which are $\sigma^2_{\hat{X}_0}$, $\sigma^2_{\hat{Y}_0}$, and $\sigma^2_{\hat{Z}_0}$. Since these vary with geometry, it would be necessary to compute each over the region of interest.

Performing the indicated operations to compute $\hat{\underline{\beta}}$, even in the most simple case of Model 3, involves some tedious matrix arithmetic. Further, if the emitter is in motion (an aircraft for example) the entire sequence of computations must be repeated for each data sample. This could place a prohibitive demand upon the data handling subsystem of a time difference net which must track a moving target in real time. Let us suggest a sub-optimum procedure for computing $\hat{\underline{\beta}}$.

Suppose the receiving sites are all fixed ground stations, and the problem is to locate the emitter only when it is within a region small enough for linearity to hold throughout with (X_0^*, Y_0^*, Z_0^*) remaining fixed. Then

$$\begin{bmatrix} \hat{X}_0 \\ \hat{Y}_0 \\ \hat{Z}_0 \end{bmatrix} = \underline{F}\ \underline{\theta} \tag{8.1}$$

where $\underline{F}$ is a 3 by K-1 constant matrix which could be pre-computed. Computation of estimated emitter location under these assumptions is a simple algebraic operation, and there would be no problem in it being accomplished in real time. When the region of interest is too large for linearity to hold, one might have several pre-computed 3 by K-1 matrices $\underline{F}_1, \underline{F}_2, \ldots, \underline{F}_n$ and have the measurement $\underline{\theta}$ automatically choose the appropriate one.

8.2 Utility of the Linear Time Difference Model. Formulating the linear model in matrix form $\underline{\lambda} = \underline{B}\ \underline{\gamma} + \underline{e}$ greatly simplifies matters. Indeed, it was the matrix model which permitted us to derive a general solution for $\hat{\underline{\beta}}$ and D $\hat{\underline{\beta}}$ in three dimensions with all the errors considered.

A similar approach could easily be applied to other types of emitter locational nets. Consider the K station range only radar problem. Let the measurement of distance from each of K radar sites to a radar target be the vector $\underline{d}$. Then

$$\underline{d} = \underline{R} + \underline{e}$$

$$\underline{d} \approx R^{*} + \underline{A}\ \underline{\beta} + \underline{e}_N \quad . \tag{8.2}$$

The solution is

$$\hat{\beta} = (\underline{A}'\ \Sigma_N^{-1}\ \underline{A})^{-1}\ \underline{A}'\ \Sigma_N^{-1}\ (\underline{d} - \underline{R}^{*}) \tag{8.3}$$

$$D\,\underline{\hat{\beta}} = (\underline{A}'\ \Sigma_N^{-1}\ \underline{A})^{-1} \qquad (8.4)$$

where $\underline{A}$, $\underline{R}^*$ and $\underline{\hat{\beta}}$ are as previously defined.

8.3 Comparison of the Radar and Time Difference Net. Assume the range at each of K radar sites is measured with error $u_0\,\sigma$. Then in the radar model

$$D\,\underline{\hat{\beta}} = u_0^2\,\sigma^2(\underline{A}'\ \underline{A})^{-1} \quad . \qquad (8.5)$$

Assume the arrival time at each of K receiving sites (at same locations as radar sites) is measured with error σ. Then for the time difference model

$$D\,\underline{\hat{\beta}} = u_0^2\,\sigma^2[\underline{A}'\ C'\ (C\ C')^{-1}\ C\ \underline{A}]^{-1} \quad . \qquad (8.6)$$

By performing the indicated operation on C as defined,

$$C'\ (C\ C')^{-1}\ C = I_K - \frac{1}{K}\,J_K^K \quad . \qquad (8.7)$$

Substituting Equation 8.7 into Equation 8.6,

$$D\,\underline{\hat{\beta}} = u_0^2\sigma^2[\underline{A}'\ A - \frac{1}{K}\,\underline{A}'\ J_1^K\ J_K^1\ \underline{A}]^{-1}$$

which may be written

$$D\,\underline{\hat{\beta}} = u_0^2\sigma^2(\underline{A}'\underline{A}) \quad + u_0^2\sigma^2\ \frac{(\underline{A}'\underline{A})^{-1}\underline{A}'J_1^K J_K^1\underline{A}(\underline{A}'\underline{A})^{-1}}{J_K^1(I_K-\underline{A}(\underline{A}'\underline{A})^{-1}\underline{A}')J_1^K} \quad .$$

Let

$$\underline{a} = u_0^2\sigma^2\ \frac{(\underline{A}'\underline{A})^{-1}\underline{A}'J_1^K J_K^1\underline{A}(\underline{A}'\underline{A})^{-1}}{J_K^1(I_K-\underline{A}(\underline{A}'\underline{A})^{-1}\underline{A}')J_1^K}$$

where $\underline{a}$ is a 3 by 3 positive semidefinite matrix. Then

Range measurements: $$D\,\hat{\underline{\beta}} = u_0^2\sigma^2\,(\underline{A}'\,\underline{A})^{-1} \tag{8.8}$$

Arrival time measurements: $$D\,\hat{\underline{\beta}} = u_0^2\sigma^2\left[(\underline{A}'\,\underline{A})^{-1} + \underline{a}\right] \quad . \tag{8.9}$$

Hence, under the simplified assumptions made here, the accuracy of a time difference net is degraded by the square root of the diagonal elements of $\underline{a}$ when compared with a range only radar net. Another way of looking at this result is that $D\,\hat{\underline{\beta}}$ for the time difference net is degraded by amount $\underline{a}$ due to not knowing t_0.

CHAPTER IX

SUMMARY AND CONCLUSIONS

9.1 Summary. The problem of estimating location of an emitter (radiator of electromagnetic energy) by a time difference hyperbolic net has been examined. The number of receiving stations in the net is the arbitrary number K from which K-1 baselines are established. A time difference hyperbolic net of K-1 baselines is an operation of K receiving stations instrumented with a capability to measure K-1 arrival time differences for a sufficient set of K-1 pairs of stations. The problem of three dimensional location (emitter and receiving stations in X - Y - Z space) was analyzed as opposed to the less general case of two dimensional operation.

Four or more receiving stations in a time difference net are required to locate an emitter. From three arrival time difference measurements acquired by cooperation of four receiving stations, the deterministic solution for emitter location is derived in Chapter II. Emitter location is computed by finding the roots of a second order polynomial. Since the second order polynomial has two roots, one may not be able to ascertain which is the applicable root and which represents a "ghost" location. Hence, location is not always unique when $K = 4$. When K is greater than four, this ambiguity is removed.

As stated above, three or more time differences are required to compute estimated emitter location. However, one time difference measurement provides some locational information, i.e., it permits one to define a curved "plane" which passes through the point of emitter location. A second time difference measurement (acquired by cooperation of one additional receiver) places the emitter on a curved "line" of position. When prior information is available in the form of assignment of the emitter position to one of M known point locations, then two or more receiving stations are sufficient to estimate emitter location. In Chapter VII, this problem is solved first by maximum likelihood and second by Bayesian decision theory techniques. The maximum likelihood estimator assigns equal weights to each of the assumed M possibilities, while the Bayes' estimator permits one to weight each of the M possibilities by an a priori probability. In both cases, a particular one of the M points is selected as the a posterior estimate of emitter location based on the time difference data.

The major effort in this study was devoted to the statistical problem of estimating the emitter location when $K > 4$ and determining the accuracy of the estimate when $K \geq 4$. Final results yielded the following:

(1) A linear estimator which defines the estimated emitter location as a function of time difference measurements, recorded receiving site coordinates, and the total error dispersion matrix of system errors.

(2) The dispersion matrix of the estimator. This problem is solved in Chapters V and VI following development of a mathematical model in Chapters III and IV. In development of the mathematical model, the assumption was made that errors were small enough that the change in distance between emitter and receiver i for $i = 1, 2, \ldots, K$ due to error variation could be approximated by a first order Taylor series.

This is equivalent to the "parallel line displacement" assumption commonly made in the literature on emitter location.

The solutions in Chapters V and VI extend the theory beyond previous work as follows:

(1) Correlated time difference measurement errors are permitted.

(2) The three dimensional location problem is solved.

(3) Errors due to true location of receiving stations being unknown are accounted for.

(4) Errors due to propagation anomalies are accounted for.

9.2 Conclusions. A linear estimator (and dispersion matrix of the estimator) has been derived which estimates location of an emitter based upon data obtained from a time difference hyperbolic net of K stations where K is equal to or greater than four. The estimator satisfies the three criteria: (1) of minimizing the weighted sum of squared errors; (2) of being the minimum variance unbiased linear estimator; and (3) of being the maximum likelihood estimator under the assumption of normality.

The approximations mentioned in Section 9.1 above which were necessary in development of the linear model reduces the precision of the linear estimator as errors become large. Analysis of this effect remains as an unsolved problem. However, for $K = 4$, this problem does not exist because solutions for emitter location is a deterministic operation. For $K = 5$, one would expect this problem to be minimal since the data from one baseline are blended with the best non-linear estimate already found by the first three baselines.

Computation of estimated emitter location according to the function defined by the linear estimator requires a lengthy sequence of arithmetical operations, including inversion of matrices. It is not feasible to make these computations except by electronic computer. For a time difference net in which there is relative motion between receivers and the emitter, the entire sequence of arithmetical operations must be repeated for each data sample. This could have major impact if the time difference net were required to track the emitter in real time. For $K \geq 5$, one might want to use some sub-optimal estimator instead of the minimum variance estimator so as to keep the computational requirement within reasonable bounds.

The one theoretical result derived in this paper of greatest utility is perhaps the dispersion matrix of the linear estimator. This dispersion matrix does not depend upon the time of arrival data. Instead, it is a function of geometry and the accuracy to which one expects to make the measurements. With expected measurement errors known (or estimated as was discussed in Chapter IV) one with an emitter location requirement may specify a K station time difference hyperbolic net and determine if its potential accuracy satisfies his requirement. It is necessary to compute only the three by three dispersion matrix of the estimator. The square root of the three diagonal elements is a measure of the expected error in estimating emitter coordinates X_0, Y_0 and Z_0 respectively.

BIBLIOGRAPHY

1. Marchand, Nathan. "Error Distribution of Best Estimate of Position From Multiple Time Difference Hyperbolic Networks." IEEE Transactions on Aerospace and Navigational Electronics. ANE-11. (June, 1964).

2. Graybill, Franklin A. An Introduction to Linear Statistical Models, Volume I. New York: McGraw-Hill, 1961.

3. Rao, C. Radhakrishna. Linear Statistical Inference and its Applications. New York: John Wiley and Sons, 1965.

4. Deutsch, Ralph. Estimation Theory. Englewood Cliffs: Prentice-Hall, Inc., 1965.

5. Papoulis, Athanasios. Probability, Random Variables, and Stochastic Processes. New York: McGraw-Hill, 1965.

6. Pierce, J. A., A. A. McKenzie, and R. H. Woodward. Loran. New York: McGraw-Hill, 1948.

7. Dutko, M. "The Theory of Hyperbolic Position Finding." Technical Report HRB-323-TP-104-67, HRB-Singer, Inc., State College, Pennsylvania. (August, 1967).

APPENDIX A

DEFINITION OF TERMS

This Appendix contains a collection of terms used consistently throughout the paper. Hopefully, it will aid the reader by providing a central source defining the many symbols used.

$$\underline{A} = \begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{A}_K \end{bmatrix}$$

$$\underline{A}_i = (a_{i1}, a_{i2}, a_{i3})$$

$$a_{i1} = \frac{X_0^* - X_i^*}{R_i^*}$$

$$a_{i2} = \frac{Y_0^* - Y_i^*}{R_i^*}$$

$$a_{i3} = \frac{Z_0^* - Z_i^*}{R_i^*}$$

$$\underline{B} = \begin{bmatrix} I_{3K} & \emptyset_3^{3K} \\ -C\,\underline{G} & C\,\underline{A} \end{bmatrix}$$

C is the time difference generating matrix when K-1 time differences are measured.

$$C = (J_1^{K-1}, -I_{K-1})$$

C_1 is the time difference generating matrix when $\binom{K}{2}$ time differences are measured. C_1 is a matrix of K-1 submatrices. The i^{th} submatrix is $C_{1i} = (\emptyset_{i-1}^{K-i}, J_1^{K-i}, -I_{K-i})$.

$$C_1 = \begin{bmatrix} \emptyset_0^{K-1}, & J_1^{K-1}, & -I_{K-1} \\ \emptyset_1^{K-2}, & J_1^{K-2}, & -I_{K-2} \\ \emptyset_2^{K-3}, & J_1^{K-3}, & -I_{K-3} \\ & \vdots & \\ \emptyset_{K-2}^{1}, & 1 & , -1 \end{bmatrix}$$

D is an expected value operator such that if $\underline{\lambda}$ is a random vector, D $\underline{\lambda}$ is the dispersion matrix (sometimes called variance-covariance matrix) of $\underline{\lambda}$. $D\,\underline{\lambda} = E[(\underline{\lambda} - E\,\underline{\lambda})(\underline{\lambda} - E\,\underline{\lambda})']$.

E is an expected value operator such that E $\underline{\lambda}$ is the mean vector of random vector $\underline{\lambda}$.

$\underline{e}$ is the total system error (a random vector).

$$\underline{e} = \begin{bmatrix} \underline{e}_s \\ \underline{e}_2 \end{bmatrix}$$

$\underline{e}_M$ is time difference measurement error (a random vector).

$\underline{e}_N$ is a component of $\underline{e}_M$ (a random vector).

$\underline{e}_p$ is a component of $\underline{e}_\delta$ (a random vector).

$\underline{e}_r$ is an instrumental error (a random vector).

$\underline{e}_s$ is the error in location of receiving stations (a random vector).

$\underline{e}_t$ is an instrumental error (a random vector).

$\underline{e}_{Td}$ is the error due to time difference measurement inaccuracies and propagation anomalies (a random vector).

$$\underline{e}_{Td} = \underline{e}_M + \underline{e}_\delta$$

$\underline{e}_\delta$ is the error due to propagation anomalies (a random vector).

$$\underline{e}_\delta = C(\underline{T}p + \underline{e}_p)$$

$$\underline{e}_2 = u_0(\underline{e}_{Td} - E\underline{e}_{Td})$$

$$\underline{e}_3 = \underline{e}_2 - C\,\underline{G}\,\underline{e}_s$$

$$\underline{G} = \begin{bmatrix} \underline{A}_1 & & & & & \\ & \underline{A}_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \underline{A}_K \end{bmatrix}$$

I is the identity matrix. The symbol I_n is used when it is desired to show the order of I is n.

J_m^n is a matrix of n rows and m columns, and every element is the number one.

K is the number of receiving stations in the time difference hyperbolic net.

$$\binom{K}{2} = \frac{K!}{2(K-2)!} = \frac{K(K-1)}{2}$$

$\emptyset$ is the null matrix. The symbol $\emptyset_m^n$ is used to show $\emptyset$ is n by m.

$$\underline{R} = (R_1, R_2, \ldots, R_K)'$$

$$\underline{R}^* = (R_1^*, R_2^*, \ldots, R_K^*)'$$

$$R_i = [(X_0 - X_i)^2 + (Y_0 - Y_i)^2 + (Z_0 - Z_i)^2]^{\frac{1}{2}}$$

$$R_i^* = [(X_0^* - X_i^*)^2 + (Y_0^* - Y_i^*)^2 + (Z_0^* - Z_i^*)^2]^{\frac{1}{2}}$$

$\underline{T}a$ is a vector of arrival times.

$\underline{T}d$ is the time difference vector.

t_i is the time of arrival at the i^{th} receiving site of a signal from the emitter.

t_0 is the time of radiation of a signal by the emitter.

u is the true speed of electromagnetic propagation.

u_0 is the assumed value of u.

X_i is the X coordinate of the i^{th} receiving site.

X_i^* is a point near X_i.

X_0 is the X coordinate of the emitter.

X_0^* is a point near X_0.

Y_i is the Y coordinate of the i^{th} receiving site.

Y_i^* is a point near Y_i.

Y_0 is the Y coordinate of the emitter.

Y_0^* is a point near Y_0.

Z_i is the Z coordinate of the i^{th} receiving site.

Z_i^* is a point near Z_i.

Z_0 is the Z coordinate of the emitter.

Z_0^* is a point near Z_0.

$$\underline{\alpha} = \begin{bmatrix} \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{\alpha}_K \end{bmatrix}$$

$$\underline{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3})'$$

$$\alpha_{i1} = X_i - X_i^*$$

$$\alpha_{i2} = Y_i - Y_i^*$$

$$\alpha_{i3} = Z_i - Z_i^*$$

$$\underline{\beta} = (\beta_1, \beta_2, \beta_3)'$$

$$\beta_1 = X_0 - X_0^*$$

$$\beta_2 = Y_0 - Y_0^*$$

$$\beta_3 = Z_0 - Z_0^*$$

$$\underline{\gamma} = \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \end{bmatrix}$$

$$\underline{Y}_{4A} = \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \\ \delta_0 \end{bmatrix}$$

$$\underline{Y}_{4B} = \begin{bmatrix} \underline{\beta} \\ \delta_0 \end{bmatrix}$$

$$\delta_0 = \frac{u_0 - u}{u}$$

$\underline{\theta}$ is the vector of time difference measurements (a random vector).

$$\underline{\lambda} = \begin{bmatrix} \underline{\lambda}_1 \\ \underline{\lambda}_2 \end{bmatrix}$$

$\underline{\lambda}_1$ is the recorded coordinates of receiving sites (a random vector).

$$\underline{\lambda}_2 = u_0 \, \underline{\theta} - C \, \underline{R}^* - u_0 \, E\underline{e}_{Td}$$

$$\underline{\lambda}_3 = u_0 \, \underline{\theta} - u_0 \, E\underline{e}_{Td} + C \, \underline{G} \, E \, \underline{\lambda}_1 - C \, \underline{R}^*$$

$$\underline{\lambda}_4 = u_0 \, \underline{\theta} - u_0 \, E\underline{e}_{Td}$$

$$\Sigma = D \, \underline{e}$$

$$\Sigma_M = D \, \underline{e}_M$$

$$\Sigma_N = D \, \underline{e}_N$$

$$\Sigma_p = D \, \underline{e}_p$$

$$\Sigma_r = D \, \underline{e}_r$$

$$\Sigma_s = D \, \underline{e}_s$$

$$\Sigma_t = D \, \underline{e}_t$$

$$\Sigma_{Td} = D \, \underline{e}_{Td}$$

$$\Sigma_\delta = D \, \underline{e}_\delta$$

$$\Sigma_2 = D \, \underline{e}_2$$

$$\Sigma_3 = D \, \underline{e}_3$$

APPENDIX B

PROOF OF THEOREMS

Some theorems were stated in Section 4.2 without proof. These theorems are proven below.

Theorem B.1. Let Σ be an n by n positive definite matrix, and let $\underline{F}$ be an p by n matrix. Then $\underline{F}\ \Sigma\ \underline{F}'$ is positive definite when $\underline{F}$ is of rank p and positive semidefinite when $\underline{F}$ is of rank less than p.

Proof: Consider the matrix product $\underline{b}'\ \underline{F}\ \Sigma\ \underline{F}'\ \underline{b}$ where $\underline{b}$ is any p by 1 vector. Let $\underline{a}' = \underline{b}'\ \underline{F}$. By hypothesis Σ is positive definite which implies $\underline{a}'\ \Sigma\ \underline{a} > 0$ for every $\underline{a} \neq \emptyset$ by Definition 4.2. Hence, $\underline{b}'\ \underline{F}\ \Sigma\ \underline{F}'\ \underline{b} \geq 0$ for every $\underline{b}$, and $\underline{b}'\ \underline{F}\ \Sigma\ \underline{F}'\ \underline{b} > 0$ for every $\underline{b}$ such that $\underline{b}'\ \underline{F} \neq \emptyset$. To prove the theorem, let us consider the two cases separately.

Case I: Assume $\underline{F}$ has rank p. This implies the p rows of $\underline{F}$ are independent. Therefore, there exists no $\underline{b} \neq \emptyset$ such that $\underline{b}'\ \underline{F} = \emptyset$. Hence, $\underline{b}'\ \underline{F}\ \Sigma\ \underline{F}'\ \underline{b} > 0$ for every $\underline{b} \neq \emptyset$ and $\underline{F}\ \Sigma\ \underline{F}'$ is positive definite by Definition 4.2.

Case II: Assume $\underline{F}$ has rank less than p. Then $\underline{F}\ \Sigma\ \underline{F}'$ is p by p and of rank less than p. This implies there exists a vector $\underline{b} \neq \emptyset$ such that $\underline{b}'\ \underline{F}\ \Sigma\ \underline{F}' = \emptyset$. For this particular $\underline{b}$, $\underline{b}'\ \underline{F}\ \Sigma\ \underline{F}'\ \underline{b} = 0$. For all $\underline{b}$, it was shown above that $\underline{b}'\ \underline{F}\ \Sigma\ \underline{F}'\ \underline{b} \geq 0$. Therefore $\underline{F}\ \Sigma\ \underline{F}'$ is positive semidefinite by Definition 4.3.

Theorem B.2: Let Σ and $\underline{F}$ be as defined above. Then every diagonal element of $\underline{F}\ \Sigma\ \underline{F}'$ is zero only if $\underline{F} = \emptyset$. Also $\underline{F}\ \Sigma\ \underline{F}' = \emptyset$ only if $\underline{F} = \emptyset$.

Proof: Let each diagonal element of $\underline{F}\ \Sigma\ \underline{F}'$ be zero. Suppose, contrary to fact, that $\underline{F} \neq \emptyset$. This implies there exists at least one non-null row vector $\underline{F}_i$ in $\underline{F}$, which by Definition 4.2, implies there exists a nonzero diagonal element of $\underline{F}\ \Sigma\ \underline{F}'$ equal to $\underline{F}_i\ \Sigma\ \underline{F}_i'$. This contradicts the initial assumption that each diagonal element of $\underline{F}\ \Sigma\ \underline{F}'$ is zero. Therefore, every diagonal element of $\underline{F}\ \Sigma\ \underline{F}'$ is zero only if $\underline{F} = \emptyset$.

To see that $\underline{F}\ \Sigma\ \underline{F}' = \emptyset$ only if $\underline{F} = \emptyset$, observe that when $\underline{F}\ \Sigma\ \underline{F}' = \emptyset$, each diagonal element of $\underline{F}\ \Sigma\ \underline{F}'$ is zero which implies $\underline{F} = \emptyset$ by the above proof.

Theorem B.3: Let Σ be positive definite and symmetric. Then Σ^{-1} exists and is positive definite symmetric.

Proof: Suppose, contrary to fact, that Σ^{-1} does not exist. Then Σ is of rank less than n, and there exists a non-null column vector $\underline{a}$ such that $\underline{a}'\ \Sigma = \emptyset$. For this particular $\underline{a}$, $\underline{a}'\ \Sigma\ \underline{a} = 0$ which contradicts the hypothesis that Σ is positive definite. Therefore Σ^{-1} exists. To show Σ^{-1} is symmetric, let Σ be expressed $\Sigma = \Sigma\ \Sigma^{-1}\ \Sigma$. Transposing, $\Sigma' = \Sigma'(\Sigma^{-1})'\Sigma'$. Replacing Σ' by Σ, $\Sigma = \Sigma(\Sigma^{-1})'\Sigma$. Hence $\Sigma\ \Sigma^{-1}\ \Sigma = \Sigma(\Sigma^{-1})'\Sigma$. Pre and post multiplying both sides by Σ^{-1}, the result follows. To show Σ^{-1} is positive definite, $\Sigma^{-1} = \Sigma^{-1}\ \Sigma\ \Sigma^{-1} = \Sigma^{-1}\ \Sigma(\Sigma^{-1})'$ = a positive definite matrix by Theorem B.1.

Theorem B.4: Let Σ_1 be positive definite and Σ_2 be positive definite or positive semidefinite. Then $\Sigma = \Sigma_1 + \Sigma_2$ is positive definite.

Proof: $\underline{a}'\ \Sigma\ \underline{a} = \underline{a}'(\Sigma_1 + \Sigma_2)\underline{a} = \underline{a}'\ \Sigma_1\ \underline{a} + \underline{a}'\ \Sigma_2\ \underline{a}$. Σ_1 positive definite implies $\underline{a}' \Sigma_1 \underline{a} > 0$ for every $\underline{a} \neq \emptyset$. Σ_2 positive semidefinite implies $\underline{a}' \Sigma_2 \underline{a} \geq 0$ for every $\underline{a}$. Therefore $\underline{a}' \Sigma \underline{a} > 0$ for every $\underline{a} \neq \emptyset$ which implies Σ is positive definite.

Theorem B.5: Let $\underline{Y}_1 = \underline{X}\ \underline{Y} + \underline{b}$ where $\underline{Y}_1$ and $\underline{Y}_2$ are random vectors, $\underline{X}$ is a constant matrix and $\underline{b}$ is a constant column vector. Then $D\ \underline{Y}_1 = \underline{X}(D\ \underline{Y})\underline{X}'$.

Proof:

$$D\ \underline{Y}_1 = E\{[(\underline{X}\ \underline{Y} + \underline{b}) - E(\underline{X}\ \underline{Y} + \underline{b})][(\underline{X}\ \underline{Y} + \underline{b}) - E(\underline{X}\ \underline{Y} + \underline{b})]'\}$$

$$= E\{[\underline{X}\ \underline{Y} - E(\underline{X}\ \underline{Y})][\underline{X}\ \underline{Y} - E(\underline{X}\ \underline{Y})]'\}$$

$$= E[\underline{X}(\underline{Y} - E\ \underline{Y})(\underline{Y} - E\ \underline{Y})'\underline{X}']$$

$$= \underline{X}\{E[(\underline{Y} - E\ \underline{Y})(\underline{Y} - E\ \underline{Y})']\}\underline{X}'$$

$$= \underline{X}(D\ \underline{Y})\underline{X}' \qquad .$$

Theorem B.6: Let $\underline{Y}_1$ and $\underline{Y}_2$ be two random vectors, $\underline{X}_1$ and $\underline{X}_2$ be two constant matrices, and $\underline{b}_1$ and $\underline{b}_2$ be two constant vectors. Then

$$\mathrm{Cov}[(\underline{X}_1\ \underline{Y}_1 + \underline{b}_1),(\underline{X}_2\ \underline{Y}_2 + \underline{b}_2)] = \underline{X}_1[\mathrm{Cov}(\underline{Y}_1,\ \underline{Y}_1)]\underline{X}_2'$$

Proof:

$$\mathrm{Cov}[(\underline{X}_1\ \underline{Y}_1 + \underline{b}_1),(\underline{X}_2\ \underline{Y}_2 + \underline{b}_2)] =$$

$$E\{[(\underline{X}_1\ \underline{Y}_1 + \underline{b}_1) - E(\underline{X}_1\ \underline{Y}_1 + \underline{b}_1)][(\underline{X}_2\ \underline{Y}_2 + \underline{b}_2) - E(\underline{X}_2\ \underline{Y}_2 + \underline{b}_2)]'\} =$$

$$E\{[(\underline{X}_1\ \underline{Y}_1) - E(\underline{X}_1\ \underline{Y}_1)][(\underline{X}_2\ \underline{Y}_2) - E(\underline{X}_2\ \underline{Y}_2)]'\} =$$

$$E[\underline{X}_1\ (\underline{Y}_1 - E\ \underline{Y}_1)(\underline{Y}_2 - E\ \underline{Y}_2)'\underline{X}_2'] =$$

$\underline{X}_1\{E[(\underline{Y}_1 - E\,\underline{Y}_1)(\underline{Y}_2 - E\,\underline{Y}_2)']\}\underline{X}_2' =$

$\underline{X}_1[\mathrm{Cov}(\underline{Y}_1, \underline{Y}_2)]\underline{X}_2'$.

Theorem B.7: Let $\underline{Y}_1$ and $\underline{Y}_2$ be two random vectors, and let $\underline{X}_1$ and $\underline{X}_2$ be two constant matrices such $\underline{X}_1\,\underline{Y}_1$ and $\underline{X}_2\,\underline{Y}_2$ are each n by 1 random vectors. Then

$$D(\underline{X}_1\,\underline{Y}_1 \pm \underline{X}_2\,\underline{Y}_2) = \underline{X}_1(D\,\underline{Y}_1)\underline{X}_1' + \underline{X}_2(D\,\underline{Y}_2)\underline{X}_2' \pm \underline{X}_1[\mathrm{Cov}(\underline{Y}_1, \underline{Y}_2)]\underline{X}_2' \pm \underline{X}_2[\mathrm{Cov}(\underline{Y}_2, \underline{Y}_1)]\underline{X}_1'$$

Proof: Let $\underline{Z}_1 = \underline{X}_1\,\underline{Y}_1 - E(\underline{X}_1\,\underline{Y}_1)$, and let $\underline{Z}_2 = \underline{X}_2\,\underline{Y}_2 - E(\underline{X}_2\,\underline{Y}_2)$. Then

$$D(\underline{X}_1\,\underline{Y}_1 \pm \underline{X}_2\,\underline{Y}_2) = D(\underline{Z}_1 \pm \underline{Z}_2) = E[(\underline{Z}_1 \pm \underline{Z}_2)(\underline{Z}_1 \pm \underline{Z}_2)']$$

$$= E(\underline{Z}_1\,\underline{Z}_1') + E(\underline{Z}_2\,\underline{Z}_2') \pm E(\underline{Z}_1\,\underline{Z}_2') \pm E(\underline{Z}_2\,\underline{Z}_1')$$

$$= D\,\underline{Z}_1 + D\,\underline{Z}_2 \pm \mathrm{Cov}(\underline{Z}_1, \underline{Z}_2) \pm \mathrm{Cov}(\underline{Z}_2, \underline{Z}_1) \quad .$$

The statement of the theorem follows by applying Theorems B.5 and B.6.

APPENDIX C

MODEL 5

As has been mentioned previously, $\binom{K}{2} = \frac{1}{2}K(K-1)$ arrival time differences may be obtainted from K arrival times. That is, one can write $\binom{K}{2}$ deterministic time difference equations, K-1 of which are independent, and the remaining $\frac{1}{2}(K-1)(K-2)$ are dependent. This was illustrated in Section 2.3. Previously, we have considered the case when one measures K-1 time differences associated with a subset of K-1 independent equations from the $\binom{K}{2}$ set. We will now develop a special model for $\binom{K}{2}$ time difference measurements.

From Equation 3.3 the arrival time vector at the receiving sites is

$$\underline{Ta} = t_0 \; J_1^K + \frac{1}{u} \underline{R} \qquad .$$

Assuming site errors are negligible, the linearized approximation to $\underline{Ta}$ is

$$\underline{Ta} = t_0 \; J_1^K + \frac{1}{u_0} \left(\underline{R}^* + \underline{A} \; \underline{\beta}\right) \qquad (C.1)$$

where $\underline{Ta} = (t_1, t_2, \ldots, t_K)'$. Let $(t_1 - t_2, t_1 - t_3, \ldots, t_1 - t_K; t_2 - t_3, t_2 - t_4, \ldots, t_2 - t_K; \ldots; t_{K-1} - t_K)'$ be arbitrarily chosen as an ordered sequence for the $\binom{K}{2}$ time differences to be measured. The generating matrix C_1 such that $C_1 \; \underline{Ta}$ is the above sequence is defined in Appendix A.

Let $\underline{\theta}_5$ be the $\binom{K}{2}$ column vector of measurements. Then

$$\underline{\theta}_5 = C_1 \underline{T}a + \underline{e}_K \tag{C.2}$$

where $\underline{e}_K$ is the total error. $E\underline{e}_K$ will not be $\emptyset$ if constant vectors $\underline{T}r$ and $\underline{T}p$ (discussed in Chapter IV) are applicable. Assume $E\underline{e}_K$ is known.

Substituting Equation C.1 into Equation C.2

$$\underline{\theta}_5 = \frac{C_1}{u_0} (\underline{R}^* + \underline{A}\ \underline{\beta}) + \underline{e}_K \quad . \tag{C.3}$$

Let

$$\underline{\lambda}_5 = u_0 (\underline{\theta}_5 - E\underline{e}_K) - C_1 \underline{R}^* \tag{C.4}$$

and

$$\underline{e}_5 = u_0 (\underline{e}_K - E\underline{e}_K) \quad . \tag{C.5}$$

Then

$$\underline{\lambda}_5 = C_1 \underline{A}\ \underline{\beta} + \underline{e}_5 \quad . \tag{C.6}$$

Equation C.6 will be called Model 5. By construction, $E\underline{e}_5 = \emptyset$, and $C_1 \underline{A}$ is $\binom{K}{2}$ by 3 and of rank 3.

If $D\underline{e}_5 = \Sigma_5$ is positive definite, the solution to Model 5 by the theory of Chapter V is

$$\hat{\underline{\beta}} = (\underline{A}' C_1' \Sigma_5^{-1} C_1 \underline{A})^{-1} \underline{A}' C_1' \Sigma_5^{-1} \underline{\lambda}_5 \tag{C.7}$$

$$D\ \hat{\underline{\beta}} = (\underline{A}' C_1' \Sigma_5^{-1} C_1 \underline{A})^{-1} \tag{C.8}$$

Equations C.7 and C.8 in two dimensions under the assumption of positive definite and diagonal Σ_5 is equivalent to results by Marchand (1) and Dutko (7). However, we are unable to permit Σ_5 to be diagonal when time differences are measured simultaneously. It was shown by Equation 6.38

that Σ_3, the K-1 by K-1 upper left-hand corner submatrix of Σ_5, is not diagonal even under the most simplifying assumptions. Should time differences (not arrival times) be measured sequentially and $\Sigma_r = \Sigma_\delta = \emptyset$ (these were defined in Chapter IV), then assumption of Σ_5 being diagonal is reasonable. By sequential measurement of time differences, we mean that from a repetitive emitter, each time difference $t_i - t_j$ is measured during non-overlapping time intervals.

When arrival times are measured (and time differences computed from arrival times), Σ_5^{-1} does not exist, and Model 5 is untenable. This statement holds whether arrival times are measured simultaneously or sequentially. The argument behind the statement was provided in Section 4.4.1.

VITA

Marcus Harold McLean

Candidate for the Degree of

Doctor of Philosophy

Thesis: EMITTER LOCATION BY A TIME DIFFERENCE HYPERBOLIC NET

Major Field: Electrical Engineering

Biographical:

Personal Data: Born in Daleville, Alabama, May 16, 1929, the son of Mr. and Mrs. John L. McLean.

Education: Graduated from Daleville High School, Daleville, Alabama, in May, 1947; received the Bachelor of Electrical Engineering degree from Auburn University in 1951; received the Master of Science degree from Ohio State University in 1956, with a major in Electrical Engineering; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in August, 1969.

Professional Experience: Regular officer in United States Air Force since 1952.

Professional Organizations: Member of the Institute of Electrical and Electronic Engineers.