HOOKER, John Wesley, 1932-
EXISTENCE AND OSCILLATION THEOREMS FOR A CLASS OF NON-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS.

The University of Oklahoma, Ph.D., 1967
Mathematics

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

EXISTENCE AND OSCILLATION THEOREMS FOR A CLASS OF NON-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

JOHN WESLEY HOOKER

Norman, Oklahoma

1967

EXISTENCE AND OSCILLATION THEOREMS FOR A CLASS OF

NON-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

APPROVED BY

William T. Reid, Chairman

John C. Brixey

George M. Ewing

Gene Levy

DISSERTATION COMMITTEE

ACKNOWLEDGMENTS

The author gratefully acknowledges the advice and assistance of Professor William T. Reid, who directed the writing of this paper.

Part of the research for this paper was done during the summer of 1965 and the academic year 1965-1966 while the author held a National Science Foundation Cooperative Graduate Fellowship. The remainder of the research, during the summer of 1966, was supported by the Air Force Office of Scientific Research under grant AF-AFOSR-749-65.

TABLE OF CONTENTS

EXISTENCE AND OSCILLATION THEOREMS FOR A CLASS OF NON-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

1. Introduction. This paper is concerned with the existence and oscillation of solutions of some initial value problems and boundary value problems associated with the real, scalar differential equation

$$(1.1) \qquad y'' + yF(t,y,y') = 0, \qquad a \le t \le b,$$

where F is a real-valued continuous function bounded above and below by non-negative continuous functions $\check{F}(y)$ and $\hat{F}(y)$ which satisfy certain monotoneity properties. Section 2 below gives a more precise formulation of the problem.

Nehari [5], [6], has established conditions under which, for any positive integer n, the boundary problem

$$(1.2) \qquad \begin{gathered} y'' + yF(t,y^2) = 0, \\ y(a) = 0 = y(b), \end{gathered}$$

has a solution which vanishes precisely $n - 1$ times in the open interval (a,b), (see [6; Thm. 3.2]). The conditions required by Nehari will be seen, in Section 6 below, to imply the conditions which we require here for equation (1.1), with the exception that Nehari does not assume a Lipschitz condition on F, as is done in the present work. Because of this assumption of a Lipschitz condition, the results of this paper do not include the corresponding results of Nehari. However, since equation (1.1) may involve y' explicitly, and since the function F of equation (1.1) need not be an even function of y, the class of equations discussed here differs considerably from that discussed by Nehari.

The differential equation

$$y'' + y\phi(t,y,\lambda) = 0, \tag{1.3}$$

where λ is a real parameter, was studied by Moroney [4], who employed a Prüfer transformation to obtain results concerning the existence and oscillation of solutions of (1.3) on the interval [0,1], for boundary conditions $y(0) = 0 = y(1)$, $y'(0) = 1$, and also for more general boundary conditions. The function ϕ was assumed to be continuous and non-negative on $0 \leq t \leq 1$, $-\infty < y < \infty$, $\alpha \leq \lambda < \infty$, and to satisfy certain conditions on behavior with respect to the variables y and λ, and extensive use was made of functions $\check{\phi}(y,\lambda)$ and $\hat{\phi}(y,\lambda)$, defined, respectively, as the supremum and infimum of $\phi(t,y,\lambda)$ on $0 \leq t \leq 1$. The functions $\check{F}(y)$ and $\hat{F}(y)$ used in this paper will be seen to have several properties in common with the functions $\check{\phi}$ and $\hat{\phi}$ of Moroney.

Section 3 below is devoted to results on existence and oscillatory behavior of solutions of (1.1) which satisfy initial conditions $y(a) = 0$, $y'(a) = \mu$. The principal theorems of this paper appear in Section 4, where it is shown that for each positive integer n the differential equation (1.1) has a solution which satisfies $y(a) = 0 = y(b)$ and vanishes exactly n - 1 times in (a,b). A similar result is given for the boundary conditions $y(a) = 0 = y'(b)$.

In Section 5, a variational problem with an inequality side condition is employed to give an alternate treatment of some of Nehari's results concerning the boundary problem (1.2), under the additional assumption that $F(t,s)$ has a continuous partial derivative $F_s(t,s)$. Section 6 gives a discussion of a particular class of differential equations of the form (1.1), which in turn includes the equation treated in Section 5. The hypotheses used in Section 6 are similar in some respects to the conditions assumed by Moroney [4], and the results of the section serve to clarify the relationship between the results of the present paper and the corresponding results of Nehari and Moroney.

2. Formulation of the problem. If $F = F(t,y,r)$ is a real-valued continuous function defined on

$$D: \quad a \le t \le b, \ -\infty < y < \infty, \ -\infty < r < \infty,$$

a real-valued function $y = y(t)$ defined on an interval $[\alpha,\beta] \subsetneq [a,b]$ will be called a solution of the differential equation

$$y'' + yF(t,y,y') = 0 \tag{2.1}$$

on $[\alpha,\beta]$ if $y \in C''[\alpha,\beta]$ and $y(t)$ satisfies (2.1) for all $t \in [\alpha,\beta]$.

We will be concerned with existence and oscillation of solutions of the initial value problem

$$I_\mu: \quad \begin{array}{ll} (2.1) & y'' + yF(t,y,y') = 0, \\ (2.2) & y(a) = 0, \ y'(a) = \mu, \end{array}$$

and of the boundary problems

$$B: \quad \begin{array}{ll} (2.1) & y'' + yF(t,y,y') = 0, \\ (2.3) & y(a) = 0 = y(b); \end{array}$$

$$B_0: \quad \begin{array}{ll} (2.1) & y'' + yF(t,y,y') = 0, \\ (2.4) & y(a) = 0 = y'(b). \end{array}$$

The symbol D will always be used below to denote the set $[a,b] \times R \times R$, as at the beginning of this section. It will be assumed throughout Sections 2, 3, and 4 that $F(t,y,r)$ is continuous on D and satisfies the following conditions:

CONDITION (I). $F = F(t,y,r)$ is defined on D, and there exist continuous functions $\hat{F} = \hat{F}(y)$ and $\check{F} = \check{F}(y)$, defined on $(-\infty,\infty)$, with the following properties:

(2.5a) $0 \leq \hat{F}(y) \leq F(t,y,r) \leq \check{F}(y)$ for all $(t,y,r) \in D$,

(2.5b) $\hat{F}(y) = 0$ if and only if $y = 0$, $\check{F}(y) = 0$ if and only if $y = 0$,

(2.5c) $y_2 > y_1 \geq 0$ or $y_2 < y_1 \leq 0$ implies $\hat{F}(y_2) \geq \hat{F}(y_1)$ and $\check{F}(y_2) \geq \check{F}(y_1)$,

(2.5d) $\lim_{y \to +\infty} \hat{F}(y) = \lim_{y \to -\infty} \hat{F}(y) = +\infty$.

CONDITION (II). $F(t,y,r)$ is locally Lipschitzian in (y,r) on D; that is, for each point $(\tau,\eta,\rho) \in D$ there is a neighborhood V: $|t-\tau| <\ $, $|y-\eta|<\delta$, $|r-\rho|<\gamma$, and a $k > 0$ such that if (t,y_1,r_1) and (t,y_2,r_2) are in $V \cap D$ then

$$|F(t,y_1,r_1)-F(t,y_2,r_2)| \leq k(|y_1-y_2| + |r_1-r_2|).$$

The functions $\hat{\phi}(y,\lambda)$ and $\check{\phi}(y,\lambda)$ which are used by Moroney [4] in discussing the equation $y'' + y\phi(t,y,\lambda) = 0$, as noted above in Section 1, also have the properties (2.5a, b, c) for fixed λ. The conditions (2.5) arise quite naturally in attempting to generalize the problem (1.2) of

Nehari. For example, if n is a positive integer, the differential equation

$$(2.6) \qquad y'' + p(t,y')\, y^{2n+1} = 0$$

satisfies the hypotheses required by Nehari in [5] and [6] if p(t,r) is a positive continuous function which is independent of r. On the other hand, (2.6) satisfies the conditions of the present paper if p(t,r) is a continuous function having uniform positive lower and upper bounds on $[a,b] \times R$ and satisfying a local Lipschitz condition with respect to r. This can be seen by noting first that (2.6) is of the form (2.1) with $F(t,y,r) = p(t,r)y^{2n}$ for all $(t,y,r) \in D$; if $\check{p}$ and $\hat{p}$ are defined as the respective supremum and infimum of p(t,r) on $[a,b] \times R$ then $\check{F}(y) = \check{p}\, y^{2n}$, $\hat{F}(y) = \hat{p}\, y^{2n}$ satisfy properties (2.5), and F(t,y,r) satisfies conditions (I) and (II).

3. <u>Properties of solutions of the initial value problem I_μ</u>. Let F = F(t,y,r) be continuous on D and satisfy the local Lipschitz condition (II). It follows from standard existence and uniqueness theorems, (e.g. [2; pp. 156-157]), that for each real number μ there is a largest half-open interval $[a,\beta_\mu) \subset [a,b]$ such that the problem I_μ has a solution defined on $[a,\beta_\mu)$. Furthermore, it follows that:

(i) $y_\mu(t)$ is the unique solution of I_μ on any subinterval $[a,\beta)$ of $[a,\beta_\mu)$, and

(ii) the functions $y(t,\mu) = y_\mu(t)$ and $y'(t,\mu) = y'_\mu(t)$ are continuous in (t,μ) on $[a,\beta_\mu)$, $-\infty < \mu < \infty$.

<u>The symbol $[a,\beta_\mu)$ will always denote the largest half-open subinterval of $[a,b]$ on which I_μ has a solution. In particular, if I_μ</u>

has a solution $y_\mu(t)$ *on the whole interval* $[a,b]$, *then* $\beta_\mu = b$ *and* $y_\mu(t)$ *is the unique solution of* I_μ *on* $[a,b]$.

It may be noted that the only use made of the local Lipschitz condition (II) in Sections 3 and 4 is to insure the uniqueness of solutions of (2.1), so that a uniqueness assumption could be substituted for condition (II). The local Lipschitz condition is retained, however, for simplicity of statement in referring to classical results.

THEOREM 3.1. *Let* $F(t,y,r)$ *be continuous on* D *and satisfy conditions* (I) *and* (II), *and let* $m > 0$ *satisfy the condition*

$$\check{F}(m(b-a)) = \frac{\pi^2}{4(b-a)^2}\,. \tag{3.1}$$

Then for all $\mu \in (0,m)$ *the solution* $y_\mu(t)$ *of the initial value problem* I_μ *exists on* $[a,b]$, *and* $y_\mu'(t) > 0$ *on* (a,b); *in particular*, $y_\mu(t) \neq 0$ *on* $(a,b]$ *for such values* μ.

For arbitrary $\mu > 0$ let $y_\mu(t)$ be the unique solution of I_μ on $[a,\beta_\mu)$. The continuity of y_μ' on $[a,\beta_\mu)$ insures the existence of an interval $[a,\tau)$ throughout which $y_\mu'(t) > 0$. Define $T = T_\mu$ by

$$T = \sup\ \{\tau \mid \tau \in [a,\beta_\mu),\ y_\mu'(t) > 0 \text{ for all } t \in [a,\tau)\}\,. \tag{3.2}$$

We note that $a < T \leq \beta_\mu$ and that T is a well-defined function of μ. Clearly $y_\mu(t) > 0$ on $a < t < T$, so (2.1), with (2.5a), implies $y_\mu''(t) < 0$ on $a < t < T$; hence $y_\mu'(t)$ is decreasing on $[a,T)$ and satisfies $0 < y_\mu'(t) < \mu$ for all $t \in (a,T)$. Therefore, $L \equiv \lim_{t \to T-} y_\mu'(t)$ exists and satisfies $0 \leq L < \mu$. Also, since $y_\mu'(t) > 0$ on $[a,T)$, $y_\mu(t)$ is increasing on $[a,T)$, and

$$0 < y_\mu(t) = \int_a^t y_\mu'(s)ds < \mu(t-a), \quad a < t < T,$$

so that $\eta \equiv \lim_{t\to T-} y_\mu(t)$ exists and satisfies

$$(3.3) \qquad 0 < \eta \leq \mu(T-a).$$

If $T = \beta_\mu = b$, and $y_\mu(T)$ and $y_\mu'(T)$ are not defined, we may define

$$(3.4) \qquad y_\mu(T) = \eta = \lim_{t\to T-} y_\mu(t), \quad y_\mu'(T) = \lim_{t\to T-} y_\mu'(t).$$

The function $y_\mu(t)$ is then a solution of (2.1) on the closed interval $[a,b]$, with $y_\mu'(t) > 0$ on $a < t < b$ and $y_\mu(t) > 0$ on $a < t \leq b$, so that $y_\mu(t)$ is the desired solution of I_μ on $[a,b]$. Thus, to complete the proof of the theorem, it suffices to show that $T = \beta_\mu = b$ holds for all $\mu \in (0,m)$, where m satisfies (3.1). That such an m exists follows from the properties of the function $\check{F}$.

In the remainder of the proof, the subscript μ on y_μ will be omitted, so that $y_\mu(t)$ is denoted simply by $y(t)$. For $\mu > 0$, we note first that if $T < \beta_\mu$ then $y'(T) = 0$, for otherwise $y'(T) > 0$, and the continuity of y' then implies that $y'(t) > 0$ in some interval $[a,T+\varepsilon)$, $\varepsilon > 0$, which contradicts the definition of T.

Now assume that $T < \beta_\mu$, so that $y'(T) = 0$. If both sides of (2.1) are multiplied by $2y'$, and use is made of the condition $y'(T) = 0$, it follows that

$$(3.5) \qquad y'^2(t) = \int_t^T F[s,y(s), y'(s)]2y(s)y'(s)ds, \quad a \leq t < \beta_\mu.$$

The function $z = y(s)$, $a \leq s \leq T$, is strictly increasing on $[a,T]$ and therefore has an inverse $s = \psi(z)$, $0 \leq z \leq \eta$, where $\eta = y(T)$.

Substituting in (3.5), one obtains

(3.6) $$y'^2(t) = \int_{y(t)}^{\eta} F[\psi(z),z,1/\psi'(z)]\, 2z\,dz.$$

Using (3.6), we write

$$T-a = \int_a^T dt = \int_a^T \frac{y'(t)dt}{\sqrt{y'^2(t)}} = \int_a^T \left(\int_{y(t)}^{\eta} F[\psi(z),z,1/\psi'(z)]\, 2z\,dz\right)^{-1/2} y'(t)dt.$$

Since $y(a) = 0$ and $y(T) = \eta$, this implies

(3.7) $$T-a = \int_0^{\eta}\left(\int_y^{\eta} F[\psi(z),z,1/\psi'(z)]\, 2z\,dz\right)^{-1/2} dy,$$

and (2.5a) and (3.7) imply

$$T-a \geq \int_0^{\eta}\left(\int_y^{\eta} \check{F}(z)2z\,dz\right)^{-1/2} dy.$$

Substitution of $y = \eta v$ gives

$$T-a \geq \int_0^1 \left(\int_{\eta v}^{\eta} \check{F}(z)2z\,dz\right)^{-1/2} \eta\,dv$$

and substituting $z = \eta s$ then yields

$$T-a \geq \int_0^1 \left(\eta^2 \int_v^1 \check{F}(\eta s)2s\,ds\right)^{-1/2} \eta\,dv = \int_0^1 \left(\int_v^1 \check{F}(\eta s)2s\,ds\right)^{-1/2} dv.$$

By (2.5c), $\check{F}(\eta s) \leq \check{F}(\eta)$ for all $s \in [v,1]$, so

$$T-a \geq \int_0^1 [\check{F}(\eta) \int_v^1 2s\,ds]^{-1/2} dv = [\check{F}(\eta)]^{-1/2} \int_0^1 (1-v^2)^{-1/2} dv,$$

and therefore

$$\text{(3.8)} \qquad T - a \geq \frac{\pi}{2}(\check{F}(\eta))^{-1/2}.$$

From (3.3), we have the inequality

$$0 < \eta \leq \mu(T-a) \leq \mu(b-a),$$

which, with (2.5c) and (3.8), implies

$$T - a \geq \frac{\pi}{2}(\check{F}[\mu(b-a)])^{-1/2}.$$

For m as in the statement of the theorem, it follows that if $0 < \mu < m$ then $T - a \geq b - a$. Since $\beta_\mu \leq b$, this contradicts the assumption made above that $T < \beta_\mu$, and hence $T = \beta_\mu \leq b$ holds if $0 < \mu < m$. The case $T = \beta_\mu < b$ is impossible, for y(T) and y'(T) could then again be defined by (3.4), and y(t) could be continued as a solution of (2.1) to an interval $[a, \beta_\mu + \delta)$, $\delta > 0$, which contradicts the definition of β_μ. Therefore $T = \beta_\mu = b$ for all μ such that $0 < \mu < m$, which, as noted above, completes the proof of the theorem.

We now prove two lemmas which will be needed for Theorem 3.2 and for the results in Section 4. In both these lemmas, the interval $[a, \beta_\mu)$ is again defined as in the remarks preceding Theorem 3.1.

LEMMA 3.1. <u>Let</u> F(t,y,r) <u>be continuous on</u> D <u>and satisfy conditions</u> (I) <u>and</u> (II), <u>and for</u> $\mu \in R$ <u>let</u> $y_\mu(t)$ <u>be the unique solution of the initial value problem</u> I_μ <u>defined on</u> $[a, \beta_\mu)$. <u>If</u> $y_\mu(t)$ <u>has only finitely many zeros on</u> $[a, \beta_\mu)$, <u>then</u> $\beta_\mu = b$, <u>and</u> $y_\mu(t)$ <u>can be extended to the closed interval</u> $[a,b]$ <u>as a solution of</u> I_μ.

We again omit the subscript on y_μ and write $y(t) = y_\mu(t)$. Given $y(t) = y_\mu(t)$ as stated, let τ be the largest zero of $y(t)$ on $[a, \beta_\mu)$. Attention will be restricted to the case in which $y(t) > 0$ on (τ, β_μ); the proof for the case $y(t) < 0$ on (τ, β_μ) is similar. From (2.1) and (2.5a) it follows that $y''(t) < 0$ on (τ, β_μ), so $y'(t)$ is decreasing on $[\tau, \beta_\mu)$. Therefore $L \equiv \lim_{t \to \beta_\mu -} y'(t)$ exists and satisfies $-\infty \leq L < y'(\tau)$. Also $y(t)$ is bounded above on (τ, β_μ), so that there exists a finite constant $M > 0$ such that $y(t) \leq M$ on (τ, β_μ).

To see that $L \neq -\infty$, note that if t_1 and t_2 are in (τ, β_μ) then

$$y'(t_2) - y'(t_1) = \int_{t_1}^{t_2} y''(t)dt = \int_{t_1}^{t_2} -y(t)F[t, y(t), y'(t)]dt,$$

and hence

$$(3.9) \qquad |y'(t_2) - y'(t_1)| \leq \left| \int_{t_1}^{t_2} y(t)\check{F}(y(t))dt \right| \leq M\check{F}(M) \cdot |t_2 - t_1| .$$

Thus $y'(t)$ is Lipschitzian on (τ, β_μ) and hence bounded, so L is finite. From (3.9) it follows also that $\lim_{t \to \beta_\mu -} y(t)$ exists. If $y(t)$ and $y'(t)$ are defined at $t = \beta_\mu$ by

$$(3.10) \qquad y(\beta_\mu) = \lim_{t \to \beta_\mu -} y(t), \quad y'(\beta_\mu) = L,$$

then $y(t)$ is a solution of (2.1) on $[a, \beta_\mu]$, and if $\beta_\mu < b$ then $y(t)$ can be extended as a solution of (2.1) to an interval $[a, \beta_\mu + \delta)$, $\delta > 0$, which contradicts the definition of β_μ. Therefore $\beta_\mu = b$, and $y(t)$ as extended by (3.10) is a solution of I_μ on $[a,b]$, which completes the proof.

LEMMA 3.2. *Let* $F(t,y,r)$ *be continuous on* D *and satisfy conditions* (I) *and* (II). *For* $\mu_0 \in R$, *let* $y_0(t)$ *be the unique solution of the initial value problem* I_{μ_0} *on some closed interval* $[a,c] \subseteq [a,b]$. *Then there exists an* $m > 0$ *such that for all* $\mu \in (\mu_0 - m, \mu_0 + m)$ *the solution* $y_\mu(t)$ *of* I_μ *is defined on the interval* $[a,c]$. *If* $y_0(t)$ *has precisely* n *zeros on the open interval* (a,c), *then:* (i) *if* $y_0(c) \neq 0$, *there exists a* $\delta \in (0,m)$ *such that for all* $\mu \in (\mu_0 - \delta, \mu_0 + \delta)$ *the solution* $y_\mu(t)$ *of* I_μ *defined on* $[a,c]$ *vanishes exactly* n *times in* (a,c), *and* $y_\mu(c) \neq 0$; (ii) *if* $y_0(c) = 0$, *there exists a* $\delta^* \in (0,m)$ *such that for all* $\mu \in (\mu_0 - \delta^*, \mu_0 + \delta^*)$ *the solution* $y_\mu(t)$ *of* I_μ *defined on* $[a,c]$ *vanishes either* n *or* $n + 1$ *times in* $(a,c]$.

The first conclusion of the lemma follows from standard embedding theorems for solutions of differential equations, (e.g. [2; pp. 163-164]).

To proceed to the proof of the second part of the lemma, let $t_0, t_1, \ldots, t_n \in [a,c)$ be such that

(i) $\quad a = t_0 < t_1 < \ldots < t_n < c,$

(ii) $\quad y_0(t_j) = 0, \quad (j = 1,\ldots,n),$

(iii) $\quad y_0(t) \neq 0$ *for* $t \in (t_{j-1}, t_j), \quad (j = 1,\ldots,n)$, *and for* $t \in (t_n, c)$.

Then $y_0'(t_j) \neq 0$, $(j = 0,1,\ldots,n)$, because the only solution of (2.1) satisfying $y(\tau) = 0$, $y'(\tau) = 0$ for some $\tau \in [a,b]$ is $y(t) \equiv 0$, $a \leq t \leq b$. Thus there exists a $\gamma > 0$ such that for each of the intervals

$$V_0 = [a, a+\gamma), \quad V_j = (t_j - \gamma, t_j + \gamma), \quad (j = 1,\ldots,n),$$

$y_0'(t)$ is of constant sign in V_j, $V_i \cap V_j = \emptyset$ for $i \neq j$, and $V_j \subset [a,c)$,

$(j = 0,1,\ldots,n)$. For $m > 0$ as in the first conclusion of the lemma, the continuity of $y_\mu(t) = y(t,\mu)$ and $y'_\mu(t) = y'(t,\mu)$ in (t,μ), as noted in the remarks preceding Theorem 3.1, insures that $y(t,\mu)$ and $y'(t,\mu)$ are uniformly continuous on any compact set S: $a \leq t \leq c$, $|\mu-\mu_0| \leq m_1$, where m_1 is an arbitrary fixed number satisfying $0 < m_1 < m$. Thus for arbitrary $\varepsilon > 0$ there exists a $\delta \in (0,m_1)$ such that if $|\mu-\mu_0| < \delta$ then $|y(t,\mu) - y_0(t)| < \varepsilon$ and $|y'(t,\mu) - y'_0(t)| < \varepsilon$ for all $t \in [a,c]$.

It follows that there exists a $\delta_1 \in (0,m_1)$ such that if $|\mu-\mu_0| < \delta_1$ then $y'_\mu(t)$ does not change sign on V_j, $(j = 0,1,\ldots,n)$, so $y_\mu(t)$ has at most one zero on each V_j, $(j = 0,1,\ldots,n)$, and, in particular, the only zero of $y_\mu(t)$ in V_0 is $t = a$. Now if $y_0(c) \neq 0$, the only zeros of $y_0(t)$ on $(a,c]$ occur in the intervals V_j, $(j = 1,\ldots,n)$, and $y_0(t)$ changes sign in each V_j, $(j = 1,\ldots,n)$; consequently there exists a $\delta_2 \in (0,m_1)$ such that if $|\mu-\mu_0| < \delta_2$ then $y_\mu(t)$ vanishes in V_j, $(j = 1,\ldots,n)$, and $y_\mu(t) \neq 0$ for $t \in [a,c] - \bigcup_{j=0}^{n} V_j$. Therefore, if $\delta = \min\{\delta_1,\delta_2\}$ then $|\mu-\mu_0| < \delta$ implies that $y_\mu(t)$ exists on $[a,c]$ and has exactly n zeros in (a,c), and $y_\mu(c) \neq 0$, which completes the proof for the case $y_0(c) \neq 0$.

If $y_0(c) = 0$, then for $c_1 \in (t_n,c)$ the preceding argument assures the existence of a $\delta_3 \in (0,m_1)$ such that if $|\mu-\mu_0| < \delta_3$ then $y_\mu(t)$ is defined on $[a,c]$ and has exactly n zeros in (a,c_1), and $y_\mu(c_1) \neq 0$. A similar type of argument shows that there exists a $\delta_4 \in (0,m_1)$ such that if $|\mu-\mu_0| < \delta_4$ then $y_\mu(t)$ has at most one zero in $(c_1,c]$. Thus if $\delta^* = \min\{\delta_3, \delta_4\}$, then $|\mu-\mu_0| < \delta^*$ implies that $y_\mu(t)$ exists on $[a,c]$ and has precisely n or n + 1 zeros in $(a,c]$, which completes the proof.

Again, let $F(t,y,r)$ be continuous on D and satisfy conditions (I) and (II), and for each $\mu \in R$ let $y_\mu(t)$ be the solution of the initial value problem I_μ on $[a,\beta_\mu)$. It will be proved inductively that <u>for every positive integer</u> n <u>there exists an</u> $M_n > 0$ <u>such that if</u> $\mu > M_n$ <u>then</u> $y_\mu(t)$ <u>has at least</u> n <u>zeros on</u> (a,β_μ). The theorem will be stated formally following the discussion for the case $n = 1$, because certain expressions in the statement of the theorem arise naturally in this discussion.

As before, the subscript μ on y_μ is omitted. Assume that $\mu > 0$, and, as in the proof of Theorem 3.1, define $T = T_\mu$ by

$$T = \sup \{ \tau \mid \tau \in [a,\beta_\mu),\ y'(t) > 0 \ \underline{\text{for all}}\ t \in [a,\tau) \} .$$

Then T is a well-defined function of μ for $\mu > 0$ and satisfies $a < T \le \beta_\mu$. As shown in the proof of Theorem 3.1, either $T < \beta_\mu \le b$ so that $y(T)$ and $y'(T)$ exist, or $T = b$ and $y(t)$ can be extended to $[a,b]$ as a solution of (2.1) by use of equations (3.4); also, if $T < \beta_\mu$, then $y'(T) = 0$.

Multiplying both sides of (2.1) by $2y'$ and integrating, we obtain

$$(3.11)\quad y'^2(t) - y'^2(a) + \int_a^t F[s,y(s),y'(s)]2y(s)y'(s)ds = 0, \quad a \le t \le b.$$

As noted following equation (3.5), one may substitute $z = y(s)$, $s = \psi(z)$, $0 \le z \le y(T)$, to obtain, with $y'(a) = \mu$, the equation

$$(3.12)\quad y'^2(t) = \mu^2 - \int_0^{y(t)} F[\psi(z),\ z,\ 1/\psi'(z)]2zdz, \quad a \le t \le T.$$

In particular, with $\eta \equiv y(T)$,

$$y'^2(T) = \mu^2 - \int_0^{\eta} F[\psi(z), z, 1/\psi'(z)]2zdz, \tag{3.12'}$$

which implies that

$$\mu^2 \geq \int_0^{\eta} F[\psi(z), z, 1/\psi'(z)]2zdz, \tag{3.13}$$

with equality holding if and only if $y'(T) = 0$. From (3.12) and (3.13), it follows that

$$y'^2(t) \geq \int_{y(t)}^{\eta} F[\psi(z), z, 1/\psi'(z)]2zdz,$$

and the use of property (2.5a) then shows that

$$y'^2(t) \geq \int_{y(t)}^{\eta} \hat{F}(z)2zdz, \quad a \leq t \leq T. \tag{3.14}$$

Now

$$T - a = \int_a^T dt = \int_a^T \frac{y'(t)dt}{\sqrt{y'^2(t)}},$$

and, in view of (3.14), it follows that

$$T - a \leq \int_a^T \left(\int_{y(t)}^{\eta} \hat{F}(z)2zdz\right)^{-1/2} y'(t)dt.$$

By a change of variable of integration, with $y(a) = 0$, $y(T) = \eta$, this takes the form

$$T - a \leq \int_0^{\eta} \left(\int_y^{\eta} \hat{F}(z)2zdz\right)^{-1/2} dy, \tag{3.15}$$

and the substitution $y = \eta v$, followed by the substitution $z = \eta s$, leads to

$$T - a \le \int_0^1 \left(\int_v^1 \hat{F}(\eta s) 2s\,ds\right)^{-1/2} dv$$

$$= \int_0^{1/2} \left(\int_v^1 \hat{F}(\eta s) 2s\,ds\right)^{-1/2} dv + \int_{1/2}^1 \left(\int_v^1 \hat{F}(\eta s) 2s\,ds\right)^{-1/2} dv$$

$$\le \int_0^{1/2} \left(\int_{1/2}^1 \hat{F}(\eta s) 2s\,ds\right)^{-1/2} dv + \int_{1/2}^1 \left(\int_v^1 \hat{F}(\tfrac{1}{2}\eta) 2s\,ds\right)^{-1/2} dv.$$

Consequently,

$$T - a \le (\hat{F}(\tfrac{1}{2}\eta))^{-1/2} \left[\int_0^{1/2} (1-(\tfrac{1}{2})^2)^{-1/2} dv + \int_{1/2}^1 (1-v^2)^{-1/2} dv\right]$$

$$= (\hat{F}(\tfrac{1}{2}\eta))^{-1/2} \left[\frac{1}{\sqrt{3}} + \sin^{-1} 1 - \sin^{-1} \frac{1}{2}\right] = (\hat{F}(\tfrac{1}{2}\eta))^{-1/2} \left(\frac{1}{\sqrt{3}} + \frac{\pi}{3}\right),$$

and thus

(3.16) $$T - a \le K(\hat{F}(\tfrac{1}{2}\eta))^{-1/2}, \qquad K = \tfrac{1}{3}(\pi + \sqrt{3}) .$$

In order to make use of this inequality, we need to show that $\eta \to \infty$ as $\mu \to \infty$. From (3.12') it follows that

(3.17) $$\mu^2 = y'^2(T) + \int_0^\eta F[\psi(z), z, 1/\psi'(z)] 2z\,dz,$$

and hence, by (2.5a),

$$\mu^2 \leq y'^2(T) + \int_0^{\eta} \check{F}(z)2z\,dz.$$

By (2.5c), $\check{F}(z)$ is non-decreasing on $[0, \eta]$, and consequently

$$\mu^2 \leq y'^2(T) + \int_0^{\eta} \check{F}(\eta)2z\,dz,$$

so that

$$\mu^2 \leq y'^2(T) + \eta^2 \check{F}(\eta). \tag{3.18}$$

As noted above, $y'(T) = 0$ if $a < T < b$, while if $T = b$, the fact that $y'(t)$ is decreasing on $[a,T]$ implies that $y'(T) < \frac{\eta}{b-a}$. Thus, in either case,

$$0 \leq y'(T) < \frac{\eta}{b-a}.$$

It then follows from (3.18) that

$$\mu^2 \leq \eta^2 \; [(b-a)^{-2} + \check{F}(\eta)],$$

and therefore $\eta \to \infty$ as $\mu \to \infty$. Now (3.16) and (2.5d) imply that $(T - a) \to 0$ as $\eta \to \infty$, and consequently that $(T - a) \to 0$ as $\mu \to \infty$. In particular, there exists an $M > 0$ such that $\mu > M$ implies $T < b$.

Assume that $\mu > M$ so that $T < b$. Then, as noted previously, $y'(T) = 0$ and $T < \beta_\mu$. If t_1 is defined as

$$t_1 = \sup \{\tau \mid \tau \in (T, \beta_\mu),\ y(t) > 0 \ \underline{\text{for}}\ \underline{\text{all}}\ t \in (T, \tau)\},$$

then, since $y(T) > 0$, it follows that $T < t_1 \leq \beta_\mu$. If $t_1 < \beta_\mu$, then $y(t)$ and $y'(t)$ are defined at $t = t_1$; if $t_1 = \beta_\mu$, then $y(t)$ has no zeros on

(a,β_μ), so that Lemma 3.1 implies that $\beta_\mu = b$ and $y(t)$ can be continued to the closed interval $[a,b]$ as a solution of (2.1), and $y(t_1)$ and $y'(t_1)$ are defined by this extension. Thus, in either case, $y(t_1)$ and $y'(t_1)$ exist; in particular, if $t_1 < \beta_\mu$, it follows from the definition of t_1 that $y(t_1) = 0$.

Multiplying both sides of (2.1) by $2y'$ and integrating, and using the fact that $y'(T) = 0$, we find that

$$(3.19) \qquad y'^2(t) + \int_T^t F[s,y(s),y'(s)]\, 2y(s)y'(s)ds = 0, \; T \le t \le t_1.$$

Since $y(t) > 0$ on (T,t_1), (2.1) implies that $y''(t) < 0$ on (T,t_1), so that $y'(t)$ is decreasing on $[T,t_1]$, and consequently, $y'(t) < 0$ on $(T,t_1]$. Thus $y(s)$ is strictly decreasing on $[T,t_1)$, so the function $z = y(s)$, $T \le s \le t_1$, has an inverse $s = \phi(z)$, $y(t_1) \le z \le \eta = y(T)$. Substitution in (3.19), followed by the use of (2.5a), then yields

$$(3.20) \qquad y'^2(t) \ge \int_{y(t)}^{\eta} \hat{F}(z)2zdz.$$

Now

$$t_1 - T = \int_T^{t_1} dt = \int_T^{t_1} \frac{-y'(t)dt}{\sqrt{y'^2(t)}} ,$$

so, from (3.20),

$$t_1 - T \le \int_T^{t_1} -\left(\int_{y(t)}^{\eta} \hat{F}(z)2zdz\right)^{-1/2} y'(t)dt .$$

Changing the variable of integration and using $y(T) = \eta$ gives

$$(3.21) \qquad t_1 - T \le \int_{y(t_1)}^{\eta} \left(\int_y^{\eta} \hat{F}(z)2z\,dz\right)^{-1/2} dy \le \int_0^{\eta} \left(\int_y^{\eta} \hat{F}(z)2z\,dz\right)^{-1/2} dy.$$

The right-hand side of inequality (3.21) is precisely the same as that of inequality (3.15), and hence the same steps that led to (3.16) now lead to the inequality

$$(3.22) \qquad t_1 - T \le K\,(\hat{F}(\tfrac{1}{2}\eta))^{-1/2}, \qquad K = \tfrac{1}{3}(\pi + \sqrt{3}).$$

Together, (3.16) and (3.22) imply that

$$(3.23) \qquad t_1 - a \le K_1\,(\hat{F}(\tfrac{1}{2}\eta))^{-1/2}, \qquad K_1 = 2K.$$

It follows from (2.5d) and (3.23) that $(t_1 - a) \to 0$ as $\eta \to \infty$, and hence that $(t_1 - a) \to 0$ as $\mu \to \infty$, since $\eta \to \infty$ as $\mu \to \infty$.

Thus, for sufficiently large positive μ, the value t_1 exists and satisfies $t_1 < b$. Because $t_1 < b$ implies that $y(t_1) = 0$, there exists an $M_1 > 0$ such that if $\mu > M_1$ then $y(t) = y_\mu(t)$ has at least one zero on (a,b).

To obtain an inequality for $t_1 - a$ directly in terms of μ, we recall that if $T < b$ then $y'(T) = 0$. With $y'(T) = 0$, (3.18) becomes

$$(3.24) \qquad \mu^2 \le \eta^2\,\check{F}(\eta).$$

Now $\check{F}(\eta)$ increases from 0 to ∞ as η increases from 0 to ∞, so for every $\mu \ge 0$ there exists a unique $U = U(\mu)$ such that

$$(3.25) \qquad \mu^2 = U^2\,\check{F}(U).$$

$U(\mu)$ is an increasing function of μ on $[0,\infty)$ and satisfies $U(0) = 0$,

$U(\mu) \to \infty$ as $\mu \to \infty$. From (3.24) and (3.25) and the monotoneity of $\check{F}$, it follows that

$$(3.26) \qquad \eta = \eta(\mu) \geq U(\mu),$$

which, with (3.23) and the monotoneity of $\hat{F}$, implies that

$$(3.27) \qquad t_1 - a \leq K_1 \ (\hat{F}[\tfrac{1}{2}U(\mu)])^{-1/2},$$

which is the desired inequality.

For $\mu > M_1$, so that $t_1 \in (a,b)$ and $y(t_1) = 0$, we also obtain an inequality for $|y'(t_1)|$ in terms of μ; indeed, if $y(t_1) = 0$, inequality (3.20) implies that

$$(3.28) \qquad y'^2(t_1) \geq \int_0^{\eta} \hat{F}(z)2zdz \equiv V^2(\eta).$$

The function $V(\eta) = \sqrt{V^2(\eta)}$ thus defined in an increasing function of η on $[0,\infty)$, with $V(0) = 0$, $V(\eta) \to \infty$ as $\eta \to \infty$. Consequently, by (3.28) and (3.26), we have

$$(3.29) \qquad |y'(t_1)| \geq V(\eta) \geq V(U(\mu)) \equiv W(\mu).$$

The function $W(\mu)$ thus defined is also increasing on $[0, \infty)$ and satisfies $W(0) = 0$, $W(\mu) \to \infty$ as $\mu \to \infty$; hence (3.29) implies $|y'(t_1)| \to \infty$ as $\mu \to \infty$.

It has been assumed above that $y'(a) = \mu > 0$. If $\mu < 0$, a similar discussion holds. In this case the results corresponding to (3.27) and (3.29) are

$$(3.27') \qquad t_1 - a \leq K_1 \ (\hat{F}[\tfrac{1}{2}U(|\mu|)])^{-1/2},$$

(3.29') $$|y'(t_1)| \geq W(|\mu|).$$

With U, W, and K_1 as above, we have the following theorem, which has just been proved for the case n = 1:

THEOREM 3.2. *Let* $F(t,y,r)$ *be continuous on* D *and satisfy conditions* (I) *and* (II). *For* $\mu \in R$, *let* $y_\mu(t)$ *be the unique solution of the initial value problem* I_μ *defined on* $[a,\beta_\mu)$. *Then, for each positive integer* n, *there exists an* $M_n > 0$ *such that if* $\mu > M_n$ *then* $y_\mu(t)$ *has at least* n *distinct zeros on* (a,β_μ).

For $\mu > M_n$, *if* $t_0, t_1, \ldots, t_n$ *are the first* n + 1 *zeros of* $y_\mu(t)$ *on* $[a,\beta_\mu)$, *arranged so that* $a = t_0 < t_1 < \ldots < t_n$, *and if* $d_j = y_\mu'(t_j)$, $(j = 1,2,\ldots,n)$, *then*

(i) $$|d_j| \geq W^j(\mu), \quad (j = 1,\ldots,n), \text{ and}$$

(ii) $$t_j - t_{j-1} \leq K_1\{\hat{F}[\tfrac{1}{2}UW^{j-1}(\mu)]\}^{-1/2}, \ (j = 1,\ldots,n),$$

where the exponent denotes repeated composition of functions and W^0 *denotes the identity function on* $[0,\infty)$.

The theorem has been proved for the case n = 1. Suppose that it holds for n = k, and for $\mu > M_k$, let y(t) be the corresponding solution of I_μ on $[a,\beta_\mu)$. If $d_k = y'(t_k)$, where t_k is the k^{th} zero of y(t) on (a,β_μ), then $d_k \neq 0$, because of the assumption that (i) holds for n = k. We restrict attention to the case in which $d_k > 0$, since the discussion for the case $d_k < 0$ is similar. With $d_k > 0$, define

$$T_{k+1} = \sup\{\tau \mid \tau \in [t_k,\beta_\mu),\ y'(t) > 0 \text{ for all } t \in [t_k,\tau)\}.$$

Then T_{k+1} is a function of μ for $\mu > M_k$, and by arguments similar to those used in the proof of Theorem 3.1, either $T_{k+1} < \beta_\mu \leq b$, so that $y(T_{k+1})$ and $y'(T_{k+1})$ exist and $y'(T_{k+1}) = 0$, or $T_{k+1} = b$ and $y(t)$ can be extended to $[a,b]$ as a solution of (2.1), so that $y(T_{k+1})$ and $y'(T_{k+1})$ can be defined by equations similar to equations (3.4).

Let $\eta_{k+1} = y(T_{k+1})$. The argument used in the case $n = 1$ to derive the inequality (3.16), with a, μ, η, and T replaced by t_k, d_k, η_{k+1}, and T_{k+1}, respectively, now shows that

$$T_{k+1} - t_k \leq K(\hat{F}[\tfrac{1}{2}\eta_{k+1}])^{-1/2}, \quad K = \tfrac{1}{3}(\pi + \sqrt{3}). \tag{3.30}$$

The argument used following (3.16) to show that $\eta \to \infty$ as $\mu \to \infty$ now shows that $\eta_{k+1} \to \infty$ as $d_k \to \infty$. From (i), with $n = k$, and the properties of W, we have $|d_k| \to \infty$ as $\mu \to \infty$, and hence $\eta_{k+1} \to \infty$ as $\mu \to \infty$. Therefore, from (3.30), $T_{k+1} - t_k \to 0$ as $\mu \to \infty$. From (ii), with $n = k$, it follows that $t_k - a \to 0$ as $\mu \to \infty$. Consequently,

$$\lim_{\mu\to\infty}(T_{k+1} - a) = \lim_{\mu\to\infty}[(T_{k+1} - t_k) + (t_k - a)] = 0,$$

so $T_{k+1} < b$ for sufficiently large $\mu > 0$.

As noted above, $y'(T_{k+1}) = 0$ if $T_{k+1} < b$, and for μ large enough so that $T_{k+1} < b$ it follows, as in the proof of the inequality (3.26), that

$$\eta_{k+1} \geq U(d_k), \tag{3.31}$$

where U is the function defined by (3.25). Then (3.30) and (3.31) imply

$$T_{k+1} - t_k \leq K(\hat{F}[\tfrac{1}{2}U(d_k)])^{-1/2}. \tag{3.32}$$

Next, for $T_{k+1} < b$, we define

$$t_{k+1} = \sup \{ \tau \mid \tau \in (T_{k+1}, \beta_\mu),\ y(t) > 0 \ \underline{\text{for}}\ \underline{\text{all}}\ t \in (T, \tau) \} .$$

It follows, as in the proof for the case n = 1, that either $t_{k+1} = \beta_\mu = b$ and y(t) can be extended to [a,b] as a solution of (2.1), or $t_{k+1} < b$, in which case $t_{k+1} < \beta_\mu$ and $y(t_{k+1}) = 0$.

It now follows as in the proof of (3.22) that

$$t_{k+1} - T_{k+1} \leq K\ (\hat{F}[\tfrac{1}{2}\eta_{k+1}])^{-1/2},$$

and hence that

$$t_{k+1} - T_{k+1} \leq K\ (\hat{F}[\tfrac{1}{2}U(d_k)])^{-1/2}. \tag{3.33}$$

From (3.32) and (3.33), it then follows that

$$t_{k+1} - t_k \leq K_1(\hat{F}[\tfrac{1}{2}U(d_k)])^{-1/2}, \qquad K_1 = 2K,$$

and application of (i), with n = k, then gives the inequality

$$t_{k+1} - t_k \leq K_1(\hat{F}[\tfrac{1}{2}UW^k(\mu)])^{-1/2},$$

where W^k denotes repeated composition of the function W defined by (3.29). Thus (ii) holds for n = k + 1.

Since (ii) holds for n = k + 1 and $UW^k(\mu) \to \infty$ as $\mu \to \infty$, there exists an $M_{k+1} > 0$ such that if $\mu > M_{k+1}$ then $t_{k+1} < b$, so that $t_{k+1} < \beta_\mu$ and $y(t_{k+1}) = 0$. Thus $\mu > M_{k+1}$ implies that the solution of I_μ has at least k + 1 zeros in (a, β_μ).

Finally, for $\mu > M_{k+1}$, so that $t_{k+1} < \beta_\mu$ and $y(t_{k+1}) = 0$, it follows as in the proof of (3.29) that

$$|d_{k+1}| \geq W(d_k). \tag{3.34}$$

Then (3.34) and (i), with n = k, and the fact that W is an increasing function, imply that

$$|d_{k+1}| \geq W^{k+1}(\mu),$$

so that (i) holds for n = k + 1. Thus we have shown that if the theorem holds for n = k, and if $d_k > 0$, then the theorem holds for n = k + 1.

Since for $d_k < 0$ the desired result follows by a similar argument, the result of the theorem follows by induction.

4. Existence theorems for the boundary problems B and B_0. The boundary problem B, restated here for reference, is

$$\text{B:} \quad \begin{aligned} &y'' + yF(t,y,y') = 0, \quad a \leq t \leq b, \\ &y(a) = 0 = y(b). \end{aligned}$$

Using the results of Section 3, we now prove the following result:

THEOREM 4.1. Let $F(t,y,r)$ be continuous on D and satisfy conditions (I) and (II). Then there exist values μ_n, (n = 1,2,...), $0 < \mu_1 < \mu_2 < \ldots < \mu_n < \ldots$, such that for each n the solution $y_{\mu_n}(t)$ of the initial value problem I_{μ_n} is defined on [a,b], has exactly n - 1 zeros on (a,b), and is a solution of the boundary problem B.

For each $\mu \in R$, let $y_\mu(t)$ be the unique solution of the initial value problem I_μ defined on the largest possible subinterval of [a,b]. If $y_\mu(t)$ is not defined for all $t \in [a,b]$, its domain is then a half-open interval $[a,\beta_\mu)$, as noted in the remarks preceding Theorem 3.1.

Given a positive integer n, we define

$$\mu_n = \sup \{ m \mid m > 0; \ \underline{\text{if}}\ 0 < \mu < m\ \underline{\text{then}}\ y_\mu(t)\ \underline{\text{is}}\ \underline{\text{defined}}\ \underline{\text{on}}\ [a,b]\ \underline{\text{and}}\ \underline{\text{has}}\ \underline{\text{at}}\ \underline{\text{most}}\ n-1\ \underline{\text{zeros}}\ \underline{\text{on}}\ (a,b) \} .$$

Such values of $m > 0$ exist, by Theorem 3.1, so $\mu_n > 0$, and Theorem 3.2 implies that $\mu_n < M_n$, where M_n is defined as in Theorem 3.2, so $0 < \mu_n < \infty$. Immediately from the definition of μ_n the following condition holds:

(C): <u>If</u> $0 < \mu < \mu_n$, <u>then</u> $y_\mu(t)$ <u>is defined on</u> $[a,b]$ <u>and vanishes at most</u> $n-1$ <u>times on</u> (a,b).

Suppose the solution $y_{\mu_n}(t)$ of I_{μ_n} is not defined on $[a,b]$. By Lemma 3.1, $y_{\mu_n}(t)$ then has infinitely many zeros on its domain $[a,\beta_\mu)$. Therefore we may choose $\omega \in [a,\beta_{\mu_n})$ such that $y_{\mu_n}(\omega) \neq 0$ and $y_{\mu_n}(t)$ vanishes n times in (a,ω). By Lemma 3.2, there exists a $\delta_1 > 0$ such that if $|\mu - \mu_n| < \delta_1$ then $y_\mu(t)$ is defined on $[a,\omega]$ and has n zeros in (a,ω). Thus if $\mu_n - \delta_1 < \mu < \mu_n$, then $y_\mu(t)$ has n zeros in (a,ω), which contradicts condition (C). Therefore $y_\mu(t)$ is defined on $[a,b]$.

Next, suppose that $y_{\mu_n}(t)$ has k zeros on (a,b) for some $k > n-1$. Then Lemma 3.2 implies the existence of a $\delta_2 > 0$ such that if $|\mu - \mu_n| < \delta_2$ then $y_\mu(t)$ has at least k zeros on (a,b). Thus, if $\mu_n - \delta_2 < \mu < \mu_n$, then $y_\mu(t)$ has at least k zeros on (a,b), which contradicts condition (C), since $k > n-1$. Therefore $y_{\mu_n}(t)$ has at most $n-1$ zeros on (a,b).

Next we show that $y_{\mu_n}(b) = 0$. Assume the contrary, and let k be the number of zeros of $y_{\mu_n}(t)$ on (a,b). By the above, $k \leq n - 1$. By Lemma 3.2, there exists a $\delta_3 > 0$ such that if $|\mu - \mu_n| < \delta_3$ then $y_\mu(t)$ is defined on [a,b] and has exactly k zeros in (a,b). For m satisfying $\mu_n < m < \mu_n + \delta_3$ it follows that if $0 < \mu < m$ then $y_\mu(t)$ is defined on [a,b] and has at most n - 1 zeros on (a,b). This contradicts the definition of μ_n, and therefore $y_{\mu_n}(b) = 0$.

Now with $y_{\mu_n}(b) = 0$, suppose that $y_{\mu_n}(t)$ has exactly k zeros in (a,b), where $k < n - 1$. Then Lemma 3.2 assures the existence of a $\delta_4 > 0$ such that if $|\mu - \mu_n| < \delta_4$ then $y_\mu(t)$ is defined on [a,b] and has either k or k + 1 zeros in (a,b). Since $k + 1 \leq n - 1$, this leads, as in the preceding paragraph, to a contradiction of the definition of μ_n. Therefore $y_{\mu_n}(t)$ has at least n - 1 zeros in (a,b).

Combining these results, we conclude that $\mu_n > 0$ and that $y_{\mu_n}(b) = 0$ and $y_{\mu_n}(t)$ has exactly n - 1 zeros in (a,b). Since n was an arbitrary positive integer, these results hold for n = 1,2,... . The fact that $\mu_n \leq \mu_m$ for $n < m$ follows from the definition of μ_n and μ_m, and since $y_{\mu_n}(t)$ has exactly n - 1 zeros on (a,b), it follows that $\mu_n < \mu_m$ for $n < m$, which completes the proof of the theorem.

The boundary problem B_0 is defined by

$$B_0: \quad \begin{aligned} & y'' + yF(t,y,y') = 0, \quad a \leq t \leq b, \\ & y(a) = 0 = y'(b). \end{aligned}$$

THEOREM 4.2. _Let_ F(t,y,r) _be continuous on_ D _and satisfy conditions_ (I) _and_ (II). _Then there exist values_ λ_n, (n = 1,2,...), $0 < \lambda_1 < \ldots < \lambda_n < \ldots$, _such that for each_ n _the solution_ $y_{\lambda_n}(t)$ _of_

<u>the initial value problem</u> I_{λ_n} <u>is defined on</u> $[a,b]$, <u>has exactly</u> $n - 1$ <u>zeros on</u> (a,b), <u>and is a solution of the boundary problem</u> B_0.

For each $\mu \in R$, let $y_\mu(t)$ be defined as in the proof of Theorem 4.1. For each positive integer n, let μ_n be as in Theorem 4.1, so that $y_{\mu_n}(t)$ is a solution of the boundary problem B with precisely $n - 1$ zeros on (a,b), and $y_\mu(t)$ exists on $[a,b]$ for all positive $\mu \leq \mu_n$. Then $y'_{\mu_n}(b) \neq 0$, and we define

$$\lambda_n = \inf \{\sigma \,|\, \sigma < \mu_n,\ y'_\mu(b)y'_{\mu_n}(b) > 0 \ \underline{\text{for}}\ \sigma < \mu \leq \mu_n\}.$$

From the continuity of $y'_\mu(t)$ as a function of (t,μ), $y'_\mu(b)$ is continuous in μ, and consequently $y'_\mu(b)y'_{\mu_n}(b) > 0$ for all μ sufficiently near μ_n, so that $\lambda_n < \mu_n$.

If $y'_{\lambda_n}(b)y'_{\mu_n}(b) > 0$, then, by the continuity of $y'_\mu(b)$ with respect to μ, there exists a $\delta_1 > 0$ such that if $|\mu - \lambda_n| < \delta_1$ then $y'_\mu(b)y'_{\mu_n}(b) > 0$. Thus $y'_\mu(b)y'_{\mu_n}(b) > 0$ for $\lambda_n - \delta_1 < \mu < \mu_n$, which contradicts the definition of λ_n, and therefore $y'_{\lambda_n}(b)y'_{\mu_n}(b) \leq 0$. If $y'_{\lambda_n}(b)y'_{\mu_n}(b) < 0$, then, by the continuity of $y'_\mu(b)$ in μ, there exists a $\delta_2 > 0$ such that $y'_\mu(b)y'_{\mu_n}(b) < 0$ for $|\mu - \lambda_n| < \delta_2$, and, in particular, for $\lambda_n < \mu < \lambda_n + \delta_2$. This contradicts the definition of λ_n, and it follows that $y'_{\lambda_n}(b)y'_{\mu_n}(b) = 0$, so that $y'_{\lambda_n}(b) = 0$.

If k is the number of zeros of $y_{\lambda_n}(t)$ on (a,b), condition (C) of the proof of Theorem 4.1 implies that $k \leq n - 1$. It will be shown that the assumption that $k < n - 1$ leads to a contradiction of the definition of λ_n, so that $k = n - 1$. If $k < n - 1$, then since $y'_{\lambda_n}(b) = 0$, we have $y_{\lambda_n}(b) \neq 0$, and it follows from Lemma 3.2 that $y_\mu(t)$ has exactly k zeros in (a,b) for all μ sufficiently near λ_n. Because $y_{\mu_n}(t)$ vanishes exactly n - 1 times in (a,b), it follows from Lemma 3.2 and condition (C) of the proof of Theorem 4.1 that $y_\mu(t)$ has exactly n - 1 zeros in (a,b) for all μ in some interval $(\mu_n - \varepsilon, \mu_n]$. Then, if γ is defined as

$$\gamma = \inf \{ \sigma \mid \lambda_n \leq \sigma \leq \mu_n,\ y_\mu(t) \text{ \underline{has} \underline{exactly} } n-1 \text{ \underline{zeros} \underline{in} } (a,b) \text{ \underline{for} } \sigma < \mu \leq \mu_n \},$$

it follows from the preceding remarks that $\lambda_n < \gamma < \mu_n$.

Let h be the number of zeros of $y_\gamma(t)$ in (a,b). If $y_\gamma(b) \neq 0$, then Lemma 3.2 implies that $y_\mu(t)$ has exactly h zeros in (a,b), for all μ sufficiently near γ, and it then follows from the definition of γ that h = n - 1. Thus there exists a $\delta_3 > 0$ such that if $\gamma - \delta_3 < \mu < \gamma$, then $y_\mu(t)$ has exactly n - 1 zeros on (a,b), which contradicts the definition of γ; therefore $y_\gamma(b) = 0$.

Since $y_\gamma(b) = 0$, Lemma 3.2 implies that for μ sufficiently near γ, $y_\mu(t)$ has precisely h or h + 1 zeros in (a,b), and it then follows from the definition of γ that either h = n - 1 or h + 1 = n - 1. If h = n - 1, then $y_\mu(t)$ has exactly n - 1 or n zeros in (a,b) for μ sufficiently near γ, and since $y_\mu(t)$ has at most n - 1 zeros in (a,b) for $\mu < \mu_n$, it follows that there exists a $\delta_4 > 0$ such that if $\gamma - \delta_4 < \mu < \gamma$ then $y_\mu(t)$ has exactly n - 1 zeros in (a,b), which contradicts the definition of γ. Therefore, h + 1 = n - 1, so that $y_\gamma(t)$ has exactly n - 2 zeros in (a,b) and satisfies $y_\gamma(b) = 0$, while $y_{\mu_n}(t)$ has exactly n - 1 zeros in (a,b) and satisfies $y_{\mu_n}(b) = 0$. Since all the zeros of $y_{\mu_n}(t)$ and $y_\gamma(t)$ are simple zeros, it follows that $y_\gamma'(b)y_{\mu_n}'(b) < 0$, and since $\gamma > \lambda_n$, this contradicts the definition of

λ_n. Thus the assumption that $k < n - 1$ has led to a contradiction, so $k = n - 1$, and the proof is complete.

It may be noted that, under the hypotheses of Theorems 4.1 and 4.2, there also exist negative values μ_n, $0 > \mu_1 > \dots > \mu_n > \dots$, and λ_n, $0 > \lambda_1 > \dots > \lambda_n > \dots$, such that the conclusions of Theorems 4.1 and 4.2 hold for the corresponding solutions y_{μ_n} and y_{λ_n} of the initial value problems I_{μ_n} and I_{λ_n}, $(n = 1,2,\dots)$. To obtain these results, we first note that, if $F(t,y,r)$ satisfies conditions (I) and (II), then $G(t,y,r) \equiv F(t,-y,-r)$ also satisfies conditions (I) and (II), and, if $y(t)$ is a solution of $y'' + y\,G(t,y,y') = 0$, then $u(t) \equiv -y(t)$ satisfies the equation $u'' + uF(t,u,u') = 0$. The application of Theorems 4.1 and 4.2 to the differential equation $y'' + y\,G(t,y,y') = 0$ then gives the desired conclusions. Similar remarks hold concerning Theorems 3.1 and 3.2.

5. <u>An alternate proof of some results of Nehari</u>. This section is devoted to the boundary problems

$$(5.1)\qquad \begin{aligned} &y'' + yF(t,y^2) = 0,\quad a \leq t \leq b,\\ &y(a) = 0 = y(b), \end{aligned}$$

and

$$(5.2)\qquad \begin{aligned} &y'' + \mu p(t)y + yF(t,y^2) = 0,\quad a \leq t \leq b,\\ &y(a) = 0 = y(b), \end{aligned}$$

where μ is a non-negative parameter and $p(t)$ and $F(t,s)$ satisfy conditions to be stated below.

The main result of this section is Theorem 5.1 below. This result is included in Theorems 2.1 and 7.1 of Nehari [6]. The matter of interest here is the alternate method of proof, which employs a variational problem with an inequality side condition and makes a somewhat more extensive use of variational ideas than does Nehari's argument.

Let $p = p(t)$ be a continuous function on $[a,b]$ satisfying $p(t) > 0$ for all $t \in [a,b]$. Let $F = F(t,s)$ be defined on Δ: $a \leq t \leq b$, $0 \leq s < \infty$, and satisfy the following conditions:

(5.3a) $F(t,s)$ _is continuous on_ Δ,

(5.3b) $F(t,s) > 0$ _for all_ $(t,s) \in \Delta$ _such that_ $s > 0$,

(5.3c) _there exists a_ $\vartheta > 0$ _such that if_ $0 < s_1 < s_2 < \infty$, _then_ $s_1^{-\vartheta} F(t,s_1) \leq s_2^{-\vartheta} F(t,s_2)$ _for all_ $t \in [a,b]$,

(5.3d) _the partial derivative_ $F_s(t,s)$ _exists and is continuous for all_ $(t,s) \in \Delta$.

The existence of $F_s(t,s)$ was not assumed by Nehari, so because of this extra hypothesis, Theorem 5.1 does not include the corresponding results of Nehari.

THEOREM 5.1. _Given_ $\mu \geq 0$, _and_ p _and_ F _satisfying the above conditions, the boundary problem_ (5.2) _has a solution which does not vanish on the open interval_ (a,b) _if and only if_ $\mu < \mu_1$, _where_ μ_1 _is the smallest proper value of the system_

$$y'' + \mu p(t) y = 0, \quad a \leq t \leq b, \qquad (5.4)$$
$$y(a) = 0 = y(b).$$

In particular, the boundary problem (5.1) has a solution which does not vanish on (a,b).

The proof of this theorem will be given following some preliminary definitions, two lemmas, and the statement of the variational problem to be used in the proof. Several of the early steps in the discussion below parallel certain stages of Nehari's argument [5], [6], and Nehari's notation has been used wherever possible.

The class of functions of integrable square on [a,b] will be denoted below by $\mathcal{L}_2[a,b]$. We define

$$G(t,s) = \int_0^s F(t,\sigma)d\sigma, \quad (t,s) \in \Delta, \tag{5.5}$$

and

$$P(t,s,\mu) = F(t,s) + \mu p(t), \quad (t,s) \in \Delta, \quad \mu \in [0,\infty). \tag{5.6}$$

For $\mu \geq 0$, let $\mathcal{B}_\mu$ denote the variational problem of minimizing the functional

$$J[y] = \int_a^b [y^2(t)F(t,y^2(t)) - G(t,y^2(t))]dt \tag{5.7}$$

subject to the conditions

(5.8a) $\quad y$ is a.c. on $[a,b]$, and $y' \in \mathcal{L}_2[a,b]$,

(5.8b) $\quad y(a) = 0 = y(b)$,

(5.8c) $\quad y(t) \not\equiv 0$ on $[a,b]$,

(5.8d) $\quad \phi[y,\mu] = \int_a^b [y'^2(t) - y^2(t)P(t,y^2(t),\mu)]dt \leq 0.$

For convenient reference, the class of functions satisfying conditions (5.8a-c) *will be denoted by* $\mathcal{D}$, *and for* $\mu \geq 0$ *the class of functions in* $\mathcal{D}$ *which also satisfy* (5.8d) *will be denoted by* $\mathcal{D}_{\mu}$.

LEMMA 5.1. *If* $F(t,s)$ *satisfies conditions* (5.3), *then for each* $t \in [a,b]$, $F(t,s)$ *is a continuous, strictly increasing function of* s *on* $0 \leq s < \infty$, *with* $\lim_{s \to 0} F(t,s) = 0$ *and* $\lim_{s \to \infty} F(t,s) = \infty$. *Also,* $F_s(t,s) > 0$ *for all* $(t,s) \in \Delta$ *with* $s > 0$.

The first statement of the lemma follows immediately from conditions (5.3a-c). To prove that $F_s(t,s) > 0$ for all $(t,s) \in \Delta$, one notes that (5.3c) implies

$$\frac{\partial}{\partial s}\left(\frac{F(t,s)}{s^{\gamma}}\right) = s^{-\gamma}F_s(t,s) - \gamma s^{-\gamma-1}F(t,s) \geq 0,$$

and hence

$$F_s(t,s) \geq \gamma s^{-1}F(t,s).$$

If $s > 0$, then $s^{-1}F(t,s) > 0$ by (5.3b), and $\gamma > 0$ by hypothesis, so that $F_s(t,s) > 0$ for all $(t,s) \in \Delta$ with $s > 0$.

LEMMA 5.2. *Let* μ_1 *be the smallest proper value of the system* (5.4). *Then there exists a continuous, strictly increasing function* $\Psi = \Psi(\eta)$, *defined on* $0 \leq \eta < \infty$ *and satisfying* $\Psi(0) = 0$, $\lim_{\eta \to \infty} \Psi(\eta) = \infty$, *such that if* $0 \leq \mu < \mu_1$ *then*

$$\int_a^b y'^2(t)dt \geq \Psi(1-\mu/\mu_1) \text{ for } y \in \mathcal{D}_{\mu}. \tag{5.9}$$

For all $y \in \mathcal{D}$, we have, by the Schwarz inequality, that

$$y^2(t) = [\int_a^t y'(\tau)d\tau]^2 \leq (t-a) \int_a^t y'^2(\tau)d\tau, \quad a \leq t \leq b,$$

and hence that

$$y^2(t) \leq (t-a) \int_a^b y'^2(t)dt, \quad a \leq t \leq b. \tag{5.10}$$

Condition (5.8d) and definition (5.6) imply that for $y \in \mathcal{D}_\mu$,

$$\int_a^b y'^2(t)dt \leq \int_a^b y^2(t)F(t,y^2(t))dt + \mu \int_a^b y^2(t)p(t)dt. \tag{5.11}$$

With the notation

$$\beta(y) = \int_a^b y'^2(t)dt,$$

inequality (5.10) may be written as

$$y^2(t) \leq \beta(y)(t-a). \tag{5.10'}$$

By Lemma 5.1, $F(t,s)$ is increasing in s for each $t \in [a,b]$, and it then follows from (5.10') and (5.11) that, if $y \in \mathcal{D}_\mu$,

$$\beta(y) \leq \int_a^b \beta(y)(t-a)F[t,\beta(y)\cdot(t-a)]dt + \mu \int_a^b y^2(t)p(t)dt. \tag{5.12}$$

To simplify (5.12), we define

$$\Phi(\lambda) \equiv \int_a^b (t-a)F[t, \lambda(t-a)]dt, \quad 0 \leq \lambda < \infty.$$

From Lemma 5.1, it is clear that $\Phi = \Phi(\lambda)$ is a continuous, strictly increasing function of λ on $[0,\infty)$, and that $\Phi(0) = 0$ and $\Phi(\lambda) \to \infty$

as $\lambda \to \infty$. The inequality (5.12) can then be written as

$$\beta(y) \leq \beta(y)\,\Phi[\beta(y)] + \mu\int_a^b y^2(t)p(t)dt. \tag{5.12'}$$

For $y \in \mathcal{D}$, condition (5.8c) implies that $\beta(y) > 0$, and division by $\beta(y)$ in (5.12') yields the inequality

$$1 \leq \Phi[\beta(y)] + [\mu/\beta(y)]\int_a^b y^2(t)p(t)dt \quad \underline{\text{if}}\ y \in \mathcal{D}_\mu. \tag{5.13}$$

By a classical result, the least proper value of (5.4) is given by

$$\mu_1 = \min_{y \in \mathcal{D}} \left[\int_a^b y'^2(t)dt\right]\left[\int_a^b y^2(t)p(t)dt\right]^{-1},$$

(see, for example, [7; §§1, 3(f)]). Therefore

$$\mu_1 \leq \beta(y)\left(\int_a^b y^2(t)p(t)dt\right)^{-1} \underline{\text{for}}\ y \in \mathcal{D},$$

and consequently

$$\int_a^b y^2(t)p(t)dt \leq \beta(y)/\mu_1 \quad \underline{\text{for}}\ y \in \mathcal{D}, \tag{5.14}$$

which, with (5.13), implies that

$$1-\mu/\mu_1 \leq \Phi[\beta(y)] \quad \underline{\text{if}}\ y \in \mathcal{D}_\mu. \tag{5.15}$$

Let $\Psi = \Psi(\eta)$, $0 \leq \eta < \infty$, be the inverse of the function Φ. Then Ψ is a continuous, increasing function on $[0,\infty)$ such that $\Psi(0) = 0$ and $\Psi(\eta) \to \infty$ as $\eta \to \infty$. Therefore, by (5.15),

$$\beta(y) \geq \Psi(1-\mu/\mu_1) \quad \underline{\text{if}}\ y \in \mathcal{D}_\mu,$$

which completes the proof of the lemma.

PROOF OF THEOREM 5.1: For some fixed $\mu \geq 0$, let $y_0 = y_0(t)$ be a solution of the boundary problem (5.2) such that $y_0(t)$ does not vanish on (a,b). We wish to show that $\mu < \mu_1$, where μ_1 is the smallest proper value of (5.4). If $\mu = 0$, this is trivial, since $\mu_1 > 0$. If $\mu > 0$ the proof is essentially the same as that given by Nehari [6; p. 228], and is repeated here for completeness. If $\mu > 0$, then y_0 is a solution of the linear system

$$y'' + \mu[p(t) + \mu^{-1}F(t,y_0^2)]y = 0, \quad a \leq t \leq b,$$

$$y(a) = 0 = y(b).$$

Suppose that $\mu \geq \mu_1$. Then

$$\mu[p(t) + \mu^{-1}F(t,y_0^2(t))] \geq \mu p(t) \geq \mu_1 p(t), \quad a \leq t \leq b,$$

with strict inequality holding except at t = a and t = b. It follows from Sturm's comparison theorem (see Ince [3, p. 228]) that $y_0(\tau) = 0$ for some $\tau \in (a,b)$, which is a contradiction, so that necessarily, $\mu < \mu_1$.

The proof of sufficiency is divided into two parts:

Part I: <u>If</u> $\mu < \mu_1$, <u>the variational problem</u> $\mathcal{B}_\mu$ <u>has a solution</u>.

Part II: <u>Every solution of the variational problem</u> $\mathcal{B}_\mu$ <u>is a solution of the boundary problem</u> (5.2), <u>and such a solution does not vanish on</u> (a,b).

In order to establish Part I, let μ satisfy $0 \leq \mu < \mu_1$, where μ_1 is the least proper value of (5.4). We note, as in Nehari [5; p. 110], that

$$(5.16) \qquad G(t,s) = \int_0^s F(t,\sigma)d\sigma = \int_0^s \sigma^{\gamma}\sigma^{-\gamma}F(t,\sigma)d\sigma,$$

and hence, by (5.3c) that

$$(5.17) \qquad G(t,s) \leq s^{-\gamma}F(t,s)\int_0^s \sigma^{\gamma}d\sigma = (1+\gamma)^{-1}sF(t,s).$$

Consequently, for all $y \in \mathcal{D}$, the functional $J[y]$ of (5.7) satisfies the inequality

$$J[y] \geq \gamma(1+\gamma)^{-1}\int_a^b y^2(t)F(t,y^2(t))\,dt,$$

and application of (5.6) then shows that

$$(5.18) \qquad J[y] \geq \gamma(1+\gamma)^{-1}\int_a^b y^2(t)\{P[t,y^2(t),\mu] - \mu p(t)\}dt.$$

From (5.8d) and (5.18) one obtains, for all $y \in \mathcal{D}_\mu$, the inequality

$$(5.19) \qquad J[y] \geq \gamma(1+\gamma)^{-1}\int_a^b [y'^2(t) - \mu y^2(t)p(t)]dt.$$

Inequality (5.17) also implies that

$$sF(t,s) - G(t,s) \geq 0 \ \underline{\text{for all}} \ (t,s) \in \Delta,$$

so that it follows from definition (5.7) that $J[y] \geq 0$ for all $y \in \mathcal{D}$, and hence $\inf\{J[y]\colon y \in \mathcal{D}_\mu\}$ exists. Let $\{y_n\}$, $(n = 1,2,\ldots)$ be a sequence of functions satisfying $y_n \in \mathcal{D}_\mu$, $(n = 1,2,\ldots)$, and

(5.20) $$\lim_{n\to\infty} J[y_n] = \inf\{J[y]: y \in \mathcal{D}_\mu\}.$$

The sequence $\{J[y_n]\}$ is bounded, so from (5.19) there exists a constant $\alpha > 0$ such that

(5.21) $$\int_a^b [y_n'^2(t) - \mu y_n^2(t)p(t)]dt \leq \alpha, \quad (n = 1,2,\ldots).$$

From (5.21) and (5.14), it then follows that

$$\int_a^b y_n'^2(t)dt \leq \alpha + \mu\beta(y_n)/\mu_1 = \alpha + (\mu/\mu_1)\int_a^b y_n'^2(t)dt,$$

which implies that

(5.22) $$\int_a^b y_n'^2(t) \leq k, \quad (n = 1,2,\ldots),$$

where $k = \alpha/(1-\mu/\mu_1)$. It follows from (5.10) and (5.22) that

$$y_n^2(t) \leq k(t-a) \leq k(b-a), \quad a \leq t \leq b, \quad (n = 1,2,\ldots),$$

and thus the sequence $\{y_n\}$ is uniformly bounded on $[a,b]$.

For arbitrary t_1 and t_2 on $[a,b]$, the Schwarz inequality, together with inequality (5.22), implies that, for $n = 1,2,\ldots,$

$$|y_n(t_2) - y_n(t_1)|^2 = \left(\int_{t_1}^{t_2} y_n'(t)dt\right)^2 \leq |t_2-t_1| \int_{t_1}^{t_2} y_n'^2(t)dt$$

$$\leq |t_2-t_1| \int_a^b y_n'^2(t)dt$$

$$\leq k\,|t_2-t_1|,$$

from which it follows that the sequence $\{y_n\}$ is equicontinuous on [a,b]. The Ascoli theorem then implies that $\{y_n\}$ has a subsequence, also denoted by $\{y_n\}$, (n = 1,2,...), which converges uniformly on [a,b] to a function y_0. This sequence has, in turn, another subsequence, which will still be denoted by $\{y_n\}$, (n = 1,2,...), such that $\{y_n'\}$ converges weakly in $\mathcal{L}_2[a,b]$ to $y_0' \in \mathcal{L}_2[a,b]$, and y_0 is a.c. on [a,b] and satisfies $y_0(a) = 0 = y_0(b)$, (see [8; §§32, 99]). By the bounded convergence theorem, $J[y_n] \to J[y_0]$ as $n \to \infty$, so, by (5.20), $J[y_0] = \inf\{J[y]: y \in \mathcal{D}_\mu\}$. Thus, to prove that y_0 is a solution of the variational problem $\mathcal{B}_\mu$, it remains to show that $y_0 \in \mathcal{D}_\mu$.

It has already been shown that y_0 satisfies (5.8a) and (5.8b). To show that y_0 satisfies (5.8d), we note first that, for n = 1,2,...,

$$\int_a^b y_n'^2(t)dt = \int_a^b [y_n'(t) - y_0'(t)]^2 dt + 2\int_a^b y_n'(t)y_0'(t)dt - \int_a^b y_0'^2(t)dt;$$

the first term on the right is non-negative for n = 1,2,..., and the second term converges to $2\int_a^b y_0'^2(t)dt$ as $n \to \infty$, by the weak convergence of y_n' to y_0'. It then follows that

(5.23) $$\liminf_{n\to\infty}\int_a^b y_n'^2(t)dt \geq \int_a^b y_0'^2(t)dt.$$

Also, by the bounded convergence theorem,

(5.24) $$\lim_{n\to\infty}\int_a^b y_2^2(t)P[t,y_n^2(t),\mu]dt = \int_a^b y_0^2(t)P[t,y_0^2(t),\mu]dt,$$

and, therefore, if $\phi[y,\mu]$ is the function defined in condition (5.8d),

(5.25) $$\liminf_{n\to\infty}\phi[y_n,\mu] \geq \phi[y_0,\mu].$$

Since $y_n \in \mathcal{D}_\mu$, $(n = 1,2,\ldots)$, y_n satisfies (5.8d), so that $\phi[y_n,\mu] \leq 0$, and thus it follows from (5.25) that $\phi[y_0,\mu] \leq 0$, and y_0 satisfies (5.8d).

Finally, since (5.8d) holds for each y_n, Lemma 5.2 implies that

$$0 < \Psi(1-\mu/\mu_1) \leq \int_a^b y_n'^2(t)dt \leq \int_a^b y_n^2(t)P[t,y_n^2(t),\mu]dt, \quad (n = 1,2,\ldots),$$

and consequently, by (5.24),

$$\int_a^b y_0^2 P[\, t,y_0^2(t),\mu]dt \geq \Psi(1-\mu/\mu_1) > 0.$$

Therefore $y_0(t) \not\equiv 0$ on $[a,b]$, and y_0 satisfies (5.8a-d), so $y_0 \in \mathcal{D}_\mu$. Consequently y_0 is a solution of the variational problem $\mathcal{B}_\mu$, which completes Part I of the proof of the theorem.

We proceed to establish Part II, that is, to prove that every solution of the variational problem $\mathcal{B}_\mu$ is a solution of the corresponding problem (5.2). The first step is to show that, if $y_0 \in \mathcal{D}_\mu$ is a solution of $\mathcal{B}_\mu$, then $\phi[y_0,\mu] = 0$. Condition (5.8d) implies $\phi[y_0,\mu] \leq 0$, and it will be shown that the assumption that $\phi[y_0,\mu] < 0$ leads to a contradiction. For every real number $q > 0$, let

$$y_{(q)}(t) = q \cdot y_0(t), \qquad a \leq t \leq b.$$

Since $y_{(1)}(t) = y_0(t)$, and

$$\phi[y_{(q)},\mu] = q^2 \int_a^b [y_0'^2(t) - y_0^2(t)P(t,q^2y_0^2(t),\mu)]dt,$$

it follows that if $\phi[y_0,\mu] < 0$ then $\phi[y_{(q)},\mu] < 0$ for all q in some neighborhood N of $q = 1$. If $q \in N$, then $y_{(q)} \in \mathcal{D}_\mu$, so $J[y_{(q)}] \geq J[y_0] = \inf\{J[y]: y \in \mathcal{D}_\mu\}$, from the minimizing property of y_0. Thus, since $J[y_{(1)}] = J[y_0]$, the derivative $\frac{d}{dq}J[y_{(q)}]$ vanishes at $q = 1$. Now

$$J[y_{(q)}] = \int_a^b [q^2y_0^2(t)F(t,q^2y_0^2(t)) - G(t,q^2y_0^2(t))]dt,$$

so that

$$\left.\frac{d}{dq}J[y_{(q)}]\right|_{q=1} = 2\int_a^b y_0^4(t)F_s(t,y_0^2(t))dt = 0.$$

Since $F_s(t,s)$ is continuous and positive for all $(t,s) \in \Delta$ satisfying $s > 0$, this can hold only if $y_0(t) = 0$ on $[a,b]$, which contradicts (5.8c), since $y_0 \in \mathcal{D}_\mu$. Therefore $\phi[y_0,\mu] = 0$, which was to be shown.

Since $\phi[y_0,\mu] = 0$, y_0 minimizes $J[y]$ in the subclass of $\mathcal{D}_\mu$ consisting of those functions y in $\mathcal{D}_\mu$ which satisfy $\phi[y,\mu] = 0$. It follows from the multiplier rule, (see [1; p. 202]), that there exist multipliers λ_1, λ_2, not both zero, such that the function defined by

$$H(t,y,r) = \lambda_1[y^2F(t,y^2) - G(t,y^2)] + \lambda_2[r^2-y^2P(t,y^2,\mu)]$$

satisfies

$$(5.26) \quad \int_a^b \{H_y[t,y_0(t),y_0'(t)]\eta(t) + H_r[t,y_0(t),y_0'(t)]\eta'(t)\}dt = 0$$

for all $\eta = \eta(t)$ satisfying (5.8a) and (5.8b), and by the usual proof of the Euler necessary condition in the calculus of variations, it follows that there exists a constant c such that

$$(5.27) \qquad H_r[t,y_0(t),y_0'(t)] - \int_a^t H_y[\tau,y_0(\tau),\ y_0'(\tau)]d\tau = c$$

for t a.e. in [a,b]. By the definition of the function H, equation (5.27) implies that

$$(5.28) \qquad 2\lambda_2 y_0'(t) = \int_a^t \{2\lambda_1 y_0^3(\tau)F_s[\tau,y_0^2(\tau)] - 2\lambda_2 y_0(\tau)P[\tau,y_0^2(\tau),\mu]$$

$$- 2\lambda_2 y_0^3(\tau)P_s[\tau,y_0^2(\tau),\mu]\}\, d\tau + c$$

for t a.e. on [a,b]. From (5.6), we have $P_s(t,s,\mu) = F_s(t,s)$ for all $(t,s) \in \Delta$, so that (5.28) simplifies to

$$(5.29) \quad \lambda_2 y_0'(t) = \int_a^t \{(\lambda_1-\lambda_2)y_0^3(\tau)F_s[\tau,y_0^2(\tau)] - \lambda_2 y_0(\tau)P[\tau,y_0^2(\tau),\mu]\}d\tau + c.$$

If $\lambda_2 = 0$, then $\lambda_1 \neq 0$, and (5.29) implies

$$\int_a^t y_0^3(\tau)F_s(\tau,y_0^2(\tau))d\tau = -c/\lambda_1 \ \underline{\text{for}}\ t\ \underline{\text{a.e.}}\ \underline{\text{on}}\ [a,b].$$

This equation can hold only if $c = 0$ and $y_0(t) \equiv 0$, $a \leq t \leq b$, which is a contradiction to $y_0 \in \mathcal{D}_\mu$, and thus we have $\lambda_2 \neq 0$.

Since the right-hand member of (5.29) is continuous and $\lambda_2 \neq 0$, it follows that y_0 is an absolutely continuous function with derivative equal a.e. on [a,b] to a continuous function, and hence y_0 is continuously differentiable. Thus (5.29) holds for all $t \in [a,b]$, which in turn implies that y_0 has a continuous second derivative, and

$$(5.30) \qquad \lambda_2 y_0''(t) = (\lambda_1-\lambda_2)y_0^3(t)F_s(t,y_0^2(t)) - \lambda_2 y_0(t)P(t,y_0^2(t),\mu)$$

holds on [a,b]. We rewrite (5.30) as

$$(5.31) \qquad \lambda_2[y_0''(t) + y_0(t)P(t,y_0^2(t),\mu)] = (\lambda_1-\lambda_2)y_0^3(t)F_s(t,y_0^2(t)).$$

Multiplication by y_0 and integration yield the equation

$$\lambda_2\left\{y_0(t)y_0'(t)\Big|_a^b - \int_a^b y_0'^2(t)dt + \int_a^b y_0^2(t)P(t,y_0^2(t),\mu)dt\right\} =$$

$$(\lambda_1-\lambda_2)\int_a^b y_0^4(t)F_s(t,y_0^2(t))dt.$$

Since $y_0(a) = 0 = y_0(b)$, this simplifies to

$$\lambda_2\int_a^b [y_0'^2(t) - y_0^2(t)P(t,y_0^2(t),\mu)]dt = (\lambda_2-\lambda_1)\int_a^b y_0^4(t)F_s(t,y_0^2(t))dt.$$

The integral on the left is $\phi[y_0,\mu]$, which has been shown to equal 0, while the integral on the right is positive for $y_0 \in \mathcal{D}_\mu$. Therefore $\lambda_2-\lambda_1 = 0$, so (5.31) implies that

$$y_0''(t) + y_0(t)P(t,y_0^2(t),\mu) = 0, \quad a \leq t \leq b.$$

From (5.6), this equation may be written as

$$y_0'' + \mu p(t)y_0 + y_0F(t,y_0^2) = 0, \quad a \leq t \leq b,$$

and y_0 is thus seen to be a solution of the boundary problem (5.2). Thus, for arbitrary $\mu \geq 0$, it has been shown that every solution in $\mathcal{D}_\mu$ of the variational problem $\mathcal{B}_\mu$ is a solution of (5.2).

Finally, we want to show that if $\mu \geq 0$ and $y_0 \in \mathcal{D}_\mu$ is a solution of $\mathcal{B}_\mu$, then $y_0(t) \neq 0$ on $a < t < b$. Suppose that y_0 is a solution of $\mathcal{B}_\mu$ and that $y_0(\tau) = 0$ for some $\tau \in [a,b]$. By (5.8c), there exists a $\tau_1 \in (a,\tau)$ or $\tau_1 \in (\tau,b)$ such that $y_0(\tau_1) \neq 0$. Assume that $\tau_1 \in (a,\tau)$ and $y_0(\tau_1) > 0$; if $\tau_1 \in (\tau,b)$ or $y_0(\tau_1) < 0$, the proof is similar. Let

$$\omega = \inf\{t \mid t \in (\tau_1,b),\ y(t) = 0\}.$$

Clearly $y_0(t) > 0$ for all $t \in [\tau_1,\omega)$, and $y_0(\omega) = 0$. It was shown above that y_0 is a solution of the boundary problem (5.2), and the form of the differential equation of (5.2), together with the hypotheses on p and F, implies that $y_0''(t) < 0$ if $y_0(t) > 0$, and hence $y_0''(t) < 0$ for

$\tau_1 \leq t < \omega$. Therefore, $y_0'(t)$ is decreasing on $[\tau_1, \omega]$, so that by the mean-value theorem there exists a $\tau_2 \in (\tau_1, \omega)$ such that $y'(\tau_2) < 0$, and consequently $y'(\omega) < 0$.

Now $y_0 \in \mathcal{D}_\mu$ implies $|y_0| \in \mathcal{D}_\mu$, and since $J[y_0] = J[|y_0|]$, the function $|y_0|$ is also a solution of the variational problem $\mathcal{B}_\mu$. Therefore $|y_0|$ is a solution of the boundary problem (5.2), so $|y_0| \in C''[a,b]$. But since $y_0'(\omega) \neq 0$ and $y_0(\omega) = 0$, $|y_0|'$ is not defined at $t = \omega$, a contradiction. Therefore $y_0(t)$ does not vanish on (a,b), which completes the proof of Part II.

From the results of Parts I and II it follows that if $0 \leq \mu < \mu_1$, then the boundary problem (5.2) has a solution which does not vanish on (a,b). For $\mu = 0$, (5.2) reduces to (5.1) and hence (5.1) has a solution which does not vanish on (a,b), which completes the proof of Theorem 5.1.

From the proof above and the necessity of the condition $\mu < \mu_1$ of Theorem 5.1, one may note the following result:

COROLLARY. *For $\mu \geq 0$, the variational problem $\mathcal{B}_\mu$ has a solution if and only if $\mu < \mu_1$, where μ_1 is the least proper value of the system* (5.4).

6. *A special case of equation* (2.1). In this section the differential equation

$$y'' + yf(t,y) = 0, \tag{6.1}$$

with $f(t,y)$ satisfying conditions given below, is shown to be a special case of equation (2.1), with $F(t,y,r)$ satisfying conditions (I) and (II).

The function $f(t,y)$ is assumed to be defined and continuous on D_0: $a \leq t \leq b$, $-\infty < y < \infty$, and to satisfy the following hypotheses:

(6.2a) $$f(t,y) \geq 0 \quad \underline{\text{for all}} \ (t,y) \in D_0,$$

(6.2b) $$f(t,y) = 0 \quad \underline{\text{if and only if}} \ y = 0,$$

(6.2c) $$y_2 > y_1 \geq 0 \ \underline{\text{or}} \ y_2 < y_1 \leq 0 \ \underline{\text{implies that}} \ f(t,y_2) \geq f(t,y_1) \quad \underline{\text{for every}} \ t \in [a,b],$$

(6.2d) $$\lim_{y \to +\infty} f(t,y) = \lim_{y \to -\infty} f(t,y) = +\infty, \ \underline{\text{for every}} \ t \in [a,b],$$

(6.2e) <u>for each</u> $(\tau,\eta) \in D_0$, <u>there is a neighborhood</u> V <u>of</u> (τ,η) <u>such that</u> $f(t,y)$ <u>satisfies a Lipschitz condition with respect to</u> y <u>in</u> $V \cap D_0$.

It is readily verified that if $f(t,y) = F(t,y^2)$ for all $(t,y) \in D_0$, where $F(t,s)$ is the function appearing in the boundary problem (5.1) and satisfying conditions (5.3), then $f(t,y)$ satisfies conditions (6.2). Therefore the problem (5.1) is a special case of the boundary problem defined by the equation (6.1) with boundary conditions $y(a) = 0 = y(b)$.

The conditions (6.2a, b, c, and e) are similar to conditions on $\phi(t,y,\lambda)$ assumed by Moroney [4] in connection with the characteristic value problem

$$y'' + y\phi(t,y,\lambda) = 0, \quad 0 \leq t \leq 1,$$
$$y(0) = 0, \quad y'(0) = 1,$$
$$y(1) = 0.$$

However, in place of condition (6.2d), Moroney assumes a condition called the "regenerative property" which involves the behavior of $\phi(t,y,\lambda)$ with respect to λ.

In order to state the result of this section, the following definition is needed:

Given $f(t,y)$ defined on D_0, define

$$(6.3) \qquad \check{f}(y) = \sup_{a \le t \le b} f(t,y), \qquad \hat{f}(y) = \inf_{a \le t \le b} f(t,y).$$

THEOREM 6.1. <u>If</u> $f = f(t,y)$ <u>is continuous on</u> D_0 <u>and satisfies conditions</u> (6.2), <u>and if</u> $\check{f}$ <u>and</u> $\hat{f}$ <u>are defined by</u> (6.3), <u>then</u> $\check{f}$ <u>and</u> $\hat{f}$ <u>are continuous on</u> $-\infty < y < \infty$ <u>and satisfy conditions</u> (2.5), <u>and the function</u> $f(t,y,r) = f(t,y)$, $(t,y) \in D_0$, $r \in R$, <u>is continuous on</u> $D = [a,b] \times R \times R$ <u>and satisfies properties</u> (I) <u>and</u> (II) <u>of Section 2</u>.

From the continuity of $f(t,y)$ and from (6.2e), it is clear that $f(t,y,r)$ is continuous on D and satisfies condition (II). Thus we need only show that $\check{f}$ and $\hat{f}$ are continuous and that conditions (2.5) are satisfied.

It follows immediately from (6.2a) and (6.3) that $0 \le \hat{f}(y) \le f(t,y,r) \le \check{f}(y)$ for all $(t,y,r) \in D$, so that (2.5a) is satisfied.

If $y = 0$, then (6.2b) implies $f(t,y) = 0$ for all $t \in [a,b]$, so that $\hat{f}(y) = 0$. Conversely, if $\hat{f}(y_0) = 0$ for some $y_0 \in R$, then the continuity of $f(t,y_0)$ on $[a,b]$ implies the existence of a value $t_0 \in [a,b]$ such that $f(t_0,y_0) = \hat{f}(y_0)$, and hence $f(t_0,y_0) = 0$, so that $y_0 = 0$ by (6.2b). Thus $\hat{f}(y) = 0$ if and only if $y = 0$. Similarly, $\check{f}(y) = 0$ if and only if $y = 0$, so (2.5b) is satisfied.

To prove that $\hat{f}$ satisfies (2.5c), we note that if $y_2 < y_1 \le 0$ or $y_2 > y_1 \ge 0$, then (6.2c) and (6.3) imply that

$$\hat{f}(y_1) \leq f(t,y_1) \leq f(t,y_2), \quad a \leq t \leq b.$$

The continuity of $f(t,y_2)$ on $[a,b]$ implies that $\hat{f}(y_2) = f(t,y_2)$ for some $t \in [a,b]$, and it follows that $\hat{f}(y_1) \leq \hat{f}(y_2)$, so that $\hat{f}$ satisfies (2.5c). The proof for $\check{f}$ is similar.

To prove that (2.5d) holds, it must be shown that $\hat{f}(y) \rightarrow +\infty$ as $y \rightarrow +\infty$ and as $y \rightarrow -\infty$. For $M > 0$, condition (6.2d) implies that for each $t \in [a,b]$ there exists a B_t such that if $y \geq B_t$ then $f(t,y) \geq M$. For fixed $\tau \in [a,b]$ and for $\varepsilon > 0$, the continuity of f implies the existence of a $\delta_\tau > 0$ (depending on ε) such that if $|t - \tau| < \delta_\tau$ and $t \in [a,b]$ then $f(t,B_\tau) > f(\tau,B_\tau) - \varepsilon$. By property (6.2c), if $y \geq B_\tau$ then

(6.4) $$f(t,y) \geq f(t,B_\tau) > f(\tau,B_\tau) - \varepsilon \geq M - \varepsilon$$

for $|t-\tau| < \delta_\tau$ and $t \in [a,b]$. Since the class of intervals $(\tau-\delta_\tau, \tau+\delta_\tau)$, $\tau \in (a,b)$, is an open covering of the compact interval $[a,b]$, there is a finite set of intervals $(\tau_i - \delta_{\tau_i}, \tau_i + \delta_{\tau_i})$, $(i = 1,\ldots,K)$, whose union contains $[a,b]$. If $y \geq B = \max\{B_{\tau_i} : i = 1,\cdots,K\}$, then (6.4) implies that $f(t,y) \geq M - \varepsilon$ for all $t \in [a,b]$, so that $\hat{f}(y) \geq M - \varepsilon$ for $y \geq B$. Since this holds for arbitrary $\varepsilon > 0$, we have $\hat{f}(y) \geq M$ for $y \geq B$, which completes the proof of (2.5d).

Finally, it must be shown that $\hat{f}$ and $\check{f}$ are continuous. Given $y_0 \in R$, let $\{y_n\}$, $(n = 1,2,\ldots,)$, be a sequence of real numbers converging to y_0. For each $n = 0, 1, 2,\ldots$, $f(t,y_n)$ is continuous on $a \leq t \leq b$, so there exists a $t_n \in [a,b]$ such that $f(t_n,y_n) = \hat{f}(y_n)$. The sequence $\{t_n\}$, $(n = 1,2,\ldots)$, has a subsequence, which we also denote by $\{t_n\}$, converging to a point t_∞ in $[a,b]$. From the continuity of $f(t,y)$

it follows that

$$\lim_{n\to\infty}\hat{f}(y_n) = \lim_{n\to\infty} f(t_n,y_n) = f(t_\infty,y_0),$$

and by definition of $\hat{f}$, $f(t_\infty,y_0) \geq \hat{f}(y_0)$, so that

$$\lim_{n\to\infty}\hat{f}(y_n) \geq \hat{f}(y_0).$$

Also, $\hat{f}(y_n) \leq f(t_0,y_n)$, $(n = 1,2,\ldots)$, hence

$$\lim_{n\to\infty}\hat{f}(y_n) \leq \lim_{n\to\infty} f(t_0,y_n) = f(t_0,y_0) = \hat{f}(y_0).$$

Therefore $\lim_{n\to\infty}\hat{f}(y_n) = \hat{f}(y_0)$, so that $\hat{f}$ is continuous. A similar argument shows that f is continuous, which completes the proof of the theorem.

Thus the equations $y'' + y\,F(t,y^2) = 0$, where $F(t,s)$ satisfies the conditions of Section 5, and $y'' + yf(t,y) = 0$, where $f(t,s)$ satisfies the conditions of the present section, are special instances of the equation (2.1) of Sections 3 and 4, so that the results of those sections apply to these equations.

BIBLIOGRAPHY

1. Bliss, G. A. Lectures on the Calculus of Variations. Chicago: University of Chicago Press, 1946.

2. Graves, L. M. The Theory of Functions of Real Variables, 2nd ed. New York: McGraw-Hill, 1956.

3. Ince, E. L. Ordinary Differential Equations. London: Longmans, Green, 1927.

4. Moroney, R. M. "A class of characteristic-value problems," Trans. Amer. Math. Soc. 102 (1962), pp. 446-470.

5. Nehari, Z. "On a class of nonlinear second-order differential equations," Trans. Amer. Math. Soc. 95 (1960), pp. 101-123.

6. Nehari, Z. "Characteristic values associated with a class of nonlinear second-order differential equations," Acta Mathematica 105 (1961), pp. 141-176.

7. Reid, W. T. "Boundary value problems of the calculus of variations," Bull. Amer. Math. Soc. 42 (1937), pp. 633-666.

8. Riesz, F. and Sz.-Nagy, B. Functional Analysis. New York: F. Ungar Publishing Co., 1955.