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BY

Dr. Noel Brady, Chair

Dr. Murad Ozaydin

Dr. Max Forester

Dr. Ralf Schmidt

Dr. Doo Hun Lim

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Chapter 1

Abstract

One of the major paradigms in geometric group theory is the idea that one can understand algebraic properties of groups by studying their actions on geometric spaces. There is a basic geometric object that one can associate to any finitely generated group: its Cayley graph with the word metric.

Every pair (G, S) of a group and its finite generating set S has an associated *word metric*. The distance $d(g, h)$ between elements $g, h \in G$ is defined to be the length of the shortest word in $S \cup S^{-1}$ which is equal to $g^{-1}h$ in G. The *Cayley graph* $\Gamma(G, S)$ of a pair (G, S) has G as its vertex set, and a vertex g_1 is connected to another vertex g_2 by an edge labeled by s if $g_2 = g_1 s$ for some $s \in S$. A word metric on a pair (G, S) extends to a metric on $\Gamma(G, S)$ provided one declares each edge in $\Gamma(G, S)$ has length 1. The action of G on itself by left multiplication extends naturally to an isometric action on the Cayley graph (with fixed generating set S).

We can use a Cayley graph $\Gamma(G, S)$ to define geometric properties of a group G. One such property is the number of ends $e(G)$ of G. Roughly speaking, the number of ends $e(G)$ of a finitely generated group G measures the number of "connected components of G at infinity". For a finitely generated group G, $e(G)$ is determined

by the number of *ends* $e(\Gamma)$, of a Cayley graph Γ of G, which can be defined as follows. To find $e(\Gamma)$, remove a compact set K from Γ, and count the number of unbounded components of $\Gamma - K$. The number $e(\Gamma)$ is defined to be the supremum of this number over all compact sets. Finally it can be shown that $e(\Gamma)$ is independent of the choice of a finite generating set S used in the construction of Γ ([21]). So it makes sense to define $e(G) = e(\Gamma)$.

In [20], Stallings showed that, for a finitely generated group G, the geometric condition of having more than one end is equivalent to the algebraic condition that G *splits* (that is, G can be written as a free product with amalgamation or HNN extension) over a finite subgroup. On the other hand, the theory of Bass-Serre relates the algebraic condition that G splits to certain types of actions of G on a tree. In view of Bass-Serre theory, Stallings' theorem can be restated as follows.

Theorem 1.1. *A finitely generated group G satisfies $e(G) > 1$ if and only if G acts on a tree (without inversion) with finite edge-stabilizers.*

There is a generalization of Theorem 1.1, which replaces the tree by a possibly higher dimensional space and uses a more general notion of ends than $e(G)$. In [12], C. H. Houghton introduced the concept of the number of ends $e(G, H)$ of a finitely generated group with respect to a subgroup $H \leq G$. Being a subgroup of G, H acts on a Cayley graph Γ of G by the left multiplication. Now define $e(G, H)$ to be the number of ends of the quotient graph Γ/H. Again, one can show that this is independent of the particular finite generating set S used in the construction of Γ ([19]). A group G is called *multi-ended* if $e(G, H) > 1$ for some subgroup H.

Roughly speaking, a piecewise Euclidean (PE) cubical complex is built from a collection of a regular Euclidean cubes by glueing their faces via isometries. A *cubing* is a 1-connected PE cubical complex satisfying some additional non-positive curvature conditions. We will make this notion precise in Section 3.3.

One dimensional cubes are just unit length line segments, and so a 1-dimensional cubical complex is simply a graph. The condition of being simply connected means that the graph is a tree. Therefore one can think of cubings as generalizations of trees.

In [18], Sageev proved the following remarkable generalization of Stallings' result.

Theorem 1.2. *A finitely generated group G is multi-ended if and only if G acts 'essentially' on a cubing.*

This theorem includes the possibility that the cubing is infinite dimensional. An essential action means that the action has an unbounded orbit provided the cubing is finite dimensional.

Non-positively curved cubical complexes play a central role in low-dimensional topology and geometric group theory. A striking example of their importance is given by Agol's recent proof [1] of the Virtual Haken Conjecture and the Virtual Fibration Conjecture in 3–manifold topology. Agol's proof relies on results of Haglund and Wise which concern fundamental groups of a *speical* class of non-positively curved cubical complexes. In [11], Haglund and Wise showed if the fundamental group of a special cubical complex is word-hyperbolic then every quasiconvex subgroup is separable.

This thesis explores geometric properties of a particular class of groups, termed Houghton's groups, introduced in [13]. Roughly speaking, Houghton's group $\mathcal{H}_n$ ($n \in \mathbb{N}$) is the group of permutations of n rays of discrete points which are eventual translations (each permutation acts as a translation along each ray outside a finite set). See Section 2.1 for details.

There are n canonical copies of $\mathcal{H}_{n-1}$ inside $\mathcal{H}_n$, and $\mathcal{H}_n$ is multi-ended with respect to each of them. The i^{th} subgroup, $1 \leq i \leq n$, is obtained by restricting to permutations which fix i^{th} ray pointwise. One of the main results of this thesis is to produce an action of $\mathcal{H}_n$ on a n-dimensional cubing X_n. Note that depending on subgroups which are taken into account, there are various cubings on which $\mathcal{H}_n$ acts. One feature of our cubing is that X_n encodes all of those subgroups $\mathcal{H}_{n-1}$ at once.

Theorem A. *For each integer $n \geq 1$, there exists a n-dimensional cubing X_n and a Morse function $h : X_n \to \mathbb{R}_{\geq 0}$ such that $\mathcal{H}_n$ acts on X_n properly (but not cocompactly) by height-preserving semi-simple isometries. Furthermore, for each $r \in \mathbb{R}_{\geq 0}$ the action of $\mathcal{H}_n$ restricted to the level set $h^{-1}(r)$ is cocompact.*

An additional feature of X_n is that it comes equipped with a height function (Morse function) to the non-negative reals. The group $\mathcal{H}_n$ acts as a height-preserving fashion, and the quotient of any level set by $\mathcal{H}_n$ is cocompact. As an application we recover Brown's results for finiteness properties of $\mathcal{H}_n$.

Corollary B. *For $n \geq 2$, $\mathcal{H}_n$ is of type FP_{n-1} but not FP_n, it is finitely presented for $n \geq 3$.*

Knowing that $\mathcal{H}_n$ is finitely presented for $n \geq 3$ prompts a natural question. What are explicit presentations for the $\mathcal{H}_n$? In [15], Johnson answered this for $\mathcal{H}_3$. Another main result of this thesis provides explicit presentations for all $\mathcal{H}_n$ ($n \geq 3$).

Theorem C. *For $n \geq 3$, $\mathcal{H}_n$ is generated by $g_1, \cdots, g_{n-1}, \alpha$ with relators*

$$\alpha^2 = 1,\ (\alpha\alpha^{g_1})^3 = 1,\ [\alpha, \alpha^{g_1^2}] = 1, \alpha = [g_i, g_j],\ \ \alpha^{g_i^{-1}} = \alpha^{g_i^{-1}} \text{ for } 1 \leq i \neq j \leq n-1.$$

Finally we determine bounds for the Dehn functions of $\mathcal{H}_n$. We give a formal definition of Dehn functions in Chapter 4. Intuitively, given a finite presentation $P = \langle A | R \rangle$ for a group G, and given a word w representing the identity in G with $|w| \leq x$, the Dehn function of a presentation P, $\delta_P(x)$, measures the least upper bound on the number of relations in term of x, which one must apply to check

$w = 1$. Although the function $\delta_P(x)$ depends on the presentation, the growth type of this function is independent of choice of a finite presentation for G ([3]). The *Dehn function* of a finitely presented group G is defined to be the growth type of $\delta_P(x)$. See Section 4.1 for details. An isoperimetric function for a group is an upper bound of the Dehn function. In Chapter 4 we establish isoperimetric inequalities for $\mathcal{H}_n$ for $n \geq 3$.

Theorem D. *For any $n \geq 3$ the Dehn function $\delta_{\mathcal{H}_n}(x)$ of $\mathcal{H}_n$ satisfies*

$$\delta_{\mathcal{H}_n}(x) \preccurlyeq e^x.$$

Chapter 2

Houghton's Groups $\mathcal{H}_n$

2.1 Definition of $\mathcal{H}_n$

Fix an integer $n \geq 1$. Let $\mathbb{N}$ be the positive integers. For each k, $1 \leq k \leq n$, let

$$R_k = \{me^{i\theta} : m \in \mathbb{N},\, \theta = \pi/2 + 2\pi(k-1)/n\} \subset \mathbb{C}.$$

and let $Y_n = \bigcup_{k=1}^{n} R_k$ be the disjoint union of n copies of $\mathbb{N}$, each arranged along a ray emanating from the origin in the plane. We shall use the notation $\{1, \cdots, n\} \times \mathbb{N}$ for Y_n, letting (k, p) denote the point of R_k with distance p from the origin. A bijection $g : Y_n \to Y_n$ is called an *eventual translation* if it acts as a translation on each R_k outside a finite set. More precisely g is an eventual translation if the following holds:

- $\star$ There is an n-tuple $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ and a finite set $K_g \subset Y_n$ such that $(k, p) \cdot g = (k, p + m_k)$ for all $(k, p) \in Y_n - K_g$.

Definition 2.1. For an integer $n \geq 1$, Houghton's group $\mathcal{H}_n$ is defined to be the group of all eventual translations of Y_n.

As indicated in the definition $\star$, Houghton's group $\mathcal{H}_n$ acts on Y_n on the right; thus gh denotes g followed by h for $g, h \in \mathcal{H}_n$. For notational convenience we denote

g^{-1} by $\overline{g}$.

Let g_i be the translation on the ray of $R_1 \cup R_{i+1}$, from R_1 to R_{i+1} by 1 for $1 \le i \le n-1$. More precisely, g_i is defined by

$$(j,p)g_i = \begin{cases} (1, p-1) & \text{if } j = 1 \text{ and } p \ge 2, \\ (i+1, 1) & \text{if } (j,p) = (1,1) \\ (i+1, p+1) & \text{if } j = i+1, \\ (j,p) & \text{otherwise.} \end{cases} \tag{2.1}$$

Figure 2.1 illustrates some examples of elements of $\mathcal{H}_n$, where points which do not involve arrows are meant to be fixed, and points of each finite set K are indicated by circles. Finite sets K_{g_i} and K_{g_j} are singleton sets. After simple computation, one can check that the commutator of g_i and g_j $(i \neq j)$ is the transposition exchanging $(1,1)$ and $(1,2)$. The last element g is rather generic and K_g consists of eight points (circles).

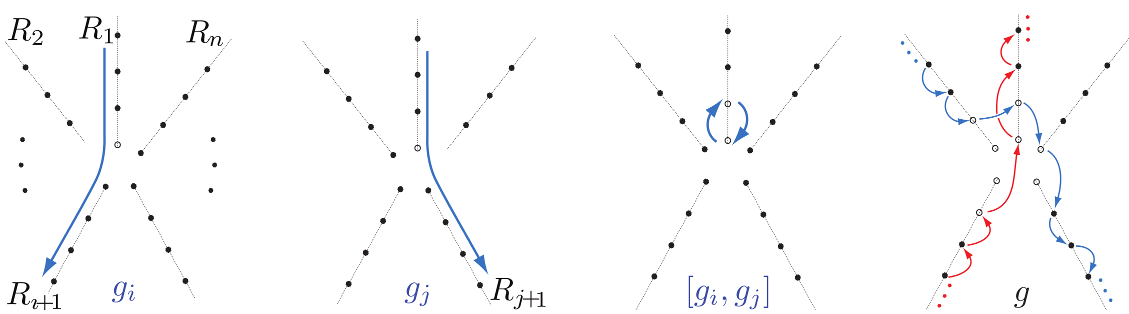

Figure 2.1: Some elements of $\mathcal{H}_n$

2.2 Abelianization of $\mathcal{H}_n$

Assigning an n-tuple $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ to each $g \in \mathcal{H}_n$ defines a homomorphism $\varphi : \mathcal{H}_n \to \mathbb{Z}^n$. Let $\Sigma_{n,\infty}$ denote the infinite symmetric group consisting of all permutations of Y_n with finite support.

Lemma 2.2. *For $n \in \mathbb{N}$, $\Sigma_{n,\infty} \leq \mathcal{H}_n$.*

Proof. Every element $f \in \Sigma_{n,\infty}$ has a finite support. This means there exists a finite set $K \subset Y_n$ such that f acts on each $R_i \subset Y_n$ as the identity outside K. So $f \in \mathcal{H}_n$. □

Lemma 2.3. *For $n \geq 3$, $ker\varphi = \Sigma_{n,\infty} = [\mathcal{H}_n, \mathcal{H}_n]$.*

Proof. Fix $n \geq 3$. We show the equalities by verifying a chain of inclusions. Each element $g \in ker\varphi$ acts on Y_n as the identity outside a finite set $K \subset Y_n$. This means that g is a permutation on K, and so $g \in \Sigma_{n,\infty}$. The claim in the proof of Lemma 2.7 implies that every element with finite support is a product of conjugations of $[g_i, g_j]$ $(i \neq j)$. Suppose $g \in [\mathcal{H}_n, \mathcal{H}_n]$. Lemma 2.7 says $\mathcal{H}_n$ is generated by $g_1, \cdots, g_{n-1}$ if $n \geq 3$. Take an expression for g in letters $g_1, \cdots, g_{n-1}$. Since $g \in [\mathcal{H}_n, \mathcal{H}_n]$, the sum of powers of g_i's in this expression is 0 for each $i = 1, \cdots, n-1$. So $g \in ker\varphi$. □

In Section 2.3, we shall see that $\mathcal{H}_n$ is generated by $g_1, \cdots, g_{n-1}$ defined in equation (2.1). Note that $\varphi(g_i) \in \mathbb{Z}^n$ has only two nonzero values -1, 1, and

$$\varphi(g_i) = (-1, 0, \cdots, 0, 1, 0, \cdots, 0)$$

where 1 occurs in $(i+1)^{th}$ component. The image of $\mathcal{H}_n$ is generated by those elements and $\varphi(\mathcal{H}_n) = \{(m_1, \cdots, m_n) | \sum m_i = 0\}$ is free abelian of rank $n - 1$.

Corollary 2.4. *For $n \geq 3$, the ablelianization of $\mathcal{H}_n$ is given by the following short exact sequence.*

$$1 \to [\mathcal{H}_n, \mathcal{H}_n] \to \mathcal{H}_n \xrightarrow{\varphi} \mathbb{Z}^{n-1} \to 1. \tag{2.2}$$

The above result was originally shown by C. H. Houghton in [13]. Note that the group $\mathcal{H}_n$ has the property that the rank of the abelianization is one less than the number of rays of the space Y_n (namely, the number of ends of Y_n with respect to the action) on which $\mathcal{H}_n$ acts transitively. This was the main reason that Houghton introduced and studied a family of groups $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ in the same paper.

Remark 2.5. For $n \geq 3$, by replacing the commutator subgroup by $\Sigma_{n,\infty}$ in the short exact sequence (2.2), we have the following short exact sequence.

$$1 \to \Sigma_{n,\infty} \to \mathcal{H}_n \overset{\varphi}{\to} \mathbb{Z}^{n-1} \to 1, \tag{2.3}$$

The embedding $\Sigma_{n,\infty} \hookrightarrow \mathcal{H}_n$ is crucial for our study in two directions. Firstly, it gives rise to finite presentation for $\mathcal{H}_n$ for $n \geq 3$ (Section 2.3). Secondly, this inclusion is used to construct exponential isoperimetric inequalities for $\mathcal{H}_n$ $n \geq 3$ (Section 4.2).

We suppress n in $\Sigma_{n,\infty}$ when the underlying set Y_n is clear from the context.

2.3 Finite Presentations for $\mathcal{H}_n$ $(n \geq 3)$

Note that every element of $\mathcal{H}_1$ acts on $Y_1 = \mathbb{N}$ by the identity outside a finite set. If the element acted as a non-trivial translation outside of a finite set, then it would fail to be a bijection of Y_1. So $\mathcal{H}_1 \subset \Sigma_{1,\infty}$. The converse inclusion is obvious, therefore $\mathcal{H}_1$ is the infinite symmetric group $\Sigma_{1,\infty}$ of all permutations of $\mathbb{N}$ with finite support. By considering elements with successively larger supports, one can argue that $\Sigma_{1,\infty}$ is not finitely generated. It is known [7] that $\mathcal{H}_2$ is finitely generated but not finitely presented, and that $\mathcal{H}_n$ is finitely presented for $n \geq 3$. Brown showed more than just finite presentedenss, and in Section 3.5, we will give a precise statement and a new proof of Brown's result about the finiteness properties of $\mathcal{H}_n$ (Corollary B). In this section we will provide explicit finite presentations for $\mathcal{H}_n$ $(n \geq 3)$.

Finite generating sets for $\mathcal{H}_n$ $(n \geq 2)$.

We begin by proving that $\mathcal{H}_2$ is generated by two elements. Many of the techniques in this argument extend to the $\mathcal{H}_n$ for $n \geq 3$. Consider the two elements $g_1, \beta \in \mathcal{H}_2$ where g_1 is defined by equation (2.1) and β is the transposition exchanging $(1,1)$ and $(2,1)$ as depicted in Figure 2.2.

Figure 2.2: $\mathcal{H}_2$ is generated by g_1 and β

Suppose $g \in \mathcal{H}_2$. Since $\varphi(g_1)$ generates the image $\varphi(\mathcal{H}_2) \cong \mathbb{Z}$ there exists $k \in \mathbb{Z}$ such that the given g and g_1^k agree on $Y_2 - K$ for some finite set $K \subset Y_2$. So we have

$$\overline{g}_1^k g = f \text{ or } g = g_1^k f$$

where $f \in Perm(K)$. One can take suitable conjugates $\overline{g}_1^{k'} \beta g_1^{k'}$ to express transpositions exchanging two consecutive points of Y_2. For example, $\overline{g}_1^3 \beta g_1^3$ is the transposition swapping $(2,3)$ and $(2,4)$. Since any permutation of a finite set K can be written as a product of those transpositions, f can be written as a product of conjugations of β. So $g = g_1^k f \in \mathcal{H}_2$.

Remark 2.6. The conjugation $\overline{h}gh$ is denoted by g^h. Note that $(i,p)g = (j,q)$ is equivalent to $(i,p)hg^h = (j,q)h$. The following observation is quite elementary but useful: If β is a transposition exchanging two points (i,p) and (j,q) then β^h is the transposition exchanging $(i,p)h$ and $(j,q)h$.

We have seen that a translation and a transposition are sufficient to generate all of $\mathcal{H}_2$. Moreover, we have seen from the third example of Figure 2.1 that $[g_i, g_j]$ is a transposition on Y_n for $1 \leq i \neq j \leq n-1$. So the following lemma should not be a surprise.

Lemma 2.7. *$\mathcal{H}_n$ is generated by $g_1, \cdots, g_{n-1}$ for $n \geq 3$.*

Proof. Let $g \in \mathcal{H}_n$ and $\varphi(g) = (m_1, \cdots, m_n)$. Note that $m_1 = -(m_2 + \cdots + m_n)$. Consider the element $g' \in \mathcal{H}_n$ given by

$$g' = g_1^{m_2} g_2^{m_3} \cdots g_{n-1}^{m_n}.$$

By construction, $\varphi(g') = \varphi(g)$, and so two element g' and g agree on $Y_n - K$ for some finite set K. This means there exists $f \in Perm(K)$ such that $g = g'f$. So it suffices to show that every finite permutation can be written as a product of the $g_i^{\pm 1}$, $i = 1, \cdots, n-1$.

Claim: Any transposition of Y_n is a conjugation of $[g_1, g_2] = g_1 g_2 \overline{g}_1 \overline{g}_2$ by a product of g_i's. First we express transpositions exchanging points of R_1 in this manner. We already saw that $\alpha = [g_1, g_2]$ is the transposition exchanging $(1, 1)$ and $(1, 2)$.

Consider the following elements $h_1, h_2 \in \mathcal{H}_n$:

$$h_1 = \alpha^{g_1 \overline{g}_2^{(p-2)} \overline{g}_1} \text{ and } h_2 = \alpha^{g_1 \overline{g}_2^{(q-2)} \overline{g}_1}$$

where $p, q \geq 2$ are integers. By the observation in Remark 2.6, h_1 is the transposition exchanging $(1, 1)$ and $(1, p)$. By the same reason, one sees h_2 is the transposition swapping $(1, 1)$ and $(1, q)$. Therefore the permutation exchanging $(1, p)$ and $(1, q)$ can be expresses by the conjugation $h_1^{h_2}$. This allows one to express transpositions of R_{i+1} for all i by conjugations $h_1^{h_2 g_i^m}$ for some $m \in \mathbb{N}$.

Next we show that transpositions exchanging points of different rays can be expressed in a similar fashion. Set $2 \leq i \neq j \leq n$ and $q, r \in \mathbb{N}$. The permutations h_3, h_4 given by

$$h_3 = \alpha^{g_j \, g_i^q \, \overline{g}_j} \text{ and } h_4 = \alpha^{g_i \, g_j^r \, \overline{g}_i}$$

are the transpositions swapping $(1, 1)$ and (i, q) and swapping $(1, 1)$ and (j, r) re-

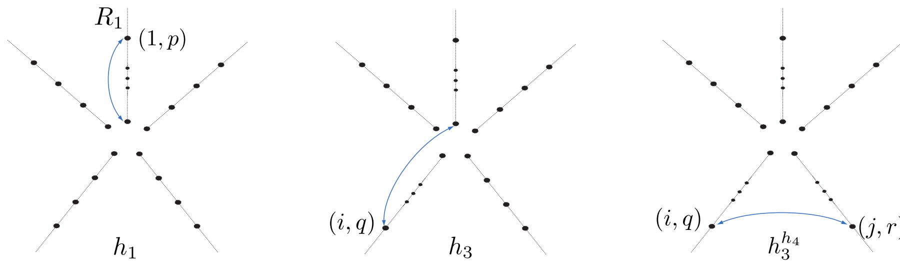

Figure 2.3: Transpositions are conjugations of $[g_i, g_j]$

spectively. Thus the conjugation $h_1^{h_3}$ represents the transposition exchanging $(1,p)$ and (i,q). Similarly the conjugation $h_3^{h_4}$ represents the transposition swapping (i,q) and (j,r). Figure 2.3 illustrates various transpositions of Y_n given by conjugations of α. Our claim follows since i, j, p, q and r are arbitrary. So we are done because any finite permutation $f \in Perm(K)$ above can be written as a product of transpositions. $\square$

Finite presentation for $\mathcal{H}_3$.

In [15], Johnson provided a finite presentation for $\mathcal{H}_3$.

Theorem 2.8. *The Houghton's group $\mathcal{H}_3$ is isomorphic to H_3 presented by*

$$H_3 = \langle g_1, g_2, \alpha \,|\, r_1, \cdots, r_5 \rangle, \tag{2.4}$$

$r_1 \;:\; \alpha^2 = 1,$

$r_2 \;:\; (\alpha\, \alpha^{\bar{g}_1})^3 = 1\,,$

$r_3 \;:\; [\alpha, \alpha^{\bar{g}_1^2}] = 1,$

$r_4 \;:\; [g_1, g_2] = \alpha,$

$r_5 \;:\; \alpha^{\bar{g}_1} = \alpha^{\bar{g}_2}.$

Note that there is a map ψ from an abstract group H_3 to $\mathcal{H}_3$ such that $\psi(g_i)$ is the translation g_i defined in equation (2.1) $(i = 1, 2)$ and $\psi(\alpha)$ is the transposition $\alpha = [g_1, g_2]$. Observe that relations r_1, r_2 and r_3 are reminiscent relations of Coxeter systems for finite symmetric groups. It is easy to check that $\mathcal{H}_3$ satisfies those five relations. Johnson showed that the group H_3 fits into the short exact sequence of (2.3) with the normal subgroup generated by α, and that $\psi : H_3 \to \mathcal{H}_3$ induces isomorphisms between the normal subgroups and the quotients. So the following diagram commute. The Five Lemma completes the proof.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle\langle \alpha \rangle\rangle & \longrightarrow & H_3 & \longrightarrow & H_3/\langle\langle \alpha \rangle\rangle & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & [\mathcal{H}_3, \mathcal{H}_3] & \longrightarrow & \mathcal{H}_3 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 1.
\end{array}$$

Finite presentations for $\mathcal{H}_n$ ($n \geq 3$).

We extend the previous argument to find finite presentations for $\mathcal{H}_n$ for all $n \geq 3$. The first step is to establish appropriate presentations for symmetric groups on finite balls of Y_n by applying Tietze transformation to the Coxeter systems for finite symmetric groups.

Fix positive integers $n, r \in \mathbb{N}$. Let $B_{n,r}$ denote the ball of Y_n centered at the origin with radius r, i.e., $B_{n,r} = \{(i,p) \in Y_n : 1 \leq i \leq n, 1 \leq p \leq r\}$. The ball $B_{n,r}$ contains nr points which can be identified to the points of $\{1, 2, \cdots, r, \cdots, 2r, \cdots, nr\} \subset \mathbb{N}$ via $\chi : B_{n,r} \to \mathbb{N}$ defined by

$$\chi(i,p) = (i-1)r + p.$$

Recall the following Coxeter system for the symmetric group S_{nr} on $\{1, 2, \cdots, nr\}$

$$S_{nr} = \langle \sigma_1, \cdots, \sigma_{nr-1} \,|\, \sigma_k^2,\ (\sigma_k\, \sigma_{k+1})^3,\ [\sigma_k, \sigma_{k'}] \text{ for } |k - k'| \geq 2 \rangle \tag{2.5}$$

Remark 2.9. One can interpret the above three types of relators respectively as follows;

- generators are involutions,

- two generators with overlapping support satisfy braid relations,
- two generators commute if they have disjoint supports.

Note that every collection of transpositions of S_{nr} satisfies the three types of relations. We shall see in Theorem 2.10 that a 'reasonable' set of transpositions satisfying the above relations provides a presentation for the symmetric group S_{nr}.

The arrangement of Y_n in the plane is different than the arrangement of nr points $\{1, 2, \cdots, nr\}$ in $\mathbb{N}$. In particular, the adjacency is different. By a pair of two *adjacent* points in Y_n, we mean either a pair of two consecutive points in a ray of Y_n or a pair of $(1, 1)$ and $(j+1, 1)$ for $1 \leq j \leq n-1$. We aim to produce a new presentation for the symmetric group $\Sigma_{n,r}$ on $B_{n,r}$ with a generating set which consists of the transpositions exchanging all pairs of adjacent points in Y_n. More precisely we need the following $nr-1$ transpositions

α_p^i, exchanging (i, p) and $(i, p+1)$ for $1 \leq i \leq n$ and $1 \leq p \leq r-1$

α_0^{j+1}, exchanging $(1, 1)$ and $(j+1, 1)$ for $1 \leq j \leq n-1$.

For an element $\sigma \in S_{nr}$ consider the element of $Perm(B_{n,r})$ given by

$$(\sigma)^{\chi^{-1}} = \chi(\sigma)\chi^{-1}$$

which stands for the composition $B_{n,r} \xrightarrow{\chi} \{1, \cdots, nr\} \xrightarrow{\sigma} \{1, \cdots, nr\} \xrightarrow{\chi^{-1}} B_{n,r}$. Let us denote the induced element by $\chi^*(\sigma)$. New generators of the first type are defined by χ^* as follows.

$$\alpha_p^i := \chi^*(\sigma_{(i-1)r+p}) \tag{2.6}$$

for $1 \leq i \leq n$ and $1 \leq p \leq r-1$. Note χ^* defines a bijection

$$\{\sigma_1, \cdots, \widehat{\sigma_r}, \cdots, \widehat{\sigma_{2r}}, \cdots, \widehat{\sigma_{(n-1)r}}, \cdots, \sigma_{nr-1}\} \leftrightarrow \{\alpha_p^i \mid 1 \leq i \leq n,\ 1 \leq p \leq r-1\},$$

($\widehat{\ }$denotes deletion).

Replace $\sigma_{(i-1)r+p}$ by α_p^i in the Coxeter system (2.5), $1 \le i \le n$ and $1 \le p \le r-1$. Then (2.5) becomes

$$S_{nr} = \langle \alpha_p^i,\ \sigma_{jr} \,|\, R \rangle \ \ 1 \le i \le n\ , 1 \le p \le r-1, \text{ and } 1 \le j \le n-1, \tag{2.7}$$

where R consisting of

R_1: $(\alpha_p^i)^2$, $(\sigma_{jr})^2$, $\forall\, i, p, j$

R_2: $[\alpha_p^i, \alpha_q^i]$, $\forall\, i$, $|p - q| \ge 2$

R_3: $[\alpha_p^i, \alpha_p^{i'}]$, $\forall\, p, q$, $i \ne i'$

R_4: $[\alpha_p^i, \sigma_{jr}]$, $\forall\, j$, $(i-1)r + p \ne jr \pm 1$

R_5: $[\sigma_{jr}, \sigma_{j'r}]$, $j \ne j'$

R_6: $(\alpha_p^i\, \alpha_{p+1}^i)^3$, $\forall\, i$, $1 \le p \le r-2$

R_7: $(\alpha_p^i\, \sigma_{jr})^3$, $\forall\, j$, $(i-1)r + p = jr \pm 1$.

Note that the difference in presentations (2.5) and (2.7) is just notation and the set R of relations exhibits the same idea of Remark 2.9. We also want to replace σ_{jr} by α_0^{j+1} for $j = 1, \cdots, n-1$ as follows. For each j, α_0^{j+1} corresponds, under χ^*, to the transposition $(1\ jr{+}1)$ of S_{nr}. Observe that the transposition $(1\ jr{+}1)$ of S_{nr} can be expressed as $(\sigma_{jr})^{\overline{w_{jr-1}}}$ where $w_k = \sigma_1\sigma_2 \cdots \sigma_k$.

As in (2.6), define new generators of the second type by

$$\alpha_0^{j+1} := \chi^*((\sigma_{jr})^{\overline{w_{jr-1}}}) \text{ for } j = 1, \cdots, n-1. \tag{2.8}$$

Figure 2.4 illustrates how one transforms the generating set of S_{nr} to a new generating set.

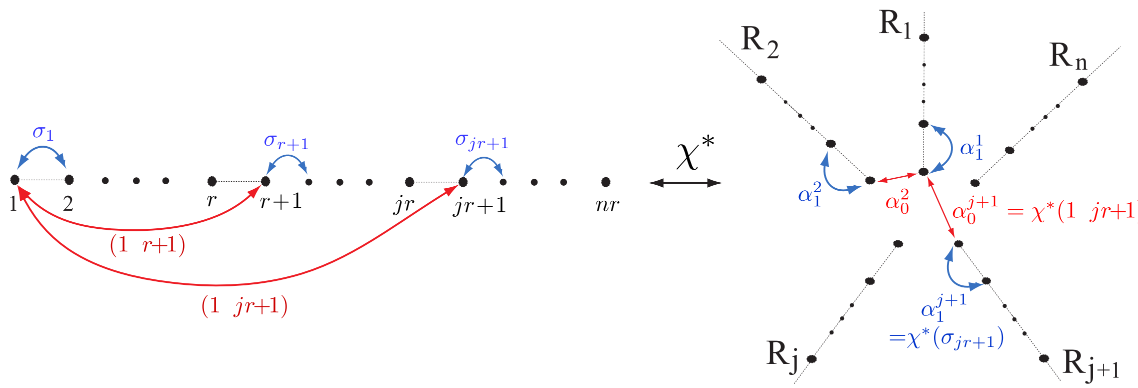

Figure 2.4: Correspondence between two generating sets via χ^*

Consider the group $\Sigma_{n,r}$ presented by

$$\Sigma_{n,r} = \langle \alpha_p^i,\, \alpha_0^{j+1} \,|\, R \,\rangle \;\; 1 \le i \le n \,, 1 \le p \le r-1, \text{ and } 1 \le j \le n-1, \tag{2.9}$$

where R' consists of

R_1': $(\alpha_p^i)^2$, $(\alpha_0^j)^2$, $\forall\, i, p, j$

R_2': $[\alpha_p^i, \alpha_q^i]$, $\forall\, i$, $|p-q| \ge 2$

R_3': $[\alpha_p^i, \alpha_p^{i'}]$, $\forall\, p, q$, $i \ne i'$

R_4': $[\alpha_p^i, \alpha_0^{j+1}]$, $\forall\, j$, $\alpha_p^i \ne \alpha_1^1$, α_1^{j+1}

R_5': $(\alpha_p^i\, \alpha_{p+1}^i)^3$, $\forall\, i$, $1 \le p \le r-2$

R_6': $(\alpha_1^1\, \alpha_0^{j+1})^3$, $(\alpha_1^{j+1}\, \alpha_0^j)^3$.

Note that R and R' share the same idea described in Remark 2.9. Expecting R' to replace the original relators of R is reasonable.

Theorem 2.10. *With the above definition, $\Sigma_{n,r} \cong S_{nr}$ for all integers n, $r \ge 1$.*

Proof. First we show $\langle\langle R \rangle\rangle \le \langle\langle R' \rangle\rangle$. It is clear $\langle\langle R_1 \rangle\rangle \le \langle\langle R_1' \rangle\rangle$. Basic idea is to use appropriate conjugation relations between σ_{jr} and α_0^{j+1} given by (2.8).

In order to proceed we need the following identities for $j = 1, 2, \cdots, n-1$

$$[\sigma_{jr}, \alpha_p^i] = [\alpha_0^{j+1}, \alpha_p^i]^{w_{jr-1}} \text{ if } (i-1)r + p \geq jr + 2, \tag{2.10}$$

$$[\sigma_{jr}, \alpha_p^i] = [\alpha_0^{j+1}, \alpha_{p+1}^i]^{w_{jr-1}} \text{ if } (i-1)r + p \leq jr - 2, \tag{2.11}$$

$$[\sigma_{jr}, \sigma_{j'r}] = [\alpha_0^{j+1}, \alpha_1^{j'+1}]^{w_{jr-1}} \text{ if } j' < j + 1, \tag{2.12}$$

$$(\alpha_1^{j+1}\sigma_{jr})^3 = ((\alpha_1^{j+1}\alpha_0^{j+1})^3)^{w_{jr-1}} \tag{2.13}$$

$$(\alpha_{r-1}^{j}\sigma_{jr})^3 = ((\alpha_1^1\alpha_0^{j+1})^3)^{w_{jr-1}w_{jr-2}} \tag{2.14}$$

$$(\alpha_1^1)^{w_{jr-1}w_{jr-2}} = \alpha_{r-1}^j. \tag{2.15}$$

First we apply an induction argument on j to verify (2.10), (2.11) and (2.13). The induction assumption for (2.10) together with relators of R_3' yields $\alpha_p^i = (\alpha_p^i)^{w_{jr-1}}$ if $(i-1)r + p \geq jr + 2$. So (2.10) follows from (2.8). Similarly one can verify (2.11) by using (2.10) together with R_2' and R_5'. The identity (2.13) follows from (2.10), R_2' and R_3'.

For (2.12), (2.14) and (2.15) we apply simultaneous induction on j together with (2.10), (2.11) and (2.13). The base case of (2.12) holds trivially. The relation (2.15) is clear when $j = 1$, which establishes the base case of (2.14). Suppose (2.12), (2.14) and (2.15) hold for $j = k$. Observe that

$$\begin{aligned}
&w_{(k+1)r-1}w_{(k+1)r-2}\\
=&(\sigma_1\cdots\sigma_{kr}\sigma_{kr+1}\cdots\sigma_{(k+1)r-1})(\sigma_1\sigma_2\cdots\sigma_{kr}\sigma_{kr+1}\cdots\sigma_{(k+1)r-2})\\
=&(\sigma_1\cdots\sigma_{kr})(\sigma_1\sigma_2\cdots\sigma_{kr-1})(\sigma_{kr+1}\cdots\sigma_{(k+1)r-1})\\
&(\sigma_{kr}\sigma_{kr+1}\cdots\sigma_{(k+1)r-2}) && (2.12),\ R_2',\ R_3'\\
=&(\sigma_1\cdots\sigma_{kr-1})(\sigma_1\sigma_2\cdots\sigma_{kr-2}\sigma_{kr}\sigma_{kr-1})(\sigma_{kr+1}\cdots\sigma_{(k+1)r-1})\\
&(\sigma_{kr}\sigma_{kr+1}\cdots\sigma_{(k+1)r-2}) && (2.11),\ (2.12)
\end{aligned}$$

Thus we have

$$\begin{aligned}
(\alpha_1^1)^{w_{(k+1)r-1}w_{(k+1)r-2}} &= (\alpha_{r-1}^k)^{(\sigma_{kr}\sigma_{kr-1}\sigma_{kr+1}\cdots\sigma_{(k+1)r-1})(\sigma_{kr}\sigma_{kr+1}\cdots\sigma_{(k+1)r-2})} && (2.14)\\
&= (\sigma_{kr})^{(\sigma_{kr+1}\cdots\sigma_{(k+1)r-1})(\sigma_{kr}\sigma_{kr+1}\cdots\sigma_{(k+1)r-2})} && (2.13)\\
&= (\sigma_{kr})^{(\sigma_{kr+1}\sigma_{kr}\sigma_{kr+2}\cdots\sigma_{(k+1)r-1})(\sigma_{kr+1}\cdots\sigma_{(k+1)r-2})} && (2.13)\\
&= (\alpha_1^{k+1})^{(\sigma_{kr+2}\cdots\sigma_{(k+1)r-1})(\sigma_{kr+1}\cdots\sigma_{(k+1)r-2})} && (2.13)\\
&= \cdots = \alpha_{r-1}^{k+1} && R_2', R_5'
\end{aligned}$$

So we have verified the statement (2.14) with $j = k + 1$ which imply (2.13) for $j = k + 1$. For (2.12) we need to check $(\alpha_1^{j'+1})^{w_{(k+1)r-1}} = \sigma_{j'r}$ if $j' < k + 2$;

$$\begin{aligned}
(\alpha_1^{j'+1})^{w_{(k+1)r-1}} &= (\alpha_1^{j'+1})^{\sigma_1\sigma_2\cdots\sigma_{j'r}\sigma_{j'r+1}\cdots\sigma_{(k+1)r-1}} && \text{definition of } w_{(k+1)r-1}\\
&= (\alpha_1^{j'+1})^{\sigma_{j'r}\sigma_{j'r+1}\cdots\sigma_{(k+1)r-1}} && (2.10),\ R_3'\\
&= (\sigma_{(j'+1)r})^{\sigma_{j'r+2}\cdots\sigma_{(k+1)r-1}} && (2.10),\ (2.13)\\
&= \cdots = \sigma_{(j'+1)r} && (2.10).
\end{aligned}$$

Now we see that identities (2.10)-(2.14) imply $\langle\langle R\rangle\rangle \leq \langle\langle R'\rangle\rangle$. On the other hand, Remark 2.9 guarantees $\langle\langle R' \rangle\rangle \leq \langle\langle R \rangle\rangle$. So we have $\langle\langle R' \rangle\rangle = \langle\langle R \rangle\rangle$

The next step is to apply Tietze transformations to the presentation (2.7) which is identical to the Coxeter presentation (2.5) up to notation for generators. First adjoin extra letters α_0^{j+1}'s to get the following presentation

$$\langle \alpha_p^i,\, \sigma_{jr},\, \alpha_0^{j+1} \,|\, R \cup R_\beta \,\rangle \tag{2.16}$$

where R_β consists of words of the form

$$\overline{(\alpha_0^{j+1})}(\sigma_{jr})^{\overline{w_{jr-1}}}$$

corresponding to (2.8). Then replace $R \cup R_\beta$ by $R' \cup R_\beta$ in (2.16). This is legitimate since two normal closures $\langle\langle R \rangle\rangle$ and $\langle\langle R' \rangle\rangle$ are the same. Finally we remove σ_{jr}'s from the generating set and R_β from the relators simultaneously to get the presentation $\Sigma_{n,r}$. In all, $\Sigma_{n,r}$ and S_{nr} are isomorphic. $\square$

Let n be a positive integer. From Theorem 2.10 we have a sequence of symmetric groups $\Sigma_{n,r} = \langle\, A_{n,r} \,|\, Q_{n,r} \,\rangle$ on $B_{n,r} \subset Y_n$, where $A_{n,r}$ and $Q_{n,r}$ denote the generators and relators in (2.9) respectively. The natural inclusion $B_{n,r} \hookrightarrow B_{n,r+1}$ induces an inclusion $i_r : \Sigma_{n,r} \hookrightarrow \Sigma_{n,r+1}$ such that

$$A_{n,r} \hookrightarrow A_{n,r+1} \text{ and } Q_{n,r} \hookrightarrow Q_{n,r+1}$$

for $r \in \mathbb{N}$. The direct limit $\varinjlim_r \Sigma_{n,r}$ with respect to inclusions i_r is nothing but the infinite symmetric group $\Sigma_{n,\infty}$ on Y_n consisting of permutations with finite support.

Lemma 2.11. *For each $n \in \mathbb{N}$, $\Sigma_{n,\infty} \cong \langle\, A_n \,|\, Q_n \,\rangle$, where $A_n = \cup_r A_{n,r}$ and $Q_n = \cup_r Q_{n,r}$.*

Proof. For each $k \in \mathbb{N}$ we have a commuting diagram

$$\begin{array}{ccc} \langle \cup_{r=1}^{k} A_{n,r} \,|\, \cup_{r=1}^{k} Q_{n,r} \rangle & \longrightarrow & \langle A_n \,|\, Q_n \rangle \\ \downarrow & & \downarrow \\ \Sigma_{n,k} & \longrightarrow & \Sigma_{n,\infty}. \end{array}$$

The isomorphism on the right follows from the fact that the left map is an isomorphism and that the two horizontal maps are inclusions. □

The previous Lemma implies that one can get a presentation for $\Sigma_{n,\infty}$ by allowing the subscript p to be any positive integer in (2.9). We keep the same notation R'_1-R'_6 to indicate relators of the same type in Q_n.

Recall elements $g_1, \cdots, g_{n-1}$ of $\mathcal{H}_n$ defined in (2.1), and it is easy to see that

$$[g_i, g_j] = \alpha_1^1$$

for $i \neq j$. Now consider the action of $g_1, \cdots, g_{n-1}$ on A_n.

$$(\alpha_p^1)^{g_k^{-m}} = \alpha_{p+m}^1, \ \ (\alpha_1^1)^{g_k} = \alpha_0^k, \ \text{ for } \ p, m \in \mathbb{N}, 1 \leq k \leq n-1$$

$$(\alpha_0^k)^{g_k} = \alpha_1^{k+1}, \ \text{for} \ 1 \leq k \leq n-1$$

$$(\alpha_0^k)^{g_{k'}^{-1}} = (\alpha_0^k)^{\alpha_1^1}, \ (\alpha_0^k)^{g_{k'}} = (\alpha_0^k)^{\alpha_0^{k'}}, \ \text{for} \ 1 \leq k \neq k' \leq n-1$$

$$(\alpha_p^i)^{g_{i-1}^m} = (\alpha_{p+m}^i), \ (\alpha_p^i)^{g_j^{-m}} = (\alpha_p^i)^{g_j^m} = (\alpha_p^i), \ \text{for} \ p, m \in \mathbb{N}, \ 2 \leq i \leq n \ \text{and} \ j \neq i-1$$

Let Q'_n be the relators corresponding to the above action. For $n \geq 3$, consider the group H_n defined by

$$H_n = \langle g_1, \cdots, g_{n-1}, A_n \,|\, Q_n, Q'_n, (\alpha_1^1)^{-1}[g_i, g_j] \rangle, \ 1 \leq i < j \leq n-1 \tag{2.17}$$

Lemma 2.12. *We have a short exact sequence for each n;*

$$1 \to \langle\, \mathcal{A}_n \,\rangle \to H_n \to \mathbb{Z}^{n-1} \to 1. \tag{2.18}$$

Proof. Note that the relations Q'_n ensure that $\langle A_n \rangle$ is a normal subgroup of H_n. Clearly the set of relations Q_n becomes trivial in the quotient $H_n/\langle A_n \rangle$. Likewise, the relations in Q'_n all become trivial in the quotient $H_n/\langle A_n \rangle$. The extra relator $(\alpha_1^1)^{-1}[g_i, g_j]$ implies that g_i and g_j commute in $H_n/\langle\, \mathcal{A}_n \,\rangle\,(i \neq j)$, so the quotient is the free abelian group generated by $g_1, \cdots, g_{n-1}$. □

Lemma 2.13. *For $n \geq 3$, $H_n \cong \mathcal{H}_n$.*

Proof. As in Johnson's argument for $n = 3$, there exists a homomorphism $\psi : H_n \to \mathcal{H}_n$ defined by

$$\psi(g_i) = g_i \text{ and } \psi(\alpha) = \alpha.$$

So we have a commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle\, \mathcal{A}_n \,\rangle & \longrightarrow & H_n & \longrightarrow & H_n/\langle\, \mathcal{A}_n \,\rangle & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & [\mathcal{H}_n, \mathcal{H}_n] & \longrightarrow & \mathcal{H}_n & \longrightarrow & \mathbb{Z}^{n-1} & \longrightarrow & 1
\end{array}$$

The existence of the top row and the isomorphism $H_n/\langle A_n \rangle \simeq \mathbb{Z}^{n-1}$ are the result of Lemma 2.12. The bottom row is given by Corollary 2.4, and the isomorphism $\langle A_n \rangle \simeq [H_n, H_n]$ follows from Lemma 2.3 and Lemma 2.11. As the outer four vertical maps are isomorphisms, so is the inner one by the Five lemma. □

Theorem 2.14. *For $n \geq 3$, H_n has a finite presentation*

$$H_n \cong \langle\, g_1, \cdots, g_{n-1}, \alpha \mid P_n \,\rangle \tag{2.19}$$

where P_n consists of

$$
\begin{aligned}
r'_1 &: \ \alpha^2=1,\\
r'_2 &: \ (\alpha\,\alpha^{g_1})^3 = 1,\\
r'_3 &: \ \alpha \sim \alpha^{\bar{g}_1^2},\\
r'_4 &: \ [g_i, g_j]=\alpha, \textit{ for } 1\le i<j \le n-1,\\
r'_5 &: \ \alpha^{\bar{g}_i} = \alpha^{\bar{g}_j}, \textit{ for } 1\le i<j \le n-1
\end{aligned}
$$

($\sim$ denotes commutation).

Proof. P_n already contains $(\alpha_1^1)^{-1}[g_i, g_j]$ for $1 \le i < j \le n-1$. We show $\langle\langle P_n\rangle\rangle$ contains $\langle\langle \mathcal{Q}'_n \cup \mathcal{Q}_n\rangle\rangle$ for all $n \ge 3$ by induction on n. The base case is established by Theorem 2.8. Assume H_n has the presentation as in 2.19 for $n \ge 3$. Obviously $g_1, \cdots, g_n$ and α generate $\mathcal{H}_{n+1}$. Consider the natural inclusion of $\iota : Y_n \to Y_{n+1}$ such that $\iota(R_i)$ is the i^{th} ray of Y_{n+1}. This induces an embedding $\iota : H_n \to H_{n+1}$, and ${\mathcal{Q}'}_{n+1}$ contains relations corresponding the action of g_n on α_0^n and α_p^{n+1}'s:

$$(\alpha_0^n)^{g_n} = \alpha_1^{n+1},\ (\alpha_0^n)^{\bar{g}_n} = \alpha_1^1,\ (\alpha_p^{n+1})^{g_n} = \alpha_{p+1}^{n+1} \tag{2.20}$$

for $p \in \mathbb{N}$. This implies $\langle\langle \iota(\mathcal{Q}_n)\rangle\rangle = \langle\langle \mathcal{Q}_{n+1}\rangle\rangle$ in H_{n+1}. So the normal closure of P_{n+1} contains $\langle\langle \mathcal{Q}_{n+1}\rangle\rangle$ by the induction assumption.

We must examine the action of $g_1, \cdots, g_n$ on $\mathcal{A}_n$'s:

$$(\alpha_p^1)^{\bar{g}_n} = (\alpha_p^1)^{\bar{g}_i},\ k \in \mathbb{N},\ 1 \le i \le n-1, \tag{2.21}$$

$$(\alpha_p^i)^{g_n} = \alpha_p^i,\ p \in \mathbb{N},\ 2 \le i \le n-1, \tag{2.22}$$

$$(\alpha_0^i)^{g_n} = (\alpha_0^i)^{\alpha_0^n},\ (\alpha_0^i)^{\bar{g}_n} = (\alpha_0^i)^{\alpha_1^1},\ 1 \le i \le n-1, \tag{2.23}$$

$$(\alpha_0^n)^{\bar{g}_i} = (\alpha_0^n)^{\alpha_1^1},\ (\alpha_0^n)^{g_i} = (\alpha_0^{n+1})^{\alpha_0^i},\ 1 \le i \le n-1, \tag{2.24}$$

$$(\alpha_p^{n+1})^{g_i} = \alpha_p^{n+1},\ k \in \mathbb{N},\ 1 \le i \le n-1, \tag{2.25}$$

together with (2.20) and $\iota(Q'_n)$. From Lemma 2.15 we see that P_{n+1} implies

$$\alpha \sim (\overline{g}_n)^{\overline{g}_i^2},\ \alpha \sim (\overline{g}_i)^{\overline{g}_n^2},\ 1 \leq i \leq n-1, \tag{2.26}$$

$$\alpha^{\overline{g}_i^k} = \alpha^{\overline{g}_n^k},\ \alpha \sim \alpha^{\overline{g}_i^{(k+1)}},\ k \in \mathbb{N},\ 1 \leq i \leq n-1 \tag{2.27}$$

We want to establish identities (2.21)-(2.25) from (2.26), (2.27) and P_{n+1}. The identity (2.21) is a part of (2.27). Observe that, by using (2.20), one can reduce (2.22) and (2.25) respectively, as follows

$$\alpha^{g_i^{k+1}} \sim g_n,\ \alpha^{g_n^{k+1}} \sim g_i,\ 2 \leq i \leq n-1,\ k \in \mathbb{N}, \tag{2.28}$$

One can verify (2.28) by inductions based on (2.26) as follows. The identity (2.26) provides the base cases. Assume $\alpha^{g_i^{k+1}} \sim g_n$ for $k \geq 1$. Then r_1 and r_2 of P_{n+1} imply

$$\alpha^{g_i^{k+2}} \sim \overline{g}_i g_n g_i = \overline{g}_i \alpha g_i g_n = \alpha^{g_i} g_n$$

On the other hand, $\alpha^{g_i^{k+2}} \sim \alpha^{g_i}$ by (2.27). Thus $\alpha^{g_i^{k+2}} \sim g_n$. An analogous argument verifies the second identity of (2.28).

Note that (2.23) and (2.24) are equivalent to

$$\alpha^{g_i} \sim \overline{g}_n \alpha,\ \alpha^{g_n} \sim \alpha g_i,\ 1 \leq i \leq n-1,$$

which are immediate consequence of (2.26).

In all, $\langle\langle P_{n+1} \rangle\rangle$ also contains $\langle\langle \mathcal{Q}'_{n+1} \rangle\rangle$. So P_{n+1} is enough for relators in H_{n+1}, and hence H_n has the presentation (2.19) for $n \geq 3$. □

Lemma 2.15. *P_n implies the following identities*

$$\alpha \sim (\overline{g}_{n-1})^{\overline{g}_i^2},\ \alpha \sim (\overline{g}_i)^{\overline{g}_{n-1}^2},\ 1 \leq i \leq n-2, \tag{2.29}$$

$$\alpha^{\overline{g}_i^k} = \alpha^{\overline{g}_{n-1}^k},\ \alpha \sim \alpha^{\overline{g}_i^{(k+1)}},\ k \in \mathbb{N},\ 1 \leq i \leq n-2 \tag{2.30}$$

Proof. From r_1, r_2 and r_5 we have

$$\alpha\alpha^{\overline{g}_i}\alpha = \alpha^{\overline{g}_i}\alpha\alpha^{\overline{g}_i} \Rightarrow \alpha^{\overline{g}_i\alpha} = \alpha^{g_i\alpha\overline{g}_i} \Rightarrow \alpha^{g_i\alpha} = \alpha^{\overline{g}_i\alpha g_i} = \alpha^{g_{n-1}^{-1}\alpha g_i} = \alpha^{g_i\overline{g}_{n-1}}.$$

Thus,

$$\alpha^{g_i} \sim \overline{g}_{n-1}\overline{\alpha} = g_i\overline{g}_{n-1}\overline{g}_i \Rightarrow \alpha \sim (\overline{g}_{n-1})^{\overline{g}_i^2}.$$

One obtains $\alpha \sim (\overline{g}_i)^{\overline{g}_{n-1}^2}$ by using an analogous argument. Next, to establish (2.30), we use simultaneous induction on k together with (2.29). Observe that r_4 and r_5 provide the base case $k = 1$ and that $\alpha^{\overline{g}_i^k} = \alpha^{\overline{g}_{n-1}^k}$ holds trivially when $k = 0$. Now assume

$$\alpha^{\overline{g}_i^{(k-2)}} = \alpha^{\overline{g}_{n-1}^{(k-2)}},\ \alpha^{\overline{g}_i^{(k-1)}} = \alpha^{\overline{g}_{n-1}^{(k-1)}},\ \alpha \sim \alpha^{\overline{g}_i^k} \tag{2.31}$$

for $k \geq 2$. From r_1 and (2.30),

$$\alpha^{\overline{g}_i^k} = \alpha^{\overline{g}_i^k\alpha} = \alpha^{\overline{g}_{n-1}^{(k-1)}\overline{g}_i\alpha} = \alpha^{\overline{g}_{n-1}^{(k-2)}\overline{g}_i\overline{g}_{n-1}} = \alpha^{\overline{g}_i^{(k-1)}\overline{g}_i\overline{g}_{n-1}} = \alpha^{\overline{g}_{n-1}^k}.$$

For the second assertion, note that r_4 and the first identity of (2.29) imply that $\alpha^{g_i^2}$ commute with both α and g_{n-1}^{-1}. So we have

$$\alpha^{g_i^2} \sim \alpha^{\overline{g}_{n-1}^{(k-1)}} = \alpha^{\overline{g}_i^{(k-1)}},$$

or equivalently,

$$\alpha \sim \alpha^{\overline{g}_i^{(k+1)}}.$$

□

As a consequence of Lemma 2.13 and Theorem 2.14 we have

Theorem C. *For $n \geq 3$, $\mathcal{H}_n$ has a finite presentation*

$$\mathcal{H}_n \cong \langle\, g_1, \cdots, g_{n-1}, \alpha \mid P_n \,\rangle \tag{2.32}$$

where P_n is the same as in the presentation (2.19) of Theorem 2.14.

2.4 Further properties of $\mathcal{H}_n$

Definition 2.16 (**Amenable and a-T-menable groups**)**.** A (discrete) group G is called *amenable* if it has a *Følner* sequence, i.e., there exists a sequence $\{F_i\}_{i\in\mathbb{N}}$ of finite subsets of G such that

$$\frac{|g \cdot F_i \triangle F_i|}{|F_i|}$$

tends to 0 for each $g \in G$.

A group G is called *a-T-menable* if there is a proper continuous affine action of G on a Hilbert space.

Recall the following known fact about amenable groups ([17]).

Theorem 2.17. *The class of amenable group contains abelian groups, and locally finite groups. An extension of an amenable group by another amenable group is again amenable.*

The above facts together with the short exact sequence (2.3) implies amenability of $\mathcal{H}_n$.

$$1 \to \Sigma_{n,\infty} \to \mathcal{H}_n \xrightarrow{\varphi} \mathbb{Z}^{n-1} \to 1,$$

Theorem 2.18. *$\mathcal{H}_n$ is amenable for all n.*

The class of amenable groups contains the class of a-T-menable groups.

Corollary 2.19. *$\mathcal{H}_n$ is a-T-amenable for all n. As a consequence, Houghton's groups satisfy the Baum-Connes conjecture.*

Chapter 3

A cubing for $\mathcal{H}_n$

3.1 Definition of the cubing X_n

In [7], Ken Brown constructed an infinite dimensional cell complex on which $\mathcal{H}_n$ acts by taking the geometric realization of a poset, which is defined using factorizations in a monoid $\mathcal{M}_n$ containing $\mathcal{H}_n$. In this section we modify his idea to construct an n-dimensional cubical complex X_n on which $\mathcal{H}_n$ acts. It turns out that X_n is a cubing for all $n \in \mathbb{N}$. We discuss this in detail in Section 3.3.

Two monoids $\mathcal{M}_n$ and $\mathcal{T}_n$ are important for the construction. Fix a positive integer $n \in \mathbb{N}$. Let $\mathcal{M}_n$ be the monoid of injective maps $Y_n \to Y_n$ which behave as eventual translations, i.e., each $\alpha \in \mathcal{M}_n$ satisfies

- $\star$ There is an n-tuple $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ and a finite set $K \subset Y_n$ such that $(k,p)\alpha = (k, p+m_k)$ for all $(k,p) \in Y_n - K$.

The group homomorphism $\varphi : \mathcal{H}_n \to \mathbb{Z}^n$ naturally extends to a monoid homomorphism $\varphi : \mathcal{M}_n \to \mathbb{Z}^n$ and $\varphi(\alpha) = (m_1, \cdots, m_n)$. A monoid homomorphism $h : \mathcal{M}_n \to \mathbb{Z}_{\geq 0}$ is defined by $h(\alpha) = \sum m_i$. This map h is called the *height function* on $\mathcal{M}_n$. For convenience let $S(\alpha)$ denote the discrete set $Y_n - (Y_n)\alpha$. One can readily check that $h(\alpha) = |S(\alpha)|$ and that $\mathcal{H}_n = h^{-1}(0)$.

Consider elements $t_1, \cdots, t_n$ of $\mathcal{M}_n$ where t_i is the *translation* by 1 on the i^{th} ray, i.e.,

$$(j,p)t_i = \begin{cases} (j, p+1) & \text{if } j = i \\ (j,p) & \text{if } j \neq i. \end{cases}$$

for all $p \in \mathbb{N}$. Let $\mathcal{T}_n \subset \mathcal{M}_n$ be the commutative submonoid generated by $t_1, \cdots, t_n$.

Figure 3.1 illustrates some examples of $\mathcal{M}_n$, as before, where points which do not involve arrows are meant to be fixed, points of each finite set K are indicated by circles. The left most one is a generator t_i of $\mathcal{T}_n$ which behaves as the translation on R_i by 1 and fixes $Y_n - R_i$ pointwise. The next example shows the commutativity of $\mathcal{T}_n$; $t_it_j = t_jt_i$. In the third figure, $\alpha = [g_1, g_2]$ as before. The last element g is rather generic one with $h(g) = 1$.

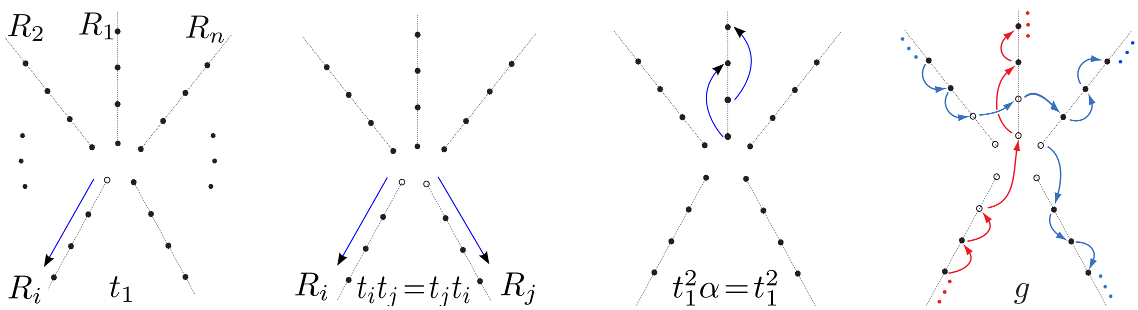

Figure 3.1: Some elements of $\mathcal{M}_n$

One crucial feature of $\mathcal{T}_n$ is that it is commutative. $\mathcal{T}_n$ acts on $\mathcal{M}_n$ by left multiplication (pre-composition). This action induces a relation $\geq$ on $\mathcal{M}_n$, i.e.,

$$\alpha_1 \geq \alpha_2 \text{ if } \alpha_1 = t\alpha_2 \text{ for some } t \in \mathcal{T}_n. \tag{3.1}$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}_n$

Proposition 3.1. *The relation $\geq$ on $\mathcal{M}_n$ is a partial order, i.e., it satisfies for all $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}_n$,*

- $\alpha_1 \geq \alpha_1$,

- *if* $\alpha_1 \geq \alpha_2$ *and* $\alpha_2 \geq \alpha_1$ *then* $\alpha_1 = \alpha_2$,

- *if* $\alpha_1 \geq \alpha_2$ *and* $\alpha_2 \geq \alpha_3$ *then* $\alpha_1 \geq \alpha_3$.

Proof. Reflexivity can by verified by using $t = 1 \in \mathcal{T}_n$ in equation 3.1. The condition that $\alpha_1 \geq \alpha_2$ and $\alpha_2 \geq \alpha_1$ implies that $h(\alpha_1) = h(\alpha_2)$ and so $t = 1$ is the only candidate for t in inequalities $\alpha_1 = t\alpha_2$ and $\alpha_2 = t\alpha_1$. So we have antisymmetry. One can check transitivity by taking an obvious composition of t's in the two inequalities.

□

Being a submonoid, $\mathcal{H}_n$ acts on $\mathcal{M}_n$. In this paper, we focus on the action of $\mathcal{H}_n$ on $\mathcal{M}_n$ by right multiplication (post-composition).

Remark 3.2. The two actions of $\mathcal{H}_n$ and $\mathcal{T}_n$ on $\mathcal{M}_n$ commute.

A chain $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k$ corresponds to a k-simplex in the the geometric realization $\|\mathcal{M}_n\|$. The lengths of chains are not bounded and hence the dimension of $\|\mathcal{M}_n\|$ is infinite. To show the finiteness properties of $\mathcal{H}_n$ (Corollary B), Brown studied the action of $\mathcal{H}_n$ on $\|\mathcal{M}_n\|$ equipped with a filtration satisfying the condition of Theorem 3.51.

Instead, for the construction of desired complex X_n, we want to start with Cayley graph of $\mathcal{M}_n$ with respect to the generating set $\{t_1, \cdots, t_n\}$ for $\mathcal{T}_n$. Recall the *Cayley graph* $\Gamma(G, S)$ of a group G with respect to a generating set S is the metric graph whose vertices are in $1-1$ correspondence with the elements of G and which has an edge (labelled by s) of length 1 joining g to gs for each $g \in G$ and $s \in S$.

Definition 3.3 (Cayley graph $\mathcal{C}_n$). Associated to the monoid $\mathcal{M}_n$, a directed graph $\mathcal{C}_n$ is defined by

- The vertex set $\mathcal{C}_n^{(0)}$ is in $1-1$ correspondence with the elements of $\mathcal{M}_n$. By abusing notation, let α denote the vertex of $\mathcal{C}_n$ corresponding the same element of $\mathcal{M}_n$.

- A vertex α is joined to another vertex β by an edge e (of length 1) labelled by t_i if $\beta = t_i\alpha$ for some t_i; we assume all edges are directed, i.e., directed from α to β.

The monoid homomorphism $h : \mathcal{M}_n \to \mathbb{Z}_{\geq 0}$ can be extended linearly to a map $h : \mathcal{C}_n \to \mathbb{R}_{\geq 0}$. Note that $h(\beta) = h(\alpha) + 1$ if $\beta = t_i\alpha$ for some t_i. In this situation, we say that the edge e_{t_i} is directed "upward" with respect to the height function h.

Remark 3.4. For a vertex β and a fixed i $(1 \leq i \leq n)$, there is unique edge labelled by t_i whose initial vertex is β; namely, the edge which ends in the uniquely determined vertex $t_i\beta$.

Least upper bound in $\mathcal{M}_n$.

Lemma 3.5. *For $\alpha_1, \alpha_2 \in \mathcal{M}_n$, there exists a unique element β such that*

- *$\beta \geq \alpha_1$ and $\beta \geq \alpha_2$, and*

- *if $\beta' \geq \alpha_1$ and $\beta' \geq \alpha_2$, then $\beta' \geq \beta$.*

Proof. Suppose $\varphi(\alpha_1) = (m_1, \cdots, m_n)$ and $\varphi(\alpha_2) = (m'_1, \cdots, m'_n)$. Consider the restrictions $\alpha_{1|R_i}$ and $\alpha_{2|R_i}$ for $i = 1, \cdots, n$. One can find smallest non-negative integers k_i, k'_i so that

$$t_i^{k_i}\alpha_{1|R_i} = t_i^{k'_i}\alpha_{2|R_i} \tag{3.2}$$

for each i as follows. If $m_i > m'_i$ then there exists smallest $p_0 \in \mathbb{N} \cup \{0\}$ such that

$$(i,p)\alpha_1 = (i,p)t_i^{m_i - m'_i}\alpha_2$$

for all $p > p_0$. The existence of p_0 is obvious. On R_i, α_1 and $t_i^{m_i - m_i'}\alpha_2$ agree outside a finite set L (this set L can be smaller than the finite set in the definition of $\mathcal{M}_n$). The integer k_0 is determined by the point in $L \cap R_i$ with the largest distance from the origin (if the intersection is trivial then $p_0 = 0$). Set $k_i = p_0$ and $k_i' = m_i - m_i' + p_0$. The integers k_i and k_i' are desired powers for t_i satisfying (3.2). One can find appropriate k_i and k_i' in a similar way in case $m_i < m_i'$. If $m_i = m_i'$ then set $k_i = k_i' = p_0$. Apply this process to get k_i's and k_i''s for all $i = 1, \cdots, n$.

Now set $\beta = t_1^{k_1} \cdots t_n^{k_n}\alpha_1 = t_1^{k_1'} \cdots t_n^{k_n'}\alpha_2$. The first condition clearly holds. The definition of $\geq$ together with the minimality of the k_i and k_i' ensure that if $\beta' \geq \alpha$ and $\beta' \geq \alpha'$, then $\beta' \geq \beta$, and hence condition two holds. Finally, the uniqueness of β follows from the second condition together with the antisymmetry of $\geq$. (Proposition 3.1). □

Definition 3.6 (**Least upper bound**)**.** An element β is called an *upper bound* of α_1 and α_2 if it satisfies the first condition of Lemma 3.5. If β is the unique element satisfying both conditions of Lemma 3.5, then it is called the *least upper bound* (simply *lub*) of α_1 and α_2 and is denoted by $\alpha_1 \wedge \alpha_2$.

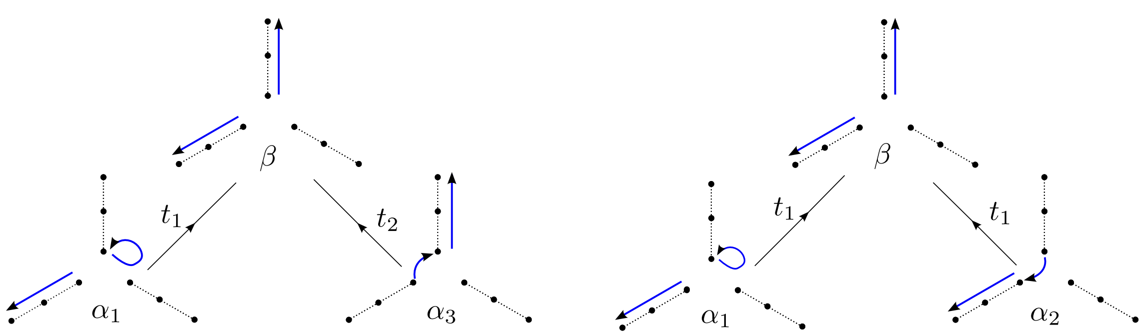

Figure 3.2: β is an upper bound of α_1, α_2 and α_3

Figure 3.2 shows some elements of $\mathcal{M}_n$ with their upper bound; β is an upper bound of α_1, α_2 and α_3. Note that, in the right figure, edges directed into the vertex β are not necessarily labelled by distinct generators of $\mathcal{T}_n$. The reader should contrast this with the situation of edges directed away from a vertex as described in Remark 3.4.

The following proposition implies that the notion of *lub* for a finite collection of $\mathcal{M}_n$ is well-defined.

Proposition 3.7. *Suppose $\{\alpha_1, \cdots, \alpha_\ell\} \subset \mathcal{M}_n$. There exists lub of those finite elements, i.e. there exists a unique element β satisfying*

- $\beta \geq \alpha_j$, $j = 1, \cdots, \ell$,
- *if $\beta' \geq \alpha_j$ for all j then $\beta' \geq \beta$.*

Proof. We extend the idea in the proof of Lemma 3.5 to find β with desired properties. Fix i ($1 \leq i \leq n$). Suppose the translation lengths of α_j 'at infinity' is $m_{i,j}$ (i.e., α_j has image $m_{i,j}$ under the composition $\mathcal{M}_n \xrightarrow{\varphi} \mathbb{Z}^n \xrightarrow{\pi_i} \mathbb{Z}$ where π_i is the projection to the i^{th} component). We want to find smallest non-negative integers $k_{i,1}, \cdots, k_{i,\ell}$ so that

$$t_i^{k_{i,j}} \alpha_{j|R_i} = t_i^{k_{i,j'}} \alpha_{j'|R_i}$$

for all $j, j' \in \{1, \cdots, \ell\}$. Let $M_i = max\{m_{i,1}, \cdots, m_{i,\ell}\}$. Observe that, for $j = 1, \cdots, \ell$, elements $t_i^{M_i - m_{i,j}} \alpha_j$ act as a translation on R_i by M_i outside a finite set K. They agree on R_i outside a finite set L (as in the proof Lemma 3.5, L can be smaller than K). So there exists smallest $p_0 \in \mathbb{N} \cup \{0\}$ such that

$$(i, p)\alpha_j = (i, p)\alpha_{j'}$$

for all $p > p_0$ and $j, j' \in \{1, \cdots, \ell\}$. Now set $k_{i,j} = M_i - m_j + p_0$ for $j = 1, \cdots, \ell$. Apply this process to find $k_{i,j}$ for all $i = 1, \cdots, n$. As before, we define

$$\beta = t_1^{k_{1,j}} t_2^{k_{2,j}} \cdots t_n^{k_{n,j}} \alpha_j \tag{3.3}$$

for some $j \in \{1, \cdots, \ell\}$. Note it is well-defined because $t_1^{k_{1,j}} \cdots t_n^{k_{n,j}} \alpha_j = t_1^{k_{1,j'}} \cdots t_n^{k_{n,j'}} \alpha_{j'}$ for any $j' \in \{1, \cdots, \ell\}$. The rest of the argument is analogous to the proof of Lemma 3.5. By the definition (3.3) of β it satisfies the first condition. The second condition is also satisfied by the minimality of powers $k_{i,j}$. Finally, the uniqueness follows from the antisymmetry of $\geq$. □

Remark 3.8. The operator *lub* preserves the inclusion between finite sets of $\mathcal{M}_n$, i.e.,

$$lub(C_1) \leq lub(C_2)$$

if $C_1 \subset C_2 \subset \mathcal{M}_n$.

Cubical structure of $\mathcal{C}_n$ and the definition of X_n.

Commutativity of $\mathcal{T}_n$ plays an important role in the construction of X_n; any permutation in the product $t_1 t_2 \cdots t_k$, $k \leq n$, represents the same element in $\mathcal{T}_n$. Yet each variation in the expression $t_1 t_2 \cdots t_k$ determines a path from α_1 to $t_1 t_2 \cdots t_k \alpha_1$. These $k!$ paths form the 1-skeleton of a k-cube. Figure 3.3 illustrates 3! paths form the 1-skeleton of a 3-cube having β and $t_i t_j t_k \beta$ as its bottom and top vertex respectively.

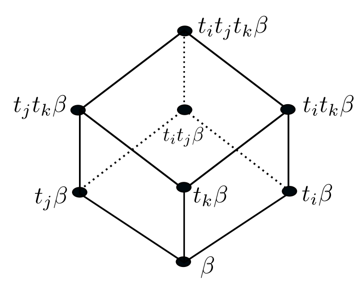

Figure 3.3: Three distinct generators t_i, t_j, t_k determine a 3-cube in X_n.

Definition 3.9 (**Cubical complex X_n**)**.** For each $n \in \mathbb{N}$ X_n is defined inductively as follows

(1) $X_n^{(1)} := \mathcal{C}_n$,

(2) for each $k \geq 2$, $X_n^{(k)}$ is obtained from $X_n^{(k-1)}$ by attaching a k-cube along every copy of the boundary of a k-cube in $X_n^{(k-1)}$.

The height function $h : \mathcal{C}_n \to \mathbb{R}_{\geq 0}$ extends linearly to a Morse function $h : X_n \to \mathbb{R}_{\geq 0}$ (see Definition 3.43). In our study, a Morse function h is roughly a height function which restricts to each k-cube to give the standard height function. Let $\square^k$ denote the standard k-cube, i.e., $\square^k = \{(x_1, \cdots, x_k) \in \mathbb{R}^k \,|\, 0 \leq x_i \leq 1\}$. The standard height function $\overline{h} : \square^k \to \mathbb{R}$ is defined by $\overline{h}(x_1, \cdots x_k) = \sum x_i$. Our Morse function h can be described as follows. Let $\sigma_j^k \subset X_n$ be a k-cube and let $\varphi_j : \square^k \to \sigma^k \subset X_n$ denote the attaching map (isometric embedding) for σ_j^k used in the construction of X_n. The image of $\sigma_j^k = \varphi_j(\square^k)$ under h is the translation of $\overline{h}(\square^k)$ by $h\varphi_j(0)$. In other words, the following diagram commutes.

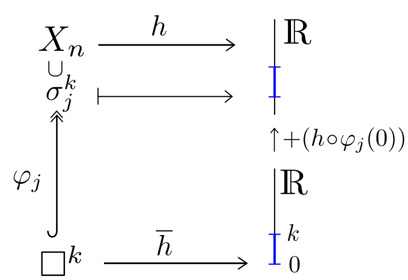

Figure 3.4: A Morse function h restricted to a cube is the standard height function up to a translation.

Two actions of $\mathcal{H}_n$ and $\mathcal{T}_n$ on $\mathcal{M}_n$ extend to actions on X_n. Note that cell(cube) structure of X_n is completely determined by $\mathcal{T}_n$ whose action commutes with the action of $\mathcal{H}_n$. So the action of $\mathcal{H}_n$ on X_n is cellular, i.e., it preserves cell structure. If a cube σ is spanned by a collection of vertices $\{\alpha_1, \cdots, \alpha_m\}$, then the cube $\sigma \cdot g$ is the same dimensional cube spanned by $\{\alpha_1 g, \cdots, \alpha_m g\}$.

Intuitively the action of $\mathcal{H}_n$ can be considered as a 'horizontal' action. Each $g \in \mathcal{H}_n$ preserves the height. By contrast, the action of $\mathcal{T}_n$ is 'vertical'. Each non trivial element of $\mathcal{T}_n$ increases the height under the action (see Remark 3.44). We discuss these actions on X_n further in section 3.4.

Lemma 3.10. *The graph $\mathcal{C}_n$ is simplicial and connected for all $n \in \mathbb{N}$.*

Proof. Connectedness is an immediate consequence of the existence of an upper bound for any pair of vertices in $\mathcal{M}_n$ (Lemma 3.5). Suppose α_1 and α_2 are vertices of $\mathcal{C}_n$. Let β be a vertex with

$$\beta = \tau_1\alpha_1 \text{ and } \beta = \tau_2\alpha_2$$

for some $\tau_1, \tau_2 \in \mathcal{T}_n$. Observe that τ_1 and τ_2 define paths p_1 and p_2 where p_i joins α_i to β, $i = 1, 2$. So α_1 and α_2 are connected by the concatenation of the two paths.

If an edge connects two vertices β_1 and β_2 then their heights differ by 1. So no edge in $\mathcal{C}_n$ starts and ends at the same vertex. Finally we want to show there is no bigon in C_n. Suppose two distinct edges share two vertices β_1 and β_2. Again,those two vertices can not have the same height and so we may assume $\beta_1 \geq \beta_2$. Our claim is that those two edges are labelled by one generater of $\mathcal{T}_n$. If two edges were labelled by two distinct generators t_i and t_j, then we would have $\beta_1 = t_i\beta_2$ and $\beta_1 = t_j\beta_2$. This contradicts the fact that β_1 is injective because

$$(i,1)\beta_1 = (i,1)t_i\beta_2 = (i,2)\beta_2 \text{ but } (i,1)\beta_1 = (i,1)t_j\beta_2 = (i,1)\beta_2.$$

Therefore the two edges are labelled by a generater of $\mathcal{T}_n$. This means

$$\beta_1 = t_i\beta_2 \tag{3.4}$$

for some t_i. By Remark 3.4, there is only one edge connecting β_1 and β_2 satisfying (3.4). So no two distinct edges share two vertices in $\mathcal{C}_n$. □

Lemma 3.11. *The graph X_1 is simply connected.*

Proof. Suppose ℓ is a loop in X_1. Consider the image of ℓ under a Morse function h. The image $h(\ell) = [a, b]$ is a compact interval. Using the homotopy constructed in Lemma 2.3 of [2] if necessary, we may assume that a and b are integers. Note that if $a = b$ then ℓ is the constant loop at a vertex of X_1, so we may also assume that $b - a \geq 1$. Let $\ell \cap h^{-1}(a) = \{\alpha_1, \cdots, \alpha_k\}$. By Remark 3.4, there exists an unique edge e_j emanating from each α_j $(1 \leq j \leq k)$. Observe that the loop ℓ contains those edges and that ℓ is homotopy equivalent to $\ell - \cup_j e_j$ (Indeed, what we remove is not a whole edge but an edge minus the top vertex). Under this homotopy equivalence one can homotope ℓ to another loop ℓ' such that $h(\ell') = [a + 1, b]$. Essentially, in the passage from ℓ to ℓ', we removed back-trackings involving vertices $\alpha_1, \cdots, \alpha_k$. Apply the same process to establish homotopy equivalence between ℓ and a vertex with height b. It can be shown that $h^{-1}(b)$ is a singleton set even if it is not clear a priori. After applying the above homotopy equivalences up to $b - a$ steps one obtains homotopy equivalence between the loop ℓ and $\ell \cap h^{-1}(b)$. If the preimage $h^{-1}(b)$ consisted of multiple number of vertices then ℓ was homotopic to a discrete set $h^{-1}(b)$. We have shown that ℓ is null homotopic. □

Lifting across a square. Fix $n \geq 2$. Suppose β is a vertex of X_n. For a pair of generators t_i and t_j $(i \neq j)$, there exists a unique square σ containing four vertices $\beta, t_i\beta, t_j\beta$ and $t_it_j\beta$. Consider the path p with length 2, joining two vertices $t_i\beta$ and $t_j\beta$, consisting of two lower edges of the square σ. We say a path p has a *turn* at β. This path p is homotopic to a path $\tilde{p}$ rel two ends vertices which consists of two upper edges of σ. Figure 3.5 illustrates this idea. We say $\tilde{p}$ is a *lifting* of p (across a square σ). The notion of lifting is useful in proving Lemma 3.12 as well as amenability of $\mathcal{C}_n$ in Section 3.3.2.

Lemma 3.12. *For all $n \geq 2$, X_n is simply connected.*

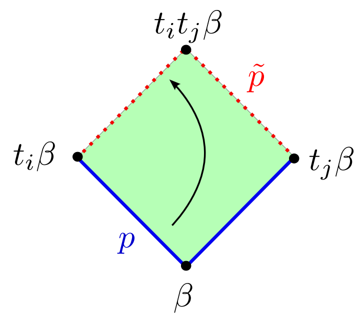

Figure 3.5: A path with a turn at β lifts to a path $\tilde{p}$.

Proof. Suppose $\ell : \mathbb{S}^1 \to X_n$ is a loop in X_n for $n \geq 2$. We may assume that ℓ lies in 1-skeleton of X_n and that $h(\ell) = [a, b]$ for some non negative integers $a < b$. We may further assume that ℓ does not contain back-trackings. As in the proof of Lemma 3.11, we may assume that ℓ lies in the 1–skeleton of X_n, and that $h(\ell) = [a, b]$ for nonnegative integers $a \leq b$. Note that if $a = b$ then ℓ is a constant loop at some vertex of X_n, so we may assume that $a < b$. Let $\alpha_1, \ldots, \alpha_k$ be the vertices of ℓ. By Proposition 3.7, there exists a unique lub $\beta = lub\{\alpha_1, \cdots, \alpha_k\}$. We want to establish a homotopy equivalence between ℓ and the trivial loop β by applying liftings across squares. We start with vertices in $\ell \cap h^{-1}(a) = \{\beta_1, \cdots, \beta_{k'}\}$. The loop ℓ may intersect multiple number of squares even for a single vertex $\beta_j \in h^{-1}(a)$. Let $\ell^{-1}(\beta_j) = \{v_{j,1}, \cdots v_{j,r_j}\} \subset \mathbb{S}^1$ for $j = 1, \cdots, k'$. From the fact that ℓ does not contain back-tracking, we have the following crucial observation for each j.

(1) Each preimage $v_{j,m}$ belongs to an interval $I_{j,m}$ such that $\ell(I_{j,m})$ is a path having a turn at β_j,

(2) the two end vertices of $\ell(I_{j,m})$ are given by $t_i\beta_j$ and $t_{i'}\beta_j$ for distinct generators t_i and $t_{i'}$ of $\mathcal{T}_n$.

So each interval $I_{j,m}$ determines a square $\sigma_{j,m}$ whose bottom vertex is β_j. Now take liftings of $\ell(I_{j,m})$ for all j and m ($1 \leq j \leq k'$, $1 \leq m \leq r$). The resulting loop ℓ' is homotopy equivalent to ℓ. Another important observation is that the top vertex

$w_{j,m}$ of a square $\sigma_{j,m}$ in which lifting occured satisfies

$$\beta \geq w_{j,m}. \tag{3.5}$$

Indeed, three vertices on $\ell(I_{j,m})$ were already vertices of the loop ℓ. The top vertex $w_{j,m}$ is the *lub* of those three vertices. By Remark 3.8, each vertex $w_{j,m}$ satisfies $\beta \geq w_{j,m}$. This means that in the passage from ℓ to ℓ' vertices of $\ell \cap h^{-1}(a)$ were replaced by some vertices satisfying equation (3.5). So β is still an upper bound of the vertices of ℓ'. Observe that $h(\ell') = [a+1, b]$.

Apply the same process consecutively to vertices with smallest height in each step. The property given by equation (3.5) ensures that this procedure stops after finitely many $(h(\beta) - a)$ steps. In all, the given loop ℓ converges to β. □

3.2 Properties of the monoid $\mathcal{M}_n$

In this section we study some algebraic properties which are useful for us to study geometry of X_n in the sequel.

Lemma 3.13. $\mathcal{M}_n = \mathcal{T}_n\mathcal{H}_n = \{tg \mid t \in \mathcal{T}_n,\ g \in \mathcal{H}_n\}$

Proof. Suppose $\beta \in \mathcal{M}_n$ and $\varphi(\beta) = (m_1, \cdots, m_n)$. There exist $t \in \mathcal{T}_n$ and $g \in \mathcal{H}_n$ such that $h(t) = h(\alpha)$ and $\varphi(\beta) = \varphi(t\,g)$. Thus β and tg agree on $Y_n - K$ for some finite set K. We want to find $f \in \mathcal{H}_n$ such that $\beta = tgf$. Existence of such f comes from the fact that being injections, β and tg have right inverses f_1 and f_2 respectively (i.e., left inverses in the composition of functions). Consider those right inverses f_1 and f_2

$$f_1 : (K)\beta \to K \text{ and } f_2 : (K)tg \to K.$$

By definition, $\beta f_1 = tgf_2$ on K. Observe that one can turn $f_2 f_1^{-1}$ into an element $f \in \mathcal{H}_n$ with $supp(f) \subset (K)tg \cup (K)\beta$. For example, one can extend $f_2 f_1^{-1} : (K)tg \to$

$(K)\beta$ to $f : (K)tg \cup (K)\beta \to (K)tg \cup (K)\beta$ by a bijection $f' : (K)\beta - (K)tg \to (K)tg - (K)\beta$. More precisely, f is defined by

$$f = \begin{cases} f_2 f_1^{-1} & \text{on } (K)tg \\ f' & \text{on } (K)\beta - (K)tg \end{cases}$$

where f' is a bijection between congruent finite sets $(K)\beta - (K)tg$ and $(K)tg - (K)\beta$. □

Definition 3.14 (Greatest lower bound). An element β is called a *lower bound* of α_1 and α_2 if it satisfies the first condition below, and is called the *greatest lower bound* (simply *glb*) if it satisfies the second condition as well.

- $\beta \leq \alpha_1$ and $\beta \leq \alpha_2$,
- If $\beta' \leq \alpha_1$ and $\beta' \leq \alpha_2$ then $\beta' \leq \beta$.

The *glb* of α_1 and α_2 is denoted by $\alpha_1 \vee \alpha_2$.

Note that not every pair of vertices admits a lower bound (and in particular greatest lower bound). For example, the right figure in Figure 3.6 describes α_1 and α_3 which do not have a lower bound. If there was a common lower bound γ' then we would be forced to have $(2,1)\gamma' = (1,1)\gamma'$, contradicting the injectivity of γ'. The geometric interpretation of this is that there is no square containing α_1 and α_3. However, if one replaces α_1 by α_2 then there exists a common lower bound (indeed $\alpha_2 \vee \alpha_3$) for α_2 and α_3 as illustrated in the left figure. Note that $(1,1)\alpha_2 \neq (2,1)\alpha_3$ but $(1,1)\alpha_1 = (1,1) = (2,1)\alpha_3$. Lemma 3.22 states that such equation determines the existence of *glb* of those pairs. In general, existence of a common lower bound guarantees the existence of *glb*.

Lemma 3.15. *For any $\alpha_1, \alpha_2 \in \mathcal{M}_n$, $\alpha_1 \vee \alpha_2$ exists and unique if there is a lower bound of α_1 and α_2.*

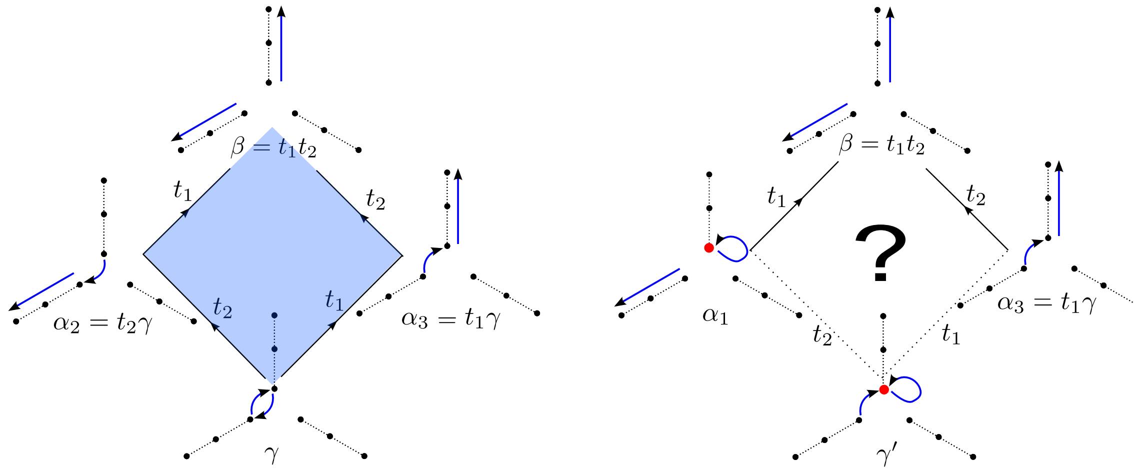

Figure 3.6: Examples $\alpha_1, \alpha_2, \alpha_3$ where $\alpha_2 \vee \alpha_3 = \gamma$ exists, but $\alpha_1 \vee \alpha_3$ does not exist.

Proof. Suppose γ is a lower bound of α_1 and α_2. Then there exist $\tau = t_1^{k_1}, \cdots, t_n^{k_n}$ and $\tau' = t_1^{k_1'}, \cdots, t_n^{k_n'}$ satisfying

$$\alpha_1 = \tau\gamma \text{ and } \alpha_2 = \tau'\gamma.$$

Claim: $\beta := t_1^{m_1} \cdots t_n^{m_n}\gamma$ is the *glb*, where $m_i = min\{k_i, k_i'\}$.

The commutativity of $\mathcal{T}_n$ implies that

$$\alpha_1 = t_1^{l_1}, \cdots, t_n^{l_n}\beta \text{ and } \alpha_2 = t_1^{l_1'}, \cdots, t_n^{l_n'}\beta$$

where $l_i = max\{k_i - k_i', 0\}$ and $l_i' = max\{k_i' - k_i, 0\}$. So β satisfies the first condition. Observe that $l_i l_i' = 0$ for each i. This means that either

$$\alpha_{1|R_i} = \beta_{|R_i} \text{ or } \alpha_{2|R_i} = \beta_{|R_i}$$

for each i. Suppose β' is another element with $\alpha_1 = \tau_1\beta'$ and $\alpha_2 = \tau_2\beta'$ for some

$\tau_1, \tau_2 \in \mathcal{T}_n$. Then we have

$$(\tau_1\beta')_{|R_i} = \beta_{|R_i} \text{ or } (\tau_2\beta')_{|R_i} = \beta_{|R_i}$$

for each i. So $\beta' \leq \beta$. Uniqueness of the glb follows from the second condition in Definition 3.14 together with antisymmetry of $\geq$ (Proposition 3.1). □

Corollary 3.16. *If $\alpha_1, \alpha_2 \in \mathcal{T}_n$ then $\alpha_1 \vee \alpha_2$ always exists.*

Proof. If two elements α_1 and α_2 belong to $\mathcal{T}_n$, then they have a common lower bound $1_{\mathcal{M}_n}$, namely the identity map on Y_n. By Lemma 3.15, $\alpha_1 \vee \alpha_2$ exists. □

Lemma 3.17. *For a pair of elements $\alpha_1, \alpha_2 \in \mathcal{M}_n$, the lub and glb of the pair can be characterized by height function as follows.*

(1) β is the lub of α_1, α_2 if β is an upper bound and β has the smallest height among the upper bounds;

(2) β is the glb of α_1, α_2 if β is a lower bound and β has the largest height among the lower bounds.

Proof. Suppose β is an element satisfying the condition (1) above. We want to show $\beta = \alpha_1 \wedge \alpha_2$. By the definition of $\alpha_1 \wedge \alpha_2$ it is obvious that $h(\beta) = h(\alpha_1 \wedge \alpha_2)$. By Lemma 3.15, $\beta \vee (\alpha_1 \wedge \alpha_2)$ exists since β and $\alpha_1 \wedge \alpha_2$ have common lower bounds α_1 and α_2. Note that $\beta \vee (\alpha_1 \wedge \alpha_2)$ is an upper bound of α_1 and α_2. If $\beta \neq \alpha_1 \wedge \alpha_2$ then $h(\beta \vee (\alpha_1 \wedge \alpha_2)) < h(\beta)$, contradicting to our choice of β. So $\beta = \alpha_1 \wedge \alpha_2$. Analogous argument can be applied to the case of *glb*. One can draw a contradiction by assuming $\beta \neq (\alpha_1 \vee \alpha_2)$ for an element β satisfying the condition of (2) above. □

The following specifies the relationship between *lub* and *glb* under assumption on existence of *glb*.

Proposition 3.18. *Suppose, for $\alpha_1, \alpha_2 \in \mathcal{M}_n$, $\tau\alpha_1 = \alpha_1 \wedge \alpha_2 = \tau'\alpha_2$ where $\tau = t_1^{k_1} \cdots t_n^{k_n}$ and $\tau' = t_1^{k'_1} \cdots t_n^{k'_n}$. Then $\alpha_1 \vee \alpha_2$ exists if and only if the following conditions are satisfied for all $i = 1, 2, \cdots, n$*

- *there is no common generater t_i in τ and τ', i.e., $k_i k'_i = 0$,*
- $(\bigcup_{p \le k_i} (i,p)\alpha_1) \cap (\bigcup_{p \le k'_i} (i,p)\alpha_2) = \emptyset$.

Moreover, we have

$$\tau'(\alpha_1 \vee \alpha_2) = \alpha_1 \text{ and } \tau(\alpha_1 \vee \alpha_2) = \alpha_2. \tag{3.6}$$

Proof. First we construct $\gamma = \alpha_1 \vee \alpha_2$ by using the conditions. Define γ by

$$(i,p)\gamma = \begin{cases} (i,p)\alpha_1 & \text{if } k'_i = 0 \\ (i,p)\alpha_2 & \text{if } k_i = 0 \end{cases}$$

for each $i = 1, \cdots, n$. This is well defined since $k_i k'_i = 0$, and if $k_1 = k'_i = 0$ then we have $(i,p)\alpha_1 = (i,p)\tau\alpha_1 = (i,p)\tau'\alpha_2 = (i,p)\alpha_2$ for all $p \in N$. One can verify γ is the desired element as follows.

Claim 1: $\tau'\gamma = \alpha_1$ and $\tau\gamma = \alpha_1$.

Fix i. Suppose $k'_i = 0$. For all $p \in N$, we have $(i,p)\tau'\gamma = (i,p)t_i^{k'_i}\gamma = (i,p)\gamma = (i,p)\alpha_1$. Suppose $k'_i \neq 0$. This means $k_i = 0$ and so we have

$$\begin{aligned}(i,p)\tau'\gamma = (i,p)t_i^{k'_i}\gamma = (i,p+k'_i)\gamma = (i,p+k'_i)\alpha_2 = (i,p)t_i^{k'_i}\alpha_2 \\ = (i,p)\tau'\alpha_2 = (i,p)\tau\alpha_1 = (i,p)t_i^{k_i}\alpha_1 = (i,p)\alpha_1,\end{aligned}$$

for all $p \in N$. Since the above identities hold for every i, we have verified $\tau'\gamma = \alpha_1$. A similar argument shows that $\tau\gamma = \alpha_1$.

Claim 2: $\gamma \in \mathcal{M}_n$. We need to verify injectivity. The image $(Y_n)\gamma$ coincides with $(Y_n)\alpha_1$ up to a finite set. We already know that $(i,p)\gamma = (i,p)\alpha_1$ if $k'_i = 0$ $(p \in \mathbb{N})$. If $k'_i \neq 0$ then $(i, p + k'_i)\gamma = (i,p)\alpha_1$ $(p \in \mathbb{N})$. If suffices to show, for i with $k'_i \neq 0$, that $(i,p)\gamma \notin (Y_n)\alpha_1$ for all $p \leq k'_i$. This is clear from the second condition of the Lemma because $(i,p)\gamma \in \bigcup_{p \leq k'_i}(i,p)\alpha_2$.

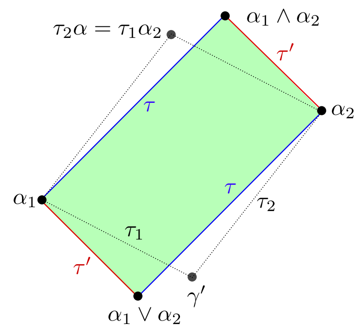

Figure 3.7: 'Big rectangle': if $\alpha_1 \vee \alpha_2$ exists $\alpha_1 \wedge \alpha_2$ determines $\alpha_1 \vee \alpha_2$, vice versa.

Claim 3: $\gamma = \alpha_1 \vee \alpha_2$. The lower bound γ has the largest height among lower bounds of α_1 and α_2. If there was another lower bound γ' of α_1 and α_2 with $h(\gamma') > h(\gamma)$ then there would be $\tau_1, \tau_2 \in \mathcal{T}_n$ such that

$$\tau_1\gamma' = \alpha_1,\ \tau_2\gamma' = \alpha_2 \text{ and } |\tau_1| < |\tau'|.$$

This implies α_1 and α_2 have an upper bound $\tau_2\alpha_1 = \tau_1\alpha_2$ such that $h(\tau_1\alpha_2) = (\tau_1) + h(\alpha_2)$ is strictly smaller than $h(\alpha_1 \wedge \alpha_2) = h(\tau') + h(\alpha_2)$ (see Figure 3.7). This contradicts the fact that $\alpha_1 \wedge \alpha_2$ has the smallest height among upper bounds of α_1 and α_2 (Lemma 3.17). So the lower bound γ attains maximum height among lower bounds of α_1 and α_2. By Lemma 3.17, $\gamma = \alpha_1 \vee \alpha_2$.

The converse statement is rather easy to check. Suppose that $\alpha_1 \vee \alpha_2$ exists and that

$$\tau_1(\alpha_1 \vee \alpha_2) = \alpha_1 \text{ and } \tau_2(\alpha_1 \vee \alpha_2) = \alpha_2. \tag{3.7}$$

for some $\tau_1, \tau_2 \in \mathcal{T}_n$. As in the proof of Lemma 3.15, we see that there is no common generator in τ_1 and τ_2. So the first condition is satisfied. Injectivity of γ together with equation (3.7) guarantees the second condition.

Finally, the argument in Claim 3 above says that $\alpha_1 \vee \alpha_2$ determines $\alpha_1 \wedge \alpha_2$. If $\alpha_1 \vee \alpha_2$ satisfies equation (3.7) then we have

$$\tau_1 \tau = \tau_2 \tau'.$$

Now the condition that there is no common generator in pairs τ and τ', and τ_1 and τ_2 enables one to conclude that $\tau_1 = \tau'$ and $\tau_2 = \tau$. Hence equation (3.6) follows. □

Definition 3.19 (**Maximal elements**)**.** For $\beta \in \mathcal{M}_n$, α is called a *maximal element* of β if it satisfies

$$\beta = t_i \alpha \tag{3.8}$$

for some generator t_i of $\mathcal{T}_n$.

The following Lemma says the number of maximal elements of β varies depending on the height of β.

Lemma 3.20. *Suppose $h(\beta) = h$. There exists $n \times h$ many maximal elements of β.*

Proof. Fix i. The identity (3.8) determines α completely except $(i,1)\alpha$. It also implies $(Y_n - (i,1))\alpha = (Y_n)\beta$. So there are precisely $|S(\beta)| = |Y_n - (Y_n)\beta| = h$ many choices for $(i,1)\alpha$. It turns out any of these choice defines an injective map α. Consider elements $\alpha_1, \cdots, \alpha_h$ defined by

$$(j,p)\alpha_k = \begin{cases} (j,p)\beta & \text{if } j \neq i,\ p \in \mathbb{N} \\ (i,p+1)\beta & \text{if } j = i,\ p \in \mathbb{N}, \end{cases}$$

and $(i,1)\alpha_k \in S(\beta)$. They are all desired h many injective maps.

□

Remark 3.21. The argument of Lemma 3.20 implies that, each maximal element α_k of β is labelled by t_i as well as a point of $S(\beta)$. Figures 3.2 and 3.6 illustrate this idea for $\beta = t_1 t_2$. Note that $S(\beta) = \{(1,1),(1,2)\}$. In those figures maximal elements α_1, α_2 and α_3 are labelled by

$$\alpha_1 \leftrightarrow (t_1,\, (1,1)),\ \alpha_2 \leftrightarrow (t_1,\, (1,2)),\ \alpha_3 \leftrightarrow (t_2,\, (1,1)).$$

In general, for $\beta \in \mathcal{M}_n$, the set of maximal elements of β is in 1-to-1 correspondence with $\{1, \cdots, n\} \times S(\beta)$.

The corollaries below follows from Proposition 3.18. They provide criterions when a collection of maximal elements form squares and k-cubes respectively.

Corollary 3.22. *Suppose $t_i\alpha_1 = t_j\alpha_2$ $(i \neq j)$. There exists $\alpha_1 \vee \alpha_2$ if and only if the first coordinates are distinct and the second coordinates are distinct (using the ordered pair notation introduced in Remark 3.21).*

Proof. This is a special case of Proposition 3.18, with the two conditions of that proposition rephrased in terms of the coordinates introduced in Remark 3.21. □

Corollary 3.23. *Suppose $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ is a collection of maximal elements of β. There exists glb$(\alpha_1, \cdots, \alpha_k)$ if and only if the first coordinates are all distinct and the second coordinates are all distinct (using the ordered pair notation introduced in Remark 3.21).*

Proof. We construct *glb* inductively with the base case $k = 2$, which is done in Corollary 3.22. Assume the statement holds for k maximal vertices of β. Say the glb of those vertices is γ. Note that the information of k vertices encoded in the coordinates completely determine γ. Now apply Proposition 3.18 to γ and α_{k+1} to check the existence of $\gamma \vee \alpha_{k+1}$. Note that $\gamma \wedge \alpha_{k+1} = \beta$. The condition on distinct first coordinates implies there is no common generater t_i in the two paths. The condition on the other coordinates implies the second condition of Proposition 3.18. So $\gamma \vee \alpha_{k+1}$ exists.

Conversely, the existence of glb directly implies there exists a cube whose top vertex is β. So the condition on coordinates is satisfied. □

3.3 X_n is CAT(0)

In this chapter we show X_n is a cubing for each $n \in N$, i.e., X_n is a 1-connected non positively curved cubical complex. There are several ways to think about this result.

By Lemma 3.12, X_n is simply connected. In Subsection 3.3.1, we prove X_n is non positively curved by using Gromov's *link condition*.

Another way to see X_n is a cubing is to use the fact ([8], [16]) that there is 1-to-1 correspondence

$$\text{the class of cubings} \leftrightarrow \text{the class of median graphs}$$

One associates a median graph to a cubing by considering the 1–skeleton of the cubing. In the reverse direction, one thinks of a median graph as the 1–skeleton of a cubing, and defines the cubing inductively on skeleta, as in Definition 3.9. In Subsection 3.3.2 we prove $\mathcal{C}_n$ is a median graph for all $n \in \mathbb{N}$.

3.3.1 Gromov link condition

Definition 3.24 (**cubical complex**)**.** Intuitively, a cubical complex is a regular CW-complex, except that it is built out of Euclidean cubes I^k instead of balls. More precisely, a *cubical complex* X is a CW-complex where for each k-cell $\sigma_j^k \subset X$ its attaching map $\varphi_j : \partial I^k \to X^{n-1}$ satisfies the following conditions:

(1) the restriction of φ_j to each face of I^k is a linear homeomorphism onto a cube of one lower dimension,

(2) φ_j is a homeomorphism onto its image.

We give X the standard CW-topology.

The non-positive curvature condition we will use is a local condition captured by conditions on the link of a vertex.

Definition 3.25 (**Link of a vertex**). The *link of a vertex v in I^k*, denoted by $Lk(v, I^k)$, is defined to be intersection of the cube I^k and the unit sphere $\mathbb{S}^{k-1}$ centered at v with respect to L^1 metric. Note that $Lk(v, I^k)$ is the standard simplex of dimension $k-1$. For a vertex $\alpha \in X$ and a cell σ_j^k containing α, the *link of α in σ_j^k* is defined to be the image $\varphi_j(Lk(v, I^k))$, where $\varphi_j(v) = \alpha$. The *link of α in X*, denoted by $Lk(\alpha, X)$, is defined to be the union of all links of α in cells containing α.

Definition 3.26 (**Ascending/Descending links of a vertex**). For a vertex α of a cubical complex X equipped with a Morse function h, the descending link $Lk_\downarrow(\alpha, X)$ is defined by

$$Lk_\downarrow(\alpha, X) = \bigcup\{\varphi_j(Lk(v, \sigma_j^k) : h\varphi_j \text{ attains maximum at } v,\ \varphi_j(v) = \alpha\}.$$

Likewise, the ascending link $Lk_\uparrow(\alpha, X)$ is defined by

$$Lk_\uparrow(\alpha, X) = \bigcup\{\varphi_j(Lk(v, \sigma_j^k) : h\varphi_j \text{ attains minimum at } v,\ \varphi_j(v) = \alpha\}.$$

Definition 3.27 (**Flag complex**). A simplicial complex L is said to be a *flag complex* if every collection of pairwise adjacent vertices of L spans a simplex of L.

Definition 3.28 (Gromov's condition). A cubical complex X satisfies the Gromov's condition if $Lk(\alpha, X)$ is a flag complex for each $\alpha \in X^{(0)}$.

Following Gromov, we say that a cubical complex X is *non positively curved* (simply NPC) if it satisfies the Gromov's condition.

From the way we constructed X_n, we see that any vertex of $\mathrm{Lk}_{\downarrow}(\alpha, X_n,)$ corresponds to a vertex β such that $\alpha = t_i\beta$ for some generator $t_i \in \mathcal{T}_n$. By Remark 3.21, there exists a bijection

$$\{\beta \mid \alpha = t_i\beta \text{ for some } t_i\} \leftrightarrow \{i \mid 1 \leq i \leq n\} \times S(\alpha)$$

Identify the later set with $\{(i,k) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq n,\ 1 \leq k \leq h(\alpha)\}$. Under this (composition of) identification, points $\{(i,k) \mid 1 \leq k \leq h(\alpha) \mid\}$ correspond to vertices in $\mathrm{Lk}_{\downarrow}(\alpha, X_n,)$ which are joined to α by edges labelled by t_i. Consider the simplicial complex $L_{n,h}$ for $h \in \mathbb{N}$ defined by

(1) vertices: $L_{n,h}^{(0)} = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq n,\ 1 \leq k \leq h\}$

(2) simplexes: A collection of vertices $\{(x_0, y_0), \cdots, (x_k, y_k)\}$ form a k-simplex if x_i's are all distinct and y_i's are all distinct.

Remark 3.29. By Corollary 3.23, we see that $L_{n,h}$ and $Lk_{\downarrow}(\alpha, X_n)$ are identical if $h = h(\alpha)$. Note that $L_{n,h}$ is flag for all $n, h \in \mathbb{N}$. If two vertices of $L_{n,h}$ are connected by an edge, their first coordinates are distinct and second coordinates are distinct. This means that any collection $C \subset L_{n,h}^{(0)}$ of pairwise adjacent vertices satisfy the condition (2) of definition of $L_{n,h}$ above. So a collection C forms a simplex of $L_{n,h}$.

Lemma 3.30. *For every vertex $\alpha \in X_n$, $Lk(\alpha, X_n)$ is flag.*

Proof. Suppose C is a collection of pairwise adjacent vertices of $Lk(\alpha, X_n)$. We may assume $C = \{z_1, \cdots, z_k\} \cup \{z'_1, \cdots, z'_{k'}\}$ where $z_i \in Lk_{\uparrow}(\alpha, X_n)$ and $z'_j \in Lk_{\downarrow}(\alpha, X_n)$.

For each vertex $\alpha \in X_n$, there exists unique n-cube having α as the bottom vertex. So $\text{Lk}_{\uparrow}(\alpha, X_n,)$ is simply the standard $(n-1)$-simplex. By Remark 3.29 $Lk_{\downarrow}(\alpha, X_n)$ is also flag. So those subcollections of C form a $(k-1)$-simplex σ and a $(k'-1)$-simplex σ' respectively in the ambient complexes.

Our claim is that there exist $(k+k')$-cube which contributes $(k+k'-1)$ simplex $\sigma * \sigma'$ to $Lk(\alpha, X_n)$. Consider the ordered pair notation introduced in Remark 3.21 for $\{z_1, \cdots, z_k\}$. Let $T_1 = \{t_{i_1}, \cdots, t_{i_k}\} \subset \{t_1, \cdots, t_n\}$ denote the set consisting of first coordinates of vertices $\{z_1, \cdots, z_k\}$. Likewise let $T_2 = \{t_{j_1}, \cdots, t_{j_{k'}}\}$ be the set corresponding to $\{z'_1, \cdots, z'_{k'}\}$. Observe that the $(k-1)$-simplex $\sigma \subset Lk_{\downarrow}(\alpha, X_n)$ corresponds to k-cube ρ with top vertex α, which is generated by T_1. Similarly, $\sigma' \subset Lk_{\uparrow}(\alpha, X_n)$ corresponds to k'-cube ρ' which is generated by T_2, whose bottom vertex is α. Since T_1 and T_2 disjoint, the k-cube ρ together with edges labelled by T_2 spans a $(k+k')$-cube. This cube is the desired cube containing ρ and ρ', and so α as well. So a collection C forms a simplex $\sigma * \sigma'$ in $Lk(\alpha, X_n)$. □

By Lemma 3.12 and Lemma 3.30 we have the following Theorem.

Theorem 3.31. *For all $n \in \mathbb{N}$, X_n is a cubing.*

The following is known facts for a cubing, and proofs can be found in [4].

Corollary 3.32. *For all $n \in \mathbb{N}$, X_n is uniquely geodesic and is contractible.*

3.3.2 Cayley graph $\mathcal{C}_n$ is a median graph

For an edge e of $\mathcal{C}_n$, let $\partial_- e$ and $\partial_+ e$ denote the initial and terminal vertices of e. Although generators of $\mathcal{T}_n$ do not have inverses, we can consider the *reverse* edge of e. Let $\bar{e}$ denote the reverse edge of an edge e, i.e., $\partial_- \bar{e} = \partial_+ e$ and $\partial_+ \bar{e} = \partial_- e$. Note that $\bar{\bar{e}} = e$.

By a *path* we mean an edge path; concatenation of edges (including reverse edges) $e_1, \cdots, e_k$ where $\partial_+(e_i) = \partial_-(e_{i+1})$ for $i = 1, 2, \cdots, k-1$. If a path p is a concatenation of edges $e_1, \cdots, e_k$ in order, we write this path as $p = e_1 \cdots e_k$ and the *reverse* path $\overline{p}$ of p is defined by $\overline{p} = \overline{e}_k \cdots \overline{e}_1$.

A path is called *ascending* if it does not contain reverse edges. Similarly a path is called *descending* it consists of only reverse edges. Obviously the reverse path of an ascending path is descending and vice versa. Suppose p is an ascending path joining α to $\tau\alpha$ for some $\tau \in \mathcal{T}_n$. One reads the edge labeling of p in order to form a word $\tau = \tau_1 \cdots \tau_k$, $\tau_i \in \{t_1, \cdots, t_n\}$. The expression for τ is not unique in $(\mathcal{T}_n)$ because of commutativity of $\mathcal{T}_n$. Indeed permutations of generators in τ produce a class of paths rel two end vertices α and $\tau\alpha$. In the sequel, a choice of such path is less relevant. Instead we will be interested in two end vertices of an ascending path. When we say a path p *given by* $\tau \in \mathcal{T}_n$, we mean a choice from the class of paths rel two end vertices α and $\tau\alpha$, and write $p \simeq \tau$.

For a pair of vertices α and β, the *distance in* $\mathcal{C}_n$ is the smallest length of paths joining them. We denote the distance by $d(\cdot, \cdot)$. A *geodesic* $[\alpha, \beta]$ joining vertices α and β is a path whose length is $d(\alpha, \beta)$.

Remark 3.33. Note that $d(\alpha, \beta) \geq |h(\alpha) - h(\beta)|$. Every ascending/descending path is a geodesic. If p is an ascending/descending path joining α to β then $d(\alpha, \beta) = |h(\beta) - h(\alpha)|$.

Standard geodesics in $\mathcal{C}_n$.

We say a path p has a *turn* if p contains $\overline{e}_{t_i} e_{t_j}$ for some i, j. One can apply liftings (finitely many times) to transform p to a path of the form

$$e_{t_{i_1}} \cdots e_{t_{i_k}} \overline{e}_{t_{j_1}} \cdots \overline{e}_{t_{j_\ell}} \quad (k, \ell \geq 0). \tag{3.9}$$

Such path is called *standard*, i.e., a path is standard if it is a concatenation of one ascending path and one descending path where the ascending path occurs first. Figure 3.8 illustrates liftings of paths. Note that the length of paths does not increase under liftings. As before, for a standard path p given by $(\tau)(\overline{\tau}')$ joining α to β, there are many expressions (determined by permutations in each parenthesis). However every such choice shares important information: initial vertex α, terminal vertex β and the top vertex. We write $p \simeq \tau \cdot \overline{\tau}'$ and we mean p is a concatenation of one ascending path and one descending path which are determined by choices in τ and τ' respectively.

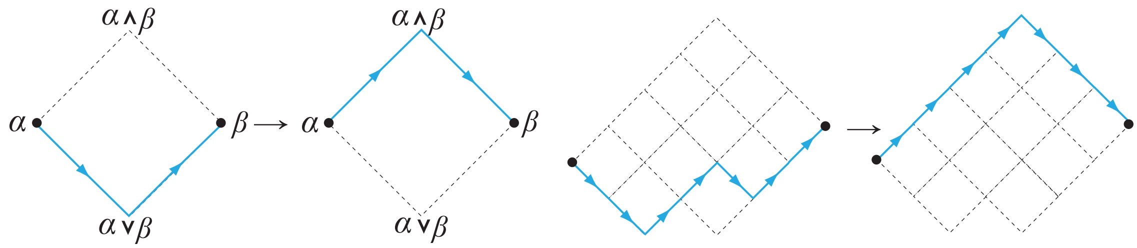

Figure 3.8: Every path can be transformed to a standard path via liftings

Remark 3.34. Note that the top vertex of a standard path is a common upper bound of the two end vertices. By Lemma 3.17, a standard path $\pi \cdot \overline{\pi'}$ joining α to β is a geodesic if and only if $\pi\alpha = \alpha \wedge \beta = \pi'\beta$.

A standard path is called a *standard geodesic* if it is a geodesic.

Definition 3.35 (Median graph). The *geodesic interval* $\mathcal{I}[\alpha, \beta]$ is the collection of vertices lying on geodesics $[\alpha, \beta]$. A graph is called a *median* if, for each triple of vertices α, β, γ, the geodesic intervals $[\alpha, \beta], [\beta, \gamma]$ and $[\gamma, \alpha]$ have a unique common point.

Proposition 3.36. *$\gamma \in \mathcal{I}[\alpha, \beta]$ if and only if $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$.*

Proof. Suppose $\gamma \in \mathcal{I}[\alpha, \beta]$. There exists a geodesic joining α and β that contains γ. So γ determines two subpaths which are geodesics joining α to γ, and γ to β respectively. Thus $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$. Conversely, if $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$ then any concatenation of two geodesics $[\alpha, \gamma]$ and $[\gamma, \beta]$ is again a geodesic containing γ and hence $\gamma \in \mathcal{I}[\alpha, \beta]$. ∎

Lemma 3.37. *If $\gamma \in \mathcal{I}[\alpha,\beta]$ then*

$$\mathcal{I}[\alpha,\beta] \cap \mathcal{I}[\beta,\gamma] \cap \mathcal{I}[\gamma,\alpha] = \{\gamma\}.$$

Proof. Obviously γ is in the intersection if $\gamma \in \mathcal{I}[\alpha,\beta]$. Suppose the intersection contains other point γ'. From Proposition 3.36, we have

$$\begin{aligned} d(\alpha,\beta) &= d(\alpha,\gamma) + d(\gamma,\beta) \\ &= d(\alpha,\gamma') + d(\gamma',\gamma) + d(\gamma,\gamma') + d(\gamma',\beta) \qquad (\gamma' \in \mathcal{I}[\alpha,\gamma], \gamma' \in \mathcal{I}[\beta,\gamma]) \\ &= d(\alpha,\gamma') + d(\gamma',\beta) + 2d(\gamma',\gamma) \qquad (\gamma' \in \mathcal{I}[\alpha,\beta]) \\ &= d(\alpha,\beta) + 2d(\gamma',\gamma). \end{aligned}$$

So $d(\gamma',\gamma) = 0$ or $\gamma' = \gamma$. □

We will need the following two lemmas.

Lemma 3.38. *Suppose $\gamma, \gamma' \in \mathcal{I}[\alpha,\beta]$. Then $\gamma \wedge \gamma' \in \mathcal{I}[\alpha,\beta]$. Moreover if there exists a lower bound of γ and γ' then $\gamma \vee \gamma' \in \mathcal{I}[\alpha,\beta]$*

Proof. Suppose $\gamma, \gamma' \in \mathcal{I}[\alpha,\beta]$. We first construct a geodesic joining α to β which passes $\gamma \wedge \gamma'$. Consider elements $\pi, \pi', \tau, \tau', \sigma, \sigma', \rho, \rho' \in \mathcal{T}_n$ to express $\alpha \wedge \gamma$, $\alpha \wedge \gamma'$, $\gamma \wedge \beta$, and $\gamma' \wedge \beta$ as follows

$\pi\alpha = \alpha \wedge \gamma = \tau\gamma$, $\pi'\alpha = \alpha \wedge \gamma' = \tau'\gamma'$, $\sigma\gamma = \gamma \wedge \beta = \rho\beta$, and $\rho'\gamma' = \gamma' \wedge \beta = \rho'\beta$.

By Remark 3.34, the following standard paths are all geodesic: $\pi \cdot \overline{\tau}$ joining α and γ; $\sigma \cdot \overline{\rho}$ joining γ and β; $\pi' \cdot \overline{\tau'}$ joining α and γ'; $\sigma' \cdot \overline{\rho'}$ joining γ' and β. Figure 3.9 illustrates those standard paths.

Claim 1: $\gamma \wedge \gamma' \leq \alpha \wedge \beta$. The standard path $\pi\overline{\tau}$ connecting α to γ is a geodesic by Remark 3.34. By the same reason, the standard path $\sigma\overline{\tau}$ connecting γ to β is a

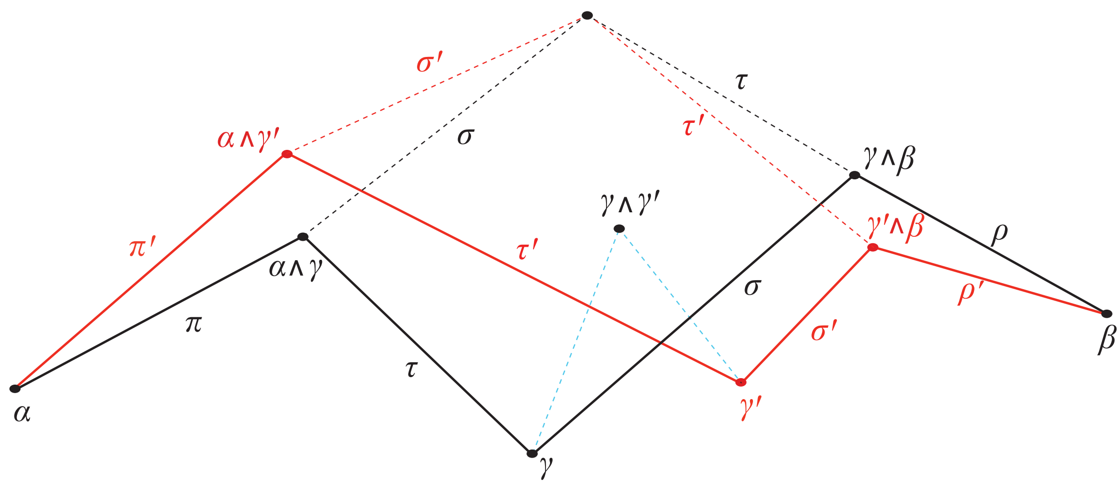

Figure 3.9: Two geodesics $p_1 \simeq \pi \cdot \overline{\tau} \cdot \sigma \cdot \overline{\rho}$ and $p_2 \simeq \pi' \cdot \overline{\tau'} \cdot \sigma' \cdot \overline{\rho'}$ of α and β passing γ and γ' respectively

geodesic. Consider the path p_1 defined by the concatenation of $\pi \cdot \overline{\tau}$ and $\sigma \cdot \overline{\rho}$,

$$p_1 \simeq \pi \cdot \overline{\tau} \cdot \sigma \cdot \overline{\rho}.$$

By Proposition 3.36, this path p_1 is a geodesic since $\gamma \in \mathcal{I}[\alpha, \beta]$. Similarly we have a geodesic p_2 of α and β which passes γ' given by

$$p_2 \simeq \pi' \cdot \overline{\tau'} \cdot \sigma' \cdot \overline{\rho'}.$$

By applying liftings to p_1 and p_2 one obtains geodesics $\tilde{p_1}$ and $\tilde{p_2}$ given by

$$\tilde{p}_1 \simeq \pi \cdot \sigma \cdot \overline{\tau} \cdot \overline{\rho},\ \tilde{p}_2 \simeq \pi' \cdot \sigma' \cdot \overline{\tau'} \cdot \overline{\rho'}.$$

From Remark 3.34, we see that the top vertex of these paths is $\alpha \wedge \beta$. Thus $\alpha \wedge \beta = \pi\sigma\alpha = \tau\sigma\gamma$ and $\alpha \wedge \beta = \pi'\sigma'\alpha = \tau'\sigma'\gamma'$. So $\alpha \wedge \beta \geq \gamma$ and $\alpha \wedge \beta \geq \gamma'$, and Claim 1 is verified.

A chain $\gamma \leq (\gamma \wedge \gamma') \leq (\alpha \wedge \beta)$ allows one to decompose τ and σ as $\tau = \tau_1\tau_2$ and

$\sigma = \sigma_1\sigma_2$ so that

$$\tau_1\sigma_1\gamma = \gamma \wedge \gamma', \;\; \tau_2\sigma_2(\gamma \wedge \gamma') = \alpha \wedge \beta. \tag{3.10}$$

Figure 3.10 illustrates this decomposition. Consider the path connecting α to $\gamma \wedge \gamma'$ and then to β defined by

$$p_3 \simeq \pi \cdot \sigma_1 \cdot \overline{\tau_2} \cdot \sigma_2 \cdot \overline{\tau_1} \cdot \overline{\rho}.$$

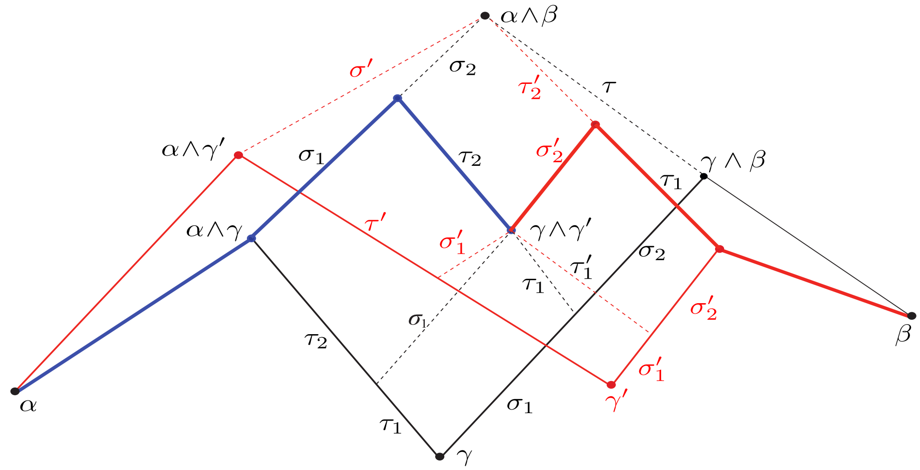

Figure 3.10: A geodesic $p_3 \simeq \pi \cdot \sigma_1 \cdot \overline{\tau_2} \cdot \sigma_2 \, \overline{\tau_1} \cdot \overline{\rho}$ of α and β passing $\gamma \wedge \gamma'$.

Apply appropriate liftings to the path p_3 to obtain the geodesic $\tilde{p_1}$. Since liftings do not increase lengths, two paths $\tilde{p_1}$ and p_3 have the same length. Thus p_3 is a geodesic joining α to β, which passes $\gamma \wedge \gamma'$. Therefore $\gamma \wedge \gamma' \in \mathcal{I}[\alpha, \beta]$.

For the second assertion of the Lemma, we assume the existence of $\gamma \vee \gamma'$. As before, use a chain $\gamma' \leq (\gamma \wedge \gamma') \leq (\alpha \wedge \beta)$ to get the decomposition of τ' and σ'; $\tau' = \tau_1'\tau_2', \; \sigma' = \sigma_1'\sigma_2'$ such that

$$\tau_1'\sigma_1'\gamma = \gamma \wedge \gamma', \;\; \tau_2'\sigma_2'(\gamma \wedge \gamma') = \alpha \wedge \beta \tag{3.11}$$

From identities (3.10) and (3.11) we have $\tau_2\sigma_2 = \tau_2'\sigma_2'$. Proposition 3.18 together with the existence of $\gamma \vee \gamma'$ implies that

$$\tau_1' \sigma_1' (\gamma \vee \gamma') = \gamma, \text{ and } \tau_1 \sigma_1 (\gamma \vee \gamma') = \gamma'.$$

Claim 2: $\tau_2 = \tau_2'$ and $\sigma_2 = \sigma_2'$. Look at the loop ℓ_1 formed by two ascending paths $\tau\sigma$ and $\sigma\tau$ emanating from γ. We first show ℓ_1 fits into the situation described in Proposition 3.18, i.e., $(\alpha \wedge \gamma) \wedge (\gamma \wedge \beta) = \alpha \wedge \beta$ and $(\alpha \wedge \gamma) \vee (\gamma \wedge \beta) = \gamma$. This loop has γ and $\alpha \wedge \beta$ as bottom and top vertices respectively. Recall that the path $p_1 \simeq \pi \cdot \overline{\tau} \cdot \sigma \cdot \overline{\rho}$ is a geodesic of α and β. The restriction p_4 of p_1 defined by

$$p_4 \simeq \overline{\tau} \cdot \sigma$$

is also a geodesic joining $\alpha \wedge \gamma$ and $\gamma \wedge \beta$. Consider the standard geodesic p_5, joining α to β, defined by

$$p_5 \simeq \pi \cdot \sigma \cdot \overline{\tau} \cdot \overline{\rho}.$$

This geodesic p_5 also passes $\alpha \wedge \gamma$ and $\gamma \wedge \beta$. So the restriction p_6 of p_5 defined by

$$p_6 \simeq \sigma \cdot \overline{\tau}$$

is a standard geodesic connecting $\alpha \wedge \gamma$ to $\gamma \wedge \beta$. By Remark 3.34, $(\alpha \wedge \gamma) \wedge (\gamma \wedge \beta) = \alpha \wedge \beta$. Moreover, by Proposition 3.18, $(\alpha \wedge \gamma) \vee (\gamma \wedge \beta) = \gamma$.

There is another loop ℓ_2 formed by two ascending paths $\tau'\sigma'$ and $\sigma'\tau'$ emanating from γ'. This loop ℓ_2 has $\alpha \wedge \beta$ as top vertex and γ' as bottom vertex. By an analogous argument that we applied to the loop ℓ_1, one can show

$$(\alpha \wedge \gamma') \wedge (\gamma' \wedge \beta) = \alpha \wedge \beta, \ (\alpha \wedge \gamma') \vee (\gamma' \wedge \beta) = \gamma'.$$

Again, one can deduce the following fact from ℓ_3, formed by two ascending paths

$\tau_1'\sigma_1'\tau\sigma$ and $\tau_1\sigma_1\sigma'\tau'$ emanating from $\gamma \vee \gamma'$.

$$(\alpha \wedge \gamma) \wedge (\gamma' \wedge \beta) = \alpha \wedge \beta,\ (\alpha \wedge \gamma) \vee (\gamma' \wedge \beta) = \gamma \vee \gamma'.$$

Now Proposition 3.18 implies that σ and τ do not share a letter t_i. Similarly there is no common letter between σ' and τ', and σ and τ'. So we have $\tau_2 = \tau_2'$ and $\sigma_2 = \sigma_2'$ from the identity $\tau_2\sigma_2 = \tau_2'\sigma_2'$.

Claim 3 $\tau_1 = \sigma_1' = 1$. Apply Proposition 3.18 to the loop ℓ_3 again to see that two diagonal edges of ℓ_3 are given by the same element of $\mathcal{T}_n$. In particular we have

$$\sigma = \tau_1\sigma_1\sigma'. \tag{3.12}$$

This means

$$\sigma = \sigma_1\sigma_2 = \tau_1\sigma_1\sigma_1'\sigma_2'$$

By Claim 2 we have $\tau_1\sigma_1' = 1$. This completes the proof since the path $p \simeq \pi\overline{\tau}\,\overline{\tau_1'}\,\sigma_1\,\sigma'\,\overline{\rho'}$ passes $\gamma \vee \gamma'$ and lifts to the geodesic p_3;

$$\begin{aligned} p &\simeq \pi\overline{\tau_2}\,\overline{\tau_1'}\,\sigma_1\,\sigma_2'\,\overline{\rho'} \\ &\simeq \pi\overline{\tau_2}\,\sigma_1\,\overline{\tau_1'}\,\sigma_2'\,\overline{\rho'} \\ &\simeq \pi\,\sigma_1\,\overline{\tau_2}\,\sigma_2\overline{\tau_1'}\,\overline{\rho'}. \end{aligned}$$

□

Corollary 3.39. *For any vertices α, β, $\mathcal{I}[\alpha, \beta]$ is convex in $\mathcal{C}_n^{(0)}$, i.e., for any $\gamma, \gamma' \in \mathcal{I}[\alpha, \beta]$, $\mathcal{I}[\gamma, \gamma'] \subset \mathcal{I}[\alpha, \beta]$.*

Proof. In case of $\gamma \leq \gamma'$, one can show that $\mathcal{I}[\gamma, \gamma'] \subset \mathcal{I}[\alpha, \beta]$ by using Proposition 3.18 and Lemma 3.38. For the general case, use induction on the distance between γ and γ' together with Lemma 3.38 and the previous observation to the pais of vertices

$$\gamma \leq \gamma \wedge \gamma', \ \gamma' \leq \gamma \wedge \gamma', \ \gamma \vee \gamma' \leq \gamma, \text{ and } \gamma \vee \gamma' \leq \gamma'$$

.

□

The following notion of orthant of a vertex is useful for us to examine the structure of $\mathcal{C}_n^{(0)}$ carefully. For a vertex $\alpha \in \mathcal{C}_n^{(0)}$, orthant $\mathcal{O}(\alpha)$ of α is defined by

$$\mathcal{O}(\alpha) = \{\beta \mid \beta \geq \alpha\}. \tag{3.13}$$

Note that $\mathcal{O}(\alpha)$ is also convex in $\mathcal{C}_n^{(0)}$ for any vertex α.

Lemma 3.40. *Suppose $\mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta] \neq \emptyset$ for some $\gamma \notin \mathcal{I}[\alpha, \beta]$. There exists unique $\delta_0 \in \mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta]$ with smallest distance from γ. Moreover $d(\alpha, \gamma) = d(\alpha, \delta_0) + d(\delta_0, \gamma)$.*

Proof. Suppose $\gamma \notin \mathcal{I}[\alpha, \beta]$ and $\mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta] = \{\delta_1, \cdots \delta_k\}$. Since γ is a common lower bound of those elements, there exists *glb* of $\{\delta_1, \cdots \delta_k\}$. Let δ_0 denote the *glb*. The Claim 1 in proof of Theorem 3.41 shows that δ_0 is the unique vertex in $\mathcal{I}[\alpha, \beta]$ with the smallest distance from γ. If $d(\alpha, \gamma) > d(\alpha, \delta_0) + d(\delta_0, \gamma)$ then it is not difficult to show there exists another vertex $\delta' \in \mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta]$ with smaller distance. So δ_0 satisfies the equality. □

Theorem 3.41. *$\mathcal{C}_n$ is a median graph for any $n \in \mathbb{N}$.*

Proof. From Lemma 3.37, it suffices to show $\mathcal{I}[\alpha, \beta] \cap \mathcal{I}[\beta, \gamma] \cap \mathcal{I}[\gamma, \alpha]$ is a singleton set for any α, β and $\gamma \notin \mathcal{I}[\alpha, \beta]$.

Case 1: $\mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta] \neq \emptyset$. From Lemma 3.40, we see that there exists unique $\delta_0 \in \mathcal{O}(\gamma)$ with smallest distance from γ. We want to show $\mathcal{I}[\alpha, \beta] \cap \mathcal{I}[\beta, \gamma] \cap \mathcal{I}[\gamma, \alpha] = \{\delta_0\}$. From Lemma 3.40 (the second assertion), a concatenation of two geodesics joining α to δ_0 and δ_0 to γ is again a geodesic. Thus the intersection of three intervals contains δ_0. Suppose the intersection contains another vertex w.

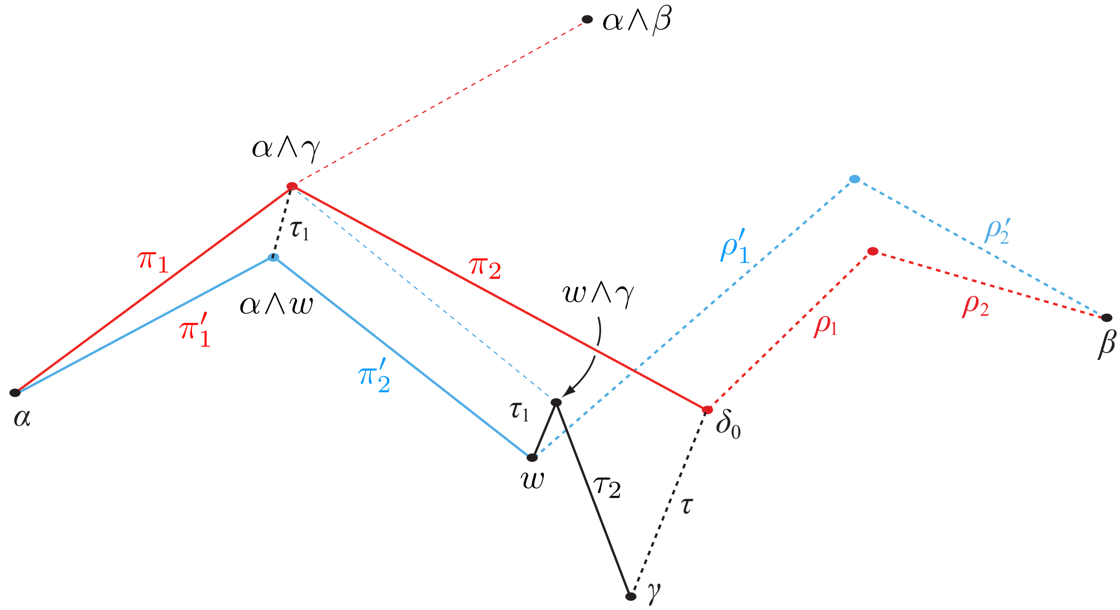

Figure 3.11: $\delta_0 \in \mathcal{O}(\gamma)$ with $d(w,\gamma) > d(\delta_0,\gamma)$

Claim 1: $d(w,\gamma) > d(\delta_0,\gamma)$. Say p_1, p_2, p_3 are standard geodesics joining α to δ_0, α to w and w to γ respectively, given by

$$p_1 = \pi_1\overline{\pi}_2,\ p_2 = \pi_1'\overline{\pi}_2',\ p_3 = \tau_1\overline{\tau}_2,$$

and $\delta_0 = \tau\gamma$ for some $\pi_1, \pi_2, \pi_1', \pi_2', \tau_1, \tau_2, \tau \in \mathcal{T}_n$. Figure 3.11 illustrates this situation for those paths p_1(solid red), p_2(solid blue) and p_3(solid black). Since the concatenation p_2p_3 is a geodesic joining α to γ, after applying liftings to p_2p_3, one obtains a standard geodesic passing $\alpha \wedge \gamma$. Consider the ascending path joining w to $\alpha \wedge \gamma$ given by $\tau_1\pi_2'$. This passes $w \wedge \gamma$ an so $w \wedge \gamma \in \mathcal{I}[\alpha,\beta]$ by Corollary 3.39. Since $w \wedge \gamma \geq \gamma$ we see that $w \wedge \gamma \geq \delta_0$ and $w \wedge \gamma \neq \delta_0$ by the definition of δ_0. Thus $d(w,\gamma) = d(w, w\wedge\gamma) + d(w\wedge\gamma,\gamma) > |\tau_2| > |\tau| = d(\delta_0,\gamma)$.

Claim 2: $w \notin\in \mathcal{I}[\alpha,\beta] \cap \mathcal{I}[\beta,\gamma] \cap \mathcal{I}[\gamma,\alpha]$. Say p_4 and p_5 are standard geodesics joining δ_0 to β, w to β respectively given by

$$p_4 = \rho_1\overline{\rho}_2,\ p_5 = \rho_1'\overline{\rho}_2'.$$

Since δ_0 belongs to the intersection the concatenation p_1p_4 is a geodesic with length $d(\alpha,\beta)$. If w belongs to the intersection then the concatenation p_2p_5 is a geodesic with the same length. So we must have

$$|\pi_1| + |\pi_2| + |\tau| = d(\alpha,\gamma) = |\pi_1'| + |\pi_2'| + |\tau_1'| + |\tau_2'|$$

and

$$|\pi_1| + |\pi_2| + |\rho_1| + |\rho_2| = |\pi_1'| + |\pi_2'| + |\rho_1'| + |\rho_2'|.$$

However claim 1 implies that $|\tau_1'| + |\tau_2'| > |\tau|$ and hence $|\pi_1| + |\pi_2| > |\pi_1'| + |\pi_2'|$. So we have $|\rho_1'| + |\rho_2'| > |\rho_1| + |\rho_2|$. This means that any standard path of the concatenation $\overline{p}_3\, p_5$ does not path $\gamma \wedge \beta$. So the path $\overline{p}_3\, p_5$ is not a geodesic. Since paths p_3 and p_5 are arbitrary, $w \notin \mathcal{I}[\gamma,\beta]$.

Case 2: $\mathcal{O}(\gamma) \cap \mathcal{I}[\alpha,\beta] = \emptyset$. Obviously $d(\mathcal{O}(\gamma), \mathcal{I}[\alpha,\beta]) = min\{d(u,v) \mid u \in \mathcal{O}(\gamma),\ v \in \mathcal{I}[\alpha,\beta]\} > 0$. Say the distance is $d > 0$. Let $D \subset \mathcal{O}(\gamma)$ denote the set of all vertices of $\mathcal{O}(\gamma)$ realizing d. Note that the cardinality of D is finite (there are finitely many elements in $\mathcal{O}(\gamma)$ whose height is less than $h(\alpha \wedge \beta)$ and, by the argument showing δ_0 is unique (Lemma 3.40), there are finitely many vertices realizing d whose height is greater than $h(\alpha \wedge \beta)$). Take glb over D and let ϵ denote this unique element. Say $\epsilon' \in \mathcal{I}[\alpha,\beta]$ is unique vertex with $d(\epsilon',\epsilon) = d$ (uniqueness follows from the argument in the proof of Lemma 3.40). See Figure 3.12.

Claim 3: $d(\alpha,\epsilon') + d + d(\epsilon,\gamma) = d(\alpha,\gamma)$ and $d(\beta,\epsilon') + d + d(\epsilon,\gamma) = d(\beta,\gamma)$. Pick any standard geodesics $q_1\overline{q_2}$ connecting α to ϵ' and ascending paths q_3 and q_4 joining ϵ' to ϵ, and γ to ϵ' respectively. Then the concatenation $q_1\overline{q}_2q_3\overline{q}_4$ is a geodesic. Consider the standard path $q_1q_2\overline{q}_3\overline{q}_4$. Observe that the ascending path q_1q_2 connecting α to $\alpha\wedge\gamma$ realizes the distance $d(\alpha, \mathcal{O}(\gamma))$. Thus any standard path given by $q_1q_2\overline{q}_3\overline{q}_4$ is a geodesic. Similar argument shows that $d(\beta,\epsilon')+d+d(\epsilon,\gamma) = d(\beta,\gamma)$. So ϵ' belongs to the intersection $\mathcal{I}[\alpha,\beta]\cap\mathcal{I}[\beta,\gamma]\cap\mathcal{I}[\gamma,\alpha]$.

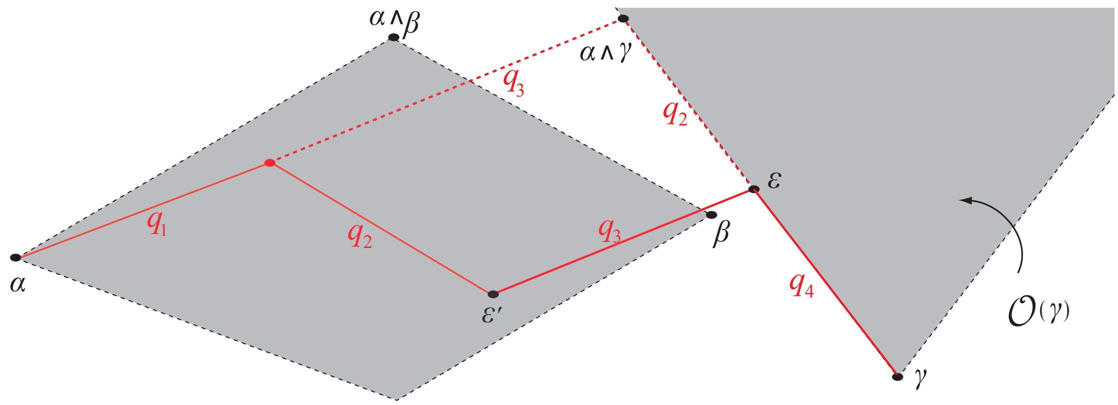

Figure 3.12: Vertices ϵ and ϵ' with $d(\epsilon',\epsilon) = d(\mathcal{O}(\gamma), \mathcal{I}[\alpha,\beta])$

Claim 4: $\mathcal{I}[\alpha,\beta]\cap\mathcal{I}[\beta,\gamma]\cap\mathcal{I}[\gamma,\alpha] = \{\epsilon'\}$. Suppose the intersection contains another vertex $w \in \mathcal{I}[\alpha,\beta]$. We apply an analogous trick as in **Case 2:** with small difference with $d(w,\gamma) > d + d(\epsilon,\gamma)$. One can show this inequality together with

$$d(\alpha,\gamma) = d(\alpha,w) + d(w,\gamma)$$

yields

$$d(\beta,\gamma) > d(\beta,w) + d(w,\gamma).$$

□

3.4 Properties of the action of $\mathcal{H}_n$ on X_n

In this section we examine the action of $\mathcal{H}_n$ on X_n. First we examine that the stabilizer of every cell is a finite symmetric group.

Lemma 3.42. *Suppose α is a vertex of X_n with $h(\alpha) = h$. Then the stabilizer of α is the finite symmetric group Σ_h on h points.*

Proof. Let $\Sigma_h \leq \mathcal{H}_n$ denote the symmetric group on a finite set $S(\alpha) = Y_n - (Y_n)\alpha$. We show the stabilizer of α is simply $\Sigma_h \leq \mathcal{H}_n$. If $g \in \Sigma$ then $\alpha g = \alpha$. Conversely, if $g \in \mathcal{H}_n$ and $\alpha g = \alpha$ then g restricted to the set $(Y_n)\alpha$ must be the identity. So $supp(g) \subset (\alpha)$ and hence $g \in \Sigma_h$. $\square$

Recall the following definition of a Morse function defined on a (affine) CW-complex complex X where $\varphi_j : \square^k \to \sigma_j^k \subset X^k$ denote the attaching map of k-cell σ_j^k.

Definition 3.43 (Morse function). A map $f : X \to \mathbb{R}$ is a *Morse function* if

- for every cell $\varphi_j(\square^k)$ of X $f\varphi_j : \square^k \to \mathbb{R}$ extends to an affine map $\mathbb{R}^m \to \mathbb{R}$ and $f\varphi_j$ is constant only when $k = 0$ and

- the image of the 0-skeleton is discrete in $\mathbb{R}$.

Note that the map h defined on a cubing X_n has the property that $h\varphi_j : \square^k \to \mathbb{R}$ extends to the standard height function $\mathbb{R}^k \to \mathbb{R}$ up to the translation by $h\varphi(0)$ (see Figure 3.4). Note also that $h\varphi_j : \square^k \to \mathbb{R}$ is trivial only when $k = 0$ for all j. Moreover, the image $h(X_n^{(0)})$ is just $h(\mathcal{C}_n) = h(\mathcal{M}_n) = \mathbb{Z}_{\geq 0}$. Therefore the map h is a Morse function.

Remark 3.44. The map $h : X_n \to \mathbb{R}_{\geq 0}$ satisfies the following

$$h(\alpha g) = h(\alpha) \text{ and } h(t\alpha) = h(t) + h(\alpha)$$

for all $g \in \mathcal{H}_n$ and $t \in \mathcal{T}_n$. In that sense, the action of $\mathcal{H}_n$ is 'horizontal' and the action of $\mathcal{T}_n$ is 'vertical'.

Let $X_{n,r}$ denote the subcomplex of X_n consisting of cubes up to height r, i.e.,

$$X_{n,r} := \{\sigma \in X_n \mid h(\sigma) \subset [0, r]\}.$$

Lemma 3.45. *Suppose a k-cube $\sigma \subset X_n$ is generated by $T = \{\tau_i, \cdots, \tau_k\} \subset \{t_1, \cdots, t_n\}$ with bottom vertex α. There exists $g \in \mathcal{H}_n$ such that $\sigma \cdot g$ is the k-cube generated by T with bottom vertex $t_1^{h(\alpha)}$.*

Proof. From the proof of Lemma 3.13, we see that there exists an element $g \in \mathcal{H}_n$ such that $\alpha g = t_1^{h(\alpha)}$. Since the action of $\mathcal{H}_n$ on X_n is cellular, a cube σg is the desired cube bottom vertex $t_1^{h(\alpha)}$. □

Corollary 3.46. *For $r \in \mathbb{Z}_{\geq 0}$, $\mathcal{H}_n$ acts on $X_{n,r}$ cocompactly.*

Proof. Fix $r \in \mathbb{Z}_{\geq 0}$. There are finitely many k-cubes with bottom vertex $t_1^{r'}$ for $1 \leq r' \leq r$ and $0 \leq k \leq n$. Lemma 3.45 implies that for any cube $\sigma \subset X_{n,r}$ there exists $g \in \mathcal{H}_n$ such that $\sigma \cdot g$ is a k-cube with bottom vertex $t_1^{r'}$ for some $r' \leq r$. So the quotient $X_{n,r}/\mathcal{H}_n$ is finite. □

Definition 3.47 (Semi-simple group action). Let X be a metric space and let g be an isometry of X. The displacement function of g is the function $d_g : X \to \mathbb{R}_{\geq 0}$ defined by $d_g(x) = d(x, x.g)$. The translation length of d_g is the number $|d_g| := inf\{d_g(x) | x \in X\}$. The set of points where d_g attains this infimum will be denoted Min(d_g). More generally, if G is a group acting by isometries on X, then Min(G):= $\cap_{g \in G}$ Min(d_g). An isometry d_g is called semi-simple if Min(d_g) is non-empty. An action of a group by isometries of X is called semi-simple if all of its elements are semi-simple.

The following theorem summarizes the action of $\mathcal{H}_n$ on X_n, see Figure 3.13 for the underlying theme.

Theorem A. *For each integer $n \geq 1$, there exists a n-dimensional cubing X_n and a Morse function $h : X_n \to \mathbb{R}_{\geq 0}$ such that $\mathcal{H}_n$ acts on X_n properly (but not cocompactly) by height-preserving semi-simple isometries. Furthermore, for each $r \in \mathbb{R}_{\geq 0}$ the action of $\mathcal{H}_n$ restricted to the level set $h^{-1}(r)$ is cocompact.*

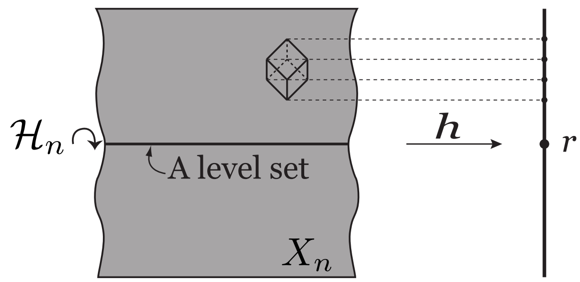

Figure 3.13: A Morse function h on a CAT(0) cubical complex X

Proof. **Proper action.** We want show that, for $\alpha, \beta \in X_n^{(0)}$ the set $S(\alpha, \beta) := \{g \in \mathcal{H}_n | \alpha g = \beta\}$ is finite. An element α has the right inverse (left inverse in composition of functions) $\alpha^{-1} : (Y_n)\alpha \to Y_n$. So if $g \in S(\alpha, \beta)$ then $g_{|(Y_n)\alpha}$ is completely determined by

$$g = \alpha^{-1}\beta.$$

This means that one can decompose g by

$$g = \begin{cases} \alpha^{-1}\beta & \text{on } (Y_n)\alpha \\ f & \text{on } S(\alpha) \end{cases}$$

for some $f \in \mathcal{H}_n$ with $supp(f) \subset S(\alpha)$. Since there are finitely many f with $supp(f) \subset S(\alpha)$, $S(\alpha, \beta)$ is finite. Note that if $\alpha = \beta$ then $S(\alpha, \beta)$ is simply the finite symmetric group on $h(\alpha) = h(\beta)$ points that we discussed in Lemma 3.42. Now suppose σ is a cube with vertices $\{\alpha_1, \cdots, \alpha_k\}$. If σg intersects σ non trivially then $g \in S(\alpha_i, \alpha_j)$ for some $1 \leq i, j \leq k$. Therefore the action is proper.

Height-preserving isometries. The action preserves height by Remark 3.44. To show the action in question is by isometries, we want to use the fact that two actions of $\mathcal{T}_n$ and $\mathcal{H}_n$ on X_n commute. Suppose $\alpha, \beta \in X_n^{(0)}$. Consider the smallest convex set H containing α and β, i.e., the intersection of all convex sets containing α and β. Observe that the cell structure of X_n is completely determined by the action of $\mathcal{T}_n$ on $X_n^{(0)}$. Since the two actions of $\mathcal{T}_n$ and $\mathcal{H}_n$ on X_n commute, Hg is a convex set with the same cell structure of H. By Corollary 3.32, α and β are joined by the unique geodesic ℓ, which lies in H. The image ℓg is the local geodesic (in Hg) joining αg and βg. Since Hg is convex ℓg is the global geodesic. So we have $d(\alpha,\beta) = d(\alpha g, \beta g)$ for $\alpha, \beta \in X_n^{(0)}$ and $g \in \mathcal{H}_n$.

Semi-simple action. For any element $f \in \mathcal{H}_n$ with finite order, it is not difficult to check that $d_f = 0$ and $\mathrm{Min}(d_g)$ is non empty. For example one can take $\alpha \in \mathcal{M}_n$ such that $supp(f) \subset S(\alpha)$ to see $d(\alpha f, \alpha) = 0$ and $\alpha \in \mathrm{Min}(d_f)$. Suppose $g \in \mathcal{H}_n$ with $\varphi(g) = (m_1, \cdots, m_n) \neq \overrightarrow{0}$ $(m_1 = -(m_2 + \cdots + m_n))$.

Claim: $d_g = \sqrt{\sum m_i^2}$. We want to find explicit element $\alpha \in X_n^{(0)}$ realizing d_g. The idea is to choose a vertex α with big enough height so that 'translation' by g is realized clearly. Consider the element $g' = g_1^{m_2} g_2^{m_3} \cdots g_{n-1}^{m_n}$. Since $\varphi(g) = \varphi(g)'$, there exists $f \in \mathcal{H}_n$ such that $g = fg'$ and that $supp(f) \subset B_{n,r}$ where $B_{n,r} \subset Y_n$ is the ball centered at the origin of radius r. Set $r' = max\{r, \sum |m_i|\}$ and consider the element $\alpha = t_1^{r'} t_2^{r'} \cdots t_n^{r'}$. The maximality of r' ensures that $\alpha, \alpha g' \in \mathcal{T}_n$. By Corollary 3.16, there exists $\alpha \vee \alpha g'$. It is possible to find explicit expression for $\alpha \wedge \alpha g'$ as follows. Define $k_i, k_i' \in Z$, $i = 1, \cdots, n$ by

$$k_1 = max\{0, -\sum_2^n m_i\},\ k_1' = max\{0, \sum_2^n m_i\} \text{ and}$$

$$k_i = max\{0, m_i\},\ k_i = max\{0, -m_i\} \text{ for } 2 \leq i \leq n.$$

Since $\varphi(\alpha) = (r', \cdots, r')$ and $\varphi(\alpha g') = (r' + m_1, r' + m_2, \cdots, r' + m_n)$, the top vertex is given by

$$\tau\alpha = \alpha \wedge (\alpha g') = \tau'(\alpha g') \tag{3.14}$$

where $\tau = t_1^{k_1} \cdots t_n^{k_n}$ and $\tau' = t_1^{k_1} \cdots t_n^{k_n}$. By Proposition 3.18, $\alpha \vee \alpha g'$ satisfies

$$\alpha = \tau'(\alpha \vee \alpha g') \text{ and } \alpha g' = \tau(\alpha \vee \alpha g'). \tag{3.15}$$

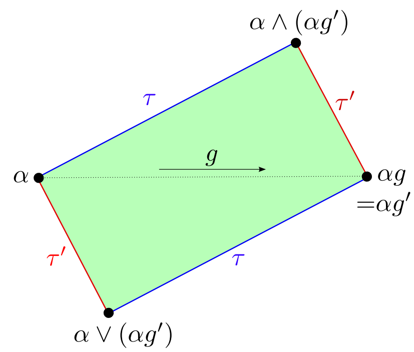

Figure 3.14: If $h(\alpha)$ is big enough, the translation by g is realized in a 'big rectangle' R

Equalities (3.14) and (3.14) imply that there exists a 'big rectangle' R which contains $\alpha \wedge \alpha g'$ and $\alpha \vee \alpha g'$ as its top and bottom vertices respectively (as described in Proposition 3.18). See Figure 3.14. Observe that R is the convex hull of two vertices α and $\alpha g'$. The diagonal of R joining α and $\alpha g'$ has length $\sqrt{\sum m_i^2}$. The diagonal is the geodesic joining α to $\alpha g'$ because no smaller cube contains those two vertices. So we have

$$d(\alpha, \alpha g) = d(\alpha, \alpha f g') = d(\alpha, \alpha g') = \sqrt{\sum m_i^2}.$$

Note that the size of R is determined by $\varphi(g) = (m_1, \cdots, m_n)$ and that R has the smallest size among convex hulls containing β and βg for $\beta \in X_n^{(0)}$. This means $d_g \geq d(\alpha, \alpha g)$. Therefore d_g attains minimum at α and $\alpha \in$Min(d_f).

Cocompact action. Note that $h^{-1}(r) \subset X_{n,r'}$ for $r \in \mathbb{R}$ and $r \leq r' \in \mathbb{Z}$. By

Remark 3.46, the action on the level set is cocompact for each $r \in Z_{\geq 0}$. $\square$

3.5 Finiteness Properties of $\mathcal{H}_n$

In this section we discuss finite properties of $\mathcal{H}_n$. We first recall properties F_n and FP_n (see [6],[9]) which are generalized concepts of being finitely generated and finitely presented of groups.

We say a group $G = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ is *finitely generated* if $\mathcal{A}$ is finite and *finitely presented* if both $\mathcal{A}$ and $\mathcal{R}$ are finite. These finite conditions can be interpreted topologically via *presentaion* 2*-complex (Cayley complex)* $K = K(\mathcal{A}; \mathcal{R})$. K has one vertex and it has one edge e_a^1 (oriented and labelled by a) for each generator $a \in \mathcal{A}$. The 2-cells e_r^2 of K are indexed by the relators $r \in \mathcal{R}$; if $r = a_1 \cdots a_k$ then σ_r is attached along the loop labelled by $a_1 \cdots a_k$. From the construction of K we see that it has finite 1-skeleton if G is finitely generated and it has finite 2-skeleton as well if G is finite presented. By the Seifert-Van Kampen theorem we have $\pi_1(K) \cong G$. Note that the existence of a complex K with finite 2-skeleton with $\pi_1(K) \cong G$ guarantees finite presentedness of a group G.

More general finiteness properties are based on $K(G,1)$ spaces (*Eilenberg-Mac Lane complex*). For a group G, a complex K is called a $K(G,1)$ complex if $\pi_1(K) \cong G$ and the universal cover $\widetilde{K}$ is contractible. It is known that for a group G there exists unique $K(G,1)$ complex having one vertex up to homotopy.

Definition 3.48 (**Property** F_n)**.** We say a group G has type F_n if there exists a $K(G,1)$ complex having finite n-skeleton.

Consider the augmented chain complex $C_*(\widetilde{K}; \mathbb{Z})$ of the universal cover of a $K(G,1)$ complex K.

$$\cdots \xrightarrow{\partial_3} C_2(\widetilde{K}; \mathbb{Z}) \xrightarrow{\partial_2} C_1(\widetilde{K}; \mathbb{Z}) \xrightarrow{\partial_1} C_0(\widetilde{K}; \mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

This is unique free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ up to chain homotopy. Note that each $C_i(\widetilde{K};\mathbb{Z})$ is finitely generated free $\mathbb{Z}G$-module if K has finite i-skeleton. A module is *projective* if it is a direct summand of a free module.

Definition 3.49 (**Property** FP_n)**.** We say a group G has type FP_n if there exists a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$ which is finitely generated in dimensions $\leq n$.

Remark 3.50. An immediate consequence is that if a group G is has type F_n then it has type FP_n for integers $n \in \mathbb{N}$.

A G-CW complex is a complex K together with a homomorphism $G \to AutK$. A *G-filtration* of a CW complex K is a countable collection of G-subcomplexes $K_0 \subset K_1 \subset \cdots$ such that $K = \bigcup K_i$.

Let K be a contractible G-CW complex which admits a G-filtration $\{K_i\}$ satisfying

- the stabilizer of every cell is finitely presented and has type FP_n for all n
- each K_i is finite mod G
- for all sufficiently large j, K_{j+1} is obtained from K_j by the adjunctions of n-cells, up to homotopy.

Theorem 3.51 (**Brown's Criterion**)**.** *With the above assumption, G has type FP_{n-1} but not FP_n. If $n \geq 3$ then G is finitely presented.*

Brown show the following finiteness properties of $\mathcal{H}_n$ by constructing a CW complexes satisfying the above criterion ([7]). By Theorem A, a cubing X_n, $n \geq 1$, satisfies the criterion above, Remark 3.46.

Corollary B. *For $n \geq 2$, $\mathcal{H}_n$ is of type FP_{n-1} but not FP_n, it is finitely presented for $n \geq 3$.*

Proof. We show our X_n satisfies the above Brown's criterion. By Corollary 3.32, X_n is contractible. Lemma 3.42 implies the first condition is satisfied since a finite symmetric group satisfies required finiteness properties. The second condition follows from Remark 3.46. With respect to a Morse function $h : X_n \to \mathbb{R}$, each descending link of a vertex $v \in X_n$ is homotopic to bouquet of spheres $\mathbb{S}^{n-1}$ if the height of $h(v) \geq 2n-1$ (Lemma 3.52). This means that the passage from $X_{n,h}$ to $X_{n,h+1}$ consists of the adjunction of n-cubes, up to homotopy. The theorem therefore follows from the criterion Theorem 3.51. □

Lemma 3.52. *If $h \geq 2n-1$ then $L_{n,h}$ homotopic to a bouquet of spheres $\mathbb{S}^{n-1}$.*

Proof. We argue by induction on n. If $n = 1$, then $L_{n,h}$ is a bouquet of spheres $\mathbb{S}^0$, provided $h \geq 2 \cdot 1 - 1$. Assume $L_{n-1,k}$ is homotopy equivalent to a bouquet of copies of $\mathbb{S}^{n-2}$ if $k \geq 2h - 1$ and $n \geq 2$.

We use Betvina-Brady Morse theory to understand the topology. Equip $L = L_{n,h}$ with a Morse function $f : L \to [0, 2] \subset \mathbb{R}$ as follows. First consider the partition of the vertex set of $L_{n,h}$: $L_0 = \{(x, y) \mid x \geq 2,\ y \geq 2\} \cup \{(1, 1)\}$, $L_1 = \{(1, y) \mid y \geq 2\}$ and $L_2 = \{(x, 1) \mid x \geq 2\}$. Arrange the vertices so that $f(x, y) = h$ if $(x, y) \in L_h$, $h = 0, 1, 2$. See Figure 3.15.

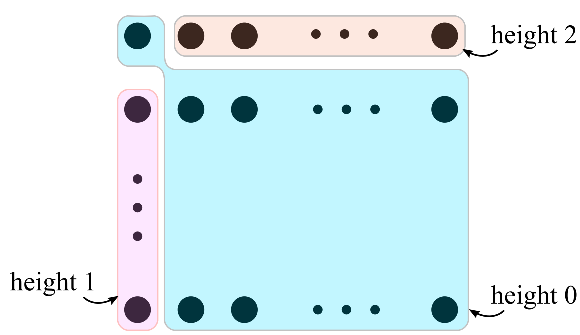

Figure 3.15: The complex $L_{n,h}$ equipped with a Morse function $f : L_{n,h} \to [0, 2]$

It is clear that $f^{-1}(0)$ is a cone on $L_{n-1,h-1}$ and so it is contractible. Note that even if there are horizontal cells at height 0 it is still true that $f^{-1}[0, 1]$ is homotopy equivalent to $f^{-1}(0)$ with the copies of $Lk_{\downarrow}((1,y),L)$ conned off $((1,y) \in L_1)$. Observe that the descending link of a vertex $(1,y) \in L_1$ (in L) is spanned by vertices $(x,y') \in L_0$ with $y' \neq y$. Now $Lk_{\downarrow}((1,y),L) \simeq L_{n-1,h-2}$, $f^{-1}[0,1]$ is obtained from $f^{-1}(0)$ by adjoining, for each $(1,y)$, a cone over $L_{n-1,h-2}$. In view of inductive hypothesis, $f^{-1}[0,1]$ is homotopy equivalent to a bouquet of spheres $\mathbb{S}^{n-1}$.

Observe that $Lk_{\downarrow}((x,1),L) \simeq L_{n-1,h-1}$ for each vertex $(x,1) \in L_2$. So, by the inductive hypothesis again, the complex $L = f^{-1}[0,2]$ is homotopy equivalent to $f^{-1}[0,1]$ with the copies of spheres $\mathbb{S}^{n-2}$ conned off. (Similar proof can be found in [7].) $\square$

Chapter 4

Isoperimetric inequalities for $\mathcal{H}_n$

4.1 Dehn functions of finitely presented groups

We recall the (algebraic) definition of Dehn function of a group from [3]. Let $P = \langle \mathcal{A} \mid \mathcal{R} \rangle$ be a finite presentation for a group G with the identity 1_G. A word w is an element of the free monoid with the generating set $\mathcal{A} \cup \mathcal{A}^{-1}$. Denote the length of w by $|w|_G$. We say w is null-homotopic when $w = 1_G$, i.e., w lies in the normal closure of $\mathcal{R}$ in the free group $F(\mathcal{A})$. We define the *area* of a null-homotopic word w to be

$$Area(w) := min\{N \mid w =^{free} \prod_{i=1}^{N} u_i^{-1} r_i u_i \text{ with } u_i \in F(\mathcal{A}),\ r_i \in \mathcal{R}^{\pm 1}\}.$$

The *Dehn function* of P is the function $\delta_P : \mathbb{N} \to \mathbb{N}$ defined by

$$\delta_P(x) := max\{Area(w) \mid w = 1_G, |w|_G \leq x\}.$$

Although the Dehn function $\delta_P(x)$ depends on the presentation P, asymptotic growth type of $\delta_P(x)$ (as x tends to infinity) only depends on G up to the equivalence relation $\simeq$ defined as follows (see [3]). Two functions $f, g : \mathbb{N} \to [0, \infty)$ are said to be $\simeq$ equivalent if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means that there exists a constant $C > 0$ such that $f(x) \leq Cg(Cx+C)+Cx+C$ for all $x \in \mathbb{N}$. Up to this equivalence relation, any finite presentation P of a group G determines the same asymptotic growth type, which is called the *Dehn function* of G. The Dehn function of G is denoted by $\delta_G(x)$. The Dehn function is an important invariant of group theory. If a group G is CAT(0), the $\delta_G(x)$ is bounded above by x^2, see [5]. A group G is hyperbolic if and only if $\delta_G(x) \simeq x$, see [10]. In particular, every finite group has a linear Dehn function.

Remark 4.1. For any $m \in \mathbb{N}$ the symmetric group Σ_m on $\{1, 2, \cdots, m\}$ satisfies $\delta_{\Sigma_k}(x) \leq Cx + C$ for some $C > 0$. Note that the constant C depends on the group Σ_m and hence on m. We need an upper bound for $\delta_{\Sigma_m}(x)$ which only depends on x (see Lemma 4.3).

4.2 Exponential isoperimetric inequalities for $\mathcal{H}_n$

In this section we aim to establish exponential upper bounds for $\delta_{\mathcal{H}_n}$ for $n \geq 3$.

Theorem D. *For $n \geq 3$, the Dehn function $\delta_{\mathcal{H}_n}(x)$ satisfies*

$$\delta_{\mathcal{H}_n}(x) \preceq e^x.$$

We start with the case $n = 3$. Let $B_{3,x} \subset Y_3$ centered at the origin with radius $x \in N$. Let $\Sigma_{3,x} \leq \mathcal{H}_3$ denote the finite symmetric group on $B_{3,x}$ given by (2.9). The following is an outline for the proof.

1. For a given word $w = 1 \in \mathcal{H}_3$ with $|w| \leq x$, rewrite w by $w' \in \Sigma_{3x}$ by using a canonical way such that

- $|w'|_{\Sigma_{3,x}} \leq x^5$,

- the gap between w and w' is filled with area$\leq x^2$.

2. Establish a cubic upper bound for $\delta_{\Sigma_{3,x}}(x)$.

3. Bound $Area_{\mathcal{H}_3}(r)$ for all relators r of $\Sigma_{3,x}$ by e^x,

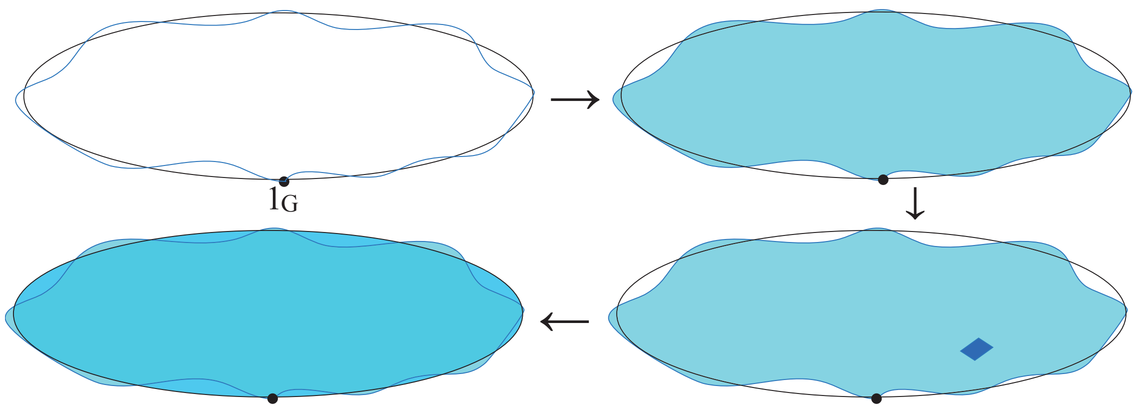

Figure 4.1: Sketch of the proof for $\delta_{\mathcal{H}_3}(x) \preccurlyeq e^x$

Figure 4.1 illustrates the strategy. Now we have a desired upper bound for the area of w since

$$Area_{\mathcal{H}_3}(w) \leq (x^5)^3 \cdot e^{Ax+A} + x^2 \leq e^{(A+18)x+A} + x^2 \preccurlyeq e^x. \tag{4.1}$$

We remark that, in each step, any bound not exceeding exponential function is good enough for us. We have established exponential upper bound for $\delta_{\mathcal{H}_3}(x)$ up to those claims.

First we establish a cubic upper bound for $\delta_{\Sigma_{3,x}}(x)$ by using the following fact. Let S_m denote the finite symmetric group on m points with the Coxeter presentation given in (2.5), where $\{\sigma_1, \cdots, \sigma_{m-1}\}$ is the generating set. Let $s_i = \sigma_{\phi(i)}$ for some $\phi : \mathbb{N} \to \{1, 2, \cdots, m-1\}$.

Theorem 4.2 (Deletion Theorem, [14])**.** *Suppose* $w = s_1 s_2 \cdots s_k$*. If* $|w| < k$ *then there exists* i *and* j *(*$1 \le i < j \le k$*) such that*

$$s_{i+1}s_{i+2}\cdots s_j = s_i s_{i+1}\cdots s_{j-1}, \tag{4.2}$$

and so

$$w = s_1 s_2 \cdots \widehat{s}_i \cdots \widehat{s}_j \cdots s_k.$$

Lemma 4.3. *For any* $m \in \mathbb{N}$*, a null-homotopic word* $w \in \S_m$ *with* $|w| \le x$ *satisfies* $Area(w) \le x^3$*.*

Proof. We first show any word $w \in \S_m$ given by the form

$$s_j = s_i^{s_{i+1}s_{i+2}\cdots s_{j-1}}$$

as in (4.2) has at most quadratic area in $|j-i|$. ($g^h := h^{-1}gh$.) Suppose $w = s_{k+1}(\overline{s}_0{}^{s_1\cdots s_k})$ represents the identity of Σ_m. We want to show $Area(w) \le k^2$ by induction on k.

The base case is obvious since $s_2(\overline{s}_0{}^{s_1}) = 1$ is a single commutation relation. We may assume no two consecutive letters in w are the same. Now suppose $w = s_{k+2}(\overline{s}_0{}^{s_1\cdots s_{k+1}})$ is null-homotopic. We consider two cases.

Case 1. There exists i_0 ($k+2 \ge i_0 \ge 1$) such that either

$$[s_{i_0}, s_i] = 1 \text{ for all } i_0 > i \ge 0 \tag{4.3}$$

or

$$[s_{i_0}, s_i] = 1 \text{ for all } k+2 \ge i > i_0. \tag{4.4}$$

If the condition (4.3) holds then one can apply commutation relations consecutively to decompose the diagram of w into a $s_{i_0}-$corridor with length $2(i_0-1)+1$

and a diagram of $w' = s_{k+1}(\overline{s}_0{}^{s_1 \cdots \widehat{s}_{i_0} \cdots s_k})$. (See Figure 4.2) Thus, by induction assumption, we have

$$Area(w) \leq Area(w') + 2(i_0 - 1) + 1 < (k+1)^2.$$

By using an analogous argument together with condition (4.4), one can draw the same upper bound.

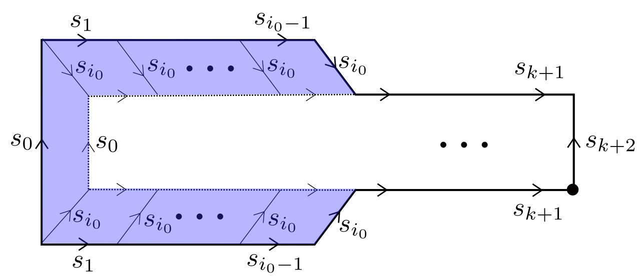

Figure 4.2: The diagram for w can be reduced by commutation relations in Case 1.

Case 2. For each $k+1 \geq i \geq 1$ there exists $i'' > i > i'$ such that

$$[s_{i''}, s_i] \neq 1 \text{ and } [s_i, s_{i'}] \neq 1. \tag{4.5}$$

The above condition simply says that $\sigma(i'') = \sigma(i) \pm 1$ and $\sigma(i') = \sigma(i) \pm 1$. We need to examine subwords of w. Let w_j denote the subword $\overline{s}_0{}^{s_1 \cdots s_j}$ for $j = 1, \cdots, k+1$. For $j = 0$, w_0 is simply defined to be s_0. Note that each w_j is a transposition since it is a conjugation of a transposition s_0. For a transposition $s \in \Sigma_m$ let $d_+(s)$ and $d_-(s)$ denote the two points of $supp(s)$ with $d_+(s) > d_-(s)$. The function $d : \{0, 1, \cdots, k+1\} \to \mathbb{N}$ measures the difference $d_+ - d_-$, i.e., $d(j) = d_+(w_j) - d_-(w_j)$ for $j = 0, 1, \cdots k+1$. Set $D(j) = \{d_-(w_j), d_-(w_j) + 1, \cdots, d_+(w_j)\}$. Note that $d(0) = d(k+1) = 1$. So there exists i such that $d(i+1) \leq d(i)$, say i_0 is the smallest such number. Observe that $|d(j) - d(j+1)| \leq 1$ for all j since conjugating by s_j introduces at most one point to $D(j-1)$. The following observation is crucial to establish desired bound;

$$d(w_{i_0}) = i_0 + 1.$$

This identity implies there is 1-to-1 correspondence between

$$D(i_0) - D(0) \text{ and } \{s_1, s_2, \cdots, s_{i_0}\}. \tag{4.6}$$

This bijection and the condition (4.5) implies

$$d_+(s_i) = d_+(s_j) + 1 \Rightarrow i > j \tag{4.7}$$

for all i with $i_0 \leq i \leq 1$.

From the 1-to-1 correspondence (4.6) and the fact that $D(s_{i_0+1}) \subset D(w_{i_0})$, there exists unique $i_0 > i' \geq 0$ such that $s_{i_0+1} = s_{i'}$. Say $s_{i_0+1} = s_{i'} = (p\; p{+}1)$, transposition exchanging p and $p{+}1$. On the other hand, if $d_-(s_{i_0+1}) \geq d_+(s_0)$ then, from (4.7), we see that there exists unique i'' with $i_0 \geq i'' > i'$ such that $s_{i''} = (p{+}1\; p{+}2)$. Observe that s_{i_0+1} commutes with s_i for all i with $i_0 \geq i > i''$ and that $s_{i'}$ commute with s_j for all j with $i'' > j \geq i'$. Apply those commutation relations to rearrange letters in the expression of $w = s_{k+2}(\overline{s}_0{}^{s_1 \cdots s_{k+1}})$ so that s_{i_0+1}, $s_{i''}$ and $s_{i'}$ show up in a row. Then apply the relation

$$s_{i_0+1} s_{i''} s_{i'} = s_{i'} s_{i''} s_{i'} = s_{i''} s_{i'} s_{i''},$$

where the second identity comes from the braid relation $((p\,p{+}1)(p{+}1\; p{+}2))^3 = 1$. Now one applies the argument of **Case 1** since $s_{i''}$ commute with all s_i for $i \leq i''$. In all, from

$$
\begin{aligned}
s_1 \cdots s_{i_0} s_{i_0+1} &= s_1 \cdots s_{i'} \cdots s_{i''} \cdots s_{i_0} s_{i_0+1} \\
&= s_1 \cdots \widehat{s}_{i'} \cdots s_{i''-1} (s_{i'} s_{i''} s_{i_0+1}) s_{i''+1} \cdots s_{i_0} \\
&= s_1 \cdots \widehat{s}_{i'} \cdots s_{i''-1} (s_{i''} s_{i'} s_{i''}) s_{i''+1} \cdots s_{i_0} \\
&= s_{i''} s_1 \cdots \widehat{s}_{i'} \cdots s_{i''-1} (s_{i'} s_{i''}) s_{i''+1} \cdots s_{i_0}
\end{aligned}
$$

we have

$$
w = s_{k+2}(\overline{s}_0{}^{s_1 \cdots s_{i''} \cdots s_{k+1}}) = s_{k+2}(\overline{s}_0{}^{s_{i''} s_1 \cdots \widehat{s}_{i''} \cdots s_{k+1}}) = s_{k+2}(\overline{s}_0{}^{s_1 \cdots \widehat{s}_{i''} \cdots s_{k+1}})
$$

The number of required relators in the above process is at most

$$
2\{(i_0 - i'' - 1) + (i'' - i' - 1) + 1 + i''\} + k^2 \leq k^2 + 2k < (k+1)^2
$$

since $i_0 \leq \frac{k}{2}$. So we have shown any word in the form of (4.2) has area at most $|j - i|^2 \leq k^2$.

This means whenever one applies the identity (4.2) to a null-homotopic word $s_1 \cdots s_x \in \Sigma_m$, the number of relators is bounded by x^2. At the same time one can reduce the number of generators in the expression by two. So we have $\delta_{\Sigma_m}(x) \leq x^3$. $\square$

As discussed in Remark 2.3, one of important feature of $\mathcal{H}_n$ is $\Sigma_{n,\infty} \hookrightarrow \mathcal{H}_n$ when $n \geq 3$. If a word $w \in \mathcal{H}_3$ is null-homotopic then $w \in \Sigma_{3,\infty} = \cup_r \Sigma_{3,r}$ (Lemma 2.11) It is natural to ask the minimum r so that $w \in \Sigma_{3,r}$. We have a reasonable bound for r as well as the length of w in $\Sigma_{3,r}$.

Lemma 4.4. *There is a canonical way so that any null-homotopic word $w \in \mathcal{H}_3$ with $|w| \leq x$ can be written as a word $w' \in \Sigma_{3,x}$ with length at most $O(x^5)$.*

Proof. Suppose $w \in \mathcal{H}_3$ is a null homotopic word with $|w|_{\mathcal{H}_3} = x$. Then w can be written as $w = g_1^{m_1} g_2^{n_1} \alpha^{\epsilon_1} \cdots g_1^{m_k} g_2^{n_k} \alpha^{\epsilon_k}$ for some $m_i, n_i \in \mathbb{Z}$, $\epsilon_i \in \{-1, 0, 1\}$.

Use the identity

$$g\alpha \equiv \alpha^{g^{-1}} g \tag{4.8}$$

to rewrite w as

$$w \equiv f\, g_1^{m_1} g_2^{n_1} \cdots g_1^{m_k} g_2^{n_k}, \tag{4.9}$$

where f is a product of at most $k(< x)$ many transpositions whose supports belong to the ball $B_{3,x}$ of radius x. The symbol $\equiv$ indicates that no relators are required in the above rewriting. The identity in free group $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1} = \alpha$ allows one to exchange g_1 and g_2;

$$g_2 g_1 = \alpha^{-1} g_1 g_2,\ g_2 g_1^{-1} = \alpha^{g_1} g_1^{-1} g_2, \tag{4.10}$$

$$g_2^{-1} g_1 = \alpha^{g_1 g_2 g_1^{-1}} g_1 g_2^{-1}, \text{ and } g_2^{-1} g_1^{-1} = (\alpha^{-1})^{g_2 g_1} g_1^{-1} g_2^{-1}. \tag{4.11}$$

Use identities (4.10) and (4.11) together with (4.8) to rewrite (4.9) again as

$$w = f\, f_1 f_2 \cdots f_\ell\, g_1^0 g_2^0 \tag{4.12}$$

where each f_i is a transposition of $B_{3,x}$. By applying identities (4.10) and (4.11) consecutively one can show that $g_2^{n_i} g_1^{m_j}$ can be written as $f' g_1^{m_j} g_2^{n_i}$ and that f' is a product of at most $|n_i m_j|$ transpositions of $B_{3,x}$. This means that the number ℓ in expression (4.12) is bounded by

$$\left(\sum |m_j|\right)\left(\sum |n_i|\right) \le x^2.$$

We need to bound the length of transpositions f_i's in (4.12), $i = 1, \cdots, \ell$. The isomorphism $\chi^* : \Sigma_{3,x} \cong S_{3x}$ (Theorem 2.10) transforms each $f_i \in \Sigma_{3,x}$ into a transposition $\chi^*(f_i)$ of the set $\{1, \cdots, 3x\}$. By Lemma 4.6, $\chi^*(f_i)$ has length at most $O(x^2)$ in S_{3x}. Since each generator of S_{3x} becomes a word of S_{3x} of length $\leq 2x$ under χ^*, $\chi^*(f_i)$ correspond to a word of $\Sigma_{3,x}$ with length at most $O(x^3)$. In all, the element $f\, f_1 f_2 \cdots f_\ell$ in the expression (4.12) has length at most $O(x^5)$.

Finally we need to calculate the area between two loops, namely one given by w and the other one given by w'. Let $B > 0$ denote the maximum number of relations that we used in (4.10) and (4.11). Note that whenever one exchanges g_1 and g_2 by applying an identity of (4.10) and (4.11), a transposition is produced. We already calculated the number of those exchanges of two generators, which is bounded by x^2. □

The argument if the proof of the above lemma extends to general cases $n \geq 4$.

Lemma 4.5. *There is a canonical way so that any null-homotopic word $w \in \mathcal{H}_n$ with $|w| \leq x$ can be written as a word in $\Sigma_{n,x}$ with length at most $O(x^5)$.*

Proof. Identities (4.8)(4.10)(4.11) holds for $g \in \mathcal{H}_n$ and for all pair of generators g_i, g_j of $\mathcal{H}_n$. Moreover the isomorphism between $\Sigma_{3,x}$ and S_{3x} (Theorem 2.10) allows one to bound the length of each transposition of $B_{n,r}$ by $O(x^3)$. As before we establish an upper bounds: $O(x^5)$ for the length of rewritten word w', and x^2 for the area between two loops w and w'. □

The following fact can be found in [14].

Lemma 4.6. *Let S_m be the finite symmetric group with the Coxeter system. For $\sigma \in S_m$, set $r_i(\sigma) = |\{j : i < j \text{ but } \sigma(i) > \sigma(j)\}|$. The length of an element σ is given by $\sum_1^m r_i(\sigma)$. In particular, there exist a unique element with the largest length $\sum_1^{m-1} i$.*

Lemma 4.7. *Any relator R of $\Sigma_{3,r}$ requires at most $O(e^x)$ relators of $\mathcal{H}_n$.*

Proof. Each generator of $\Sigma_{3,r}$ is an involution. Observe that this can be expressed as appropriate conjugation of $\alpha^2 = 1$ in $\mathcal{H}_3$. Similarly a braid relator $(\sigma_i\sigma_{i+1})^3 = 1$ can be expressed by conjugation of the relator $(\alpha\alpha^{g_1})^3$. Thus each relator of two types of $\Sigma_{3,r}$ requires area only 1 in $\mathcal{H}_n$. For commutation relator of $S_{n,r}$ we need to examine two sequences of words $u_k = \alpha^{\bar{g}_1^k}\alpha^{\bar{g}_2^k}$ and $v_k = [\alpha, \alpha^{\bar{g}_1^{k+1}}]$. Note that we already check u_k and u_k represent the identity in Lemma 2.15. One can use simultaneous induction on k together with the argument in the proof of Lemma 2.15 to show areas of u_k and v_k are bounded above by $O(e^x)$. The fillings depicted in Figure 4.3 require $O(e^x)$ relators. Since one can produce all commutation relation of $\Sigma_{3,r}$ by taking appropriate conjugation of v_k, all the relators of $\Sigma_{3,r}$ require at most $\mathcal{O}(e^x)$ relators of $\mathcal{H}_n$. □

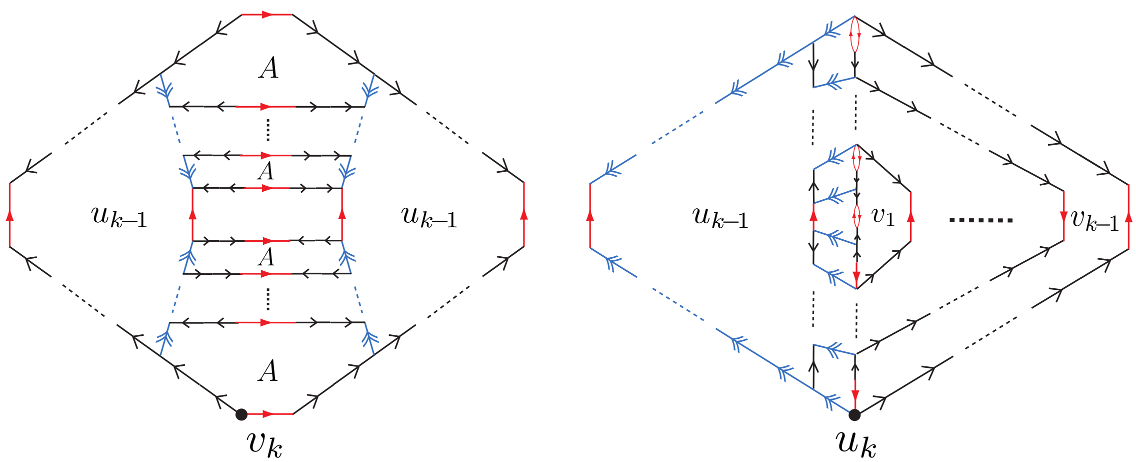

Figure 4.3: Diagrams of $u_k = \alpha^{\bar{g}_1^k}\alpha^{\bar{g}_2^k}$ and $v_k = [\alpha, \alpha^{\bar{g}_1^{k+1}}]$ ($A : \alpha^{g_1^{-2}g_2} = \alpha^{g_1^{-2}}$)

Proof of Theorem D. Fix $n \geq 4$. We follow the 3-step plan that we applied to $\mathcal{H}_3$. For Step 1, Lemma 4.5 provides the same upper bounds: $O(x^5)$ for the length of rewritten word, and $O(x^2)$ for the gap between two words. The upper bound established in Lemma 4.3 does not depend on the subscript. This means we still have the same cubic upper bound for $\delta_{\Sigma_{n,x}}(x)$. One can show first two types of relations of $\Sigma_{n,x}$ require area 1 by taking appropriate conjugations as before. Note that there are variations of words u_k and v_k in $\mathcal{H}_n$. Instead of using g_1 and g_2 one can use g_i and g_i $(1 \leq i < j \leq n-1)$ to generate commutation relations of $\Sigma_{n,x}$. Again, exponential upper bounds for those words can be established by simultaneous induction on the length of two sequences of words. In all, we establish exponential upper bound for $\mathcal{H}_n$.

Remark 4.8. We remark that if words u_k and v_k (and their variations for general cases) have exponential areas then exponential upper bounds for $\delta_{\mathcal{H}_3}$ $(\delta_{\mathcal{H}_3})$is sharp.

Chapter 5

Related problems of $\mathcal{H}_n$

Level sets of cubings X_n associated to $\mathcal{H}_n$ are typically $(n-2)$-connected. There are a number of higher dimensional Dehn functions $\delta^k_{\mathcal{H}_n}(x)$ of $\mathcal{H}_n$ yet to explore $(k \leq n-2)$.

Question 1. Find $\delta^k_{\mathcal{H}_n}(x)$ of $\mathcal{H}_n$.

The following is from M. Bestvina's list of open questions

Question 2. Are there groups of type F_n (FP_n) but not F_{n+1} (FP_{n+1}) which do not contain $\mathbb{Z}^2$ subgroup $(n \geq 3)$?

All known examples contain $\mathbb{Z}^2$. Houghton's groups $\mathcal{H}_n$ have potential to contain interesting subgroups providing a positive answer to this question. It is easy to find subgroups of $\mathcal{H}_n$ $(n \geq 3)$ which do not contain $\mathbb{Z}^2$ subgroups. $\mathcal{H}_n$ also has many subgroups with restrictive finiteness properties. A good goal would be to find subgroups belonging to both families.

Bibliography

[1] Ian Agol, Daniel Groves, and Jason Manning. The virtual haken conjecture. *arXiv:1204.2810*, 2012.

[2] Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.

[3] Martin R. Bridson. The geometry of the word problem. In *Invitations to geometry and topology*, volume 7 of *Oxf. Grad. Texts Math.*, pages 29–91. Oxford Univ. Press, Oxford, 2002.

[4] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[5] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[6] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[7] Kenneth S. Brown. Finiteness properties of groups. In *Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985)*, volume 44, pages 45–75, 1987.

[8] Victor Chepoi. Graphs of some CAT(0) complexes. *Adv. in Appl. Math.*, 24(2):125–179, 2000.

[9] Ross Geoghegan. *Topological methods in group theory*, volume 243 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[10] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[11] Frédéric Haglund and Daniel T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.

[12] C. H. Houghton. Ends of locally compact groups and their coset spaces. *J. Austral. Math. Soc.*, 17:274–284, 1974. Collection of articles dedicated to the memory of Hanna Neumann, VII.

[13] C. H. Houghton. The first cohomology of a group with permutation module coefficients. *Arch. Math. (Basel)*, 31(3):254–258, 1978/79.

[14] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[15] D. L. Johnson. Embedding some recursively presented groups. In *Groups St. Andrews 1997 in Bath, II*, volume 261 of *London Math. Soc. Lecture Note Ser.*, pages 410–416. Cambridge Univ. Press, Cambridge, 1999.

[16] M Roller. Poc sets, median algebras and group actions. *Preprint*, 1998.

[17] Volker Runde. *Lectures on amenability*, volume 1774 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[18] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)*, 71(3):585–617, 1995.

[19] Peter Scott. Ends of pairs of groups. *J. Pure Appl. Algebra*, 11(1-3):179–198, 1977/78.

[20] John Stallings. *Group theory and three-dimensional manifolds*. Yale University Press, New Haven, Conn., 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1969, Yale Mathematical Monographs, 4.

[21] John R. Stallings. On torsion-free groups with infinitely many ends. *Ann. of Math. (2)*, 88:312–334, 1968.