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SOME QUESTIONS ON DISTRIBUTIONS, NETS, AND DUALITY

APPROVED BY

DISSERTATION COMMITTEE

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CHAPTER I

INTRODUCTION

The contributions of this thesis are the following.

Chapter II gives a new method by which distributions may be constructed. The essential idea is to use nets to form a weak completion of a dual space. The theory is then developed to a stage where it parallels other methods of constructing distributions, [6],[1] [18], [22].

The novelty of Chapter III is twofold. First is the introduction of a topology on the sum set (Definition 3.10). Using this definition, a new proof is given to a theorem of Birkhoff, [1]. The second contribution is the proof of several theorems on double, iterated, and partial limits of nets and subnets. These are the proofs of theorems 3.11, 3.13, 3.14, 3.18, and corollary 3.14.

Chapter IV is based on a result of Ellis, [5]. One of the problems considered by Ellis was the duality of an arbitrary number of groups with a topology. It is first shown that his upper bound topology does not make the direct sum of an arbitrary number of groups, a topological group. His result is then modified by introducing a weak topology

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[1]Numbers in brackets refer to the bibliography.

on the direct sum. This weak topology is shown to be an upper bound for his topologies. The direct sum with the weak topology is then proven to be a topological group.

The following heuristic discussion will make explicit these purposes.

In the late nineteenth and early twentieth centuries, mathematics was invaded by some strange objects. These objects have been labeled the operational calculus. Technically they are methods of solving different types of problems with no apparent mathematical justification. The origin of the problem is usually in the field of applied mathematics and the solution is any mathematical chicanery which "gets the answer". On the other hand these methods have two outstanding features. First, they have a labor saving formalism, and second, and more important, they work. Consequently, any mathematical justification of these techniques would only enhance their use. To this end, the work of Schwartz [18], Mikusinski [17], and Bochner [3] deserve special mention.

To make this concise description more intelligible, we consider an example. Operational calculus utilizes many improper functions of which the Dirac delta function, $\delta(x)$, is probably the most famous [6], [22], [23]. It is said to arise in the following manner.

$$\text{Let } Y(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

The derivative of this function is the Dirac delta function, $\delta(x)$, which has the following (mathematically impossible!) properties [6]: it vanishes everywhere except at the origin where its value is so large that

$$\int_{-\infty}^{\infty}\delta(x)dx = 1.$$

moreover

$$\int_{-\infty}^{\infty}\delta(x)\phi(x)dx = \phi(0),$$

$$\int_{-\infty}^{\infty}\delta'(x)\phi(x)dx=\delta(x)\phi(x)\Big]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\delta(x)\phi'(x)dx=-\int_{-\infty}^{\infty}\delta(x)\phi'(x)dx=-\phi'(0),$$

$$\int_{-\infty}^{\infty}\delta''(x)\phi(x)dx=\int_{-\infty}^{\infty}\delta(x)\phi''(x)dx=\phi'(0), \; . \; . \; . \; ,$$

where prime denotes differentiation and the function $\phi(x)$ is assumed to possess the requisite number of derivatives [22]. There are several methods by which we can make the ill-defined delta function into a well-defined mathematical entity. One is using a limiting procedure involving ordinary functions. For instance let

$$f_n(x)= \frac{1}{\pi}\frac{n}{1+n^2x^2}$$

or

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{1}{\pi}\frac{n}{1+n^2x^2}\,dx=1=\int_{-\infty}^{\infty}\delta(x)dx.$$

Another would be to define the delta function as a measure, that is, a set function instead of an ordinary point function. A third and more elegant way would be to use the generalized functions as originated by Sobolev [20], [21] and later developed in a somewhat extended form by Schwartz [18], [19]. This method is characterized by the fact that the delta function generates a certain linear functional, say L, such that

$$L\{\phi(x)\}=\phi(0),$$

while $\delta'(x)$ generates another linear functional, say M, such that

$$M\{\phi'(x)\}=-\phi'(0).$$

Schwartz's method is to replace the delta function and its derivatives by linear functionals. The functions $\phi(x)$ are chosen in a manner so that they will satisfy certain continuity and differentiability properties [22]. Following Bochner [2] the $\phi(x)$ are called testing functions while the linear functionals are called distributions by Schwartz [18]. Using Schwartz distributions all the improper functions of mathematical physics can be replaced by appropriate linear functionals [22].

Now, Mikusinski [17] has generalized the Schwartz theory in the following manner. He noted that in the Schwartz theory you start with three sets, F, the locally summable functions, ϕ, the infinitely differentiable functions with compact support, and C the complex numbers. You have a mapping from $F \times \phi$ (cartesian product) with values in C. Moreover a sequence $f_n \epsilon F$ is said to converge if and only if $f_n \cdot \phi$ converges in C for all $\phi \epsilon \phi$. That is, he forms a weak closure of F, by means of convergence in C in order to obtain the desired distributions (up to identification). He further defined an operation on F, which can be extended to $\overline{F}$ so that it preserves certain properties possessed by F. However, in the generalization F, ϕ and C are chosen to be any sets.

Using Mikusinski's results, Temple [22] indicated how this point of view might better be exploited. The work of Korevaar [9], [10], [11], [12], Lighthill [14], and Love [15] can be mentioned as a few who adopted this point of view. Temple indicates a complete equivalence between the Schwartz theory and Mikusinski's general theory. There are several shortcomings to this. Schwartz utilizes filters to obtain his completion whereas Mikusinski, Temple, and others [23] use sequences. In a general topological space, sequences are not adequate for describing the topology whereas filters may be [8]. In particular one can not show the closure of the closure is closed which is tatamount to showing completion. Furthermore if one obtains a completion by means of sequences, it may not be the same for filters. That is, one obtains limits by means of filters which can not be had by means of sequences. However, from the point of view of applications, sequences or nets, are more appealing. The tools and techniques of sequences are well known and in practice many problems resolve themselves to sequential analysis. Neighborhoods, open sets, etc., are not in general use by applied mathematicians, and since nets have both the generality and the formalism of sequences, they will be used in preference to filters.

But, even within this setting it is important that some form of completion exist or one is immediately confronted with many pathologies. And, since the completion of F depends on the completeness of C, it seems that a natural requirement would be that C be endowed with certain completeness properties.

We start with Mikusinski's general method with some variations. First, we let C be a complete metric space and complete F by means of nets. Next, we endow ϕ with a topology, and, following Schwartz, define continuity of elements of F and show that we get the Mikusinski theory by letting ϕ have the discrete topology. We further make several observations concerning completeness using sequences and ϕ with or without a topology. Finally, we define a regular operation on F, and discuss some of its properties relative to nets and sequences.

Returning to the problem of completion, there are two standard methods which overcome much of the inadequacies of sequences. They are filters and Moore-Smith convergence [8]. It is part of the folk-lore of mathematics that these two ideas are equivalent [8]. The Moore-Smith approach generalizes the concept of a sequence to that of nets, but its application to the problem of completion does not seem to be available in the literature.

The Cantor approach to completion replaces an iterated limit of sequences by a single sequence. With this as a model, the completion used in Chapter II is accomplished by means of nets.

A justification for the use of this model is the proof of Theorem 3.9. The proof has a further advantage. It gives a proper setting for limits of nets in a function space. Consequently, theorems on iterated, and double limits of nets can be proven within this context. This is the primary content of Chapter III.

One of the properties enjoyed by Schwartz's distribution is infinite differentiability. This property coupled with the infinite differentiability of the elements of ϕ reflect a certain duality [18]. It was hoped that this property might also be generalized. Only minor results were obtained and are also contained in Chapter II. However, in order to exploit this idea, results of Ellis [5] and Kaplan [7] concerning Pontryagin duality were studied. It was found that the approach of Ellis did offer some generalization. This is the primary content of Chapter IV. The result of Ellis is concerned with the Pontryagin duality of the direct sum and direct product of a family of groups. He showed that the complement of the direct sum, if its topology lies within a certain interval, is the cartesian product of complement. In Chapter IV we show that the strongest topology which lies in this interval does not make it a topological group, and we replace it by one which does.

CHAPTER II

DISTRIBUTIONS

As stated in the introduction, this chapter is based on Mikusinski's generalization [17] of the Schwartz theory of distributions [18], [19]. Our method will parallel the one suggested by Temple [22], [23] but with several generalizations. The problem of completion will be accomplished by means of nets instead of sequences and hence will be more general. Moreover the sets involved will be endowed with the topological properties necessary to actually accomplish this. Both Temple [22], [23] and Mikusinski [17] are not too clear on this point. The space ϕ will then be given a topology and from this we get Mikusinski's general approach as a special case. Finally, from the point of view of applications, sequences (or nets) are more appealing and better known. To this end, our general method includes the work of Korevaar [9], [10], [11], [12], [13], Lighthill [14], Temple [22], [23], and Love [15], [16] where the sets F, ϕ, and C have been specialized.

2.1) Let F, ϕ, and C be non-empty sets and $(\cdot)$ a map: $F x \phi \rightarrow C : (f,\phi) \rightarrow f\cdot\phi$ satisfying

2.2) $f\cdot\phi = g\cdot\phi$ for all $\phi \in \phi$ if and only if $f=g$, where $f, g \in F$. Since no confusion is likely to arise we will suppress the $(\cdot)$ and merely write $f\phi$. We introduce a topology in F in the following manner. Let C be a complete metric space with metric D, a family of pseudo-metrics can be introduced on F as follows.

For each $\phi \in \Phi$ define

2.3) $d_\phi(f,g)=D(f\phi,g\phi)$ where f, $g \in F$.

We show that this definition provides us with a family of pseudo-metrics.

Now for all ϕ Φ, f, g, h in F we have

2.4) $d_\phi(f,g)=D(f\phi,g\phi)\geq 0$

2.4a) $d_\phi(f,g)=D(f\phi,g\phi)=D(g\phi,f\phi)=d_\phi(g,f)$

2.4b) $d_\phi(f,f)=D(f\phi,f\phi)=0$

2.4c) $d_\phi(f,g)=D(f\phi,g\phi)\leq D(f\phi,h\phi)+D(h\phi,g\phi)=d_\phi(f,h)+d_\phi(h,g)$

In most of the applications [18], C is usually the real or complex numbers. But from Kelly [8] we have that this family of pseudo-metrics generate a subbase for a uniformity of F and thus a topology. More specifically the following definitions taken from Kelly justify this claim.

A uniformity for a set X is a non-void family of subsets of XxX such that

a) each member of $\mathcal{U}$ contains the diagonal elements (x,x) where $x \in X$;

b) If $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$ where U^{-1} is the set of (y,x) such that $(x,y) \in U$;

c) If $U \in \mathcal{U}$, then $V \cdot V \in U$ for some V in $\mathcal{U}$;

d) If U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$; and

e) If $U \in \mathcal{U}$ and $U \subset V \in XxX$, then $V \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is a uniform space.

A subfamily B of a uniformity $\mathcal{U}$ is a base if and only if each member of $\mathcal{U}$ contains a member of B, or what is the same thing, satisfy a, b, c, d of the foregoing.

If $(X,\mathcal{U})$ is a uniform space the topology $\mathcal{J}$ of the uniformity

$\mathcal{U}$, or the uniform topology is the family of all subsets T of X such that for each x in T there is $U \in \mathcal{U}$ such that $U[x] \subset T$.

Each pseudo-metric d_ϕ for the set F generates a uniformity in the following way. For each positive number r let $V_{\phi,r} = (f,g)\ d_\phi(f,g) < r$. Clearly $(V_{\phi,r})^{-1} = V_{\phi,r}$, $V_{\phi,r} \cap V_{\phi,s} = V_{\phi,t}$ where $t = \min(r,s)$, and $V_{\phi,r} \cdot V_{\phi,r} \subset V_{\phi,2r}$. It follows that the family of all sets of the form $V_{\phi,r}$ is a base for a uniformity for F. This is called the uniformity generated by d_ϕ. Every family of pseudo-metrics generates a uniformity; it will also be said to generate the gage of this uniformity. The following theorem 6.18 from Kelly will relate these concepts somewhat better.

2.5) Theorem 6.18 Kelly [8]. Let $(X,\mathcal{U})$ be a uniform space and let P be the gage of . In our case X=F, P=Φ. Then:

a) The family of all sets $V_{\phi,r}$ for $\phi \in \Phi$ and r positive is a base for the uniformity $\mathcal{U}$.

b) Suppose Φ' is a subfamily of Φ which generates Φ. A net $\{f_n, n \in D\}$ in F converges to a point $f \in F$ if, and only if, $\{d_\phi(f_n,f), n \in D\}$ converges to zero for each $\phi \in \Phi$.

Now 2.5b plays an important part in our development and deserves special notice. The following companion theorem will be of the same usefulness.

Theorem 2.6. Let $\{f_n\}_{n \in D}$ be a net in F then $\{f_n\}_{n \in D}$ converges to f if, and only if, $\{f_n\phi\}_{n \in D}$ converges to $f\phi$ for all $\phi \in \Phi$.

Proof: By the definition of d_ϕ, $d_\phi(f_n,f) = D(f_n\phi, f\phi)$. But by 2.5b $\{f_n\}_{n \in D}$ converges to f if, and only if, $d_\phi(f_n,f)$ converges to $f\phi$ for all $\phi \in \Phi$. It should be noted that at this stage nothing is said relative to the location of f but is implicit that $f\phi \in C$.

As a consequence of 2.2 one has the following trivial theorem.

Theorem 2.7. $d_\phi(f,g)=0$, for all $\phi \varepsilon \Phi$ if, and only if, $f=g$. Moreover we say Φ is total with respect to F if, and only if, this condition is satisfied.

Proof: $d_\phi(f,g)=D(f\phi,g\phi)$ for all $\phi \varepsilon \Phi$. Therefore $d_\phi(f,g)=0$ if, and only if, $D(f\phi,g\phi)=0$. But this implies $f\phi=g\phi$ if, and only if, $f=g$, for all $\phi \varepsilon \Phi$.

Due to the type of convergence considered limits of nets in F may not be contained in F. So we complete F in a manner very similar to the completion of a metric space. Recall that in a metric space every sequence need not converge but every Cauchy sequence does. Moreover the completion of a metric space is accomplished by considering equivalence classes of Cauchy sequences and identifying elements of F with constant sequences in respective equivalence classes. The family of equivalence classes is the completion of F (denoted by $\overline{F}$) having the properties that F is dense in $\overline{F}$ and $\overline{F}$ is complete. For limits of nets we do the same thing. One considers equivalence classes of so called Cauchy nets. These are nets which possess certain convergence properties. That is, $\overline{F}$ is equivalence classes of Cauchy nets. Specifically we have at our disposal a space F, a uniformity $\mathcal{U}$, and a gage Φ of $\mathcal{U}$. That is, we have a uniform space. Now a uniform space is said to be complete if, and only if, each Cauchy net in the space converges to a point in the space [8]. Moreover, a net $\{f_n\}_{n \varepsilon D}$ in the uniform space $(f,\mathcal{U})$ is a Cauchy net if, and only if, for each member U of $\mathcal{U}$ there is N in D such that $(f_m,f_n) \varepsilon U$ whenever m and n follow N in the ordering of D. Better yet the family of all sets of the form $V_{\phi,r}$, for $\phi \varepsilon \Phi$, r positive, is a base for the uniformity $\mathcal{U}$, from which it follows that $\{f_n\}_{n \in D}$ is a Cauchy net if, and only if, $(f_m,f_n)_{(m,n) \in D \times D}$ is eventually in each set of the form $V_{\phi,r}$. That is, for all $\phi \in \Phi$ and $r > 0$, there exists an $N \in D$ such that $d_\phi(f_m,f_n) < r$ for $m,n > D$. In terms of pseudo-metrics the foregoing is equivalent to " for all $\phi \in \Phi, d_\phi(f_n,f_m)_{(n,m) \in D \times D}$ converges to zero" [8]. We will have occasion to call on each of the foregoing formulations of a Cauchy net. So let us denote limits of Cauchy nets in F by $\bar{f}$, and the completion of F by $\bar{F}$ where $\bar{F}$ is the family of all equivalence classes of Cauchy nets in F indexed by a sufficiently small set. The necessity of having a small index set is that in the completion of F we are faced with a logical problem. For a given topological space there is no universal directed set in general. Consequently we are forced to take the class of all directed sets which have Cauchy nets in F. This is usually an extremely large family and we need to replace it by a set. This is accomplished in two steps. First the family of directed sets is partitioned into equivalence classes (def. 2.8) and second the neighborhood system of the point is used to select a representative of an equivalence class thus assuring us that the power set of $\bar{F}$ is adequate for indexing purposes.

Definition 2.8. A Cauchy net $\{f_n\}_{n \in D}$ in F is equivalent to a Cauchy net $\{g_m\}_{m \in S}$, (notation $f_n \sim g_m$,) if and only if for all $\phi \in \Phi, d_\phi(f_n,g_m)_{(n,m) \in D \times S}$ converges to 0, when DxS is given the product ordering. We show that "$\sim$" provides us with an equivalence relation. Now for all $\phi \in \Phi$ and Cauchy nets $\{f_n\}_{n \in D}, \{g_m\}_{m \in S}, \{h_i\}_{i \in I}$ we have

2.8a) $d_\phi(f_n,f_m) = D(f_n\phi, f_m\phi) \rightarrow 0$ since $\{f_n\}_{n \in D}$ is a Cauchy net.

Therefore $f_n \sim f_n$ and "$\sim$" is reflexive.

2.8b) $d_\phi(f_n,g_m)=D(f_n\phi,g_m\phi)=D(g_m\phi,f_n\phi)=d_\phi(g_m,f_n)$ for all $n\in D$, $m\in S$. But DxS and SxD have the product ordering so if $d_\phi(f_n,g_m)\underset{(n,m)\in DxS}{\longrightarrow}0$ this implies $d_\phi(g_m f_n)\longrightarrow 0$ by the properties of (C,D). That is "$\sim$" is symmetric.

2.8c) $d_\phi(f_n,g_m)=D(f_n\phi,g_m\phi)\leq D(f_n\phi,g_i\phi)+D(h_i\phi,g_m\phi)=d_\phi(f_n,h_i)+d_\phi(h_i,g_m)$ for all $n\in D$, $m\in S$, $i\in I$. And since DxS, DxI, IxS have the product ordering then by the properties of (C,D) if $d_\phi(f_n,h_i)\underset{(n,i)\in DxI}{\longrightarrow}0$ and $d_\phi(h_i,g_m)\underset{(c,m)\in IxS}{\longrightarrow}0$ then $d_\phi(f_n,g_m)\underset{(n,m)\in DxS}{\longrightarrow}0$. That is "$\sim$" transitive and hence an equivalence relation. Thus "$\sim$" partitions the family of Cauchy nets into equivalence classes (possibly a much smaller class than the one started with). We next show the neighborhood system of a point is adequate for indexing purposes and permit a class to be a set. Since we will be working with representatives of equivalence classes we need to show that any representative will be sufficient.

2.8d) Let $\{f_n\}_{n\in D}\sim\{f'_m\}_{m\in S}$ and $\{g_p\}_{p\in P}\sim\{g'_i\}_{i\in I}$. Then for all $n\in D$, $m\in S$, $p\in P$, $i\in I$, $\phi\in\emptyset$ we have $|d_\phi(f_n,g_p)-d_\phi(f'_m,g'_i)| = |d_\phi(f_n,g_p)-d_\phi(f_n,g'_i)+d_\phi(f_n,g'_i)-d_\phi(f'_m,g'_i)| \leq |d_\phi(f_n,g_p)-d_\phi(f_n,g'_i)| + |d_\phi(f_n,g'_i)-d_\phi(f'_m,g'_i)| \leq |d_\phi(g_p,g_i)+d_\phi(f_n,f'_m)|$. . .

But $\{f_n\}_{n\in D}\sim\{f'_m\}_{m\in S}$ and $\{g_p\}_{p\in P}\sim\{g'_i\}_{i\in I}$ implies that $d_\phi(f_n,g_p)=\lim_{(n,p)\in DxP\,(m,i)\in SxI} d_\phi(f'_m,g'_i)$. Thus showing that the choice of representative is immaterial. Also $d_\phi(f_n,f_m)\underset{(n,m)\in DxD}{\longrightarrow}0$ implies $D(f_n\phi,f_m\phi)\longrightarrow 0$ for all $\phi\in\emptyset$. But $\{f_n\phi\}_{n\in D}$ is a net in a metric space and hence has a limit. Denote this limit by $\bar{f}\cdot\phi$. From 2.5b, we have that convergence in C implies convergence in F. Or we write $\lim_{n \in D} f_n = f$. Moreover the choice of representative of an equivalence class is immaterial in obtaining $\overline{f}$, since it is true for obtaining $\overline{f}\phi$ in C. We are now justified in writing: for all $\phi \in \Phi$, $\lim_{(n,m) \in D \times S} d_\phi(f_n, g_m) = \lim D(f_n\phi, g_m\phi) = D(f\phi, g\phi) = d_\phi(f,g)$. Finally, our previous statement that $\overline{F}$ is the family of equivalence classes of Cauchy nets in F has a more precise meaning.

Returning to the problem of completion, if we define a map H: $F \to C^\Phi$; $f \to H_f$; $H_f(\phi) = f(\phi)$ and note that $H_f = H_g$ if, and only if, $f=g$ then we have a one to one map of F into C^Φ. Hence we can identify F with a subset of C^Φ where C^Φ has the topology of pointwise convergence. We further note that $\overline{F} \subset C^\Phi$. C^Φ is a uniform space since C is, [8], and for each $\overline{f} \in \overline{F}$, $U \in \mathcal{U}$, $U[\overline{f}]$ is identical with the neighborhood system of $\overline{f}$. Since $\overline{f} \in U[\overline{f}]$ one can form a directed set in the following way: $U[\overline{f}] \geq V[\overline{f}]$ if, and only if, $U \subset V$. From the property of set inclusion this is clearly a directed set. Define a net S on this directed set in the following way: $S(U[\overline{f}]) = S_{U[\overline{f}]} = \overline{f}$. This net is Cauchy since $(S_{U[\overline{f}]}, S_{U[\overline{f}]}) = (\overline{f}, \overline{f}) \in U$ for all $U \in \mathcal{U}$, and is thus eventually in U, for all $U \in \mathcal{U}$. But we have shown that any representative from an equivalence class is sufficient. So, we can use the neighborhood system of each element $\overline{f} \in \overline{F} \subset C^\Phi$ for indexing purposes. That is, our family of directed sets is a set of cardinality at most the power set of C^Φ. For convenience we will continue to use our former notation for nets and limits.

To be precise F is not a subset of $\overline{F}$ but we can identify F with a subset of $\overline{F}$. Let $T: F \to \overline{F}: f \to \{f_n\}_{n \in D}$, $f_n = f$ for all $n \in D$. Clearly T is one to one from F to T[F] and to finish the completion problem we show T[F] is dense in $\overline{F}$, that T is a uniform isomorphism and $\overline{F}$ is complete.

Theorem 2.9. T[F] is dense in $\overline{F}$.

Proof: Let $\overline{f} \in F$ then there exists a net $\{f_n\}_{n \in D} \in F$ such that $\{f_n\}_{n \in D}$ converges to $\overline{f}$, or $\{T(f_n)\}_{n \in D}$ converges to $T(\overline{f})$. Hence T[F] is dense in $\overline{F}$.

Theorem 2.10. T is a uniform isomorphism.

Proof: If U is a member of the uniformity of F then by definition of T, T[U] is a member of the uniformity of $\overline{F}$. Hence if $(T(f),T(g)) \in T[U]$ then $(f,g) \in U$ and conversely. This satisfies the definition of uniform isomorphism.

Theorem 2.11. $\overline{F}$ is complete.

Proof: To show $\overline{F}$ complete, first observe that it is sufficient to show that each Cauchy net in T[F] converges to a point in $\overline{F}$ because T[F] is dense in $\overline{F}$. Since each Cauchy net in T[F] is of the form $T\cdot\overline{f}=T(f_n)$ where $\overline{f}$ is a Cauchy net in F one sees that $T\cdot\overline{f}$ converges in $\overline{F}$ to the member $\overline{f}$ of $\overline{F}$.

In what follows let us agree to identify F with T[F] and merely think of F as a subset of $\overline{F}$. Recall that we defined $\emptyset$ to be total with respect to F. From Kelly [8] we have that this is equivalent to F being Hausdorff or separated and hence $\overline{F}$ enjoys this property. Moreover, as noted earlier, d_ϕ has a natural and obvious extension to $\overline{F}$. For all $\overline{f}$, $\overline{g}$, $\overline{h} \in \overline{F}$ and for all $\phi \in \emptyset$ we have

2.12) $d_\phi(\overline{f},\overline{f})=0$

2.12a) $d_\phi(\overline{f},\overline{g})=d_\phi(\overline{g},\overline{f})$

2.12b) $d_\phi(\overline{f},\overline{g})=d_\phi(\overline{f},\overline{h})+d_\phi(\overline{h},\overline{g})$.

All of the foregoing discussion parallels the ideas set forth by Mikusinski [17]. Specifically, Mikusinski assumes 2.1, 2.2, and that C possess a topology by which one can define a limit of certain sequences in C. He defines a weak convergence in F by $f_n \to f$ if, and only if, $f_n\phi \to f\phi$ in C. He defines $\overline{F}$ to be the weak closure F and further assumes that ϕ is total with respect to $\overline{F}$. As an example of this general technique he cites the Schwartz theory of distributions. That is, his weak limits are distributions in the sense of Schwartz. Temple [22],[23], indicates an equivalence between these two methods in obtaining distributions as well as a further exploitation of the Mikusinski point of view. However, Schwartz uses filters and shows the space of distributions is complete. Moreover, his distributions are continuous linear functionals which imply the space ϕ has a topology. Since sequences are inadequate to describe the topology in a general topological space this suggested the possibility of a counter example to Temple's proof of the equivalence of the two methods. We will do this shortly but first we have noted that by means of nets (or filters) F can be completed. In particular cases one can show that sequences are adequate. This is true without regard to the continuity of the elements of F, but one does not obtain the Schwartz distributions in either case.

Our example will consider only real distributions and our basic compact set will be the interval [-1,1]. Let $\delta(x-\frac{n}{n+1})$ be a sequence of delta functions. These are continuous linear functionals in the sense of Schwartz. That is, for any sequences of testing functions $\phi_K(x)$ converging uniformly to 0, $\delta(x-\frac{n-1}{n})\phi_K(x)$ converges uniformly to 0 for each n and for any closed interval contained in [-1,1]. Moreover,

$\lim_{K\to\infty} \sum_{n=1}^{K} \delta(x-\frac{n}{n-1})\phi_K(x)$ converges since for any closed interval (compact set) properly contained in [-1,1] the foregoing sum has only a finite number of summands. But, consider the following sequence of testing functions:

$$\phi_K(x)=1/2\left\{\frac{1}{\sqrt{K}}+\frac{1}{\sqrt{K+1}}\right\}\exp\left\{(2K^2-1)/K^2(K+1)^2\right\}\ln\left\{1/2(1+\sqrt{(K+1)/K})\right\}/\left\{x^2-\frac{(K)^2}{K+1}\right\},$$

for $\frac{-K}{K+1}<x<\frac{K}{K+1}$,

and $\phi_K(x)=0$, for all other x.

These functions have the property that their maximum is at x=0 and minimum at $x=\pm\frac{K}{K+1}$. Furthermore if $\frac{-K+1}{K}\leq x\leq\frac{K-1}{K}$ then $\frac{1}{\sqrt{K+1}}=\phi_K(x)<\frac{1}{K}$ which implies they converge uniformly to 0. A somewhat tedious calculation will show $\phi_K^{(n)}(x)$ converges uniformly to 0 for each derivative n. But for these $\phi_K(x)$, $\sum_{n=1}^{K}\delta(x-\frac{n-1}{n})\phi_K(x)\geq K\frac{(1)}{\sqrt{K}}=\sqrt{K}\to\infty$. Hence the sum is not a continuous linear functional. The essential idea is to pick a variable compact set within the fixed compact set [-1,1]. If one redefines the foregoing sequence on the fixed compact set [-1,1], then a classical theorem tells us that a uniformly convergent sequence of continuous function on a compact converges to a continuous function. This is the situation with which Temple worked, hence, his space of distributions. Next we introduce a topology in ϕ and consider the completion relative to continuous f, by starting with those which are, or, better yet, start by assuming all of the elements of F are continuous.

<u>Definition</u> <u>2.13</u>. $\bar{f}$ is continuous at ϕ if, and only if, for all $\epsilon>0$,

there exists a neighborhood V of ϕ such that for all $x \in V$, $D(\overline{f}(x),\overline{f}(\phi))<\epsilon$. This is a standard definition of continuity and the purpose of the next three theorems is to rephrase the definition of continuity of elements of $\overline{F}$ in terms of elements of F.

Theorem 2.14. $\overline{f}$ is continuous at $\phi \in \Phi$ if, and only if, for all $\epsilon > 0$ and all $N \in D$ there exists an $n \in N$ and a neighborhood V of ϕ such that for all $x \in V$, $(D(f(x),f_n(x))<\epsilon$.

Proof: Let $\overline{f}$ be continuous, $\epsilon > 0$, and N be given, then $D(\overline{f}(x),f_n(x))<\epsilon/3$ for all x in some $V(\epsilon/3,N)$ due to the continuity of $\overline{f}$. Moreover, $D(\overline{f}((\phi),f_n(\phi))<\epsilon/3$ for some $n > N(\phi, \epsilon/3)$ since $\lim_{m \in D} f_m=\overline{f}$. There exists a neighborhood $V_2(n,\phi,\epsilon/3)$ for this choice of n such that for all $x \in V_2$, $D(f_n(\phi),f_n(x))<\epsilon/3$ since f_n is continuous. Or we have for $\epsilon > 0$, $n > N$ and all $x \in V=V_1 \cap V_2$, $D(\overline{f}(x),f_n(x)) \leq D(\overline{f}(x),f(\phi))+D(\overline{f}(\phi), f_n(\phi))+D(f_n(\phi),f_n(x))<\epsilon$.

Conversely, for $\epsilon > 0$, we must show there exists a neighborhood V_1 of ϕ such that $D(\overline{f}(x),\overline{f}(\phi))<\epsilon$. Now, there exists an N such that for some $n > N$, $D(\overline{f}(x),f_n(x))<\epsilon/3$ for all $x \in V_1(n,\epsilon/3)$, in particular, if $x=\phi$ we also have $D(\overline{f}(\phi),f_n(\phi))<\epsilon/3$. But, since f_n is continuous there exists a $V_2(n,\phi,\epsilon/3)$ such that for all $x \in V_2$, $D(f_n(x),f_n(\phi))<\epsilon/3$. Therefore, for some $n > N$ and all $x \in V_1 \cap V_2$ we have $D(\overline{f}(x),\overline{f}(\phi)) \leq D(\overline{f}(x),f_n(x))+D(f_n(x),f_n(\phi))+D(f_n(\phi),\overline{f}(\phi))<\epsilon$.

Theorem 2.15. $\overline{f}$ is continuous at $\phi \in \Phi$ if, and only if, for all $\epsilon > 0$ and for all $N \in D$, there exists $n \in N$ and a neighborhood V of ϕ such that $\lim_{m \in D} D(f_m(x),f_n(x))<\epsilon$.

Proof: $\epsilon > D(f(x),f_n(x))=\lim_{m \in D} D(f_m(x),f_n(x))$ for $n > N$, and $x \in V(n,\epsilon)$.

Conversely, $\epsilon > \lim_{m \in D} D(f_m(x), f_n(x)) = D(\overline{f}(x), f_n(x))$ for $x \in V(n, \epsilon)$.

Theorem 2.16. $\overline{f}$ is continuous at $\phi \in \emptyset$ if, and only if, for all $\epsilon > 0$ and for all $N \in D$, there exists an $n \in N$ and there exists $V(n, \phi)$ such that if $m \geq M_x$, then $D(f_m(x), f_n(x)) < \epsilon$ for all $x \in V$.

Proof: Assume $D(f_m(x), f_n(x)) < \epsilon$ for all $x \in V(n, \phi)$ and $m \geq M_x$. Then

$\lim_{m \geq M_x \in D} D(f_m(x), f_n(x)) < \epsilon$ for each $x \in V(n, \epsilon)$ since $\lim f_m(x)$ exists.

If $\lim_{m \in D} D(f_m(x), f_n(x)) < \epsilon$ for each $x \in V(n, \epsilon)$, then for each $x \in V$ there exists an M_x such that if $m \geq M_x$, then $D(f_m(x), f_n(x)) < \epsilon$ for all $x \in V$.

Definition 2.17. A net $\{f_n\}_{n \in D}$ is a strong Cauchy net if, and only if,

a. $\{f_n\}_{n \in D}$ is Cauchy in F and

b. $f = \lim_{n \in D} f_n$ is continuous at $\phi \in \emptyset$

Notation: $\overline{F}^s$ is the set of all strong Cauchy nets in $\overline{F}$.

Theorem 2.18. $\overline{F}^s$ is complete with respect to the strong completion.

Proof: Let $\{\overline{f}_n\}_{n \in D}$ be a strong Cauchy net. It has a limit since $\overline{F}$ is complete and in particular a continuous limit since it is a strong Cauchy net.

Corollary 2.19. If $\emptyset$ has the discrete topology then theorem 2.18 is equivalent to Theorem 2.11.

Summarizing, we see that F can be completed by means of nets with or without a topology in F. But, to obtain a generalization of the Schwartz theory, $\emptyset$ should have a topology. By restricting F, $\emptyset$, C, and $(\cdot)$, sequences can be used to obtain a completion of F which is not equivalent to the Schwartz theory. From the point of view of application this has definite advantages.

It should be further noted that if $\emptyset$ is a vector space, all of the foregoing is a dual space completion.

CHAPTER III

NETS

In general topology there are three ideas from which nearly all other ideas arise [1]. They are closure, neighborhood, and sequential convergence. For specialized spaces such as metric or Hausdorff and first countable these ideas are essentially the same. However, for general spaces closure and neighborhoods are roughly equivalent, but not so for sequential convergence. We owe to Moore and Smith the idea of a net, which generalizes the concept of sequence and overcomes the lack of equivalence of closure and convergence.

Since the concept of sequence is the basic object to be generalized, it is accomplished in the following manner. The domain of any sequence is the positive integers, and this is true irrespective of its range [1]. Consequently, it is not surprising that all of the properties of the integers are not used. So we replace the integers by a set whose properties are precisely those needed for sequential convergence. Specifically it is their order. This set will be called a directed set. Then the concept of a sequence is replaced by that of a net. Namely, a net is a function whose domain is a directed set. Of course, additional definitions must be given to cover such properties as convergence, etc.

With this as a motivation for the use of nets, we turn to a theorem of Birkhoff's [1] as found in Kelly [8]. This is a theorem on iterated limits of nets. Iterated limits are closely related to the closure of the closure of a set, and hence the completion of space. Recall that in the completion of the real number system (Cantor approach) an iterated limit of sequences is used. And in Chapter II, we used it to complete the space F.

In this theorem, an iterated limit is replaced by a single limit in much the same manner as a sequence of sequences is replaced in a single sequence. A natural question arises, is this setting adequate for describing the corresponding generalization of a net of nets? Another question is, to what extent are partial and iterated limits related to double limits? And further, what is the relation of partial, iterated, and double limits of subnets to nets? The answer to these questions is the content of this chapter. As usual, we start with several definitions, and then we enlarge somewhat on Kelly's proof of the iterated limit theorem. Having this, we then either prove or exhibit counter-examples to show the relationship of the different types of limits of nets and subnets in a general topological space. For the sake of completeness we next prove two more theorems due to Birkhoff [1] and conclude with a remark on a net of nets. The final part of the chapter contains two theorems which are natural generalizations of sequences to nets in a complete metric space (more generally a complete uniform space).

Definition 3.1. A binary relation $\geq$ directs a set D if D is non-void and

a) if m, n, and p are members of D such that $m \geq n$ and $n \geq p$ then $m \geq p$;

b) if $m \in D$, then $m \geq m$; and

c) if m and n are members of D, then there is a $p \geq D$ such that $p \geq m$ and $p \geq n$.

We say that m follows n in the ordering $\geq$ and that n precedes m if, and only if, $m \geq n$.

Definition 3.2. A directed set is a pair $(D, \geq)$ such that $\geq$ directs D.

Definition 3.3. A net is a pair $(S, \geq)$ such that S is a function and $\geq$ directs the domain of S.

Definition 3.4. A net $\{S_n, n \in D, \geq\}$ is in a set A if, and only if, $S_n \in A$ for all $n \in D$.

Definition 3.5. A net $\{S_n, n \in D, \geq\}$ is eventually in A if, and only if, there is an element $m \in D$ such that, if $n \in D$ and $n \geq m$, then $S_n \in A$.

Definition 3.6. A net is frequently in A if, and only if, for each $m \in D$ there is an n in D such that $n \geq m$ and $S_n \in A$.

Definition 3.7. A net $(S, \geq)$ in a topological space (X,T) converges to s relative to T if, and only if, it is eventually in each T neighborhood of s.

The notion of convergence depends on the function S, the topology T, and the ordering $\geq$. However, in cases where no confusion is likely to result we may omit mention of T or of $\geq$ or of both and simply say "the net converges to s" or even better the sequence notation $\lim_{n \in D} S_n = s$.

Definition 3.8. If the net S is frequently in a set A, then the set E of all members $n \in D$ such that $S_n \in A$ has the property: for each $m \in E$ there is $p \in E$ such that $p \geq m$. Such subnets of D are called confinal.

At first glance this definition might appear to be adequate for the definition of a subnet. However, examples can be constructed to the contrary [8].

Definition 3.9. A net $\{T_m\}_{m \in D}$ is a subnet of a net $\{S_n\}_{n \in E}$ if, and only if, there is a function N on D with values in E such that

a) $T = S \cdot N$ or equivalently $T_i = S_{N_i}$ for each i in D; and

b) for each m E there is n in D with the property if $p \geq n$, then $N_p \geq m$.

It is important to note that in general the set D is much larger than E.

As promised, we enlarge somewhat on Kelly's iterated limit theorem. First we define the sum set of a family of sets X_α, which when the X_α are mutually disjoint is merely the union of the X_α's.

Definition 3.10. Let $\{X_\alpha\}_{\alpha \in I}$ be a family of sets and $X = \bigcup_{\alpha \in I} X_\alpha$ and consider the product set IxX. Let $X'_\alpha = \alpha \times X_\alpha \subset I \times X$. (We note that $X'_\alpha \cap X_\beta' = \emptyset$ if $\alpha \neq \beta$ and that X'_α and X_α have a natural and obvious 1-1 correspondence.) We define the sum set of the X_α to be the $\sum_{\alpha \in I} X = \bigcup_{\alpha \in I} X'_\alpha$.

Thus $\sum_{\alpha \in I} X_\alpha$ $I \times (\bigcup_{\alpha \in I} X_\alpha)$ and (α, x) $\sum_{\alpha \in I} X_\alpha$ if, and only if, $\alpha \in I$ and $x \in X_\alpha$.

If I and each X_α is partially ordered then $\sum_{\alpha \in I} X_\alpha$ is partially ordered by $(\alpha, x) \leq (\beta, y)$ if, and only if, $\alpha < \beta$ or $\alpha = \beta$ and $x \leq y$ in $X_\alpha = X_\beta$. Clearly if each X_α and I is a directed set then $\sum_{\alpha \in I} X_\alpha$ is directed.

Now consider the situation in Kelly's theorem. We have a directed set D, a family of directed sets $\{E_p\}_{p \in D}$ and a topological space X. We also have a function $S: \sum_{p \in D} E_p \rightarrow X : (m,n) \rightarrow S(m,n)$. In view of our defining $\sum_{p\in D} E_p$ with an ordering, we can even call S a net in X. (Hereafter, to help fix notation let us agree to call $\sum_{p\in D} E_p, E$). We further consider the set $F = D \times \prod_{p\in D} E_p$ and define $R: F \to E: (m,f) \to (m, f(m))$ and we consider the map $T: F \to X: (m,f) \to S(m, f(m))$. But $F = D \times \prod_{p\in D} E_p$ can begin the product ordering. Namely $(m,f) \geq (p,g)$ if, and only if, $m \geq p$ and $f \geq g$ where $f \geq g$ if, and only if, $f(p) \geq g(p)$ for all $p \in D$. Hence T is a net.

But the definition of T implies that $T = S \cdot R$ or definition 3.9a is satisfied. To show 3.9b consider any $(m_1, n_1) \in E$. Pick $f_1 \in \prod_{p\in D} E_p$ so that for $m = m_1$, $f_1(m) \geq n_1$ in the ordering of E_m, otherwise let $f_1(m)$ be arbitrary. Then for any $(m,f)F$ and $f(m) \geq f_1(m)$ we certainly have $R(m,f) = (m, f(m)) \geq (m, f_1(m)) \geq (m_1, n_1)$ in the ordering of E. For if $(m,f) \geq (m_1, f_1)$ in the ordering of F then either $m > m_1$ and $R(m,f) = (m, f(m)) > (m_1, n_1)$ in the ordering of E or $m = m_1$ and $f(m) \geq f_1(m) \geq n_1$ in the ordering of E_m. In either case $R(m,f) = (m, f(m)) \geq (m_1, n_1)$ in the ordering of E. That is, 2.9b is satisfied and we have proven

<u>Theorem 3.11</u>. T is a net and a subnet of S. A result of the foregoing proof was that R is a map of F onto E. We will need this fact later. Concerning the convergence of the nets just described we have, if $\lim_{n\in E_m} S(m,n)$ exists for each $m \in D$, and we define $S\#(m) = \lim_{m\in E_m} S(m,n)$ and if the net S# converges then $\lim_{m\in D} S\#(m) = \lim_{m\in D} \lim_{n\in E_m} S(m,n)$. Furthermore, the net $T = S \cdot R$ converges and the $\lim_{(m,f)\in R} T(p,f) = \lim_{m\in D} S\#(m) = \lim_{m\in D} \lim_{n\in E_m} S(m,n) = a \in X$.

Following Kelly [8] another formulation would be: For each neighborhood U of s in X there exists a $(m,f) \in F$ such if $(p,g) \geq (m,f)$ in the ordering of F then $S \cdot R(p,g) = T(p,g) \in U \subset X$. In the following we will have occasion to call on both of these formulations.

Now this theorem states that the existence of an iterated limit of a net implies the existence and equality of a double limit of a subnet. Of course, a rather natural question is, how about the converse? Since sequences are nets, it is rather easy to construct a subsequence which converges and its sequence does not. Better yet, see the discussion following Definition 3.12. Of course, another question might be, does the existence of the iterated limit of a net help in any way to establish the equality of the double and iterated limits of subnets? The following definition and two theorems answer this in the affirmative if you start with S first and in the negative if you start with T first. Moreover, the two theorems give a better relationship between existence and equality of partial, iterated, and double limits of S and T.

Definition 3.12. $R^*: X^E \to X^F : S \to T$ or $R^*(S) = G = S \cdot R$.

Now given R and T we ask if we can find an S such that $T = S \cdot R$. As commented earlier, R is a mapping of F onto E. Consequently, R^* may not be onto or what is the same thing given, $T \in X^F$ and R there may not exist an S such that $T = S \cdot R$. Therefore, the existence and equality of the iterated and double limits of T have no relationship to S if we start with T first.

Theorem 3.13. If $\lim_{n \in E_m} S(m,n)$ exists for each $m \in D$ then $T^* = \lim T(\cdot, f)$ exists in X^D with its topology.

Proof: To facilitate the proof we start with some definitions.

$$T: D \times \prod_{p \in D} E_p \to X : (p,f) \to S(p,f(p)).$$

$$T_1 : \prod_{p \in D} E_p \to X^D : f \to T_1(f) = F(\cdot, f) \text{ where } T_1(f)(m) = T(m,f).$$

$\pi_m : X^D \to X^m = X : g \to g(m)$ so that $\pi_m(g) = g(m)$ and in particular

$$\pi_m \cdot T_1(f) = T(m,f).$$

Now to show $T^* = \lim_{f \in \pi E_p} T(\cdot, f) = \lim_{f \in \pi E_p} T_1(f)$ exists we need only show that for each $m \in D$, the $\lim_{f \in \pi E_p} \pi_m \cdot T_1(f)$ exists. That is, since T_1 is a function from a directed set $(\prod_{p \in D} E_p)$ to a product space (X^D), it converges if, and only if, its projection into each coordinate space converges.

So, let $\lim_{n \in E_m} S(m,n) = S\#(m) = x_m$ and let U_m be any open neighborhood of x_m. Since $\lim_{n \in E_m} S(m,n) = x_m$, there exists $n_m \in E_m$ such that if $n \geq n_m$ in the ordering of E_m then $S(m,n) \in U_m$. Pick $f_m \in \prod_{p \in D} E_p$ such that $f_m(p) \geq n_m$ for $p=m$ and arbitrary otherwise. But if $f \geq f_m$ in the ordering of $\prod_{p \in D} E_p$ then $f(m) \geq f_1(m) \geq n_m$ in the ordering of E_m so $\pi_m \cdot T(f) = T(m,f) = S(m,f(m)) \in U_m$ since $f(m) \geq n_m$. Thus $\lim_{f \in \pi E_p} \pi_m \cdot T_1(f) = x_m = S\#(m)$. Hence $\lim_{f \in \pi E_p} T_1(f) = S\#$.

Now, the foregoing theorem can be restated in the following way. Since $S \in X^E$, $(\#)$ is a mapping of X^E into X^D. Specifically it is a mapping of those S for which $\lim_{n \in E_m} S(m,n)$ exists. That is,

$$\#: X^E \to X^D : S \to S\#, \text{ where } S\#(m) = \lim_{n \in E_m} S(m,n), S\# : D \to X : m \to S\#(m).$$

But, from Kelly [8] we have $X^{D \times E_p}$ is homeomorphic to $(X^D)^{\pi E_p}$.

So we define

$R^*: X^E \to (X^D)^{\pi E_p}: S \to R^*(S)$ where

$R^*(S) = S \cdot R = T$ and $R^*(S)(f)(m) = S \cdot R(m,f) = S(m,f(m)) = T(m,f)$.

Our theorem then becomes "to each $S \in X^E$ in the domain of #

(that is, $\lim_{n \in E_m} S(m,n)$ exists) we have a convergent net in

X^D(viz. $T_1 \in (X^D)^{\pi E_p}$) such that

$S\# = \lim_{f \in \pi E_p} T_1(f) = T^*$." And

$*: (X^D)^{\pi E_p} \to X^D; T_1 \to \lim_{f \in \pi E_p} T_1(f) = T^*$ where $T^* D \to X; m \to T^*(m)$.

Moreover, if $\lim_{m \in D} \lim_{n \in E_m} S(m,n)$ exists we have the following

theorem.

<u>Theorem 3.14</u>. If $\lim_{m \in D} \lim_{n \in E_m} S(m,n)$ exists then $T^* \in X^D$ is a convergent net

in X and $\lim T^*(m) = \lim_{m \in D} \lim_{f \in \pi E_p} T_1(f)(m) = \lim_{m \in D} \lim_{f \in \pi E_p} T(m,f) = \lim_{m \in D} \lim_{n \in E_m} S(m,n) =$

$\lim_{(p,f) \in F} T(p,f)$.

<u>Proof</u>: By the previous theorem $T^*(m) = \lim_{f \in \pi E_p} T_1(f)(m) = \lim_{f \in \pi E_p} T(m,f) =$

$\lim_{n \in E_m} S(m,n)$. Hence the iterated limits exists if anyone of them does.

This establishes the first three equalities. The last equality is Kelly's

iterated limit theorem.

In particular, this theorem states that if $\lim_{m \in D} \lim_{n \in E_m} S(m,n)$ ex-

ists then $\lim_{p \in D} \lim_{f \in \pi E_p} T(p,f) = \lim_{(p,f) \in F} T(p,f)$, or what is the same thing the iterated limit of the subnet T of the net S, equals the double limit of T. And even further since $S\#=T^*$, S or equivalently T can be thought of as a net of nets since $S\#=T^*$ is a net.

All of the foregoing discussion has been concerned with relationship of iterated and double limits of nets to iterated and double limits of subnets in a general topological space. With one exception, the foregoing theorems and discussion take care of all possible implications. This one exception is due to Birkhoff and is listed for completeness.

Theorem 3.15. The space X is T_3 (regular) if, and only if, for every net $S:E \to X$ the existence of the limits;

1) $\lim_{n \in E_m} S(m,n)=x_m=S\#(m)$ exists for every m.

2) $\lim_{(m,f)\in F} T(m,f)=x,$

implies $\lim_{m\in D} S\#(m)=\lim_{m\in D} \lim_{n\in E_m} S(m,n)$ exists and equals x.

Proof: Let A be any closed set in X and x any point not in A and assume X not T_3 so that x and A cannot be separated by disjoint open neighborhoods. Then every neighborhood $U\in\mathcal{U}_x$ has its closure intersecting A in a point s_U so that by Kelly's theorem 2.2 there exists a net in U converging to x_U, say the net $S_U:E_U\to U$. For D take the directed (downward) set U_x and define $S(U,n)=S_U(n)\in X$. Then $\lim_{n\in E_U} S(U,n)=x_U=S\#(U)$ exists.

Moreover, $T(U,f)=S(U,f(U))=S_U(f(U))\in U$ and hence (since $(U,f)\leq(V,g)$ if, and only if, $V\subset U$ and $f(U)\leq g(V)$)T converges to x. However, $S\#(U)\in A$ and since A is closed, $x\notin A$ and $\lim_{U\in\mathcal{U}_x} S(U)\neq x$ by Kelly's theorem 2.2c.

Conversely if X is T_3 and if $\lim_{n \in E_m} S(m,N)=x_m=S\#(m)$ and $\lim_{(m,f)\in F} T(m,f)=x$, the closure of the set for any $m_0 \geq D$, $\overline{\{x_m\}}_{m \geq m_0}$ contains x. Otherwise, since X is regular we can pick a set $U \supset \overline{\{x_m\}}_{m \geq m_0}$ and V a neighborhood of x which are disjoint and pick for m an f(m) such that for $m \geq m_0$, $T(m,f)=S(m,f(m))$ a neighborhood of x_m. This is possible since $\lim_{n} S(m,n) \longrightarrow x_m$ for each $m \geq m_0$ there exist n such that $S(m,n) \in U$ for $n \geq$ some n_0. So for $m \geq m_0$ pick $f(m) \geq n_0$ and for $m < m_0$ pick f(m) arbitrary. But then T cannot be eventually in V, contrary to the assumption $\lim_{(m,f)\in F} T(m,f)=x$.

Thus $x \in \overline{\{x\}}_{m \geq m_0}$ for each $m_0 \in D$ and by Kelly's theorem 2.7 the net S# converges to x.

<u>Theorem</u> <u>3.16</u>. Let X be T_3 and suppose $\lim_{(m,n)\in E} S(m,n)$ exists and that $\lim_{m \in D} S(m,n)=x_m=S(m)$ exists for every m then

1) $\lim_{m \in D} S\#(m)$ exists and equals x and

2) $\lim_{m \in D} \lim_{n \in E_m} S(m,n)=x$.

<u>Proof</u>: (2) is trivial from (1) and (1) is true from the previous theorem and the theorem that a subnet of a convergent net converges and to the same limit. Thus if S converges and $\lim_{n \in E_m} S(m,n)$ exists then since T is a subnet of S, $\lim_{(m,f)\in F} T(m,f)=x$ and by Theorem 3.15, $\lim_{m \in D} S\#(m)=x$.

It should be noted that in the last two theorems we used only that $\lim_{n \in E_m} S(m,n)$ exists for $m \geq m_0$, but not for all m. Theorems asserting the existence of double limits in terms of partial limits require the concept of uniform convergence and thus we need the concept of uniform space. Our next theorem and corollary is of this nature. We first give two examples to show that the iterated and double limits of a net are not in general equal, to further motivate the necessity of uniform convergence.

Example 1. Let Z be the integers and $f: Z\times Z \to R$ = reals, be defined by $f(m,n)=\frac{mn}{m^2+n^2}$. Order ZxZ by $(m_1,n_1)\geq(m,n)$ if, and only if, $m_1\geq m$ and $n_1\geq n$. Then $\lim_{n\to\infty}\lim_{m\to\infty}\frac{mn}{m^2+n^2}=\lim_{m\to\infty}\lim_{n\to\infty}\frac{mn}{m^2+n^2}=0$.

But $\lim_{(m,n)\to\infty}\frac{mn}{m^2+n^2}=1/2$ for m=n.

Example 2. Let ZxZ be ordered as in 1 and $f: Z\times Z\to R\times R$ be defined by $f(m,n)=\left\{\frac{1}{m}, \frac{(-1)^n}{m}\right\}$ then $\lim_{m\to\infty}\lim_{n\to\infty}\left\{\frac{1}{m}, \frac{(-1)^n}{m}\right\}$ does not exist and $\lim_{(m,n)\to\infty}\left(\frac{1}{m}, \frac{(-1)^n}{m}\right)=(0,0)$.

Definition 3.17. Let P and Q be directed sets, PxQ the product directed set, $(X,\mathcal{U})$ a complete uniform space such that $S: P\times Q\to X$, $T: Q\to X$ are nets. Then the $\lim_{p\in P} S(p,q)$ is said to be uniform in q if, and only if, $\lim_{p\in P} S(p,q)=T(q)$ for all $q\in Q$. Or equivalently, if, and only if, for all $U\in\mathcal{U}$ there exists a $P_U\in P$ such that if $p\geq P_U$, then $S(p,q)\in U[T(q)]$ for all q.

Theorem 3.18. Let $(X,\mathcal{U})$ be a complete uniform space, $S: P\times Q\to X$, $T: Q\to X$. If $\lim_{p\in P} S(p,q)=T(q)$ exists for each $q\in Q$ uniformly and if $\lim_{q\in Q} S(p,q)$ exists for every p then $\lim_{p\in P}\lim_{q\in Q} S(p,q)$, $\lim_{q\in Q}\lim_{p\in P} S(p,q)$ and $\lim_{(p,q)\in P\times Q} S(p,q)$

all exists and are equal.

Proof: For all $W[T(q)]$, $W \in \mathcal{U}$, there exists a $U \in \mathcal{U}$ such that $U \cdot U \subset W$. And if $\lim_{p \in P} S(p,q)$ exists for all p then $S(p_1,q) \in V[S(p_1,q_1)]$ for some $V \in \mathcal{U}$ and $q \geq q_V$, $q_1 \geq q_V$. Moreover, $S(p_1,q) \in U[T(q)]$ for all q and $p_1 \geq p_U$. Hence $U \cdot V \cdot U \subset W$ and $T(q_1) \in W[T(q)]$ where q, $q_1 \geq q_V$, $W \in \mathcal{U}$. But this implies $(T(q_1), T(q)) \in W$ for all W. Or, T is a Cauchy net which converges to a point $x \in X$ (recall we assumed X was a complete uniform space). Next consider $U[x]$, then, there exists a $V \in \mathcal{U}$ such that $V \cdot V \subset U$ and $S(p,q) \in V[T(q)]$, $T(q) \in V[x]$, $p \geq p_U$, $q \geq q_V$. Hence, $S(p,q) \in U$ for $p \geq p_U$, $q \geq q_V$ and S converges to x. Furthermore, this proves the existence and equality of all three limits.

Corollary 3.19. The foregoing theorem is true if the word uniform is replaced by the word metric.

Proof: A metric space is a uniform space. Kelly [8].

CHAPTER IV

DUALITY

In this chapter we will concern ourselves with Pontryagin duality and the relation of a certain "weak" topology to it. As a starting point we consider the following result of Ellis [5]. "The complement of a cartesian product of spaces, with the usual product topology is the direct sum of their complements; and the complement of a direct sum of spaces, if its topology lies within a certain interval, is the cartesian product of the complement". However, the upper bound of these topologies (that is, the strongest topology which lies in this interval) does not make the direct sum a topological group. Since Pontryagin duality is primarily concerned with topological groups, it was felt that Ellis's results could be strengthened by replacing his upper bound topology group. This is the primary result of this chapter. Specifically, we show that our so-called "weak" topology lies within this interval of topologies and does indeed make the direct sum a topological group.

We start by giving several theorems and definitions which are standard results of topological groups each of which is germane to our interest. Complete proofs are not given, but references and, or comments are.

Definition 4.1. A triple (G,•,T) is a topological group if, and only if, (G,•) is a group, (G,T) is a topological space, and the function whose value at (x,y) of GxG is x-y is continuous relative to the product topology.

The reason for using additive notation is that in what follows all groups will be assumed to be abelian. Following Kelly [8], we shall further assume that all topological groups are Hausdorff.

Theorem 4.2. The family $\mathcal{U}$ of neighborhoods of the identity 0 of a topological group has the properties.

1) If U and V belong to $\mathcal{U}$, then $U \cap V \epsilon \mathcal{U}$;

2) If $U \epsilon \mathcal{U}$ and $U \subset V$, then $V \epsilon \mathcal{U}$;

3) If $U \epsilon \mathcal{U}$, and some $V \epsilon \mathcal{U}$, then $V - V \epsilon U$; and

4) For each $U \epsilon \mathcal{U}$ and each $x \epsilon G$, $x+U-x \epsilon \mathcal{U}$.

Proof: Kelly [8].

Moreover, x+G is a homeomorphism of G into G. Hence, the neighborhoods of the identity are adequate for describing the topology of the topological group G.

Definition 4.3. A neighborhood of the identity 0 of a topological group G is called a nucleus.

Let $\{X_\alpha\}_{\alpha \epsilon I}$, $\{Y_\alpha\}_{\alpha \epsilon I}$ each be families of topological groups indexed by some set I.

Definition 4.4. $X = \underset{\alpha \epsilon I}{\pi} X = [x | x(\alpha) \epsilon X_\alpha \text{ for all } \alpha \epsilon I]$ is called the direct product of the $\{X_\alpha\}_{\alpha \epsilon I}$.

Definition 4.6. The members of the defining subbase for the product topology for the space $X = \underset{\alpha \epsilon I}{\pi} X_\alpha$ are of the form $[x \epsilon X | x_\alpha \epsilon U$, where U is open in $X_\alpha]$.

They are, intuitively, cylinders over open sets in the coordinate spaces [8].

We will use π to denote this topology.

Definition 4.7. A base for the box topology $\mathcal{L}$ for the space X can be described as the family of all sets $\pi_{\alpha \in I} U_\alpha$ where U_α is open in X_α for all $\alpha \in I$.

Definition 4.8. Let σ, T be two topologies on a topological space X. Then $\sigma \subset T$ if, and only if, every σ open set is T open in which case we say T is stronger than σ.

Theorem 4.9. $\pi \subset \mathcal{L}$

Proof: The product of open sets need not be open relative to π but they are $\mathcal{L}$ open.

Theorem 4.10. X with the product topology (notation (X, π)) is Hausdorff if, and only if, each X_α is Hausdorff.

Proof: Kelly [8].

Theorem 4.11. $(X, \mathcal{L})$ is Hausdorff if, and only if, each X_α is Hausdorff.

Proof: Kelly [8].

Theorem 4.12. (X, π) and $(X, \mathcal{L})$ are each topological groups.

Proof: The proof is a routine verification of the group postulates where addition in X is defined by $(x+x')(\alpha)=x(\alpha)+x'(\alpha)$.

Corollary 4.13. The direct sum is a subgroup of the direct product and is a topological subgroup with the induced topology.

Definition 4.14. If G is a topological group then the group H of all continuous homomorphisms of G into the reals mod.(1) with the induced topology (notation S) is called the character group of G, where H is given the compact-open topology. Notation H=G*.

Definition 4.15. A base for the compact-open topology is the set of $W[K,U]=[h \in H | h[K] \subset U$, where K is compact in G and U open in S].

Definition 4.16. Two groups G and H are said to be Pontryagin dual if they are isomorphic and homeomorphic to the character groups of each other.

Having these definitions, we note that two of the big questions in Pontryagin duality are: (1) What pairs of topological groups are Pontryagin dual? (2) What topological groups are character groups of their character groups? Kaplan [7] has shown that the family of groups which are character groups of their character groups is closed under the formation of cartesian products.

To effect this duality, both Kaplan [7] and Ellis [5] show an algebraic isomorphism and then (whenever possible) a homeomorphism. The result of Ellis that we are interested in is that part concerned with the algebraic isomorphism. Specifically, his theorem which states; "for each $\alpha \in I$, let X_α and Y_α be topological groups which are character groups of each other. If π is the product topology for $X = \underset{\alpha \in I}{\pi} X_\alpha$ and T any topology for $Y = \sum_{\alpha \in I} \oplus Y_\alpha$ such that for all $\alpha \in I$, $T(W) \subset T \subset T(\mathfrak{M})$, then each of the groups (X, π) and (Y,T) is algebraically isomorphic to the character group of the other." [5]. We further note that $T(W) \subset T \subset T(\mathfrak{M})$ is the interval of topologies originally stated and $T(\mathfrak{M})$ is the so-called upper bound of Ellis's interval of topologies.

The general plan of attack is to show the following: (1) the weak topology $\subset T(\mathfrak{M})$, (2) $T(\mathfrak{M})$ does not make Y a topological group, (3) the weak topology is the strongest topology which makes Y a topological group, and (4) Y with the weak topology is algebraically isomorphic to $(X, \pi)^*$.

Definition 4.17. Let $\mathfrak{M} = [M \in Y \mid$ for each $\alpha \in I$, there exist a nucleus $V_\alpha \subset Y_\alpha$ such that $i_\alpha(V_\alpha)$ M, where i_α is the injection of Y_α into Y$]$. Let $\mathfrak{M}_F = [M_F \in \mathfrak{M} \mid M_F = \bigcap_{i=1}^{n} M_i$ where $\bigcap_{i=1}^{n} M_i$ denotes any finite intersection of $M_i \in \mathfrak{M}]$.

The topology $T(\mathfrak{M})$ is characterized by defining a set $U \in Y$ to be $T(\mathfrak{M})$ open if, and only if, for each $y \in U$ there exists an $M_F \in \mathfrak{M}_F$ such that $y + M_F \in U$. A trivial verification will show that this is a topology for Y.

Definition 4.18. If $y_\alpha \in Y_\alpha$, then $(y_\alpha \mid Y_\alpha)$ is the smallest $\frac{1}{2n}$, n an integer, such that $2^n y_\alpha \in Y_\alpha$. If there is no smallest then $(y_\alpha \mid Y_\alpha) = 0$.

The set of $y \in Y$ such that $\sum_{\alpha \in I} (y_\alpha \mid Y_\alpha) < 1$ is a base for the asterisk (*) topology. Note $\sum_{\alpha \in I} (y_\alpha \mid Y_\alpha)$ is a finite sum since $Y = \{y_\alpha\}$ has finitely non-zero terms [7].

Theorem 4.19. The * topology is equivalent to the compact-open topology.

Proof: Ellis [5].

Theorem 4.20. $T(W) \subset \pi \subset \mathcal{L} \subset * \subset T(\mathfrak{M})$.

Proof: Since T(W) is not germane to our interest we have not defined it, but merely note that it can be found in Ellis's paper as well as the fact $T(W) \subset \pi$. $\pi \subset \mathcal{L}$ by Theorem 4.9. Kaplan [7] shows $\mathcal{L} \subset *$ and Ellis [5] shows that $* \subset T(\mathfrak{M})$.

The purpose in introducing this theorem is that when we introduce our weak topology, we will use $\mathcal{L}$ and $T(\mathfrak{M})$ to prove the weak topology is in Ellis's interval of topologies. Of course, Theorem 4.19 is fundamental to Pontryagin duality.

Theorem 4.21. $(Y,T(\mathfrak{M}))$ is not a topological group.

Proof: Let $Y=R\oplus R=R\times R$ where R is the field of real numbers. A nucleus in R is any open set containing the origin. Moreover, $\mathfrak{M}$ is made up of sets of the form $(0,y)\cup(x,0)$ where $|y|<\epsilon$ and $|x|<\epsilon$, $\epsilon>0$. $\mathfrak{M}_F$ is trivially equal to $\mathfrak{M}$. Consider the set of (r,θ) defined by the polar inequality $r<\cos 2\theta$. This is just a four leaf rose, whose petals lie on the X and Y axes, and which intersects the lines $y=\pm x$ only at the origin. This set is $T(\mathfrak{M})$ open since translates of the sets $(0,y)\cup(x,0)$, $|y|<\epsilon$, $|x|<\epsilon$ can be made to lie completely in the rose by proper choice of $\epsilon>0$. But by Theorem 4.2, if this set is to be an open set of a topological group, it must contain a set U such that V-V is contained in it. But this would imply for some $x\neq 0\in R$ and $x=y$, $(x,0)-(0,y)=(x,-y)=\left[(r,\theta)\,\middle|\, r<\cos(2\theta)\right]$. But this is impossible since this point lies on the line $y=-x$, which only intersects this set at the origin. Hence, $T(\mathfrak{M})$ does not make Y a topological group.

Definition 4.22. A set $W\subset Y = \sum_{\alpha\in I}\oplus Y_\alpha$ is weakly open if, and only if, $W\cap(\sum_{\alpha\in J}\oplus Y_\alpha)$ is open in $\sum_{\alpha\in J}\oplus Y_\alpha$ for all finite subsets J of I. We call this the weak topology for Y and denote it by ω.

Theorem 4.23. (Y,ω) is a topological group.

Proof: Any nucleus W in Y satisfies Theorem 4.2 if, and only if, $W\cap(\sum_{\alpha\in J}\oplus Y_\alpha)$ is a nucleus satisfying Theorem 4.2 in $\sum_{\alpha\in J}\oplus Y_\alpha$ for all finite $J\subset I$. But, $\sum_{\alpha\in J}\oplus Y_\alpha$ is a topological group (with the product topology) hence $W \cap \sum_{\alpha \in J} \oplus Y_\alpha$ is a nucleus in $\sum_{\alpha \in J} \oplus Y_\alpha$ and (Y, ω) is a topological group.

The following two lemmas help to better characterize the topology and establish the fact that $\mathcal{L} \subset \omega \subset T(\mathfrak{M})$.

Lemma 4.25. If W is an ω nucleus, then W contains elements of the form $y = [\{y_\alpha\} \mid y_\alpha = 0_\beta ;\ \alpha = \beta$ and $y_\alpha \neq 0_\beta$ if $\alpha \neq \beta]$.

Proof: $W \cap Y_\alpha$ is open and $i_\alpha(W \cap Y_\alpha) \subset W$. Hence W contains elements of the stated form.

Theorem 4.26. $\mathcal{L} \subset \omega \subset T(\mathfrak{M})$.

Proof: Consider any $\mathcal{L}$ open set. It is merely the product of nuclei from each Y_α. But, the intersection of each $\mathcal{L}$ open set with any $\sum_{\alpha \in J} \oplus Y_\alpha$ is the product of a finite number of nuclei in $\sum_{\alpha \in J} \oplus Y_\alpha$. But, the set is open in $\sum_{\alpha \in J} \oplus Y_\alpha$. Hence the set is ω open and $\mathcal{L} \subset \omega$.

Next, let W be an ω nucleus. From Lemma 4.24 we have for each $\alpha \in I$, $W_\alpha = W \cap Y_\alpha$ is an open neighborhood of 0_α in Y_α if each Y_α is non-discrete. And for each discrete Y_α, $W_\alpha = W \cap Y_\alpha$ is trivially open. Thus by Lemma 4.25 $i_\alpha(W_\alpha) = W \cap Y_\alpha \subset W$. Hence $\omega \subset T(\mathfrak{M})$.

Summarizing, we have shown that (Y, ω) is a topological group, $T(W) \subset \mathcal{L} \subset T(\mathfrak{M})$, and $T(\mathfrak{M})$ is not a topological group. What remains to be shown is that is the strongest topology which makes Y a topological group and $(Y, \omega)^*$ algebraically isomorphic to (X, π), and $(X, \pi)^*$ algebraically isomorphic (Y, ω). We do this in Theorem 4.30 and Corollary 4.31. To do this we need two preliminary theorems and a corollary.

Theorem 4.27. Let $Y = \sum_{\alpha \in K} \oplus Y_\alpha$ with the ω-topology. Then for all groups

G, the map f where $f:Y\to G$ is continuous if, and only if, $f|Y_\alpha$ is continuous for all $\alpha\in I$.

Proof: We first note that the word group will mean topological abelian Hausdorff group and the word map will mean a continuous homomorphism.

Let $f_\alpha = f|Y_\alpha$ be continuous for all G, O any open set contained in G, and let Y have the ω-topology. Now, $Y_\alpha \cap f^{-1}[O] = f_\alpha^{-1}[O]$ is relatively open in Y_α for all $\alpha\in I$, and all open O in all G. But, Y has the ω-topology which implies that $Y_\alpha \cap f^{-1}[O]$ is open in Y_α for each open O in each G. Hence f_α is continuous for all $\alpha\in I$.

To show f is continuous whenever f_α is continuous, for all $\alpha\in I$, we need to show $f^{-1}[O]\cap \sum_{\alpha\in J} \oplus Y_\alpha$ is open in $\sum_{\alpha\in J}\oplus Y_\alpha$ for all finite $J\subset I$. To do this we show it true for $Y_\alpha \oplus Y_\beta = Y_\alpha x Y_\beta$ and hence, by induction, it will be true for all finite $J\subset I$.

Define $f_\alpha x f_\beta : Y_\alpha x Y_\beta \to G:(y_\alpha, y_\beta)\to f_\alpha(y_\alpha)+f_\beta(y_\beta)$, $f_\alpha \boxtimes f_\beta : Y_\alpha x Y_\beta \to GxG:(y_\alpha, y_\beta)\to(f_\alpha(y_\alpha), f_\beta(y_\beta))$ and $h:GxG\to G: (g_1,g_2)\to g_1+g_2$.

Clearly, we have $f_\alpha x f_\beta = h\cdot f_\alpha \boxtimes f_\beta$. Moreover, since the product of open sets in GxG is a subbase for the product topology of GxG, the continuity of $f_\alpha \boxtimes f_\beta$ can be established in terms of these sets. But, if UxV is open in GxG, then $[f_\alpha \boxtimes f_\beta]^{-1}[UxV] = f_\alpha^{-1}[U]\boxtimes f_\beta^{-1}[V]$ is open in $Y_\alpha x Y_\beta$ with the product topology. Hence, $f_\alpha x f_\beta$ is continuous. Also, h is continuous since the group operation, +, is continuous. Therefore, $f_\alpha x f_\beta = h\cdot f_\alpha \boxtimes f_\beta$ is continuous.

However, if O is open in G, then $[f_\alpha x f_\beta]^{-1}[O]=$
$[(y_\alpha, y_\beta)\in Y_\alpha x Y_\beta \mid f_\alpha(y_\alpha)+f_\beta(y_\beta)\in O] = [(y_\alpha, y_\beta)\in Y_\alpha x Y_\beta \mid (f|Y_\alpha)(y_\alpha)+ (f|Y_\beta)(y_\beta)\in O] = [(y_\alpha, y_\beta)\in Y_\alpha x Y_\beta \mid (f|Y_\alpha x Y_\beta)(y_\alpha, 0)+(f|Y_\alpha x Y_\beta)$

$(0,y_\beta)\varepsilon\, 0] = [(y_\alpha, y_\beta)\varepsilon Y_\alpha x Y_\beta \mid (f|Y_\alpha)(y_\alpha, y_\beta)\varepsilon\, 0] =$
$[(y_\alpha, y_\beta)\varepsilon Y_\alpha x Y_\beta \mid (f|Y_\alpha x Y_\beta)]\varepsilon\, 0$. Thus, $f|Y_\alpha x Y_\beta$ is continuous and, by induction, $f \mid \sum_{\alpha\varepsilon J} \oplus Y_\alpha$ is continuous. Hence, f is continuous.

Theorem 4.28. Let X be a family of maps of $Y \to G$, G a fixed group. Furthermore, let $x|Y_\alpha = x_\alpha$ be continuous for all $\alpha\varepsilon I$, $x\varepsilon X$. If T is the smallest topology on Y making all such maps continuous, then $T \subset \omega$.

Proof: If O is any open set in G, then $x^{-1}[0]$ is T open in Y, for all $x\varepsilon X$, $O\varepsilon G$. Furthermore, $x^{-1}[0]\ Y_\alpha = x_\alpha{}^{-1}[0]$ is open in Y_α for all $\alpha\varepsilon I$, $x\varepsilon X$ since $x_\alpha = x|Y_\alpha$ is continuous. But, by the same construction as in Theorem 4.27, we have $x|Y_\alpha x Y_\beta$ continuous on $Y_\alpha x Y_\beta$ and hence continuous on $\sum_{\alpha\varepsilon J} \oplus Y_\alpha$ for all finite $J \subset I$. Therefore,

$x^{-1}[0] \cap \sum_{\alpha\varepsilon J} \oplus Y_\alpha = [x \mid \sum_{\alpha\varepsilon J} \oplus Y_\alpha]^{-1}[0]$ is open in $\sum_{\alpha\varepsilon J} \oplus Y_\alpha$ and $T \subset \omega$.

Corollary 4.29. Let X be a family of maps of $Y \to G$, G a fixed group. Furthermore, let $x|Y_\alpha = x_\alpha$ be continuous for all $\alpha\varepsilon I$, $x\varepsilon X$ and T any topology on Y making all such maps continuous. Then $T \subset \omega$.

Proof: The same as Theorem 4.28 since the proof does not depend on T being the smallest topology which makes all maps $X:Y \to G$ continuous.

Theorem 4.30. Let $X_\alpha = Y_\alpha{}^*$, $Y_\alpha = X_\alpha{}^*$. $X = \prod_{\alpha\varepsilon I} X_\alpha$, $Y = \sum_{\alpha\varepsilon I} \oplus Y_\alpha$. Then $(Y,\omega)^*$ is algebraically isomorphic to (X,π) and $(X,\pi)^*$ is algebraically isomorphic to (Y,ω).

Proof: From Ellis [5] we have the following theorem. "For each $\alpha\varepsilon I$, let X_α and Y_α be topological groups which are character groups of each other. If π is the product topology for X and T a topology for Y such that $T(W) \subset T \subset T(\mathfrak{M})$, then each of the groups (X,π) and (Y,T) is algebraically isomorphic to the character group of the other". But, from Theorem 4.26, we have $T(W)\subset\omega\subset T(\mathfrak{m})$. Hence, the theorem is true.

The following corollary is the main result of this chapter.

Corollary 4.31. For any topology T such that $(Y,T)^*$ is algebraically ismorphic to (X,π), then $T\subset\omega$. In particular, the compact open topology is contained in ω.

Proof: If $x\in(Y,T)^*$, then x is continuous by the definition of a character group. By Theorem 4.30, $X_\alpha=Y_\alpha{}^*$ implies $x|Y_\alpha$ is continuous. Hence, from Theorem 4.29 we have $T\subset\omega$ and, in particular, the compact open topology is contained in ω.

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