MULTIPLIERS IN HARDY AND BERGMAN SPACES, AND RIESZ DECOMPOSITION

By

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During all my time at Oklahoma State, he has been posing several challenging problems. For some of them, I even had no idea of how to approach them at the beginning. For example, I failed to get a non-trivial estimate for the Mahler measure of the Hadamard product of two polynomials at first. However, I got some estimates for more general case – the Hadamard product of two analytic or harmonic functions in Hardy H^p and Bergman A^p spaces. These are considered in Section 1.2 and Chapter 2. Unfortunately, they become inefficient when the exponent p is approaching 0. At that time, switched to another problem. But then I realized how to approach the problem on the Mahler measure, and I figured out why the estimates I was trying to obtain in H^p for $p < 1$ were not as good as I expected. The problem is that the Hadamard product operator may easily be unbounded in H^0. Finally, I obtained the estimates I needed. They are in Section 1.3 and Chapter 3. I would also like to thank Dr. Pritsker for his help with the proof of the limit relation (1.15). His help simplified the proof dramatically.

I would also like to thank my advisor for many useful suggestions he has given me. Some of them opened approaches I was completely unaware of, e.g., discrepancy results. But the main thing I am grateful to Dr. Pritsker for is the following. I

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learned how to take a problem in an area I had no essential experience with, learn it, really understand the problem, and solve it. This is what happened, e.g., with the Riesz decomposition for poly-superharmonic functions.

In other words, my advisor prepared me for doing serious independent research.

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Multipliers' methods have proven to be an efficient tool in virtually any area of Analysis. Many linear operators act as multipliers on Taylor series, Fourier series, Fourier integrals, etc., of a function. This means the operators introduce some multiplicative factors to the series or integrals. As a consequence, conditions on boundedness of multipliers imply important inequalities in Analysis, in particular, in Approximation Theory.

We consider series multipliers in Hardy and Bergman spaces in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, as well as multipliers of Fourier integrals in Hardy spaces in tubes over open cones (in $\mathbb{C}^n$). Obtained conditions are used to derive some inequalities, e.g., Bernstein and Nikolskiĭ type inequalities for entire functions.

Some of the multiplier conditions are surprisingly sharp. As an example, a critical index for Bochner-Riesz means of Fourier integrals in Hardy spaces in tubes has been found.

For the Hadamard product of two polynomials (again, a multiplier-type operator), we obtain sharp inequalities for its Mahler measure. They imply several sharp inequalities used in Approximation Theory.

We conclude the thesis by the Riesz Decomposition result for m-superharmonic functions in $\mathbb{R}^n$, $2m < n$, which generalizes work of K. Kitaura and Y. Mizuta for super-biharmonic functions.

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LIST OF SYMBOLS

$\mathbb{N}$	Set of natural numbers: $1, 2, 3, \ldots$
$\mathbb{Z}$	Set of integers
$\mathbb{Z}_+$	Set of non-negative integers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of non-negative real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}^n$	n-dimensional real vector space
$\mathbb{R}^n_+$	$\{x = (x_1, \ldots, x_n) : x_j \geq 0,\ j = 1, \ldots, n\}$
$\mathbb{C}^n$	n-dimensional complex vector space
$\mathbb{D}$	The open unit disk in the complex plane: $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\mathbb{D}_R$	The disk $\{z \in \mathbb{C} : \lvert z\rvert < R\}$
T_Γ	Tube (subset of $\mathbb{C}^n$) over open cone $\Gamma \subset \mathbb{R}^n$
$V_n(\Gamma)$	The maximum possible volume of a simplex built on n linearly independent unit vectors contained in an open cone $\Gamma \subset \mathbb{R}^n$
H^p	Hardy space
h^p	Harmonic Hardy space
A^p	Bergman space
a^p	Harmonic Bergman space
$\lVert f\rVert_{H^0}$	Mahler measure of a function f
H^0	Space of functions analytic in $\mathbb{D}$ and having finite Mahler Measure
U^μ	Potential of a Borel measure μ
$B(a, r)$	The open ball in $\mathbb{R}^n$: $\{x \in \mathbb{R}^n : \lvert x - a\rvert < r\}$
$S(a, r)$	The sphere in $\mathbb{R}^n$: $\{x \in \mathbb{R}^n : \lvert x - a\rvert = r\}$
σ_n	The surface measure of the unit sphere in $\mathbb{R}^n$: $\sigma_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$

CHAPTER 1

Introduction and Main Results

1.1 Overview of Main Topics

There are many forms of representation of a function by series or integrals. For example, a function f analytic in a disk $\{z : |z - a| < R\}$ of the complex plane can be represented by its Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k .$$

A 2π-periodic function f of a real variable from the $L^2[0, 2\pi]$ space can be written as its Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}. \tag{1.1}$$

In addition to power and trigonometric systems, there are many other orthogonal systems that can serve as bases for other expansions.

Let us note that the series on the right-hand side of (1.1) converges in L^2-norm to f. An immediate question is if it converges at least a.e. to the generating function f. In 1966, L. Carleson [19] gave a positive answer to this question. In 1967, R. Hunt [48] generalized this result for functions from L^p-spaces, $p > 1$.

Note that if $f \in L^p[0, 2\pi]$, $p \geq 1$, its Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} \, dx$$

are well defined. However, when $p = 1$, the Carleson-Hunt type result is not valid. Moreover, in 1926, A. N. Kolmogorov constructed an example of a function from L^1 whose Fourier series diverges everywhere [53].

The same questions could be asked about convergence of the Fourier series in norm. And the answer is: When $1 < p < \infty$, the partial sums $\sum_{|k|\le N} c_k e^{ikx}$ converge to f in L^p as $N \to \infty$. For $p = 1, \infty$, norm convergence fails. For $p = 1$, an example is again due to A. N. Kolmogorov.

Thus to approximate a periodic function from L^1 by trigonometric polynomials, we need a modification of the Fourier sums. Various methods of summation work here. For example, it is known that if $f \in L^p$ for $1 \le p < \infty$, then the Abel-Poisson means $\sum_{k\in\mathbb{Z}} e^{-2\pi|k|t} c_k e^{2\pi ikx}$ converge to f in L^p-norm as $t \to 0+$. Another example is the Bochner-Riesz means $\sum_{|k|<R} \left(1 - \frac{k^2}{R^2}\right)^{\alpha} c_k e^{2\pi ikx}$, $\alpha > 0$, which converge to f as $R \to \infty$. It is also known that for a continuous function f, convergence of the above means is uniform. These summation methods are examples of Fourier multipliers.

Multipliers of Fourier series and integrals have been investigated and widely used since 1923, when they were introduced by M. Fekete [30]. The idea is to introduce some multiplicative factors λ_n into the Fourier series, i.e., to consider a modified Fourier series

$$\Lambda f \sim \sum_{n\in\mathbb{Z}} c_n \lambda_n e^{2\pi inx}$$

that has better properties than the original one. This approach has been successfully applied to problems of approximation theory, differential equations, numerical analysis, etc., provided Λ defines a bounded linear operator on the corresponding function space. The first effective sufficient condition for boundedness of Λ in $L^p(\mathbb{T})$, $p \in (1, \infty)$, and its applications, were found by J. Marcinkiewicz [62]. Later, for the non-periodic case of multipliers of Fourier integrals, these conditions were obtained by S. G. Michlin [63, 64] and L. Hörmander [45] (see also [85, Ch. IV]). The most investigated cases are $p = 1, 2, \infty$, which is not a surprise. Employing the Riesz-Thorin theorem, it is easy to transfer such results to the case $p \in (1, \infty)$. These results and techniques became classical and are well described, e.g., in [86].

For $p \in (0, 1)$, $L^p(\mathbb{T})$ spaces are only pre-normed, and there are no linear continuous functionals, and no Fourier series in these spaces. This is the reason for considering the $H^p(\mathbb{D})$ spaces of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and having their boundary values in $L^p(\mathbb{T})$. Namely, $H^p(\mathbb{D})$ consists of all functions f holomorphic in $\mathbb{D}$, such that

$$\|f\|_{H^p} := \begin{cases} \sup_{r\in[0,1)} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, & p \in (0,\infty), \\ \sup_{z\in\mathbb{D}} |f(z)|, & p = \infty, \end{cases} \tag{1.2}$$

is finite. We often write H^p for $H^p(\mathbb{D})$.

Any function $f \in H^p(\mathbb{D})$, $p > 0$, has the Taylor series expansion in $\mathbb{D}$. If $p \geq 1$, then the Taylor series coincides with the Fourier series of the non-tangential (or radial) limit values of f on the unit circle. For $0 < p < 1$, one can consider multipliers of Taylor series instead of Fourier series. One special case of these multipliers, namely, the Hadamard product of two functions, is considered in Chapter 2.

The disseration is structured as follows. In Section 1.2 and Chapter 2, we investigate the Hadamard product of two analytic or harmonic functions as a linear operator acting between Hardy spaces (H^p to H^q), with $p, q \geq 1$. We also obtain estimates for the norm of this operator in Bergman spaces of analytic or harmonic functions, as well as consider the case of the operator acting from Hardy to Bergman spaces.

For the Hadamard product operator acting from H^p to H^p with $p < 1$, the estimates like those obtained in Theorem 1.2.1 are not valid. In fact, dependence on p becomes crucial (see Theorem 2.2.1 in Section 2.2), and the constants blow up when p approaches 0. Since this is unavoidable for analytic functions, we restrict our attention to polynomials. In Section 1.3 and Chapter 3 we obtain estimates for the Mahler measure of the Hadamard product of two polynomials (Mahler measure is a limiting case for the H^p-pre-norms when $p \to 0+$). A sharp estimate we obtain is also used to get corresponding estimates in H^p-norm/pre-norm. The aforementioned estimates are also used for proving some sharp inequalities. For example, estimates for the odd and even parts of a polynomial in H^p pre-norm ($p < 1$) are derived in Section 3.2.

In Section 1.4 and Chapter 4, we study multipliers of Fourier integrals acting between the Hardy spaces $H^p(T_\Gamma)$ and $H^q(T_\Gamma)$, where $0 < p \le q \le 1$ and $T_\Gamma \subset \mathbb{C}^n$ is a tube over an open cone $\Gamma \subset \mathbb{R}^n$ (for precise definitions, see Section 1.4). We obain efficient sufficient conditions for the multipliers, which in some cases are also necessary. One of the most interesting cases is that of radial kernels. In particular, we obtain the critical index for the Bochner-Riesz means of Fourier integrals, i.e., the index when they define a bounded linear operator from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$.

Note that for $p \ge 1$, there is no difference between multipliers in H^p and in L^p since these spaces could be identified. Moreover, for $p \ge 1$, the conditions for multipliers of Fourier integrals and Fourier series are the same in view of the well-known result due to K. de Leeuw.

Despite the fact that the conditions we obtained work for $0 < p \le q \le 1$, it is possible to derive more general results, for $0 < p \le q \le \infty$, using a proper "scaling of powers". Such a technique is used in Section 4.3, where we obtain Bernstein-type and Nikol'skiĭ-type inequalities for entire functions of exponential type.

Section 1.5 and Chapter 5 are devoted to the Riesz decomposition for super-polyharmonic functions in $\mathbb{R}^n$. This decomposition generalizes the one recently discovered by K. Kitaura and Y. Mizuta [52] for super-biharmonic functions.

As a general principle, Sections 1.2 – 1.5 contain the main definitions and some of the main results. Detailed explanations, other results, proofs, and more historical references are postponed to the forthcoming chapters.

The main results of the dissertation are published in [99, 100, 101, 102], and also [103], which is submitted for consideration for publication and had been still under consideration at the time of the thesis preparation.

1.2 Hadamard Product in Hardy and Bergman Spaces

The *Hadamard Product*, or the *Hadamard Convolution*, of two harmonic functions f and g in $\mathbb{D}$:

$$f\left(re^{i\theta}\right)=\sum_{n=-\infty}^{\infty}a_nr^{|n|}e^{in\theta},\quad g\left(re^{i\theta}\right)=\sum_{n=-\infty}^{\infty}b_nr^{|n|}e^{in\theta},\quad r\in[0,1),\,\theta\in\mathbb{R},$$

is defined by

$$(f*g)\left(re^{i\theta}\right)=\sum_{n=-\infty}^{\infty}a_nb_nr^{|n|}e^{in\theta},\quad r\in[0,1),\,\theta\in\mathbb{R}. \tag{1.3}$$

If we fix one of the functions, say, f, we can consider the Hadamard product as a linear operator $f*$ on a space of harmonic (or analytic) functions in $\mathbb{D}$. Thus, one can think about it as a coefficient multiplier that introduces coefficients a_n into the series decomposition of g:

$$f*:\ \sum_{n=-\infty}^{\infty}b_nr^{|n|}e^{in\theta}\mapsto\sum_{n=-\infty}^{\infty}a_nb_nr^{|n|}e^{in\theta}.$$

There are many results devoted to coefficient multipliers in various function spaces (see e.g., [26, 107, 94], [27, Ch. 3, § 3.4]). We restrict our attention to the case when the a_n's are taken from the Fourier series of f.

As usual, $h^p=h^p(\mathbb{D})$ denotes the *Harmonic Hardy Space*, i.e., the set of all functions f harmonic in $\mathbb{D}$, such that

$$\|f\|_{h^p}:=\begin{cases}\sup_{r\in[0,1)}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{it}\right)\right|^p\,dt\right)^{1/p}, & p\in(0,\infty),\\ \sup_{z\in\mathbb{D}}|f(z)|, & p=\infty,\end{cases} \tag{1.4}$$

is finite. Let us note that the holomorphic Hardy space H^p is a subset of h^p in view of (1.2). We also consider Hardy spaces in a disk of an arbitrary radius $R>0$:

$$\mathbb{D}_R:=\{z\in\mathbb{C}:|z|<R\}.$$

The corresponding Hardy spaces $H^p(\mathbb{D}_R)$, $p\in(0,\infty]$, consist of all functions f holomorphic in $\mathbb{D}_R$, such that

$$\|f\|_{H^p(\mathbb{D}_R)}:=\begin{cases}\sup_{r\in[0,R)}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{it}\right)\right|^p\,dt\right)^{1/p}, & p\in(0,\infty),\\ \sup_{z\in\mathbb{D}_R}|f(z)|, & p=\infty,\end{cases}$$

is finite. We will use them in Section 2.2.

Following [27], for $0 < p < \infty$, the *Bergman Space* $A^p = A^p(\mathbb{D})$ consists of all functions f analytic in $\mathbb{D}$, for which

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, d\sigma(z)\right)^{1/p} = \left(\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left|f\left(re^{i\theta}\right)\right|^p r\, dr\, d\theta\right)^{1/p} < \infty. \quad (1.5)$$

(Here and in the sequel, $d\sigma(z)$ denotes the Lebesgue area measure in $\mathbb{D}$ normalized by the condition $\sigma(\mathbb{D}) = 1$.)

The *Harmonic Bergman Spaces* $a^p = a^p(\mathbb{D})$ consist of functions f harmonic in $\mathbb{D}$, such that $\|f\|_{a^p}$ given by the same expression as in (1.5) is finite (see [6, Ch. 8]).

If T is a bounded linear operator mapping a space X into a space Y (normed or pre-normed), we will use the notation $T \in \mathcal{L}(X, Y)$.

The results of this section (and Chapter 2) were motivated by Theorem 4.2.6 of the monograph [82] by T. Sheil-Small, which states that if F is any harmonic function in $\mathbb{D}$, then the Hadamard product operator $F*$ it defines has the operator norm

$$\|F*\|_{h^\infty \to h^\infty} = \|F\|_{h^1}. \quad (1.6)$$

In fact, the operator $F*$ of h^1 into h^1 also has the same norm (see Theorem 1.2.1 (c) below). However, if we replace the harmonic Hardy space h^∞, or h^1, by their holomorphic visa-vis, H^∞, and H^1, respectively, then (1.6) is no longer true (see Proposition 1.2.1).

The following theorem is a generalization of Theorem 4.2.6 from [82].

Theorem 1.2.1 *(a) For $1 \le p \le q \le \infty$, and $F \in h^{q/p}(\mathbb{D})$, the operator $F* \in \mathcal{L}(h^p, h^q)$ with the norm at most $\|F\|_{h^{q/p}}$ (assuming $q/p = 1$ if $p = q = \infty$). Moreover, for any function g harmonic in $\mathbb{D}$, and $r \in [0, 1)$,*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left|(F*g)\left(re^{it}\right)\right|^q dt\right)^{1/q} \le$$

$$\|F\|_{h^{q/p}} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|g\left(re^{it}\right)\right|^p dt\right)^{1/p}, \quad 1 \le p \le q < \infty, \quad (1.7)$$

$$\max_{\theta\in\mathbb{R}} \left|(F*g)\left(re^{i\theta}\right)\right| \le \|F\|_{h^\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|g\left(re^{it}\right)\right|^p dt\right)^{1/p}, \quad 1 \le p < \infty, \tag{1.8}$$

and

$$\max_{\theta\in\mathbb{R}} \left|(F*g)\left(re^{i\theta}\right)\right| \le \|F\|_{h^1} \max_{\theta\in\mathbb{R}} \left|g\left(re^{i\theta}\right)\right|. \tag{1.9}$$

(b) If F is a positive harmonic function, and $F* \in \mathcal{L}(h^p, h^p)$ for some $p \in [1,\infty]$, then $F \in h^1(\mathbb{D})$, and

$$\|F\|_{h^1} = \|F*\|_{h^p \to h^p}.$$

Thus, $F* \in \mathcal{L}(h^p, h^p)$ for all $p \in [1,\infty]$, and the operator norm does not depend on p.

(c) If $F \in h^1(\mathbb{D})$ then

$$\|F*\|_{h^1\to h^1} = \|F*\|_{h^\infty \to h^\infty} = \|F\|_{h^1}.$$

Remark 1.2.1 *If $p \ge q$ then Hölder's inequality implies $h^p \subset h^q$, and $\|f\|_{h^q} \le \|f\|_{h^p}$. Thus, Theorem 1.2.1 applied with $p = q$ yields $\|F*g\|_{h^q} \le \|F*g\|_{h^p} \le \|F\|_{h^1}\|g\|_{h^p}$, whence $\|F*\|_{h^p\to h^q} \le \|F\|_{h^1}$. For a positive harmonic F, and $g(z) \equiv 1$, the mean-value property implies $\|F*g\|_{h^q} = F(0) = \|F\|_{h^1}$. Hence, $\|F*\|_{h^p\to h^q} = \|F\|_{h^1}$ in this case. Considering aforementioned, the only interesting case is when $p \le q$.*

Parts (a) and (b) of Theorem 1.2.1 could be restated for holomorphic function g to give estimates of $F*$ acting from H^p to H^q (and, in fact, for $p = q = 1$ the result follows immediately from the estimate for q-means proven by M. Pavlović in [72]). However, Part (c) has no holomorphic analogue because of the following result:

Proposition 1.2.1 *For any $M > 0$, there exists a function $\mathcal{F} \in H^1(\mathbb{D})$ such that*

$$\|\mathcal{F}*\|_{H^p\to H^p} = 1, \quad \forall p \in [1,\infty],$$

but $\|\mathcal{F}\|_{H^1} > M$.

Other results of Chapter 2 deal with the Hadamard product operator in Bergman spaces, and from Hardy to Bergman spaces.

1.3 Mahler Measure of the Hadamard Product of Two Polynomials

The estimates of the previous section become more specific if we consider polynomials instead of general analytic or harmonic functions. As we already mentioned, an estimate of the form

$$\|F*\|_{H^p \to H^p} \leq C$$

where C does not depend on $p < 1$ is not valid. The reason is that the operator $F*$ becomes unbounded as an operator on the "space of Mahler measure" H^0 (see below), which is a limiting case for H^p as $p \to 0+$. Considering polynomials helps to explain the reason for this, and to obtain some unexpected inequalities.

For a function f holomorphic in the unit disk $\mathbb{D}$, its **Mahler Measure** is defined by

$$\|f\|_{H^0} := \exp\left(\sup_{r\in[0,1)} \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(re^{it}\right)\right| \, dt\right).$$

Note that if $f \in H^{p_0}(\mathbb{D})$, for some $p_0 > 0$, then $\|f\|_{H^0} = \lim_{p\to 0+} \|f\|_{H^p}$.

For $n \in \mathbb{Z}_+$, let $\mathbb{C}_n[z]$ denote the set of all polynomials in the complex variable z with complex coefficients of degree at most n.

So, for a polynomial $P \in \mathbb{C}_n[z]$, we have

$$\|P\|_{H^0} := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln\left|P\left(e^{it}\right)\right| \, dt\right).$$

The Mahler measure has proven to be an efficient tool in obtaining sharp inequalities for polynomials. For example, K. Mahler [60] proved that if $P(z) = \sum_{k=0}^{n} a_k z^k$, then $|a_k| \leq \binom{n}{k} \|P\|_{H^0}$. Obtaining Bernstein's inequality in Hardy spaces H^p:

$$\|P'\|_{H^p} \leq n \|P\|_{H^p},$$

had been a difficult problem for $p < 1$ (see, e.g. [50, 90]). In an important paper [17], N. G. de Bruijn and T. A. Springer proved that if $\deg(P) \leq n$, then $\|P'\|_{H^0} \leq n \|P\|_{H^0}$. This was a corollary of a much more powerful result ([17, Theorem 7]), which is also a cornerstone for our considerations. For more historical remarks, see Chapter 3.

It is clear that $\|\cdot\|_{H^0}$ is not a norm or even a pre-norm (the triangle inequality fails even in a weak form). However, it has an important multiplicative property

$$\|PQ\|_{H^0} = \|P\|_{H^0}\|Q\|_{H^0}. \tag{1.10}$$

Furthermore, if $P(0) \neq 0$ and has zeros $\{\alpha_k\}$, then Jensen's formula implies that

$$\|P\|_{H^0} = \frac{|P(0)|}{\prod\limits_{|\alpha_k|<1} |\alpha_k|}.$$

Using the multiplicative property (1.10) for a polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ of degree exactly n, we obtain

$$\|P\|_{H^0} = |a_n| \prod_{|\alpha_k|>1} \alpha_k. \tag{1.11}$$

(As usual, if a product is empty, we assume its value is 1.)

In [17], N. G. de Bruijn and T. A. Springer obtained several sharp results on a different kind of product of two polynomials – the Schur-Szegő product. This product is well studied because it enjoys a very powerful apolarity property, that is not available for the Hadamard product. Fortunately, it is possible to reduce the Hadamard product to the Schur-Szegő product and obtain the sharp estimates we need.

The **Schur-Szegő Product** of $R(z) = \sum\limits_{k=0}^{n} \binom{n}{k} r_k z^k$ and $W(z) = \sum\limits_{k=0}^{n} \binom{n}{k} w_k z^k$ is given by

$$(R *_S W)(z) := \sum_{k=0}^{n} \binom{n}{k} r_k w_k z^k.$$

Note that

$$(R *_S W)(z) = (R * W * L)(z),$$

where $L(z) = \sum_{k=0}^{n} \binom{n}{k}^{-1} z^k$.

It follows immediately from [17, Theorem 7] that for two polynomials R and W,

$$\|R *_S W\|_{H^0} \leq \|R\|_{H^0}\|W\|_{H^0}. \tag{1.12}$$

Clearly, this inequality is sharp. Taking, for example, $W_0(z) = (1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$, we obtain $\|R *_S W_0\|_{H^0} = \|R\|_{H^0} \|W_0\|_{H^0}$, for any polynomial $R \in \mathbb{C}_n[z]$.

Using the proof of [17, Theorem 7], V. V. Arestov [3] obtained sharp estimates for the Schur-Szegő product in more general spaces. In particular, they are valid in any H^p, $p \in [0, \infty]$. Specifically, [3, Theorem 1] implies that for any two polynomials $R, W \in \mathbb{C}_n[z]$,

$$\|R *_S W\|_{H^p} \leq \|R\|_{H^0} \|W\|_{H^p}, \quad 0 \leq p \leq \infty. \tag{1.13}$$

Using (1.13), we obtained a sharp estimate for the Hadamard product. The main result of Chapter 3 is given by the following statement.

Theorem 1.3.1 *(a) For any polynomials P and Q of degree at most n with complex coefficients, the following estimate holds:*

$$\|P * Q\|_{H^p} \leq \|\Theta_n\|_{H^0} \|P\|_{H^0} \|Q\|_{H^p}, \quad 0 \leq p \leq \infty, \tag{1.14}$$

where

$$\Theta_n(z) := \sum_{k=0}^{n} \binom{n}{k}^2 z^k.$$

For $p = 0$, equality in (1.14) is achievable, e.g., taking $P(z) = Q(z) = (1+z)^n$.

(b) $\|\Theta_n\|_{H^0} \leq 4^n$, for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|\Theta_n\|_{H^0}^{1/n} = \exp\left(\frac{4G}{\pi}\right) \approx 3.20991230072\cdots, \tag{1.15}$$

where G is Catalan's constant and $G = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \approx 0.915965594177219\cdots$.

Moreover, there is an absolute constant $a > 0$ such that

$$\left| \ln \|\Theta_n\|_{H^0}^{1/n} - \frac{4G}{\pi} \right| \leq a \frac{\ln^2 n}{n}, \quad n \geq 2. \tag{1.16}$$

It is an interesting fact that the constant in (1.15) has already appeared in some sharp estimates unrelated to the Hadamard product. For example, P. B. Borwein considered factoring polynomials on $[-1, 1]$ into products of polynomials of smaller degrees, and got the same constant in the estimate of the product of uniform norms of two factors [11, Corollary 1]. Later, I. E. Pritsker [74] considered the problem of the best constant M_E in the inequality

$$\prod_{k=1}^{m} \|p_k\|_{C(E)} \leq (M_E)^n \|p\|_{C(E)},$$

where p_k's are some complex polynomials, $p(z) = \prod_{k=1}^{m} p_k(z)$, and $n = \deg(p)$. It was shown in [74, § 3.2] (see also [75]) that for $E = [-1, 1]$, M_E is exactly the constant we obtained in (1.15).

1.4 Multipliers of Fourier Integrals

Multipliers of Fourier integrals have the same motivation as multipliers of Fourier series. Now, the multiplicative factor is some Lebesgue measurable function. For a function f with Fourier transform $\widehat{f}$, we can consider the linear operator defined in the following way

$$F_\varphi [f] (x) = \int_{\mathbb{R}^n} \varphi (t) \widehat{f} (t) e^{2\pi i(x,t)} dt,$$

where (x, t) denotes the usual inner product of two vectors in $\mathbb{R}^n$.

Owing to the K. de Leeuw theorem [56], the case of multipliers for Fourier integrals in $L^p (\mathbb{R}^n)$, $p \in [1, \infty]$, may be reduced to the case of multipliers of Fourier series in $L^p (\mathbb{T}^n)$. A detailed explanation of this fact and related results could be found in [86, Ch. VII, § 3].

For $p \in (0, 1)$, the situation is quite different. We need to investigate the multipliers for Fourier integrals separately. Moreover, as in the case of series' multipliers, one needs to study the Hardy spaces H^p instead of L^p. For the univariate case, it is H^p in the upper half-plane. Several useful sufficient conditions for such multipliers were obtained by A. A. Soljanik in [84]. They were also successfully applied to obtaining several two-sided estimates of approximation of a function by some means of its Fourier integrals.

Several efficient conditions for multipliers of Fourier series in H^p spaces in polydisk $\mathbb{D}^m$, and their applications to various problems of approximation theory, were obtained by R. M. Trigub in [94]. Later, the results of [94] were extended to the case of H^p spaces in the Reinhardt domains by Vit. V. Volchkov [105]. We will also use some crucial ideas of the proofs of [94].

Let B be an open set in $\mathbb{R}^n$, $n \in \mathbb{N}$. Following [86, Chapter III], the *tube* with base B is

$$T_B = \{z \in \mathbb{C}^n, z = x + iy : \, x \in \mathbb{R}^n, y \in B\}.$$

Despite the fact that this definition is related to an open set B, we will also use the same notation for not necessarily open B when proving some technical results in Section 4.1. We will also use the notation E^o for the interior of a set E.

A nonempty open set $\Gamma \subset \mathbb{R}^n$ is called an *open cone* in $\mathbb{R}^n$ if $0 \notin \Gamma$ and whenever $x, y \in \Gamma$ and $\alpha, \beta > 0$, the linear combination $\alpha x + \beta y \in \Gamma$. The closure of an open cone is called a *closed cone*.

For any open cone Γ, the set

$$\Gamma^* = \{x \in \mathbb{R}^n : \, (x, t) \geq 0, \, \forall t \in \Gamma\}$$

is closed. If Γ^* has nonempty interior, then it is a closed cone, and Γ is called a *regular cone*. The closed cone Γ^* is called the cone *dual* to Γ.

It is obvious that in the univariate case, the only possible open cones are $(0, \infty)$ and $(-\infty, 0)$, which are also regular. Their dual cones are $[0, \infty)$ and $(-\infty, 0]$, respectively. For $n = 2$, open cones are sectors of angular measure at most π. If the angle is strictly less than π, then we have a regular cone.

A holomorphic in T_B function belongs to the *Hardy space* $H^p(T_B)$, $p \in (0, \infty]$, if

$$\|f\|_{H^p} := \|f\|_{H^p(T_B)} := \begin{cases} \sup_{y\in B} \left(\int_{\mathbb{R}^n} |f(x+iy)|^p \, dx\right)^{1/p}, & p \in (0,\infty), \\ \sup_{z \in T_B} |f(z)|, & p = \infty, \end{cases}$$

is finite. It is clear that the latter expression defines a norm in $H^p(T_B)$ for $p \in [1, \infty]$, and a pre-norm for $p \in (0, 1)$.

We will also use the following notation $f_y(\cdot) := f(\cdot + iy)$, $y \in B$. Using it, we have $\|g\|_{H^p(T_B)} = \sup_{y \in B} \|f_y\|_p$, where $\|\cdot\|_p$ is a standard norm (or pre-norm) in $L^p(\mathbb{R}^n)$.

Since the general case of an arbitrary open set B is too cumbersome and heavily dependent on the geometry of B even for $H^2(T_B)$ (see, e.g., [86, Ch. III, § 2]), it is reasonable to restrict the investigation to the case of an open cone Γ.

If $f \in L^1(\mathbb{R}^n)$, its Fourier transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(t)\, e^{-2\pi i(\xi,t)}\, dt, \quad \xi \in \mathbb{R}^n.$$

We will also use the following notation $\widetilde{f}(\xi) := \widehat{f}(-\xi)$.

For a function from $H^p(T_\Gamma)$, $p \in [1, \infty)$, its Fourier transform may be defined as an L^p Fourier transform of a limit function, the existence of which is guaranteed by Theorem 5.6 in [86, Ch. III, § 5]. For $p < 1$ and a general cone, it does not work, and we need to consider the limit function using tempered distributions. First of all, we need the following statement, which follows from the proof of [86, Ch. III, § 2, Lemma 2.12].

Lemma 1.4.1 ([95, Lemma 1]) *Let Γ be an open cone in $\mathbb{R}^n$, $p \in (0, \infty]$, and $q \in [p, \infty]$. If $f \in H^p(T_\Gamma)$, then for any $\delta \in \Gamma$, we have $f_\delta \in H^q(T_\Gamma)$ and*

$$\|f_\delta\|_{H^q} \le \left(\frac{\Omega_n}{\Omega_{2n}}\right)^{\frac{1}{p}-\frac{1}{q}} D_{\delta,\Gamma}^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{H^p},$$

where Ω_m is the volume of the unit ball in R^m, i.e., $\Omega_m = \pi^{m/2}/\Gamma(m/2+1)$, and $D_{\delta,\Gamma} = \operatorname{dist}(\delta, \mathbb{R}^n \setminus \Gamma)$.

The following result is in fact a modification of Lemma 4 from [29].

Theorem 1.4.1 ([95, Th. 1]) *Let Γ be an open cone in $\mathbb{R}^n$, $p \in (0, 1]$, and $f \in H^p(T_\Gamma)$. Then the limit $\lim_{t \to 0, t \in \Gamma} f(x + it)$ exists in the sense of tempered distributions, i.e., there exists a tempered distribution L, such that for any test function*

φ,

$$\lim_{t\to 0, t\in\Gamma} \int_{\mathbb{R}^n} f\left(x+it\right)\varphi(x)\,dx = L\left(\varphi\right).$$

Moreover, the Fourier transform of the tempered distribution L is a regular tempered distribution generated by an ordinary function given by the formula (the right-hand side does not depend on $\delta \in \Gamma$):

$$\widehat{f_0}\left(\xi\right) = e^{2\pi(\xi,\delta)}\widehat{f_\delta}\left(\xi\right), \quad \xi \in \mathbb{R}^n,$$

where $\widehat{f_\delta}$ is the classical Fourier transform of the function $f_\delta\left(x\right)$.

Lemma 1.4.1 implies $f_\delta \in L^1\left(\mathbb{R}^n\right)$, which means that $\widehat{f_0}$ is well-defined. Let us also note that for $p = 1$, our $\widehat{f_0}$ coincides with the classical Fourier transform of the limit function $f\left(x\right) = \lim_{\zeta\to 0,\, \zeta\in\Gamma} f_\zeta\left(x\right)$.

Therefore, the following definition of the Fourier transform is justified.

Definition 1.4.1 *The Fourier transform of a function $f \in H^p\left(T_\Gamma\right)$, $p \in (0,1]$, is defined by*

$$\widehat{f}\left(\xi\right) = e^{2\pi(\xi,\delta)}\widehat{f_\delta}\left(\xi\right), \quad \xi \in \mathbb{R}^n \quad (\delta \in \Gamma \; - \; \textit{arbitrary}). \tag{1.17}$$

Furthermore, if $f \in H^p\left(T_\Gamma\right)$ for some $p \in (0,1]$, then the following *inversion formula* holds true (see [95])

$$f\left(z\right) = \int_{\Gamma^*} \widehat{f}\left(t\right)e^{2\pi i(z,t)}\,dt, \quad z \in T_\Gamma. \tag{1.18}$$

Therefore, for any $p \in (0,1]$, the space $H^p\left(T_\Gamma\right)$ contains nonzero functions if and only if the cone Γ is regular (in fact, this is true for $p \in (0,\infty)$ since $f \in H^p$ implies $\left(f\right)^{[p]+1} \in H^s$ with $s = p/\left([p]+1\right) \in (0,1]$). So, we will investigate only the case of a regular cone.

Since there are no nontrivial translation-invariant linear bounded operators from $H^p\left(T_\Gamma\right)$ to $H^q\left(T_\Gamma\right)$, $p > q$ (see [96, Theorem 2]), we also assume $p \leq q$.

Definition 1.4.2 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. A Lebesgue measurable function $\varphi : \Gamma^* \to \mathbb{C}$ is called a multiplier from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$ (notation: $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$), $0 < p \le q \le 1$, if for any function $f \in H^p(T_\Gamma)$, the function $\varphi \widehat{f}$ coincides almost everywhere on Γ^* with the Fourier transform of a function $F_\varphi[f] \in H^q(T_\Gamma)$, and*

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} := \sup_{\|f\|_{H^p} \neq 0} \frac{\|F_\varphi[f]\|_{H^q}}{\|f\|_{H^p}} < \infty.$$

It follows immediately from (1.18) that the function $F_\varphi[f]$ is defined uniquely as

$$F_\varphi[f](z) = \int_{\Gamma^*} \varphi(t) \widehat{f}(t) e^{2\pi i (z,t)} \, dt, \quad z \in T_\Gamma.$$

Our first theorem in this section deals with the case of a compactly supported multiplier only. In fact, the most popular kernels are radial and compactly supported. Our theorem is sharp in this case (see Theorems 4.2.1 and 4.2.2 in Chapter 4).

Theorem 1.4.2 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. Assume that a function $\varphi \in C(\mathbb{R}^n)$ satisfies $\operatorname{supp} \varphi \subset [-\sigma, \sigma]^n$ for some $\sigma > 0$. If $\widehat{\varphi} \in L^q(\mathbb{R}^n)$ for some $q \in (0, 1]$, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$ for any $p \in (0, q]$, and*

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \le \frac{\gamma(n,p,q)}{(V_n(\Gamma))^{1/p-1}} \sigma^{n(1/p-1)} \|\widehat{\varphi}\|_q, \tag{1.19}$$

where

$$\gamma(n,p,q) = 2^{n\left(\frac{2}{p}+\frac{1}{q}-2\right)} \left(\frac{\pi^{\frac{n}{2}} n^{n\left(\frac{1}{2}+\frac{1}{q}\right)}}{\Gamma\left(\frac{n}{2}+1\right)} \right)^{\frac{1}{p}-1} \left(\sum_{m=0}^{\infty} \frac{1}{(m!)^q} \right)^{\frac{1}{q}}.$$

Here and in the sequel, by γ, we denote some positive constants depending only on the parameters in parentheses. The following geometric characteristics of the cone Γ is also used throughout the thesis:

$$V_n(\Gamma) = \frac{1}{n!} \max \left\{ \left| \det \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} \right| : a_1, \dots, a_n \in \overline{\Gamma}, \ |a_1| = \dots = |a_n| = 1 \right\} \tag{1.20}$$

(here a_{kl} denotes the l-th component of the vector a_k). Geometrically, $V_n(\Gamma)$ is the maximum possible volume of a simplex that could be built on n unit vectors contained in $\overline{\Gamma}$.

It is worth noting that the requirement $\widehat{\varphi} \in L^q(\mathbb{R}^n)$ is essential and in some cases is also necessary (see Theorems 4.2.1 and 4.2.2 in Chapter 4).

It is also possible to avoid the restriction that φ has to be compactly supported. We can require some smoothness instead. Using the method from [94], we can decompose our function into a sum of compactly supported functions whose Fourier transforms are in $L^q(\mathbb{R}^n)$. Owing to the Local Property of a multiplier (Lemma 4.2.1), this approach seems very natural. The result is given by the following statement

Theorem 1.4.3 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $q \in (0,1]$.*

(a) If $\varphi \in C^r(\mathbb{R}^n)$ for some natural $r > n\left(\frac{1}{q} - \frac{1}{2}\right)$, and for some $p \in (0,q]$, $\alpha, \beta \geq 0$, the inequalities

$$|\varphi(x)| \leq \frac{A}{1+|x|^\alpha}; \quad \sum_{j=1}^{n} \left| \frac{\partial^r \varphi}{\partial x_j^r}(x) \right| \leq \frac{B}{1+|x|^\beta}, \quad \forall x \in \mathbb{R}^n,$$

$$\min\{\beta - \alpha - r, 0\} + \frac{2qr\alpha}{n(2-q)} - \frac{2rq}{2-q}\left(\frac{1}{p} - \frac{1}{q}\right) > 0,$$

hold true, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, and

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \leq \frac{\gamma(n,p,q,r,\alpha,\beta)}{(V_n(\Gamma))^{1/p-1}}(A+B).$$

In particular, if $\alpha = \beta > n\left(\frac{1}{p} - \frac{1}{2}\right)$, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$.

(b) Suppose that $\varphi \in C^s(\mathbb{R}^n)$ for $s = \left[\frac{n}{q} - \frac{n+1}{2}\right]$, and $\operatorname{supp}\varphi \subset [-1,1]^n$. If

$$\max_{j=1,\ldots,n} \sup_{t_j \neq 0} \sup_{x \in \mathbb{R}^n} \frac{\left| \frac{\partial^s \varphi}{\partial x_j^s}(x_1,\ldots,x_n) - \frac{\partial^s \varphi}{\partial x_j^s}(x_1,\ldots,x_{j-1},x_j+t_j,x_{j+1},\ldots,x_n) \right|}{|t_j|^\alpha} < \infty,$$

for some $\alpha > \frac{n}{q} - \frac{n+1}{2} - s$, and for any $j = 1,\ldots,n$, the segment $[-1,1]$ could be split into finite number of segments (bounded with regard to the rest of variables) such that, on any of these segments, the real and imaginary parts of $\frac{\partial^s \varphi}{\partial x_j^s}$ (as functions of x_j) are convex or concave, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$ for any $p \in (0,q]$.

Recent work of Y. Heo, F. Nazarov and A. Seeger [42, 43] is devoted to Fourier multipliers in $L^p(\mathbb{R}^n)$, $p \geq 1$, and Lorentz spaces. The main results of their articles are efficient estimates for the norms of Fourier multipliers from L^p to L^p and to Lorentz spaces $L^{p,\nu}$. The authors deal with general radial kernels. One of the most popular applications of these results is the Bochner-Riesz means.

Employing the above results, we answer the question: When the Bochner-Riesz means of the Fourier integral

$$R_h^{r,\alpha}(f;z) = \int_{|x|\leq 1/h} \widehat{f}(x)\left(1 - h^{2r}|x|^{2r}\right)^{\alpha} e^{2\pi i(z,x)}\,dx, \quad z \in T_\Gamma,$$

define a bounded linear operator from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$?

Let us note that in L^p, with $1 \leq p \leq \infty$, the Bochner-Riesz means are investigated well (see, e.g., [20], [57], [21, Ch. 5], or [86, Ch. IV, § 4; Ch. VII, § 5]). For approximation of functions in H^p spaces, $0 < p \leq 1$, by their Bochner-Riesz means see, e.g., [84, § 3], [94, § 2], [95, § 4]. In our case, the following statement holds true.

Proposition 1.4.1 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. Assume $\alpha > 0$, $r \in \mathbb{N}$, and $0 < p \leq q \leq 1$. The function*

$$\varphi_{r,\alpha}(x) := \begin{cases} \left(1 - |x|^{2r}\right)^{\alpha}, & |x| \leq 1, \\ 0, & |x| > 1, \end{cases}$$

belongs to $\mathcal{M}_{p,q}(T_\Gamma)$ if and only if

$$\alpha > \frac{n}{q} - \frac{n+1}{2}.$$

It may seem surprising, that the critical index does not depend on p. However, this is easily justified by Theorem 4.2.2.

It is interesting to find the critical index for Bochner-Riesz means for the case of fractional powers r. Unfortunately, the proof of Proposition 1.4.1 does not work since in that case, $\varphi_{r,\alpha}$ looses its smoothness at the origin.

To show that the obtained sufficient conditions are relatively sharp, we provide careful investigation of the local behavior of multipliers. Special attention is paid to compactly supported radial functions. One of the tools here is non-increasing rearrangements. In particular, we prove Lemma 4.1.7 that generalizes the following well-known equality $\int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_0^\infty (f^*(t))^p \, dt$, where f^* denotes the non-increasing rearrangement of f. Another auxiliary result – Nikol'skiĭ type inequality given by Proposition 4.1.1 – is of an independent interest.

Moreover, in Chapter 4, we obtain Bernstein and Nikol'skiĭ type inequalities for entire functions of exponential type belonging to a Hardy space $H^p(T_\Gamma)$ (see Theorems 4.3.2 and 4.3.3).

1.5 Riesz Decomposition for m-Superharmonic Functions in $\mathbb{R}^n$

We complete the thesis with a result on Riesz Decomposition for super-polyharmonic functions. Despite the fact that some books on Potential Theory do not emphasize this, such type of problems have strong relation to Harmonic Analysis. Let us start with the classical Laplacian $\Delta f = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} f$. If the function f is sufficiently smooth, then the Fourier transform of Δf is $\widehat{\Delta f}(y) = -4\pi^2 |y|^2 \widehat{f}(x)$. Furthermore, for $m \in \mathbb{N}$,

$$\widehat{\Delta^m f}(y) = \left(-4\pi^2\right)^m |y|^{2m} \widehat{f}(y). \tag{1.21}$$

It is also known (see, e.g., [55, Ch. 1, § 1, Formula (1.1.1)]) that for $n \geq 2$ and $\alpha < n/2$,

$$\widehat{|x|^{\alpha-n}}(y) = \frac{\pi^{\frac{n}{2}-\alpha}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} |y|^{-\alpha}. \tag{1.22}$$

Relations (1.21) and (1.22) suggest two important ideas. The first one is to replace $2m$ (or m) in (1.21) by a fractional α. To make this idea suitable for a wide range of functions f, it is also conceivable to consider (1.21) in distributional sense. This leads us to the notion of distributional Laplacian $\int f(x) (-\Delta)^m \varphi(x) \, dx$ (see below).

The second idea is that if $\widehat{f}$ could be written in the form

$$\widehat{f}(y) = c_{m,n} \left((-\Delta)^m f \right)\widehat{\ }(y) \, \widehat{|x|^{2m-n}}(y) \, ,$$

where

$$c_{m,n} := \frac{\Gamma\left(\frac{n}{2} - m\right)}{4^m \pi^{\frac{n}{2}} \Gamma(m)},$$

then f must be the convolution of $c_{m,n} |x|^{2m-n}$ and $(-\Delta)^m f$ (or a measure related to $(-\Delta)^m f$). This idea (even with fractional α instead of $2m$) is developed in [55, Ch. 1]. In fact, this motivation is a good starting point for considering the Riesz decomposition from the point of view of Harmonic Analysis.

In this section, we will assume $2m < n$, and hence[1]

$$c_{m,n} = \left(2^{m-1}(m-1)! \sigma_n \prod_{j=0}^{m-1} (n - 2m + 2j) \right)^{-1} , \tag{1.23}$$

where σ_n is the surface measure of the unit sphere in $\mathbb{R}^n$, i.e.,

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

There are several equivalent definitions of a superharmonic function on an open subset $\Omega \subset \mathbb{R}^n$ (see, e.g., [40, Ch. 2], [4, Ch. 3], [46, Ch. III]). Let us mention the most popular two.

Definition 1.5.1 ([4, § 3.1]) *A function $s : \Omega \to [-\infty, \infty)$ is called subharmonic on Ω if:*

(i) s is upper semicontinuous on Ω,

(ii) $s(x) \le M(s; x, r)$ whenever the closure of a ball $B(x, r)$ centered at x and of radius r is contained in Ω; here, $M(s; x, r)$ denote the normalized spherical means of s over the spheres $S(x, r)$ of radius r centered at x:

$$M(s; x, r) = \frac{1}{\sigma_n r^{n-1}} \int_{S(x,r)} f(y) \, d\sigma(y),$$

[1] In fact, equality (1.22) is valid for $\alpha = 2m < \frac{n+1}{2}$. Nevertheless, we will use formula (1.23) as the definition of $c_{m,n}$ for $2m < n$.

where $d\sigma$ is the surface measure in $\mathbb{R}^n$;

(iii) $s \not\equiv -\infty$ on each component of Ω.

(Note that the last condition is sometimes omitted.)

Definition 1.5.2 ([46, § 3.2]) *A function $s : \Omega \to [-\infty, \infty)$ is called subharmonic on Ω if:*

(i) s is upper semicontinuous on Ω,

(ii) for every compact subset K of Ω and every every continuous function h on K which is harmonic in the interior of K, if the inequality $s \leq h$ holds on ∂K, it holds in K.

A function $u : \Omega \to (-\infty, +\infty]$ is called *superharmonic* on Ω if $-u$ is subharmonic on Ω. The set of functions subharmonic on Ω is denoted by $\mathcal{S}(\Omega)$, and the class of superharmonic functions by $\mathcal{SH}(\Omega)$.

Let us note that if $s \in C^2(\Omega)$, then it is subharmonic if and only if its Laplacian Δs is non-negative in Ω. Moreover, for an arbitrary $s \in \mathcal{S}(\Omega)$, and an open subset ω such that $\overline{\omega} \subset \Omega$, there exists a decreasing sequence of functions $s_k \in \mathcal{S}(\omega) \cap C^\infty(\omega)$ convergent to s pointwise on ω (see, e.g., [4, Th. 3.3.3]). In fact, the sequence s_k is constructed explicitly as a convolution of s and some fixed smooth function. This result and Green's formula suggest to consider Laplacian in the distributional sense to give an equivalent definition of a subharmonic function.

For an open set $\Omega \subset \mathbb{R}^n$, we use $C_0(\Omega)$ to denote the vector space (over $\mathbb{R}$) of all real-valued functions continuous on Ω and having compact support in Ω. Furthermore, $C_0^\infty(\Omega) := C_0(\Omega) \cap C^\infty(\Omega)$. [2]

[2]Some textbooks, e.g., [31] use another notation, namely, $C_c(\Omega)$ and $C_c^\infty(\Omega)$, respectively. The ones with index 0 are used to denote corresponding spaces of functions vanishing at infinity, not necessarily compactly supported. However books on Potential Theory use $C_0(\Omega)$ and $C_0^\infty(\Omega)$, as we do.

Definition 1.5.3 ([4, § 4.3]) *Let Ω be an open subset of $\mathbb{R}^n$. If $u : \Omega \to [-\infty, +\infty]$ is locally integrable on Ω, then the linear functional*

$$L_u(\varphi) := \int_\Omega u(x)\Delta\varphi(x)\,dx, \quad \varphi \in C_0^\infty(\Omega), \tag{1.24}$$

is called the distributional Laplacian of u.

Using Green's formula, it is easy to conclude (see, e.g., [4, § 4.3]) that if $u \in C^2(\Omega)$, then $L_u(\varphi) = \int_\Omega \varphi(x)\Delta u(x)\,dx$. Furthermore, if $s \in \mathcal{S}(\Omega)$, then L_s is a positive linear functional on $C_0^\infty(\Omega)$, and there is a unique measure μ_s on Ω, such that

$$a_n^{-1} L_s(\varphi) = \int_\omega \varphi(x)\,d\mu_s(x), \quad \varphi \in C_0^\infty(\Omega),$$

where $a_n = \sigma_n \max\{1, n-2\}$. The measure μ_s is called *the Riesz measure associated with s*. For a superharmonic function u, the Riesz measure is defined to be the one associated with the subharmonic function $-u$. In both cases, Riesz measure is a non-negative measure. This measure characterizes a subharmonic (or superharmonic) function. Namely, if $u, v \in \mathcal{S}(\Omega)$, (or $\mathcal{SH}(\Omega)$) are such that $L_u = L_v$ on $C_0^\infty(\Omega)$, then $u - v$ is harmonic in Ω (see, e.g., [40, Ch. 3, Lemma 3.7]).

The Riesz Decomposition Theorem in various forms and for various underlying sets could be found in any book on Potential Theory (see, e.g., [4, Th. 4.4.1]). We cite it from [40, Ch. 3, Th. 3.9] (see Theorem 1.5.1 below). The classical definition of the *potential* of a finite and compactly supported Borel measure μ in $\mathbb{R}^n$, $n \geq 2$, is given by (see, e.g., [4, Ch. 4, § 4.2])

$$U^\mu(x) = \int_{\mathbb{R}^n} K(x-y)\,d\mu(y),$$

where

$$K(x) = \begin{cases} -\log|x|, & n = 2, \\ |x|^{2-n}, & n \geq 3. \end{cases}$$

We will also consider potentials with slightly different kernels, and the measure μ does not have to be finite or compactly supported.

Theorem 1.5.1 (Riesz Decomposition Theorem, 'Local Version') *Let u be superharmonic in a domain $D \subset \mathbb{R}^n$, $n \geq 2$, and $u \not\equiv \infty$. Then there exists a unique Borel measure μ in D, such that for any compact subset $E \subset D$,*

$$u(x) = \int_E K(x-y)\, d\mu(y) + h_E(x),$$

where h_E is harmonic in the interior of E.

There are several versions of the Riesz Decomposition Theorem for functions superharmonic in a ball, half-space, etc. (see, e.g., [4, Ch. 4, § 4.4]). However, we are interested in generalizations of the following global type of result (see, e.g., [55, Ch. I, § 5, Ths. 1.24 and 1.25]).

Theorem 1.5.2 (Riesz Decomposition Theorem, 'Global Version') *Suppose u is superharmonic in $\mathbb{R}^n$, $n \geq 3$. Then, there is a harmonic function h in $\mathbb{R}^n$ such that*

$$u(x) = c_{1,n} \int_{\mathbb{R}^n} K_2\,(x-y)\; d\mu_u(y) + h(x),$$

if and only if

$$\lim_{r\to\infty} M\,(r,u) > -\infty.$$

Here and in what follows we use the following notations.

For a measurable function g, the *spherical mean* over the sphere $S(0,r)$ of radius $r > 0$ centered at the origin is defined by

$$M(r,g) = \frac{1}{\sigma_n r^{n-1}} \int_{S(0,r)} g(x)\, d\sigma(x),$$

The *Riesz Kernels* are given by:

$$K_\alpha(x) := |x|^{\alpha-n}, \quad \alpha > 0.$$

As a corollary of Theorem 1.5.2, one can obtain (see [4, Cor. 4.4.2]) that if u is superharmonic in $\mathbb{R}^n$, $n \geq 3$, $u \geq 0$, and $u \not\equiv \infty$, then

$$u(x) = c_{1,n} \int_{\mathbb{R}^n} K_2\,(x-y)\; d\mu_u(y) + c, \quad x \in \mathbb{R}^n,$$

where c is a non-negative constant.

We are interested in a generalization of the Riesz Decomposition Theorem for m-superharmonic functions (see Definition 1.5.5 below). Recently, for $m = 2$ (superbiharmonic functions) the generalization we are looking for was obtained by K. Kitaura and Y. Mizuta [52]. Let us introduce precise definitions first.

Definition 1.5.4 *Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. A function $u : \Omega \to \mathbb{R}$ is called $m-$harmonic ($m \in \mathbb{N}$), or polyharmonic of order m, in Ω if $u \in C^{2m}(\Omega)$, and $\Delta^m u \equiv 0$ in Ω. The set of all functions m-harmonic in Ω is denoted by $\mathcal{H}^m(\Omega)$.*

Polyharmonic functions have many interesting properties. The monograph [5] is an excellent source of information about them.

Definition 1.5.5 *Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. A function $u : \Omega \to (-\infty, \infty]$ is called m-superharmonic if*

(i) u is locally integrable on Ω (with respect to the Lebesgue measure in $\mathbb{R}^n$);

(ii) u is lower semicontinuous in Ω;

(iii) $\mu_u := (-\Delta)^m u$ is a positive Radon measure in Ω in the sense of distributions, i.e.,

$$\int_\Omega \varphi(x)\, d\mu_u(x) = \int_\Omega u(x)\, (-\Delta)^m \varphi(x)\, dx \geq 0, \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0;$$

(iv) For every point $x \in \Omega$,

$$u(x) = \lim_{r \to 0+} \frac{1}{m(B(x,r))} \int_{B(x,r)} u(t)\, dt,$$

where $B(x,r)$ denotes the open ball centered at x and of radius r, and m denotes its Lebesgue measure, i.e., $m(B(x,r)) = r^n \pi^{n/2} / \Gamma(n/2 + 1)$.[3]

The class of all m-superharmonic functions in Ω is denoted by $\mathcal{SH}^m(\Omega)$. If $m = 2$, we have the class of super-biharmonic functions.

[3] Note that this condition is weaker than the requirement on x to be the Lebesgue point of u.

The generalization of Theorem 1.5.2 for super-biharmonic functions in $\mathbb{R}^n$ is given by the following result.

Theorem 1.5.3 (K. Kitaura, Y. Mizuta [52, Theorem 1.2]) *Let $n \geq 5$, $u \in \mathcal{SH}^2(\mathbb{R}^n)$, and $\mu_u = \Delta^2 u$. Then $M(2r,u) - 4M(r,u)$ is bounded when $r > 1$ if and only if u is of the form*

$$u(x) = c_{2,n} \int_{\mathbb{R}^n} K_4(x-y)\, d\mu_u(y) + h(x),$$

where $h \in \mathcal{H}^2(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} (1+|y|)^{4-n}\, d\mu_u(y) < \infty.$$

Moreover, in [52], the authors consider the case of lower dimensions too. However, they use some modification of the Riesz kernels in the latter case.

The main point is that the possibility for a superbiharmonic function to possess a Riesz decomposition is given in terms of boundedness of a linear combination of spherical means. For the m-superharmonic case, the appropriate linear combination of spherical means is more complicated. It is defined in Proposition 1.5.1 below.

Let us mention another generalization of Theorem 1.5.2 obtained by N. S. Landkof [55, Chap. 1, § 6].

Definition 1.5.6 *A function $u : \mathbb{R}^n \to [0,\infty]$ is called α-superharmonic in $\mathbb{R}^n$ (here $0 < \alpha < 2$) if*

(i) $u \not\equiv \infty$;

(ii) u is lower semicontinuous in $\mathbb{R}^n$;

(iii) u satisfies the condition

$$\int_{|x|>1} \frac{|u(x)|}{|x|^{n+\alpha}}\, dx < \infty;$$

(iv) For any $x \in \mathbb{R}^n$,

$$\varepsilon_\alpha^{(r)} * u(x) = \varepsilon_{\alpha x}^{(r)}(u) \leq u(x), \quad r > 0,$$

where

$$\varepsilon_{\alpha}^{(r)}(x) := \begin{cases} 0, & |x| < r, \\ \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}+1}} \sin\left(\frac{\pi\alpha}{2}\right) r^{\alpha} \left(|x|^2 - r^2\right)^{-\frac{\alpha}{2}} |x|^{-n}, & |x| > r, \end{cases}$$

and

$$\varepsilon_{\alpha}^{(r)}(u) := \int_{\mathbb{R}^n} u(x)\varepsilon_{\alpha}^{(r)}(x)\, dx.$$

It is interesting (see [55, Ch. I, § 6, Formula (1.6.1)]) that

$$\int_{\mathbb{R}^n} K_{\alpha}(x-y)\, \varepsilon_{\alpha}^{(r)}(x)\, dx \le K_{\alpha}(x), \quad |x| < 1,$$

and

$$\int_{\mathbb{R}^n} K_{\alpha}(x-y)\, \varepsilon_{\alpha}^{(r)}(x)\, dx = K_{\alpha}(x), \quad |x| \ge 1.$$

Let also

$$A(n,\alpha) := \pi^{\alpha-\frac{n}{2}} \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}.$$

The following result gives the Riesz decomposition for α-superharmonic functions.

Theorem 1.5.4 ([55, Ch. 1, § 6, Th. 1.30]) *Assume that u is α-superharmonic in $\mathbb{R}^n$, $n \ge 3$. Then $u(x)$ has a unique decomposition in the form*

$$u(x) = U_{\alpha}^{\mu}(x) + A = A(n,\alpha) \int_{\mathbb{R}^n} K_{\alpha}(x-y)\; d\mu(y) + A,$$

where μ is a positive Borel measure in $\mathbb{R}^n$, which is finite on every compact subset $K \subset \mathbb{R}^n$, and the constant $A \ge 0$. Furthermore, if u is α-harmonic in some open subset $\Omega \subset \mathbb{R}^n$, then $\mu(\Omega) = 0$.

(f is called α-harmonic at the point x_0 if it is continuous in a neighborhood of x_0, satisfies condition (iii) of Definition 1.5.6, and for sufficiently small r

$$f(x_0) = \varepsilon_{\alpha}^{(r)} * f(x_0) = \varepsilon_{\alpha x_0}^{(r)}(f).$$

If f is α-harmonic at each point of Ω, it is called α-superharmonic in Ω.)

Unfortunately, powerful tools developed in [55, Ch. 1] to prove Theorem 1.5.4 seem to be applicable only for $0 < \alpha < 2$. We will use another approach (closer to the work of K. Kitaura and Y. Mizuta [52]) to get the result for $\alpha = 2m < n$, $m \in \mathbb{N}$. We start with the proposition that helps to find a linear combination of spherical means whose boundedness is necessary for the Riesz decomposition of u.

Proposition 1.5.1 *Let $m \in \mathbb{N}$, $m \geq 2$, and let $\alpha_{m,1} = 1$. Then there are unique real constants $\alpha_{m,2}, \ldots, \alpha_{m,m}$ such that for every polynomial of the form*

$$F_m(r) := \sum_{k=0}^{m-1} a_k r^{2k},$$

we have

$$\sum_{j=1}^{m} \alpha_{m,j} F_m\left(2^{m-j} r\right) = a_0 \sum_{j=1}^{m} \alpha_{m,j}, \quad r \in \mathbb{R}. \tag{1.25}$$

The constants are given by

$$\alpha_{m,k+1} = (-1)^{k+\frac{m-1}{2}(m-2)} 4^{\frac{m}{2}(m-1)-(m-k-1)} \frac{\prod\limits_{1\leq l<j\leq m-1} (\theta_{m,j,k} - \theta_{m,l,k})}{\prod\limits_{1\leq l<j\leq m-1} (4^j - 4^l)}, \tag{1.26}$$

where

$$\theta_{m,j,k} = \begin{cases} 4^{m-j}, & 1 \leq j \leq k, \\ 4^{m-1-j}, & k+1 \leq j \leq m-1, \end{cases} \qquad 1 \leq k \leq m-1.$$

To formulate the main result of this section (and Chapter 5), we need to introduce the class $\mathcal{R}$ of functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying:

(i) $\varphi(x) \equiv 1$ in $\overline{B(0,1)}$ (as usual, $B(0,r)$ denotes the ball in $\mathbb{R}^n$ of radius r centered at the origin);

(ii) $\operatorname{supp}\varphi \subset \overline{B(0,2)}$;

(iii) $0 \leq \varphi(x) \leq 1$, $x \in \mathbb{R}^n$.

Such functions are often used for regularization purposes.

Our main result is given by the following theorem.

Theorem 1.5.5 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, $\mu_u = (-\Delta)^m u$, and $\varphi \in \mathcal{R}$ is chosen arbitrarily. Furthermore, let $\alpha_{m,j}$ be the absolute constants from Proposi-*

tion 1.5.1. Then

$$\sup_{r>1}\left|\sum_{j=1}^{m}\alpha_{m,j}M\left(2^{m-j}r,u\right)\right|<\infty \quad \textit{and} \quad \sup_{r>1}\int_{1\leq|t|\leq 2}u(rt)\,(-\Delta)^m\,\varphi(t)\,dt<\infty \tag{1.27}$$

if and only if

$$\int_{\mathbb{R}^n}(1+|y|)^{2m-n}\;d\mu_u(y)<\infty, \tag{1.28}$$

and u is of the form

$$u(x)=c_{m,n}\int_{\mathbb{R}^n}K_{2m}(x-y)\,d\mu_u(y)+h(x), \tag{1.29}$$

where $h\in\mathcal{H}^m\left(\mathbb{R}^n\right)$.

Note that (1.28) is the condition for existence of the potential in (1.29). Furthermore, the normalizing coefficients $c_{m,n}$ are chosen so that $c_{m,n}\left(-\Delta\right)^m K_{2m}$ is the delta-function δ_0 (see [41] and [32, § 3]).

Comparing Theorems 1.5.3 and 1.5.5, one can observe that the first condition in (1.27) for $m=2$ is exactly the condition on the boundsdness of $M\left(2r,u\right)-4M\left(r,u\right)$ used in Theorem 1.5.3. The second one is an extra condition. However, for $m=2$, the second condition in (1.27) follows from the first one. This seems to be false for $m\geq 3$.

Moreover, for the case $2m\geq n$, one needs to consider different kernels. For example, K. Kitaura and Y. Mizuta [52] considered special kernels wich are products of the Riesz kernels and $\ln\frac{1}{|x|}$. It was shown that if $u\in\mathcal{SH}^2\left(\mathbb{R}^n\right)$ and $n\leq 4$, then the linear combination of spherical means $M\left(2r,u\right)-4M\left(r,u\right)$ is bounded on $r>1$ if and only if $u\in\mathcal{H}^2\left(\mathbb{R}^n\right)$. The authors investigate the case for each n between 2 and 4 separately. The Riesz decomposition for superharmonic functions in $\mathbb{R}^n$ $(m=1)$ is also proven in [52].

The following corollary gives an easy to use sufficient condition for an m-superharmonic function to have the representation (1.29).

Corollary 1.5.1 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, $\mu_u = (-\Delta)^m u$. If*

$$\sup_{r>1} \left| \sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) \right| < \infty,$$

and one of the conditions

$$(a) \quad \sup_{r>1} \frac{1}{r^n} \int\limits_{r \le |x| \le 2r} |u(x)|^p \, dt < \infty, \quad \textit{for some } p \in [1, \infty);$$

$$(b) \quad \frac{u(x)}{|x|^{n/p}} \in L^p\left(\mathbb{R}^n \setminus B(0,1)\right), \qquad \textit{for some } p \in [1, \infty],$$

is satisfied, then (1.28) and (1.29) hold.

Open Problem. It would be interesting to generalize Theorem 1.5.5 to the case of α-superharmonic functions in $\mathbb{R}^n$, $\alpha > 2$. We have already mentioned a formula for spherical means of Riesz kernels obtained in [16], which could be a good starting point. Although it is unclear what should be a condition replacing the boundedness of the linear combination of spherical means $\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right)$ in the case of a fractional power of Laplacian $\alpha/2$ instead of m.

CHAPTER 2

Estimates for the Hadamard Product on Hardy and Bergman Spaces

In Section 1.2, we gave the necessary definitions and stated the norm boundedness problem for the Hadamard product operator. Let us outline some references related to coefficient multipliers and to the Hadamard product.

P. L. Duren and A. L. Shields [26] obtained several conditions for multipliers of H^p $(0 < p < 1)$ into l^q $(p \leq q \leq \infty)$, and into H^q $(1 \leq q \leq \infty)$. They also discovered that these multipliers, in the majority of cases, are the same as multipliers of larger spaces B^p into l^q and H^q, respectively. Their conditions are often given in terms of asymptotics of the integral means.

In [18], J. Caveny discovered interesting relations between inclusions of functions in some Hardy spaces and boundedness of their Hadamard product.

Since there exist very convenient convolution representations for the Hadamard product (the first one was obtained by J. Hadamard in [39]), it is possible to estimate the Hadamard product operator norm in terms of integral norms of the functions involved. Moreover, these relations are useful for obtaining several beautiful integral representations and unexpected relations (see, e.g., [15, 73]).

Let us also note that the coefficient multipliers from H^p to H^q (including exponents below 1) were also investigated by P. L. Duren in [23]. In contrast to [26] cited above, the conditions are given in terms of estimates of the growth of the multiplier sequence $\{\lambda_n\}$. Other effective sufficient (and some necessary) conditions for multipliers of H^p in a polydisc with $p \in (0, 1]$, given in terms of growth of λ_n, were obtained by R. M. Trigub in [94]. More general questions of characterization of linear functionals in these spaces are considered by P. L. Duren, B. W. Romberg and A. L. Shields in [25].

Several efficient results about general coefficient multipliers in Bergman spaces A^p were obtained by D. Vukotić in [107]. The conditions are given in terms of asymptotics of the sequence $\{a_n\}$ defining the multiplier, convergence of some weighted series, as well as in terms of asymptotics of weighted partial sums like $\sum_{n=1}^{N} n^{2q/p-q}|a_n|^q$.

As we already mentioned in Section 1.2, we consider the Hadamard product operator acting in Hardy spaces $H^p(\mathbb{D})$ of analytic, or $h^p(\mathbb{D})$ of harmonic functions, as well as Bergman spaces $a^p(\mathbb{D})$, or $A^p(\mathbb{D})$. In particular, the case of an operator acting from H^p to H^q with arbitrary exponents p and q is studied. We do not require the exponents to be conjugate since the technique we use does not involve Hausdorff-Young inequalities.

2.1 Hadamard Product in Hardy Spaces

Lemma 2.1.1 *Let $(X, \mathfrak{G}, \mu)$ be a measure space with positive measure μ, and $f, g : X \to [0, \infty)$. If g is μ-measurable and $f \in L^1(X, \mu)$, then for any $p \in [1, \infty)$, the following inequality holds true*

$$\left(\int_X fg\,d\mu\right) \le \left(\int_X fg^p\,d\mu\right)^{1/p} \left(\int_X f\,d\mu\right)^{1-1/p}. \tag{2.1}$$

In particular, for any Lebesgue measurable set $E \subset \mathbb{R}$, if $f \in L^1(E)$, then for any function g, Lebesgue measurable on E,

$$\int_E |f(t)||g(t)|\,dt \le \left(\int_E |f(t)||g(t)|^p\,dt\right)^{1/p} \left(\int_E |f(t)|\,dt\right)^{1-1/p}, \quad p \in [1, \infty). \tag{2.2}$$

Proof. Assume that $fg \in L^1(X, \mu)$. Let us consider the following measure

$$\nu(\Omega) = \int_\Omega |f|\,d\mu = \int_\Omega f\,d\mu, \quad \Omega \in \mathfrak{G}.$$

Then $(X, \mathfrak{G}, \nu)$ is also a measure space, and $\int_X g\,d\nu = \int_X fg\,d\mu$.

If $\int_X f\,d\mu = 1$, Jensen's inequality applied with a convex function $\varphi(t) = |t|^p$ and the measure space $(X, \mathfrak{G}, \nu)$ implies

$$\left(\int_X fg\,d\mu\right)^p = \left|\int_X g\,d\nu\right|^p \leq \int_X |g|^p\,d\nu = \int_X fg^p\,d\mu,$$

since f and g are both non-negative.

If $\int_X f\,d\mu =: a \neq 1$, applying the last inequality to f/a, we get (2.1).

Now, let $\int_X fg\,d\mu = \infty$. Since g is μ-measurable, the set $A := \{x \in X : g(x) > 1\}$ is μ-measurable. Using the fact that $f, g \geq 0$, we obtain

$$\int_X fg\,d\mu \leq \int_{X\setminus A} f\,d\mu + \int_A fg^p\,d\mu \leq \int_X f\,d\mu + \int_X fg^p\,d\mu.$$

Since $f \in L^1(X, \mu)$, this implies $\int_X fg^p\,d\mu = \infty$. Thus, (2.1) holds. □

Proof of Theorem 1.2.1. It is shown in [82, Ch. 4, § 4.1.2, formula (4.8)] that if f and g are harmonic in $\mathbb{D}$, then

$$(f * g)\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{r}{R}e^{i(\theta-t)}\right) g\left(Re^{it}\right) dt, \quad 0 \leq r < R < 1,\ \theta \in \mathbb{R}. \tag{2.3}$$

Moreover, if g is harmonic in $\overline{\mathbb{D}}$, this formula is valid with $R = 1$. It is also clear that $f * g = g * f$.

(a) Employing (2.3), we have

$$I := \frac{1}{2\pi}\int_0^{2\pi} \left|(F * g)\left(re^{i\theta}\right)\right|^q d\theta =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{2\pi}\int_0^{2\pi} F\left(\frac{r}{R}e^{i(\theta-t)}\right) g\left(Re^{it}\right) dt\right|^q d\theta, \quad 0 \leq r < R < 1.$$

Applying (2.2) to $f(t) := F\left(\frac{r}{R}e^{i(\theta-t)}\right)$ and $g\left(Re^{it}\right)$ with $E = [0, 2\pi]$, and considering 2π-periodicity of f, we obtain

$$I \leq \frac{1}{(2\pi)^{q+1}}\left(\int_0^{2\pi} \left|F\left(\frac{r}{R}e^{it}\right)\right| dt\right)^{q-q/p} \times$$

$$\int_0^{2\pi}\left(\int_0^{2\pi} \left|F\left(\frac{r}{R}e^{i(\theta-t)}\right)\right| \left|g\left(Re^{it}\right)\right|^p\,dt\right)^{q/p} d\theta.$$

Minkowski's integral inequality with the power $q/p \geq 1$ applied to the second factor yields

$$I \leq \left(\frac{1}{2\pi}\int_0^{2\pi}\left|F\left(\frac{r}{R}e^{it}\right)\right|\,dt\right)^{q-q/p}\times$$
$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|F\left(\frac{r}{R}e^{i\theta}\right)\right|^{q/p}\,d\theta\right)\left(\frac{1}{2\pi}\int_0^{2\pi}\left|g\left(Re^{it}\right)\right|^p\,dt\right)^{q/p}.$$

Since $|\cdot|^{q/p}$ is convex, applying Jensen's inequality to the first integral, we obtain

$$I \leq \left(\frac{1}{2\pi}\int_0^{2\pi}\left|F\left(\frac{r}{R}e^{it}\right)\right|^{q/p}\,dt\right)^p\left(\frac{1}{2\pi}\int_0^{2\pi}\left|g\left(Re^{it}\right)\right|^p\,dt\right)^{q/p}.$$

Hence, for any $0 \leq r < R < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|(F*g)\left(re^{it}\right)\right|^q\,dt \leq \|F\|_{h^{q/p}}^q\left(\frac{1}{2\pi}\int_0^{2\pi}\left|g\left(Re^{it}\right)\right|^p\,dt\right)^{q/p}.$$

Since $F*g$ is continuous in $\overline{\mathbb{D}_R}$, passing to the limit as $r \to R-$, we get

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|(F*g)\left(Re^{it}\right)\right|^q\,dt\right)^{1/q} \leq \|F\|_{h^{q/p}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|g\left(Re^{it}\right)\right|^p\,dt\right)^{1/p},$$

for any $0 < R < 1$.

For $R = 0$, the estimate follows from (1.3) and subgarmonicity of $|g|^p$.

To get (1.8), we use (2.3) and Jensen's inequality.

For $p = q = \infty$, inequality (1.9) follows from (2.3) immediately.

Thus, $F* \in \mathcal{L}\left(h^p, h^q\right)$ for any $1 \leq p \leq q \leq \infty$, and

$$\|F*g\|_{h^q} \leq \|F\|_{h^{q/p}}\|g\|_{h^p}, \quad g \in h^p\left(\mathbb{D}\right).$$

(b) Let us take $g_0(z) \equiv 1$. Then, $\left(F*g_0\right)(z) = F(0)$, $z \in \mathbb{D}$. So,

$$\|F*g_0\|_{h^p} = |F(0)|, \quad p \in [1,\infty].$$

If for some $p \in [1,\infty]$, $F* \in \mathcal{L}\left(h^p, h^p\right)$, then the mean value property implies

$$\|F*\|_{h^p\to h^p} \geq |F(0)| = \left|\frac{1}{2\pi}\int_0^{2\pi}F\left(re^{it}\right)\,dt\right|, \quad r \in [0,1).$$

Taking $\sup_{r\in[0,1)}$ in this inequality, and considering that $F \geq 0$, we conclude by part (a) that $\|F*\|_{h^p\to h^p} = \|F\|_{h^1}$.

(c) For $R \in (0,1)$, let us take $g_R\left(re^{i\theta}\right) := P_{Rr}(\theta) = \sum_{n=-\infty}^{\infty} (Rr)^{|n|} e^{in\theta}$, the Poisson kernel. It is harmonic in $\mathbb{D}$, and $\|g_R\|_{h^1} = 1$. Moreover, $F * g_R(z) = F(Rz)$, $z \in \mathbb{D}$. Thus,

$$\sup_{\|g\|_{h^1}=1} \|F * g\|_{h^1} \geq \sup_{R\in(0,1)} \|F * g_R\|_{h^1} = \sup_{R\in(0,1)} \|F(R\cdot)\|_{h^1} = \|F\|_{h^1}.$$

Part (a) now implies $\|F *\|_{h^1 \to h^1} = \|F\|_{h^1}$.

For $p = \infty$, let us take $\mathcal{G}$ defined on the unit circle by

$$\mathcal{G}\left(e^{it}\right) = \begin{cases} \overline{\mathcal{F}\left(e^{-it}\right)} / \left|\mathcal{F}\left(e^{-it}\right)\right|, & \mathcal{F}\left(e^{-it}\right) \neq 0 \\ 1, & \mathcal{F}\left(e^{-it}\right) = 0, \end{cases} \qquad t \in \mathbb{R},$$

where $\mathcal{F}\left(e^{it}\right)$ is the function of boundary values of F (it exists and belongs to $L^1[0,2\pi]$; see, e.g., [24, Ch. 2, Th. 2.2, 2.6]). Therefore, $\mathcal{G} \in L^\infty(\mathbb{T})$. Hence (see, e.g., [44, Ch. 3, Corollary on p. 38]), its Poisson integral

$$G\left(re^{it}\right) := \frac{1}{2\pi} \int_0^{2\pi} \mathcal{G}\left(e^{i\theta}\right) P_r(t-\theta)\, d\theta, \quad r \in [0,1),\, \theta \in [0,2\pi),$$

is harmonic in $\mathbb{D}$, converges to $\mathcal{G}\left(e^{it}\right)$ as $r \to 1-$ for almost all t, and

$$\|G\|_{h^\infty} = \|\mathcal{G}\|_{L^\infty(\mathbb{T})} = 1 \equiv \left|\mathcal{G}\left(e^{it}\right)\right|. \tag{2.4}$$

Let us fix $r \in [0,1)$, and take a sequence $\{R_n\}_{n=1}^\infty$ such that $R_n \in (r,1)$ for any n, and $R_n \to 1$ as $n \to \infty$. Applying (2.3) with $\theta = 0$, we obtain

$$(F * G)(r) = \frac{1}{2\pi} \int_0^{2\pi} F\left(\frac{r}{R_n} e^{-it}\right) G\left(R_n e^{it}\right) dt, \quad n \in \mathbb{N}. \tag{2.5}$$

Now, the triangle inequality and (2.4) yield

$$\left| \int_0^{2\pi} F\left(\frac{r}{R_n} e^{-it}\right) G\left(R_n e^{it}\right) dt - \int_0^{2\pi} F\left(re^{-it}\right) \mathcal{G}\left(e^{it}\right) dt \right| \leq$$

$$\int_0^{2\pi} \left| F\left(\frac{r}{R_n} e^{-it}\right) - F\left(re^{-it}\right) \right| dt + \int_0^{2\pi} \left|F\left(re^{-it}\right)\right| \left|G\left(R_n e^{it}\right) - \mathcal{G}\left(e^{it}\right)\right| dt.$$

Clearly, $F\left(\frac{r}{R_n} e^{-it}\right)$ converges to $F\left(re^{-it}\right)$ uniformly on $t \in [0,2\pi]$ as $n \to \infty$, whence the first integral converges to 0. Relation (2.4) also yields

$$\left|F\left(re^{-it}\right)\right| \left|G\left(R_n e^{it}\right) - \mathcal{G}\left(e^{it}\right)\right| \leq 2\left|F\left(re^{-it}\right)\right| \in L^1[0,2\pi].$$

Since $G\left(R_n e^{it}\right) \to \mathcal{G}\left(e^{it}\right)$ a.e., the Lebesgue Dominated Convergence Theorem implies that the second integral also converges to 0 as $n \to \infty$.

Thus, passing to the limit in (2.5) as $n \to \infty$, we conclude

$$(F * G)(r) = \frac{1}{2\pi} \int_0^{2\pi} F\left(re^{-it}\right) \mathcal{G}\left(e^{it}\right)\, dt, \quad r \in [0,1). \tag{2.6}$$

Now, let us take a sequence $\{r_n\}_{n=1}^{\infty}$, such that $r_n \in [0,1)$ for any n, and $r_n \to 1$ as $n \to \infty$. Then, let us denote

$$f_n(t) := \frac{1}{2\pi} F\left(r_n e^{-it}\right) \mathcal{G}\left(e^{it}\right), \quad f(t) := \frac{1}{2\pi} \left|\mathcal{F}\left(e^{-it}\right)\right|.$$

Clearly, $f_n, f \in L^1[0,2\pi]$, and $f_n(t) \to f(t)$ a.e. on $[0,2\pi]$. Since $|\mathcal{G}\left(e^{it}\right)| \equiv 1$,

$$\begin{aligned}\lim_{n\to\infty} \int_0^{2\pi} |f_n(t)|\, dt &= \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \left|F\left(r_n e^{-it}\right)\right| dt = \|F\|_{h^1} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left|\mathcal{F}\left(e^{-it}\right)\right| dt = \int_0^{2\pi} |f(t)|\, dt.\end{aligned}$$

This implies that $\lim_{n\to\infty} \|f_n - f\|_{L^1[0,2\pi]} = 0$. Now, we may pass to the limit as $r \to 1-$ in (2.6) and obtain

$$\lim_{r\to 1-} (F * G)(r) = \frac{1}{2\pi} \int_0^{2\pi} \left|\mathcal{F}\left(e^{-it}\right)\right| dt = \|F\|_{h^1}.$$

Hence $\|F * G\|_{h^\infty} \geq \liminf_{r\to 1-} |(F * G)(r)| = \|F\|_{h^1} = \|F\|_{h^1}\|G\|_{h^\infty}$. Therefore, $\|F *\|_{h^\infty\to h^\infty} \geq \|F\|_{h^1}$, and part (a) implies $\|F *\|_{h^\infty\to h^\infty} = \|F\|_{h^1}$. □

Proof of Proposition 1.2.1. Let us consider the following sequence of functions:

$$\mathcal{F}_m(z) = \mathcal{F}_m\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} \left(\frac{m}{m+1}\right)^n r^n e^{in\theta} = \frac{1}{1 - \frac{mz}{m+1}}, \quad z \in \mathbb{D},\, m \in \mathbb{N}.$$

If g is holomorphic in $\mathbb{D}$ with the Taylor expansion $g\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} b_n r^n e^{in\theta}$, then, according to (1.3),

$$(\mathcal{F}_m * g)\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} \left(\frac{m}{m+1}\right)^n b_n r^n e^{in\theta} = g\left(\frac{m}{m+1} re^{i\theta}\right),\ r \in [0,1),\ \theta \in \mathbb{R}. \tag{2.7}$$

It is easy to see that (sharpness could be verified on $g(z) \equiv 1$)

$$\|\mathcal{F}_m *\|_{H^p\to H^p} = 1, \quad m \in \mathbb{N},\, p \in [1,\infty]. \tag{2.8}$$

If the sequence $\{\|\mathcal{F}_m\|_{H^1}\}_{m=1}^{\infty}$ were bounded, then since

$$\lim_{m\to\infty} \mathcal{F}_m(z) = \frac{1}{1-z}, \quad z \in \mathbb{D},$$

Fatou's lemma would imply

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{1-e^{i\theta}}\right| d\theta \leq \liminf_{m\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left|\mathcal{F}_m\left(e^{i\theta}\right)\right| d\theta < \infty.$$

But the integral on the left-hand side is divergent.

Therefore, for an arbitrary $M > 0$, there exists $m_M \in \mathbb{N}$ such that $\|\mathcal{F}_{m_M}\|_{H^1} > M$. At the same time, equalities (2.8) are valid for any $m \in \mathbb{N}$, and the statement follows with $\mathcal{F} = \mathcal{F}_{m_M}$. $\square$

Using the proof of [108, Ch. 8, § 8.1, Th. 8.1.5] or applying the M. Riesz Theorem on the norm of conjugate harmonic function (see, e.g., [36, Ch. III, § 2, Th. 2.3]) directly, we get that if $p \in (1,\infty)$, and A is a linear operator defined on $h^p(\mathbb{D})$ that vanishes on anti-analytic functions g with $g(0) = 0$ and the restriction of A to $H^p(\mathbb{D})$ belongs to $\mathcal{L}(H^p, H^p)$, then

$$\|A\|_{h^p\to h^p} \leq C(p)\,\|A\|_{H^p\to H^p}.$$

(Note that this statement is not true for $p = 1$ or $p = \infty$. For example, consider the operator $\mathcal{F}*$ with $\mathcal{F}(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.)

Thus, if we take the function $\mathcal{F}_{m_M}$ from Proposition 1.2.1, then $\mathcal{F}_{m_M}*$ satisfies the conditions of the last statement. Moreover, $\|\mathcal{F}_{m_M}*\|_{H^p\to H^p} = 1$. Multiplying by corresponding constant (depending of p), one can easily deduce

Corollary 2.1.1 *For any $M > 0$, and any $p \in (1,\infty)$, there exists an analytic in $\mathbb{D}$ function $\mathcal{G}$ such that*

$$\|\mathcal{G}*\|_{h^p\to h^p} = 1,$$

but $\|\mathcal{G}\|_{h^1} = \|\mathcal{G}\|_{H^1} > M$.

2.2 Hadamard Product from $H^p(\mathbb{D})$ to $H^q(\mathbb{D})$ with Arbitrary $p, q \in (0, \infty]$

As we already noticed in Section 1.3, the estimates similar to those obtained in Theorem 1.2.1 are impossible when $p < 1$. The results of the present section give some estimates for the latter case, however they are not sharp.

Let us remind that we use the following notation:

$$\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}, \quad R > 0.$$

Jensen's inequality could be easily applied to show that if $f \in H^p(\mathbb{D})$ for some $p \in (0, \infty]$, then $f \in H^q(\mathbb{D})$, for any $q \in [0, p)$, and

$$\|f\|_{H^q(\mathbb{D})} \leq \|f\|_{H^p(\mathbb{D})}.$$

We need an inverse inequality of some kind given by the following lemma.

Lemma 2.2.1 *If $f \in H^p(\mathbb{D}_2)$, for some $p \in (0, \infty)$, then $f \in H^\infty(\mathbb{D})$, and*

$$\|f\|_{H^\infty(\mathbb{D})} \leq 4^{1/p} \|f\|_{H^p(\mathbb{D}_2)}.$$

Proof. Take an arbitrary $z_0 \in \mathbb{D}$, $z_0 \neq 0$, and let $R := |z_0|$. Since f is holomorphic in $\mathbb{D}_2$, $|f(z)|^p$ is subharmonic there, for any $p \in (0, \infty)$. Using the submean property, for any $\rho \in (0, 2 - R)$, we obtain

$$|f(z_0)|^p \leq \frac{1}{\pi\rho^2} \int_{D(z_0,\rho)} |f(z)|^p \, dA \leq \frac{1}{\pi\rho^2} \int_{R-\rho \leq |z| \leq R+\rho} |f(z)|^p \, dA =$$

$$\frac{1}{\pi\rho^2} \int_{R-\rho}^{R+\rho} \left(\int_0^{2\pi} \left|f\left(te^{i\varphi}\right)\right|^p d\varphi \right) t\, dt = \frac{2}{\rho^2} \int_{R-\rho}^{R+\rho} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|f\left(te^{i\varphi}\right)\right|^p d\varphi \right) t\, dt =$$

$$\leq \frac{2}{\rho^2} \|f\|^p_{H^p(\mathbb{D}_2)} \int_{R-\rho}^{R+\rho} t\, dt = \frac{4R}{\rho} \|f\|^p_{H^p(\mathbb{D}_2)}. \tag{2.9}$$

Taking $\rho = R$. we obtain

$$|f(z_0)|^p \leq 4 \|f\|^p_{H^p(\mathbb{D}_2)}.$$

For $z_0 = 0$, the submean property yields

$$|f(z_0)|^p \leq \|f\|^p_{H^p(\mathbb{D})} \leq \|f\|^p_{H^p(\mathbb{D}_2)}$$

immediately. Thus, for any $z_0 \in \mathbb{D}$,

$$|f(z_0)| \leq 4^{1/p} \|f\|_{H^p(\mathbb{D}_2)}. \tag{2.10}$$

□

Note. It is clear from the proof, that (2.10) is true for any $z_0 \in \overline{\mathbb{D}}$. Indeed, (2.9) is obviously true for $R = 1$ and any $\rho \in (0, R)$, because we only need $\overline{D(z_0, \rho)} \subset \mathbb{D}_2$ that is the case for such ρ. Passing to the limit as $\rho \to 1-$, we obtain

$$|f(z)| \leq 4^{1/p} \|f\|_{H^p(\mathbb{D}_2)}, \quad z \in \overline{\mathbb{D}}. \tag{2.11}$$

Thus,

$$\|f\|_{H^p(\mathbb{D})} \leq \|f\|_{H^\infty(\mathbb{D})} \leq 4^{1/p} \|f\|_{H^p(\mathbb{D}_2)}.$$

Lemma 2.2.2 *If $f \in H^p(\mathbb{D}_2)$, for some $p \in (0, \infty)$, then for any $q \in (p, \infty]$, $f \in H^q(\mathbb{D})$, and*

$$\|f\|_{H^q(\mathbb{D})} \leq 4^{\frac{1}{p}-\frac{1}{q}} \|f\|_{H^p(\mathbb{D}_2)}^{1-\frac{p}{q}} \|f\|_{H^p(\mathbb{D})}^{\frac{p}{q}} \leq 4^{\frac{1}{p}-\frac{1}{q}} \|f\|_{H^p(\mathbb{D}_2)} \tag{2.12}$$

Proof. If $q = \infty$, this is just Lemma 2.2.1. For $q \in (p, \infty)$, using Lemma 2.2.1, we get

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(te^{i\varphi}\right) \right|^q d\varphi &\leq \sup_{z \in \mathbb{D}} |f(z)|^{q-p} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(te^{i\varphi}\right) \right|^p d\varphi \\ &\leq 4^{\frac{q}{p}-1} \|f\|_{H^p(\mathbb{D}_2)}^{q-p} \|f\|_{H^p(\mathbb{D})}^p, \quad t \in (0,1). \end{aligned}$$

Taking power $1/q$ and passing to $\sup_{t \in (0,1)}$, we get

$$\|f\|_{H^q(\mathbb{D})} \leq 4^{\frac{1}{p}-\frac{1}{q}} \|f\|_{H^p(\mathbb{D}_2)}^{1-\frac{p}{q}} \|f\|_{H^p(\mathbb{D})}^{\frac{p}{q}}.$$

The last inequality in (2.12) is obvious. □

Theorem 2.2.1 *If $F \in H^1(\mathbb{D}_2)$, then for any $p, q \in (0, \infty]$ and $g \in H^p(\mathbb{D})$, $F * g \in H^q(\mathbb{D})$, and*

$$\|F * g\|_{H^q(\mathbb{D})} \leq 4^{1/p} \|F\|_{H^1(\mathbb{D}_2)} \|g\|_{H^p(\mathbb{D})}. \tag{2.13}$$

Proof. Consider the following functions:

$$\mathcal{G}(z) := g\left(\frac{z}{2}\right), \quad \mathcal{F}(z) := F(2z).$$

Since $g \in H^p(\mathbb{D})$, we deduce (for the case $0 < p < \infty$).

$$\begin{aligned}\frac{1}{2\pi}\sup_{t\in(0,2)}\int_0^{2\pi}\left|\mathcal{G}\left(te^{i\varphi}\right)\right|^p d\varphi &= \frac{1}{2\pi}\sup_{t\in(0,2)}\int_0^{2\pi}\left|g\left(\frac{t}{2}e^{i\varphi}\right)\right|^p d\varphi \\ &= \frac{1}{2\pi}\sup_{t\in(0,1)}\int_0^{2\pi}\left|g\left(te^{i\varphi}\right)\right|^p d\varphi = \|g\|^p_{H^p(\mathbb{D})}.\end{aligned}$$

Thus, $\mathcal{G} \in H^p(\mathbb{D}_2)$, and

$$\|\mathcal{G}\|_{H^p(\mathbb{D}_2)} = \|g\|_{H^p(\mathbb{D})}. \tag{2.14}$$

For $p = \infty$, this relation is just trivial.

The same consideration shows that $\mathcal{F} \in H^1(\mathbb{D})$, and

$$\|\mathcal{F}\|_{H^1(\mathbb{D})} = \|F\|_{H^1(\mathbb{D}_2)}. \tag{2.15}$$

Since F and g are holomorphic in $\mathbb{D}$, we could consider their Taylor series expansions in $\mathbb{D}$

$$F\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}, \quad g\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} b_n r^n e^{in\theta}.$$

But then

$$\mathcal{F}\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} a_n 2^n r^n e^{in\theta}, \quad \mathcal{G}\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} b_n 2^{-n} r^n e^{in\theta}.$$

Now, using (1.3) for convolutions $F * g$ and $\mathcal{F} * \mathcal{G}$, we get

$$(\mathcal{F} * \mathcal{G})\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} a_n 2^n b_n 2^{-n} r^n e^{in\theta} = (f * g)\left(re^{i\theta}\right).$$

Assume $q < \infty$. Since G is harmonic in $\mathbb{D}_2$, whence in $\overline{\mathbb{D}}$, we can apply (2.3) with $R = 1$, and obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|(F * g)\left(re^{i\theta}\right)\right|^q d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left|(\mathcal{F} * \mathcal{G})\left(re^{i\theta}\right)\right|^q d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{2\pi}\int_0^{2\pi}\mathcal{F}\left(re^{i(\theta-t)}\right)\mathcal{G}\left(e^{it}\right)dt\right|^q d\theta.\end{aligned}$$

Changing variables $\theta - t = u$ and considering 2π-periodicity of the functions involved, we can proceed

$$\begin{aligned} &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{2\pi}\int_{\theta-2\pi}^{\theta}\mathcal{F}\left(re^{iu}\right)\mathcal{G}\left(e^{i(\theta-u)}\right)\,du\right|^q d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{2\pi}\int_{0}^{2\pi}\mathcal{F}\left(re^{iu}\right)\mathcal{G}\left(e^{i(\theta-u)}\right)\,du\right|^q d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_{0}^{2\pi}\left|\mathcal{F}\left(re^{iu}\right)\right|\left|\mathcal{G}\left(e^{i(\theta-u)}\right)\right|\,du\right)^q d\theta. \end{aligned}$$

Applying inequality (2.11) to $\mathcal{G}$, and considering relations (2.14) and (2.15), we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}\left|(F*g)\left(re^{i\theta}\right)\right|^q d\theta &\leq 4^{q/p}\left\|\mathcal{G}\right\|^q_{H^p(\mathbb{D}_2)}\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\mathcal{F}\left(re^{iu}\right)\right|\,du\right)^q dt \\ &\leq 4^{q/p}\left\|\mathcal{G}\right\|^q_{H^p(\mathbb{D}_2)}\left\|\mathcal{F}\right\|^q_{H^1(\mathbb{D})} = 4^{q/p}\left\|g\right\|^q_{H^p(\mathbb{D})}\left\|F\right\|^q_{H^1(\mathbb{D}_2)}. \end{aligned}$$

Passing to $\sup_{r\in(0,1)}$ and taking power $1/q$ of both sides, we obtain (2.13).

For $q = \infty$, using (2.3) with $R = 1$ and the triangle inequality, we conclude

$$\left|(F*g)\left(re^{i\theta}\right)\right| \leq \frac{1}{2\pi}\int_0^{2\pi}\left|\mathcal{F}\left(re^{i(\theta-t)}\right)\right|\left|\mathcal{G}\left(e^{it}\right)\right|\,dt.$$

Applying (2.11) to $\mathcal{G}$, and considering (2.14) and (2.15), we get

$$\left|(F*g)\left(re^{i\theta}\right)\right| \leq 4^{1/p}\left\|\mathcal{G}\right\|_{H^p(\mathbb{D}_2)}\frac{1}{2\pi}\int_0^{2\pi}\left|\mathcal{F}\left(re^{i(\theta-t)}\right)\right|\,dt$$

$$\leq 4^{1/p}\left\|\mathcal{G}\right\|_{H^p(\mathbb{D}_2)}\left\|\mathcal{F}\right\|_{H^1(\mathbb{D})} = 4^{1/p}\left\|g\right\|_{H^p(\mathbb{D})}\left\|F\right\|_{H^1(\mathbb{D}_2)}, \quad r\in[0,1),\ \theta\in\mathbb{R},$$

which completes the proof. □

2.3 Hadamard Product in Bergman Spaces

The following statement is a generalization of Theorem 1.2.1 for the norm of harmonic Bergman spaces a^p.

Theorem 2.3.1 *(a) Let $F \in h^1(\mathbb{D})$. For any $p \in [1,\infty)$, the operator $F* \in \mathcal{L}(a^p, a^p)$ with the norm at most $\|F\|_{h^1}$, and for any g harmonic in $\mathbb{D}$ and $R \in [0,1)$, we have*

$$\left(\int_{\mathbb{D}_R}\left|(F*g)(z)\right|^p d\sigma(z)\right)^{1/p} \leq \|F\|_{h^1}\left(\int_{\mathbb{D}_R}|g(z)|^p\,d\sigma(z)\right)^{1/p}$$

$$\leq \|F\|_{h^1} R^{2/p} \left(\int_{\mathbb{D}} |g(z)|^p \, d\sigma(z) \right)^{1/p}. \tag{2.16}$$

(b) If F is a positive harmonic function, and $F* \in \mathcal{L}(a^p, a^p)$ for some $p \in [1, \infty)$, then $F \in h^1(\mathbb{D})$, and

$$\|F\|_{h^1} = \|F*\|_{a^p \to a^p}.$$

Thus, $F* \in \mathcal{L}(a^p, a^p)$ for any $p \in [1, \infty)$, and the operator norm does not depend on p.

The same result holds for the case of holomorphic Bergman spaces. Unfortunately, as in the case of Hardy spaces, the estimate of the norm of $F*$ operator by $\|F\|_{h^1}$ is not sharp in general. The details are outlined in the following statement.

Proposition 2.3.1 *For any $M > 0$, there exists a function $\mathcal{F} \in H^1(\mathbb{D})$ such that*

$$\|\mathcal{F}*\|_{A^p \to A^p} = 1, \quad \forall p \in [1, \infty),$$

but $\|\mathcal{F}\|_{H^1} > M$.

Proofs of Theorem 2.3.1 and Proposition 2.3.1 are similar to the proofs of Theorem 1.2.1 and Proposition 1.2.1, respectively, so, we will omit them.

Despite the fact that there are relations between Hardy and Bergman spaces (see, e.g., [27, Ch. 3, § 3.2]), the form of Theorem 2.3.1 may seem artificial, since the norm in h^1 is involved in the estimate. To make it more natural, we will consider a different Hadamard-type product.

If f and g are two analytic functions in $\mathbb{D}$ with Taylor expansions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

the operator $\star$ is defined by

$$(f \star g)(z) = \sum_{n=0}^{\infty} \frac{a_n b_n}{n+1} z^n, \quad z \in \mathbb{D}. \tag{2.17}$$

Clearly, this operator is well-defined since the series in the right-hand side converges in $\mathbb{D}$. It is also obvious that

$$(f \star g)(z) = (g \star f)(z).$$

The following statement is an analogue of (2.3) for Bergman spaces.

Lemma 2.3.1 *Let f and g be holomorphic in $\mathbb{D}$. Then, for $0 < R < 1$,*

$$(f \star g)(z) = \int_{\mathbb{D}} f\left(\frac{z\overline{\zeta}}{R}\right) g(R\zeta)\, d\sigma(\zeta), \quad z \in \mathbb{D}_R, \tag{2.18}$$

i.e., for $0 \le r < R < 1$, and $\theta \in \mathbb{R}$,

$$(f \star g)\left(re^{i\theta}\right) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 f\left(\frac{r\rho}{R}e^{i(\theta-t)}\right) g\left(R\rho e^{it}\right) \rho\, d\rho\, dt. \tag{2.19}$$

If g is holomorphic in $\overline{\mathbb{D}}$, then (2.18) and (2.19) are valid with $0 < R \le 1$.

Proof. Let us fix an arbitrary $r \in [0, R)$. For any $n \in \mathbb{N}$, the orthogonality of exponentials on $[0, 2\pi]$ implies

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \left(\sum_{k=0}^n a_k \left(\frac{r\rho}{R}\right)^k e^{ik(\theta-t)}\right)\left(\sum_{k=0}^n b_k R^k \rho^k e^{ikt}\right)\rho\, d\rho\, dt =$$

$$2\sum_{k=0}^n \left(e^{ik\theta} a_k b_k r^k \int_0^1 \rho^{2k+1}\, dt\right) = \sum_{k=0}^n \frac{a_k b_k}{k+1} r^k e^{ik\theta}.$$

Since the series $\sum_{k=0}^n a_k (r\rho/R)^k e^{ik(\theta-t)}$ and $\sum_{k=0}^n b_k R^k \rho^k e^{it}$ converge absolutely and uniformly on $(\rho, t) \in [0,1] \times [0, 2\pi]$, we can pass to the limit as $n \to \infty$ to obtain

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 f\left(\frac{r\rho}{R}e^{i(\theta-t)}\right) g\left(R\rho e^{it}\right)\rho\, d\rho\, dt = \sum_{k=0}^\infty \frac{a_k b_k}{k+1} r^k e^{ik\theta} = (f \star g)\left(re^{i\theta}\right).$$

□

Using Lemma 2.3.1 instead of the integral representation for the Hadamard product given by (2.3), one can prove the following theorem by repeating arguments from the proof of Theorem 1.2.1

Theorem 2.3.2 *Let $F \in A^1(\mathbb{D})$. For any $p \in [1, \infty)$, the operator $F\star \in \mathcal{L}(A^p, A^p)$ with the norm at most $\|F\|_{A^1}$, and for any g holomorphic in $\mathbb{D}$ and $R \in [0, 1)$, we have*

$$\left(\int_{\mathbb{D}_R} |(F \star g)(z)|^p \, d\sigma(z)\right)^{1/p} \leq \|F\|_{A^1} \left(\int_{\mathbb{D}_R} |g(z)|^p \, d\sigma(z)\right)^{1/p}$$
$$\leq \|F\|_{A^1} R^{2/p} \left(\int_{\mathbb{D}} |g(z)|^p \, d\sigma(z)\right)^{1/p}. \qquad (2.20)$$

Remark 2.3.1 *It is easy to see that if $F \in A^1$, then the operator $F\star \in \mathcal{L}(H^p, H^p)$, and*

$$\|F\star\|_{H^p \to H^p} \leq \|F\|_{A^1}.$$

Now, we return to the Hadamard product. Let us note that if F is holomorphic in $\mathbb{D}$, and has Maclauren series expansion $F(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$\mathcal{F}(z) := (zF(z))' = \sum_{k=0}^{\infty} a_k (k+1) z^k, \quad z \in \mathbb{D}.$$

From (1.3) and (2.17), for an arbitrary g holomorphic in $\mathbb{D}$, we get

$$(F * g)(z) = (\mathcal{F} \star g)(z), \quad z \in \mathbb{D}.$$

Thus, the following statement follows from Theorem 2.3.2 immediately.

Corollary 2.3.1 *Let F be such that $\mathcal{F}(z) := (zF(z))' \in A^1(\mathbb{D})$. Then, for any $p \in [1, \infty)$, $F* \in \mathcal{L}(A^p, A^p)$, and*

$$\|F*\|_{A^p \to A^p} \leq \|\mathcal{F}\|_{A^1}.$$

The following statement gives a norm estimate for multipliers of Hardy into Bergman spaces in terms of integral norms of the generating function.

Theorem 2.3.3 *(a) Let $1 \leq p \leq q < \infty$, and $F \in a^q(\mathbb{D})$. Then, for any g harmonic in $\mathbb{D}$, and $0 \leq R < 1$, the following estimate holds true*

$$\left(\int_{\mathbb{D}_R} |(F * g)(z)|^q \, d\sigma(z)\right)^{1/q} \leq \|F\|_{a^q} R^{2/q} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|g\left(Re^{it}\right)\right|^p \, dt\right)^{1/p}. \qquad (2.21)$$

In particular, $F* \in \mathcal{L}(h^p, a^q)$*, and* $F* \in \mathcal{L}(H^p, A^q)$ *with*

$$\|F*\|_{h^p \to a^q} \le \|F\|_{a^q}\,, \quad \|F*\|_{H^p \to A^q} \le \|F\|_{a^q}\,. \tag{2.22}$$

(b) When $q = p$*, the requirement* $F \in a^q(\mathbb{D})$ *should be weakened to* $F \in a^1(\mathbb{D})$*, and the norm* $\|F\|_{a^q}$ *should be replaced by* $\|F\|_{a^1}$ *in* (2.22)*. Moreover, if* F *is a positive harmonic function, and* $F* \in \mathcal{L}(h^p, a^p)$*, for some* $p \in [1, \infty)$*, then* $F \in a^1(\mathbb{D})$*, and* $\|F\|_{a^1} = \|F*\|_{h^p \to a^p}$ *for any* $p \in [1, \infty)$*.*

Proof. Choosing $R_1 \in (R, 1)$, and applying (2.3), we get

$$I := \int_{\mathbb{D}_R} |(F * g)(z)|^q \, d\sigma(z) =$$

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^R \left| \frac{1}{2\pi} \int_0^{2\pi} F\left(\frac{r}{R_1} e^{i(\theta - t)}\right) g\left(R_1 e^{it}\right) dt \right|^q r \, dr \, d\theta.$$

Note that if $p \in [1, \infty)$, and $G \in L^p[0, 2\pi]$, then Hölder's inequality implies

$$\frac{1}{2\pi} \int_0^{2\pi} |G(u)| \, du \le \left(\frac{1}{2\pi} \int_0^{2\pi} |G(u)|^p \, du \right)^{1/p}.$$

Applying this inequality to $G(t) = F\left(\frac{r}{R_1} e^{i(\theta - t)}\right) g\left(R_1 e^{it}\right)$, and using Minkowski's integral inequality with the power $q/p > 1$, we conclude

$$I \le \frac{1}{\pi} \int_0^{2\pi} \int_0^R \left(\frac{1}{2\pi} \int_0^{2\pi} \left| F\left(\frac{r}{R_1} e^{i(\theta - t)}\right) \right|^p \left| g\left(R_1 e^{it}\right) \right|^p dt \right)^{q/p} r \, dr \, d\theta$$

$$\le \frac{1}{\pi (2\pi)^{q/p}} \left(\int_0^{2\pi} \left(\int_0^{2\pi} \int_0^R \left| F\left(\frac{r}{R_1} e^{i(\theta - t)}\right) \right|^q \left| g\left(R_1 e^{it}\right) \right|^q r \, dr \, d\theta \right)^{p/q} dt \right)^{q/p}$$

$$= \frac{1}{\pi (2\pi)^{q/p}} \left(\int_0^{2\pi} \left| g\left(R_1 e^{it}\right) \right|^p dt \right)^{q/p} \int_0^{2\pi} \int_0^R \left| F\left(\frac{r}{R_1} e^{i\theta}\right) \right|^q r \, dr \, d\theta.$$

Changing variable $r/R_1 = \rho$ in the last integral, we deduce

$$I \le \left(\frac{1}{2\pi} \int_0^{2\pi} \left| g\left(R_1 e^{it}\right) \right|^p dt \right)^{q/p} R_1^2 \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left| F\left(\rho e^{i\theta}\right) \right|^q \rho \, d\rho \, d\theta.$$

Passing to the limit as $R_1 \to R+$ gives (2.21). Now, estimates (2.22) follow immediately.

For Part (b), we can apply the Minkowski's inequality with power q to

$$\int_0^{2\pi} \int_0^R \left(\int_0^{2\pi} \left| F\left(\frac{r}{R_1} e^{i(\theta - t)} \right) \right| \left| g\left(R_1 e^{it} \right) \right| \, dt \right)^q r \, dr \, d\theta,$$

and repeat the same estimates.

To get the last statement, we need to repeat the reasoning from the proof of Theorem 1.2.1 (b) using the mean value property $F(0) = \int_{\mathbb{D}} F(z) \, d\sigma(z)$, which implies $F(0) = \|F\|_{a^1}$ in our case. $\square$

CHAPTER 3

Mahler Measure of the Hadamard Product of Two Polynomials

As we already mentioned in Section 1.3, the Mahler measure is an efficient tool in obtaining sharp inequalities for polynomials. Bernstein's inequality in Hardy spaces $H^p(\mathbb{D})$:

$$\|P'\|_{H^p} \leq n \|P\|_{H^p},$$

for $p < 1$ has an interesting history. As we already noticed, for $p = 0$, it is an immediate corollary of [17, Theorem 7] published in 1947. However, K. Mahler proved the same inequality in [61] (published in 1961) using another method. V. V. Arestov obtained the Bernstein inequality $\|P'\|_{H^p} \leq n \|P\|_{H^p}$, $p \in (0, 1)$, in [2], and then gave a much simpler proof in [3]. The latter approach was based on the proof of [17, Theorem 7]. See Example 3.2 below for details and a reverse Bernstein inequality.

In [76], I. E. Pritsker obtained several sharp estimates for the Mahler measure, which imply corresponding estimates in H^p-norm immediately. In particular, he answered the question of what happens to the Mahler measure of a polynomial after removing a specific power term. The article also contains an extended survey of the results in this area.

Other applications of the Mahler measure are in Number Theory. For example, if a monic polynomial Q with complex coefficients is cyclotomic, then $\|Q\|_{H^0} = 1$. An exciting open question is about the smallest possible Mahler measure of an irreducible non-cyclotomic polynomial with integer coefficients – the Lehmer conjecture [58]. Moreover, the Mahler measure is related to the theory of Salem-Vijayaraghavan numbers (see [12]). For more relations, history, and applications of the Mahler measure, see the survey article [83] by C. Smyth.

There is an interesting analog of the Mahler measure – the areal Mahler measure introduced by I. E. Pritsker in [77]. It has the same close relation to Bergman spaces as the standard Mahler measure has to Hardy spaces, and allows to obtain many interesting inequalities for Bergman spaces as well as to establish several useful relations between norms of polynomials in Hardy and Bergman spaces.

We will restrict our attention to the Mahler measure of the Hadamard product of two polynomials, and employ V. V. Arestov's result [3] to get estimates in H^p-norm (or pre-norm) for this product.

3.1 Estimates for the Norm of the Hadamard Product Operator

Let $\{\lambda_{n,k}\}_{k=0}^n$ be a finite sequence of complex numbers. For two polynomials $P(z) = \sum_{k=0}^n a_k z^k$ and $Q(z) = \sum_{k=0}^n b_k z^k$, consider the following coefficient multiplier

$$\Lambda_n[P,Q](z) := \sum_{k=0}^n \lambda_{n,k} a_k b_k z^k.$$

We may fix P and consider Λ_n as a linear operator acting on Q.

The following lemma follows from (1.13), and may be useful for obtaining sharp estimates for various coefficient multipliers.

Lemma 3.1.1 *For an arbitrary polynomial $P(z) = \sum_{k=0}^n a_k z^k$ with complex coefficients, and a finite sequence $\{\lambda_{n,k}\}_{k=0}^n$, define*

$$P_\lambda(z) := \sum_{k=0}^n \binom{n}{k} \lambda_{n,k} a_k z^k.$$

(a) For every $p \in [0,\infty]$,

$$\|\Lambda_n[P,Q]\|_{H^p} \le \|P_\lambda\|_{H^0} \|Q\|_{H^p}.$$

(b) We have

$$\sup_{\deg(Q)\le n, \|Q\|_{H^0}=1} \|\Lambda_n[P,Q]\|_{H^0} = \|P_\lambda\|_{H^0}. \tag{3.1}$$

The supremum is achievable, e.g., taking $Q(z) = \alpha(1+\beta z)^n$, where $|\alpha| = |\beta| = 1$.

Proof. For (a), expressing Q as

$$Q(z) = \sum_{k=0}^{n} b_k z^k = \sum_{k=0}^{n} \binom{n}{k} \frac{b_k}{\binom{n}{k}} z^k,$$

we notice that $\Lambda_n[P,Q](z) = (P_\lambda *_S Q)(z)$. Therefore, estimate (1.13) implies

$$\|\Lambda_n[P,Q]\|_{H^p} \le \|P_\lambda\|_{H^0} \|Q\|_{H^p}, \quad 0 \le p \le \infty.$$

For (b), it is also immediate that if $Q(z) = \alpha(1+\beta z)^n$ and $|\alpha| = |\beta| = 1$, then $\|Q\|_{H^0} = 1$, and $\Lambda_n[P,Q](z) = \alpha P_\lambda(\beta z)$. Using (1.11), we also obtain $\|\Lambda_n[P,Q]\|_{H^0} = \|P_\lambda\|_{H^0}$. $\square$

Let us also note that the weighted Hadamard product could be useful for problems on Bombieri norms considered, e.g., in [7, 8, 13, 14]. There are relations between the Mahler measure and Bombieri norms. For instance, B. Beauzamy [7, Proposition 4] proved that for a polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ of degree n, its Bombieri norm $[P]_2 := \left(\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} |a_k|^2\right)^{1/2}$ can be estimated as

$$\binom{n}{\lfloor n/2 \rfloor}^{-1/2} \|P\|_{H^0} \le [P]_2 \le 2^{n/2} \|P\|_{H^0}.$$

(Here and in the sequel, $\lfloor \alpha \rfloor$ denotes the integer part, or the floor, of α.)

Proof of Theorem 1.3.1. **(a)** The statement follows from Lemma 3.1.1 applied with $\lambda_{n,k} = 1$, by using estimate (1.12). Alternatively, one can notice that $(P * Q)(z) = (\Theta_n *_S P *_S Q)(z)$ and apply (1.13) twice. Since the operation $*_S$ is associative, we may apply (1.13) in various ways, and get a bit more:

$$\|P * Q\|_{H^p} \le$$

$$\min\left\{\|\Theta_n\|_{H^0} \|P\|_{H^0} \|Q\|_{H^p}, \|\Theta_n\|_{H^0} \|P\|_{H^p} \|Q\|_{H^0}, \|\Theta_n\|_{H^p} \|P\|_{H^0} \|Q\|_{H^0}\right\}.$$

(b) It is shown in [82, Ch. 4, § 4.1.2] that if f is harmonic in $\mathbb{D}$ and g is harmonic in $\overline{\mathbb{D}}$, then

$$(f * g)\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} f\left(re^{i(\theta - t)}\right) g\left(e^{it}\right) dt, \quad r \in [0,1),\ \theta \in \mathbb{R}. \tag{3.2}$$

Since, $\Theta_n(z) = (1+z)^n * (1+z)^n$, the integral representation (3.2) yields

$$\left|\Theta_n\left(re^{i\theta}\right)\right| = \frac{1}{2\pi}\left|\int_0^{2\pi}\left(1+re^{i(\theta-t)}\right)^n\left(1+e^{it}\right)^n\,dt\right| \le 4^n,$$

whence $\|\Theta_n\|_{H^0} \le \|\Theta_n\|_{H^\infty} \le 4^n$.

Let us note that the Legendre polynomial $P_n(x)$ has the following representation (see, e.g., [1, 22.3.1 and 22.5.35]):

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x-1)^{n-k}(x+1)^k. \tag{3.3}$$

So, we get

$$P_n(x) = \frac{1}{2^n}(x-1)^n\sum_{k=0}^{n}\binom{n}{k}^2\left(\frac{x+1}{x-1}\right)^k = \frac{1}{2^n}(x-1)^n\Theta_n\left(\frac{x+1}{x-1}\right).$$

Thus, if we change the variable to $z = (x+1)/(x-1)$, we obtain

$$\Theta_n(z) = (z-1)^n P_n\left(\frac{z+1}{z-1}\right). \tag{3.4}$$

Since all the zeros of P_n are simple and belong to $[-1,1]$, and $(x+1)/(x-1)$ maps $(-\infty,-1)$ onto $(0,1)$ and $(-1,0)$ onto $(-1,0)$, equality (3.4) implies:

(i) All zeros of Θ_n are simple;

(ii) All zeros of Θ_n belong to $(-\infty,0)$;

(iii) If $\Theta_n(z_0) = 0$, then $\Theta_n(1/z_0) = 0$.

Since Θ_n is a monic polynomial, we obtain that $\|\Theta_n\|_{H^0} = \prod_{|\gamma_k|\ge 1}|\gamma_k|$, where γ_k are the zeros of Θ_n. Hence,

$$\|\Theta_n\|_{H^0} = \prod_{\alpha_k\in[0,1)}\left|\frac{\alpha_k+1}{\alpha_k-1}\right|, \tag{3.5}$$

where α_k are the zeros of the Legendre polynomial P_n.

Let us express P_n as the product of its linear terms:

$$P_n(x) = a_n\prod_{k=1}^{n}(x-\alpha_k).$$

It follows from (3.3) that

$$P_n(1) = 1, \quad a_n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 = \frac{\Theta_n(1)}{2^n}.$$

Therefore,

$$\prod_{k=1}^{n}|\alpha_k - 1| = \frac{|P_n(1)|}{|a_n|} = \frac{2^n}{\Theta_n(1)}. \tag{3.6}$$

Since all the zeros of P_n are simple and symmetric about the origin, we deduce from (3.5) and (3.6) that

$$\|\Theta_n\|_{H^0} = \prod_{\alpha_k\in[0,1)}\left(\left|\frac{\alpha_k+1}{\alpha_k-1}\right|\left|\frac{-\alpha_k-1}{-\alpha_k-1}\right|\right) = \frac{\prod\limits_{\alpha_k\in[0,1)}|\alpha_k+1|^2}{\prod\limits_{k=1}^{n}|\alpha_k-1|}$$

$$= \frac{\Theta_n(1)}{2^n}\prod_{\alpha_k\in[0,1)}|\alpha_k+1|^2. \tag{3.7}$$

Let τ_n be the counting measure for the roots of P_n, assigning the value $1/n$ to each root, i.e.,

$$\tau_n([a,b)) = \frac{\text{number of zeros of } P_n \text{ in } [a,b)}{n},$$

and let

$$f(x) := \begin{cases} 0, & x\in[-1,0], \\ \ln|x+1|, & x\in[0,1]. \end{cases}$$

Then,

$$\int_{[-1,1]} f(x)\,d\tau_n(x) = \frac{1}{n}\sum_{\alpha_k\in[0,1)}\ln|\alpha_k+1|.$$

Therefore, (3.7) yields

$$\ln\|\Theta_n\|_{H^0}^{1/n} = \frac{\ln(\Theta_n(1))}{n} - \ln 2 + \frac{2}{n}\sum_{\alpha_k\in[0,1)}\ln|\alpha_k+1| =$$

$$\frac{\ln(\Theta_n(1))}{n} - \ln 2 + 2\int_{[-1,1]} f(x)\,d\tau_n(x). \tag{3.8}$$

Applying the formula

$$\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}, \quad r+s\geq n$$

(see, e.g., [1, 24.1.1]) with $r = s = n$, we conclude that

$$\Theta_n(1) = \binom{2n}{n} = \frac{\Gamma(2n+1)}{(\Gamma(n+1))^2} = \frac{2n\Gamma(2n)}{(\Gamma(n+1))^2}.$$

Using the duplication formula for the Gamma function [1, 6.1.18], we get

$$\Theta_n(1) = \frac{2n\,(2\pi)^{-1/2}\,2^{2n-1/2}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)}{(\Gamma(n+1))^2} = \frac{4^n\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}. \tag{3.9}$$

There are several representations of Catalan's constant. One of them is

$$G = 2\int_0^{\pi/4} \ln\left(2\cos u\right)\,du = \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln\left(\cos u\right)\,du.$$

Using the substitution $x = \cos(2u)$, it is easy to show that

$$\int_0^1 \frac{\ln(1+x)}{\pi\sqrt{1-x^2}}\,dx = \frac{2G}{\pi} - \frac{\ln 2}{2}.$$

Applying (3.8) and (3.9), we now obtain

$$\ln\|\Theta_n\|_{H^0}^{1/n} - \frac{4G}{\pi} = \ln\|\Theta_n\|_{H^0}^{1/n} - \ln 2 - 2\int_0^1 \frac{\ln(x+1)}{\pi\sqrt{1-x^2}}\,dx =$$

$$\frac{1}{n}\ln\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}\right) + 2\left(\int_{[-1,1]} f(x)\,d\tau_n(x) - \int_{[-1,1]} f(x)\,d\mu(x)\right), \tag{3.10}$$

where

$$d\mu(x) = \frac{dx}{\pi\sqrt{1-x^2}}$$

is the equilibrium measure on $[-1,1]$.

It is well-known that the polynomials P_n are orthogonal on $[-1,1]$ with respect to the Lebesgue measure, $\|P_n\|_{L^2[-1,1]} = \sqrt{\frac{2}{2n+1}}$, and $\|P_n\|_{C[-1,1]} = 1$ (see, e.g., [1, 22.2.1 and 22.14.7]). Hence, the polynomials $Q_n(z) := \sqrt{n+\frac{1}{2}}P_n(z)$ are orthonormal on $[-1,1]$, and have the same zeros as P_n. Applying H.-P. Blatt's discrepancy result [9, Corollary 1], we deduce that there exists an absolute constant $c > 0$ such that

$$\left|(\tau_n - \mu)\left([a,b]\right)\right| \le c\frac{\ln n}{n}\left(\ln\|Q_n\|_{C[-1,1]} + \ln n\right) \le 2c\frac{\ln^2 n}{n} \tag{3.11}$$

for any interval $[a,b] \subset [-1,1]$ and any $n \ge 2$.

Since

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1\cdot 2\cdot 3\cdot \cdots \cdot (2n-1)}{2^n}\sqrt{\pi}$$

(see, e.g., [1, 6.1.12]), it is clear that

$$\frac{1}{n}\le \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}\le 1.$$

Therefore, from (3.10) and (3.11), we obtain

$$\left|\ln\|\Theta_n\|_{H^0}^{1/n}-\frac{4G}{\pi}\right|\le \frac{\ln n}{n}+4\max_{x\in[-1,1]}|f(x)|\,c\frac{\ln^2 n}{n}=\frac{\ln n}{n}+4c\ln 2\frac{\ln^2 n}{n},\quad n\ge 2.$$

This proves (1.16). Now (1.15) follows immediately. □

3.2 Examples

Example 3.1. (Bernstein's Inequality) As an illustration, we can deduce Bernstein's inequality from Lemma 3.1.1. Let $Q(z)=\sum_{k=0}^n b_k z^k$, $n\in\mathbb{N}$. Then,

$$Q'(z)=\frac{1}{z}\sum_{k=1}^n kb_k z^k=\frac{(P*Q)(z)}{z},$$

where $P(z):=\sum_{k=0}^n kz^k$. Using the multiplicative property (1.10) of the Mahler measure, we get $\|Q'\|_{H^0}=\|P*Q\|_{H^0}$. Futhermore, in view of Lemma 3.1.1,

$$P_\lambda(z)=\sum_{k=0}^n\binom{n}{k}kz^k=z\frac{d}{dz}(1+z)^n=zn(1+z)^{n-1}.$$

Using the multiplicative property again, we get $\|P_\lambda\|_{H^0}=\|z\|_{H^0}\,n\,\|1+z\|_{H^0}^{n-1}=n$. Finally, applying Lemma 3.1.1, we obtain

$$\|Q'\|_{H^p}\le n\,\|Q\|_{H^p},\quad 0\le p\le\infty.$$

The sharpness is verified on $Q(z)=z^n$.

There is also reverse Bernstein Inequality (see [87] and also [76]). If $Q(0)=0$, then

$$Q(z)=\sum_{k=1}^n\frac{1}{k}kb_k z^k=z\left(\sum_{k=0}^{n-1}\frac{z^k}{k+1}*Q'(z)\right).$$

Lemma 3.1.1 implies

$$\|Q\|_{H^0} \leq \left\| \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{z^k}{k+1} \right\|_{H^0} \|Q'\|_{H^0} . \tag{3.12}$$

Now

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{z^k}{k+1} = \frac{1}{nz} \sum_{k=0}^{n-1} \binom{n}{k+1} z^{k+1} = \frac{1}{nz} \sum_{k=1}^{n} \binom{n}{k} z^k = \frac{(1+z)^n - 1}{nz}.$$

As is shown in [87],

$$A_n := \|(1+z)^n - 1\|_{H^0} = \prod_{\pi/6<k<5\pi/6} 2 \sin \frac{\pi k}{n} \approx (1.4)^n.$$

Hence, (3.12) implies

$$\|Q\|_{H^0} \leq \frac{A_n}{n} \|Q'\|_{H^0} .$$

Equality is attained, e.g., for $Q(z) = (1+z)^n - 1$.

Example 3.2. Let us take $\lambda_{n,k} = \binom{n}{k}^{-2}$. This, in fact, corresponds to the Schur-Szegő product of $P *_S Q$ and $\sum_{k=0}^{n} z^k$. Lemma 3.1.1 implies

$$\|\Lambda_n[P, Q]\|_{H^p} \leq \left\| P(z) *_S \sum_{k=0}^{n} z^k \right\|_{H^0} \|Q\|_{H^p} , \quad 0 \leq p \leq \infty. \tag{3.13}$$

Since $\sum_{k=0}^{n} z^k = (z^{n+1} - 1)/(z-1)$, the multiplicative property (1.10) immediately implies $\left\|\sum_{k=0}^{n} z^k\right\|_{H^0} = 1$. Now, applying (1.12) to the first term in the right hand side of (3.13), we deduce

$$\|\Lambda_n[P, Q]\|_{H^p} \leq \|P\|_{H^0} \|Q\|_{H^p} , \quad 0 \leq p \leq \infty. \tag{3.14}$$

For $P(z) = Q(z) = (1+z)^n$, the last inequality becomes an equality.

Using induction on m, it is easy to see that (3.14) holds for $\lambda_{n,k} = \binom{n}{k}^{-m}$ with any $m \in \mathbb{N}$ (see [76, Corollary 1.6]).

Other interesting examples of coefficient multipliers used to obtain sharp polynomial inequalities could be found in, e.g., [76, 88, 89]. They essentially use the de Bruijn-Springer-Arestov inequalities. However, if we look at the Schur-Szegő product of $P(z) = \sum_{k=0}^{n} a_k z^k$ and $Q(z) = \sum_{k=0}^{n} b_k z^k$,

$$\left(P *_S Q\right)(z) = \sum_{k=0}^{n} \frac{a_k b_k}{\binom{n}{k}} z^k,$$

we may notice that the binomial coefficients in the denominator may introduce computational difficulties. In this case, our Theorem 1.3.1 should be more useful.

Example 3.3. (The Odd and Even Parts of a Polynomial) It is often easier to obtain some result under the assumption that a function is even, or odd, and then consider the general case. Thus, it is useful to have a good estimate for the norm of the even and odd parts of the function. The triangle inequality in H^p, $p \in (0,1)$, gives only $\left\|\left(f(z)+f(-z)\right)/2\right\|_{H^p} \le 2^{1/p-1}\left\|f\right\|_{H^p}$. In H^0, there is no general triangle inequality. Nevertheless, Lemma 3.1.1 and Theorem 1.3.1 allow us to obtain some sharp estimates for polynomials.

Let $Q(z) = \sum_{k=0}^{n} b_k z^k \in \mathbb{C}_n[z]$. For its even part, we have

$$\frac{Q(z)+Q(-z)}{2} = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{2k} z^{2k} = \left(P * Q\right)(z), \tag{3.15}$$

where $P(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} z^{2k} = \frac{z^{2(\lfloor n/2 \rfloor+1)}-1}{z^2-1}$. Since $\left\|P\right\|_{H^0} = 1$, Theorem 1.3.1 and the triangle inequality in H^p for $p \ge 1$ imply

$$\left\|\frac{Q(z)+Q(-z)}{2}\right\|_{H^p} \le \begin{cases} \left\|\Theta_n\right\|_{H^0}\left\|Q\right\|_{H^p}, & 0 \le p < 1, \\ \left\|Q\right\|_{H^p}, & 1 \le p \le \infty. \end{cases}$$

This estimate may be good enough, since we know sharp asymptotics for $\left\|\Theta_n\right\|_{H^0}$ given by Theorem 1.3.1. However, if we need a sharper estimate, we can employ Lemma 3.1.1 directly, and get the following statement.

Proposition 3.2.1 *Let $n \in \mathbb{N}$ and $Q \in \mathbb{C}_n[z]$. Then,*

$$\left\|\frac{Q(z)+Q(-z)}{2}\right\|_{H^p} \le \alpha_n \left\|Q\right\|_{H^p}, \quad 0 \le p < 1, \tag{3.16}$$

where

$$\alpha_n := \begin{cases} \prod\limits_{0\le j\le \frac{n-2}{4}} \cot^2\left(\frac{\pi}{2n}+\frac{\pi j}{n}\right), & \text{if } n \text{ is even,} \\ n\prod\limits_{1\le j\le \frac{n}{4}} \cot^2\left(\frac{\pi j}{n}\right), & \text{if } n \text{ is odd.} \end{cases}$$

and

$$\left\| \frac{Q(z)-Q(-z)}{2} \right\|_{H^p} \le \beta_n \|Q\|_{H^p}, \quad 0\le p<1, \tag{3.17}$$

where

$$\beta_n := \begin{cases} n\prod\limits_{\frac{n}{4}\le j\le \frac{3n}{4},\, j\ne\frac{n}{2}} \left|\tan\left(\frac{\pi j}{n}\right)\right|, & \text{if } n \text{ is even,} \\ \prod\limits_{\frac{n}{4}\le j\le \frac{3n}{4}} \left|\tan\left(\frac{\pi j}{n}\right)\right|, & \text{if } n \text{ is odd.} \end{cases}$$

For $p=0$, estimates (3.16) and (3.17) become equalities for, e.g., $Q(z)=(1+z)^n$.

Proof. Considering (3.15), to obtain (3.16), we may apply Lemma 3.1.1 with

$$P_\lambda(z) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} z^{2k} = \frac{(1+z)^n+(1-z)^n}{2}.$$

Now, we need to find $\|P_\lambda\|_{H^0}$. Using (1.11), we get

$$\|P_\lambda\|_{H^0} = \begin{cases} \prod\limits_{|\gamma_j|\ge 1} |\gamma_j|, & \text{if } n \text{ is even,} \\ n\prod\limits_{|\gamma_j|\ge 1} |\gamma_j|, & \text{if } n \text{ is odd,} \end{cases} \tag{3.18}$$

where γ_j's are the zeros of $(1+z)^n+(1-z)^n$ (counting multiplicities).

If n is even, then γ_j's are the solutions of the equation

$$\left(\frac{z+1}{z-1}\right)^n = -1,$$

i.e.,

$$\frac{\gamma_j+1}{\gamma_j-1} = \exp\left(i\frac{(\pi+2\pi j)}{n}\right), \quad j=0,\dots,n-1.$$

Thus, we obtain

$$\gamma_j = \frac{\exp\left(i\frac{(\pi+2\pi j)}{n}\right)+1}{\exp\left(i\frac{(\pi+2\pi j)}{n}\right)-1} = -i\cot\left(\frac{\pi}{2n}+\frac{\pi j}{n}\right), \quad j=0,\dots,n-1.$$

Since also $|\gamma_{n-1-j}| = |\gamma_j|$, we have from (3.18) that

$$\|P_\lambda\|_{H^0} = \prod_{0\le j\le \frac{n-2}{4}} \cot^2\left(\frac{\pi}{2n} + \frac{\pi j}{n}\right). \tag{3.19}$$

If n is odd, γ_j's satisfy the equation

$$\frac{\gamma_j + 1}{\gamma_j - 1} = \exp\left(i\frac{2\pi j}{n}\right), \quad j = 1, \dots, n-1.$$

Hence

$$\gamma_j = \frac{\exp\left(i\frac{2\pi j}{n}\right) + 1}{\exp\left(i\frac{2\pi j}{n}\right) - 1} = -i\cot\left(\frac{\pi j}{n}\right), \quad j = 1, \dots, n-1,$$

and (3.18) implies

$$\|P_\lambda\|_{H^0} = n \prod_{1\le j\le \frac{n}{4},\ \frac{3n}{4}\le j\le n-1} \left|\cot\left(\frac{\pi j}{n}\right)\right| = n\prod_{1\le j\le\frac{n}{4}} \cot^2\left(\frac{\pi j}{n}\right). \tag{3.20}$$

Using (3.19), (3.20) and Lemma 3.1.1, we get (3.16).

Estimate (3.17) follows essentially in the same way. First of all, note that if $Q(z) = \sum_{k=0}^{n} b_k z^k$, then in notations of Lemma 3.1.1, we obtain

$$\frac{Q(z) - Q(-z)}{2} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} b_{2k+1} z^{2k+1} = \Lambda_n\left[R, Q\right](z), \tag{3.21}$$

where

$$R(z) := \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} z^{2k+1}, \quad R_\lambda(z) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n}{2k+1} z^{2k+1} = \frac{(1+z)^n - (1-z)^n}{2}.$$

Now, from Lemma 3.1.1 and (3.21) we conclude that

$$\left\|\frac{Q(z) - Q(-z)}{2}\right\|_{H^p} \le \|R_\lambda\|_{H^0}\, \|Q\|_{H^p}\,, \quad p\in[0,\infty]. \tag{3.22}$$

Note that the leading coefficient of R_λ is 1 when n is odd, and it is equal to n when n is even. Hence, (1.11) implies

$$\|R_\lambda\|_{H^0} = \begin{cases} n\prod\limits_{|\delta_j|\ge 1} |\delta_j|\,, & \text{if } n \text{ is even,}\\ \prod\limits_{|\delta_j|\ge 1} |\delta_j|\,, & \text{if } n \text{ is odd,}\end{cases} \tag{3.23}$$

where δ_j are the zeros of $(1+z)^n-(1-z)^n$ (counting multiplicities). In other words, δ_j's are solutions of one of the equations

$$\frac{1+\delta_j}{1-\delta_j}=\exp\left(i\frac{2\pi j}{n}\right),\quad j=0,\dots,n-1.$$

Each of these equations has a solution if and only if $j\neq n/2$. It is unique and given by

$$\delta_j=\frac{\exp\left(i\frac{2\pi j}{n}\right)-1}{\exp\left(i\frac{2\pi j}{n}\right)+1},\quad j=0,\dots,n-1,\, j\neq\frac{n}{2}.$$

Using Euler's formula for the exponential, it is easy to see that

$$|\delta_j|^2=\frac{\left|\cos\left(\frac{2\pi j}{n}\right)+i\sin\left(\frac{2\pi j}{n}\right)-1\right|^2}{\left|\cos\left(\frac{2\pi j}{n}\right)+i\sin\left(\frac{2\pi j}{n}\right)+1\right|^2}=\frac{1-\cos\left(\frac{2\pi j}{n}\right)}{1+\cos\left(\frac{2\pi j}{n}\right)}=\tan^2\left(\frac{\pi j}{n}\right).$$

Thus,

$$|\delta_j|=\left|\tan\left(\frac{\pi j}{n}\right)\right|,\quad j=0,\dots,n-1,\, j\neq\frac{n}{2}.\tag{3.24}$$

Clearly $|\delta_j|\geq 1$ if and only if $\frac{n}{4}\leq j\leq\frac{3n}{4}$.

Thus, from (3.23) and (3.24), we obtain

$$\|R_\lambda\|_{H^0}=\begin{cases} n\displaystyle\prod_{\frac{n}{4}\leq j\leq\frac{3n}{4},\, j\neq\frac{n}{2}}\left|\tan\left(\frac{\pi j}{n}\right)\right|, & \text{if } n \text{ is even,}\\ \displaystyle\prod_{\frac{n}{4}\leq j\leq\frac{3n}{4}}\left|\tan\left(\frac{\pi j}{n}\right)\right|, & \text{if } n \text{ is odd.}\end{cases}$$

Finally, (3.22) implies (3.17). $\square$

CHAPTER 4

Fourier Multipliers in Hardy Spaces in Tubes over Open Cones

Our main results on multipliers of Fourier integrals in Hardy spaces were formulated in Section 1.4 (see Theorems 1.4.2, 1.4.3, and Proposition 1.4.1). For definitions and historical remarks, please also see Section 1.4. We will start the chapter with the *basic properties of multipliers.*

1) $\|\varphi+\psi\|^q_{\mathcal{M}_{p,q}(T_\Gamma)} \leq \|\varphi\|^q_{\mathcal{M}_{p,q}(T_\Gamma)} + \|\psi\|^q_{\mathcal{M}_{p,q}(T_\Gamma)}$.

2) If $p \leq q \leq r$, then $\|\varphi\psi\|_{\mathcal{M}_{p,r}(T_\Gamma)} \leq \|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \|\psi\|_{\mathcal{M}_{q,r}(T_\Gamma)}$.

3) For any real number $\alpha > 0$, $\|\varphi(\alpha\cdot)\|_{\mathcal{M}_{p,q}(T_\Gamma)} = \alpha^{n(1/q-1/p)} \|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)}$.

4) **Local Property.** If for any point of Γ^*, including the point at infinity, there exists a neighborhood in which $\varphi : \Gamma^* \to \mathbb{C}$ coincides with a function from $\mathcal{M}_{p,q}(T_\Gamma)$, then $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$.

Properties 1)–3) easily follow from Definition 1.4.2, while the Local Property will be proven later in Lemma 4.2.1. Moreover, Property 1) can also be extended to the case of an infinite sum. The precise statement is given in Proposition 4.2.1.

4.1 Some Auxiliary Results

For two vectors $a, b \in \mathbb{R}^n$ such that $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, and $-\infty < a_j < b_j < \infty$, we will consider the open and closed rectangles in $\mathbb{R}^n$:

$$(a,b)_n := \prod_{j=1}^n (a_j, b_j), \quad [a,b]_n := \prod_{j=1}^n [a_j, b_j].$$

We will also use the following notation

$$\mathcal{V}(a,b) := \{\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n : \nu_j = a_j \text{ or } b_j,\ j = 1, \ldots, n\}.$$

For $p \in (0, \infty]$, let us consider the p-th means of a function $f : T_B \to \mathbb{C}$:

$$m_p(f, y) := \|f(\cdot + iy)\|_p = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x+iy)|^p \, dx\right)^{1/p}, & p \in (0, \infty), \\ \sup_{x \in \mathbb{R}^n} |f(x+iy)|, & p = \infty, \end{cases} \quad y \in B.$$

We will need several statements of Hadamard three-lines-theorem type.

Lemmas 4.1.2 – 4.1.3 below are obtained through personal communications with Professor David C. Ullrich.

For an arbitrary set $E \subset \mathbb{R}^n$, let us denote $A(E)$ as the set of all functions continuous and bounded in E, and holomorphic in its interior, E^o. $SH(E)$ denotes the set of all functions continuous and bounded in E, and plurisubharmonic in E^o. Further, we will consider harmonic, subharmonic and plurisubharmonic functions in $\mathbb{C}^n$ assuming that they are so, as functions of two independent variables:

$$u(z) = u(x+iy) = u(x, y), \quad z = x+iy, \, x, y \in \mathbb{R}^n.$$

The following lemma is the Three-Lines Theorem for subharmonic functions. The proof could be found, e.g., in [81, Ch. 2, § 2.3, Corollary 2.3.6].

Lemma 4.1.1 *Suppose that u is continuous in $T_{[0,1]}$, subharmonic in $T_{(0,1)}$, and for some $C \in \mathbb{R}$ and $\alpha \in [0, \pi)$,*

$$u(x+iy) \leq Ce^{\alpha|x|}, \quad x \in \mathbb{R}, \, y \in [0,1]. \tag{4.1}$$

Then,

$$u(z) \leq \max\left(\sup_{t \in \mathbb{R}} u(t), \sup_{t \in \mathbb{R}} u(t+i)\right), \quad z \in T_{[0,1]}.$$

Note. The function $u(x, y) = e^{\pi x} \sin(\pi y)$ shows that Lemma 4.1.1 fails for $\alpha = \pi$.

We need a multivariate analog of this lemma.

Lemma 4.1.2 *Suppose that B is a convex set in $\mathbb{R}^n$ with nonempty interior. Assume that u is continuous, plurisubharmonic and bounded above in T_B. For $y_0, y_1 \in B$ and $t \in [0,1]$, set $y_t := (1-t)\, y_0 + t y_1$. Then,*

$$\sup_{x \in \mathbb{R}^n} u(x+iy_t) \leq \max\left(\sup_{x \in \mathbb{R}^n} u(x+iy_0), \sup_{x \in \mathbb{R}^n} u(x+iy_1)\right).$$

Proof. Fix $x_0 \in \mathbb{R}^n$ and for $s + it \in T_{[0,1]}$ define

$$F(s+it) = u(x_0 + iy_0 + (s+it)(y_1 - y_0)) = u(x_0 + s(y_1 - y_0) + iy_t).$$

It is obvious that F is continuous and bounded above in $T_{[0,1]}$. Furthermore, if $y_0, y_1 \in B^o$, then F is subharmonic in $T_{[0,1]}$. If one (or both) of y_0, y_1 belongs to ∂B, then (considering that B is convex) there exist sequences $\{y_{0,j}\}_{j=1}^{\infty}$ and $\{y_{1,j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} y_{k,j} = y_k, \quad y_{k,j} \in B^o, \quad k = 0, 1.$$

Since u is continuous in T_B, the functions

$$F_j(s+it) := u(x_0 + iy_{0,j} + (s+it)(y_{1,j} - y_{0,j}))$$

converge to F uniformly on any compact subset of $T_{[0,1]}$. This implies that F is subharmonic in $T_{(0,1)}$.

Applying Lemma 4.1.1 to F, we get

$$u(x_0 + iy_t) = F(it) \le \max\left(\sup_{s\in\mathbb{R}} F(s), \sup_{s\in\mathbb{R}} F(s+i)\right)$$
$$\le \max\left(\sup_{x\in\mathbb{R}^n} u(x+iy_0), \sup_{x\in\mathbb{R}^n} u(x+iy_1)\right).$$

Since $x_0 \in \mathbb{R}^n$ was chosen arbitrarily, the lemma is proven. □

Lemma 4.1.3 *Suppose that B is a convex set in $\mathbb{R}^n$ with nonempty interior. For $y_0, y_1 \in B$ and $t \in [0,1]$ set $y_t := (1-t)y_0 + ty_1$. If $f \in A(T_B)$, then*

$$m_p(f, y_t) \le \max(m_p(f, y_0), m_p(f, y_1)), \quad p \in (0, \infty].$$

Proof. Let us first suppose $p \in (0, \infty)$. For $0 < N < \infty$, define

$$u_N(z) = \int_{\{s\in\mathbb{R}^n : |s|<N\}} |f(z+s)|^p \, ds.$$

It is clear that $u_N(x+iy) \le (m_p(f,y))^p$, for any $x \in \mathbb{R}^n$, $y \in B$. Now $f \in A(T_B)$ implies that $|f|^p$ is subharmonic in T_B, and hence u_N is plurisubharmonic (in fact,

u_N is subharmonic in T_{B^o} – see, e.g., [81, Ch. 2, § 2.4, Th. 2.4.8]). Since f is bounded in T_B, u_N is also bounded there. As soon as f is also continuous in T_B, employing the Lebesgue Dominated Convergence Theorem, it is clear that u_N is continuous in T_B. Using Lemma 4.1.2, we get

$$u_N(iy_t) \le \max\left(\sup_{x\in\mathbb{R}^n} u_N(x+iy_0), \sup_{x\in\mathbb{R}^n} u_N(x+iy_1)\right)$$
$$\le \max\left((m_p(f,y_0))^p, (m_p(f,y_1))^p\right).$$

Since $(m_p(f,y_t))^p = \lim_{N\to\infty} u_N(iy_t)$, we are done.

For $p=\infty$, we should apply Lemma 4.1.2 to the function $u(z) := |f(z)|$. □

Note. Lemma 4.1.3 and hence previous statements cannot be considered new. It was mentioned in [86, Ch. III, § 6.1] that if $f \in H^p(T_B)$, then $\log \|f(\cdot + iy)\|_p$ is a convex function of $y \in B$. However, this source contains no references on the proof of this fact. This is the reason of why the lemma is proven here. Note that other results of such type for holomorphic and subharmonic functions could be found in [4, Ch. 3, § 3.5].

Now, we easily obtain

Corollary 4.1.1 *Suppose that B is a convex set in $\mathbb{R}^n$ with nonempty interior, and $f \in A(T_B)$. If K is a convex hull of a set $E \subset B$, then*

$$\sup_{y\in K} m_p(f,y) = \sup_{y\in E} m_p(f,y), \quad p\in(0,\infty].$$

Lemma 4.1.4 *Suppose that B is a convex set in $\mathbb{R}^n$ with nonempty interior, and $f \in A(T_B)$. If K is a convex hull of a set $E \subset B$, then for any $y_0 \in K^o$ and any p and q such that $0 < p \le q \le \infty$,*

$$m_q(f,y_0) \le \left(\frac{n!}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)(\operatorname{dist}(y_0,\partial K))^n}\right)^{\frac{1}{p}-\frac{1}{q}} \sup_{y\in E} m_p(f,y). \tag{4.2}$$

Proof. For $q = p$, the statement is just Corollary 4.1.1. Since $p = \infty$ implies $q = \infty$, whence $p = q$ again, we will consider the case $0 < p < q \leq \infty$. We will also suppose that the supremum in the right-hand side of (4.2) is finite, and that $K^o \neq \emptyset$. Otherwise, the statement is void.

Let us fix $y_0 \in K^0$. We can use the approach of [86, Ch. III, § 2, Lemma 2.12]. Let us fix an arbitrary x_0 in $\mathbb{R}^n$ and let $\varepsilon := \operatorname{dist}(y_0, \partial K) > 0$. Then, $B_n(y_0, \varepsilon) \subset K^o$ (here $B_n(y_0, \varepsilon)$ is the ball in $\mathbb{R}^n$ with the center at y_0 and of radius ε). If Ω_m denotes the volume of the unit ball in $\mathbb{R}^m$, then using the subharmonicity of $|f|^p$, we get

$$\begin{aligned} |f(x_0 + iy_0)|^p &\leq \frac{1}{\varepsilon^{2n}\Omega_{2n}} \int_{B_{2n}(x_0,y_0,\varepsilon)} |f(x+it)|^p \, dx\, dt \\ &\leq \frac{1}{\varepsilon^{2n}\Omega_{2n}} \int_{T_{B_n(y_0,\varepsilon)}} |f(x+it)|^p \, dx\, dt. \end{aligned} \tag{4.3}$$

Corollary 4.1.1 justifies changing the order of integration in (4.3), and we obtain

$$|f(x_0 + iy_0)|^p \leq \frac{(\max_{y\in E} m_p(f,y))^p}{\varepsilon^{2n}\Omega_{2n}} \int_{B_n(y_0,\varepsilon)} dt = \frac{(\max_{y\in E} m_p(f,y))^p \, \Omega_n}{\varepsilon^n \Omega_{2n}}.$$

Since $x_0 \in \mathbb{R}^n$ was taken arbitrarily, we get

$$m_\infty(f, y_0) \leq \left(\frac{\Omega_n}{\varepsilon^n \Omega_{2n}}\right)^{\frac{1}{p}} \sup_{y\in E} m_p(f,y).$$

Now, for $q > p$, using the last inequality, we have

$$m_q(f,y_0) \leq (m_\infty(f,y_0))^{\frac{q-p}{q}} \left(\int_{\mathbb{R}^n} |f(x+iy_0)|^p \, dx\right)^{\frac{1}{q}} \leq \left(\frac{\Omega_n}{\varepsilon^n \Omega_{2n}}\right)^{\frac{1}{p}-\frac{1}{q}} \sup_{y\in E} m_p(f,y).$$

Since $\varepsilon = \operatorname{dist}(y_0, \partial K)$, and $\Omega_m = \frac{\pi^{m/2}}{\Gamma(m/2+1)}$, inequality (4.2) follows immediately. □

Applying Lemma 4.1.4 to $B = [a,b]_n$ and $E = \mathcal{V}(a,b)$, we obtain

Corollary 4.1.2 *Assume f is holomorphic in $T_{(a,b)_n}$ as well as bounded and continuous in $T_{[a,b]_n}$. Then, for any $0 < p \leq q \leq \infty$, the following inequality folds*

$$\sup_{y\in(a,b)_n} \|f(\cdot + iy)\|_q$$

$$\leq \left(\frac{n!}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)\left(\min_{j=1,\ldots,n}\left(\min\left(y_j - a_j, b_j - a_j\right)\right)\right)^n}\right)^{\frac{1}{p}-\frac{1}{q}} \max_{\nu\in\mathcal{V}(a,b)} \|f(\cdot + i\nu)\|_p.$$

Let us return to $V_n(\Gamma)$ introduced in Section 1.4 (see (1.20)). As soon as the set $\left\{(x_1,\ldots,x_n)\in\mathbb{R}^{n^2}:\,x_j\in\overline{\Gamma},\,|x_j|=1,\,\forall j=1,\ldots,n\right\}$ is compact in $\mathbb{R}^{n^2}$, then the maximum in (1.20) is attained on some set of vectors $e_1,\ldots,e_n$. Since Γ is also open and nonempty, then $V_n(\Gamma)>0$. Although the set of vectors $e_1,\ldots,e_n$ may not be unique, let us fix one such set $e:=\{e_1,\ldots,e_n\}$. We will consider only this fixed set in the following argument. Consider the linear map

$$\Psi_e:=\begin{pmatrix} e_{11} & \ldots & e_{n1}\\ \vdots & \ddots & \vdots\\ e_{1n} & \ldots & e_{nn}\end{pmatrix},$$

and denote $\Gamma^e:=\Psi_e\left(\left(\mathbb{R}^n_+\right)^o\right)$ (here $\mathbb{R}^n_+$ is the first octant in $\mathbb{R}^n$, as usual, i.e., $\mathbb{R}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n:\,x_j\geq 0,\,\forall j=1,\ldots,n\}$). Since $|\det\Psi_e|=n!V_n(\Gamma)>0$, this map is a bijection of $\mathbb{R}^n$ onto $\mathbb{R}^n$, $\left(\mathbb{R}^n_+\right)^o$ onto Γ^e, and $\mathbb{R}^n_+$ onto $\overline{\Gamma^e}$. It is also clear that $\Gamma^e\subset\Gamma$, and it is also an open cone.

Let us denote a *translation* of a cone Γ by a vector ζ by $\Gamma_\zeta:=\{x+\zeta:\,x\in\Gamma\}$.

Lemma 4.1.5 *Let Γ be an open cone in $\mathbb{R}^n$, $n\in\mathbb{N}$. Assume that r and R are some points in $\left(\mathbb{R}^n_+\right)^o$ such that $r_j<R_j$, $\forall j=1,\ldots,n$. If a function F is holomorphic in $T_{\Psi_e\left((r,R)_n\right)}$ as well as bounded and continuous in $T_{\Psi_e\left([r,R]_n\right)}$, then for any $y\in\Psi_e\left((r,R)_n\right)$, and for any p and q such that $0<p\leq q\leq\infty$, the following inequality holds true*

$$\|F(\cdot+iy))\|_q$$
$$\leq\left(\frac{1}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)V_n(\Gamma)\left(\min_{j=1,\ldots,n}\left(\min\left(\left(\Psi_e^{-1}y\right)_j-r_j,R_j-\left(\Psi_e^{-1}y\right)_j\right)\right)\right)^n}\right)^{\frac{1}{p}-\frac{1}{q}}$$
$$\times\max_{\nu\in\mathcal{V}(r,R)}\|F(\cdot+i\Psi_e\nu)\|_p. \tag{4.4}$$

To prove the lemma, we only need to apply Corollary 4.1.2 to the function $G(z)=F(\Psi_e z)$, with $a=r$, $b=R$, $y=\Psi_e^{-1}y$, and get back to F.

Lemma 4.1.6 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $\varphi : \Gamma^* \to \mathbb{C}$ be a Lebesgue measurable function. Assume that there exists a Lebesgue measurable function $\varphi^* : \mathbb{R}^n \to \mathbb{C}$ such that*

(i) $\varphi^*(x) = \varphi(x)$ almost everywhere on Γ^*;

(ii) $\varphi^*(\cdot)\, e^{2\pi(\delta,\cdot)} \in L^1(\mathbb{R}^n)$, for some $\delta \in \Gamma$.

Then, for any function f, belonging to $H^p(T_\Gamma)$ with some $p \in (0,1]$, the following equality holds true

$$M_\varphi(f;x) := \int_{\Gamma^*} \varphi(t)\, \widehat{f}(t)\, e^{2\pi i(x,t)}\, dt = \int_{\mathbb{R}^n} f(x+t+i\delta)\, \widehat{\varphi^*}(t+i\delta)\, dt, \quad x \in \mathbb{R}^n. \tag{4.5}$$

Proof. Let us fix an arbitrary $x \in \mathbb{R}^n$. Since $\varphi^* = \varphi$ a.e. on Γ^*, and $\operatorname{supp} \widehat{f} \subset \Gamma^*$, then

$$M_\varphi(f;x) = \int_{\mathbb{R}^n} \varphi^*(t)\, \widehat{f}(t)\, e^{2\pi i(x,t)}\, dt.$$

As soon as $\varphi^*(\cdot)\, e^{2\pi(\delta,\cdot)} \in L^1(\mathbb{R}^n)$, and $f_\delta \in L^1(\mathbb{R}^n)$ (as we already noticed), using Tonelli's theorem, it is easy to see that the function $G(t,u) := \varphi^*(t)\, e^{2\pi(\delta,t)} f_\delta(u)$ belongs to $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore the function $\mathcal{G}(t,u) := G(t,u)\, e^{-2\pi i(u-x,t)}$ is also there. Furthermore, let us write the Fourier transform (see Definition 1.4.1) of f with our δ:

$$\widehat{f}(t) = e^{2\pi(\delta,t)} \widehat{f_\delta}(t) = e^{2\pi(\delta,t)} \int_{\mathbb{R}^n} f_\delta(u)\, e^{-2\pi i(u,t)}\, du, \quad t \in \mathbb{R}^n.$$

An application of Fubini's theorem to $\mathcal{G}$ shows that $\varphi^* \widehat{f} \in L^1(\mathbb{R}^n)$, and allows us to change the order of integration in the equation below:

$$M_\varphi(f;x) = \int_{\mathbb{R}^n} \left(\varphi^*(t)\, e^{-2\pi i(-x+i\delta,t)} \int_{\mathbb{R}^n} f_\delta(u)\, e^{-2\pi i(u,t)}\, du \right) dt$$

$$= \int_{\mathbb{R}^n} \left(f_\delta(u) \int_{\mathbb{R}^n} \varphi^*(t)\, e^{-2\pi i(u-x+i\delta,t)}\, dt \right) du = \int_{\mathbb{R}^n} f_\delta(u)\, \widehat{\varphi^*}(u-x+i\delta)\, du$$

$$= \int_{\mathbb{R}^n} f_\delta(t+x)\, \widehat{\varphi^*}(t+i\delta)\, dt = \int_{\mathbb{R}^n} f(t+x+i\delta)\, \widehat{\varphi^*}(t+i\delta)\, dt.$$

Since $x \in \mathbb{R}^n$ was chosen arbitrarily, (4.5) holds. □

In the univariate case, (4.5) was proven in [84, Proof of Proposition 1].

Following [85, Appendix B.2], for a Lebesgue measurable function h on $\mathbb{R}^n$, we will consider its *distribution function*

$$\lambda_h(\alpha) := m\{x \in \mathbb{R}^n : |h(x)| > \alpha\}, \quad \alpha \geq 0,$$

with m – the Lebesgue measure on $\mathbb{R}^n$, as well as the *non-increasing rearrangement* of h given by

$$h^*(t) := \inf\{\alpha : \lambda_h(\alpha) \leq t\}, \quad t \geq 0.$$

As shown in [85, Appendix B.2], both functions λ_h and h^* are non-negative, non-increasing and right continuous. Moreover, h and h^* have the same distribution function, and

$$\int_{\mathbb{R}^n} |h(x)|^p \, dx = \int_0^\infty (h^*(t))^p \, dt, \quad p \in (0,\infty). \tag{4.6}$$

For a function $\varphi \in L^2(\mathbb{R}^n)$, let us denote

$$a_\sigma(\varphi)_2 := \inf\left\{\|\varphi - \psi\|_2 : \psi \in L^2(\mathbb{R}^n), m\left(\operatorname{supp}\widehat{\psi}\right) \leq \sigma\right\}.$$

Since the Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$, then $\|\varphi - \psi\|_2 = \left\|\widehat{\varphi} - \widehat{\psi}\right\|_2$, whence

$$\begin{aligned} a_\sigma(\varphi)_2 &= \inf\left\{\left(\int_{\mathbb{R}^n \setminus E} |\widehat{\varphi}(x)|^2 \, dx\right)^{\frac{1}{2}} : m(E) \leq \sigma\right\} \\ &\leq \left(\int_{\mathbb{R}^n \setminus \left[\frac{-\sigma^{1/n}}{2}, \frac{\sigma^{1/n}}{2}\right]_n} |\widehat{\varphi}(x)|^2 \, dx\right)^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

We also need a refined version of (4.6) that is given by the following statement. Although I am not sure, to the best of my knowledge, this result is new.

Lemma 4.1.7 *Let $f \in L^p(\mathbb{R}^n)$ for some $p \in (0,\infty)$, and f^* be its non-increasing rearrangement. Then, for any $\sigma > 0$,*

$$\sup_{E:\, m(E) \leq \sigma} \int_E |f(x)|^p \, dx = \int_0^\sigma (f^*(t))^p \, dt. \tag{4.8}$$

Proof. Let us take an arbitrary measurable set E so that $m(E) \le \sigma$, and consider $h(x) := f(x)\chi_E(x)$, where χ_E is the indicator of E. Obviously, $h^*(t) \le f^*(t)$, $t \ge 0$. It is also clear that $\lambda_h(\alpha) \le \sigma$, for any $\alpha \ge 0$. Hence, $h^*(t) = 0$, $t \ge \sigma$. Now, from (4.6) we obtain

$$\begin{aligned} \int_E |f(x)|^p \, dx &= \int_{\mathbb{R}^n} |h(x)|^p \, dx = \int_0^\infty (h^*(t))^p \, dt \\ &= \int_0^\sigma (h^*(t))^p \, dt \le \int_0^\sigma (f^*(t))^p \, dt. \end{aligned} \tag{4.9}$$

Since E was chosen arbitrarily with the only requirement $m(E) \le \sigma$, then

$$\sup_{E:\, m(E) \le \sigma} \int_E |f(x)|^p \, dx \le \int_0^\sigma (f^*(t))^p \, dt. \tag{4.10}$$

Let us construct a set on which the supremum is attended. First, assume that f is bounded. Define

$$A := \sup\{\alpha : \, m(\{x \in \mathbb{R}^n : \, |f(x)| \ge \alpha\}) \ge \sigma\}.$$

If $A = 0$, then $m(\mathcal{B}_m) < \sigma$ for each $\mathcal{B}_m := \{x \in \mathbb{R}^n : \, |f(x)| \ge 1/m\}$, $m \in \mathbb{N}$. Hence,

$$m(\operatorname{supp} f) = m\left(\bigcup_{m=1}^\infty \mathcal{B}_m\right) = \lim_{m\to\infty} m(\mathcal{B}_m) \le \sigma.$$

Thus, we could take $E = \operatorname{supp} f$, so that (4.9) becomes an equality, and (4.8) follows immediately.

Now, we will consider the case $A > 0$. Let us denote

$$M_f := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| = \inf\{a : \, m(\{x \in \mathbb{R}^n : \, |f(x)| > a\}) = 0\}.$$

It is clear that $A \le M_f$, and if

$$m(\{x \in \mathbb{R}^n : \, |f(x)| = M_f\}) < \sigma, \tag{4.11}$$

then $A < M_f$.

Let us denote

$$\mathcal{U}_\alpha := \{x \in \mathbb{R}^n : \, |f(x)| \ge \alpha\}, \ \alpha > 0, \quad \widetilde{\mathcal{U}}_A := \bigcup_{\alpha \in (A, M_f]} \mathcal{U}_\alpha = \{x \in \mathbb{R}^n : \, |f(x)| > A\}.$$

Then $\alpha_1 > \alpha_2$ implies $\mathcal{U}_{\alpha_1} \subset \mathcal{U}_{\alpha_2}$, and all $\mathcal{U}_\alpha$'s are Lebesgue measurable. Since $A = \sup\{\alpha : m(\mathcal{U}_\alpha) \geq \sigma\}$, then $\alpha > A$ implies $m(\mathcal{U}_\alpha) < \sigma$. Therefore, using the continuity of Lebesgue measure from below, we get $m\left(\widetilde{\mathcal{U}}_A\right) \leq \sigma$.

Since $m\left(\widetilde{\mathcal{U}}_A\right) \leq \sigma$ and $m(\mathcal{U}_A) \geq \sigma$, we can use the continuity of Lebesgue measure to choose a Lebesgue measurable set E so that $\widetilde{\mathcal{U}}_A \subset E \subset \mathcal{U}_A$, and $m(E) = \sigma$.

If requirement (4.11) is not satisfied, then take E to be any subset of $\mathcal{U}_{M_f}$ with $m(E) = \sigma$.

Let us consider $g := f\chi_E$, and take an arbitrary $\alpha \geq A$ (remember that $A > 0$). Then

$$\lambda_f(\alpha) = \lambda_g(\alpha), \quad \alpha \geq A. \tag{4.12}$$

Also note that $\lambda_f(\alpha) = \lambda_g(\alpha) = 0$, $\alpha > M_f$. Moreover, from the definition of A, we get

$$\lambda_f(\alpha) = m(\{x \in \mathbb{R}^n : |f(x)| > \alpha\}) \geq m(\{x \in \mathbb{R}^n : |f(x)| \geq A\}) \geq \sigma, \quad \alpha < A. \tag{4.13}$$

Since $\widetilde{\mathcal{U}}_A \subset E \subset \mathcal{U}_A$, then for any $\alpha \in (0, A)$, $x \in E$ implies $|g(x)| = |f(x)| \geq A > \alpha$. From another side, if $|g(x)| > \alpha$ then $|f(x)| > \alpha$. Hence, $x \in \widetilde{\mathcal{U}}_A \subset E$. Thus,

$$\{x \in \mathbb{R}^n : |g(x)| > \alpha\} = E, \quad 0 < \alpha < A.$$

Therefore,

$$\lambda_g(\alpha) = m(E) = \sigma, \quad 0 < \alpha < A. \tag{4.14}$$

Considering (4.12), (4.13) and (4.14), for $t \in [0, \sigma)$, we obtain

$$g^*(t) = \inf\{\alpha : \lambda_g(\alpha) \leq t\} = (4.14) = \inf\{\alpha : \alpha \geq A, \lambda_g(\alpha) \leq t\} = (4.12)$$

$$= \inf\{\alpha : \alpha \geq A, \lambda_f(\alpha) \leq t\} = (4.13) = \inf\{\alpha : \lambda_f(\alpha) \leq t\} = f^*(t). \tag{4.15}$$

Since also $\lambda_g(\alpha) \leq m(\operatorname{supp} g) = \sigma$, for any $\alpha \geq 0$, then $g^*(t) = 0$ when $t \geq \sigma$. Now, using (4.15) and (4.6), we get

$$\begin{aligned}\int_E |f(x)|^p \, dx &= \int_{\mathbb{R}^n} |g(x)|^p \, dx = \int_0^\infty (g^*(t))^p \, dt \\ &= \int_0^\sigma (g^*(t))^p \, dt = \int_0^\sigma (f^*(t))^p \, dt.\end{aligned}$$

Hence,

$$\sup_{E:\, m(E)\le\sigma} \int_E |f(x)|^p \, dx \ge \int_0^\sigma (f^*(t))^p \, dt,$$

which completes the proof for a bounded function f.

Let us get rid of this restriction. Since $f \in L^p(\mathbb{R}^n)$, then

$$m(\{x \in \mathbb{R}^n : |f(x)| > f^*(\varepsilon)\}) \le \varepsilon, \quad \varepsilon > 0. \tag{4.16}$$

Let us consider functions

$$f_{(\varepsilon)}(x) := \min(|f(x)|, f^*(\varepsilon)\}, \quad x \in \mathbb{R}^n,\, \varepsilon > 0.$$

Clearly, they are in $L^p(\mathbb{R}^n)$ and also bounded. Moreover, (4.16) implies that $f_{(\varepsilon)}$ coincides with $|f|$ everywhere except some set of Lebesgue measure not more than ε.

Since for any $\alpha > 0$,

$$\{x \in \mathbb{R}^n : |f(x)| > \alpha\} \subset \left\{x \in \mathbb{R}^n : f_{(\varepsilon)}(x) > \alpha\right\} \cup \left\{x \in \mathbb{R}^n : |f(x)| > f_{(\varepsilon)}(x)\right\},$$

then $\lambda_f(\alpha) \le \lambda_{f_{(\varepsilon)}}(\alpha) + \varepsilon$. Hence,

$$f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \le t\} \le \inf\left\{\alpha : \lambda_{f_{(\varepsilon)}}(\alpha) \le t - \varepsilon\right\} = f^*_{(\varepsilon)}(t-\varepsilon), \quad t \ge \varepsilon.$$

Thus, applying (4.8) to the bounded function $f_{(\varepsilon)}$, and considering that $f^* \ge 0$, $\left|f_{(\varepsilon)}\right| \le |f|$, we get

$$\begin{aligned}\int_\varepsilon^\sigma (f^*(t))^p \, dt &\le \int_\varepsilon^{\sigma+\varepsilon} (f^*(t))^p \, dt = \int_\varepsilon^{\sigma+\varepsilon} \left(f^*_{(\varepsilon)}(t-\varepsilon)\right)^p dt = \int_0^\sigma \left(f^*_{(\varepsilon)}(t)\right)^p dt \\ &= \sup_{E:\, m(E)\le\sigma} \int_E \left|f_{(\varepsilon)}(x)\right|^p dx \le \sup_{E:\, m(E)\le\sigma} \int_E |f(x)|^p \, dx.\end{aligned} \tag{4.17}$$

Applying Fatou's lemma, we can pass to the limit as $\varepsilon \to 0+$ to conclude

$$\int_0^\sigma (f^*(t))^p \, dt \leq \sup_{E:\, m(E) \leq \sigma} \int_E |f(x)|^p \, dx.$$

Since the inverse inequality (4.10) was obtained without any assumption on boundedness of f, this completes the proof. □

The next result follows immediately from the previous lemma, (4.7), and (4.6).

Corollary 4.1.3 *If $f \in L^2(\mathbb{R}^n)$, then*

$$a_\sigma(f)_2 = \left(\int_\sigma^\infty \left(\widehat{f}^*(t) \right)^2 dt \right)^{\frac{1}{2}}, \quad \sigma \geq 0.$$

Note that this statement is contained in [92, Proof of Theorem 2]. However, the source does not contain its detailed proof.

Corollary 4.1.4 *Let $f \in L^2(\mathbb{R}^n)$. Then, for any $p \in (0, \infty)$, the following inequality holds true*

$$\int_{\mathbb{R}^n} \left| \widehat{f}(x) \right|^p dx \leq 2 \int_0^\infty \left(\frac{a_\sigma(f)_2}{\sqrt{\sigma}} \right)^p d\sigma.$$

Proof. The reasoning of this proof is the same as used in the proof of Theorem 2 in [92] just mentioned.

Since $f \in L^2(\mathbb{R}^n)$, then $\widehat{f}$ also belongs to the same space. Applying (4.7), Corollary 4.1.3, and considering that $\widehat{f}^*$ is non-increasing and non-negative, we get

$$\widehat{f}^*(2t) \leq \frac{1}{\sqrt{t}} \left(\int_t^{2t} \left(\widehat{f}^*(u) \right)^2 du \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{t}} \left(\int_t^\infty \left(\widehat{f}^*(u) \right)^2 du \right)^{\frac{1}{2}} = \frac{a_t(f)_2}{\sqrt{t}}, \quad t > 0.$$

Therefore, (4.6) implies

$$\int_{\mathbb{R}^n} \left| \widehat{f}(x) \right|^p dx = \int_0^\infty \left(\widehat{f}^*(t) \right)^p dt = 2 \int_0^\infty \left(\widehat{f}^*(2\sigma) \right)^p d\sigma \leq 2 \int_0^\infty \left(\frac{a_\sigma(f)_2}{\sqrt{\sigma}} \right)^p d\sigma$$

that completes the proof. □

We also need a Nikol'skiĭ type inequality in a pointwise form. Unfortunately, it is not true without additional assumptions. The following statement is one of such 'constrained' forms.

Proposition 4.1.1 *Assume $\varphi \in C^r(\mathbb{R}^n)$, $r, n \in \mathbb{N}$, and there exist non-increasing functions $F, G : \mathbb{R}_+ \to \mathbb{R}_+$ such that for some $j = 1, \ldots, n$,*

$$|\varphi(x)| \leq F(|x|), \quad \left|\frac{\partial^r \varphi}{\partial x_j^r}(x)\right| \leq G(|x|), \quad x \in \mathbb{R}^n.$$

Then

$$\left|\frac{\partial^k \varphi}{\partial x_j^k}(x)\right| \leq \left(\frac{C_0 r}{k}\right)^k (F(|x|))^{1-\frac{k}{r}} (G(|x|))^{\frac{k}{r}}, \quad x \in \mathbb{R}^n,\ k = 1, \ldots, r-1,$$

where C_0 is an absolute constant.

Proof. Suppose $g \in C^r(\mathbb{R})$, and for some $a, b \geq 0$,

$$|g(x)| \leq F\left(\sqrt{a + x^2}\right), \quad \left|g^{(r)}(x)\right| \leq G\left(\sqrt{b + x^2}\right), \quad x \in \mathbb{R}. \tag{4.18}$$

Then, fixing some $x \geq 0$ and applying the Nikol'skiĭ type inequality on $\mathbb{R}_+$ (see, e.g., [91, Chapter 3, § 3.10.2, Estimate (9)]) to the function $h(t) := g(t + x)$, we obtain

$$\begin{aligned}
\sup_{t \geq x}\left|g^{(k)}(t)\right| &= \sup_{t \geq 0}\left|h^{(k)}(t)\right| \leq \left(\frac{C_0 r}{k}\right)^k \left(\sup_{t \geq 0}|h(t)|\right)^{1-\frac{k}{r}} \left(\sup_{t \geq 0}\left|h^{(r)}(t)\right|\right)^{\frac{k}{r}} \\
&= \left(\frac{C_0 r}{k}\right)^k \left(\sup_{t \geq x} F\left(\sqrt{a + t^2}\right)\right)^{1-\frac{k}{r}} \left(\sup_{t \geq x} G\left(\sqrt{b + t^2}\right)\right)^{\frac{k}{r}}.
\end{aligned}$$

Since F and G are non-increasing, then $F\left(\sqrt{a + x^2}\right)$ and $G\left(\sqrt{b + x^2}\right)$ are non-increasing on $\mathbb{R}_+$, whence, for $x \geq 0$,

$$\left|g^{(k)}(x)\right| \leq \left(\frac{C_0 r}{k}\right)^k \left(F\left(\sqrt{a + x^2}\right)\right)^{1-\frac{k}{r}} \left(G\left(\sqrt{b + x^2}\right)\right)^{\frac{k}{r}}. \tag{4.19}$$

If $x < 0$, then considering $\mathcal{G}(t) := g(-t)$, we deduce that (4.19) holds for $x \in \mathbb{R}$.

Now, take any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and consider

$$g(t) := \varphi(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n).$$

Applying (4.19) to this function with $a = b = \sum_{l=1,\ldots,n;\, l \neq j} x_l^2$, we get

$$\left|\frac{\partial^k \varphi}{\partial x_j^k}(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)\right| = \left|g^{(k)}(t)\right|$$

$$\le \left(\frac{C_0 r}{k}\right)^k \left(F\left(\sqrt{x_1^2+\cdots+x_{j-1}^2+t^2+x_{j+1}^2+\cdots+x_n^2}\right)\right)^{1-\frac{k}{r}}$$

$$\times \left(G\left(\sqrt{x_1^2+\cdots+x_{j-1}^2+t^2+x_{j+1}^2+\cdots+x_n^2}\right)\right)^{\frac{k}{r}}, \quad t \in \mathbb{R}.$$

Taking $t = x_j$ completes the proof. □

Corollary 4.1.5 *Assume $\varphi \in C^r(\mathbb{R}^n)$, $r, n \in \mathbb{N}$. If for some non-negative α, β, A and B, the following growth estimates*

$$|\varphi(x)| \le \frac{A}{1+|x|^\alpha}; \quad \left|\frac{\partial^r \varphi}{\partial x_j^r}(x)\right| \le \frac{B}{1+|x|^\beta}, \quad x \in \mathbb{R}^n,\ j = 1, \ldots, n,$$

are satisfied, then for $k = 1, \ldots, r-1$,

$$\left|\frac{\partial^k \varphi}{\partial x_j^k}(x)\right| \le \left(\frac{C_0 r}{k}\right)^k \left(\frac{A}{1+|x|^\alpha}\right)^{1-\frac{k}{r}} \left(\frac{B}{1+|x|^\beta}\right)^{\frac{k}{r}}, \quad x \in \mathbb{R}^n, \tag{4.20}$$

where C_0 is an absolute constant.

Note that (4.20) is used in [94, Proof of Theorem 3b], but its justification is absent there.

Equipped with these statements, we can proceed to the proofs of the main results of this chapter (and Section 1.4).

4.2 Conditions for Fourier Multipliers

4.2.1 Multipliers with Compactly Supported Kernel

The goal of this subsection is to prove Theorem 1.4.2. First, we need Proposition 4.2.2, which is rather technical, but it can be used for obtaining various conditions for Fourier multipliers. Let us start with a generalization of the basic Property 1) of a multiplier given by the following statement.

Proposition 4.2.1 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, $0 < p \le q \le 1$, and let $\{\varphi_m\}_{m=1}^\infty$ be a sequence of Fourier multipliers, $\varphi_m \in \mathcal{M}_{p,q}(T_\Gamma)$. Assume that $\sum_{m=1}^\infty |\varphi_m| \in L^\infty(\Gamma^*)$ and $\varphi(x) = \sum_{m=1}^\infty \varphi_m(x)$ almost everywhere on Γ^*.*

If $\sum_{m=1}^{\infty} \|\varphi_m\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q < \infty$ *then* $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, *and*

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \leq \left(\sum_{m=1}^{\infty} \|\varphi_m\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q \right)^{\frac{1}{q}}.$$

Proof. Since $\sum_{m=1}^{\infty} \varphi_m(x)$ converges to φ almost everywhere, and the Lebesgue measure in $\mathbb{R}^n$ is complete, then φ is measurable. Let us take an arbitrary $f \in H^p(T_\Gamma)$ and fix an arbitrary $y \in \Gamma$. Then, since the inversion formula (1.18) is true, we have $\widehat{f}(t)\, e^{-2\pi(y,t)} \in L^1(\mathbb{R}^n)$. Using the assumption that $\sum_{m=1}^{\infty} |\varphi_m| \in L^\infty(\Gamma^*)$, we can apply the Lebesgue Dominated Convergence Theorem to derive

$$\begin{aligned} \sum_{m=1}^{\infty} \int_{\Gamma^*} \varphi_m(t)\, \widehat{f}(t)\, e^{2\pi i(x+iy,t)}\, dt &= \int_{\Gamma^*} \varphi(t)\, \widehat{f}(t)\, e^{2\pi i(x+iy,t)}\, dt \\ &= F_\varphi[f](x+iy), \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.21}$$

Again, since $\varphi \in L^\infty(\Gamma^*)$ and $\widehat{f}(t)\, e^{-2\pi(y,t)} \in L^1(\mathbb{R}^n)$, then the Lebesgue Dominated Convergence Theorem implies $F_\varphi[f](\cdot + iy)$ is continuous on $\mathbb{R}^n$, whence Lebesgue measurable.

Since all φ_m's belong to $\mathcal{M}_{p,q}(T_\Gamma)$, then $|F_{\varphi_m}[f](\cdot + iy)|^q \in L^1(\mathbb{R}^n)$, for any m. As soon as

$$\sum_{m=1}^{\infty} \int_{\mathbb{R}^n} |F_{\varphi_m}[f](x+iy)|^q\, dx \leq \sum_{m=1}^{\infty} \|\varphi_m\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q \|f\|_{H^p}^q < \infty,$$

the Dominated Convergence Theorem implies that the series $\sum_{m=1}^{\infty} |F_{\varphi_m}[f](x+iy)|^q$ converges almost everywhere on $\mathbb{R}^n$ to a function from $L^1(\mathbb{R}^n)$ (see, e.g., [31, Ch. 2, § 2.3, Theorem 2.25]).

Using the triangle inequality for the power q, (4.21) implies

$$\sum_{m=1}^{\infty} |F_{\varphi_m}[f](x+iy)|^q \geq \left| \sum_{m=1}^{\infty} F_{\varphi_m}[f](x+iy) \right|^q = |F_\varphi[f](x+iy)|^q,$$

and we immediately conclude that $|F_\varphi[f](\cdot + iy)|^q \in L^1(\mathbb{R}^n)$, and

$$\|F_\varphi[f](\cdot + iy)\|_q^q \leq \sum_{m=1}^{\infty} \|\varphi_m\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q \|f\|_{H^p}^q.$$

Passing to $\sup_{y\in\Gamma}$ in the last inequality, we get the statement. □

Proposition 4.2.2 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. Assume $f \in H^p(T_\Gamma)$ for some $p \in (0,1]$, and $\varphi(\cdot)\, e^{2\pi\alpha\sqrt{n}|\cdot|} \in L^1(\mathbb{R}^n)$, for some $\alpha > 0$. Then, for any $q \in [p,1]$, and $r, R \in \mathbb{R}_+^n$ such that $0 < r_j < R_j$, $j = 1, \dots, n$, and $|R| \le \alpha$, the following inequality holds*

$$\begin{aligned} \|M_\varphi(f)\|_{H^q} &\le 2^{n\left(\frac{1}{p}+\frac{1}{q}-1\right)} \left(\frac{1}{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right) V_n(\Gamma) \left(\min_{j=1,\dots,n}(R_j - r_j)\right)^n} \right)^{\frac{1}{p}-1} \\ &\times \max_{\nu \in \mathcal{V}(r,R)} \|\widehat{\varphi}(\cdot + i\Psi_e \nu)\|_q \|f\|_{H^p}. \end{aligned} \tag{4.22}$$

Proof. For any $y \in [r,R]_n$, $(\Psi_e y)_j = \sum_{k=1}^n e_{kj} y_k$. Since $e_1, \dots, e_n$ are unit vectors, and $y \in [r,R]_n \subset \mathbb{R}_+^n$, applying Cauchy-Schwartz inequality, we get

$$|\Psi_e y| \le \sqrt{n}\,|y| \le \sqrt{n}\,|R| \le \sqrt{n}\alpha. \tag{4.23}$$

As soon as $\varphi(\cdot)\, e^{2\pi\alpha\sqrt{n}|\cdot|} \in L^1(\mathbb{R}^n)$, the function

$$\widehat{\varphi}(\Psi_e(x+iy)) = \int_{\mathbb{R}^n} \varphi(t)\, e^{2\pi(\Psi_e y, t)} e^{-2\pi i(\Psi_e x, t)}\, dt$$

is holomorphic in $T_{(r,R)_n}$ as well as continuous and bounded in $T_{[r,R]_n}$. Since Ψ_e is a nonsingular linear transformation, $\widehat{\varphi}$ is holomorphic in $T_{\Psi_e\left((r,R)_n\right)}$, continuous and bounded in $T_{\Psi_e\left([r,R]_n\right)}$.

We will also use the fact that if $f \in H^p(T_\Gamma)$ for some p, then, for any $w \in \Gamma$, $f_w \in H^{p_0}(T_\Gamma)$ with any $p_0 \in [p, \infty]$ (see Lemma 1.4.1). Hence, $f_w \in H^p(T_\Gamma)$, and using the definition of Fourier transform (1.17) with $\delta = w$, we have

$$\begin{aligned} M_\varphi(f; x+iw) &= \int_{\Gamma^*} \varphi(t)\, \widehat{f}(t)\, e^{2\pi i(x+iw,t)}\, dt = \int_{\Gamma^*} \varphi(t)\, \widehat{f}(t)\, e^{-2\pi(w,t)} e^{2\pi i(x,t)}\, dt \\ &= \int_{\Gamma^*} \varphi(t)\, \widehat{f_w}(t)\, e^{2\pi i(x,t)}\, dt = M_\varphi(f_w; x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.24}$$

Let us choose an arbitrary $\rho \in (r,R)_n$. Then, (4.23) and Cauchy-Schwarz inequality imply

$$|\varphi(t)|\, e^{2\pi(\Psi_e \rho, t)} \le |\varphi(t)|\, e^{2\pi|\Psi_e \rho||t|} \le |\varphi(t)|\, e^{2\pi\sqrt{n}\alpha|t|} \in L^1(\mathbb{R}^n).$$

Using (4.24) and applying Lemma 4.1.6 with $\delta = \Psi_e \rho$ and $\varphi^* = \varphi$ to the function f_w, we conclude

$$M_{\varphi}(f; x + iw) = \int_{\Gamma^*} f_w(x + u + i\Psi_e\rho)\, \widehat{\varphi}(u + i\Psi_e\rho)\, du, \quad x \in \mathbb{R}^n.$$

In the following, we will suppose that the maximum on the right-hand side of (4.22) is finite (otherwise, (4.22) is trivial). Under this assumption, we have that

$$\|M_{\varphi}(f; \cdot + iw)\|_q^q = \int_{\mathbb{R}^n} \|g_{\Psi_e\rho}(w, x; \cdot)\|_1^q\, dx,$$

where $g(w, x; \cdot) := f_w(x + \cdot)\, \widehat{\varphi}(\cdot)$ (recall that $g_\beta(z) = g(z + i\beta)$). If we consider this function as a function of the last argument with fixed x and w, then it obviously satisfies the conditions of Lemma 4.1.5. Applying this statement with $q = 1$, $p = p$, $f(\cdot) = g(w, x; \cdot)$, we continue our estimates with

$$\|M_{\varphi}(f; \cdot + iw)\|_q^q \le \Theta \int_{\mathbb{R}^n} \max_{\nu \in \mathcal{V}(r,R)} \|g(w, x; \cdot + i\Psi_e\nu)\|_p^q\, dx,$$

where

$$\Theta := \left(\frac{1}{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right) V_n(\Gamma) \left(\min_{j=1,\ldots,n} \left(\min\left(\rho_j - r_j, R_j - \rho_j\right)\right)\right)^n} \right)^{\frac{q}{p} - q}.$$

Now, let us note that if $F_1, \ldots, F_N \in L^+(X, \mu)$, then

$$\int_X \max_{j=1,\ldots,N} F_j\, d\mu \le \int_X (F_1 + \cdots + F_N)\, d\mu \le N \max_{j=1,\ldots,N} \int_X F_j\, d\mu.$$

Using this fact and changing variables ($x + u = t$), we get

$$\|M_{\varphi}(f; \cdot + iw)\|_q^q \le 2^n \Theta \max_{\nu \in \mathcal{V}(r,R)} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f_w(t + i\Psi_e\nu)\, \widehat{\varphi}(t - x + i\Psi_e\nu)|^p\, dt \right)^{q/p} dx. \tag{4.25}$$

Since $q/p > 1$, we can employ Minkovskii's integral inequality and obtain:

$$\|M_{\varphi}(f; \cdot + iw)\|_q^q$$

$$\le 2^n \Theta \max_{\nu \in \mathcal{V}(r,R)} \left(\int_{\mathbb{R}^n} |f_w(t + i\Psi_e\nu)|^p \left(\int_{\mathbb{R}^n} |\widehat{\varphi}(t - x + i\Psi_e\nu)|^q\, dx \right)^{p/q} dt \right)^{q/p}$$

$$= 2^n \Theta \max_{\nu \in \mathcal{V}(r,R)} \left\| \widehat{\varphi} \left(\cdot + i \Psi_e \nu \right) \right\|_q^q \left\| f_w \left(\cdot + i \Psi_e \nu \right) \right\|_p^q \leq 2^n \Theta \max_{\nu \in \mathcal{V}(r,R)} \left\| \widehat{\varphi} \left(\cdot + i \Psi_e \nu \right) \right\|_q^q \left\| f \right\|_{H^p}^q .$$

Since, the maximum in the right-hand side is assumed finite, taking $\sup_{w \in \Gamma}$, we have

$$\begin{aligned} \left\| M_\varphi (f) \right\|_{H^q} & \leq & 2^{n/q} \left(\frac{1}{\pi^{\frac{n}{2}} \Gamma \left(\frac{n}{2} + 1 \right) V_n (\Gamma) \left(\min_{j=1,\ldots,n} \left(\min \left(\rho_j - r_j, R_j - \rho_j \right) \right) \right)^n } \right)^{\frac{1}{p} - 1} \\ & \times & \max_{\nu \in \mathcal{V}(r,R)} \left\| \widehat{\varphi} \left(\cdot + i \Psi_e \nu \right) \right\|_q \left\| f \right\|_{H^p} . \end{aligned}$$

Since the left hand side of this inequality does not depend on ρ, we could take $\rho = \frac{1}{2} (r + R)$, and the last inequality yields (4.22). $\square$

Following [86, Ch. III, § 4], a convex, compact and symmetric with respect to the origin set $K \subset \mathbb{R}^n$ with nonempty interior is called a *symmetric body*. Its *polar set* is defined by $K^* = \{ t \in \mathbb{R}^n : (x,t) \leq 1, \ \forall x \in K \}$. Let us also set

$$\| z \| := \sup_{t \in K^*} \left| (z,t) \right| = \sup_{t \in K^*} \left| (z_1 t_1 + \cdots + z_n t_n) \right| .$$

Note that K^* is again a symmetric body, and $(K^*)^* = K$ (see, e.g., [86, Ch. III, § 4, Lemma 4.7]).

It is said that an entire function f defined in $\mathbb{C}^n$ is of *exponential type* K, where K is a symmetric body, if for any $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that

$$\left| f(z) \right| \leq A_\varepsilon e^{2\pi (1+\varepsilon) \| z \|}, \quad \forall z \in \mathbb{C}^n.$$

The class of all entire functions of exponential type K is denoted by $\mathcal{E}(K)$.

Proof of Theorem 1.4.2. Since φ is compactly supported on convex body $K := [-\sigma, \sigma]^n = [-\sigma, \sigma] \times \cdots \times [-\sigma, \sigma]$, then, according to the multivariate Paley-Wiener theorem [86, Ch. III, § 4, Th. 4.9],

$$\widehat{\varphi}(z) = \int_{[-\sigma,\sigma]^n} \varphi(t) \, e^{-2\pi (z,t)} \, dt$$

is a function of $\mathcal{E}(K^*)$ class. Therefore,

$$\left| \widehat{\varphi}(z) \right| \leq A_\varepsilon e^{2\pi (1+\varepsilon) \| z \|}, \tag{4.26}$$

where $\|z\| = \sup_{y\in K} |z_1 y_1 + \dots z_n y_n|$. If we fix all the other variables except j-th, then, clearly, the function

$$\Phi_j(\xi) := \widehat{\varphi}(z_1, \dots, z_{j-1}, \xi, z_{j+1}, \dots, z_n)$$

is a univariate entire function of exponential type $2\pi\sigma$.

Applying Bernstein inequality in L^p-metric (for $p \in (0,1)$, the result is due to Q. I. Rahman and G. Schmeisser [78, Corollary 1]), we get

$$\left\| \frac{\partial \widehat{\varphi}}{\partial x_j} (z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_n) \right\|_{L^q(\mathbb{R})} = \left\| \Phi_j' \right\|_{L^q(\mathbb{R})} \le 2\pi\sigma \left\| \Phi_j \right\|_{L^q(\mathbb{R})}.$$

Thus,

$$\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \left| \frac{\partial \widehat{\varphi}}{\partial x_j} \right|^q dx_j \right) dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n \le (2\pi\sigma)^q \left\| \widehat{\varphi} \right\|_q^q < \infty.$$

Applying Tonelli's theorem, we obtain that $\frac{\partial \widehat{\varphi}}{\partial x_j} \in L^q(\mathbb{R}^n)$, and

$$\left\| \frac{\partial \widehat{\varphi}}{\partial x_j} \right\|_q \le 2\pi\sigma \left\| \widehat{\varphi} \right\|_q, \quad j = 1, \dots, n. \tag{4.27}$$

Expanding exponential to the Taylor series, we have

$$e^{2\pi(y,t)} = \sum_{m=0}^{\infty} \frac{(2\pi)^m}{m!} \left(\sum_{j=1}^{n} y_j t_j \right)^m, \quad y, t \in \mathbb{R}^n.$$

The following equality could be easily checked by induction

$$\left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \left\{ e^{-2\pi i(x,t)} \right\} i^m = \left(\sum_{j=1}^{n} y_j t_j \right)^m e^{-2\pi i(x,t)} (2\pi)^m. \tag{4.28}$$

Now,

$$\widehat{\varphi}(x+iy) = \int_{[-\sigma,\sigma]^n} \varphi(t) \sum_{m=0}^{\infty} \frac{(2\pi)^m}{m!} \left(\sum_{j=1}^{n} y_j t_j \right)^m e^{-2\pi i(x,t)} dt. \tag{4.29}$$

Since

$$\left| \frac{(2\pi)^m}{m!} \left(\sum_{j=1}^{n} y_j t_j \right)^m \right| \le \frac{(2\pi)^m \sigma^m |y|^m n^{m/2}}{m!}, \quad t \in [-\sigma,\sigma]^n,$$

the series on the right hand side of (4.29) converges uniformly (with respect to t) and absolutely on $[-\sigma,\sigma]^n$. Since $\varphi \in L^1([-\sigma,\sigma]^n)$, applying the Lebesgue Dominated

Convergence Theorem, we can put the integral sign inside the series. Thus, using (4.28), we get

$$\widehat{\varphi}(x+iy) = \sum_{m=0}^{\infty} \frac{i^m}{m!} \int_{[-\sigma,\sigma]^n} \varphi(t) \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \left\{ e^{-2\pi i (x,t)} \right\} dt.$$

Since $\varphi \in C(\mathbb{R}^n)$ and is compactly supported, then $|t|^k \varphi(t) \in L^1(\mathbb{R}^n)$, for any $k \in \mathbb{N}$, and we can take the differentiation operators outside of the integral. Hence,

$$\widehat{\varphi}(x+iy) = \sum_{m=0}^{\infty} \frac{i^m}{m!} \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x). \tag{4.30}$$

Now, (4.27) implies

$$\int_{\mathbb{R}^n} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right) \widehat{\varphi}(x) \right|^q dx \le \sum_{j=1}^{n} |y_j|^q \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_j} \widehat{\varphi}(x) \right|^q dx \le (2\pi\sigma)^q \sum_{j=1}^{n} |y_j|^q \|\widehat{\varphi}\|_q^q.$$

Hence, by induction,

$$\int_{\mathbb{R}^n} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \right|^q dx \le (2\pi\sigma)^{mq} \left(\sum_{j=1}^{n} |y_j|^q \right)^m \|\widehat{\varphi}\|_q^q. \tag{4.31}$$

From (4.30), we obtain

$$|\widehat{\varphi}(x+iy)|^q \le \sum_{m=0}^{\infty} \frac{1}{(m!)^q} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \right|^q. \tag{4.32}$$

Considering (4.31),

$$\sum_{m=0}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(m!)^q} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \right|^q dx \le \sum_{m=0}^{\infty} \frac{(2\pi\sigma)^{mq}}{(m!)^q} \left(\sum_{j=1}^{n} |y_j|^q \right)^m \|\widehat{\varphi}\|_q^q < \infty.$$

Therefore, the series on the right-hand side of (4.32) converges to a function from $L^1(\mathbb{R}^n)$, and its L^1-norm is (see [31, Ch. 2, § 2.3, Theorem 2.25])

$$\sum_{m=0}^{\infty} \int_{\mathbb{R}^n} \frac{1}{(m!)^q} \left| \left(\sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j} \right)^m \widehat{\varphi}(x) \right|^q dx.$$

Now, (4.32) implies that $\widehat{\varphi}(\cdot + iy) \in L^q(\mathbb{R}^n)$, and

$$\|\widehat{\varphi}(\cdot + iy)\|_q \le \left(\sum_{m=0}^{\infty} \frac{(2\pi\sigma)^{mq}}{(m!)^q} \left(\sum_{j=1}^{n} |y_j|^q \right)^m \right)^{\frac{1}{q}} \|\widehat{\varphi}\|_q, \quad y \in \mathbb{R}^n. \tag{4.33}$$

Take

$$\tau := 2\pi\sigma n^{\frac{1}{2}+\frac{1}{q}}, \quad R := \left(\frac{1}{\tau}, \ldots, \frac{1}{\tau}\right), \quad r := \left(\frac{\varepsilon}{\tau}, \ldots, \frac{\varepsilon}{\tau}\right), \tag{4.34}$$

where $\varepsilon\,(0,1)$. If $\nu \in \mathcal{V}\,(r, R)$, then

$$\left|(\Psi_e \nu)_j\right| \leq \frac{\sqrt{n}}{\tau} = \frac{1}{2\pi\sigma n^{\frac{1}{q}}}.$$

Using (4.33) with $y = \Psi_e \nu$, we get

$$\|\widehat{\varphi}\,(\cdot + i\Psi_e \nu)\|_q \leq \left(\sum_{m=0}^{\infty} \frac{1}{(m!)^q}\right)^{1/q} \|\widehat{\varphi}\|_q \,.$$

Having applied Proposition 4.2.2 with r and R as in (4.34), we obtain

$$\|M_\varphi\,(f)\|_{H^q} \leq 2^{n\left(\frac{1}{p}+\frac{1}{q}-1\right)} \left(\frac{\left(2\pi\sigma n^{\frac{1}{2}+\frac{1}{q}}\right)^n}{\pi^{n/2}\Gamma\left(\frac{n}{2}+1\right) V_n\,(\Gamma)\,(1-\varepsilon)^n}\right)^{\frac{1}{p}-1} \left(\sum_{m=0}^{\infty} \frac{1}{(m!)^q}\right)^{\frac{1}{q}} \|f\|_{H^p} \,.$$

Passing to the limit as $\varepsilon \to 0+$ completes the proof. $\square$

It is clear that if $\varphi \in C^\infty\,(\mathbb{R}^n)$ and is compactly supported, then it belongs to the Schwartz space $\mathcal{J}$. Applying Theorem 3.2 from [86, Ch. 1, § 3], we get $\widehat{\varphi} \in \mathcal{J}$. Integrating in polar coordinates, we conclude $\widehat{\varphi} \in L^q\,(\mathbb{R}^n)$, for any $p \in (0, \infty]$. Applying Theorem 1.4.2, we easily deduce

Corollary 4.2.1 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. If $\varphi \in C^\infty\,(\mathbb{R}^n)$ and is compactly supported, then $\varphi \in \mathcal{M}_{p,q}\,(T_\Gamma)$, for any $0 < p \leq q \leq 1$.*

4.2.2 Local Property

The following lemma was mentioned as one of the basic multiplier's properties. Now, we are ready to present its proof.

Lemma 4.2.1 (Local Property) *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $0 < p \leq q \leq 1$. Assume a function $\varphi : \Gamma^* \to \mathbb{C}$ has the following property: for any point $t \in \Gamma^*$, including the point at infinity, there exists a neighborhood V_t such that, in $V_t \cap \Gamma^*$, φ coincides with some function $\varphi_t \in \mathcal{M}_{p,q}\,(T_\Gamma)$. Then $\varphi \in \mathcal{M}_{p,q}\,(T_\Gamma)$.*

Proof. Without any loss of generality, we will consider that V_t's are open balls $V_t = B_n(t, r_t) = \{x \in \mathbb{R}^n : |x - t| < r_t\}$ of radius $r_t > 0$, and $V_\infty := \{x \in \mathbb{R}^n : |x| > r_\infty\}$.

Since $\Gamma^* \setminus V_\infty$ is a compact in $\mathbb{R}^n$, there exists a finite subcover of $\Gamma^* \setminus V_\infty$ by V_t's, i.e., $\Gamma^* \setminus V_\infty \subset \cup_{k=1}^m V_{t_k}$. For simplicity, let us denote $V_{t_{m+1}} := V_\infty$. Then, $\Gamma^* \subset \cup_{k=1}^{m+1} V_{t_k}$.

Using, e.g., [67, Ch. 1, § 1.2, Th. 1.2.3], it is clear that there exists a partition of unity subordinate to the open covering $\{V_{t_k}\}_{k=1}^{m+1}$ that is a family of C^∞-functions $\{\zeta_{(t_k)}\}_{k=1}^{m+1}$ such that

$$0 \le \zeta_{(t_k)} \le 1, \quad \operatorname{supp} \zeta_{(t_k)} \subset V_{t_k}, \quad k = 1, \dots, m+1,$$

the family $\{\operatorname{supp} \zeta_{(t_k)}\}$ is locally finite, and

$$\sum_{k=1}^{m+1} \zeta_{(t_k)}(x) = 1, \quad \forall x \in \Gamma^*. \tag{4.35}$$

It is clear that $\zeta_{(\infty)} = \zeta_{(t_{m+1})}$ is equal to 1 on $\Gamma^* \setminus \cup_{k=1}^m V_{t_k}$. Hence, $\eta_{(\infty)} := 1 - \zeta_{(\infty)}$ is also from $C^\infty(\mathbb{R}^n)$ class, and

$$\eta_{(\infty)}(x) = 0, \quad \forall x \in \Gamma^* \setminus \bigcup_{k=1}^m V_{t_k}.$$

Since $\eta_{(\infty)}$ is compactly supported, Corollary 4.2.1 implies $\eta_{(\infty)} \in \mathcal{M}_{p,p}(T_\Gamma)$.

As soon as $\operatorname{supp} \zeta_{(t_k)} \subset V_{t_k}$ and $\varphi = \varphi_{(t_k)}$ on V_{t_k}, for $k = 1, \dots, m+1$, we have

$$\zeta_{(t_k)}(x)\, \varphi(x) = \zeta_{(t_k)}(x)\, \varphi_{(t_k)}(x), \quad x \in \Gamma^*,\ k = 1, \dots, m+1.$$

Multiplying (4.35) by $\varphi(x)$, we get

$$\varphi(x) = \sum_{k=1}^{m+1} \zeta_{(t_k)}(x)\, \varphi_{(t_k)}(x), \quad x \in \Gamma^*. \tag{4.36}$$

This implies that φ is Lebesgue measurable, since all $\varphi_{(t_k)}$ are multipliers, whence measurable, and $\zeta_{(t_k)}$ are continuous.

Since functions $\zeta_{(t_k)}$ are infinitely differentiable on $\mathbb{R}^n$ and compactly supported for any $k = 1, \dots, m$, Corollary 4.2.1 implies that $\zeta_{(t_k)} \in \mathcal{M}_{p,p}(T_\Gamma)$. Hence, using Property 2) of a multiplier, $\zeta_{(t_k)}\varphi_{(t_k)} \in \mathcal{M}_{p,q}(T_\Gamma)$.

Now, $\varphi_{(\infty)}\zeta_{(\infty)} = \varphi_{(\infty)} - \varphi_{(\infty)}\eta_{(\infty)} \in \mathcal{M}_{p,q}(T_\Gamma)$, because $\eta_{(\infty)} \in \mathcal{M}_{p,p}(T_\Gamma)$ and $\varphi_{(\infty)} \in \mathcal{M}_{p,q}(T_\Gamma)$.

Thus all the summands in (4.36) belong to $\mathcal{M}_{p,q}(T_\Gamma)$, whence $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$. □

4.2.3 Necessary Conditions

The Local Property of a multiplier and Theorem 1.4.2 allow us to get efficient necessary conditions and even criteria for a function to be a multiplier. These conditions are especially usefull for radial functions. In particular, we can easily obtain the critical index for Bochner-Riesz means (Proposition 1.4.1). The key point is the condition $\widehat{\varphi} \in L^q$, which is illustrated by the following statement.

Theorem 4.2.1 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $\varphi \in C(\Gamma^*)$. If $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$ for some $0 < p \leq q \leq 1$, then for any point $x \in (\Gamma^*)^o$, and its every bounded neighborhood V_x such that $\overline{V_x} \subset (\Gamma^*)^o$, the function φ coincides in $\overline{V_x}$ with a compactly supported continuous function whose Fourier transform belongs to $L^q(\mathbb{R}^n)$.*

To prove Theorem 4.2.1, we need a couple of lemmas that may be of independent interest.

Lemma 4.2.2 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, $\varphi \in L^1_{loc}(\Gamma^*)$, and $0 < p \leq q \leq 1$. If $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, and ψ is a compactly supported function such that $\widetilde{\psi}(\cdot) = \widehat{\psi}(-\cdot) \in H^p(T_\Gamma)$, then $\widehat{\varphi\psi} \in L^q(\mathbb{R}^n)$.*

Proof. Let us consider the function

$$g(z) := \int_{\Gamma^*} \varphi(t)\,\psi(t)\,e^{2\pi i(z,t)}\,dt, \quad z \in T_\Gamma. \tag{4.37}$$

Since $\widetilde{\psi}(\cdot) \in H^p(T_\Gamma)$, the inversion formula implies $\operatorname{supp}\psi \subset \Gamma^*$. As soon as $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, we also deduce

$$\|g\|_{H^q} \leq \|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \left\|\widetilde{\psi}\right\|_{H^p}. \tag{4.38}$$

Since $\varphi \in L^1_{loc}(\Gamma^*)$, and ψ is continuous and compactly supported, then $\varphi\psi \in L^1(\Gamma^*)$. Moreover, $\left|e^{2\pi i(z,t)}\right| \leq 1$, for $z \in T_\Gamma$, $t \in \mathbb{R}^n$. Hence, applying the Lebesgue Dominated Convergence Theorem, we obtain from (4.37) that

$$\widehat{\varphi\psi}(-x) = g(x) := \lim_{y\to 0,\, y\in\Gamma} g(x+iy) = \int_{\Gamma^*} \varphi(t)\,\psi(t)\,e^{2\pi i(x,t)}\,dt, \quad x \in \mathbb{R}^n.$$

Note that $|g(x)|^q$ is also Lebesgue measurable on $\mathbb{R}^n$ as a limit of Lebesgue measurable functions $|g(x+iy)|^q$. Hence, using Fatou's Lemma and (4.38), we get

$$\left\|\widehat{\varphi\psi}\right\|_q \leq \liminf_{y\to 0,\, y\in\Gamma} \|g(\cdot+iy)\|_q \leq \|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)} \left\|\widetilde{\psi}\right\|_{H^p} < \infty.$$

□

Lemma 4.2.3 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, $\varphi \in L^1_{loc}(\Gamma^*)$, and $0 < p \leq q \leq 1$. If $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, and $\psi \in C^\infty(\mathbb{R}^n)$ is compactly supported with* $\operatorname{supp}\psi \subset (\Gamma^*)^o$, *then $\widehat{\varphi\psi} \in L^q(\mathbb{R}^n)$.*

Proof. Let us consider

$$\widetilde{\psi}(z) = \widetilde{\psi}(x+iy) = \int_{\mathbb{R}^n} \psi(t)\,e^{-2\pi(y,t)}e^{2\pi i(x,t)}\,dt, \quad z = x+iy \in T_\Gamma. \tag{4.39}$$

We need to prove that $\widetilde{\psi} \in H^p(T_\Gamma)$. Since $\psi \in L^2(\mathbb{R}^n)$ and is compactly supported, the Paley-Wiener Theorem implies that $\widetilde{\psi}$ is an entire function of exponential type.

Since ψ is compactly supported, then it is clear that for any $y \in \Gamma$, $x \in \mathbb{R}^n$, we have $\psi(\cdot)\,e^{-2\pi(y,\cdot)}e^{2\pi i(x,\cdot)} \in L^1(\mathbb{R}^n)$. According to Fubini's theorem, we can choose the order of integration in (4.39) as we need.

If $g \in C^\infty(\mathbb{R}^n)$ and is compactly supported, then Lebesgue integral is, in fact, Riemann integral, and using integration by parts in the iterated integrals, and applying Leibnitz differentiation formula, we arrive at

$$\int_{\mathbb{R}^n} g(t)\,e^{-2\pi(y,t)}e^{2\pi i(x,t)}\,dt$$
$$= \frac{i^k}{(2\pi)^k x_j^k} \sum_{l=0}^{k} \binom{k}{l} (-2\pi y_j)^{k-l} \int\limits_{\operatorname{supp} g} \left(\frac{\partial^l}{\partial t_j^l} g(t)\right) e^{-2\pi(y,t)}e^{2\pi i(x,t)}\,dt. \tag{4.40}$$

Since $\operatorname{supp}\psi \subset (\Gamma^*)^o$, then for any $t \in \operatorname{supp}\psi$ and $y \in \overline{\Gamma}$ we have $(y,t) > 0$. As soon as $\operatorname{supp}\psi$ is compact, then $\inf\left\{(y,t) \,|\, y \in \overline{\Gamma}, |y| = 1, t \in \operatorname{supp}\psi\right\}$ is attained at some couple, y_0 and t_0. Therefore,

$$a := \min\{(y,t) \,|\, y \in \Gamma, |y| = 1, t \in \operatorname{supp}\psi\} = (y_0, t_0) > 0,$$

whence

$$(y,t) \geq a\,|y|\,, \quad y \in \Gamma,\, t \in \operatorname{supp}\psi.$$

Applying standard calculus to the function $h(\xi) := \xi^m e^{-2\pi a\xi}$, $m \in \mathbb{Z}_+$, we deduce that $h(\xi) \leq \frac{m^m}{(2\pi a)^m} e^{-m}$ on $(0,\infty)$. Thus, for $y \in \Gamma$, $t \in \operatorname{supp}\psi$, we have

$$|y_j|^m e^{-2\pi(y,t)} \leq |y|^m e^{-2\pi a|y|} \leq \gamma_1(m,a) := \begin{cases} \frac{m^m}{(2\pi a)^m} e^{-m}, & m \in \mathbb{N}, \\ 1, & m = 0. \end{cases}$$

Now, applying (4.40) to $\psi(\cdot + iy)$, and considering the last estimate, we obtain

$$\left|\widetilde{\psi}(x+iy)\right| \leq \frac{\gamma_2(n,k,\psi)}{|x_j|^k}, \quad x_j \neq 0,\, y \in \Gamma, \tag{4.41}$$

where

$$\gamma_2(n,k,\psi) := \frac{1}{(2\pi)^k} \sum_{l=0}^{k} \binom{k}{l} (2\pi)^{k-l} \gamma_1(k-l,a) \int\limits_{\operatorname{supp}\psi} \left|\frac{\partial^l}{\partial t_j^l}\psi(t)\right| dt < \infty$$

does not depend on x and y.

Using Hölder's inequality, we also have

$$|x|^{2m} = \left(\sum_{j=1}^{n} x_j^2\right)^m \leq \sum_{j=1}^{n} x_j^{2m} \left(\sum_{j=1}^{n} 1\right)^{1-\frac{1}{m}} = n^{1-\frac{1}{m}} \sum_{j=1}^{n} x_j^{2m}, \quad m \in \mathbb{N}.$$

Hence, from (4.41), we clearly get

$$\begin{aligned} |x|^{2m}\left|\widetilde{\psi}(x+iy)\right| &\leq n^{1-\frac{1}{m}} \sum_{j=1}^{n} x_j^{2m} \left|\widetilde{\psi}(x+iy)\right| \leq n^{1-\frac{1}{m}} \sum_{j=1}^{n} \gamma_2(n,2m,\psi) \\ &= n^{2-\frac{1}{m}} \gamma_2(n,2m,\psi)\,, \quad x \in \mathbb{R}^n,\, y \in \Gamma,\, m \in \mathbb{N}. \end{aligned} \tag{4.42}$$

It is also obvious that

$$\left|\widetilde{\psi}(x+iy)\right| \leq \|\psi\|_1 < \infty, \quad x \in \mathbb{R}^n,\, y \in \Gamma. \tag{4.43}$$

Integrating in polar coordinates and considering (4.42) and (4.43), we easily deduce that $\widetilde{\psi} \in H^p(T_\Gamma)$. Finally, Lemma 4.2.2 implies that $\widehat{\varphi\psi} \in L^q(\mathbb{R}^n)$. $\square$

Proof of Theorem 4.2.1. Let us take an arbitrary $x \in (\Gamma^*)^o$ and its bounded neighborhood V_x such that $\overline{V_x} \subset (\Gamma^*)^o$. Consider a function $\psi_{(x)}$ with the following properties:

1). $\psi_{(x)} \in C^\infty(\mathbb{R}^n)$;

2). $\psi_{(x)}$ is compactly supported and $\operatorname{supp}\psi_{(x)} \subset (\Gamma^*)^o$;

3). $\psi_{(x)} \equiv 1$ on $\overline{V_x}$.

To prove that it is possible, let us first note that since $\mathbb{R}^n$ is a normal topological space, there exists an open set U such that $\overline{V_x} \subset U \subset \overline{U} \subset (\Gamma^*)^o$. Then, [67, Ch. 1, § 1.2, Corollary 1.2.6] guarantees the existence of a function $\psi_{(x)}$ with desired properties.

Now, the function

$$G(t) := \varphi(t)\,\psi_{(x)}(t)$$

is continuous, compactly supported and coincides with φ on $\overline{V_x}$. Moreover, according to Lemma 4.2.3, $\widehat{G} \in L^q(\mathbb{R}^n)$, which completes the proof. $\square$

As we can see, the requirement on the Fourier transform of a multiplier to be in $L^q(\mathbb{R}^n)$ is essential. If our kernel is radial and compactly supported, then the requirement $\widehat{\varphi} \in L^q$ is crucial. Moreover, using the Local Property (Lemma 4.2.1), it is often easier to show that a radial function is a multiplier, and then conclude that its Fourier transform is in L^q (see, e.g., Corollary 4.2.3). Such an approach is justified by the following theorem.

Theorem 4.2.2 *Let $\varphi : \mathbb{R}^n \to \mathbb{C}$, $n \in \mathbb{N}$, be a continuous compactly supported radial function. Assume that in some neighborhood of the origin, φ coincides with a continuous compactly supported function whose Fourier transform belongs to $L^q(\mathbb{R}^n)$, for some $q \in (0,1]$. If $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, for some regular cone Γ and $p \in (0,q]$, then*

$\widehat{\varphi} \in L^q(\mathbb{R}^n)$.

To prove this theorem, we need the following statement.

Lemma 4.2.4 *Let $\psi \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, and is compactly supported. Assume that $\varphi \in C(\mathbb{R}^n)$, is also compactly supported and $\widehat{\varphi} \in L^q(\mathbb{R}^n)$, for some $q \in (0,1]$. Then, $\widehat{\psi\varphi} \in L^q(\mathbb{R}^n)$.*

Proof. Since ψ is compactly supported, then there exists $R > 0$ such that $\operatorname{supp} \psi \subset B(0,R)$. Take $a := (R, \dots, R) \in \left(\mathbb{R}^n_+\right)^o$. Then the function

$$\tau_a \psi(x) = \psi(x-a), \quad x \in \mathbb{R}^n,$$

also belongs to $C^\infty(\mathbb{R}^n)$-class, and $\operatorname{supp} \tau_a\psi \subset \left(\mathbb{R}^n_+\right)^o$. Obviously, $\tau_a\varphi$ is also continuous and compactly supported.

Since $\varphi \in L^1(\mathbb{R}^n)$, using the property of the Fourier transform of a translation, we get $\widehat{\tau_a\varphi}(x) = e^{-2\pi i(a,x)}\widehat{\varphi}(x)$, and hence $\|\widehat{\tau_a\varphi}\|_q = \|\widehat{\varphi}\|_q < \infty$.

According to Theorem 1.4.2, $\tau_h\varphi \in \mathcal{M}_{p,q}\left(T_{\left(\mathbb{R}^n_+\right)^o}\right)$, for any $p \in (0,q]$. Now, Lemma 4.2.3 applied to $\tau_a\varphi$, $\tau_a\psi$ and the cone $\left(\mathbb{R}^n_+\right)^o$ implies $\widehat{\tau_a(\psi\varphi)} \in L^q(\mathbb{R}^n)$. Hence $\widehat{\psi\varphi} \in L^q(\mathbb{R}^n)$. □

Proof of Theorem 4.2.2. Let us take an arbitrary $x \in \mathbb{R}^n$, $x \neq 0$. Since $(\Gamma^*)^o \neq \emptyset$, there exists a rotation T such that $Tx \in (\Gamma^*)^o$.

Since $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$, according to Theorem 4.2.1, in any closed ball $\overline{B(Tx,r)} \subset (\Gamma^*)^o$, the function φ coincides with some continuous compactly supported $\varphi_{(Tx)}$ such that $\widehat{\varphi_{(Tx)}} \in L^q(\mathbb{R}^n)$.

Since T is a rotation, then T maps $B(x,r)$ onto $B(Tx,r)$, and considering that φ is radial and T preserves the norm in $\mathbb{R}^n$, we have

$$\varphi(\xi) = \varphi(T\xi) = \varphi_{(Tx)}(T\xi), \quad \xi \in B(x,r).$$

Since Fourier transform commutes with rotation, $\left(\varphi_{(Tx)} \circ T\right)^{\widehat{}} \in L^q(\mathbb{R}^n)$. Thus, in some open ball $B(t,r)$ of any point $t \in \mathbb{R}^n$ (the condition on the origin is given explicitly in the theorem), φ coincides with some function $\varphi_{(t)}$ that is continuous, compactly supported and with $\widehat{\varphi_{(t)}} \in L^q(\mathbb{R}^n)$.

Since $\operatorname{supp}\varphi$ is a compact set in $\mathbb{R}^n$, we can choose a finite number of the balls under consideration so that

$$\operatorname{supp}\varphi \subset \bigcup_{k=0}^{m} B(t_k, r_k).$$

Let us denote $B_k := B(t_k, r_k)$, $k = 0, \ldots, m$, and let $B_{m+1} := \mathbb{R}^n \setminus \operatorname{supp}\varphi$. Thus, $\cup_{k=0}^{m+1} B_k$ is an open covering of $\mathbb{R}^n$.

According to [67, Ch. 1, § 1.2, Th. 1.2.3], for the open set $\cup_{k=0}^{m} B_k$, there exists a partition of unity subordinate to $\{B_k\}_{k=0}^{m+1}$ that is a family of C^∞-functions $\left\{\zeta_{(k)}\right\}_{k=0}^{m+1}$ such that

$$0 \le \zeta_{(k)} \le 1, \quad \operatorname{supp}\zeta_{(k)} \subset B_k, \quad k = 0, \ldots, m+1,$$

the family $\left\{\operatorname{supp}\zeta_{(k)}\right\}$ is locally finite, and

$$\sum_{k=0}^{m+1} \zeta_{(k)}(x) = 1, \quad x \in \bigcup_{k=0}^{m} B_k.$$

Multiplying both sides by $\varphi(x)$ and considering that $\operatorname{supp}\zeta_{(m+1)} \subset B_{m+1}$, and $\varphi \equiv 0$ in B_{m+1}, we obtain

$$\varphi(x) = \sum_{k=0}^{m} \zeta_{(k)}(x)\, \varphi(x) = \sum_{k=0}^{m} \zeta_{(k)}(x)\, \varphi_{(t_k)}(x), \quad x \in \mathbb{R}^n. \tag{4.44}$$

Lemma 4.2.4 implies $\widehat{\zeta_{(k)}\varphi_{(t_k)}} \in L^q(\mathbb{R}^n)$, $k = 0, \ldots, m$. Hence, (4.44) yields $\widehat{\varphi} \in L^q(\mathbb{R}^n)$. $\square$

From Theorems 1.4.2 and 4.2.2, we easily obtain

Corollary 4.2.2 *Let $\varphi : \mathbb{R}^n \to \mathbb{C}$, $n \in \mathbb{N}$, be a continuous compactly supported radial function. Assume that in some neighborhood of the origin, φ belongs to $C^\infty(\mathbb{R}^n)$-class. Then, for any $0 < p \le q \le 1$ and any regular cone $\Gamma \subset \mathbb{R}^n$, $\varphi \in \mathcal{M}_{p,q}(T_\Gamma)$ if and only if $\widehat{\varphi} \in L^q(\mathbb{R}^n)$.*

4.2.4 Sufficient Conditions Involving Growth of Partial Derivatives

Proof of Theorem 1.4.3. Our proof is very similar to [94, Proof of Theorem 3].

Proof of (b). It is clear that $\varphi \in L^2(\mathbb{R}^n)$. If $K := \left[-\frac{\sigma^{1/n}}{2}, \frac{\sigma^{1/n}}{2}\right]_n$, then the estimate (4.7), Paley-Wiener's and Plancherel's theorems imply that

$$\inf\left\{\|\varphi - \psi\|_2 : \psi \in \mathcal{E}(K^*)\right\} = \inf\left\{\left\|\widehat{\varphi} - \widehat{\psi}\right\|_2 : \psi \in \mathcal{E}(K^*)\right\} \geq a_\sigma(\varphi)_2 .$$

Applying the direct theorem on approximation by entire functions of exponential type [71, Ch. 5, § 5.2, Theorem 5.2.4 (see Estimate (5))], we obtain

$$a_\sigma(\varphi)_2 \leq \frac{\gamma_0(s,n)}{\sigma^{s/n}} \max_{j=1,\ldots,n} \omega_2\left(\frac{\partial^s \varphi}{\partial x_j^s}; \frac{1}{\sigma^{1/n}}\right)_{2,j}, \tag{4.45}$$

where $\omega_2(g,h)_{2,j}$ denotes the partial (on j-th variable) modulus of smoothness of g with the step h in $L^2(\mathbb{R}^n)$-norm.

Lemma 6 from [93] asserts that if g is bounded and piecewise convex function on $\mathbb{R}^n$, then for any $h > 0$ and $p \geq 1$, $\|\Delta_h^2 g\|_p \leq M h^{1/p} \omega(g;h)_\infty$, where $\Delta_h^2 g$ is the forward difference of second order and step h (i.e., $\Delta_h^2 g(x) = g(x+2h) - 2g(x+h) + g(x)$), and where M depends only on the number of points dividing the intervals on which g is convex. In fact, the proof of this lemma only requires g to be convex or concave on each of the intervals, i.e., it may be convex on some of them and concave on the others.

Under our assumptions, we can apply the lemma with $p = 2$, and obtain

$$\omega_2\left(\frac{\partial^s \varphi}{\partial x_j^s}; h\right)_{2,j} \leq M h^{\frac{1}{2}} \omega\left(\frac{\partial^s \varphi}{\partial x_j^s}; h\right)_\infty \leq M C h^{\frac{1}{2}+\alpha},$$

where

$$C := \max_{j=1,\ldots,n} \sup_{t_j \neq 0} \sup_{x \in \mathbb{R}^n} \frac{\left|\frac{\partial^s \varphi}{\partial x_j^s}(x_1,\ldots,x_n) - \frac{\partial^s \varphi}{\partial x_j^s}(x_1,\ldots,x_{j-1},x_j+t_j,x_{j+1},\ldots,x_n)\right|}{|t_j|^\alpha} < \infty,$$

which is finite according to the assumption of our theorem. Therefore, (4.45) implies

$$\int_1^\infty \left(\frac{a_\sigma(\varphi)_2}{\sqrt{\sigma}}\right)^q d\sigma \leq (\gamma_0(s,n) M C)^q \int_1^\infty \left(\frac{1}{\sigma^{s/n} \sigma^{(\alpha+1/2)/n} \sigma^{1/2}}\right)^q d\sigma.$$

Since $\alpha > \frac{n}{q} - \frac{n+1}{2} - s$, then $s/n + (\alpha + 1/2)/n + 1/2 > 1/q$ and the last integral is finite. Also considering that $a_\sigma(\varphi)_2 \le \|\varphi\|_2$, and applying Corollary 4.1.4, we obtain

$$\begin{aligned}\|\widehat{\varphi}\|_q^q &\le 2\int_0^1 \frac{\|\varphi\|_2^q}{\sigma^{q/2}}\,d\sigma + 2\int_1^\infty \left(\frac{a_\sigma(\varphi)_2}{\sqrt{\sigma}}\right)^q d\sigma \\ &\le 2\|\varphi\|_2^q \frac{1}{1-q/2} + 2(\gamma_0(s,n)MC)^q \int_1^\infty \left(\frac{1}{\sigma^{s/n}\sigma^{(\alpha+1/2)/n}\sigma^{1/2}}\right)^q d\sigma < \infty.\end{aligned}$$

Now, application of Theorem 1.4.2 completes the proof.

Proof of (a). Let us show that if φ and all $\frac{\partial^r\varphi}{\partial x_j^r}$, $j = 1, \ldots, n$, belong to $L^2(\mathbb{R}^n)$ with some $r > n\left(\frac{1}{q} - \frac{1}{2}\right)$, then there exists some constant $\gamma_1(r,q,n)$ such that

$$\|\widehat{\varphi}\|_q^q \le \gamma_1(r,q,n)\,\|\varphi\|_2^{q-\frac{n}{r}\left(1-\frac{q}{2}\right)} \max_{j=1,\ldots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2^{\frac{n}{r}\left(1-\frac{q}{2}\right)}. \tag{4.46}$$

Indeed, (4.45) implies $a_\sigma(\varphi)_2 \le \frac{4\gamma_0(r,n)}{\sigma^{r/n}} \max_{j=1,\ldots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2$. Choosing σ_0 so that $\|\varphi\|_2\,\sigma_0^{r/n} = \max_{j=1,\ldots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2$, applying the last inequality, employing the condition $r > n\left(\frac{1}{q} - \frac{1}{2}\right)$, i.e., $q\left(\frac{1}{2} + \frac{r}{n}\right) > 1$, and considering that $a_\sigma(\varphi)_2 \le \|\varphi\|_2$, we get

$$\begin{aligned}\int_0^\infty \left(\frac{a_\sigma(\varphi)_2}{\sqrt{\sigma}}\right)^q d\sigma &\le \|\varphi\|_2^q \int_0^{\sigma_0} \frac{d\sigma}{\sigma^{q/2}} + (4\gamma_0(r,n))^q \max_{j=1,\ldots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2^q \int_{\sigma_0}^\infty \frac{d\sigma}{\sigma^{q/2+rq/n}} \\ &= \left(\frac{1}{1-\frac{q}{2}} + \frac{(4\gamma_0(r,n))^q}{\frac{q}{2}+\frac{rq}{n}-1}\right)\|\varphi\|_2^{q-\frac{n}{r}\left(1-\frac{q}{2}\right)} \max_{j=1,\ldots,n}\left\|\frac{\partial^r\varphi}{\partial x_j^r}\right\|_2^{\frac{n}{r}\left(1-\frac{q}{2}\right)}.\end{aligned}$$

Now, Corollary 4.1.4 implies (4.46) immediately with

$$\gamma_1(r,q,n) := 2\left(\frac{1}{1-\frac{q}{2}} + \frac{(4\gamma_0(r,n))^q}{\frac{q}{2}+\frac{rq}{n}-1}\right).$$

Let us consider the following partition of unity. Take an arbitrary function $h_{(0)} \in C^\infty(\mathbb{R})$ satisfying the following three conditions: (i) $h_{(0)}(t) = 0$ for $t \le -1/2$; (ii) $\|h_{(0)}\|_\infty = 1$; (iii) $h_{(0)}(t) + h_{(0)}(-t) \equiv 1$, i.e., $h_{(0)} - 1/2$ is odd. For $\nu \in \mathbb{N}$, we also set

$$h_{(\nu)}(t) := h_{(0)}\left(\frac{t+1}{2^{\nu-1}} - \frac{3}{2}\right) h_{(0)}\left(\frac{3}{2} - \frac{t+1}{2^\nu}\right).$$

It is clear that $\operatorname{supp} h_{(\nu)} \subset [2^{\nu-1}-1, 2^{\nu+1}-1]$. Using the Leibnitz differentiation formula, we get

$$\left|h_{(\nu)}^{(s)}(t)\right| \le \frac{3^s}{2^{\nu s}} \max_{k=0,\ldots,s}\left\|h_{(0)}^{(k)}\right\|_\infty^2, \quad \nu \in \mathbb{N},\ s \in \mathbb{Z}_+,\ t \in \mathbb{R}. \tag{4.47}$$

Let us also observe that

$$h_{(0)}\left(\frac{1}{2}-t\right)+\sum_{\nu=1}^{\infty}h_{(\nu)}\left(t\right)=1,\quad t\geq 0. \tag{4.48}$$

Therefore, considering $\varphi_{(0)}\left(x\right):=\varphi\left(x\right)h_{(0)}\left(\frac{1}{2}-|x|^2\right)$, $\varphi_{(\nu)}\left(x\right):=\varphi\left(x\right)h_{(\nu)}\left(|x|^2\right)$ $x\in\mathbb{R}^n$, $\nu\in\mathbb{N}$, we obtain the following decomposition

$$\varphi\left(x\right)=\sum_{\nu=0}^{\infty}\varphi_{(\nu)}\left(x\right),\quad x\in\mathbb{R}^n. \tag{4.49}$$

Obviously,

$$\operatorname{supp}\varphi_{(\nu)}\subset\left\{x\in\mathbb{R}^n:\,2^{\nu-1}-1\leq|x|^2\leq 2^{\nu+1}-1\right\},\ \nu\in\mathbb{N},\quad \operatorname{supp}\varphi_{(0)}\subset\overline{B\left(0,1\right)}. \tag{4.50}$$

It is also clear that the series in (4.49) converges absolutely (for any x, it is a finite sum) to $|\varphi\left(x\right)|$ that is bounded on $\mathbb{R}^n$ since φ is continuous and compactly supported.

If all the $\varphi_{(\nu)}$ belong to $\mathcal{M}_{p,q}\left(T_\Gamma\right)$, then Proposition 4.2.1 implies

$$\|\varphi\|^q_{\mathcal{M}_{p,q}(T_\Gamma)}\leq\sum_{\nu=0}^{\infty}\left\|\varphi_{(\nu)}\right\|^q_{\mathcal{M}_{p,q}(T_\Gamma)}, \tag{4.51}$$

whereas the series in the right-hand side of this inequality converges. To prove that, we need to estimate the norms $\left\|\widehat{\varphi}_{(\nu)}\right\|_q$.

Note that if $a\geq 0$, $\nu\in\mathbb{N}$, and $x\in\operatorname{supp}\varphi_{(\nu)}$, then (4.51) yields

$$|x|^a\geq\left(2^{\nu-1}-1\right)^{a/2}\geq\left(2^{\nu-2}\right)^{a/2}=2^{-a}\left(\sqrt{2}\right)^{\nu a}.$$

Since also $\operatorname{supp}\varphi_{(0)}\subset\overline{B\left(0,1\right)}$, we obtain that

$$1+|x|^a\geq 2^{-a}\left(\sqrt{2}\right)^{\nu a},\quad \nu\in\mathbb{Z}_+,\,x\in\operatorname{supp}\varphi_{(\nu)}. \tag{4.52}$$

Since $\alpha\geq 0$, the condition on the growth of φ yields $|\varphi\left(x\right)|\leq\frac{A}{1+|x|^\alpha}\leq\frac{A}{2^{-\alpha}\left(\sqrt{2}\right)^{\nu\alpha}}$. Using (4.50), we also get

$$m\left(\operatorname{supp}\varphi_{(\nu)}\right)=\begin{cases}\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\left(\left(2^{\nu+1}-1\right)^{\frac{n}{2}}-\left(2^{\nu-1}-1\right)^{\frac{n}{2}}\right), & \nu\in\mathbb{N},\\ \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}, & \nu=0\end{cases}\leq 2^{\frac{\nu n}{2}}\frac{2^{\frac{n}{2}}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{4.53}$$

Hence, for $\nu \in \mathbb{Z}_+$,

$$\begin{aligned}\left\|\varphi_{(\nu)}\right\|_2 &\leq \max_{x\in\operatorname{supp}\varphi_{(\nu)}} |\varphi(x)| \left(m\left(\operatorname{supp}\varphi_{(\nu)}\right)\right)^{\frac{1}{2}} \left\|h_{(\nu)}\right\|_\infty \\ &\leq \frac{\left(\sqrt{2}\right)^{\frac{n}{2}} \pi^{\frac{n}{4}}}{2^{-\alpha}\left(\Gamma\left(\frac{n}{2}+1\right)\right)^{\frac{1}{2}}} \frac{A}{\left(\sqrt{2}\right)^{\nu\alpha}} \left(\sqrt{2}\right)^{\frac{\nu n}{2}}. \qquad (4.54)\end{aligned}$$

Applying the Faá di Bruno's formula for derivatives of a composition (see, e.g., [51] or [47]), we have

$$\frac{\partial^s}{\partial x_j^s}\left(h_{(\nu)}\left(|x|^2\right)\right) = \sum_{k_1+2k_2=s;\, k_1,k_2\in\mathbb{Z}_+} \frac{s!}{k_1!k_2!} h_{(\nu)}^{(k_1+k_2)}\left(|x|^2\right)\left(2x_j\right)^{k_1}.$$

Since $h_{(\nu)}\left(|x|^2\right) \equiv 0$ when $|x| \geq \left(\sqrt{2}\right)^{\nu+1} \geq \sqrt{2^{\nu+1}-1}$, considering also (4.47), we get

$$\left|\frac{\partial^s}{\partial x_j^s}\left(h_{(\nu)}\left(|x|^2\right)\right)\right| \leq \frac{\gamma_2\left(s, h_{(0)}\right)}{\left(\sqrt{2}\right)^{\nu s}}, \quad x \in \mathbb{R}^n,\ \nu \in \mathbb{Z}_+, \qquad (4.55)$$

where

$$\gamma_2\left(s, h_{(0)}\right) := \max_{l=0,\dots,s} \left\|h_{(0)}^{(l)}\right\|_\infty^2 \sum_{k_1+2k_2=s;\, k_1,k_2\in\mathbb{Z}_+} \frac{s!}{k_1!k_2!} 3^{(k_1+k_2)} \left(2\sqrt{2}\right)^{k_1}.$$

Applying the Leibnitz rule for differentiation of a product, from (4.55), we derive that for any $r, \nu \in \mathbb{Z}_+$, $j = 1, \dots, n$,

$$\left|\frac{\partial^r}{\partial x_j^r}\varphi_{(\nu)}(x)\right| \leq 2^r \max_{k=0,\dots,r} \left(\frac{\left|\frac{\partial^k}{\partial x_j^k}\varphi(x)\right| \gamma_2\left(r-k, h_{(0)}\right)}{\left(\sqrt{2}\right)^{\nu(r-k)}}\right), \quad x \in \mathbb{R}^n.$$

Now, Corollary 4.1.5 implies

$$\begin{aligned}\left|\frac{\partial^r}{\partial x_j^r}\varphi_{(\nu)}(x)\right| &\leq 2^r \max_{k=0,\dots,r} \left(\left(\frac{A}{1+|x|^\alpha}\right)^{1-\frac{k}{r}} \left(\frac{B}{1+|x|^\beta}\right)^{\frac{k}{r}} \frac{\gamma_2\left(r-k, h_{(0)}\right)}{\left(\sqrt{2}\right)^{\nu(r-k)}}\right) \\ &\times \max\left(\max_{k=1,\dots,r-1}\left(\frac{C_0 r}{k}\right)^k, 1\right), \quad x \in \mathbb{R}^n,\ r, \nu \in \mathbb{Z}_+,\ j = 1, \dots, n.\end{aligned}$$

Using (4.52), the last inequality, and

$$\max_{k=0,\dots,r} \frac{1}{\left(\sqrt{2}\right)^{\nu\alpha(1-k/r)+\nu\beta k/r+\nu(r-k)}} \leq \frac{1}{\left(\sqrt{2}\right)^{\nu(\alpha+r)}} + \frac{1}{\left(\sqrt{2}\right)^{\nu\beta}},$$

it is easy to conclude that

$$\left|\frac{\partial^r}{\partial x_j^r}\varphi_{(\nu)}(x)\right| \leq \gamma_3\left(\alpha,\beta,r,h_{(0)}\right)(A+B)\left(\frac{1}{\left(\sqrt{2}\right)^{\nu(\alpha+r)}}+\frac{1}{\left(\sqrt{2}\right)^{\nu\beta}}\right), \quad x\in\mathbb{R}^n,$$

where $r,\nu\in\mathbb{Z}_+$, $j=1,\ldots,n$, and

$$\gamma_3\left(\alpha,\beta,r,h_{(0)}\right) := 2^r\left(2^\alpha+2^\beta\right)\max\left(\max_{k=1,\ldots,r-1}\left(\frac{C_0 r}{k}\right)^k,1\right)\max_{k=0,\ldots,r}\gamma_2\left(k,h_{(0)}\right).$$

Similarly to (4.54), considering (4.53), for any $\nu\in\mathbb{Z}_+$ and $j=1,\ldots,n$, we obtain

$$\begin{aligned}\left\|\frac{\partial^r}{\partial x_j^r}\varphi_{(\nu)}\right\|_2 &\leq \gamma_3\left(\alpha,\beta,r,h_{(0)}\right)\frac{\left(\sqrt{2}\right)^{\frac{n}{2}}\pi^{\frac{n}{4}}}{\left(\Gamma\left(\frac{n}{2}+1\right)\right)^{\frac{1}{2}}}(A+B)\\ &\times\left(\frac{1}{\left(\sqrt{2}\right)^{\nu(\alpha+r)}}+\frac{1}{\left(\sqrt{2}\right)^{\nu\beta}}\right)\left(\sqrt{2}\right)^{\frac{\nu n}{2}}. \end{aligned} \tag{4.56}$$

From (4.46), (4.54) and (4.56), we get

$$\begin{aligned}\left\|\widehat{\varphi}_{(\nu)}\right\|_q^q &\leq \gamma_4\left(\alpha,\beta,n,r,q,h_{(0)}\right)(A+B)^q\left(\sqrt{2}\right)^{\frac{\nu nq}{2}-\nu\alpha q}\\ &\times\left(\frac{1}{\left(\sqrt{2}\right)^{\nu r}}+\frac{1}{\left(\sqrt{2}\right)^{\nu(\beta-\alpha)}}\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)}, \end{aligned}$$

where

$$\begin{aligned}\gamma_4\left(\alpha,\beta,n,r,q,h_{(0)}\right) &:= \gamma_1(r,q,n)\left(\gamma_3\left(\alpha,\beta,r,h_{(0)}\right)\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)}\\ &\times\left(\frac{\left(\sqrt{2}\right)^{\frac{n}{2}}\pi^{\frac{n}{4}}}{\sqrt{\Gamma\left(\frac{n}{2}+1\right)}}\right)^q 2^{\alpha\left(q-\frac{n}{r}\left(1-\frac{q}{2}\right)\right)}. \end{aligned}$$

Applying Theorem 1.4.2 with $\sigma=\sqrt{2^{\nu+1}-1}$, we obtain that $\varphi_{(\nu)}\in\mathcal{M}_{p,q}\left(T_\Gamma\right)$, and

$$\left\|\varphi_{(\nu)}\right\|_{\mathcal{M}_{p,q}(T_\Gamma)}^q \leq \frac{\gamma_5(n,p,q)}{\left(V_n(\Gamma)\right)^{q\left(\frac{1}{p}-1\right)}}\left(\sqrt{2^{\nu+1}-1}\right)^{nq\left(\frac{1}{p}-1\right)}\left\|\widehat{\varphi}_{(\nu)}\right\|_q^q$$

$$\begin{aligned}&\leq \frac{\gamma_5(n,p,q)\,\gamma_4\left(\alpha,\beta,n,r,q,h_{(0)}\right)\left(\sqrt{2}\right)^{nq\left(\frac{1}{p}-1\right)}}{\left(V_n(\Gamma)\right)^{q\left(\frac{1}{p}-1\right)}}(A+B)^q\left(\sqrt{2}\right)^{\frac{\nu nq}{p}-\nu\alpha q-\nu n}\\ &\quad\times\left(1+\frac{1}{\left(\sqrt{2}\right)^{\nu(\beta-\alpha-r)}}\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)}, \end{aligned}$$

where γ_5 is the constant from the estimate in Theorem 1.4.2. Since the series

$$\sum_{\nu=0}^{\infty}\left(\sqrt{2}\right)^{\frac{\nu nq}{p}-\nu\alpha q-\nu n}\left(1+\frac{1}{\left(\sqrt{2}\right)^{\nu(\beta-\alpha-r)}}\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)} \tag{4.57}$$

converges if and only if

$$\min\left(\beta-\alpha-r,0\right)>\frac{2rq}{2-q}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{2qr\alpha}{n\left(2-q\right)},$$

considering (4.51) and fixing some $h_{(0)}$ satisfying aforementioned conditions, we conclude that $\varphi\in\mathcal{M}_{p,q}\left(T_\Gamma\right)$, and

$$\|\varphi\|_{\mathcal{M}_{p,q}(T_\Gamma)}\leq\frac{\gamma\left(n,p,q,r,\alpha,\beta\right)}{\left(V_n\left(\Gamma\right)\right)^{\frac{1}{p}-1}}\left(A+B\right),$$

with

$$\begin{aligned}\gamma\left(n,p,q,r,\alpha,\beta\right) \;:=\;& \left(\sqrt{2}\right)^{n\left(\frac{1}{p}-1\right)}\Bigg(\gamma_5\left(n,p,q\right)\gamma_4\left(\alpha,\beta,n,r,q,h_{(0)}\right)\\ &\times\sum_{\nu=0}^{\infty}\left(\sqrt{2}\right)^{\frac{\nu nq}{p}-\nu\alpha q-\nu n}\left(1+\frac{1}{\left(\sqrt{2}\right)^{\nu(\beta-\alpha-r)}}\right)^{\frac{n}{r}\left(1-\frac{q}{2}\right)}\Bigg)^{\frac{1}{q}}.\end{aligned}$$

□

4.2.5 Bochner-Riesz Means

Applying Theorem 1.4.3 (b), it is easy to show that the Bocher-Riesz means of the Fourier integral belongs to $\mathcal{M}_{p,q}\left(T_\Gamma\right)$ under the assumptions of Proposition 1.4.1. However, we will give more elegant proof of this statement based only on Theorem 4.2.2 and some known estimates.

Proof of Proposition 1.4.1. Let us show that for any $r\in\mathbb{N}$, the function

$$\varphi_{r,\alpha}\left(x\right):=\left(1-|x|^{2r}\right)_+^{\alpha}=\begin{cases}\left(1-|x|^{2r}\right)^{\alpha}, & |x|\leq 1,\\ 0, & |x|>1,\end{cases}$$

belongs to $\mathcal{M}_{p,q}\left(T_\Gamma\right)$ if and only if $\varphi_{2,\alpha}\in\mathcal{M}_{p,q}\left(T_\Gamma\right)$. Indeed, the formula for geometric progression yields

$$\left(1-|x|^{2r}\right)_+=\left(1-|x|^{2}\right)_+\sum_{j=0}^{r-1}|x|^{2j}. \tag{4.58}$$

Having taken some $h \in C^\infty(\mathbb{R}^n)$ so that $h \equiv 1$ in $\overline{B(0,1)}$ and $h \equiv 0$ outside of $B(0,2)$ (such a function exists due to [67, Ch. 1, § 1.2, Corollary 1.2.6]), the equation (4.58) implies that

$$\varphi_{r,\alpha}(x) = \varphi_{2,\alpha}(x)\,\zeta(x), \quad x \in \mathbb{R}^n,$$

where

$$\zeta(x) := \left(\sum_{j=0}^{r-1} |x|^{2j}\right)^{\alpha} h(x).$$

Obviously, $\zeta \in C^\infty(\mathbb{R}^n)$ and is compactly supported. According to Corollary 4.2.1, $\zeta \in \mathcal{M}_{p,p}(T_\Gamma)$, for any $0 < p \le 1$, and any regular cone $\Gamma \subset \mathbb{R}^n$. If $\varphi_{2,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$, then Property 2) of a multiplier yields $\varphi_{r,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$.

From another side, (4.58) also implies

$$\varphi_{2,\alpha}(x) = \varphi_{r,\alpha}(x)\,\eta(x), \quad x \in \mathbb{R}^n,$$

where

$$\eta(x) := \left(\sum_{j=0}^{r-1} |x|^{2j}\right)^{-\alpha} h(x).$$

Using the same reasonings, $\varphi_{r,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$ implies $\varphi_{2,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$.

Now, since $\varphi_{2,\alpha}$ is radial and belongs to $C^\infty(B(0,1))$, then, according to Corollary 4.2.2, $\varphi_{2,\alpha} \in \mathcal{M}_{p,q}(T_\Gamma)$ if and only if its Fourier transform belongs to $L^q(\mathbb{R}^n)$.

As shown in [38, Appendix B.5], for any $\alpha > 0$,

$$\widehat{\varphi_{2,\alpha}}(t) = \frac{\Gamma(\alpha+1)}{\pi^\alpha |t|^{n/2+\alpha}} J_{n/2+\alpha}(2\pi|t|),$$

where J_ν is the Bessel function. An asymptotic behavior of J_ν is also well-known. Lemma 3.11 from [86, Ch. IV, § 3] asserts that

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}} \cos\left(s - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{s^{3/2}}\right), \quad s \to \infty.$$

Thus,

$$\begin{aligned} \widehat{\varphi_{2,\alpha}}(t) &= \frac{\Gamma(\alpha+1)}{\pi^{\alpha+1}|t|^{n/2+\alpha+1/2}} \cos\left(2\pi|t| - \frac{\pi n}{4} - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) \\ &+ O\left(\frac{1}{|t|^{n/2+\alpha+3/2}}\right), \quad |t| \to \infty. \end{aligned}$$

Therefore, it is clear that $\widehat{\varphi_{2,\alpha}} \in L^q(\mathbb{R}^n)$ if and only if $n/2 + \alpha + 1/2 > n/q$. $\square$

The following statement follows immediately from Proposition 1.4.1 and Corollary 4.2.2.

Corollary 4.2.3 *Let $\alpha > 0$, $r, n \in \mathbb{N}$, and $q \in (0,1]$. The Fourier transform of the function $\varphi_{r,\alpha}(x) = \left(1 - |x|^{2r}\right)_+^{\alpha}$ belongs to $L^q(\mathbb{R}^n)$ if and only if $\alpha > \frac{n}{q} - \frac{n+1}{2}$.*

4.3 Bernstein and Nikol'skiĭ Type Inequalities for Entire Functions of Exponential Type

Univariate Bernstein type inequalities for entire functions of exponential type σ are extremely useful tools of Approximation Theory. Usually, they have the following form

$$\|f'\| \leq \sigma \|f\|.$$

Initially formulated by S. N. Bernstein for trigonometric polynomials in uniform norms, the inequality have been obtained for many other normed and pre-normed spaces as well. We have already discussed such type of inequalities in $H^p(\mathbb{D})$ spaces in Section 1.3 and Chapter 3 (see Example 3.2 in Section 3.2).

In $L^p(\mathbb{R})$, $p \geq 1$, the Bernstein inequality can be found in the classical monograph by R. Boas [10, Ch. 11, § 11.3, Theorem 11.3.3]. For $p \in (0,1)$, the result is due to Q. Rahman and G. Schmeisser [78, Corollary 1]. There are also multivariate analogs. For example, in [33], M. Ganzburg obtained an estimate for the norm (4.59) of the gradient of an entire function. The estimate is given in terms of a supremum-norm of the function. There are more Bernstein-type inequalities in his paper [34]. One of them establishes a Bernstein type inequality for trigonometric polynomials in more general setup than L^p-norm ($p \geq 1$). Another interesting Bernstein type inequality for star-like domains in $\mathbb{R}^n$ was obtained by A. Kroó in [54]. There are several Bernstein type inequalities for entire functions of exponential type satisfying some additional assumptions (see, e.g., [80, 37]).

The proof of the original Bernstein inequality has its own history, and new results on this subject have still been appearing. For example, P. Nevai [68] recently proved that the Schur inequality stating that for any algebraic polynomial P of degree at most $m-1$,

$$\|P\|_{C[-1,1]} \leq m \left\| \sqrt{1-(\cdot)^2} P(\cdot) \right\|_{C[-1,1]},$$

and the Bernstein inequality for trigonometric polynomials are equivalent in the sense that they could be easily obtained from each other. Moreover, [68] contains an interesting story and references on the history of the Bernstein inequality.

Our Theorem 4.3.2 establishes a Bernstein type inequality for entire functions of exponential type, which belong to Hardy spaces $H^p(T_\Gamma)$ in tubes over open cones. The precise definitions and the result are given in Section 4.3.1.

Another family of inequalities heavily used not only in Approximation Theory, but also in virtually every area of classical Analysis, is Nikol'skiĭ type inequalities. An alternative name is "Different Metrics Inequalities". The idea is to compare norms of a function (or its derivatives) in different spaces usually under additional assumptions on the function (see, e.g., [69, 70, 28, 22, 65], just to name a few). Very powerful Nikolskiĭ (and Berstein) type inequalities were obtained by I. I. Ibragimov [49]. In [35, Sect. 5.3], M. Ganzburg obtained some Nikol'skiĭ type estimates for entire functions of exponential type in several variables. Another interesting subject where Nikol'skiĭ type inequalities in L^p or in H^p could be useful is Nikol'sliĭ constants (see the article by E. Levin and D. Lubinsky [59]).

Our Theorem 4.3.3 establishes a Nikol'skiĭ type inequality for entire functions of exponential type belonging to Hardy spaces in tubes.

Finally, let us note that Theorems 4.3.2 and 4.3.3 in a weaker form were announced in [98] and published in a virtually unavailable author's paper [97]. For example, the Bernstein type estimate was obtained using the Fourier multipliers approach, which, in particular, brought an additional multiplicative constant in the right-hand side. The direct proof we give below allows us to obtain a better estimate.

4.3.1 Definitions and Main Results

Following [86, Ch. III, § 4], we remind some notions on multivariate entire functions of exponential type.

A set $K \subset \mathbb{R}^n$ is called a *symmetric body* if it is convex, compact, symmetric with respect to the origin, and has a nonempty interior. In fact, any symmetric body is a closed unit ball with respect to some norm. Its *polar set* is defined by

$$K^* = \{t \in \mathbb{R}^n : (x,t) \leq 1, \ \forall x \in K\},$$

where (x,t) denotes the usual inner product of two vectors in $\mathbb{R}^n$.

Note that if $K \subset \mathbb{R}^n$ is convex, closed, and $0 \in K$, then $K^{**} = (K^*)^* = K$ (see [86, Ch. III, § 4, Lemma 4.7]). It is also clear that if K is a symmetric body, so is K^*.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, let us also set

$$\|z\| := \sup_{t \in K^*} |z_1 t_1 + \cdots + z_n t_n| . \tag{4.59}$$

An entire function f defined in $\mathbb{C}^n$ is of *exponential type* K, where K is a symmetric body, if for any $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that

$$|f(z)| \leq A_\varepsilon e^{2\pi(1+\varepsilon)\|z\|}, \quad \forall z \in \mathbb{C}^n. \tag{4.60}$$

The class of all entire functions of exponential type K is denoted by $\mathcal{E}(K)$. One of the most interesting results of L^2 theory for these functions is the Paley-Wiener theorem that describes the support of the Fourier transform of a function from $L^2(\mathbb{R}^n) \cap \mathcal{E}(K^*)$. Let us recall the multivariate version of this theorem.

Theorem 4.3.1 (E. M. Stein, G. Weiss [86, Ch. III, § 4, Th. 4.9]) *Suppose $F \in L^2(\mathbb{R}^n)$. Then F is the Fourier transform of a function vanishing outside a symmetric body K if and only if F is the restriction to $\mathbb{R}^n$ of a function in $\mathcal{E}(K^*)$.*

Let us note that considering aforementioned relations about K^{**}, Theorem 4.3.1 holds true with K and K^* switched.

For a multi-index $k = (k_1, \ldots, k_n)$, $k_j \in \mathbb{N} \cup \{0\}$, we denote $|k| = \sum_{j=1}^{n} k_j$, and for a function $f : \mathbb{C}^n \to \mathbb{C}$, we let

$$D^k f = \frac{\partial^{|k|} f}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}.$$

The following theorem establishes a Bernstein type inequality for entire functions of exponential type K in H^p-norm (or pre-norm).

Theorem 4.3.2 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, $p \in (0, \infty)$, and let K be a symmetric body in $\mathbb{R}^n$. Then, for any function $f \in \mathcal{E}(K) \cap H^p(T_\Gamma)$ and any multi-index $k = (k_1, \ldots, k_n)$, the following inequality holds*

$$\left\| D^k f \right\|_{H^p} \leq (2\pi)^{|k|} \prod_{j=1}^{n} \sigma_j^{k_j} \left\| f \right\|_{H^p}, \tag{4.61}$$

where $\sigma_j := \max_{t \in K^* \cap \Gamma^*} |t_j|$, $j = 1, \ldots, n$.

It is easy to see that $H^p(T_\Gamma)$ spaces are not included one into another. Thus, the inequalities comparing the H^p norms for different exponents p do not exist. However, if we require that the functions involved belong to $\mathcal{E}(K)$, then the following Nikol'skiĭ type inequality holds true.

Theorem 4.3.3 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and K be a symmetric body in $\mathbb{R}^n$. If a function f belongs to the class $\mathcal{E}(K) \cap H^p(T_\Gamma)$ for some $p \in (0, \infty)$, then it also belongs to $H^q(T_\Gamma)$ for any $q \in (p, \infty]$, and*

$$\|f\|_{H^q} \leq \lceil p/2 \rceil^{n(1/p - 1/q)} \left(m\left(K^* \cap \Gamma^*\right) \right)^{1/p - 1/q} \|f\|_{H^p}. \tag{4.62}$$

Here m denotes the Lebesgue measure in $\mathbb{R}^n$, and $\lceil a \rceil$ denotes the ceiling of a real number a, i.e., $\lceil a \rceil = \min\{m \in \mathbb{Z} : m \geq a\}$.

4.3.2 Proofs of Bernstein and Nikolskiĭ Type Inequalities

Proof of Theorem 4.3.2. Let us fix an arbitrary $\delta \in \Gamma$. If $p \in (0,1]$ and $f \in H^p(T_\Gamma)$, then $f_\delta \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap H^2(T_\Gamma)$ (see Lemma 1.4.1). It is also clear that $f_\delta \in \mathcal{E}(K)$. According to Theorem 4.3.1, $\operatorname{supp} \widehat{f_\delta} \subset K^*$. But since $f \in H^p(T_\Gamma)$ for some $p \in (0,1]$, we have that $\operatorname{supp} \widehat{f_\delta} \subset \Gamma^*$. Hence, $\operatorname{supp} \widehat{f_\delta} \subset K^* \cap \Gamma^* \subset \Omega$, where

$$\Omega := [-\sigma_1, \sigma_1] \times \cdots \times [-\sigma_n, \sigma_n]$$

is again a symmetric body. According to Theorem 4.3.1, $f_\delta \in \mathcal{E}(\Omega^*)$.

Now, let us consider the case $p \in (1, \infty)$. Take $r := [p] + 1$, where $[p]$ denotes the integer part of p, and consider the function

$$g(z) := (f(z))^r, \quad z \in \mathbb{C}^n.$$

It is clear that $g \in H^{p/r}(T_\Gamma)$ and $p/r \in (0,1)$. Moreover, $f \in \mathcal{E}(K)$ implies $g \in \mathcal{E}\left(\frac{1}{r}K\right)$.

Let us note that for any $r > 0$,

$$rK^* = \left(\frac{1}{r}K\right)^*.$$

Indeed,

$$rK^* = \{rt : (x,t) \leq 1, \forall x \in K\} = \left\{t : \left(x, \frac{1}{r}t\right) \leq 1, \forall x \in K\right\}$$

$$= \left\{t : \left(\frac{1}{r}x, t\right) \leq 1, \forall x \in K\right\} = \left\{t : (x,t) \leq 1, \forall x \in \frac{1}{r}K\right\} = \left(\frac{1}{r}K\right)^*.$$

Since now $g_\delta \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap H^2(T_\Gamma)$, Theorem 4.3.1 implies that $\operatorname{supp} \widehat{g_\delta} \subset \left(\frac{1}{r}K\right)^* \cap \Gamma^* = rK^* \cap \Gamma^* \subset r\Omega$. And hence, $g_\delta \in \mathcal{E}((r\Omega)^*)$. But then, for any $\varepsilon > 0$, there exists A_ε such that

$$\begin{aligned} |f_\delta(z)| = |g_\delta(z)|^{\frac{1}{r}} &\leq A_\varepsilon^{\frac{1}{r}} e^{2\pi(1+\varepsilon)\sup_{t \in r\Omega}\left|z_1 \frac{t_1}{r} + \cdots + z_n \frac{t_n}{r}\right|} \\ &= A_\varepsilon^{\frac{1}{r}} e^{2\pi(1+\varepsilon)\sup_{t \in \Omega}|z_1 t_1 + \cdots + z_n t_n|}. \end{aligned}$$

Thus, we obtain that $f_\delta \in \mathcal{E}(\Omega^*)$ for any $p \in (0,\infty)$ and any fixed $\delta \in \Gamma$.

Let us consider the function of one complex variable ζ,

$$\mathcal{F}(\zeta) := f_\delta(z_1,\ldots,z_{j-1},\zeta,z_{j+1},\ldots,z_n)$$

with all z_k's fixed ($k=1,\ldots,n$, $k\neq j$). Since $f_\delta \in \mathcal{E}(\Omega^*)$, then for any $\varepsilon>0$, there exists a constant A_ε such that

$$|\mathcal{F}(\zeta)| \le A_\varepsilon e^{2\pi(1+\varepsilon)\sup_{t\in\Omega}\left(\left|\sum_{k=\overline{1,n},k\neq j} z_k t_k\right|+|\zeta t_j|\right)} \le$$

$$A_\varepsilon e^{2\pi(1+\varepsilon)\sum_{k=\overline{1,n},k\neq j}|z_k|\sigma_k} e^{2\pi\sigma_j(1+\varepsilon)|\zeta|}.$$

Since $\mathcal{F}$ is a function of only ζ, it means that it is an entire function of exponential type at most $2\pi\sigma_j$. Applying the Bernstein inequality in $L^p(\mathbb{R})$ (for $p\ge 1$, see, e.g., [10, Ch. 11, § 11.3, Theorem 11.3.3]; for $p\in(0,1)$ – [78, Corollary 1]), we obtain

$$\|\mathcal{F}'\|_p \le 2\pi\sigma_j \|\mathcal{F}\|_p. \tag{4.63}$$

But the derivative of an entire function of exponential type is also an entire function of the same type (see, e.g., [10, Ch. 2, § 2.4, Theorem 2.4.1]). Thus, applying (4.63) k_j times and considering that z_k's, $k\neq j$, are arbitrary, we get the inequality

$$\int_{\mathbb{R}} \left| \left(\frac{\partial^{k_j}}{\partial z_j^{k_j}} f_\delta \right)(x_1,\ldots,x_n) \right|^p dx_j = \left\| \frac{d^{k_j}}{d\zeta^{k_j}} \mathcal{F} \right\|_p^p \le$$

$$(2\pi\sigma_j)^{k_j p} \|\mathcal{F}\|_p^p = (2\pi\sigma_j)^{k_j p} \int_{\mathbb{R}} |f_\delta(x_1,\ldots,x_n)|^p \, dx_j$$

that holds true for any $x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n \in \mathbb{R}$. Since $f_\delta \in L^p(\mathbb{R}^n)$, applying Tonelli-Fubini's theorem, we conclude that $\frac{\partial^{k_j}}{\partial z_j^{k_j}} f_\delta \in L^p(\mathbb{R}^n)$, and

$$\left\| \frac{\partial^{k_j}}{\partial z_j^{k_j}} f_\delta \right\|_p \le (2\pi\sigma_j)^{k_j} \|f_\delta\|_p \le (2\pi\sigma_j)^{k_j} \|f\|_{H^p} < \infty.$$

Since $\left(\frac{\partial^{k_j}}{\partial z_j^{k_j}} f_\delta\right)(x) = \left(\frac{\partial^{k_j}}{\partial z_j^{k_j}} f\right)(x+i\delta)$, and $\delta\in\Gamma$ was taken arbitrarily, passing to $\sup_{\delta\in\Gamma}$ in the last inequality, we obtain

$$\left\| \frac{\partial^{k_j}}{\partial z_j^{k_j}} f \right\|_{H^p} \le (2\pi\sigma_j)^{k_j} \|f\|_{H^p}. \tag{4.64}$$

It is also clear that in the multivariate case, partial differentiation in any variable preserves the function in the same class $\mathcal{E}(\Omega^*)$. Thus, applying (4.64) when differentiating with respect to other variables, we get (4.61). □

Note. The constant $(2\pi)^{|k|}$ in (4.61) is nothing else but a consequence of the definition of an entire function of exponential type in several variables given in [86, Ch. III, § 4]. For a classical univariate entire function of exponential type at most σ (the definition does not contain 2π in the exponent) belonging to H^p space in the upper half-plane, inequality (4.61) will have the following form

$$\left\|f^{(k)}\right\|_{H^p} \le \sigma^k \|f\|_{H^p}.$$

Proof of Theorem 4.3.3. **Step 1:** $q=\infty$**,** $p=2$**.** Since $f \in H^2(T_\Gamma)$, Theorem 3.1 from [86, Ch. III, § 3] implies that

$$f(z) = \int_{\Gamma^*} e^{2\pi i(z,t)} F(t)\, dt, \quad z \in T_\Gamma, \tag{4.65}$$

where $F \in L^2(\Gamma^*)$, and

$$\|f\|_{H^2} = \left(\int_{\Gamma^*} |F(t)|^2\, dt\right)^{1/2}. \tag{4.66}$$

Since $f \in \mathcal{E}(K)$, then clearly $f_\delta \in \mathcal{E}(K)$, for any $\delta \in \Gamma$. Hence, $f_\delta(x)$ is a restriction on $\mathbb{R}^n$ of a function from the class $\mathcal{E}(K)$, and $f_\delta \in L^2(\mathbb{R}^n)$. According to Theorem 4.3.1, f_δ is a Fourier transform of a function $\mathcal{F}$ vanishing outside K^*, i.e.,

$$f(x+i\delta) = f_\delta(x) = \int_{K^*} e^{-2\pi i(x,t)} \mathcal{F}(t)\, dt = \int_{K^*} e^{2\pi i(x,t)} \mathcal{F}(-t)\, dt. \tag{4.67}$$

From (4.65) and (4.67) we have that

$$f(x+i\delta) = \int_{\Gamma^*} e^{2\pi i(x,t)} e^{-2\pi(\delta,t)} F(t)\, dt = \int_{K^*} e^{2\pi i(x,t)} \mathcal{F}(-t)\, dt, \quad x \in \mathbb{R}^n.$$

Hence,

$$e^{-2\pi(\delta,t)} F(t)\, \chi_{\Gamma^*}(t) = \mathcal{F}(-t)\, \chi_{K^*}(t),$$

for a.e. $t \in \mathbb{R}^n$. Since Γ is a regular cone, $K^* \cap \Gamma^*$ has a non-empty interior. Therefore, the last equality implies

$$\mathcal{F}(-t) = e^{-2\pi(\delta,t)} F(t) \chi_{K^* \cap \Gamma^*}(t),$$

for a.e. $t \in \mathbb{R}^n$. Thus, from (4.67), we get

$$f(x + i\delta) = \int_{K^* \cap \Gamma^*} e^{2\pi i(x,t)} e^{-2\pi(\delta,t)} F(t)\, dt, \quad x \in \mathbb{R}^n.$$

Since $(\delta, t) \geq 0$ for any $\delta \in \Gamma$, $t \in \Gamma^*$ by the definition of the conjugate cone, using the Cauchy-Schwartz inequality and (4.66), we obtain

$$\begin{aligned} \|f_\delta\|_\infty &\leq \left(\int_{K^* \cap \Gamma^*} \left| e^{-2\pi(\delta,t)} F(t) \right|^2 dt \right)^{1/2} (m(K^* \cap \Gamma^*))^{1/2} \\ &\leq \|f\|_{H^2} (m(K^* \cap \Gamma^*))^{1/2}, \quad y \in \Gamma. \end{aligned}$$

Passing to $\sup_{\delta \in \Gamma}$ in the last inequality, we have

$$\|f\|_{H^\infty} \leq \|f\|_{H^2} (m(K^* \cap \Gamma^*))^{1/2}. \tag{4.68}$$

Step 2: $p \in (0, \infty)$, $q = \infty$. Let us denote $r := \lceil p/2 \rceil$. Then $p \leq 2r < p + 2$. Consider the following function

$$g(z) := \left(f\left(\frac{z}{r} \right) \right)^r, \quad z \in \mathbb{C}^n.$$

Since $f \in \mathcal{E}(K)$, then for any $\varepsilon > 0$ there exists a constant A_ε such that

$$|f(z)| \leq A_\varepsilon e^{2\pi(1+\varepsilon)\sup_{t \in K^*}|z_1 t_1 + \cdots + z_n t_n|}, \quad z \in \mathbb{C}^n.$$

Hence

$$|g(z)| \leq A_\varepsilon^r e^{2\pi(1+\varepsilon)\sup_{t \in K^*}|z_1 t_1 + \cdots + z_n t_n|}, \quad z \in \mathbb{C}^n.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this implies $g \in \mathcal{E}(K)$. It is also clear that for any $\zeta \in \Gamma$, $g_\zeta \in \mathcal{E}(K)$. Furthermore, $f \in H^p(T_\Gamma)$ implies $f_{\zeta/r} \in H^\infty(T_\Gamma)$ (see Lemma 1.4.1), and for an arbitrary $y \in \Gamma$, we have

$$\|g_\zeta(\cdot + iy)\|_2 = \left(\int_{\mathbb{R}^n} \left| f\left(\frac{x + i\zeta + iy}{r} \right) \right|^{2r} dx \right)^{1/2}$$

$$\leq \left\| f\left(\cdot + i\frac{\zeta}{r} + i\frac{y}{r}\right)\right\|_{\infty}^{\frac{2r-p}{2}} \left(\int_{\mathbb{R}^n} \left| f\left(\frac{x}{r} + i\frac{\zeta+y}{r}\right)\right|^p dx\right)^{1/2}$$

$$\leq \left\| f_{\zeta/r}\right\|_{H^\infty}^{r-p/2} r^{n/2} \left\| f\right\|_{H^p}^{p/2}.$$

Since $y \in \Gamma$ was chosen arbitrarily, we conclude that $g_\zeta \in H^2(T_\Gamma)$, and

$$\left\| g_\zeta\right\|_{H^2} \leq r^{n/2} \left\| f_{\zeta/r}\right\|_{H^\infty}^{r-p/2} \left\| f\right\|_{H^p}^{p/2}.$$

From (4.68), we now deduce

$$\left\| g_\zeta\right\|_{H^\infty} \leq \left(m\left(K^* \cap \Gamma^*\right)\right)^{1/2} r^{n/2} \left\| f_{\zeta/r}\right\|_{H^\infty}^{r-p/2} \left\| f\right\|_{H^p}^{p/2}.$$

From the definition of g, we have $\left\| g_\zeta\right\|_{H^\infty} = \left\| f_{\zeta/r}\right\|_{H^\infty}^{r}$. If $f \not\equiv 0$ (in which case, the statement is trivial), then the last inequality implies

$$\left\| f_{\zeta/r}\right\|_{H^\infty}^{p/2} \leq \left(m\left(K^* \cap \Gamma^*\right)\right)^{1/2} r^{n/2} \left\| f\right\|_{H^p}^{p/2}.$$

Since $\zeta \in \Gamma$ was chosen arbitrarily, taking the $\sup_{\zeta\in\Gamma}$, we obtain

$$\left\| f\right\|_{H^\infty} \leq \left(m\left(K^* \cap \Gamma^*\right)\right)^{1/p} r^{n/p} \left\| f\right\|_{H^p}. \tag{4.69}$$

Step 3. If $q \in (p, \infty)$, then

$$\left\| f\right\|_{H^q}^{q} \leq \left\| f\right\|_{H^\infty}^{q-p} \left\| f\right\|_{H^p}^{p},$$

whence, (4.62) follows from (4.69) immediately. □

Note that the function $(p/2+1)^{1/p}$ is strictly decreasing on $(0, \infty)$. Indeed, $(p/2+1)^{1/p} = e^{h(p)}$, where

$$h(p) := \frac{\ln(p/2+1)}{p}.$$

The function $h(p)$ is strictly decreasing on $(0, \infty)$, which can easily be proven using elementary Calculus.

Hence, $(p/2+1)^{1/p} \leq \sqrt{2}$, for any $p \in [2, \infty)$, and thus

$$\lceil p/2\rceil^{n(1/p-1/q)} \leq \left(\frac{p}{2}+1\right)^{\frac{n}{p}\left(1-\frac{p}{q}\right)} \leq 2^{\frac{n}{2}\left(1-\frac{p}{q}\right)}, \quad 2 \leq p < q.$$

This leads us to the following corollary.

Corollary 4.3.1 *Let Γ be a regular cone in $\mathbb{R}^n$, $n \in \mathbb{N}$, and K be a symmetric body in $\mathbb{R}^n$. If a function f belongs to the class $\mathcal{E}(K) \cap H^p(T_\Gamma)$ for some $p \in (0, \infty)$, then it also belongs to $H^q(T_\Gamma)$ for any $q \in (p, \infty]$, and*

$$\|f\|_{H^q} \leq 2^{\frac{n}{2}\left(1-\frac{p}{q}\right)} \left(m\left(K^* \cap \Gamma^*\right)\right)^{1/p-1/q} \|f\|_{H^p}.$$

Here m denotes the Lebesgue measure in $\mathbb{R}^n$.

Further Remarks

It is interesting to know if inequalities (4.61) and (4.62) are sharp. If $p = \infty$, Γ is the interior of the first octant $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \geq 0,\ j = 1, \ldots, n\}$, and $K = \prod_{j=1}^n \left[-\frac{2\pi}{\tau_j}, \frac{2\pi}{\tau_j}\right]$, the Bernstein inequality has the form

$$\left\|D^k f\right\|_{H^\infty} \leq \prod_{j=1}^{n} \tau_j^{k_j} \|f\|_{H^\infty},$$

which is obviously sharp. The equality is achieved, for example, on $f(z) = \prod_{j=1}^n e^{i\tau_j z_j}$.

If $p < \infty$, then the problem of sharpness is open even in the univariate case. Let us cite one of the results due to Q. I. Rahman and Q. M. Tariq.

Theorem 4.3.4 ([80, Th. 3]) *Let f be an entire function of exponential type τ satisfying the condition $f(z) = e^{i\tau z} f(-z)$. Furthermore, let f belong to L^2 on the real axis. Then*

$$\int_{-\infty}^{\infty} \left|f'(x)\right|^2 dx \leq \frac{\tau^2}{2} \int_{-\infty}^{\infty} \left|f'(x)\right|^2 dx. \tag{4.70}$$

The coefficient $\tau^2/2$ of $\int_{-\infty}^{\infty} |f(x)|^2\, dx$ in (4.70) cannot be replaced by a smaller number.

Note that for $p = \infty$, the condition $f(z) = e^{i\tau z} f(-z)$ does not help to decrease the constant, i.e., the constant τ in the inequality

$$\sup_{x \in \mathbb{R}} \left|f'(x)\right| \leq \tau \sup_{x \in \mathbb{R}} |f(x)|$$

is sharp (see [79]). The problem of the smallest possible constant for other p's is stated in [80] as open.

Despite the fact that we do not claim sharpness of the constant in (4.62), it is an improvement of the result of I. I. Ibragimov [49]. It deals with functions from the class $W_{\sigma}^{(p)}$ of entire functions of exponential type σ having finite L^p norm on the real axis. Its multivariate analog $W_{\sigma_1,\ldots,\sigma_n}^{(p)}$ is, in our notations, $\mathcal{E}(K) \cap L^p(\mathbb{R}^n)$, where $K = \prod_{j=1}^{n} \left[-\frac{2\pi}{\sigma_j}, \frac{2\pi}{\sigma_j}\right]$. The following estimate was obtained.

Theorem 4.3.5 ([49, Th. 1*]) *If $f(z_1, \ldots, z_n) \in W_{\sigma_1,\ldots,\sigma_n}^{(p)}$ and $1 \leq p < q \leq \infty$, then*

$$\|f(x_1, \ldots, x_n)\|_q \leq \begin{cases} \prod\limits_{j=1}^{n} \left(\frac{\sigma_j}{\pi}\right)^{\frac{1}{p}-\frac{1}{q}} \|f(x_1, \ldots, x_n)\|_p, & 1 \leq p \leq 2, \\ \prod\limits_{j=1}^{n} \left(\frac{p\sigma_j}{\pi}\right)^{\frac{1}{p}-\frac{1}{q}} \|f(x_1, \ldots, x_n)\|_p, & p > 2, \end{cases} \tag{4.71}$$

where $\|f(x_1, \ldots, x_n)\|_p^p = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} |f(x_1, \ldots, x_n)|^p \, dx_1 \ldots dx_n$.

For $K = \prod_{j=1}^{n} \left[-\frac{2\pi}{\sigma_j}, \frac{2\pi}{\sigma_j}\right]$ and Γ being the interior of $\mathbb{R}_+^n$, our Theorem 4.3.3 yields

$$\|f\|_{H^q} \leq \prod_{j=1}^{n} \left(\lceil p/2 \rceil \frac{\sigma_j}{2\pi}\right)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{H^p},$$

which is better than (4.71) and is valid for any $0 < p < q \leq \infty$. However, the class $W_{\sigma_1,\ldots,\sigma_n}^{(p)}$ is larger than $\mathcal{E}(K) \cap H^p(T_\Gamma)$.

CHAPTER 5

Riesz Decomposition for Poly-Superharmonic Functions in $\mathbb{R}^n$

The proof of Theorem 1.5.5 follows, in general, the idea of the proof of [52, Th. 1.2]. But the general case of m-superharmonic functions is more complicated, whence we need to develop appropriate tools first. This is done in Sections 5.1 and 5.3.

5.1 Lemmas on Riesz Kernels

We will assume that x, y are vectors in $\mathbb{R}^n$, $m, n, L \in \mathbb{N}$, $n \geq 2$, and that $2m < n$ or $2m - n$ is a positive odd integer.

Following [52], we consider the *generalized Riesz kernels*

$$K_{2m,L}(x,y) := \begin{cases} K_{2m}(x-y), & |y| < 1, \\ K_{2m}(x-y) - \sum\limits_{|\nu| \leq L} \frac{x^\nu}{\nu!} \left(D^\nu K_{2m}\right)(-y), & |y| \geq 1, \end{cases} \qquad L \in \mathbb{Z}_+.$$

Let us recall that for a multi-index $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_j \in \mathbb{Z}_+$,

$$x^\nu = x_1^{\nu_1} \cdot \cdots \cdot x_n^{\nu_n}, \quad \nu! = \nu_1! \cdot \cdots \cdot \nu_n!, \quad |\nu| = \nu_1 + \cdots + \nu_n, \quad D^\nu f(x) = \frac{\partial^{|\nu|} f}{\partial x_1^{\nu_1} \ldots \partial x_n^{\nu_n}}.$$

We will also use Δ_x to denote the n-dimentional Laplace operator applied with respect to the variable $x \in \mathbb{R}^n$.

Lemma 5.1.1 *If $2m < n$ or $2m - n$ is a positive odd integer, then*

$$\Delta_x K_{2m}(x) = (2m-n)(2m-2)K_{2(m-1)}(x), \tag{5.1}$$

$$\Delta_x K_{2m,2(m-1)}(x,y) = (2m-n)(2m-2)K_{2(m-1),2(m-2)}(x,y). \tag{5.2}$$

Proof. Since $2m < n$ or $2m-n$ is a positive odd integer, then

$$\frac{\partial}{\partial x_j}\left(|x|^{2m-n}\right) = \frac{\partial}{\partial x_j}\left(\left(x_1^2+\cdots+x_n^2\right)^{m-n/2}\right) = (2m-n)x_j\left(x_1^2+\cdots+x_n^2\right)^{m-n/2-1}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x_j^2}\left(|x|^{2m-n}\right) &= (2m-n)\left(\left(x_1^2+\cdots+x_n^2\right)^{m-n/2-1}\right.\\ &+ \left. x_j^2(2m-n-2)\left(x_1^2+\cdots+x_n^2\right)^{m-n/2-2}\right). \qquad (5.3)\end{aligned}$$

Hence

$$\begin{aligned}\Delta_x\left(|x|^{2m-n}\right) &= (2m-n)\left(n|x|^{2m-n-2}+(2m-n-2)|x|^{2m-n-2}\right)\\ &= (2m-n)(2m-2)|x|^{2m-n-2}.\end{aligned}$$

This gives (5.1). Now, for $|y| \geq 1$, we get

$$\frac{\partial}{\partial x_j}\left(\sum_{|\nu|\leq 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)$$

$$= \sum_{|\nu|\leq 2m-2}\left(\frac{1}{\nu!}\left(D^\nu K_{2m}\right)(-y)\nu_j x^{\nu_j-1}\prod_{k=\overline{1,n},\, k\neq j} x_k^{\nu_k}\right).$$

Therefore,

$$\frac{\partial^2}{\partial x_j^2}\left(\sum_{|\nu|\leq 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)$$

$$= \sum_{|\nu|\leq 2m-2}\left(\frac{1}{\nu!}\left(D^\nu K_{2m}\right)(-y)\nu_j(\nu_j-1)x^{\nu_j-2}\prod_{k=\overline{1,n},\, k\neq j} x_k^{\nu_k}\right)$$

$$= \sum_{\nu_1+\cdots+\nu_n\leq 2(m-1),\, \nu_j\geq 2}\frac{x_1^{\nu_1}}{\nu_1!}\cdots\frac{x_{j-1}^{\nu_{j-1}}}{\nu_{j-1}!}\frac{x_j^{\nu_j-2}}{(\nu_j-2)!}\frac{x_{j+1}^{\nu_{j+1}}}{\nu_{j+1}!}\cdots\frac{x_n^{\nu_n}}{\nu_n!}\left(D^{\nu_1\ldots\nu_n}K_{2m}\right)(-y).$$

Replacing the multi-index ν by $\tilde{\nu} = (\tilde{\nu}_1,\ldots,\tilde{\nu}_n)$ with

$$\tilde{\nu}_k = \begin{cases}\nu_k, & k\neq j,\\ \nu_j-2, & k=j,\end{cases}$$

we obtain

$$\frac{\partial^2}{\partial x_j^2}\left(\sum_{|\nu|\le 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)=\sum_{|\tilde\nu|\le 2(m-2)}\frac{x^{\tilde\nu}}{\tilde\nu!}\left(D^{\tilde\nu_1\ldots\tilde\nu_{j-1}(\tilde\nu_j+2)\tilde\nu_{j+1}\ldots\tilde\nu_n}K_{2m}\right)(-y). \tag{5.4}$$

From (5.3), it is clear that

$$\begin{aligned}\left(D^{\tilde\nu_1\ldots\tilde\nu_{j-1}(\tilde\nu_j+2)\tilde\nu_{j+1}\ldots\tilde\nu_n}K_{2m}\right)(-y)&=D^{\tilde\nu}\left(\frac{\partial^2}{\partial y_j^2}K_{2m}\right)(-y)\\&=(2m-n)D^{\tilde\nu}\left(K_{2(m-1)}+y_j^2(2m-n-2)K_{2(m-2)}\right)(-y).\end{aligned}$$

Setting $\nu := \tilde\nu$ on the right-hand side of (5.4), we see that

$$\begin{aligned}&\frac{\partial^2}{\partial x_j^2}\left(\sum_{|\nu|\le 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)\\=&\ (2m-n)\sum_{|\nu|\le 2(m-2)}\frac{x^\nu}{\nu!}\left(D^\nu K_{2(m-1)}\right)(-y)\\+&\ (2m-n)(2m-n-2)\sum_{|\nu|\le 2(m-2)}\frac{x^\nu}{\nu!}\left(D^\nu\left(y_j^2K_{2(m-2)}\right)\right)(-y).\end{aligned}$$

Taking the sum over $j=1,\ldots,n$, we deduce

$$\begin{aligned}&\Delta_x\left(\sum_{|\nu|\le 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)\\=&\ (2m-n)n\sum_{|\nu|\le 2(m-2)}\frac{x^\nu}{\nu!}\left(D^\nu K_{2(m-1)}\right)(-y)\\+&\ (2m-n)(2m-n-2)\sum_{|\nu|\le 2(m-2)}\frac{x^\nu}{\nu!}\left(D^\nu K_{2(m-1)}\right)(-y)\\=&\ (2m-n)(2m-2)\sum_{|\nu|\le 2(m-2)}\frac{x^\nu}{\nu!}\left(D^\nu K_{2(m-1)}\right)(-y).\end{aligned}\tag{5.5}$$

Thus, considering (5.1), we obtain (5.2). □

Corollary 5.1.1 *If $2m < n$ or $2m-n$ is a positive odd integer, then for any $k \in \mathbb{Z}_+$*

$$\Delta_x^k K_{2m}(x) = c_{m,n,k}K_{2(m-k)}(x), \tag{5.6}$$

and

$$\Delta_x^k K_{2m,2(m-1)}(x,y) = c_{m,n,k}K_{2(m-k),2(m-k-1)}(x,y), \tag{5.7}$$

where

$$c_{m,n,k} := \begin{cases} 1, & k = 0, \\ 2^k \prod_{j=0}^{k-1} \left((2(m-j)-n)(m-j-1) \right), & 1 \leq k \leq m-1, \\ 0, & k \geq m. \end{cases}$$

In particular, $K_{2m}(x)$ and $K_{2m,2(m-1)}(x,y)$ (with y as a parameter) are m-harmonic functions in $\mathbb{R}^n \setminus \{0\}$.

Proof. Formulas (5.6) and (5.7) just follow from Lemma 5.1.1. Let us check the 'boundary case', $k = m$. Clearly,

$$\Delta_x^m K_{2m,2(m-1)}(x,y) = \Delta_x \left(\Delta_x^{m-1} K_{2m,2(m-1)} \right)(x,y) = c_{m,n,m-1} \Delta_x K_{2,0}(x,y).$$

Now,

$$K_{2,0}(x,y) = \begin{cases} K_2(x-y), & |y| < 1, \\ K_2(x-y) - K_2(-y), & |y| \geq 1. \end{cases}$$

Thus, $\Delta_x K_{2,0}(x,y) = \Delta_x K_2(x-y)$. Furthermore,

$$\frac{\partial}{\partial x_j} K_2(x) = \left(1 - \frac{n}{2}\right) 2x_j \left(x_1^2 + \cdots + x_n^2\right)^{-n/2},$$

$$\frac{\partial^2}{\partial x_j^2} K_2(x) = (2-n) \left(\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - n x_j^2 \left(x_1^2 + \cdots + x_n^2\right)^{-n/2-1} \right).$$

Hence,

$$\Delta_x K_2(x) = (2-n) \left(n \left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - n \left(x_1^2 + \cdots + x_n^2\right)^{-n/2} \right) = 0.$$

Thus,

$$\Delta_x^m K_{2m,2(m-1)}(x,y) = 0.$$

□

Lemma 5.1.2 *If $2m < n$ or $2m - n$ is a positive odd integer, then for any $r > 0$,*

$$M\left(r, K_{2m}(\cdot - y)\right) = \begin{cases} \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \left(\frac{|y|}{2}\right)^{2k} \frac{c_{m,n,k}}{k! \Gamma\left(\frac{n}{2}+k\right)} r^{2(m-k)-n}, & |y| \leq r, \\ \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \left(\frac{r}{2}\right)^{2k} \frac{c_{m,n,k}}{k! \Gamma\left(\frac{n}{2}+k\right)} |y|^{2(m-k)-n}, & |y| > r, \end{cases}$$

where $c_{m,n,k}$ are defined in Corollary 5.1.1.

Moreover, for any $y \neq 0$ and $r > 0$,

$$\begin{aligned} & \frac{1}{\sigma_n r^{n-1}} \int_{S(0,r)} \sum_{|\nu| \leq 2m-2} \frac{x^\nu}{\nu!} \left(D^\nu K_{2m}\right)(-y)\, d\sigma(x) \\ = & \ \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \left(\frac{r}{2}\right)^{2k} \frac{c_{m,n,k}}{k!\Gamma\left(k+\frac{n}{2}\right)} |y|^{2(m-k)-n}. \end{aligned} \tag{5.8}$$

Proof. We will use formula (7.11) from [106, Ch. 1.7]:

$$\int_{B(0,r)} f(x+y)\, dx = \sum_{k=0}^{m-1} \frac{\pi^{n/2} r^{2k+n} \left(\Delta^k f\right)(y)}{2^{2k} k! \Gamma\left(k+\frac{n}{2}+1\right)}, \tag{5.9}$$

which is valid for any function $f \in C^{2m}(\mathcal{U}) \cap \mathcal{H}^m(\mathcal{U})$ for some domain $\mathcal{U}$, $y \in \mathcal{U}$, and any $r \in (0, \operatorname{dist}(y, \partial\mathcal{U}))$.

Assume $|y| > r$. Applying (5.9) with $f = K_{2m}$, $\mathcal{U} = \mathbb{R}^n \setminus \{0\}$, and using Corollary 5.1.1, we get

$$\begin{aligned} \int_{B(0,r)} K_{2m}(x-y)\, dx &= \sum_{k=0}^{m-1} \frac{\pi^{n/2} r^{2k+n} c_{m,n,k} K_{2(m-k)}(-y)}{2^{2k} k! \Gamma\left(k+\frac{n}{2}+1\right)} \\ &= \sum_{k=0}^{m-1} \frac{\pi^{n/2} r^{2k+n} c_{m,n,k} |y|^{2(m-k)-n}}{2^{2k} k! \Gamma\left(k+\frac{n}{2}+1\right)}. \end{aligned} \tag{5.10}$$

If we fix y and let $r < |y|$ be arbitrary, then differentiating the last equality with respect to r, we obtain

$$\begin{aligned} M\left(r, K_{2m}(\cdot - y)\right) &= \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2} r^{n-1}} \sum_{k=0}^{m-1} \frac{\pi^{n/2}(2k+n) r^{2k+n-1} c_{m,n,k} |y|^{2(m-k)-n}}{2^{2k} k! \Gamma\left(k+\frac{n}{2}+1\right)} \\ &= \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \frac{r^{2k} c_{m,n,k} |y|^{2(m-k)-n}}{2^{2k} k! \Gamma\left(k+\frac{n}{2}\right)}. \end{aligned} \tag{5.11}$$

Now, let $0 < |y| < r$. We cannot apply the above approach since we have a singularity in $B(0,r)$. To get rid of it, we will use the reflection technique as in Kelvin transform, described in [4, Ch. 1, § 1.6]. For $w \neq 0$, we will consider its inverse with respect to the unit sphere $S(0,1)$:

$$w^* := \frac{1}{|w|^2} w.$$

If $x \in S(0,1)$, and $y \neq 0$, then

$$|y|\,|x-y^*| = |x-y|. \tag{5.12}$$

Indeed,

$$\begin{aligned} |y|^2\,|x-y^*|^2 &= |y|^2\left(|x|^2+|y^*|^2-2x\cdot y^*\right) = |y|^2\left(1+\frac{1}{|y|^2}-\frac{2x\cdot y}{|y|^2}\right) \\ &= |y|^2+1-2x\cdot y = |y|^2+|x|^2-2x\cdot y = |x-y|^2. \end{aligned}$$

Changing variable $w = x/r$, we obtain

$$M\left(r, K_{2m}\left(\cdot - y\right)\right) = \frac{1}{\sigma_n}\int_{S(0,1)} |rw-y|^{2m-n}\,d\sigma(w) = \frac{r^{2m-n}}{\sigma_n}\int_{S(0,1)}\left|x-\frac{y}{r}\right|^{2m-n} d\sigma(x).$$

Using (5.12), we get

$$\begin{aligned} M\left(r, K_{2m}\left(\cdot - y\right)\right) &= \frac{|y|^{2m-n}}{\sigma_n}\int_{S(0,1)}\left|x-\left(\frac{y}{r}\right)^*\right|^{2m-n} d\sigma(x) \\ &= |y|^{2m-n} M\left(1, K_{2m}\left(\cdot - \left(\frac{y}{r}\right)^*\right)\right). \end{aligned}$$

Since $\left|\left(\frac{y}{r}\right)^*\right| = \frac{r}{|y|} > 1$, we can apply (5.11) with $r = 1$ to get

$$\begin{aligned} M\left(r, K_{2m}\left(\cdot - y\right)\right) &= |y|^{2m-n}\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\frac{c_{m,n,k}}{2^{2k}k!\Gamma\left(k+\frac{n}{2}\right)}\left(\frac{r}{|y|}\right)^{2(m-k)-n} \\ &= \Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\left(\frac{|y|}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(k+\frac{n}{2}\right)}r^{2(m-k)-n}. \end{aligned} \tag{5.13}$$

For $|y| = r$, let $y_l := \left(1+\frac{1}{l}\right)y$, $l \in \mathbb{N}$. Note that $|x-y| < |x-y_l|$ provided $|x| = r$. Indeed, since $|x| = |y| = r > 0$, we get

$$\begin{aligned} |x-y_l|^2 &= |x|^2 + \left(1+\frac{1}{l}\right)^2|y|^2 - 2\left(1+\frac{1}{l}\right)x\cdot y \\ &= |x-y|^2 + \frac{2}{l}\left(r^2 - x\cdot y\right) + \frac{1}{l^2}r^2 > |x-y|^2. \end{aligned}$$

Thus, if $2m < n$, we obtain $|x-y_l|^{2m-n} \leq |x-y|^{2m-n}$. Considering that the function $|x-y|^{2m-n}$ (as a function of x) is in $L^1\left(S\left(0,r\right)\right)$, we can apply the Lebesgue Dominated Convergence Theorem to get

$$M\left(r, K_{2m}\left(\cdot - y\right)\right) = \lim_{l\to\infty} M\left(r, K_{2m}\left(\cdot - y_l\right)\right).$$

If $2m-n\geq 0$, then $|x-y_l|^{2m-n}$ converges to $|x-y|^{2m-n}$ uniformly on $S(0,r)$, and the last equality obviously justified. Therefore, in either case, applying (5.11) with $y=y_l$, we deduce that

$$M\left(r,K_{2m}\left(\cdot-y\right)\right)=\lim_{l\to\infty}\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\frac{r^{2k}c_{m,n,k}|y_l|^{2(m-k)-n}}{2^{2k}k!\Gamma\left(k+\frac{n}{2}\right)}$$

$$=\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\frac{r^{2k}c_{m,n,k}|y|^{2(m-k)-n}}{2^{2k}k!\Gamma\left(k+\frac{n}{2}\right)}=\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\left(\frac{|y|}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(\frac{n}{2}+k\right)}r^{2(m-k)-n},\ y\neq 0.$$

If $y=0$, then (5.13) is obvious.

To obtain (5.8), we should use (5.5) to conclude that

$$\Delta_x^k\left(\sum_{|\nu|\leq 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\right)=c_{m,n,k}\sum_{|\nu|\leq 2(m-k-1)}\frac{x^\nu}{\nu!}\left(D^\nu K_{2(m-k)}\right)(-y),$$

and then apply (5.9) with $\mathcal{U}=\mathbb{R}^n$, $y=0$, $f(x)=\sum_{|\nu|\leq 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)$, where y is considered as a constant. Thus, we get

$$\int_{B(0,r)}\sum_{|\nu|\leq 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\,dx=\sum_{k=0}^{m-1}\frac{\pi^{n/2}r^{2k+n}c_{m,n,k}}{2^{2k}k!\Gamma\left(k+\frac{n}{2}+1\right)}|y|^{2(m-k)-n}.$$

This equality is valid for any $y\neq 0$ and $r>0$. Differentiating with respect to r, we obtain

$$\frac{1}{\sigma_n r^{n-1}}\int_{S(0,r)}\sum_{|\nu|\leq 2m-2}\frac{x^\nu}{\nu!}\left(D^\nu K_{2m}\right)(-y)\,d\sigma(x)=\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\frac{r^{2k}c_{m,n,k}}{4^k k!\Gamma\left(k+\frac{n}{2}\right)}|y|^{2(m-k)-n}.$$

□

Note. There is even more general result on spherical means of the Riesz kernels due to J. S. Brauchart, P. D. Dragnev, E. B. Saff [16, Th. 2]. Their statement covers fractional powers of $|x-y|$, but the answer is given in terms of a hypergeometric function, which makes it more complicated to apply in our proofs.

Lemma 5.1.3 *If $2m<n$ or $2m-n$ is a positive odd integer, then for any $R>0$,*

$$\int_{B(0,R)}K_{2m}\left(x-y\right)\,dx$$

$$= \begin{cases} 2\pi^{n/2} \sum_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k! \Gamma\left(\frac{n}{2}+k\right)} \left(|y|^{2m} \left(\frac{1}{2k+n} - \frac{1}{2(m-k)} \right) + \frac{|y|^{2k} R^{2(m-k)}}{2(m-k)} \right), & |y| \leq R, \\ \pi^{n/2} \sum_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k! \Gamma\left(\frac{n}{2}+k+1\right)} |y|^{2(m-k)-n} R^{2k+n}, & |y| > R, \end{cases}$$

where $c_{m,n,k}$ are as in Corollary 5.1.1.

Proof. If $|y| \leq R$, then using Lemma 5.1.2, we get

$$\int_{B(0,R)} K_{2m}(x-y)\, dx = \int_0^R \left(\int_{S(0,r)} K_{2m}(x-y)\, d\sigma(x) \right) dr$$

$$= \int_0^R \sigma_n r^{n-1} M\left(r, K_{2m}(\cdot - y)\right) dr$$

$$= \sigma_n \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k! \Gamma\left(\frac{n}{2}+k\right)} \left(|y|^{2(m-k)-n} \int_0^{|y|} r^{2k+n-1}\, dr + |y|^{2k} \int_{|y|}^R r^{2(m-k)-1}\, dr \right)$$

$$= 2\pi^{n/2} \sum_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k! \Gamma\left(\frac{n}{2}+k\right)} \left(\frac{|y|^{2m}}{2k+n} + |y|^{2k} \frac{R^{2(m-k)} - |y|^{2(m-k)}}{2(m-k)} \right)$$

$$= 2\pi^{n/2} \sum_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k! \Gamma\left(\frac{n}{2}+k\right)} \left(|y|^{2m} \left(\frac{1}{2k+n} - \frac{1}{2(m-k)} \right) + \frac{|y|^{2k} R^{2(m-k)}}{2(m-k)} \right).$$

For $|y| > R$, the statement is just (5.10). □

5.2 Proof of Proposition 1.5.1

Proof of Proposition 1.5.1. Note that for $\alpha_{m,1} = 1$ and any $\alpha_{m,j}$, $j \geq 2$, we have

$$\begin{aligned} \sum_{j=1}^m \alpha_{m,j} F_m\left(2^{m-j} r\right) &= \sum_{k=0}^{m-1} a_k 4^{(m-1)k} r^{2k} + \sum_{k=0}^{m-1} \alpha_{m,2} a_k 4^{(m-2)k} r^{2k} + \cdots + \sum_{k=0}^{m-1} \alpha_{m,m} a_k r^{2k} \\ &= \sum_{k=0}^{m-1} a_k r^{2k} \left(4^{(m-1)k} + \alpha_{m,2} 4^{(m-2)k} + \cdots + \alpha_{m,m} \right) \\ &= \sum_{k=0}^{m-1} a_k r^{2k} \left(4^{(m-1)k} + \sum_{j=2}^m 4^{(m-j)k} \alpha_{m,j} \right). \end{aligned} \tag{5.14}$$

Let us show that there is the only set of $\alpha_{m,2}, \ldots, \alpha_{m,m}$, such that

$$4^{(m-1)k} + \sum_{j=2}^m 4^{(m-j)k} \alpha_{m,j} = 0, \quad k = 1, \ldots, m-1,$$

which is equivalent to (1.25) holding for every r and $a_0, \dots, a_m$. As we will also see, these $\alpha_{m,j}$'s satisfy (1.26).

We can rewrite the last system as

$$\sum_{j=2}^{m} 4^{(m-j)k} \alpha_{m,j} = -4^{(m-1)k}, \quad k = 1, \dots, m-1. \tag{5.15}$$

This is a linear system of $(m-1)$ equations for $(m-1)$ unknowns, whose matrix is

$$\left(\begin{array}{ccccccc|c} 4^{m-2} & 4^{m-3} & \dots & 4^{m-1-j} & \dots & 4 & 1 & -4^{m-1} \\ 4^{2(m-2)} & 4^{2(m-3)} & \dots & 4^{2(m-1-j)} & \dots & 4^2 & 1 & -4^{2(m-1)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \vdots \\ 4^{l(m-2)} & 4^{l(m-3)} & \dots & 4^{l(m-1-j)} & \dots & 4^l & 1 & -4^{l(m-1)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \vdots \\ 4^{(m-1)(m-2)} & 4^{(m-1)(m-3)} & \dots & 4^{(m-1)(m-1-j)} & \dots & 4^{m-1} & 1 & -4^{(m-1)(m-1)} \end{array}\right). \tag{5.16}$$

To evaluate the main determinant of this matrix, let us make a reflection in horizontal direction, so that the last column becomes first, next to the last becomes second, etc.:

$$A := \begin{pmatrix} 1 & 4 & \dots & 4^{j-1} & \dots & 4^{m-3} & 4^{m-2} \\ 1 & 4^2 & \dots & 4^{2(j-1)} & \dots & 4^{2(m-3)} & 4^{2(m-2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots \\ 1 & 4^l & \dots & 4^{l(j-1)} & \dots & 4^{l(m-3)} & 4^{l(m-2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots \\ 1 & 4^{m-1} & \dots & 4^{(m-1)(j-1)} & \dots & 4^{(m-1)(m-3)} & 4^{(m-1)(m-2)} \end{pmatrix}.$$

The main determinant D of the system (5.16) and the determinant of A are related by

$$D = (-1)^{\frac{m-1}{2}(m-2)} \det(A),$$

and the matrix A is a Vandermonde matrix, whose determinant is well known. Thus,

we obtain

$$D = (-1)^{\frac{m-1}{2}(m-2)} \prod_{1\le l<j\le m-1} \left(4^j - 4^l\right). \tag{5.17}$$

Since $D \neq 0$, the system (5.15) has a solution $\alpha_{m,2}, \ldots, \alpha_{m,m}$, and this solution is unique.

Now, for $k = 1, \ldots, m-1$, let us evaluate the determinant of the left-hand side of the matrix in (5.16) with k-th column replaced by the right-hand side of (5.16):

$$D_k = \begin{vmatrix} 4^{m-2} & \ldots & 4^{m-k} & -4^{m-1} & 4^{m-k-2} & \ldots & 4 & 1 \\ 4^{2(m-2)} & \ldots & 4^{2(m-k)} & -4^{2(m-1)} & 4^{2(m-k-2)} & \ldots & 4^2 & 1 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 4^{l(m-2)} & \ldots & 4^{l(m-k)} & -4^{l(m-1)} & 4^{l(m-k-2)} & \ldots & 4^l & 1 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 4^{(m-1)(m-2)} & \ldots & 4^{(m-1)(m-k)} & -4^{(m-1)(m-1)} & 4^{(m-1)(m-k-2)} & \ldots & 4^{m-1} & 1 \end{vmatrix}.$$

Multiplying the k-th column by -1 and then each column by the reciprocal of its first entry (i.e., multiplying j-th column by the reciprocal of $(1, j)$-entry), we get

$$D_k = -4^{m-2} \ldots 4^{m-k} 4^{m-1} 4^{m-k-2} \ldots 4$$

$$\times \begin{vmatrix} 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 \\ 4^{m-2} & \ldots & 4^{m-k} & 4^{m-1} & 4^{m-k-2} & \ldots & 4 & 1 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 4^{(l-1)(m-2)} & \ldots & 4^{(l-1)(m-k)} & 4^{(l-1)(m-1)} & 4^{(l-1)(m-k-2)} & \ldots & 4^{l-1} & 1 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 4^{(m-2)(m-2)} & \ldots & 4^{(m-2)(m-k)} & 4^{(m-2)(m-1)} & 4^{(m-2)(m-k-2)} & \ldots & 4^{m-2} & 1 \end{vmatrix}.$$

Since also the determinant of a transposed matrix is the same as the determinant of the initial one, taking transpose, we obtain

$$D_k = -4^{\frac{m}{2}(m-1)-(m-k-1)} \times \begin{vmatrix} 1 & 4^{m-2} & \dots & 4^{(j-1)(m-2)} & \dots & 4^{(m-2)(m-2)} \\ 1 & 4^{m-3} & \dots & 4^{(j-1)(m-3)} & \dots & 4^{(m-3)(m-2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 1 & 4^{m-k} & \dots & 4^{(j-1)(m-k)} & \dots & 4^{(m-k)(m-2)} \\ 1 & 4^{m-1} & \dots & 4^{(j-1)(m-1)} & \dots & 4^{(m-1)(m-2)} \\ 1 & 4^{m-k-2} & \dots & 4^{(j-1)(m-k-2)} & \dots & 4^{(m-k-2)(m-2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1 & \dots & 1 \end{vmatrix}.$$

Moving the k-th row to the first place, we conclude

$$D_k = (-1)^k 4^{\frac{m}{2}(m-1)-(m-k-1)} \times \begin{vmatrix} 1 & 4^{m-1} & \dots & 4^{(j-1)(m-1)} & \dots & 4^{(m-1)(m-2)} \\ 1 & 4^{m-2} & \dots & 4^{(j-1)(m-2)} & \dots & 4^{(m-2)(m-2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 1 & 4^{m-k} & \dots & 4^{(j-1)(m-k)} & \dots & 4^{(m-k)(m-2)} \\ 1 & 4^{m-k-2} & \dots & 4^{(j-1)(m-k-2)} & \dots & 4^{(m-k-2)(m-2)} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1 & \dots & 1 \end{vmatrix}.$$

Applying the formula for a Vandermonde determinant, we deduce

$$D_k = (-1)^k 4^{\frac{m}{2}(m-1)-(m-k-1)} \prod_{1\le l<j\le m-1} \left(\theta_{m,j,k} - \theta_{m,l,k}\right). \tag{5.18}$$

Finally, using Kramer's rule, $\alpha_{m,k+1} = \frac{D_k}{D}$, $k = 1, \dots, m-1$, whence (1.26) follows immediately from (5.17) and (5.18).

Conversely, if $\alpha_{m,2}, \dots, \alpha_{m,m}$ satisfy (5.15), representation (5.14) yields

$$\sum_{j=1}^{m} \alpha_{m,j} F_m\left(2^{m-j} r\right) = a_0 \left(1 + \sum_{j=2}^{m} \alpha_{m,j}\right) = a_0 \sum_{j=1}^{m} \alpha_{m,j}.$$

□

Note. We can give an explicit representation in (1.25) for some values of m:

$$m = 2: \qquad F_2(2r) - 4F_2(r) = -3a_0;$$

$$m = 3: \qquad F_3(4r) - 20F_3(2r) + 64F_3(r) = 45a_0;$$

$$m = 4: \quad F_4(8r) - 84F_4(4r) + 1344F_4(2r) - 4096F_4(r) = -2835a_0.$$

5.3 Spherical Means of m-Superharmonic Functions

The key ingredient to the proof of Theorem 1.5.5 is the following formula for spherical means.

Lemma 5.3.1 *Let $u \in \mathcal{SH}^m(\mathbb{R}^n)$, and let $\mu_u = (-\Delta)^m u$. Then for $r > 1$,*

$$M(r,u) = \int_{B(0,r)} f(r,y)\, d\mu_u(y) + \sum_{k=0}^{m-1} a_k r^{2k},$$

where a_k's are constants independent of r,

$$f(r,y) = c_{m,n}\Gamma\left(\frac{n}{2}\right) \begin{cases} \sum\limits_{k=0}^{m-1} \left(\frac{|y|}{2}\right)^{2k} \frac{c_{m,n,k}}{k!\Gamma\left(\frac{n}{2}+k\right)} r^{2(m-k)-n}, & |y| < 1, \\ \sum\limits_{k=0}^{m-1} \frac{c_{m,n,k}}{4^k k!\Gamma\left(\frac{n}{2}+k\right)} \left(|y|^{2k} r^{2(m-k)-n} - r^{2k}|y|^{2(m-k)-n}\right) & 1 \le |y| < r, \\ 0, & |y| \ge r, \end{cases}$$

$c_{m,n,k}$ are as in Corollary 5.1.1, and and $c_{m,n}$ are given by (1.23), so that

$$c_{m,n} (-\Delta)^m K_{2m,L}(\cdot, y) = \delta_y. \tag{5.19}$$

Proof. It follows from the Riesz decomposition that (see [32, Representation (3.1)]) if $v \in \mathcal{SH}^m(\mathbb{R}^n)$, then

$$v(x) = c_{m,n} \int_{B(0,R)} K_{2m,2(m-1)}(x,y)\, d\mu_v(y) + h_R(x), \quad x \in B(0,R),$$

where $h_R \in \mathcal{H}^m(B(0,R))$. (For (5.19), see [32, § 3].) Indeed, let us consider the following positive linear functional on $C_0^\infty(B(0,R))$:

$$L_p(\varphi) := \int_{B(0,R)} p(x) (-\Delta)^m \varphi(x)\, dx, \quad \varphi \in C_0^\infty(B(0,R)),$$

where

$$p(x) := c_{m,n} \int_{B(0,R)} K_{2m,2(m-1)}(x,y)\, d\mu_v(y).$$

Using Fubini's theorem and (5.19), we have

$$\begin{aligned} L_p(\varphi) &= \int_{B(0,R)} \left(c_{m,n} \int_{B(0,R)} K_{2m,2(m-1)}(x,y)\, d\mu_v(y) \right) (-\Delta)^m \varphi(x)\, dx \\ &= \int_{B(0,R)} \left(c_{m,n} \int_{B(0,R)} K_{2m,2(m-1)}(x,y)\, (-\Delta)^m \varphi(x)\, dx \right) d\mu_v(y) \\ &= \int_{B(0,R)} \varphi(y)\, d\mu_v(y) = L_v(\varphi). \end{aligned}$$

This implies that for a.e. $x \in B(0,R)$, $v(x) - p(x)$ coincides with a function from $\mathcal{H}^m(B(0,R))$. Let us call it $h_R(x)$. Thus, $v(x) = p(x) + h_R(x)$ a.e.

Note that two m-superharmonic functions, which are equal a.e., are equal identically. This follows from Property (iv) in Definition 1.5.5 (the definition of m-superharmonic function).

Now, we conclude that $v(x) = p(x) + h_R(x)$ everywhere in $B(0,R)$.

Therefore, since $u \in \mathcal{SH}^m(\mathbb{R}^n)$, then for any $r_2 > r_1 > 0$

$$u(x) = c_{m,n} \int_{B(0,r_j)} K_{2m,2(m-1)}(x,y)\, d\mu_u(y) + h_{r_j}(x), \quad x \in B(0,r_j), \quad j = 1,2, \tag{5.20}$$

where $h_{r_j} \in \mathcal{H}^m(B(0,r_j))$.

Let us fix two arbitrary r_1 and r_2 (assume $r_1 < r_2$), and take an arbitrary r with $1 < r < r_1 < r_2$. Integrating the last equality over the sphere of radius r, we obtain

$$\begin{aligned} M(r,u) &= \frac{c_{m,n}}{\sigma_n r^{n-1}} \int_{S(0,r)} \int_{B(0,r_j)} K_{2m,2(m-1)}(x,y)\, d\mu_u(y)\, d\sigma(x) \\ &+ \frac{1}{\sigma_n r^{n-1}} \int_{S(0,r)} h_{r_j}(x)\, d\sigma(x). \end{aligned} \tag{5.21}$$

Since $h_{r_j} \in \mathcal{H}^m(\mathbb{R}^n)$, the Almansi expansion (see, e.g., [5, Ch. I, Prop. 1.3]) implies that there exist functions $g_{0,j}, \ldots, g_{m-1,j}$ harmonic in $B(0,r_j)$, such that

$$h_{r_j}(x) = \sum_{k=0}^{m-1} |x|^{2k} g_{k,j}(x), \quad x \in B(0,r_j). \tag{5.22}$$

The mean-value property for harmonic functions yields

$$\frac{1}{\sigma_n r^{n-1}} \int_{S(0,r)} h_{r_j}(x)\, d\sigma(x) = \sum_{k=0}^{m-1} r^{2k} g_{k,j}(0). \tag{5.23}$$

Changing the order of integration in the first summand of (5.21) using Fubini's theorem, we get

$$\begin{aligned} &\frac{c_{m,n}}{\sigma_n r^{n-1}} \int_{S(0,r)} \int_{B(0,r_j)} K_{2m,2(m-1)}(x,y)\, d\mu_u(y)\, d\sigma(x) \\ &= c_{m,n} \int_{B(0,r_j)} M\left(r, K_{2m,2(m-1)}(\cdot,y)\right)\, d\mu_u(y). \end{aligned}$$

From Lemma 5.1.2, we conclude immediately that

$$c_{m,n} M\left(r, K_{2m,2(m-1)}(\cdot,y)\right) = f(r,y), \quad r > 0.$$

From (5.21), (5.23) and the last equality, we obtain

$$M(r,u) = \int_{B(0,r_j)} f(r,y)\, d\mu_u(y) + \sum_{k=0}^{m-1} r^{2k} g_{k,j}(0).$$

Since $f(r,y) = 0$ when $|y| > r$, the last equality can be rewritten as

$$M(r,u) - \int_{B(0,r)} f(r,y)\, d\mu_u(y) = \sum_{k=0}^{m-1} r^{2k} g_{k,j}(0). \tag{5.24}$$

Since the left-hand side is independent of $j \in \{1,2\}$, so is the right-hand side. But, for each $j \in \{1,2\}$, the expression in the right-hand side is a polynomial in r. Thus, we conclude that

$$\sum_{k=0}^{m-1} r^{2k} g_{k,1}(0) = \sum_{k=0}^{m-1} r^{2k} g_{k,2}(0), \quad r \in (1, r_1).$$

This immediately implies that the coefficients of this polynomial do not depend on j. So, taking any $r_1 > 1$, we may denote

$$a_k := g_{k,1}(0), \quad k = 0, \ldots, m-1, \tag{5.25}$$

and rewrite (5.24) as

$$M(r,u) = \int_{B(0,r)} f(r,y)\, d\mu_u(y) + \sum_{k=0}^{m-1} a_k r^{2k}.$$

□

It is clear that if $h \in \mathcal{H}^m(\mathbb{R}^n)$, then μ_h is a zero measure. Thus, for any $r > 1$,

$$M(r,h) = \sum_{k=0}^{m-1} a_k r^{2k},$$

and

$$\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j}r, h\right) = a_0 \sum_{j=1}^{m} \alpha_{m,j} = h(0) \sum_{j=1}^{m} \alpha_{m,j}.$$

Corollary 5.3.1 *Let $u \in \mathcal{SH}^m(\mathbb{R}^n)$, $2m < n$, and let $\mu_u = (-\Delta)^m u$. Then for any $r > 1$,*

$$\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j}r, u\right) = \int_{B(0,r)} \left(\sum_{j=1}^{m} \alpha_{m,j} f\left(2^{m-j}r, y\right)\right) d\mu_u(y)$$
$$+ \sum_{\nu=1}^{m-1} \int_{B(0,2^\nu r)\setminus B(0,2^{\nu-1}r)} \left(\sum_{j=1}^{m-\nu} \alpha_{m,j} f\left(2^{m-j}r, y\right)\right) d\mu_u(y) + a_0 \sum_{j=1}^{m} \alpha_{m,j}, \qquad (5.26)$$

where $f(r,y)$ is defined in Lemma 5.3.1, $\alpha_{m,1} = 1$, $\alpha_{m,2}, \ldots, \alpha_{m,m}$ are given by (1.26) in Proposition 1.5.1, and a_0 is from Lemma 5.3.1.

Furthermore, if $u(0) \neq \infty$, then

$$a_0 = u(0) - c_{m,n} \int_{B(0,1)} |y|^{2m-n} \, d\mu_u(y), \qquad (5.27)$$

where $c_{m,n}$ are given by (1.23).

Proof. Since $f(R,y) = 0$ when $|y| \geq R$, then representation (5.26) follows immediately from Lemma 5.3.1 and Proposition 1.5.1.

To get a_0, we need to refer to the proof of Lemma 5.3.1. Using (5.20) with some $r_1 > 1$, we conclude that

$$u(0) = c_{m,n} \int_{B(0,r_1)} K_{2m,2(m-1)}(0,y) \, d\mu_u(y) + h_{r_1}(0).$$

Since

$$K_{2m,2(m-1)}(0,y) = \begin{cases} |y|^{2m-n}, & |y| < 1 \\ 0, & |y| \geq 1, \end{cases}$$

we obtain

$$u(0) = c_{m,n} \int_{B(0,1)} |y|^{2m-n} \, d\mu_u(y) + h_{r_1}(0).$$

Now, (5.27) follows from (5.22) and (5.25). $\square$

Note. It is clear that if $h \in \mathcal{H}^m(\mathbb{R}^n)$, then μ_h is a zero measure. Moreover, using the same reasoning as in the proof of Lemma 5.3.1, we obtain that for any $r > 0$, $M(r,h) = \sum_{k=0}^{m-1} a_k r^{2k}$. Therefore, Proposition 1.5.1 and (5.27) imply

$$\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, h\right) = h(0) \sum_{j=1}^{m} \alpha_{m,j}, \quad r > 0. \tag{5.28}$$

5.4 Proof of the Riesz Decomposition

Lemma 5.4.1 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, $\mu_u = (-\Delta)^m u$, $k = 0, \ldots, m-1$, and*

$$\sup_{r>1} r^{2m-n} \mu_u(B(0,r)) < \infty. \tag{5.29}$$

Let also $1 \le a \le b$ and

$$c_1(b,r,m,n,k) := \int_{B(0,br)\setminus B(0,1)} |y|^{2k} r^{2(m-k)-n} \, d\mu_u(y),$$

$$c_2(a,b,r,m,n,k) := \int_{B(0,br)\setminus B(0,ar)} |y|^{2k} r^{2(m-k)-n} \, d\mu_u(y),$$

$$c_3(a,b,r,m,n,k) := \int_{B(0,br)\setminus B(0,ar)} |y|^{2(m-k)-n} r^{2k} \, d\mu_u(y).$$

Then

$$\sup_{r>1} |c_1(b,r,m,n,k)| < \infty, \ \sup_{r>1} |c_2(a,b,r,m,n,k)| < \infty, \ \sup_{r>1} |c_3(a,b,r,m,n,k)| < \infty.$$

Proof. It is clear that for any $k = 0, \ldots, m-1$,

$$|y|^{2k} r^{2(m-k)-n} \le b^{2k} r^{2m-n}, \quad y \in B(0,br);$$

$$|y|^{2(m-k)-n} r^{2k} \le a^{2(m-k)-n} r^{2m-n}, \quad y \in \mathbb{R}^n \setminus B(0,ar).$$

Therefore, the statement follows from (5.29) immediately. For example,

$$c_1\left(b,r,m,n,k\right) \le b^{2k} r^{2m-n} \int_{B(0,br)} d\mu_u(y)$$

$$= b^{-2(m-k)+n} \left(br\right)^{2m-n} \int_{B(0,br)} d\mu_u(y) \le b^{-2(m-k)+n} \sup_{r>1} \left(r^{2m-n} \int_{B(0,r)} d\mu_u(y) \right).$$

□

Lemma 5.4.2 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m\left(\mathbb{R}^n\right)$, and $\mu_u = \left(-\Delta\right)^m u$. Furthermore, let $\alpha_{m,j}$ be the absolute constants from Proposition 1.5.1. If*

$$\sup_{r>1} \left| \sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) \right| < \infty \quad \textit{and} \quad \sup_{r>1} \left| r^{2m-n} \mu_u\left(B(0,r)\right) \right| < \infty,$$

then

$$\sup_{r>1} \int_{B(0,r)\setminus B(0,1)} |y|^{2m-n} \, d\mu_u(y) < \infty.$$

Proof. Corollary 5.3.1 implies that

$$\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) = \int_{B(0,r)} \left(\sum_{j=1}^{m} \alpha_{m,j} f\left(2^{m-j} r, y\right) \right) d\mu_u(y)$$

$$+ \sum_{l=1}^{m-1} \int_{B\left(0,2^l r\right)\setminus B\left(0,2^{l-1} r\right)} \left(\sum_{j=1}^{m-l} \alpha_{m,j} f\left(2^{m-j} r, y\right) \right) d\mu_u(y) + a_0 \sum_{j=1}^{m} \alpha_{m,j}.$$

Let us denote

$$\beta_{m,n,k} := \Gamma\left(\frac{n}{2}\right) \frac{c_{m,n,k}}{4^k k! \Gamma\left(\frac{n}{2}+k\right)}, \tag{5.30}$$

where $c_{m,n,k}$ are defined in Corollary 5.1.1, i.e.

$$\beta_{m,n,k} := \begin{cases} 1, & k=0, \\ \frac{\Gamma\left(\frac{n}{2}\right)}{2^k k! \Gamma\left(\frac{n}{2}+k\right)} \prod_{j=0}^{k-1} \left(\left(2(m-j)-n\right)\left(m-j-1\right) \right), & 1 \le k \le m-1, \\ 0, & k \ge m. \end{cases}$$

Let us also remind that according to Proposition 1.5.1, $\alpha_{m,1} = 1$ and

$$\alpha_{m,k} = (-1)^{k+\frac{m}{2}(m-3)} 4^{\frac{m}{2}(m-1)-(m-k)} \frac{\prod_{1\le l<j\le m-1} \left(\theta_{m,j,k-1} - \theta_{m,l,k-1}\right)}{\prod_{1\le l<j\le m-1} \left(4^j - 4^l\right)},$$

where

$$\theta_{m,j,k-1} = \begin{cases} 4^{m-j}, & 1 \le j \le k-1, \\ 4^{m-1-j}, & k \le j \le m-1, \end{cases} \qquad 2 \le k \le m.$$

Using the representation of $f(r,y)$ given by Lemma 5.3.1, we get

$$\begin{aligned}
\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j}r, u\right) = & \int\limits_{B(0,1)} \left(\sum_{j=1}^{m} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} |y|^{2k} \left(2^{m-j}r\right)^{2(m-k)-n} \right) d\mu_u(y) \\
& + \int\limits_{B(0,r)\setminus B(0,1)} \left(\sum_{j=1}^{m} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} |y|^{2k} \left(2^{m-j}r\right)^{2(m-k)-n} \right) d\mu_u(y) \\
& - \int\limits_{B(0,r)\setminus B(0,1)} \left(\sum_{j=1}^{m} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} \left(2^{m-j}r\right)^{2k} |y|^{2(m-k)-n} \right) d\mu_u(y) \\
& + \sum_{l=1}^{m-1} \int\limits_{B\left(0,2^{l}r\right)\setminus B\left(0,2^{l-1}r\right)} \left(\sum_{j=1}^{m-l} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} |y|^{2k} \left(2^{m-j}r\right)^{2(m-k)-n} \right) d\mu_u(y) \\
& - \sum_{l=1}^{m-1} \int\limits_{B\left(0,2^{l}r\right)\setminus B\left(0,2^{l-1}r\right)} \left(\sum_{j=1}^{m-l} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} \left(2^{m-j}r\right)^{2k} |y|^{2(m-k)-n} \right) d\mu_u(y) \\
& + a_0 \sum_{j=1}^{m} \alpha_{m,j}. \qquad (5.31)
\end{aligned}$$

Now,

$$\begin{aligned}
& \sum_{j=1}^{m} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} \left(2^{m-j}r\right)^{2k} |y|^{2(m-k)-n} \\
& = \sum_{k=0}^{m-1} \beta_{m,n,k} r^{2k} |y|^{2(m-k)-n} \left(\sum_{j=1}^{m} \alpha_{m,j} 4^{(m-j)k} \right).
\end{aligned}$$

According to (5.15), $\sum_{j=1}^{m} \alpha_{m,j} 4^{(m-j)k} = 0$, $k = 1, \ldots, m-1$. Hence,

$$\sum_{j=1}^{m} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} \left(2^{m-j}r\right)^{2k} |y|^{2(m-k)-n} = |y|^{2m-n} \sum_{j=1}^{m} \alpha_{m,j}. \qquad (5.32)$$

Using (5.32), the linear combination of means (5.31) can be rewritten as

$$\sum_{j=1}^{m}\alpha_{m,j}M\left(2^{m-j}r,u\right)=\int\limits_{B(0,1)}\left(\sum_{j=1}^{m}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\right)d\mu_u(y)$$

$$+\int\limits_{B(0,r)\setminus B(0,1)}\left(\sum_{j=1}^{m}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\right)d\mu_u(y)$$

$$-\sum_{j=1}^{m}\alpha_{m,j}\int\limits_{B(0,r)\setminus B(0,1)}|y|^{2m-n}\,d\mu_u(y)$$

$$+\sum_{l=1}^{m-1}\int\limits_{B\left(0,2^{l}r\right)\setminus B\left(0,2^{l-1-1}r\right)}\left(\sum_{j=1}^{m-l}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\right)d\mu_u(y)$$

$$-\sum_{l=1}^{m-1}\int\limits_{B\left(0,2^{l}r\right)\setminus B\left(0,2^{l-1}r\right)}\left(\sum_{j=1}^{m-l}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}\left(2^{m-j}r\right)^{2k}|y|^{2(m-k)-n}\right)d\mu_u(y)$$

$$+\ a_0\sum_{j=1}^{m}\alpha_{m,j}. \tag{5.33}$$

It is easy to see that

$$\left|\int\limits_{B(0,1)}\left(\sum_{j=1}^{m}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\right)d\mu_u(y)\right|$$

$$\leq r^{2m-n}\mu_u\left(B(0,1)\right)\sum_{j=1}^{m}|\alpha_{m,j}|\sum_{k=0}^{m-1}|\beta_{m,n,k}|\left(2^{m-j}\right)^{2(m-k)-n}\to 0,\quad r\to\infty.$$

Hence

$$c_0\left(r,m,n,k\right):=\int\limits_{B(0,1)}\left(\sum_{j=1}^{m}\alpha_{m,j}\sum_{k=0}^{m-1}\beta_{m,n,k}|y|^{2k}\left(2^{m-j}r\right)^{2(m-k)-n}\right)d\mu_u(y)$$

is bounded as a function of r for $r>1$.

In terms of Lemma 5.4.1, we can rewrite (5.33) as

$$\begin{aligned}
\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) &= c_0\left(r, m, n, k\right) \\
&+ \sum_{j=1}^{m} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} 2^{(m-j)(2(m-k)-n)} c_1\left(1, r, m, n, k\right) \\
&- \sum_{j=1}^{m} \alpha_{m,j} \int\limits_{B(0,r)\setminus B(0,1)} |y|^{2m-n}\, d\mu_u(y) \\
&+ \sum_{l=1}^{m-1} \sum_{j=1}^{m-l} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} 2^{(m-j)(2(m-k)-n)} c_2\left(2^{l-1}, 2^{l}, r, m, n, k\right) \\
&- \sum_{l=1}^{m-1} \sum_{j=1}^{m-l} \alpha_{m,j} \sum_{k=0}^{m-1} \beta_{m,n,k} 4^{k(m-j)} c_3\left(2^{l-1}, 2^{l}, r, m, n, k\right) \\
&+ a_0 \sum_{j=1}^{m} \alpha_{m,j}.
\end{aligned}$$

Thus, Lemma 5.4.1 and boundedness of $c_0\left(r, m, n, k\right)$ imply that

$$\sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) = c\left(r, m, n, k\right) - \sum_{j=1}^{m} \alpha_{m,j} \int\limits_{B(0,r)\setminus B(0,1)} |y|^{2m-n}\, d\mu_u(y),$$

where $\sup_{r>1} |c\left(r, m, n, k\right)| < \infty$.

It is clear from (1.26) that for any fixed m, $\alpha_{m,j}$'s alternate in sign and grow in absolute value when j increases. Hence $\sum_{j=1}^{m} \alpha_{m,j} \neq 0$. Therefore, the condition $\sup_{r>1} \left| \sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) \right| < \infty$ implies that

$$\sup_{r>1} \int_{B(0,r)\setminus B(0,1)} |y|^{2m-n}\, d\mu_u(y) < \infty.$$

□

Lemma 5.4.3 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m\left(\mathbb{R}^n\right)$, and $\mu_u = \left(-\Delta\right)^m u$. Furthermore, let $\alpha_{m,j}$ be the absolute constants from Proposition 1.5.1. If*

$$\sup \left| \sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) \right| < \infty \quad \text{and} \quad \sup_{r>1} r^{2m-n} \mu_u\left(B(0,r)\right) < \infty,$$

then

$$\int_{\mathbb{R}^n} \left(1 + |y|\right)^{2m-n}\, d\mu_u(y) < \infty.$$

Proof. It is clear that

$$\int_{B(0,1)} (1+|y|)^{2m-n} \, d\mu_u(y) \leq \mu_u\left(B(0,1)\right) < \infty. \tag{5.34}$$

Furthermore,

$$\sup_{r>1} \int_{B(0,r)\setminus B(0,1)} (1+|y|)^{2m-n} \, d\mu_u(y) \leq \sup_{r>1} \int_{B(0,r)\setminus B(0,1)} |y|^{2m-n} \, d\mu_u(y). \tag{5.35}$$

The last expression is finite because of Lemma 5.4.2, whence the statement follows from (5.34) and (5.35). □

Theorem 5.4.1 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, and $\mu_u = (-\Delta)^m u$. Furthermore, let $\alpha_{m,j}$ be the absolute constants from Proposition 1.5.1. The conditions*

$$\sup \left| \sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) \right| < \infty \quad \text{and} \quad \sup_{r>1} r^{2m-n} \mu_u\left(B(0,r)\right) < \infty,$$

hold if and only if

$$\int_{\mathbb{R}^n} (1+|y|)^{2m-n} \, d\mu_u(y) < \infty, \tag{5.36}$$

and u is of the form

$$u(x) = c_{m,n} \int_{\mathbb{R}^n} K_{2m}(x-y) \, d\mu_u(y) + h(x), \quad x \in \mathbb{R}^n, \tag{5.37}$$

where $h \in \mathcal{H}^m(\mathbb{R}^n)$, and $c_{m,n}$ are given by (1.23).

Proof. Suppose that

$$\sup_{r>1} \left| \sum_{j=1}^{m} \alpha_{m,j} M\left(2^{m-j} r, u\right) \right| < \infty, \quad \text{and} \quad \sup_{r>1} \left(r^{2m-n} \int_{B(0,r)} d\mu_u(y) \right) < \infty.$$

Consider the following function

$$U_{2m}^{\mu_u}(x) := \int_{\mathbb{R}^n} |x-y|^{2m-n} \, d\mu_u(y).$$

Let us show that $U_{2m}^{\mu_u}$ is locally integrable in $\mathbb{R}^n$. Indeed, let us choose an arbitrary $R > 0$, and show that

$$\int_{\mathbb{R}^n} \left(\int_{B(0,R)} |x-y|^{2m-n} \, dx \right) d\mu_u(y) < \infty.$$

It follows from Lemma 5.1.3 that $\int_{B(0,R)} |x-y|^{2m-n}\ dx$ is continuous on $\mathbb{R}^n$, and

$$\int_{B(0,R)} |x-y|^{2m-n}\ dx \leq \begin{cases} R^{2m} 2\pi^{n/2} \sum_{k=0}^{m-1} \frac{|c_{m,n,k}|}{4^k k! \Gamma\left(\frac{n}{2}+k\right)} \left(\frac{2}{2(m-k)} - \frac{1}{2k+n}\right), & |y| < R, \\ |y|^{2m-n} R^n \pi^{n/2} \sum_{k=0}^{m-1} \frac{|c_{m,n,k}|}{4^k k! \Gamma\left(\frac{n}{2}+k+1\right)}, & |y| > R. \end{cases}$$

Lemma 5.4.3 also implies that

$$\int_{\mathbb{R}^n} (1+|y|)^{2m-n}\ d\mu_u(y) < \infty.$$

Hence, for any $R > 0$,

$$\int_{\mathbb{R}^n} \left(\int_{B(0,R)} |x-y|^{2m-n}\ dx \right) d\mu_u(y) < \infty. \tag{5.38}$$

Now, Tonelli-Fubini's Theorem yields that $U_{2m}^{\mu_u} \in L^1_{loc}(\mathbb{R}^n)$. In particular, we have that $U_{2m}^{\mu_u}(x) \neq \infty$ a.e. (in the Lebesgue measure sense) in $\mathbb{R}^n$.

Theorem 1.2 of [66, Ch. 2, § 2.1] implies that $U_{2m}^{\mu_u}$ is lower semicontinuous on $\mathbb{R}^n$.

Furthermore, if $\varphi \in C_0^\infty(\mathbb{R}^n)$, then considering (5.38) and using Fubini-Tonelli's theorem, we conclude

$$\begin{aligned} & \int_{\mathbb{R}^n} U_{2m}^{\mu_u}(x) \, (-\Delta)^m \, \varphi(x) \, dx \\ = & \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |x-y|^{2m-n} \, d\mu_u(y) \right) (-\Delta)^m \, \varphi(x) \, dx \\ = & \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |x-y|^{2m-n} \, (-\Delta)^m \, \varphi(x) \, dx \right) d\mu_u(y) \geq 0. \end{aligned} \tag{5.39}$$

(Since $|\cdot - y|^{2m-n} \in \mathcal{SH}^m(\mathbb{R}^n)$, the internal integral is nonnegative for any y.) Let us also note that the final integral is always finite because of (5.38). Hence, $(-\Delta)^m U_{2m}^{\mu_u}$ is a positive measure on $\mathbb{R}^n$.

Moreover, since the Riesz kernel $|\cdot - y|^{2m-n} \geq 0$ is superharmonic in $\mathbb{R}^n$ and $U_{2m}^{\mu_u} \not\equiv \infty$, we have that $U_{2m}^{\mu_u}$ is superharmonic in $\mathbb{R}^n$ (see [55, Ch. I, § 2, Th. 1.2]). But then it follows from lower semicontinuity and superharmonicity that

$$U_{2m}^{\mu_u}(x) = \lim_{r \to 0+} \frac{1}{m(B(x,r))} \int_{B(x,r)} U_{2m}^{\mu_u}(t)\ dt, \quad x \in \mathbb{R}^n$$

(see, e.g., [55, Ch. I, § 2, Formula (1.2.4)]).

Thus, we conclude that $U_{2m}^{\mu_u} \in \mathcal{SH}^m(\mathbb{R}^n)$.

Furthermore, since $c_{m,n}(-\Delta)^m K_{2m} = \delta_0$ (see [41] and [32, § 3]), we have that

$$c_{m,n}(-\Delta)^m K_{2m}(\cdot - y) = \delta_y.$$

Hence, we may proceed with (5.39), and obtain

$$c_{m,n}\int_{\mathbb{R}^n} U_{2m}^{\mu_u}(x)\,(-\Delta)^m \varphi(x)\,dx = \int_{\mathbb{R}^n}\varphi(y)\,d\mu_u(y) = \int_{\mathbb{R}^n} u(x)\,(-\Delta)^m\varphi(x)\,dx.$$

Thus, we have two functions, $U_{2m}^{\mu_u}$ and u, from the class $\mathcal{SH}^m(\mathbb{R}^n)$, such that the relation $(-\Delta)^m\left[c_{m,n}U_{2m}^{\mu_u}\right] = (-\Delta)^m u$ holds in distributional sense. Using the same reasoning as in the proof of Lemma 5.3.1, we conclude that $h := u - c_{m,n}U_{2m}^{\mu_u} \in \mathcal{H}^m(\mathbb{R}^n)$. Thus, (5.37) follows.

Conversely, let $u \in \mathcal{SH}^m(\mathbb{R}^n)$ be of the form (5.37), where μ_u satisfies

$$\int_{\mathbb{R}^n}(1+|y|)^{2m-n}\,d\mu_u(y) < \infty.$$

Then, applying Tonelli-Fubini's Theorem, and Lemma 5.1.2, we obtain

$$\begin{aligned}
M\left(r, U_{2m}^{\mu_u}\right) &= \frac{1}{\sigma_n r^{n-1}}\int_{S(0,r)}\left(\int_{\mathbb{R}^n}|x-y|^{2m-n}\,d\mu_u(y)\right)dx\\
&= \int_{\mathbb{R}^n} M\left(r, K_{2m}(\cdot - y)\right)\,d\mu_u(y)\\
&= \int_{B(0,r)}\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\left(\frac{|y|}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(\frac{n}{2}+k\right)}r^{2(m-k)-n}\,d\mu_u(y)\\
&+ \int_{\mathbb{R}^n\setminus B(0,r)}\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m-1}\left(\frac{r}{2}\right)^{2k}\frac{c_{m,n,k}}{k!\Gamma\left(\frac{n}{2}+k\right)}|y|^{2(m-k)-n}\,d\mu_u(y)\\
&\leq r^{2m-n}\sum_{k=0}^{m-1}|\beta_{m,n,k}|\int_{B(0,r)}d\mu_u(y)\\
&+ \sum_{k=0}^{m-1}|\beta_{m,n,k}|\int_{\mathbb{R}^n\setminus B(0,r)}|y|^{2m-n}\,d\mu_u(y), \qquad (5.40)
\end{aligned}$$

where $\beta_{m,n,k}$ are defined in (5.30). Now, if $r > 1$, we get

$$\int_{B(0,r)\setminus B(0,1)} r^{2m-n}\,d\mu_u(y) \leq \int_{B(0,r)\setminus B(0,1)}|y|^{2m-n}\,d\mu_u(y).$$

Hence,

$$r^{2m-n}\int_{B(0,r)} d\mu_u(y) \le r^{2m-n}\mu_u\left(B(0,1)\right) + \int_{B(0,r)\setminus B(0,1)} |y|^{2m-n}\, d\mu_u(y),$$

and (5.40) implies

$$M\left(r, U_{2m}^{\mu_u}\right) \le \left(r^{2m-n}\mu_u\left(B(0,1)\right) + \int_{\mathbb{R}^n\setminus B(0,1)} |y|^{2m-n}\, d\mu_u(y)\right)\sum_{k=0}^{m-1}\left|\beta_{m,n,k}\right|.$$

Since $\int_{\mathbb{R}^n}(1+|y|)^{2m-n}\, d\mu_u(y) < \infty$, we conclude that

$$\sup_{r>1} M\left(r, U_{2m}^{\mu_u}\right) \le \left(\mu_u\left(B(0,1)\right) + 2^{n-2m}\int_{\mathbb{R}^n}(1+|y|)^{2m-n}\, d\mu_u(y)\right)\sum_{k=0}^{m-1}\left|\beta_{m,n,k}\right| < \infty.$$

This yields

$$\sup_{r>1}\left|\sum_{j=1}^{m}\alpha_{m,j} M\left(2^{m-j}r, U_{2m}^{\mu_u}\right)\right| < \infty. \tag{5.41}$$

Now, from (5.37), (5.41) and (5.28), we deduce that

$$\sup_{r>1}\left|\sum_{j=1}^{m}\alpha_{m,j} M\left(2^{m-j}r, u\right)\right| < \infty.$$

Finally, for any $r>1$ we have

$$\begin{aligned} 0 \le r^{2m-n}\int_{B(0,r)} d\mu_u(y) &\le \int_{B(0,1)} d\mu_u(y) + \int_{B(0,r)\setminus B(0,1)} |y|^{2m-n}\, d\mu_u(y) \\ &= \mu_u\left(B(0,1)\right) + 2^{n-2m}\int_{B(0,r)\setminus B(0,1)} (|y|+|y|)^{2m-n}\, d\mu_u(y) \\ &\le \mu_u\left(B(0,1)\right) + 2^{n-2m}\int_{B(0,r)\setminus B(0,1)} (1+|y|)^{2m-n}\, d\mu_u(y) \\ &\le \mu_u\left(B(0,1)\right) + 2^{n-2m}\int_{\mathbb{R}^n} (1+|y|)^{2m-n}\, d\mu_u(y) < \infty. \end{aligned}$$

□

To prove Theorem 1.5.5, it remains to replace the condition

$$\sup_{r>1} r^{2m-n}\mu_u\left(B(0,r)\right) < \infty$$

by another one that should be easy to check having a particular function $u \in \mathcal{SH}^m\left(\mathbb{R}^n\right)$. The replacement is given by the following lemma.

Lemma 5.4.4 *Let $m, n \in \mathbb{N}$, $2m < n$, $u \in \mathcal{SH}^m(\mathbb{R}^n)$, and $\mu_u = (-\Delta)^m u$. The following are equivalent:*

$$(a)\quad \sup_{r>1} r^{2m-n}\mu_u(B(0,r)) < \infty;$$

$$(b)\quad \sup_{r>1} \int_{1\le|t|\le 2} u(rt)\,(-\Delta)^m\varphi(t)\,dt < \infty, \quad \textit{for some } \varphi \in \mathcal{R};$$

$$(c)\quad \sup_{r>1} \int_{1\le|t|\le 2} u(rt)\,(-\Delta)^m\varphi(t)\,dt < \infty, \quad \textit{for any } \varphi \in \mathcal{R}.$$

Proof. Since $u \in \mathcal{SH}^m(\mathbb{R}^n)$, it is locally integrable, and $d\mu_u(x)$ is a positive Borel measure on $\mathbb{R}^n$. Take any $\varphi \in \mathcal{R}$, $r > 0$, and let $\Phi(x) := \varphi(x/r)$. Since $\Phi \in C_c^\infty(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\mu_u(B(0,r)) &= \int_{B(0,r)} \Phi(x)\,d\mu_u(x) \le \int_{B(0,2r)} \Phi(x)\,d\mu_u(x)\\ &= \int_{\mathbb{R}^n} u(x)\,(-\Delta)^m\Phi(x)\,dx = r^{-2m}\int_{\mathbb{R}^n} u(x)\,[(-\Delta)^m\varphi]\left(\frac{x}{r}\right)dx\\ &= r^{-2m}\int_{r\le|x|\le 2r} u(x)\,[(-\Delta)^m\varphi]\left(\frac{x}{r}\right)dx.\end{aligned}$$

Making the substitution $t := x/r$ in the last integral, we get

$$r^{2m-n}\mu_u(B(0,r)) \le \int_{1\le|t|\le 2} u(rt)\,(-\Delta)^m\varphi(t)\,dt, \quad r > 0. \tag{5.42}$$

Analogously, since $0 \le \Phi(x) \le 1$,

$$\mu_u(B(0,2r)) \ge \int_{B(0,2r)} \Phi(x)\,d\mu_u(x) = r^{-2m}\int_{r\le|x|\le 2r} u(x)\,[(-\Delta)^m\varphi]\left(\frac{x}{r}\right)dx.$$

Making the substitution $t := x/r$ in the last integral, we arrive at

$$(2r)^{2m-n}\mu_u(B(0,2r)) \ge 2^{2m-n}\int_{1\le|t|\le 2} u(rt)\,(-\Delta)^m\varphi(t)\,dt, \quad r > 0. \tag{5.43}$$

Now, assume (a) holds. Taking an arbitrary $\varphi \in \mathcal{R}$, we conclude from (5.43) that

$$\sup_{r>1/2}\int_{1\le|t|\le 2} u(rt)\,(-\Delta)^m\varphi(t)\,dt \le \sup_{r>1} r^{2m-n}\mu_u(B(0,r)) < \infty,$$

which implies (c), and then, trivially, (b).

If (b) holds with some $\varphi \in \mathcal{R}$, then (5.42) yields (a) immediately. $\square$

Thus, Theorem 1.5.5 follows from Theorem 5.4.1 and Lemma 5.4.4.

Furthermore, we may use (5.42) to get easy-to-check sufficient conditions on u to have Riesz representation (1.28).

Proof of Corollary 1.5.1. Applying Hölder's inequality to the right-hand side of (5.42), we have that for any $p \in [1, \infty)$ and q, such that $1/p + 1/q = 1$,

$$r^{2m-n}\mu_u\left(B(0,r)\right) \leq \|(-\Delta)^m \varphi\|_{L^q\left(\overline{B(0,2)}\setminus B(0,1)\right)} \left(\int_{1\leq|t|\leq 2} |u(rt)|^p \, dt\right)^{1/p}$$

$$= \|(-\Delta)^m \varphi\|_{L^q\left(\overline{B(0,2)}\setminus B(0,1)\right)} \left(\frac{1}{r^n}\int_{r\leq|x|\leq 2r} |u(x)|^p \, dt\right)^{1/p}$$

$$\leq 2^{n/p}\|(-\Delta)^m \varphi\|_{L^q\left(\overline{B(0,2)}\setminus B(0,1)\right)} \left(\int_{r\leq|x|\leq 2r} \frac{|u(x)|^p}{|x|^n} \, dt\right)^{1/p}$$

$$\leq 2^{n/p}\|(-\Delta)^m \varphi\|_{L^q\left(\overline{B(0,2)}\setminus B(0,1)\right)} \left(\int_{|x|\geq 1} \frac{|u(x)|^p}{|x|^n} \, dt\right)^{1/p}.$$

If $p = \infty$, then clearly,

$$r^{2m-n}\mu_u\left(B(0,r)\right) \leq \|(-\Delta)^m \varphi\|_{L^1\left(\overline{B(0,2)}\setminus B(0,1)\right)} \operatorname*{ess\,sup}_{r\leq|x|\leq 2r} |u(x)|.$$

Thus, if either condition, (a) or (b) is satisfied, then $\sup_{r>1} r^{2m-n}\mu_u\left(B(0,r)\right) < \infty$. Applying Theorem 5.4.1, we get representation (1.28), and relation (1.29). □

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