COMPUTER APPLICATION TO THE STATISTICAL ANALYSIS OF CONTROL SYSTEMS

By

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PREFACE

In the past few years there has been an increase in the application of statistical methods to system analysis. Statistical functions, such as the correlation functions, can be used to obtain certain facts about a system which would otherwise remain unknown. One of the applications of the statistical functions is the analysis of systems which are subject to random or noise inputs. By sampling certain variables and numerically calculating the required statistical functions, a good deal of information can be obtained about the system and the variables connected with it. Numerical calculation and especially numerical integration is, however, a very tedious, time consuming process. The purpose of this paper is to present a set of digital computer programs which will perform the tedious numerical calculations involved in the calculation of four such functions.

Indebtedness is acknowledged to Professor Paul A. McCollum for his guidance and advice in the preparation of the computer programs; to the Oklahoma State University Engineering department for the use of their computer and supplies and; to Mrs. John Youngblood who corrected the gramatical errors and typed the paper.

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CHAPTER I

SYSTEMS AND CONTROL SYSTEMS

An electrical system can be defined as an interconnection of elements and devices which act together to establish a desired relationship between an input variable (or variables) and an output variable (or variables). The integrator circuit, shown in Figure one, can be described as a system. It is composed of elements (a resistor and a capacitor) and a device (an operational amplifier), the interconnection of which establishes a relationship (Integration) between an input variable (x) an output variable (y).

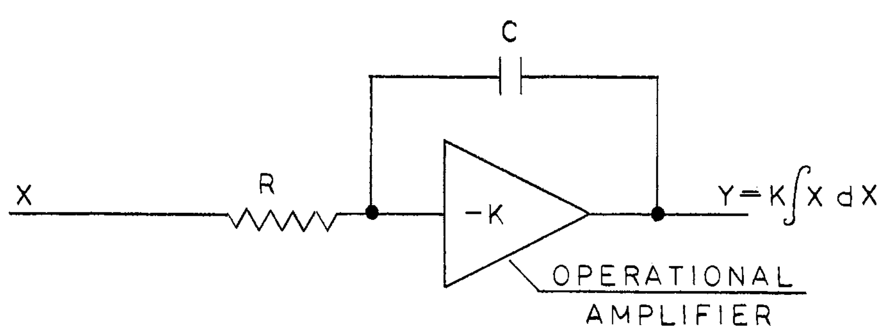

Figure 1
Integrator Circuit as a System

The device of any system can be viewed as a system within itself (Figure 2) and, any system can be considered as a device or an element in a larger system (Figure 3). Whether a circuit is to be considered as a system, a device, or an element will depend on the circuit variables, the relationships to be established and, the method of establishement chosen by the designer.

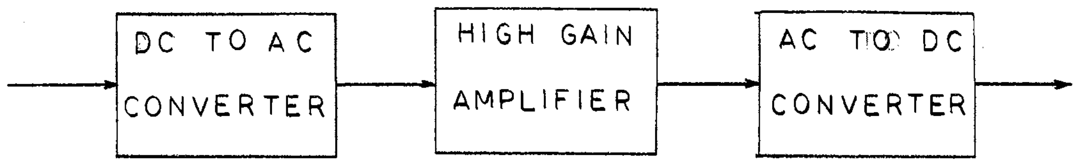

Figure 2
The Operational Amplifier as a System

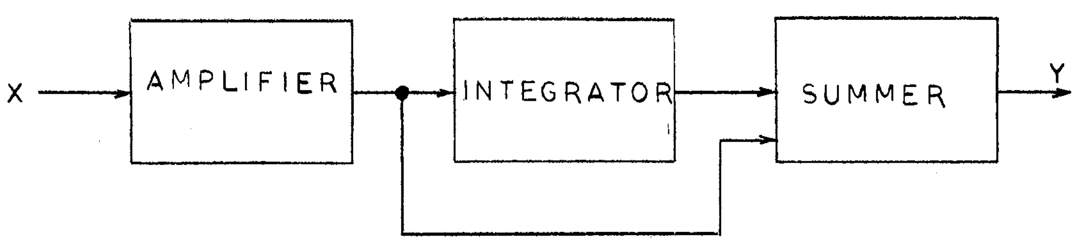

Figure 3
A System Utilizing an Integrator Circuit

Control Systems

A system which is designed to control one variable (or variables) with another variable (or variables) is defined as a control system. There are two types of control systems; open-loop control systems and closed-loop control systems.

In an open-loop control system the output variable has no effect on the input variable. An open-loop control system can be represented symbolically by a functional block diagram as shown in Figure 4.

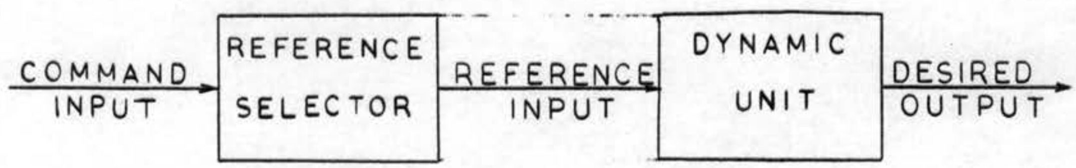

Figure 4
Functional Block Diagram
of an Open-Loop Control System.

If the output of a control system does have an effect on the input of the system, it is referred to as an closed-loop control system. Figure 5 is a functional block diagram representation of a closed loop control system.

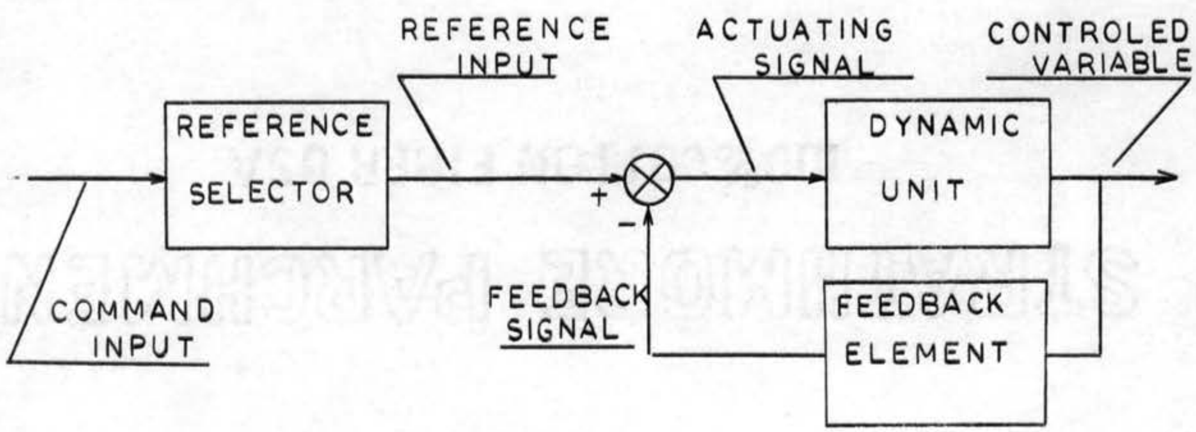

Figure 5
Functional Block Diagram
of a Closed-Loop Control System

It is evident from Figures 4 and 5 that the fundamental difference between the open-loop and closed-loop control systems is the feedback loop. The variables, devices and elements which are shown in Figures 4 and 5 are defined by the AIEE Subcommittee on Terminology and Nomenclature of the Feedback Control Systems Committee as follows:[1]

The "command" is the input which is established by some means external to and independent of the control system.

The "reference input" is derived from the command and is the actual signal input to the system.

The "controlled variable" is the quantity that is directly measured and controlled. It is the output of the controlled system.

The "primary feedback" is a signal which is a function of the controlled variable and which is compared with the reference input to obtain the actuating signal.

The "actuating signal" is obtained from a comparison measuring device and is the reference input minus the primary feedback.

The "reference input elements" produce a signal proportional to the command.

The "control elements" produce the manipulated variable from the actuating signal.

The "controlled system" is the device that is to be controlled.

The "feedback elements" produce the primary feedback from the con-

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[1]John J. D'Azzo and Constantine Houpis, Feedback Control Systems Analysis and Synthesis (New York, 1960), pp. 505-507

trolled variable.

Sampled-data Control Systems

A sampled-data control system is one in which the control signal (the command, reference input, actuating signal or, manipulated variable) is supplied intermittently and at a constant rate. In a sampled-data control system the data signal is a sequence of pulses, the magnitudes of which are determined by the signal from which the samples are derived (Figure 6).[2]

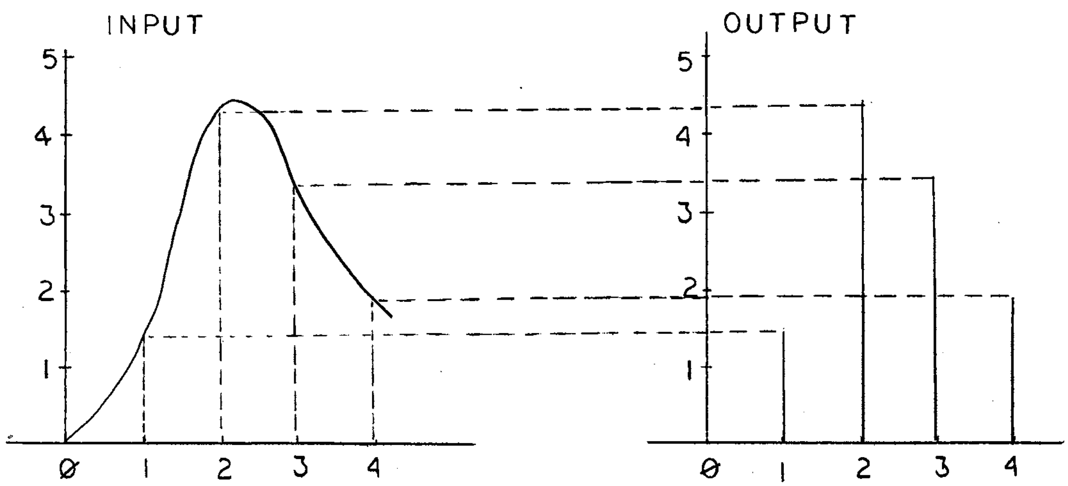

Figure 6
Sampler Input and Output Waveforms

Figure 7 illustrates a basic sampled-data feedback control system in which the actuating signal is the sampled signal.

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[2] Julius T. Tou, Digital & Sampled-Data Control Systems. (New York, 1959), p.5

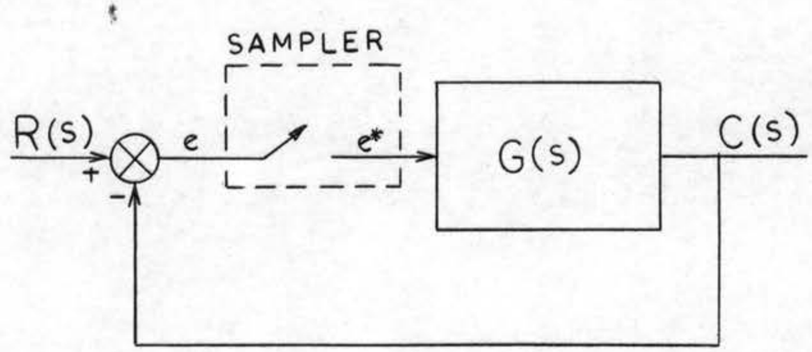

Figure 7
A Sampled-Data Feedback Control System

Digital Control Systems

A digital control system is basically a special type of sampled-data control system. The digital control system can be defined as a control system in which the control signal, in one or more sections, is expressed in a numerical code for the digital data processing and decision making equipment of the control system.[3] Figure 8 shows a typical digital feedback control system. A digital control system can be reduced to a sampled-data control system if the numerically coded data signal in the digital system is decoded into amplitude modulated signals (sampled-data) and the operation of the digital computer is represented by the transfer function of an equivalent pulsed data network.

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[3]Ibid., p. 6

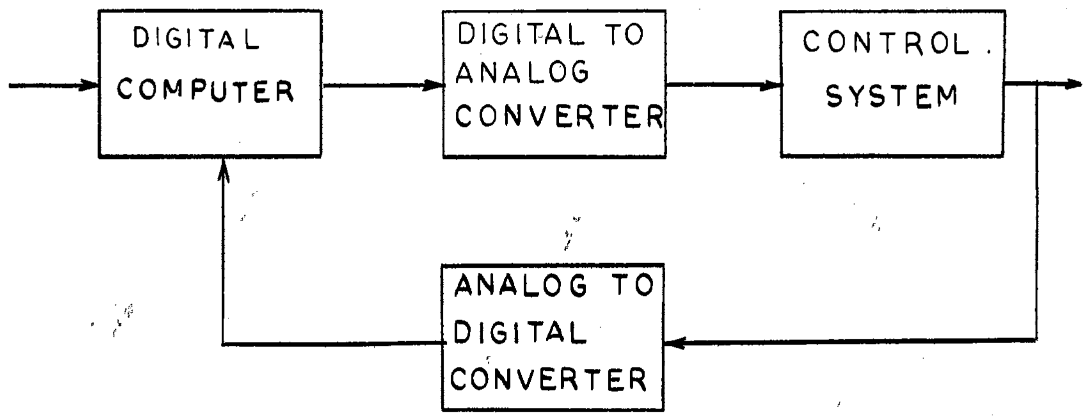

Figure 8
A Digital Feedback Control System

CHAPTER II

STATISTICAL ANALYSIS

The Fourier series (Equation 1) and the complex Fourier integrals have, for some time, been the primary tools of analysis for the engineer. Statisticians on the other hand have been making use of the properities of correlation functions. The two concepts of analysis were not associated by the two schools until the Fourier transform, which establishes

$$x(t) = \frac{a_o}{2} + \sum a_n \cdot \cos(2\pi f_n t) + b_n \cdot \sin(2\pi f_n t) \qquad (1)$$

a relation between the real time and the frequency domains, was applied (Equation 2)[1].

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad (2)$$

Convolution

One of the useful tools of analysis is the convolution theorem. A function, f(x), is known as the convolution of two other functions,

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[1]Julius Bendat, Principles and Applications of Random Noise Theory (New York, 1958).

$f_1(x)$ and $f_2(x)$, when

$$f(x) = \int_{-\infty}^{\infty} f_1(y) f_2(x-y)\,dy \qquad (3)$$

The most important property of the convolution theorem is observed when the Fourier transform is calculated as shown in Equations 4, 5, 6, 7 and 8.

$$f_1(t) \longleftrightarrow F_1(\omega) \qquad f_2(t) \longleftrightarrow F_2(\omega) \qquad (4)$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, F(t)\,dt \qquad (5)$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} \left(\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,dt \qquad (6)$$

$$= \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} * F_2(\omega)\,d \qquad (7)$$

$$= F_1(\omega) F_2(\omega) \qquad (8)$$

Results of the calculation show that convolution in the time domain transforms to multiplication in the frequency domain.

It can also be shown that multiplication in the time domain transforms to convolution in the frequency domain (frequency convolution).

$$f_1(t) f_2(t) \longleftrightarrow \frac{1}{2\pi} \int F_1(y)\, F_2(\omega-y)\,dy \qquad (9)$$

As an example of the use of the convolution integral, consider a simple system as shown in Figure 9. It is apparent that, in the frequency domain, the indirectly controlled variable is equal to the product of the command variable (or reference input) and the control element. The convolution integral finds its application in the calculation of c(t). Since

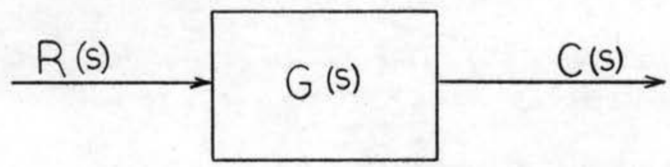

Figure 9
Open-Loop Control System

$$C(s) = R(s)\ G(s) \tag{10}$$

C(s) is the product of R(s) and G(s), by the convolution theorem, c(t) is the convolution of r(t) and g(t).

Autocorrelation

The autocorrelation function, as defined by Equation 11, shows a statistical relation between the mean square value of the function and

$$\phi_{11}(\tau) = \lim \frac{1}{T} \int_{0}^{\infty} x(t)\ x(t + \tau)\ dt \tag{11}$$

the value of the function τ units away. More precisely, it shows the dependence of the future function value on the present function value. For example, assume that the autocorrelation measurement of f(t) results in the curve shown in Figure 10. At point a on the curve τ is equal to zero and the autocorrelation function is the mean square value of f(t).

This indicates that the function, zero units away, is one hundred percent dependent upon the value of original function. At point b the function τ_b units in the future is less than one hundred percent dependent on the original function. At point c the function is independent of the function at τ equal to zero.

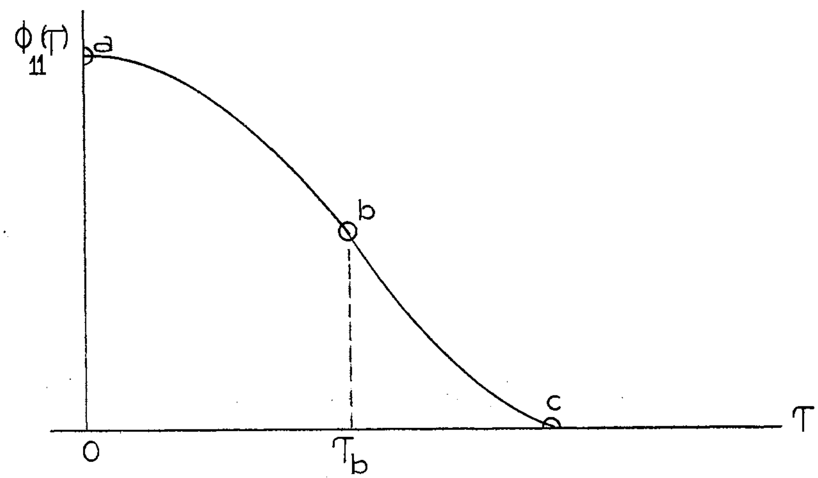

Figure 10
An Autocorrelation Curve

In the actual autocorrelation integral, the interval zero to T is suppose to be infinitely large so that the autocorrelation function will be independent of T. In practice however, the value of T is simply taken large enough so that further increases in its value do not effect the outcome.[2]

Crosscorrelation

The crosscorrelation integral is very similar to the autocorrela-

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[2]E. L. Peterson, Statistical Analysis and Optimization of System (New York, 1961), p. 40.

tion integral. As shown by Equation 12, which defines the crosscorrelation function, the integral involves two functions of time rather than one.

$$\phi_{12}(\tau) = \frac{1}{T}\int_0^\infty x(t)\ y(t+\tau)\ dt \qquad (12)$$

The crosscorrelation integral can be used to determine the unit impulse response of a linear system. This is obtained, as shown in Figure 11, by a crosscorrelation measurement of the reference input and controlled variables.

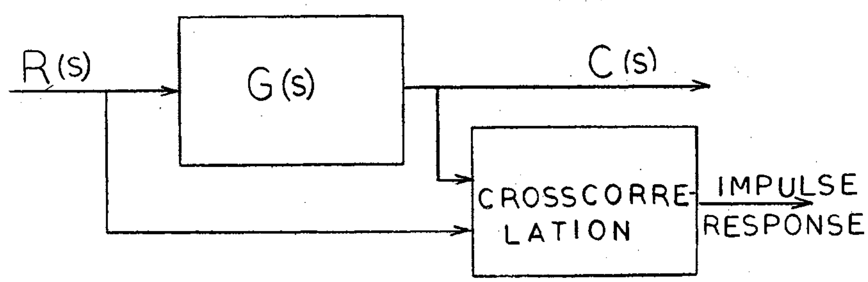

Figure 11
An Application of
the Crosscorrelation Integral

Power Density Spectrum

Especially useful when considering systems with noise or random inputs is the frequency composition of the input, and the effects of the various frequency components. Provided that the input in question can be represented as an ergodic ensemble, the Fourier transform of the time correlation function will result in a function of frequency which describes the distribution of the function with respect to frequency. Equations 13 and 14 show the relation between the power density spectrum and the autocorrelation function.

$$\Phi_{11}(\omega) = \int_{-\infty}^{\infty} \phi_{11}(\tau) e^{-j\omega\tau}\, d\tau \tag{13}$$

$$\phi_{11}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{11}(\omega) e^{j\omega\tau}\, dw \tag{14}$$

Since the autocorrelation function is a real and even function, the power density spectrum can be written as a cosine transform (as in equation 15). This relation between the autocorrelation function and the

$$\Phi_{11}(\omega) = \int \phi_{11}(\tau) \cos(\omega\tau)\, d\tau \tag{15}$$

power density spectrum is known as the "Wiener Theorem for Autocorrelation".[3]

The power density spectrum of a function shows how much power is contributed by components of the function at a given frequency. Assuming that Figure 12 represents a power density spectrum of some f(t), we can calculate the power contributed by components of f(t) of all frequencies from zero to ω_a as

$$\int_{-\omega_a}^{\omega_a} \Phi_{11}(\omega)\, d\omega \tag{16}$$

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[3] Y. W. Lee, Statistical Theory of Communication (New York, 1963) p. 56.

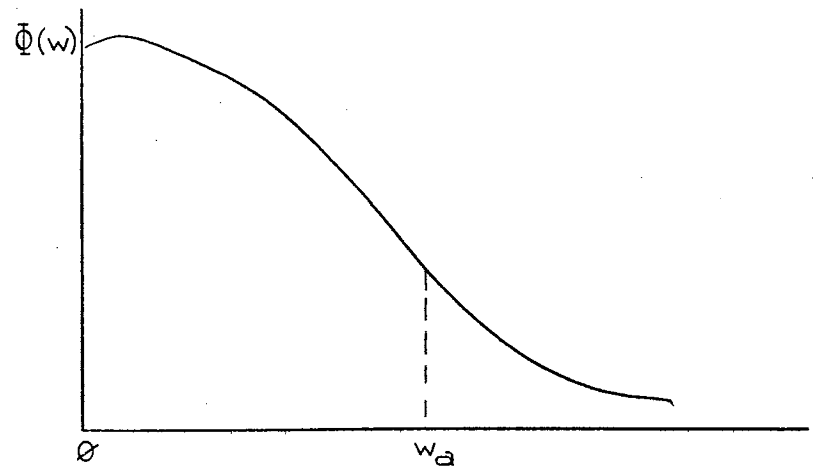

Figure 12
A Power Density
Spectrum of a Function of Time

Since the power density spectrum is actually a spectrum, the value of $\Phi(\omega)$ at omega equal to zero does not indicate a definite d.c. component (as shown in Equation 17).

$$\Phi_{11}(w) = \int_{\emptyset}^{\emptyset} \Phi_{11}(\emptyset)\ d\omega = \emptyset \qquad (17)$$

The cross-power density spectrum bears the same relation to the crosscorrelation measurement of $f_1(t)$ and $f_2(t)$ as the power density spectrum does to the autocorrelation of $f(t)$ (as shown in Equation 18).

$$\Phi_{12}(\omega) = 2\int_{\emptyset}^{\infty} \Phi_{12}(\tau)\cos(\omega\tau)\ d\tau \qquad (18)$$

CHAPTER III

COMPUTER SOLUTION

The calculation of the convolution and correlation integrals can be accomplished analytically when the time function (functions) in question can be expressed as continuous, integrable functions of time.

When the functions under investigation are arbitrary time functions (a function which is randomly distributed in time) the calculation of the statistical integrals is best done by numerical methods. Since the numerical method of analysis is a step-by-step process, which may require a formidable amount of manipulation, a digital computer is almost essential in the solution of all but the simplest of problems.

The following programs are designed for use on the IBM 1620 digital computer. They are designed to calculate the statistical function of any time function which has been sampled a known number of times at a constant, known rate. All the programs are limited to two hundred values of the input function and to values of τ from nine hundred ninety nine to one thousandth (999-.001).

Autocorrelation

The program described in this section is designed to compute the autocorrelation function of a sampled function of time, as defined by

Figure 13 and Equation 11b.

$$\phi_{11}(\tau) = 2 \cdot \frac{1}{T} \int_{0}^{T/2} x(t)\; x(t + \tau)dt \tag{11b}$$

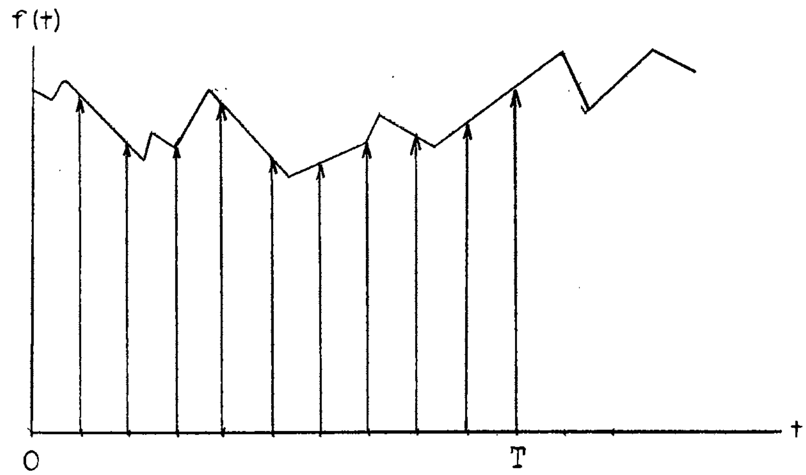

Figure 13
A Random Function of Time

The value of T, a number which corresponds to the number of samples in T units of time and the values of the sampled function of time are the required inputs for the program. The value of the autocorrelation function and τ are the output variables. τ is considered to be an output variable rather than an input variable because, due to the use of a finite amount of sampled data, there are only a limited number of values of τ for which a corresponding value of $Y(t + \tau)$ exists.

Program	Problem
Y(t)	Sampled Variable
A	T
C	τ
NO	Number of Samples
K	t

Table 1
Relation between Program
and Problem variables--Autocorrelation

Figure 14 is a flow chart for the program in Figure 15. Operation number 10 reads the value of T or A and NO, the number of samples of Y(t). The format for the READ statement, operation number 15, reserves the first 10 positions on a card for the value of the variable A and positions 11 through 13 for the variable NO. A is written in floating point notation and NO is written in fixed point notation.

Operation number 20 dimensions the subscripted problem variables. The statement limits NO to two hundred maximum which, in turn, limits the number of autocorrelation and τ values to one hundred and one each.

The titles for the output data are punched by operations 25 and 30. A numerical value for τ is computed by operations 31 and 32. This value is equal to the value of T divided by the number of increments into which T is divided by the NO samples (see Figure 14).

Operations 35, 40, and 45 read the values of the NO samples into memory as Y(1) through Y(NO)

To compute the integrand of the autocorrelation integral, Equation 6, the value of the function τ units in the future must be known . This means that less than NO values of τ can be considered because, t plus τ cannot be greater than T. Operation 46 redefines NO as NO divided by two. There will therefore be one plus NO divided by two valid values of τ.

A set of values which are analogous to the values of τ is established for the purpose of subscripting, by operations 49, 50, and 51. Operation number 55 establishes a set of values which are analogous to

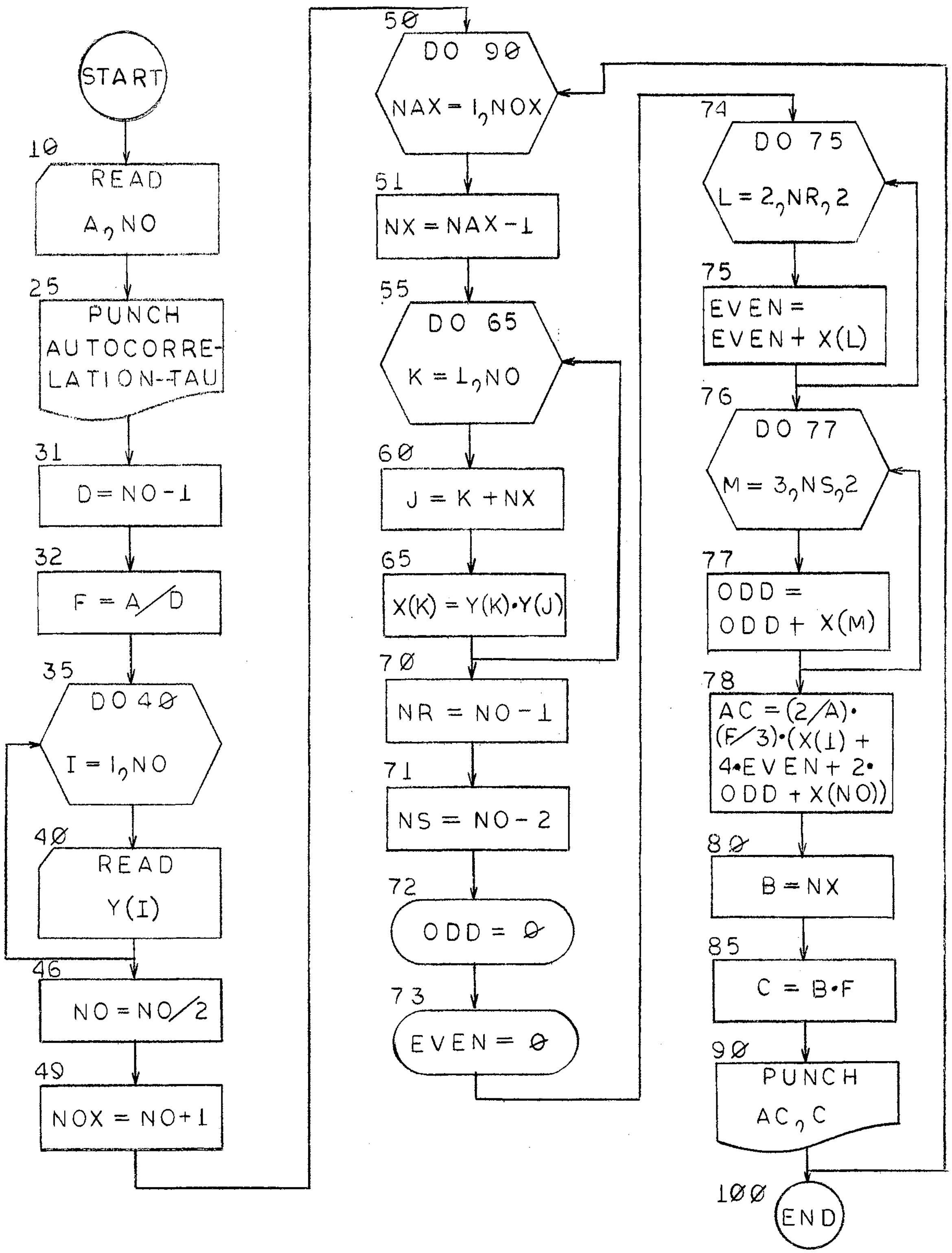

Figure 14
Flow Chart for Autocorrelation Program

```
C    1 CARMI HUMES   AUTOCORRELATION PROGRAM
    10 READ 15,A,NO
    15 FORMAT (F10.2,I3)
    20 DIMENSION Y(200),X(100)
    25 PUNCH 30
    30 FORMAT (30H    AUTOCORRELATION         TAU)
    31 D = NO-1
    32 F = A/D
    35 DO 40 I = 1,NO
    40 READ 45,Y(I)
    45 FORMAT (F7.3)
    46 NO = NO/2
    49 NOX = NO+1
    50 DO 90 NAX = 1,NOX
    51 NX = NAX-1
    55 DO 65 K = 1,NO
    60 J = K+NX
    65 X(K) = Y(K)*Y(J)
    70 NR = NO-1
    71 NS = NO-2
    72 ODD = 0
    73 EVEN = 0
    74 DO 75 L = 2,NR,2
    75 EVEN = EVEN+X(L)
    76 DO 77 M = 3,NS,2
    77 ODD = ODD+X(M)
    78 AC = (2./A)*(F/3.)*(X(1)+4.*EVEN+2.*ODD+X(NO))
    80 B = NX
    85 C = B*F
    90 PUNCH 95,AC,C
    95 FORMAT (E17.8,4H     E14.5)
   100 STOP
   105 END
```

Figure 15
Fortran Program for
the Autocorrelation Function

the NO values of time (t). The subscript value which is analogous to the quantity $(t + \tau)$ is calculated by operation number 60 and the integrand of the autocorrelation integral is calculated by operation number 65.

Operations 70 through 78, inclusive, perform the autocorrelation integral, Equation 6, by a numerical method known as Simpson's one-third rule.[7]

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[7]Daniel D. McCracken, A Guide to Fortran Programming (New York, 1963), p.42.

The actual numerical value of τ used in the previously calculated autocorrelation integral is calculated by operations 80 and 85.

The results of the calculations are punched in a format which is compatable with the headings punched in operations 25 and 30 by operations 90 and 95.

After punching the results of the program based on one value of τ, the computer will return to operation number 50 to obtain a new value of τ and recalculate the value of the autocorrelation function. This repetitious process will continue until all valid values of τ have been processed. After the last value of autocorrelation has been calculated the program will proceed to operation number 100 which is a stop command.

For optimum results, the input variable NO should be a number which satisfies Equation 19a. This optimization restriction is a result of

$$NO = 2 \cdot G \; (\; G = \text{odd} \;) \tag{19a}$$

operations 46, 70, 71, 74 and 76.

Crosscorrelation

The program described in this section is designed to compute the crosscorrelation function of a pair of sampled functions of time as defined by Equation 13 (repeated below).

$$CC = (2/T) \int_{0}^{T/2} Y_1(t) Y_2(t+\tau) dt \tag{13}$$

It is obvious that with a few minor changes, the previously described autocorrelation program can be used to perform the crosscorrelation integral. There are two major changes which must be made. The second function of time must be read into memory and, the integrand must be changed to the product of two different functions of time as opposed to a single function of time.

The flow chart in Figure 16 is a flow chart for the crosscorrelation program shown in Figure 17. Operations 10, 15, 25, 31, 32, 35, 40, 45, 46, 49, 50, 51, 55, 60, 70, through 78, 80, 85, and 95 are exactly the same as the corresponding operations in the autocorrelation program.

The dimension statement, operation number 20, has an additional variable, Z(t), corresponding to the second function of time involved in the crosscorrelation integral.

Operations 21, 22, and 23 read into memory the NO values of Z(t) just as operations 35, 40, and 45 read the NO values of Y(t).

The titles for the crosscorrelation output data, rather than the autocorrelation data, are punched by operation number 26.

The integrand of the crosscorrelation integral is calculated by operation number 65. The integrand is the product of $Y(t_x)$ and a delayed $Z(t_x)$ rather than $Y(t_x)$ and a delayed $Y(t_x)$ as in the autocorrelation program.

Operation 90 punches the output data resulting from the crosscorrelation measurement.

As in the autocorrelation program, once a value for the integral has been obtained and punched, the computer returns to operation number 50 to obtain a new value of τ for additional calculations. The process will continue until all possible values of τ have been considered.

The optimization restriction which was placed on the variable NO in the autocorrelation program still applies in the crosscorrelation case.

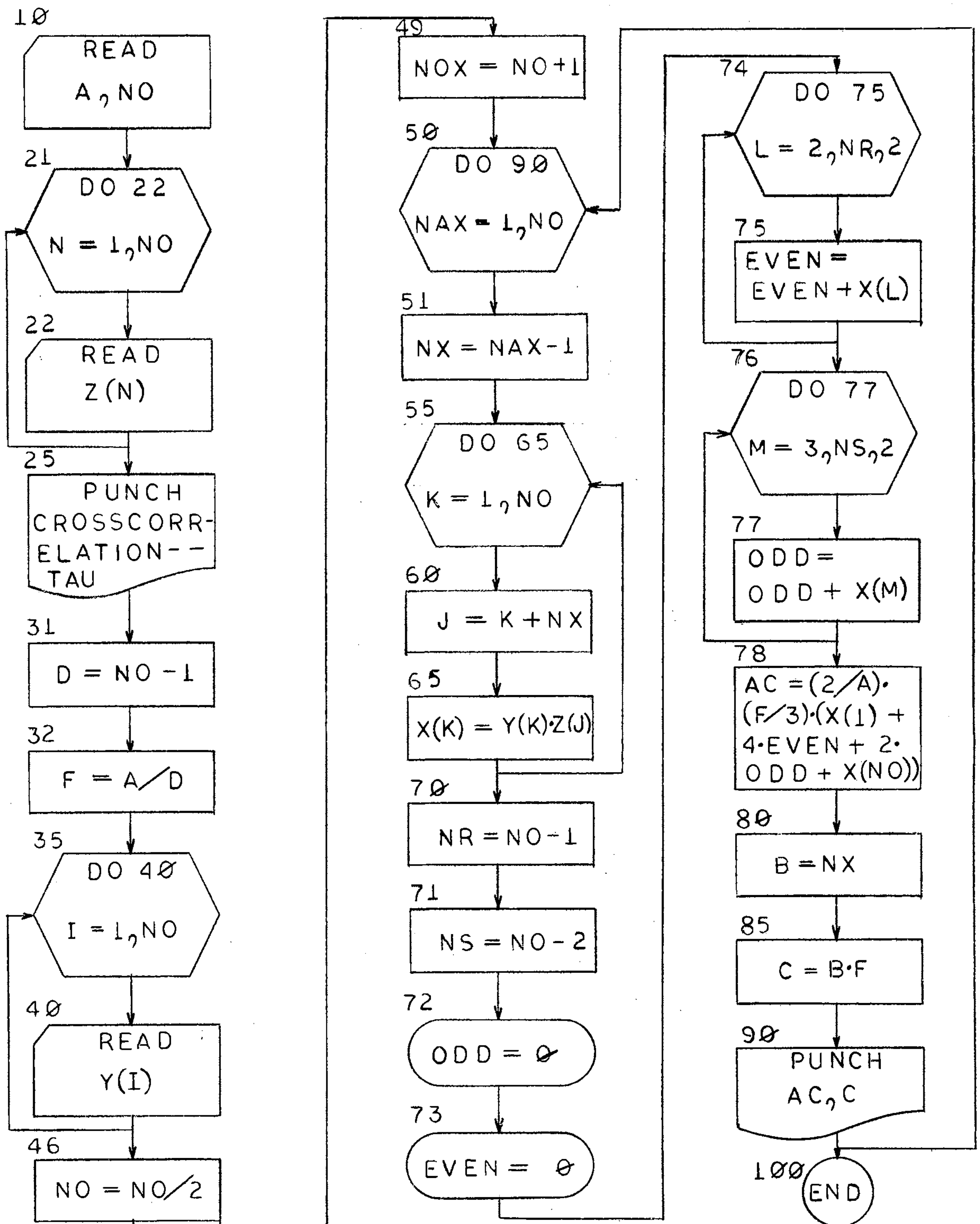

Figure 16
Flow Chart for Crosscorrelation Program

```
C     1 CARMI HUMES CROSSCORRELATION
     10 READ 15,A,NO
     15 FORMAT (F10.2,I3)
     20 DIMENSION Y(100),X(100),Z(200)
     21 DO 22 N = 1,NO
     22 READ 23,Z(N)
     23 FORMAT (F7.3)
     25 PUNCH 26
     26 FORMAT (32H    CROSSCORRELATION         TAU )
     31 D = NO-1
     32 F = A/D
     35 DO 40 I = 1,NO
     40 READ 45,Y(I)
     45 FORMAT (F7.3)
     46 NO = NO/2
     49 NOX = NO+1
     50 DO 90 NAX = 1,NOX
     51 NX = NAX-1
     55 DO 65 K = 1,NO
     60 J = K+NX
     65 X(K) = Y(K)*Z(J)
     70 NR = NO-1
     71 NS = NO-2
     72 ODD = 0
     73 EVEN = 0
     74 DO 75 L = 2,NR,2
     75 EVEN = EVEN+X(L)
     76 DO 77 M = 3,NS,2
     77 ODD = ODD+X(M)
     78 AC = (2./A)*(F/3.)*(X(1)+4.*EVEN+2.*ODD+X(NO))
     80 B = NX
     85 C = B*F
     90 PUNCH 95,AC,C
     95 FORMAT (E17,8,4H     E14.5)
    100 CONTINUE
    101 STOP
    105 END
```

Figure 17
Fortran Program for
the Crosscorrelation Function

Convolution

Like the crosscorrelation integral, the convolution integral (Equation 3b) can be calculated by the autocorrelation program with a few minor alterations.

$$F(x) = \int_{T/2}^{T} f_1 (y)\ f_2(x-y)dy \qquad (3b)$$

The flow chart in Figure 18 is a flow chart for the convolution program shown in Figure 19. There are three primary differences between the convolution and crosscorrelation programs. First, the punch statements, operations 25 and 90, are changed to produce the convolution data. Second, operation 60 is changed to calculate t minus τ rather than t plus τ. Third, operations 47, 48, 49, 50, 55, 70, 71, 74, 76, and 78 are changed to provide a past history, rather than a known future for the delayed time function. The change in the integrand in the convolution integral, which is the direct reason for the change in operation 60, is the indirect reason for the change in the third group of operations.

For optimum results, the value of NO in the convolution program should satisfy the value of NO in Equation 19b.

$$NO = 2 \cdot G\ (G \text{ even}) \qquad (19b)$$

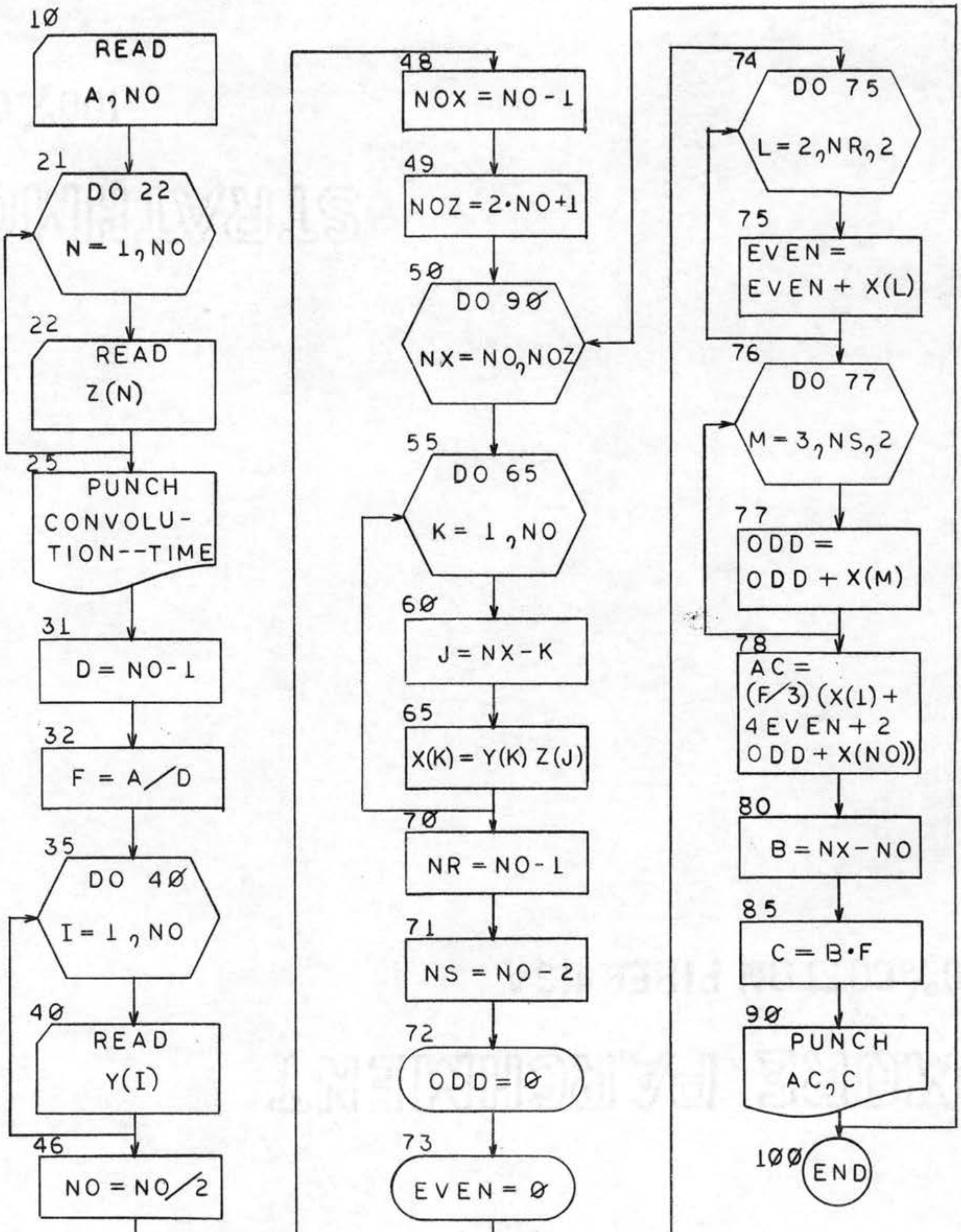

Figure 18
Flow Chart for Convolution Program

```
 10 READ 15,A,NO
 15 FORMAT (F10.2,I3)
 20 DIMENSION Y(200),X(100),Z(200)
 21 DO 22 N = 1,NO
 22 READ 23,Z(N)
 23 FORMAT (F7.3)
 25 PUNCH 26
 26 FORMAT (30H          CONVOLUTION          TIME)
 31 D = NO-1
 32 F = A/D
 35 DO 40 I = 1,NO
 40 READ 45,Y(I)
 45 FORMAT (F7.3)
 46 NO = NO/2
 48 NOX = NO-1
 49 NOZ = (2*NO)+1
 50 DO 90 NX = NO,NOZ
 55 DO 65 K = 1,NOX
 60 J = NX-K
 65 X(K) = Y(K)*Z(J)
 70 NR = NOX-1
 71 NS = NOX-2
 72 ODD = 0
 73 EVEN = 0
 74 DO 75 L = 2,NR,2
 75 EVEN = EVEN+X(L)
 76 DO 77 M = 3,NS,2
 77 ODD = ODD+X(M)
 78 AC = (F/3.)*(X(1)+4.*EVEN+2.*ODD+X(NOX))
 80 B = NX-NO
 85 C = B*F
 90 PUNCH 95,AC,C
 95 FORMAT (F17.8,4H     E14.5)
100 CONTINUE
101 STOP
105 END
```

Figure 19
Fortran Program for Convolution Function

Power Density Spectrum

The power density spectrum is essentially the Fourier transform of the autocorrelation function as defined by Equation 15. The input data for the power density spectrum program consists of the output data from the autocorrelation program (value of autocorrelation and corresponding value of τ), a number corresponding to the number of autocorrelation values, a pair of numbers corresponding to the maximum omega and the incremental omega. Power density and omega are the output variables.

$$\Phi(\omega) = 2 \int_0^{T/2} \phi(\tau) \cos(\omega\tau)\, d\tau \qquad (15)$$

Cross-power density spectrum can also be calculated with the power density spectrum program by reading the crosscorrelation measurements rather than the autocorrelation measurements as input data.

Program	Problem
NF	Number of autocorrelation values
NJ	Omega (variable)
BPD	Power density
WMAX	Maximum value of omega
DELW	Incrimental value of omega

Table 2
Relation between Program and
Problem variables--Power Density Spectrum

The number of autocorrelation values to be used, the maximum value of omega to be considered and, the incremental value of omega are read into memory by operation number 10. Operation 15 is a format for the read statement. The format assigns positions 1, 2, and 3 for the value

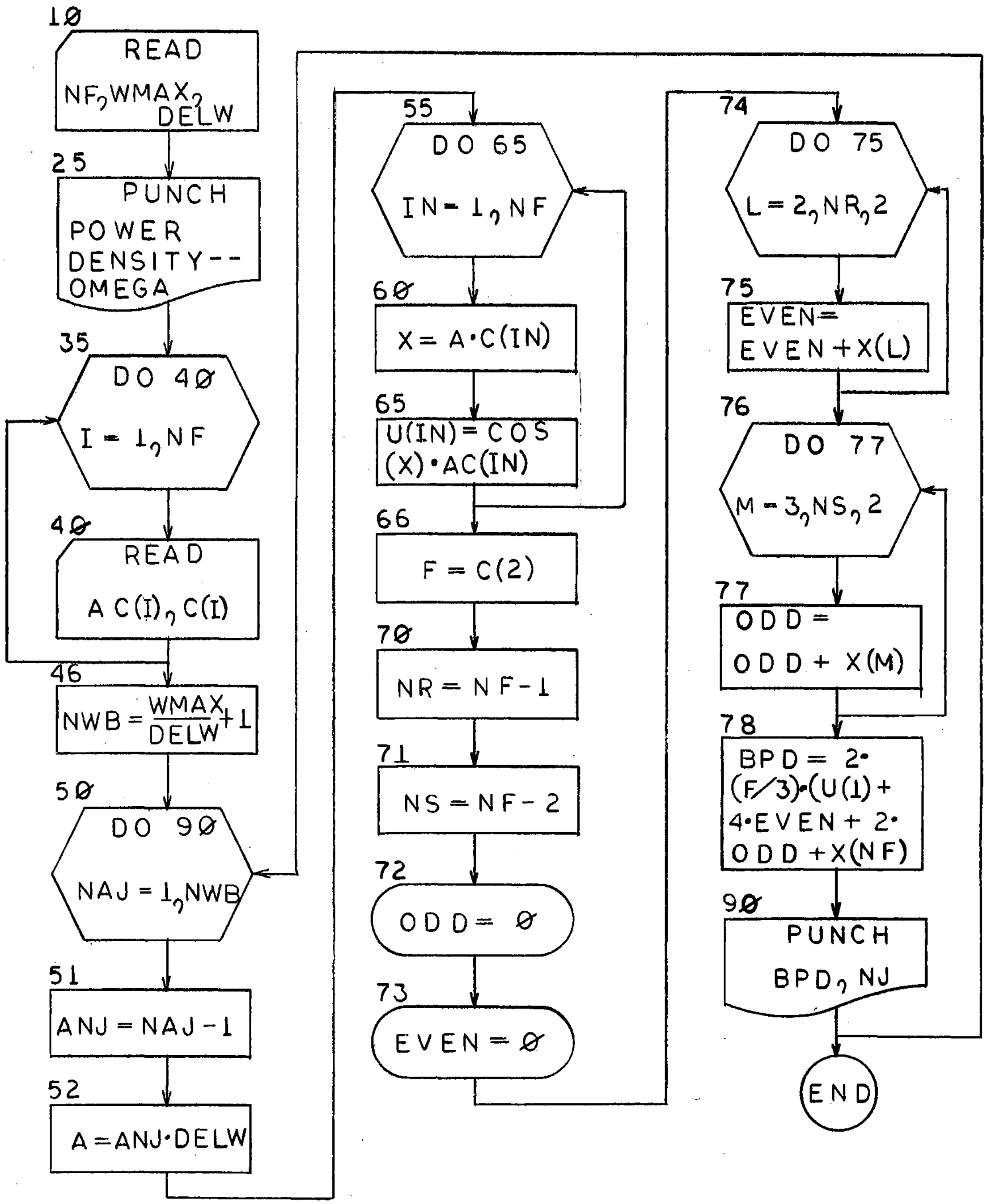

Figure 20
Flow Chart for Fourier Transform Program

of NF, positions 4 through 13 for the maximum value of omega and, positions 14 through 23 for the incremental value of omega.

Operation number 20 is a dimension statement which limits the problem variables to coincide with the limits of the crosscorrelation and autocorrelation programs.

The titles for the output data are punched by operations 25 and 30.

Operations 35, 40 and 45 read into memory the NF cards which contain the results of the autocorrelation program.

A value analogous to the variable omega is established and varied by operations 48, 50, and 51. Operation 55 establishes a variable to be used as a subscript when considering functions of t or τ.

Operations 60, and 65 establish the integrand of the Fourier transform integral.

The value of time between samples of autocorrelation is calculated by operation number 66.

Operations 70 through 78 perform the integration by a numerical process known as Simpson's one-third rule.

The results of the program are punched by operations 80 and 85.

Once the Fourier transform has been calculated for one value of omega, the process will return to operation number 50 and increase the value of omega by an amount equal to DELW. This repetitious process will continue until omega equal to WMAX has been utilized.

For optimum results, the input variable NF should be odd. This optimum condition is dictated by operations 70, 71, 74, and 76.

```
10 READ 15,NF,WMAX,DELW
15 FORMAT (I3,F10.2,F10.5)
20 DIMENSION U(200),AC(100),C(100)
25 PUNCH 30
30 FORMAT (30H    POWER DENSITY        OMEGA)
35 DO 40 I=1,NF
40 READ 45, AC(I),C(I)
45 FORMAT (E17.8,4X E14.5)
48 NWB = (WMAX/DELW)+1
50 DO 90 NAJ = 1,NWB
51 ANJ = NAJ-1
52 A = ANJ*DELW
55 DO 65 IN=1,NF
60 X = A*C(IN)
65 U(IN)=COS(X)*AC(IN)
66 F = C(2)
70 NR=NF-1
71 NS=NF-2
72 ODD=0
73 EVEN=0
74 DO 75 L=2,NR,2
75 EVEN=EVEN+U(L)
76 DO 77 M=3,NS,2
77 ODD=ODD+U(M)
78 BPD=2.*(F/3.)*(U(1)+4.*EVEN+2.*ODD+U(NF))
80 PUNCH 85, BPD,A
85 FORMAT (E17.8,4H    E14.5)
90 CONTINUE
91 STOP
95 END
```

Figure 21
Fortran Program for the Fourier Transform

CHAPTER IV

RESULT OF SAMPLE PROBLEMS

The use of the programs described in the previous chapter is illustrated by using them to calculate the correlations, convolutions, and power spectrums of a few simple functions. The results of the computations, with respect to accuracy, are discussed and the means of reducing error are pointed out.

Autocorrelation

The autocorrelation of a step function is perhaps the easiest autocorrelation measurement to make analytically. Assume a step function as defined by Equation 19.[1]

$$f(t) = \begin{cases} E & 0 < t < \\ 0 & t < 0 \end{cases} \tag{20}$$

Applying the definition, the autocorrelation of a step function is

$$\phi_{11}(\tau) = \frac{1}{T}\int_0^T f(t)f(t+\tau)\ dt \tag{21}$$

$$= \frac{1}{T}\int_0^T f^2(t)\ dt \tag{22}$$

$$= E^2 \tag{23}$$

[1] Y. W. Lee, Statistical Theory of Communication (New York, 1963), p. 91.

The autocorrelation program, as outlined in the previous chapter, was used to calculate the autocorrelation of a step function of magnitude 4. The calculations were made from 3 different sets of input conditions to show the effect of varying the input variables. The first calculation was made from twenty two samples taken over a nineteen second span, the second from sixty two samples taken over a nineteen second time span and, the third from sixty two samples taken over a thirty eight second time span.

The results of the three sets of conditions are shown in Figures 22, 23, and 24. Comparison of Figures 22 and 23 indicates that an increase in the number of readings improves results. This would be a logical assumption since the function is numerically rather than analytically integrated.

The effect of increasing the time span over which the readings are taken is shown by Figures 23 and 24. As indicated in Chapter II, increasing the value of T will reduce the net effect of the interval and thus improve the results of the problem.

The fact that the function is numerically rather than analytically integrated accounts for the majority of the error incurred in the problem. The error can be minimized by taking a large number of readings over a long period of time. In all three cases the results were within five percent of the desired results. Case number three, Figure 24, indicates results within one and two tenths percent of the desired.

AUTOCORRELATION	TAU
.15238094E+02	.00000E-99
.15238094E+02	904.76190E-03
.15238094E+02	180.95238E-02
.15238094E+02	271.42857E-02
.15238094E+02	361.90476E-02
.15238094E+02	452.38095E-02
.15238094E+02	542.85714E-02
.15238094E+02	633.33333E-02
.15238094E+02	723.80952E-02
.15238094E+02	814.28571E-02
.15238094E+02	904.76190E-02
.15238094E+02	995.23809E-02

Figure 22
Results of Autocorrelation
Measurement of $f(t) = 4U_{-1}(t)$ with NO = 22 and T = 19

AUTOCORRELATION	TAU
.15737702E+02	.00000E-99
.15737702E+02	311.47540E-03
.15737702E+02	622.95080E-03
.15737702E+02	934.42620E-03
.15737702E+02	124.59016E-02
.15737702E+02	155.73770E-02
.15737702E+02	186.88524E-02
.15737702E+02	218.03278E-02
.15737702E+02	249.18032E-02
.15737702E+02	280.32786E-02
.15737702E+02	311.47540E-02
.15737702E+02	342.62294E-02
.15737702E+02	373.77048E-02
.15737702E+02	404.91802E-02
.15737702E+02	436.06556E-02
.15737702E+02	467.21310E-02
.15737702E+02	498.36064E-02
.15737702E+02	529.50818E-02
.15737702E+02	560.65572E-02
.15737702E+02	591.80326E-02
.15737702E+02	622.95080E-02
.15737702E+02	654.09834E-02
.15737702E+02	685.24588E-02
.15737702E+02	716.39342E-02
.15737702E+02	747.54)96E-02
.15737702E+02	778.68850E-02
.15737702E+02	809.83604E-02
.15737702E+02	840.98358E-02
.15737702E+02	872.13112E-02
.15737702E+02	903.27866E-02
.15737702E+02	934.42620E-02
.15737702E+02	965.57374E-02

Figure 23
Results of Autocorrelation Measurement
of $f(t) = 4U_{-1}$ with NO = 62 and T = 19

AUTOCORRELATION	TAU
.15737704+02	.00000E-99
.15737704+02	622.95081E-03
.15737704+02	124.59016E-02
.15737704+02	186.88524E-02
.15737704+02	249.18032E-02
.15737704+02	311.47541E-02
.15737704+02	373.77049E-02
.15737704+02	436.06557E-02
.15737704+02	498.36065E-02
.15737704+02	560.65573E-02
.15737704+02	622.95081E-02
.15737704+02	685.24589E-02
.15737704+02	747.54097E-02
.15737704+02	809.83605E-02
.15737704+02	872.13113E-02
.15737704+02	934.42622E-02
.15737704+02	996.72130E-02
.15737704+02	105.90164E-01
.15737704+02	112.13115E-01
.15737704+02	118.36065E-01
.15737704+02	124.59016E-01
.15737704+02	130.81967E-01
.15737704+02	137.04918E-01
.15737704+02	143.27869E-01
.15737704+02	149.50819E-01
.15737704+02	155.73770E-01
.15737704+02	161.96721E-01
.15737704+02	168.19672E-01
.15737704+02	174.42623E-01
.15737704+02	180.65573E-01
.15737704+02	186.88524E-01
.15737704+02	193.11475E-01

Figure 24
Results of Autocorrelation Measurement
of $f(t) = 4U_{-1}(t)$ with NO = 62 and T = 38

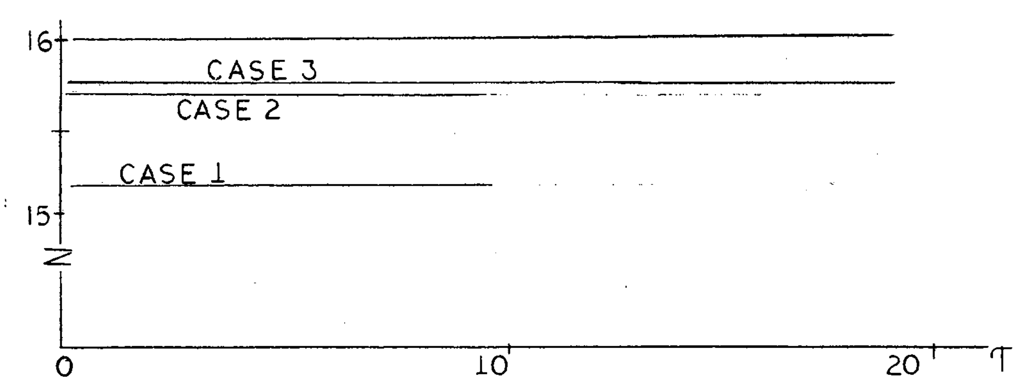

Figure 25
Graphical comparison of results of Three
Autocorrelation Measurements and expected Results

Power Density Spectrum

Analytically the power density spectrum of a unit step is an impulse at omega equal zero with a magnitude equal to the magnitude of the autocorrelation of the step function.[2]

The power density spectrum program as outlined in the previous chapter was used to calculate the power density spectrum of a step function of magnitude four. The input data for the power density spectrum was the output data from the autocorrelation program shown in Figure 24. Figures 26 and 27 illustrate the results of the power density spectrum calculation.

The desired result is an impulse at omega equals zero but, since the results of the autocorrelation measurement contained some error and the power density measurement itself contains error in its numerical method, the results are somewhat less than ideal. The results of the analytical derivation in Equation 24, 25 and 26 show the results expected when the errors due to the τ variable and the error due to the autocorrelation program are removed. Thus the Fourier transform program is in error by only about .8 percent, which is, once again, the result of the numerical method of integration

---

[2] Y. W. Lee, Statistical Theory of Communication (New York, 1963), p. 91.

POWER DENSITY	OMEGA
.58822882E+03	.00000E-99
.58481065E+03	100.00000E-04
.57462781E+03	200.00000E-04
.55789269E+03	300.00000E-04
.53495394E+03	400.00000E-04
.50628703E+03	500.00000E-04
.47248298E+03	600.00000E-04
.43423384E+03	700.00000E-04
.39231532E+03	800.00000E-04
.34756775E+03	900.00000E-04
.30087631E+03	100.00000E-03
.25314879E+03	110.00000E-03
.20529473E+03	120.00000E-03
.15820346E+03	130.00000E-03
.11272368E+03	140.00000E-03
.69644400E+03	150.00000E-03
.29675563E+03	160.00000E-03
-.65633744E+03	170.00000E-03
-.38559909E+03	180.00000E-03
-.65916240E+03	190.00000E-03
-.88356224E+03	200.00000E-03
-.10572864E+03	210.00000E-03
-.11800717E+03	220.00000E-03
-.12528714E+03	230.00000E-03
-.12777920E+03	240.00000E-03
-.12580002E+03	250.00000E-03
-.11976069E+03	260.00000E-03
-.11015257E+03	270.00000E-03
-.97531716E+02	280.00000E-03
-.82501882E+02	290.00000E-03
-.65696716E+02	300.00000E-03
-.47761659E+02	310.00000E-03
-.29336216E+02	320.00000E-03
-.11036773E+02	330.00000E-03
-.65593236E+01	340.00000E-03
-.22928071E+02	350.00000E-03

Figure 26
Results of the Power Density
Measurement of the Output Data Shown in Figure 25

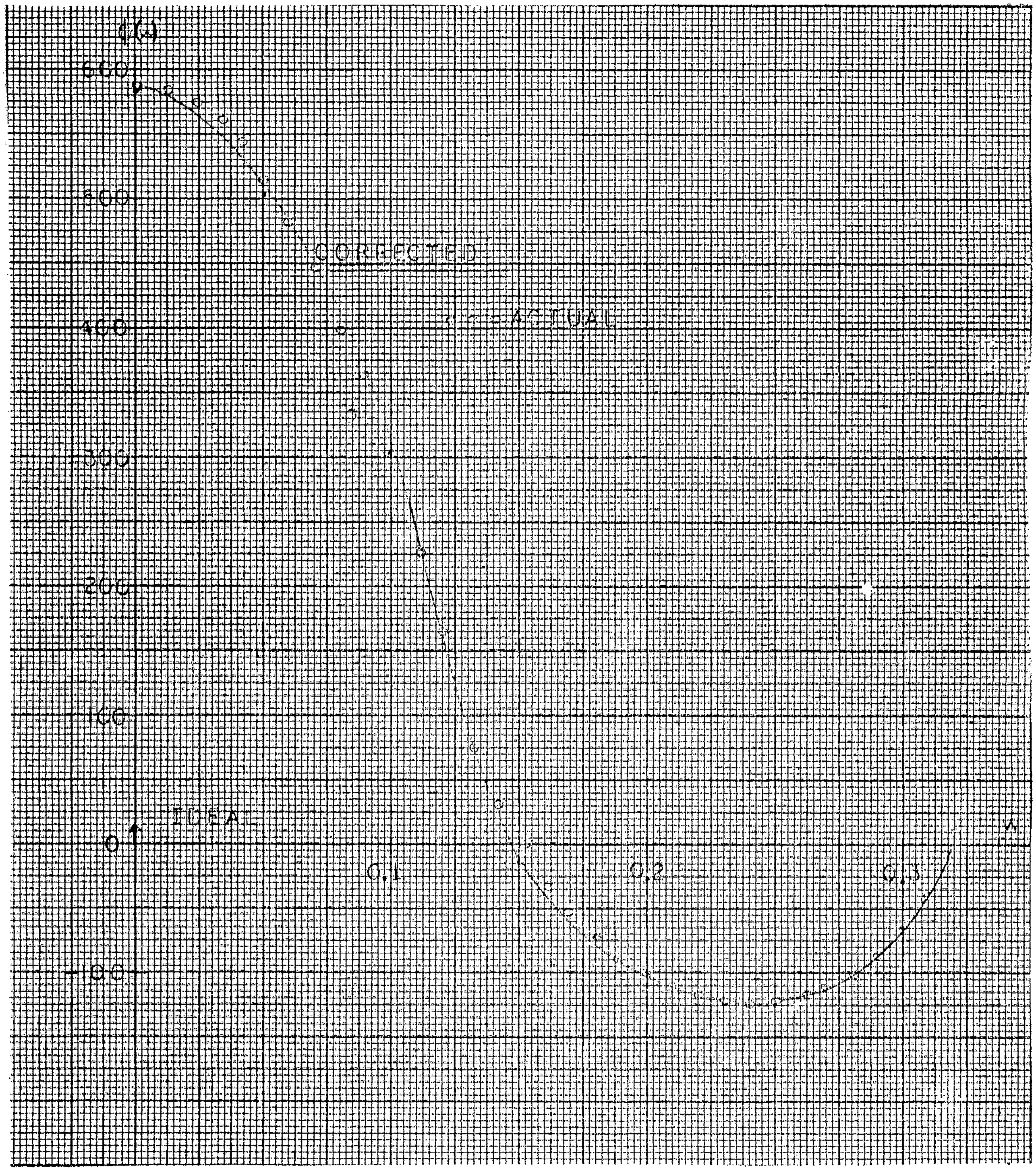

Figure 27
Comparison of the Ideal, the Corrected and the
Actual results of the Power Density Measurement

$$\Phi(\omega) = \int 15.737 \cdot \cos(\omega\tau)\, d\tau \tag{24}$$

$$\Phi(\omega) = 2 \int_0^{18.6} 15.737 \cos(\omega\tau)\, d\tau \tag{25}$$

$$\Phi(\omega) = 31.474 \frac{\sin (18.6\ \omega)}{\omega} \tag{26}$$

Crosscorrelation

The crosscorrelation measurement of two step functions is a very simple analytical calculation yet it will provide a very accurate check of the crosscorrelation program.

From the definition of the crosscorrelation integral, the crosscorrelation of two step functions is a step function whose magnitude is equal to the product of the magnitudes of the two time functions.

$$\phi(\tau) = \frac{1}{T} \int_0^T f_1(t) f_2(t+\ )\, dt \tag{27}$$

$$\phi(\tau) = \frac{1}{T} \int_0^T AU_{-1}(t) \cdot BU_{-1}(t+\ )\, dt \tag{28}$$

$$\phi(\tau) = (AB) \cdot U_{-1}(t) \tag{29}$$

The crosscorrelation between a step function of magnitude four and one of magnitude two was performed by the program described in Chapter III. The results of the program are shown in Figure 28. A comparison between the calculated results and the analytic result is shown in Figure 29. Results of the comparison indicate that the crosscorrelation program is just as accurate as the autocorrelation program. In both cases the majority of the error incurred is the direct result of the numerical method of integration. Since the error is constant with τ, the result obtained can be interpreted for their statistical implications, in both the autocorrelation and crosscorrelation cases, as if there were zero error.

CROSSCORRELATION	TAU
.78688519E+01	.00000E-99
.78688519E+01	622.95081E-03
.78688519E+01	124.59016E-02
.78688519E+01	186.88524E-02
.78688519E+01	249.18032E-02
.78688519E+01	311.47541E-02
.78688519E+01	373.77049E-02
.78688519E+01	436.06557E-02
.78688519E+01	498.36065E-02
.78688519E+01	560.65573E-02
.78688519E+01	622.95081E-02
.78688519E+01	685.24589E-02
.78688519E+01	747.54097E-02
.78688519E+01	809.83605E-02
.78688519E+01	872.13113E-02
.78688519E+01	934.42622E-02
.78688511E+01	996.72130E-02
.78688519E+01	105.90164E-01
.78688519E+01	112.13115E-01
.78688519E+01	118.36065E-01
.78688519E+01	124.59016E-01
.78688519E+01	130.81967E-01
.78688519E+01	137.04918E-01
.78688519E+01	143.27869E-01
.78688519E+01	149.50819E-01
.78688519E+01	155.73770E-01
.78688519E+01	161.96721E-01
.78688519E+01	168.19672E-01
.78688519E+01	174.42623E-01
.78688519E+01	180.65573E-01
.78688519E+01	186.88524E-01
.78688519E+01	193.11475E-01

Figure 28
Results of a Crosscorrelation
Measurement of $4U_{-1}(t)$ and $2U_{-1}(t)$

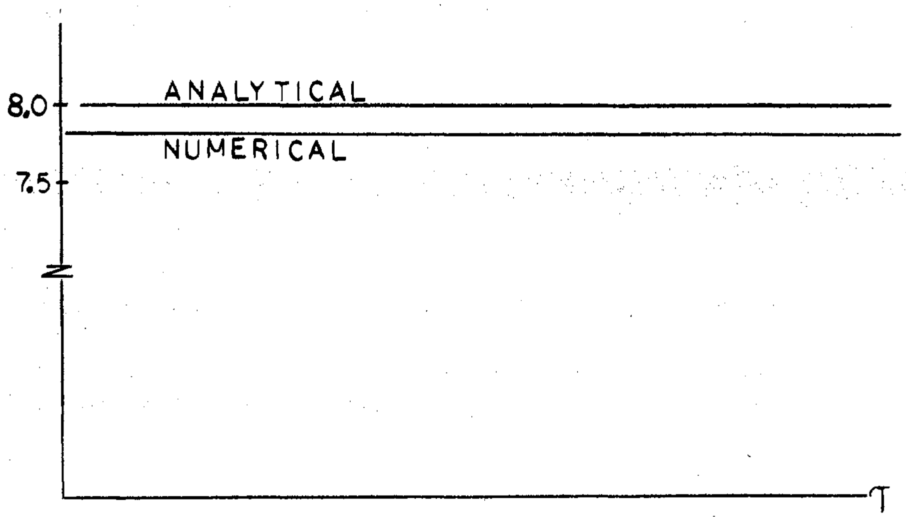

Figure 29
Comparison between the Analytical
and Numerical Results of a Crosscorrelation Measurement

Convolution

As illustrated in Chapter II, Figure 9 and Equation 5, one of the primary applications of the convolution integral and theorem is in the determination of the time function output of a system. By applying the convolution theorem, the output of an open loop control system with a transfer function of 1/s and a transformed input 1/s would be a ramp function.

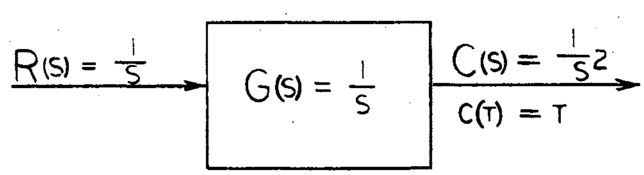

Figure 30
Open-Loop Control System with an
Input of 1/s and a Transfer Function of 1/s

The convolution program was used to calculate the convolution of two causal functions, one of magnitude four and the other of magnitude two. The results of the problem, shown in Figure 31 and 32, indicate that the error caused by the program is approximately one and fifteen one hundreds times time percent. Figure 32 gives a graphical comparison of the expected results and the actual results.

CONVOLUTION	TIME
.00000000E-99	.00000E-99
.85875704E-00	322.03389E-03
.42937852E+01	644.06778E-03
.60112993E+01	966.10167E-03
.94463274E+01	128.81356E-02
.11163842E+02	161.01695E-02
.14598870E+02	193.22033E-02
.16316384E+02	225.42372E-02
.19751412E+02	257.62711E-02
.21468926E+02	289.83050E-02
.24903954E+02	322.03389E-02
.26621468E+02	354.23728E-02
.30056496E+02	386.44067E-02
.31774010E+02	418.64406E-02
.35209039E+02	450.84745E-02
.36926553E+02	483.05084E-02
.40361581E+02	515.25422E-02
.42079095E+02	547.45761E-02
.45514123E+02	579.66100E-02
.47231637E+02	611.86439E-02
.50666665E+02	644.06778E-02
.52384179E+02	676.27117E-02
.55819208E+02	708.47456E-02
.57536722E+02	740.67795E-02
.60971750E+02	772.88134E-02
.62689264E+02	805.08473E-02
.66124292E+02	837.28811E-02
.67841806E+02	869.49150E-02
.71276834E+02	901.69489E-02
.72135591E+02	933.89828E-02
.72135591E+02	966.10167E-02
.72135591E+02	998.30506E-02

Figure 31
Results of the Convolution
Measurement of $f_1(t) = 4U_{-1}(t)$ and $f_2 = 2U_{-1}(t)$

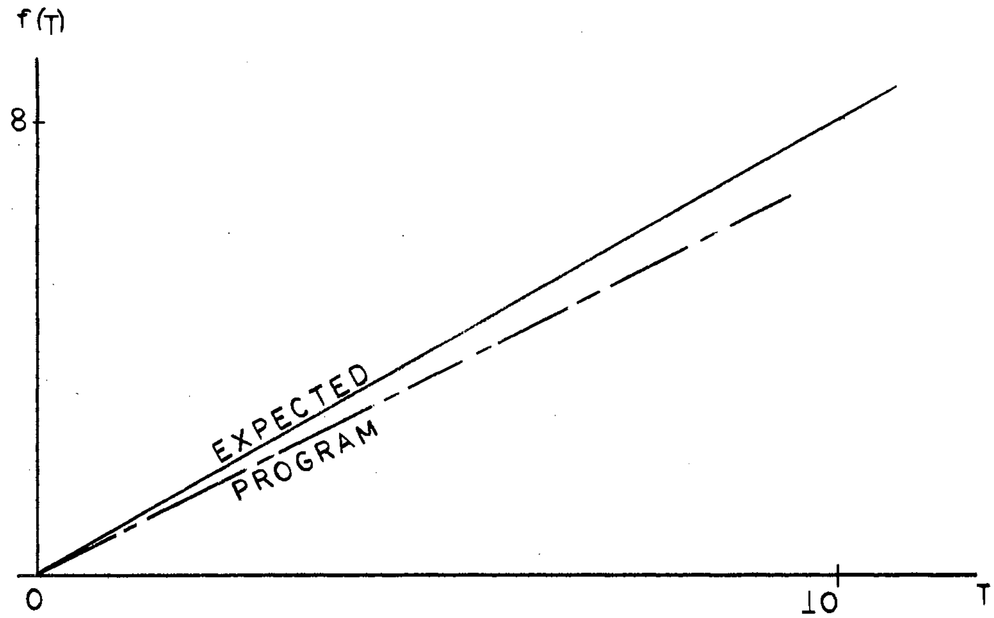

Figure 32
Comparison of Program and
Expected Results of Convolution Measurement

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