OPTIMUM PASS-FAIL TESTING DECISIONS

by

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CHAPTER I

INTRODUCTION

Success or failure? Accept or reject? Yes or no? These are but three of the innumerable dichotomous decisions made daily in every field of endeavor. While the greatest percentage of these decisions are made almost instanteously by humans (and, for the most part, almost as quickly forgotten), an increasing number of the more repetitive of them are being quantified and the results made the basis for policy determination, at times being incorporated directly into equipment.

This investigation is concerned with such decisions. More particularly, it examines methods for most economically determining the distribution of the random variable which governs the outcome of the problem. Knowledge of this distribution is considered to be available from two sources: first, from the extant experience pertaining to it, a priori knowledge, and secondly, from samples drawn from the process itself. The a priori knowledge is assumed to include not only estimates of the initial state of the governing variable but also predictions of its future behavior. It must be possible to categorize the samples taken into a dichotomy which parallels the decision space, i.e.: acceptable or unacceptable. To this knowledge must be added a third essential, a model of the problem which includes the gain or loss associated with each of the decisions.

The problem which germinated the investigations of this paper was that of an adaptive communication system involving a binary symmetric channel (1). The decision involved was the choice of one of two decoding schemes. The probability of correct transmission was the random variable concerned and the loss functions were determined by the channel entropy. While not treated specifically herein, this problem is embedded in the class of problems considered and the developments of this study can, with simple modification, be used in its solution.

From the above, it is evident that the decision problems investigated are subject to the following restrictions:

a) The process must be amenable to meaningful sampling.

b) The decision space and the samples must be dichotomous.

c) The loss functions must be determinable.

d) A priori knowledge of the governing state of nature, including possible change, must be available.

Within this framework, we will consider first the problem of determining a single optimum sample size when only a priori information is available and the state of nature remains the same throughout the period of consideration. This problem has been frequently considered for special cases. Here the method is generalized allowing adaptation to many problems and providing a foundation for the subsequent developments.

The second major development considers an adaptive decision maker. That is, there is continuous feedback during the sampling process to the decision maker who, after evaluating each sample, can direct the taking of another sample prior to the final accept or reject decision. A dynamic programming approach is used and again, the state of nature is time invariant. The use of dynamic programming in adaptive systems has been suggested previously (2), (3) and the sequential sampling problem has been widely studied since Wald's initial work in the area (4).

The natural meld of dynamic programming with statistical decision theory in the sampling problem has also been suggested (5), (6). While neither sequential sampling nor the use of dynamic programming in adaptive systems is unique, nothing has been found in the literature regarding the use of dynamic programming for sequential sampling decisions of the type considered in this paper.

The last portion of this study involves consideration of the state of nature as a stochastic process. The effect of time on the random variable involved is described by a difference equation model and the resulting distributions of the random variable studied. Finally, these results are used in the decision theory formulation to determine practical optimum sampling plans.

Throughout this paper an example problem from the operations research area is included to illustrate the use of the techniques developed. The problem, while relatively simple to facilitate the following of the techniques, is general enough to be directly adaptable to a large class of extant physical situations and, with minor modification, to many other situations both in operations research and other areas.

Where appropriate, FORTRAN programs for a digital computer have been written and used. These programs and certain results from them, appear as appendices.

It is assumed that the reader is familiar with the basic concepts of statistical decision theory such as those discussed in Weiss (7). While not a prerequisite, an understanding of the rudiments of dynamic programming is helpful (8).

CHAPTER II

TIME INVARIANT, SINGLE SAMPLE SIZE DETERMINATION

The problem of determining the optimum sample size in the time-invariant, non-sequential case is merely one of application of statistical decision theory techniques.

For solution, three essentials must be known:

(1) Set of possible decisions

(2) Loss function

(3) Description of nature

In the binary non-sequential sampling problem, only two decisions are possible. These may be called yes - no, go - no go, accept or reject, 1 or 0, etc, but in every case, the two decisions constitute the entire decision space and are mutually exclusive.

The loss functions involved are completely determined by and normally unique to the particular problem under consideration. They may or may not be determined by the problem solver and, in many cases, involve subjective judgements on the part of the individuals tasked with making such a determination. For our purposes, the example chosen for illustration will attempt to avoid controversy in the assignment of the loss functions.

Distributions of the Random Variables

In this investigation, it is considered that nature is completely described by some distribtuion of a random variable, P, the probability of one sample being "acceptable". This distribtuion is determined by considering the method of sampling. Each sample is discrete, statistically independent, and can be placed in one of two definite categories, say "favorable" or "unfavorable". Letting the random variable A represent the number of favorable results, the distribution of A becomes the familiar Binomial:

$$P\ (A = a|p;x) = f_{A|P;x}(a|p;x) = \binom{x}{a} p^{a}(1 - p)^{x-a} \tag{2.1}$$

where x is the total sample size. Note that this considers P a known quantity between zero and one and represents the distribution of A given P. Since we are attempting to determine P having sampled x items and finding a favorable ones (the sampling experience being denoted $\mathcal{E}$), we make use of Bayes Rule:

$$f_{P|\mathcal{E}}(p|a;x) = \frac{f_{A|P;x}(a|p;x)\ f_P(p)}{f_A(a)}.$$

Since

$$f_A = \int_{-\infty}^{\infty} f_{A|P;x}(a|\theta;x)\ f_p(\theta)d\theta$$

this a posteriori density becomes

$$f_{P|\mathcal{E}}(p|a;x) = \frac{f_{A|P;x}(a|p;x)\ f_P(p)}{\int_{-\infty}^{\infty} f_{A|P;x}(a|\theta;x)\ f_P(\theta)d\theta}. \tag{2.2}$$

The problem now becomes one of selecting an appropriate a priori distribution, $f_P(p)$.

Since the number of favorable samples is distributed binomially, a reasonable candidate is the Beta distribution. It fulfills the criteria of being a continuous distribution as well as having many similarities with the discrete Binomial.

Letting

$$f_{P|\mathcal{E}}(p;\lambda,\psi) = \frac{\Gamma(\lambda+2)}{\Gamma(\psi+1)\Gamma(\lambda-\psi+1)} p^{\psi}(1-p)^{\lambda-\psi} \tag{2.3}$$

with $0 \leq P \leq 1$, $\psi > -1$, $\lambda > \psi - 1$, and $\mathcal{E}$ indicating the best extant prior knowledge about P, equation 2.2 becomes

$$f_{P|\mathcal{E}}(p|a;x,\psi,\lambda) = \frac{p^{a+\psi}(1-p)^{x+\lambda-a-\psi}}{\int_0^1 \theta^{a+\psi}(1-\theta)^{x+\lambda-a-\psi}d\theta}. \tag{2.4}$$

The denominator of 2.4 is recognized as a complete Beta function yielding

$$\int_0^1 \theta^{\alpha}(1-\theta)^{\beta}d\theta = B(\alpha+1,\beta+1) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{2.5}$$

If α and β are non-negative integers, say a and b, the integral becomes

$$\int_0^1 \theta^{a}(1-\theta)^{b}\, d\theta = \frac{a!b!}{(a+b+1)!} \tag{2.6}$$

if 0! is defined as one. From 2.4 and 2.5,

$$f_{P|\mathcal{E}}(p|a,x,\psi,\lambda) = \frac{\Gamma(x+\lambda+2)}{\Gamma(a+\psi+1)\Gamma(x+\lambda-a-\psi+1)} p^{a+\psi}(1-p)^{x+\lambda-a-\psi} \tag{2.7}$$

for $0 \leq p \leq 1$, $\psi > -1$, $\lambda > \psi - 1$, a and x non-negative integers. (These restrictions on the values of the parameters pertain throughout this paper and will no longer be explicitly stated except where necessary for clarity.)

The density of equation 2.7 is again recognized as showing a Beta distribution for P which is further, and perhaps the strongest, argument for choosing the Beta as the a priori.

When the parameters of the a priori distribution, λ and ψ, are both zero, it reduces to the equally - likely or rectangular distribution

$$f_{P|\mathcal{E}}(p;o,o) = \begin{cases} 1, & o \le p \le 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{2.8}$$

For this initial development, the rectangular form for the a priori distribution will be assumed. Chapter IV will relax this restriction by allowing λ and ψ to take values other than zero.

Since a and x are non-negative integers, equation 2.6 pertains and equation 2.7 becomes

$$f_{P|\mathcal{E}}(p|a;x,o,o) = \frac{(x+1)!}{a!(x-a)!}\, p^{a}(1-p)^{x-a}\,. \tag{2.9}$$

Risk Determination

Having determined the probability distributions involved in the sampling problem, we are now prepared to formulate the risk functions which we will be considering.

The following notation will be adopted for use throughout this paper:

z = lot size

x = sample size

y = nr. remaining after sampling

a = nr. of acceptable samples

b = nr. of unacceptable samples

p = probability of good

q = (1-p), probability of bad

D_A = decision A (accept, etc)

D_B = decision B (reject, etc)

Since sampling occurs prior to the decision the risk incurred during the sampling or testing process is normally the same regardless of the eventual decision. If we know L_T, the loss during sampling, we can determine the sampling risk, R_T, by calculating the expected value of the loss. Since R_T must involve the number of samples taken, it is a function of x as well as P. Thus

$$R_T\ (x,p) = E\ [L_T]. \tag{2.10}$$

We are now prepared to calculate the risks incurred after sampling. Two different risks must be calculated here; one under decision A (accept, go, 1, yes, etc) or B (reject, no-go, zero, no, etc). Again, the losses, L_{D_A} and L_{D_B}, are determined by the problem and the associated risks are the expected values of these losses.

$$R_{D_A} = E\left[L_{D_A}\right] \quad \text{and} \quad R_{D_B} = E\left[L_{D_B}\right]. \tag{2.11}$$

These risks are functions of the variable y and the random variable P. Since, in general, y is a function of x (and possibly p), these risks can be expressed as functions of p and x.

Having determined R_{D_A} and R_{D_B}, we can determine what sampling results will be used to choose the best decision by setting up inequalities. For decision A,

$$E\left[R_{D_A}\right] < E\left[R_{D_B}\right]$$

and, for B,

$$E\left[R_{D_A}\right] \geq E\left[R_{D_B}\right].$$

Since the only random variable involved in these risks is P (a, b, x, and y are, after testing, known integers, not random variables), and we have, from equation 2.8, the distribution of p given a favorables from x samples, these inequalities can be solved for the values of a in terms of x which will form the decision boundary. These will be of the form

$$a \lesseqgtr g(x), \qquad (2.12)$$

the direction of the inequality indicating the decision. For this development, with no loss in generality, $a > g(x)$ will be used to choose decision A, accept, and $a \leq g(x)$ will choose B, reject, noting that when $a = g(x)$, $R_{D_A} = R_{D_B}$.

We can now write the expected value of the summation of these losses as follows:

$$\overline{R}\,(x,p|\mathcal{E}) = R_T P\,(o \leq a \leq x) + R_{D_A} P\,[g(x) < a \leq x] + R_{D_B} P\,[o \leq a \leq g(x)] \qquad (2.13)$$

where $\mathcal{E}$ indicates the a priori estimate on P. Since, from equation 2.1, A is discrete and binomially distributed,

$$P\,[g(x) < a \leq x] = 1 - P\,[o \leq a \leq g(x)]$$

and

$$P\,[o \leq a \leq g(x)] = \sum_{a=o}^{w} \binom{x}{a} p^a (1-p)^{x-a}$$

where w is the "greatest integer function"[1] of g(x). Equation 2.13 becomes

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[1]Apostol, T. M., Mathematical Analysis (Reading, Mass., 1957), p. 201: "The value of the 'greatest-integer function' of x is the greatest integer which is less than or equal to x, denoted by [x]."

$$\overline{R}\,(x,p|\mathcal{E}) = R_T + R_{D_A} + \left(R_{D_B} - R_{D_A}\right)\sum_{a=o}^{w}\binom{x}{a}p^a(1-p)^{x-a}. \qquad (2.14)$$

The expected value of this expression must now be considered. Since P is the only random variable involved, this expected value is

$$E\left[\overline{R}(x,p|\mathcal{E})\right] = \overline{R}(x|\mathcal{E}) = \int_{-\infty}^{\infty}\overline{R}(x,\theta|\mathcal{E})\, f_{P|\mathcal{E}}(\theta|\mathcal{E})\, d\theta$$
$$= \int_{-\infty}^{\infty}\left[R_T + R_{D_A} + \left(R_{D_B} - R_{D_A}\right)\sum_{a=o}^{w}\binom{x}{a}\ \theta^a(1-\theta)^{x-a}\right] d\theta. \qquad (2.15)$$

Since $\theta^a(1-\theta)^{x-a}$ is continuous in the closed interval zero to one when x and a are non-negative integers with $a \leq x$, the integration and summation in equation 2.15 can be interchanged[2] yielding

$$\overline{R}\,(x|\mathcal{E}) = \int_0^1\left(R_T + R_{D_A}\right)d\theta + \sum_{a=o}^{w}\binom{x}{a}\int_0^1\left(R_{D_B} - R_{D_A}\right)\theta^a(1-\theta)^{x-a}\, d\theta. \qquad (2.16)$$

Performing the indicated integrations and summation results in an expression for the total risk as a function of x and w. Since w is a function of x, the range of x in terms of w can be found by solution of the following inequality:

$$w \leq g(x_o) < w+1. \qquad (2.17)$$

Selection of a value, x_o, within this range will give, upon substitution, an $\overline{R}(x_o|\mathcal{E})$.

The value of x_o which minimizes $\overline{R}(x_o|\mathcal{E})$ can usually be found by simple techniques of differential calculus. Recalling that the optimum

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[2]Ibid., p. 221.

value of x, x_{opt}, must be a non-negative integer, allowable values of x near x_o should be substituted in equation 2.16 [not in the expression for $\overline{R}(x_o|\mathcal{E})$] until that which produces a minimum for $\overline{R}(x|\mathcal{E})$ is found. This then is the optimum sample size in the time-invariant non-sequential case if sampling is done. This minimum expected risk with sampling must be compared with the appropriate risk when no sampling is accomplished. If the latter risk is less than the minimum sampling risk, no samples should be taken.

In summary, the procedure is as follows:

a) Determine the appropriate losses, L_T, L_{D_A} and L_{D_B}.

b) Calculate associated conditional risks, $R = E\,[L|P]$.

c) Determine the decision rule.

d) Find the total risk, $\overline{R}(x,P|\mathcal{E}) = \sum_a E[R]$.

e) Find the expected value of total risk;

$$\overline{R}\,(x|\mathcal{E}) = E\,[\overline{R}(x,p|\mathcal{E})] = \int_{-\infty}^{\infty} \overline{R}(x,p|\mathcal{E})\; f_{P|\mathcal{E}}(p|\mathcal{E})\; dp$$

$$= \int_0^1 \sum_a E\,[E(L)]\; dp. \qquad (2.18)$$

f) Find the integer value of x which minimizes $\overline{R}(x|\mathcal{E})$.

An Alternate Approach

Howard (5) has developed a general model for solution of problems of this nature which could also be used in determining $\overline{R}(x|\mathcal{E})$. His model is based upon two equivalent trees, the "decision tree", and "nature's tree". For this problem, these trees take the forms shown in Figures 1 and 2.

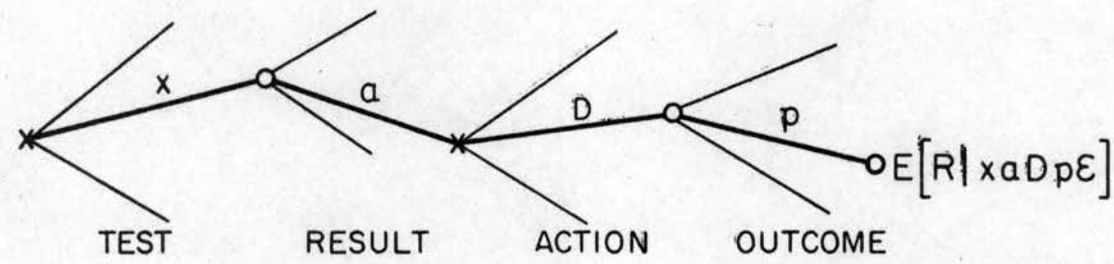

Figure 1. Decision Tree

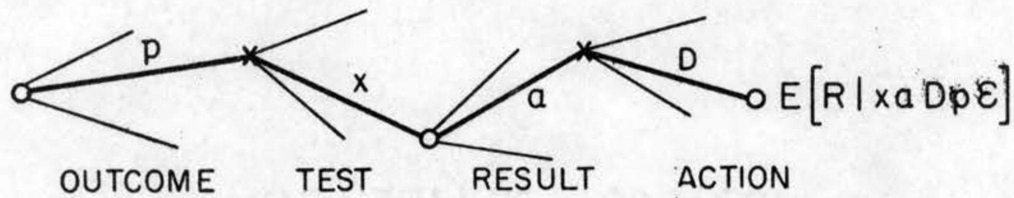

Figure 2. Nature's Tree

The script E symbol, ℰ, again denotes previous experience which, in this case, is the a priori distribution of P, f_P. The "test" is the selection of the x items to be sampled, the "result" is the number of acceptable items, a, of the x, the "action" is the decision, A or B, selected as a result of the sampling, and the "outcome" is determined by the random variable P.

Howard shows that the expected risk, given only the a priori of P, can be expressed as

$$E\,[R|\mathcal{E}] = \underset{x}{S}\;\underset{a}{S}\;\underset{D}{S}\;\underset{P}{S}\; E\,[R|x,a,D,P,\mathcal{E}]\; f_{x,A,D,P|\mathcal{E}}(x,a,D,p|\mathcal{E}) \tag{2.19}$$

$$= \underset{x}{S} f_{x|\mathcal{E}}(x) \;\underset{A}{S} f_{A|x\mathcal{E}} \;\underset{D}{S} f_{D|x,A,\mathcal{E}} \;\underset{P}{S} f_{P|x,A,D,\mathcal{E}}\; E\,[R|x,a,D,p,\mathcal{E}]$$

where $\mathfrak{S}$ is a general summation operator over the set or variable on which it operates comparable to a Reimann Stieltjes integral.

His procedure involves the assigning of probabilities to $P|\mathcal{E}$ and to $A|Px\mathcal{E}$, in this case $f_{P|\mathcal{E}}$ equally likely and $f_{A|xP\mathcal{E}}$ binomial as shown in equation 2.1. He then finds $f_{P|xA\mathcal{E}}$ by use of Bayes Theorem as shown in equations 2.2 thru 2.8. By selecting with probability one the best test - optimum x, and the best action, decision A or B, dictated by the test results, he simplifies the decision tree to that shown in Figure 3.

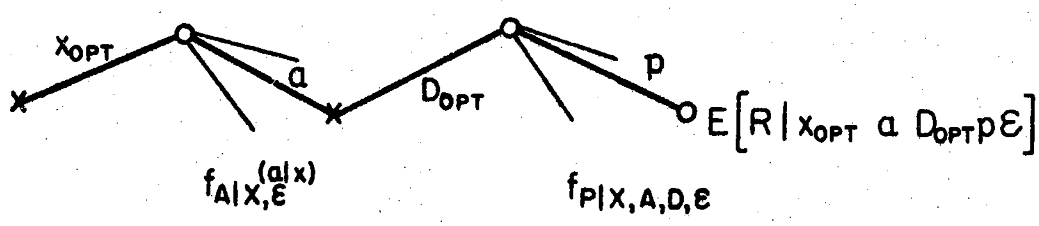

Figure 3. Modified Decision Tree

Arguing that $f_{P|xAD\mathcal{E}} = f_{P|xA\mathcal{E}}$ when the outcome, P, is governed by nature rather than an opponent, he reduces equation 2.19 to

$$E\,[R|\mathcal{E}] = \underset{A}{\mathfrak{S}} f_{A|x,\mathcal{E}}\ \mathfrak{S}\ f_{P|xA\mathcal{E}}\ E\,[R|x_{opt},a,D_{opt},p,\mathcal{E}] \qquad (2.20)$$

Comparison of equation 2.20 with equation 2.18 shows the same result if R_T is substituted for $E\,[R|x_{opt},a,D_{opt},p,\mathcal{E}]$ since

$$\underset{A}{\mathfrak{S}} f_{A|x,\mathcal{E}}\ \underset{P}{\mathfrak{S}}\ f_{P|x,A,\mathcal{E}} = \int_0^1 \sum_A f_{A|xP\mathcal{E}}\ f_{P|x,\mathcal{E}}\ dp$$

An Operations Research Example

An example illustrating the procedures developed in this chapter and also to be used in subsequent chapters has been chosen from the operations research field.

The problem is as follows: An item is to be manufactured or procured in lots of size z. The total cost of one item, including material, labor, overhead, etc., is C. The item is to be sold or released for use for a gain equal to $(1+\alpha)C$, where α represents the markup or other gain factor. A penalty of γ times the total gain is forfeited for each defective item which remains after sampling. The items can be destructively tested prior to deciding whether or not to release the remainder of the lot at a testing cost of βC per item tested. The number to be tested is x, y is the number remaining after testing, (z-x), and F is the number of defectives of the y. The random variable P is the probability of a good item, q is (1-p), a is the number of good samples, and b is the number of bad samples. The a priori distribution of P is equally likely between zero and one. Decision A is the decision to accept the lot, i.e. market y of the items; decision B is to reject the lot. Salvage value is considered negligible.

Since the loss incurred during testing is independent of whether the untested items are accepted or rejected - decision A or B - it will be designated as L_T. The losses after testing are dependent on the decision made and will be designated as L_{D_A} and L_{D_B} for the accept and reject decisions respectively.

From the statement of the problem, these losses are as follows:

$$L_T = x(1+\beta)C$$

$$L_{D_A} = y[C-(1+\alpha)C] + F\gamma(1+\alpha)C = -\alpha yC + F\gamma(1+\alpha)C$$

$$L_{D_B} = yC.$$

The expected values of the losses, the conditional (upon p) risks, are

$$R_T = E\,[L_T] = E\,[x(1+\beta)C] = x(1+\beta)C \tag{2.21}$$

$$\begin{aligned} R_{D_A} = E\left[L_{D_A}\right] &= E\,[C(F\gamma(1+\alpha) - \alpha y)] = -\alpha yC + C\gamma(1+\alpha)(1-p)y \\ &= Cy\,[\gamma(1+\alpha)-\alpha-\gamma(1+\alpha)p] \end{aligned} \tag{2.22}$$

$$R_{D_B} = E\left[L_{D_B}\right] = E\,[Cy] = Cy\ . \tag{2.23}$$

Since these risks are functions of the random variable P, the expected value of the risks, $R_{\underline{D}}$, when no testing is done (x=0, y=z) and the a priori of P is equally likely, can be easily calculated.

$$R_T\Big|_{x=o} = \int_{-\infty}^{\infty} R_T\, f_{P|\mathcal{E}}(\theta)\, d\theta = \int_{o}^{1} x(1+\beta)C\, d\theta = 0.$$

$$R_{D_A}\Big|_{x=o} = \int_{o}^{1} Cy[\gamma(1+\alpha)-\alpha-\gamma\theta(1+\alpha)]\, d\theta = Cz\left[\frac{\gamma(1+\alpha)}{2} - \alpha\right]. \tag{2.24}$$

$$R_{D_B}\Big|_{x=o} = \int_{o}^{1} Cy d\theta = Cz. \tag{2.25}$$

Thus, when no testing is done, the best decision depends on the value of gamma. The lot should be accepted if

$$R_{D_A}\Big|_{x=o} < R_{D_B}\Big|_{x=o}.$$

Substituion from equations 2.24 and 2.25 gives

$$\frac{\gamma(1+\alpha)}{2} - \alpha < 1$$

or

$$\gamma < 2. \tag{2.26}$$

Similarly, with no testing, the lot would be rejected if gamma is greater than two, and either decision would yield the same expected risk, Cz, when gamma is two.

To determine which decision is best when we have performed some sampling to help determine the distribution of p, we set up a similar inequality as the criteria for choosing decision A:

$$E\left[R_{D_A}\right] < E\left[R_{D_B}\right].$$

Substituting from equation 2.22 and 2.23 gives

$$\begin{aligned} E\,[Cy(\gamma+\alpha\gamma-\alpha-(1+\alpha)\gamma p)] &< E\,[Cy] \\ \gamma(1+\alpha)-\alpha-\gamma(1+\alpha)E[p] &< 1 \\ \gamma(1+\alpha)E[p] &> (1+\alpha)(\gamma-1) \\ E[p] &> \frac{\gamma-1}{\gamma}. \end{aligned} \tag{2.27}$$

Since the expected value of P desired here is that after sampling, equation 2.27 becomes

$$\begin{aligned} \frac{a+1}{x+2} &> \frac{\gamma-1}{\gamma} \\ a &> \frac{(\gamma-1)x + (\gamma-2)}{\gamma} = g(x). \end{aligned} \tag{2.28}$$

Note that this equation indicates that, for $o \le a \le x$, γ must be greater than one. Similarly, the criteria for choosing the reject decision, B, is $a \le g(x)$.

The total risk, $\overline{R}(x,p|\mathcal{E})$ can be written

$$\overline{R}\,(x,p|\mathcal{E}) = R_T + R_{D_A} P[a>g(x)] + R_{D_B} P[a\le g(x)]$$

$$= R_T + R_{D_A} + \left(R_{D_B} - R_{D_A}\right) P[a \leq g(x)]$$

$$= C \left\{ (1+\beta)x + y[\gamma(1+\alpha)-\alpha-\gamma(1+\alpha) p] \right.$$

$$\left. + y(1+\alpha)(1-\gamma+\gamma p) P[a \leq g(x)] \right\}.$$

Since $P[a \leq g(x)] = \sum_{a=o}^{w} \binom{x}{a} p^a (1-p)^{x-a}$, where w is [g(x)], i.e.: greatest integer function of g(x), the risk becomes

$$\bar{R}(x,p|\mathcal{E}) = C \left\{ (1+\beta)x + y(\gamma(1+\alpha)-\alpha-\gamma(1+\alpha)p] \right. \tag{2.29}$$

$$\left. + y(1+\alpha)(1-\gamma+\gamma P) \sum_{a=o}^{w} \binom{x}{a} p^a (1-p)^{x-a} \right\}.$$

To determine the expected value of this risk as a function of the sample size, x, we proceed as in equation 2.15

$$\bar{R}(x|\mathcal{E}) = C \int_o^1 \left\{ (1+\beta)x + y[\gamma(1+\alpha)-\alpha-\gamma(1+\alpha)\theta] \right.$$

$$\left. + y(1+\alpha)(1-\gamma+\gamma\theta) \sum_{a=o}^{w} \binom{x}{a} \theta^a (1-\theta)^{x-a} \right\} f_{P|\mathcal{E}}(\theta; o, o) d\theta.$$

Substitution of the appropriate values, interchanging summation and integration, and performing the integration (See Appendix A), gives

$$\bar{R}(x|\mathcal{E}) = C \left\{ z + \beta x + \frac{y(1+\alpha)}{2} \right. \tag{2.30}$$

$$\left. \left[\gamma - 2 + \frac{w+1}{(x+1)(x+2)} (2x + 4 - 2\gamma x - 2\gamma + \gamma w) \right] \right\}.$$

Since $g(x) = \frac{(\gamma-1)x + (\gamma-2)}{\gamma}$, there exists a range of interger values for x such that

$$w \leq \frac{(\gamma-1)x + (\gamma-2)}{\gamma} < w+1$$

for every integer w. Thus

$$\frac{\gamma w + 2}{\gamma-1} - \frac{\gamma}{\gamma-1} \leq x < \frac{\gamma w + 2}{\gamma-1} .$$

Choosing, as a trial value for x, the value

$$x_o = \frac{\gamma w + 2}{\gamma-1} - \frac{1}{\gamma-1} , \qquad (2.31)$$

we satisfy the inequality for all allowable values of γ.

Solving for w gives

$$w = \frac{(\gamma-1)x_o-1}{\gamma} . \qquad (2.32)$$

Substitution in equation 2.30 gives, after some algebra,

$$\overline{R}\,(x_o|\delta) = C\left\{z + \beta x_o + \frac{(z-x_o)(1+x)(\gamma-x_o-3)}{2\gamma(x_o+2)}\right\} . \qquad (2.33)$$

The value of x_o greater than or equal to zero which produces a minimum for equation 2.33 is

$$x_o = \left[\frac{(1+\alpha)(\gamma-1)(z+2)}{2\beta\gamma+\alpha+1}\right]^{1/2} -2 . \qquad (2.34)$$

The value of x_o found by equation 2.34 is an approximation only. To find the value of x which minimizes the risk requires substitution of integer values of x near x_o into equation 2.30 choosing, as x_{opt}, that which produces the minimum value of $\overline{R}(x|\delta)$. If this minimum is less than the risk when no sampling is done (from equation 2.24 if gamma is less than two, or 2.25 when equal to or greater than two), a sample size x_{opt} should be taken; otherwise, no samples should be drawn and the lot accepted or rejected on the basis of the results of equations 2.24 and 2.25.

A computer program for determining the optimum sample size has been written to investigate the effects of varying the parameters of this example, α, β, γ, and z. C was not varied as it has no effect on the sample size but only on the magnitude of the resulting expected risk.

Table I and Figure 4 show the effects of varying alpha and gamma when beta and z are constant at two and fifty respectively. The values of optimum sample size appear in the table and the concomitant risks are plotted in the figure. Note that with γ = 1.1, 7, 8, 9, or 10, no sampling would be done regardless of α and the risk is that from equation 2.24 for γ = 1.1 and from equation 2.25 for the other gamma values.

Varying beta and gamma produces the results of Table II and Figure 5 for x_{opt} and risk respectively. Alpha is held constant at 5.0 and z is 50. Again, Υ = 1.1 dictates no sampling for all beta as do certain other combinations of beta and gamma.

Table III shows the values of optimum sample size when alpha and beta are constant (5 and 2 respectively) and gamma and z are varied. Figure 6 shows the expected risks for certain lot sizes resulting when these optimums are used and gamma is varied. In this figure, the risks for the various lot sizes appear to be nearly equal in the neighborhood of gamma equal 2.3. This area was investigated to determine the actual values of gamma and risk where the various lot size curves intersected. The results of this investigation are tabulated in Table IV.

The final figures, 7 and 8, show the effects on the risk value when the lot size is varied, 8 being merely an expansion of the lower end of Figure 7. On both figures, the upper line represents the risk for gamma greater than two when no testing is done and gives a pictorial representation of the improvement in expected risk which can be attained when

TABLE I

X_{opt} FOR VARIOUS α AND γ

($\beta = 2$, $Z = 50$)

α \ γ	1.1	1.5	2	3	4	5	6	7	8	9	10
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	1	2	0	0	0	0	0	0	0
2	0	2	1	3	4	0	0	0	0	0	0
3	0	2	3	3	4	0	0	0	0	0	0
4	0	2	3	3	4	5	0	0	0	0	0
5	0	2	3	3	4	5	0	0	0	0	0
6	0	2	3	5	4	5	0	0	0	0	0
7	0	2	3	5	4	5	6	0	0	0	0
8	0	2	3	5	4	5	6	0	0	0	0
9	0	2	3	5	4	6	7	0	0	0	0
10	0	2	3	5	4	6	7	0	0	0	0

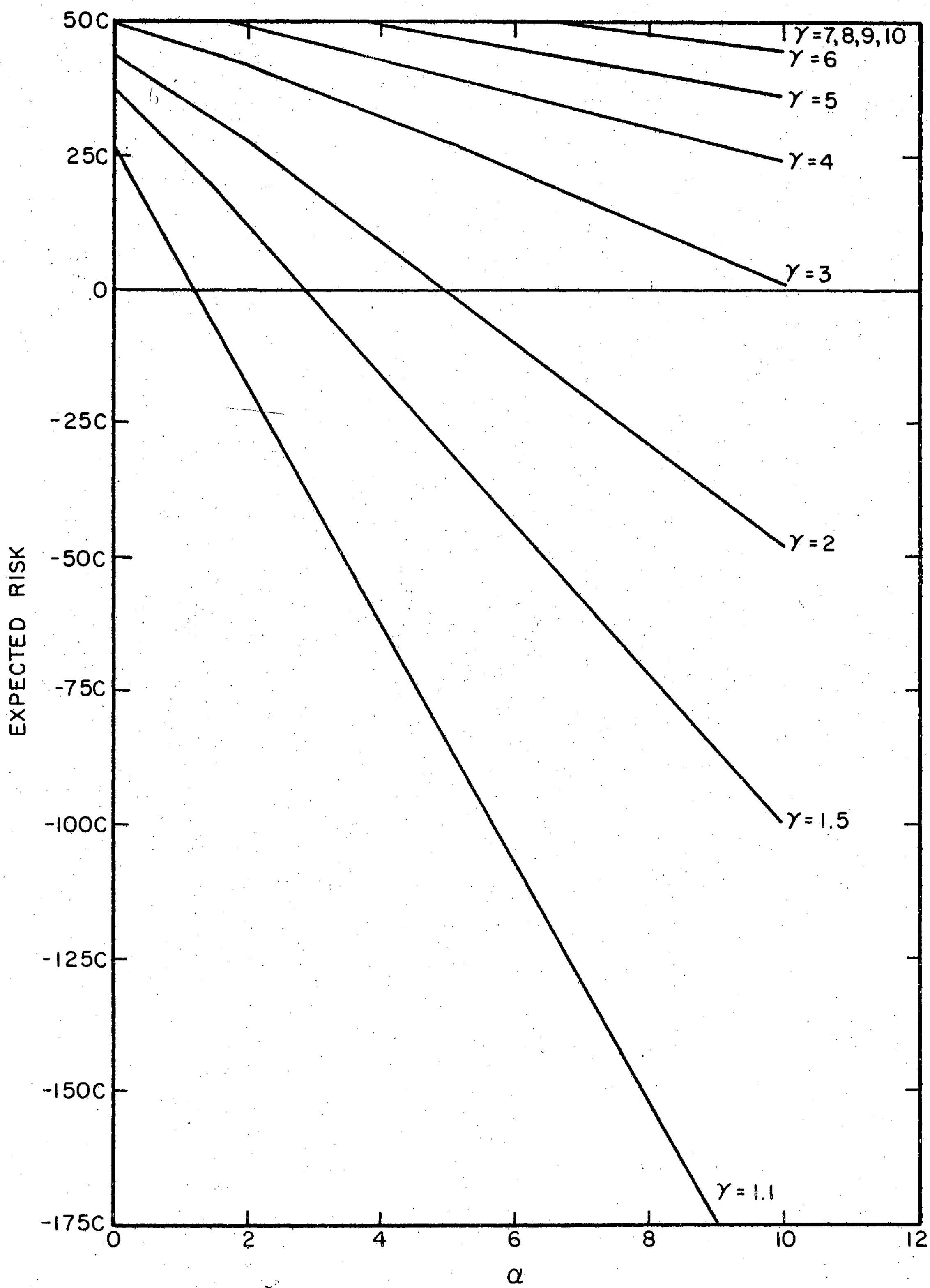

Figure 4. Expected Risk Versus α for Various γ Using Optimum Sampling (β = 2, Z = 50)

TABLE II

X_{opt} FOR VARIOUS β AND γ

(γ = 5, Z = 50)

β \ γ	1.1	1.5	2	3	4	5	6	7	8	9	10
0	0	3	5	8	12	11	13	15	17	19	21
1	0	2	3	5	7	6	7	8	0	0	0
2	0	2	3	3	4	5	0	0	0	0	0
3	0	2	3	3	4	0	0	0	0	0	0
4	0	2	1	3	4	0	0	0	0	0	0
5	0	0	1	2	0	0	0	0	0	0	0
6	0	0	1	2	0	0	0	0	0	0	0
7	0	0	1	2	0	0	0	0	0	0	0
8	0	0	1	2	0	0	0	0	0	0	0
9	0	0	1	2	0	0	0	0	0	0	0
10	0	0	1	2	0	0	0	0	0	0	0

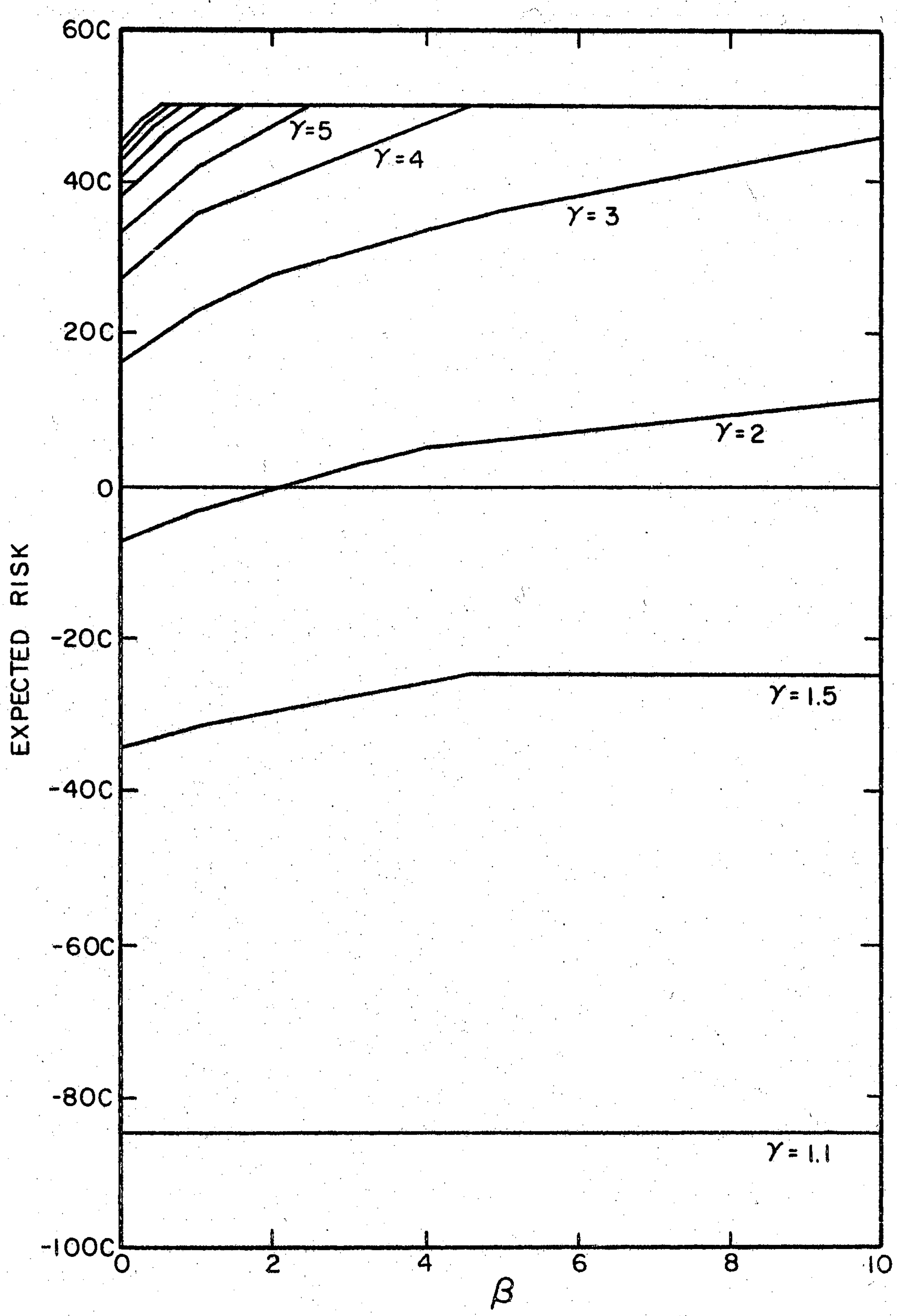

Figure 5. Expected Risk Versus β for Various γ with Optimum Sampling (α = 5, Z = 50)

TABLE III

X_{opt} FOR VARIOUS γ AND Z

(α = 5, β = 2)

Z \ γ	1.1	1.5	2	3	4	5	6	7	8	9	10
25	0	0	1	2	3	0	0	0	0	0	0
50	0	2	3	3	4	5	0	0	0	0	0
75	0	2	3	5	4	6	7	0	0	0	0
100	0	3	5	6	8	6	7	8	0	0	0
150	0	5	7	8	8	10	8	9	10	0	0
200	0	5	7	9	12	11	13	9	10	11	12
300	0	6	9	12	12	15	13	15	17	12	13
400	0	8	11	14	16	16	19	16	18	20	22
500	0	9	13	17	20	20	19	22	18	20	22
600	0	11	15	18	20	21	19	22	25	21	23
700	0	11	15	20	20	21	25	23	26	21	23
800	0	12	17	21	24	25	25	29	26	29	24
900	0	14	17	23	24	26	25	29	26	29	32
1000	0	14	19	24	28	26	31	29	33	30	32

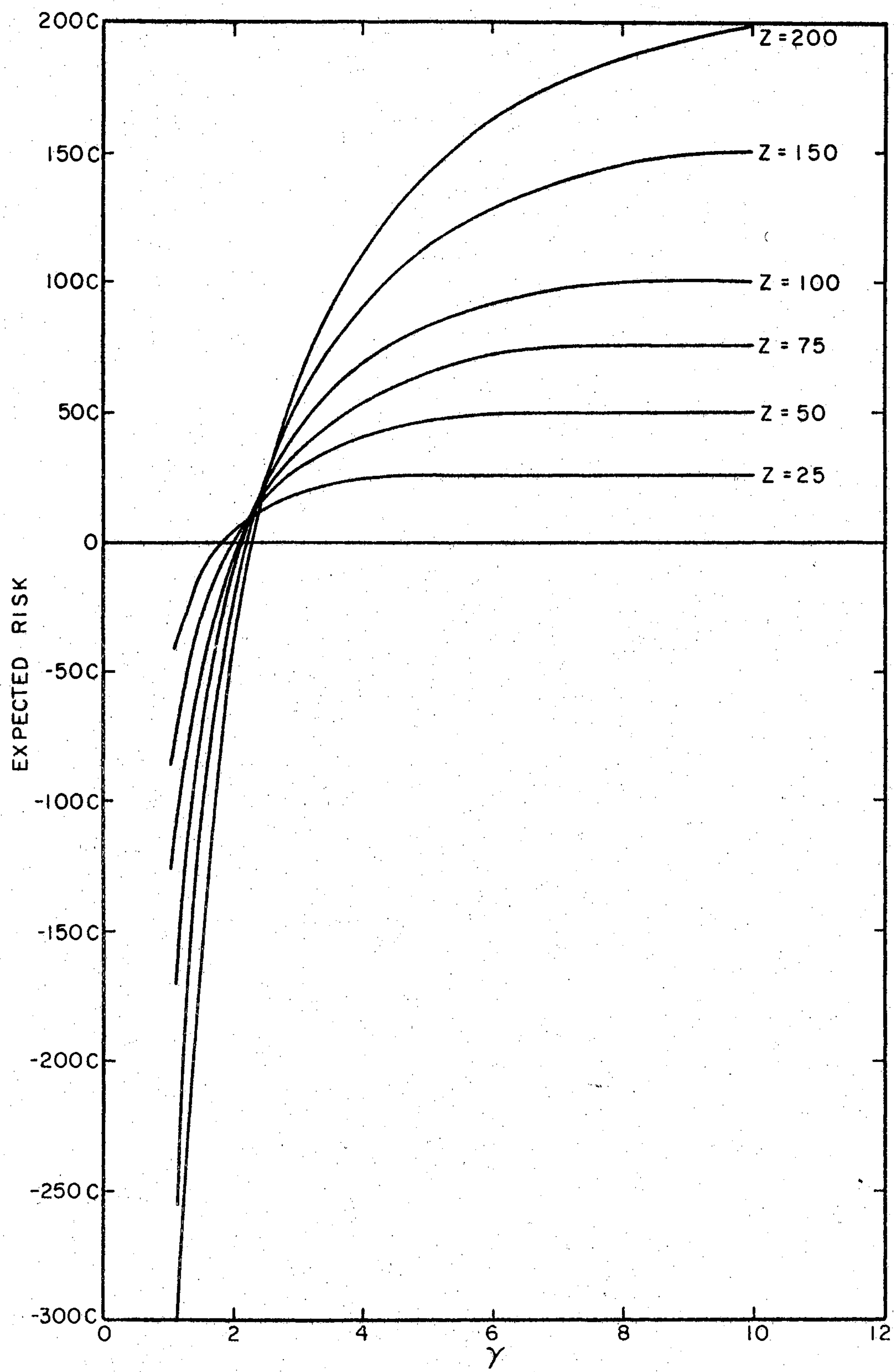

Figure 6. Expected Risk Versus γ for Various Z Using Optimum Sampling ($\alpha = 5$, $\beta = 2$)

sampling is used.

TABLE IV

VALUES OF γ AND RISK AT CERTAIN Z

INTERSECTIONS OF FIGURE 6

Z	50	75	100	150	
25	2.185	2.263	2.304	2.359	γ
	7.44	9.03	9.54	10.10	$\bar{R}$
50	--	2.334	2.356	2.399	γ
	--	12.06	12.70	13.82	$\bar{R}$
75	--	--	2.371	2.420	γ
	--	--	13.57	15.71	$\bar{R}$
100	--	--	--	2.442	γ
	--	--	--	17.74	$\bar{R}$

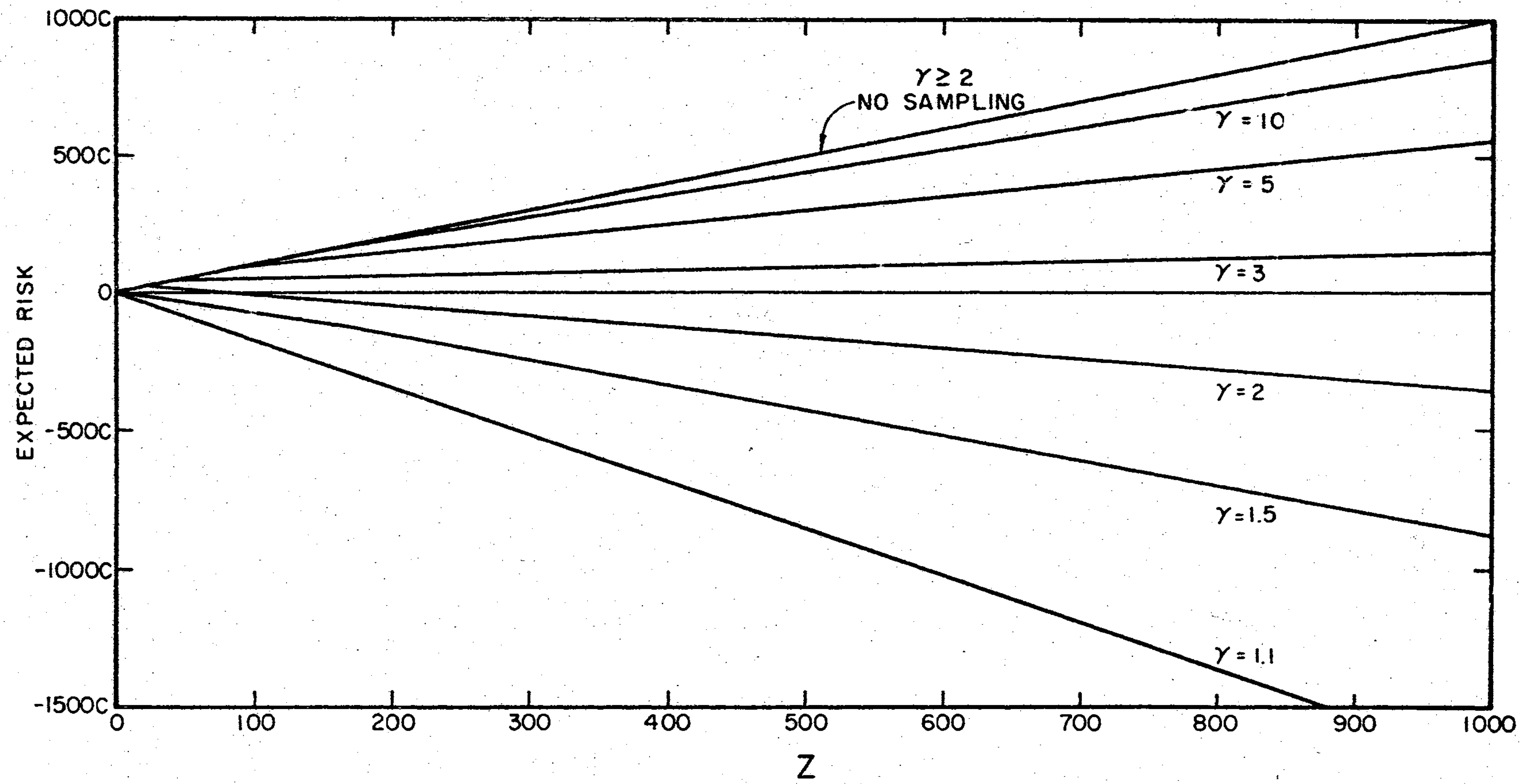

Figure 7. Expected Risk Versus Lot Size for Various Values of γ Using Optimum Sampling (α = 5, β = 2)

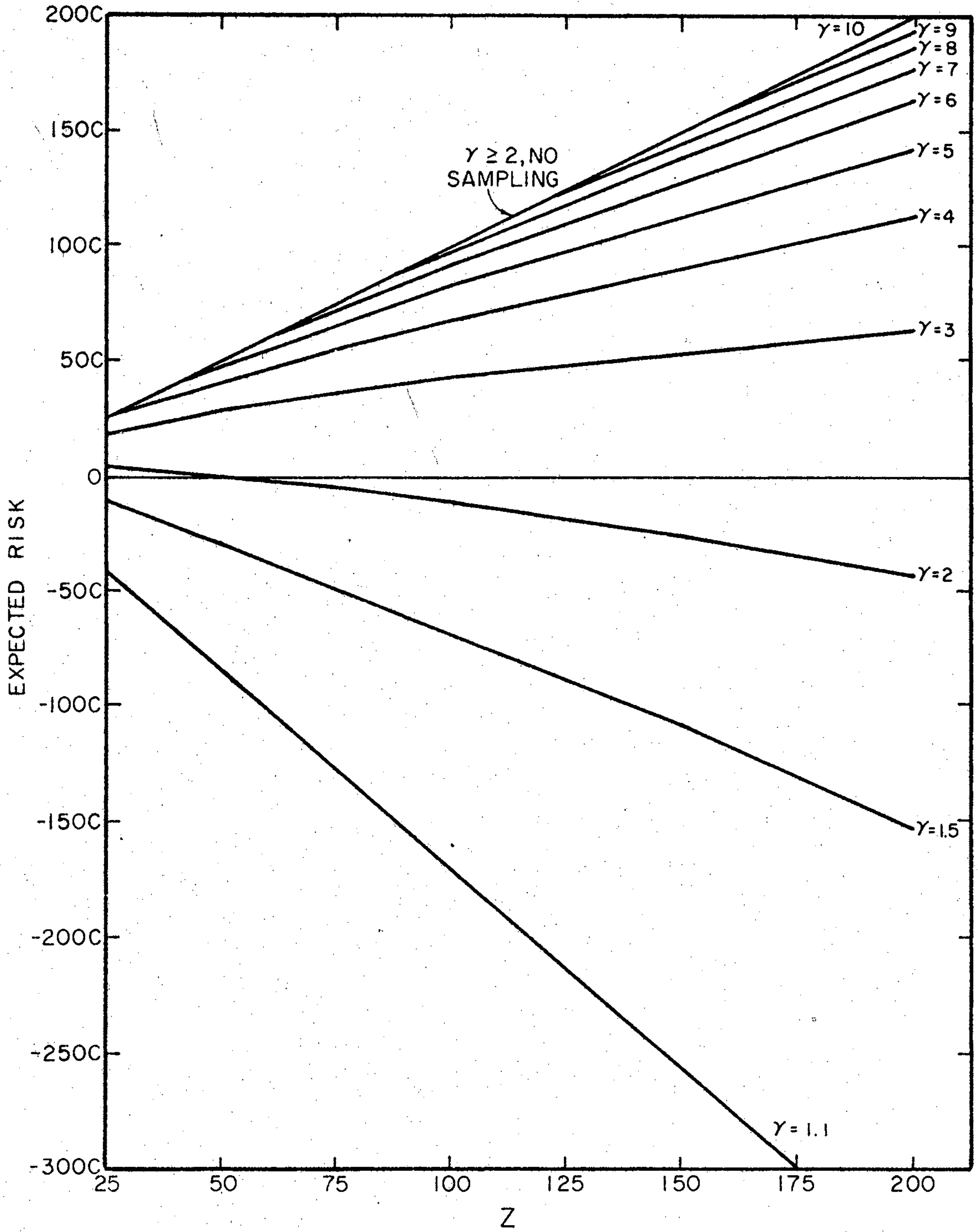

Figure 8. Expected Risk Versus Z for Various γ Using Optimum Sampling ($\alpha = 5$, $\beta = 2$)

CHAPTER III

SEQUENTIAL SAMPLING IN THE TIME INVARIANT CASE

Having established the method of determining optimum sample size in the case where the random variable P is time invariant and when the sample size must be determined before any samples are taken, we are now ready to consider the situation when the decision maker has the option of another decision prior to making his final accept or reject decision. This other decision can be made after each individual sample has been drawn, if desired, and is to either continue sampling or to stop sampling. The latter decision of course implies a choice at that sampling point of either of the previously described decisions, A or B, accept or reject.

Risk as a Function of a and x

In order to examine this problem, we must first be able to determine the risk involved as a function not only of the sample size, x, but also of the number of favorable or unfavorable samples, a or b, encountered in the x samples. This can be done for each of the final decisions, A or B, by taking the expected values of R_{D_A} and R_{D_B} (as shown in equations 2.13 and 2.14) after the sampling experience. Thus

$$\bar{R}\,(a;x|D_A) = \int_{-\infty}^{\infty} R_T\, f_{P|\mathcal{E}}\,(\theta|a;x)\, d\theta + \int_{-\infty}^{\infty} R_{D_A}\, f_{P|\mathcal{E}}\,(\theta|a;x)\, d\theta \qquad (3.1)$$

and

$$f_{P|\varepsilon}(p|a,x) = \frac{(x+1)!}{a!\ (x-a)!} p^{a}(1-p)^{x-a} \tag{3.2}$$

so that

$$\overline{R}\ (a,x|D_A) = \int_{o}^{1} \frac{(x+1)!}{a!\ (x-a)!} (R_T + R_{D_A})\ \theta^{a}(1-\theta)^{x-a}\ d\theta \tag{3.3}$$

Performing the indicated integration and similar operation for $\overline{R}(a,x|D_B)$ will yield the desired risks as functions of x and a. Having determined, by the method described in Chapter II, the values of a which would result in the choice of decision A or B, we can calculate the appropriate risk for each discrete value of a, given the value of x.

$$\overline{R}\ (a,x) = \begin{cases} \overline{R}\ (a,x|D_B), & o \le a \le g(x) \\ \overline{R}\ (a,x|D_A), & g(x) < a \le x \end{cases} \tag{3.4}$$

Probability That Next Sample Is Favorable

Consider the case where we have sampled m items and found k favorables. If no more samples were taken, the risk incurred would be $\overline{R}(k,m)$ as shown in equation 3.4. We are interested now, however, in the risk incurred if one additional sample is drawn knowing that we have experienced k favorables of m samples.

To determine this, the probability that one sample will be favorable must first be calculated. The distribution governing this single sample is the point binomial:

$$P\ (\Omega=\omega) = \phi^{\omega}(1-\phi)^{1-\omega} = \begin{cases} \phi, & \omega=1 \\ 1-\phi, & \omega=0 \end{cases} \tag{3.5}$$

Since the ϕ in this case is the same as the random variable, P, we can write its density function directly from equation 2.8 as

$$f_{P|\mathcal{E}}(\phi|k;m) = \frac{(m+1)!}{k!\ (m-k)!}\ \phi^{k}(1-\phi)^{m-k}$$

We are now prepared to find the probability the Ω will equal one given the sampling experience.

$$P\ (\Omega=1|\mathcal{E}) = \int_{-\infty}^{\infty} P\ (\Omega=1)\ f_{P|\mathcal{E}}(\phi|k;m)\ d\phi = \frac{k+1}{m+2} \tag{3.6}$$

Similarly, the probability of Ω being zero given the same sampling experience can be calculated as

$$P\ (\Omega=0|\mathcal{E}) = \frac{m-k+1}{m+2} \tag{3.7}$$

The Recursion Relationship

Since the value of $\overline{R}(a,x)$ from equation 3.4 is the risk associated with discontinuing sampling after x samples, the decision to continue or stop sampling can be made by merely comparing the values of risk associated with each decision and choosing that decision which minimizes the risk. Having made this "best" decision, the final value of the risk at any sampling point is established.

$$R(a,x) = \min \left\{ \overline{R}(a,x);\ R(a,x|\text{continue}) \right\}$$

The value of risk associated with continuing sampling, $R(a,x|\text{continue})$, can be calculated as follows:

$$R(a,x|\text{continue}) = P(\Omega=1|\mathcal{E})\ R(a+1,x+1) + P(\Omega=0|\mathcal{E})\ R(a,x+1) \tag{3.9}$$

The decision tree applicable to the determination of R(a,x) is shown in Figure 9 for one sampling point, a of x.

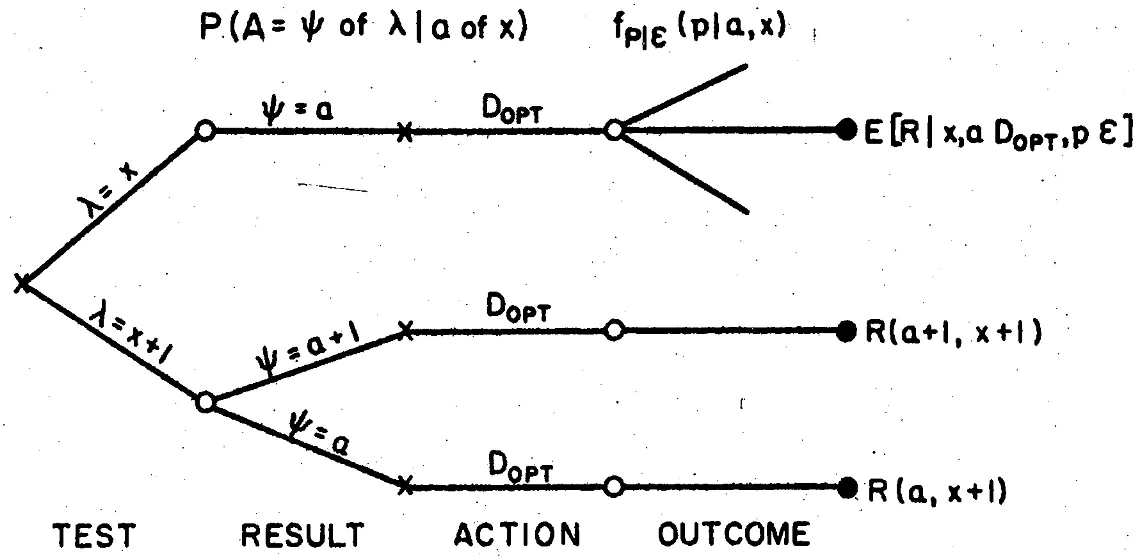

Figure 9. Sequential Decision Tree

In the figure, the value of λ is dependent on the previous decision used to optimize R(a,x), being x if that decision is to be stop sampling and x+1 if to continue sampling. When $\lambda = x$, D_{opt} is the accept or reject decision and the distribution is $f_{P|\varepsilon}(p|a,x)$. When $\lambda = x+1$, D_{opt} is the continue or stop sampling decision which yields the indicated risk with probability one.

The Dynamic Programming Solution

Equation 3.8 is a simplified recursion relation which is amendable to solution by dynamic programming techniques (2).

Consider the case when the sample size, x, is equal to the lot size, z. At this point, it is impossible to continue sampling so that equation

3.8 reduces to

$$R(a,z) = \overline{R}(a,z) = \begin{cases} \overline{R}_{D_B}(a,z), & o \le a \le g(z) \\ \overline{R}_{D_A}(a,z), & g(z) < a \le z \end{cases} \qquad (3.10)$$

Solution of this equation for all integer values of a between 0 and z gives a starting point for successive solutions of equations 3.9 and 3.8. As an example, the next step would be calculation of

$$R(a,z-1|\text{continue}) = \frac{a+1}{z+1} R(a+1,z) + \frac{z-a}{z+1} R(a,z) \qquad (3.11)$$

for every a = 0, 1, 2, ..., z-1, followed by calculation of R(a,z-1) from equation 3.8.

The calculations are continued for z-2, z-3, etc., until the sample size, x, is zero, recording the appropriate sampling decision, stop or continue, after each R(a,x) is determined. This set of decisions together with the appropriate accept or reject decision at each stop sampling point constitutes a complete policy yielding minimum risk for the problem at hand.

While the individual calculations involved in this type solution are quite simple, a very large number of them are required when the lot size, z, is appreciable. The use of a digital computer to assist in the policy determination is highly desirable. Including only the probability computations of the next sample being acceptable and ignoring the comparisons involved, the number of individual computations required is in excess of z(z+2).

The results of the calculations, the optimum sequential sampling policy, can best be shown by a graph of the policy. Such a sequential sampling diagram plots the sample size, x, versus the number of favorable results, a, for the reject and accept decision boundaries, the area between representing the continue sampling decision. Such graphs are shown in the example which follows.

The Operations Research Example

The operations research problem described in Chapter II is amenable to the dynamic programming solution of sequential sampling. From equation 2.23 the risk associated with rejecting the lot is $R_{D_B} = Cy$ while that associated with acceptance is, from equation 2.22,

$$R_{D_A} = Cy\left\{\gamma(1+\gamma) -\alpha -\gamma\ (1+\alpha)p\right\} \tag{3.14}$$

Using these, the sampling risk, and the post-sampling distribution of P from equation 3.2 gives the following risks as functions of a and x:

$$\begin{aligned} \overline{R}(a,x|D_B) &= \frac{C(x+1)!}{a!(x-a)!}\int_o^1 [x(1+\beta) + y]\ \theta^a(1-\theta)^{x-a}\ d\theta \\ &= C(z+\beta x) \end{aligned} \tag{3.15}$$

$$\begin{aligned} \overline{R}(a,x|D_A) &= \frac{C(x+1)!}{a!(x-a)!}\int_o^1 \left\{x(1+\beta) + y\ [\gamma(1+\alpha) -\alpha -\gamma\ (1+\alpha)\theta]\right\}\ \theta^a(1-\theta)^{x-a}\ d\theta \\ &= C\left\{z+\beta x + \frac{(z-x)(1+\alpha)}{x+2}\ [\gamma(x-a+1) - (x+2)]\right\} \end{aligned} \tag{3.16}$$

Using the results of equation 2.30, equation 3.4 becomes

$$\overline{R}(a,x) = \begin{cases} \overline{R}(a,x|D_B), & o\leq a\leq \dfrac{(\gamma-1)x + (\gamma-2)}{\gamma} \\ \overline{R}(a,x|D_A), & \dfrac{(\gamma-1)x + (\gamma-2)}{\gamma} < a\leq x \end{cases} \tag{3.17}$$

Using these equations with 3.6 and 3.7, a computer program was written to produce data for determination of the optimum sequential sampling policy. (See Appendix B). Values of z=50 and 100 with γ = 2 and 5 were used as inputs to this program with alpha constant at 5 and beta constant at 2. The complete results are shown in Appendix C. Summaries of the results are shown in Figures 10, 11, 12, and 13.

Discussion of Results

Of interest is a comparison of the expected risks when a sequential sampling plan is used as opposed to the expected risks when an optimum sized single sample is taken as described in Chapter II. These values are tabulated in Table V. The values for the sequential case are those which result when the sample size becomes zero. In each of the examples, sequential sampling indicated an improvement in the expected risk.

TABLE V

RISK COMPARISON - SEQUENTIAL

VERSUS SINGLE SAMPLE

Z γ	Expected Risk	
	Single Sample	Sequential Sampling
50 2	- .40000C	- 4.67833C
50 5	47.14289C	39.95562C
100 2	-12.14285C	-21.02375C
100 5	81.78579C	68.90867C

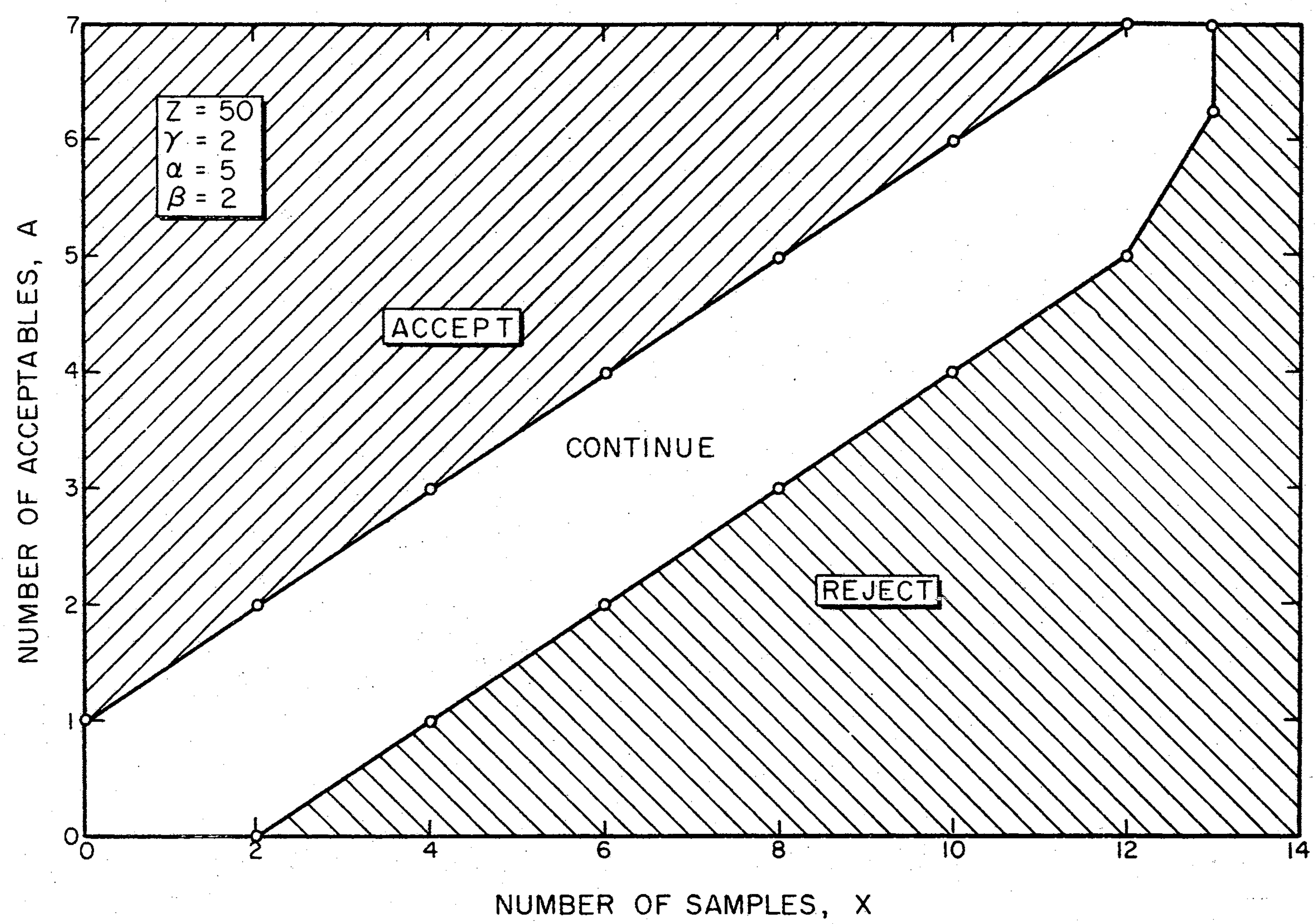

Figure 10. Sequential Sampling Policy (Z = 50, γ = 2)

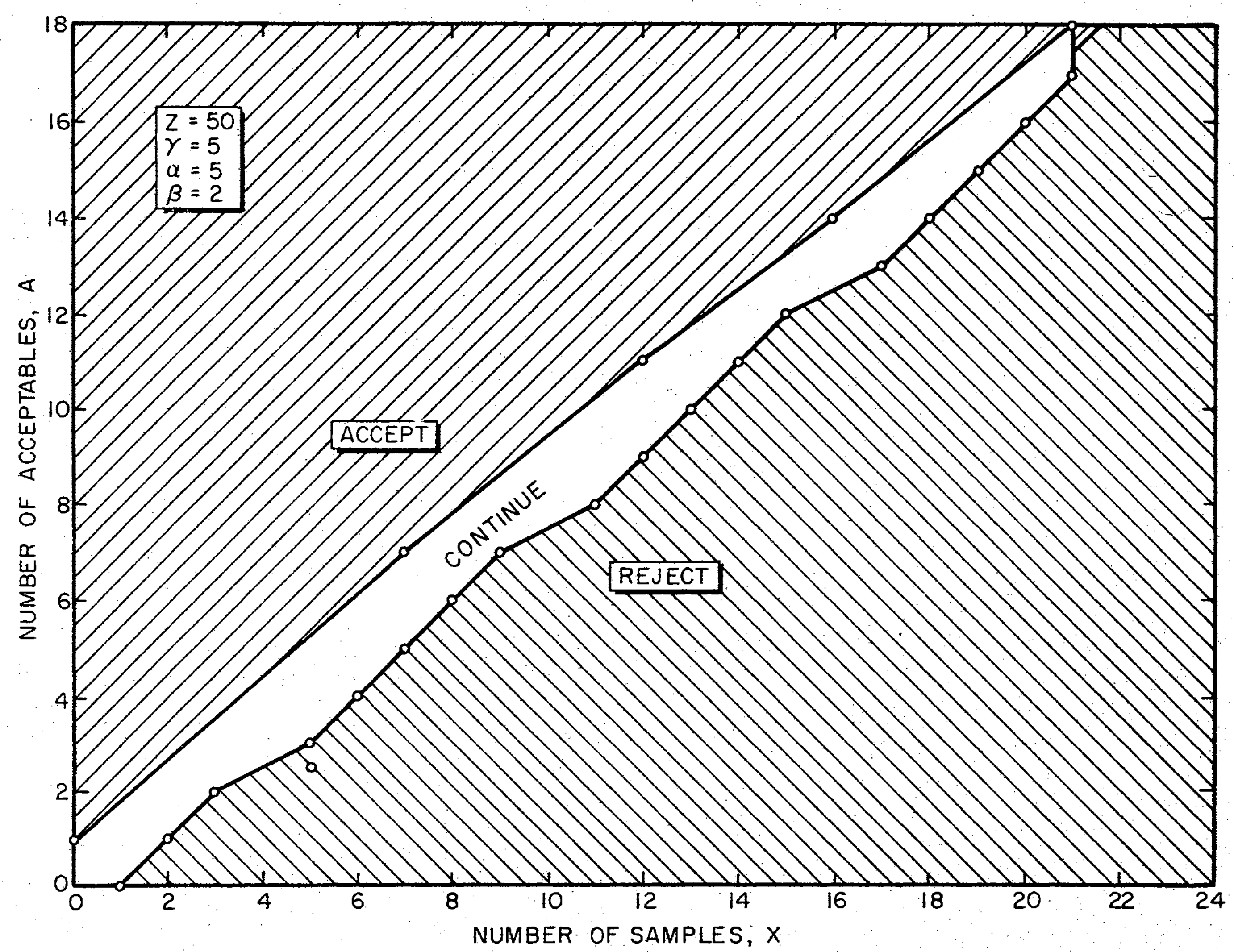

Figure 11. Sequential Sampling Policy (Z = 50, γ = 5)

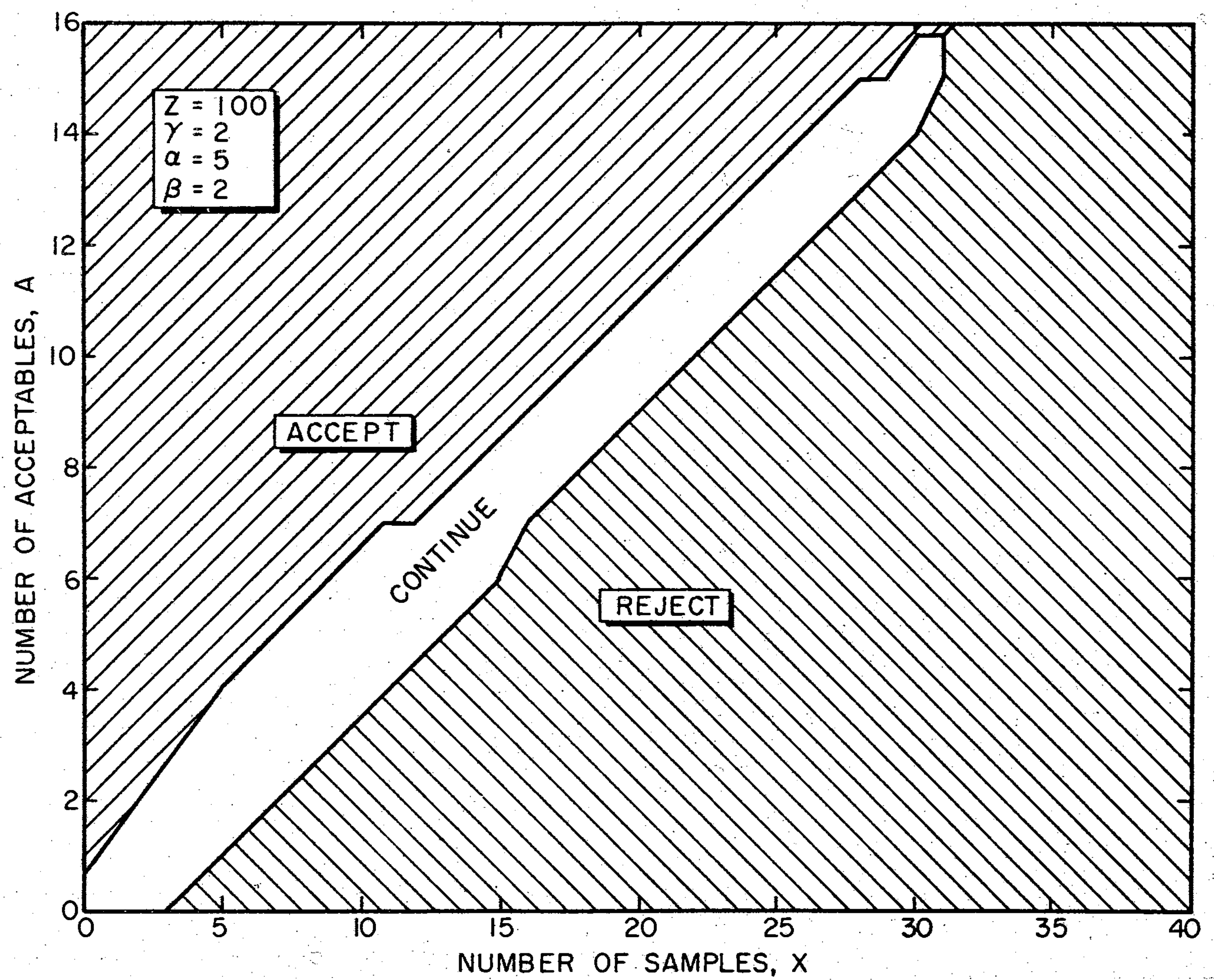

Figure 12. Sequential Sampling Policy ($Z = 100$, $\gamma = 2$)

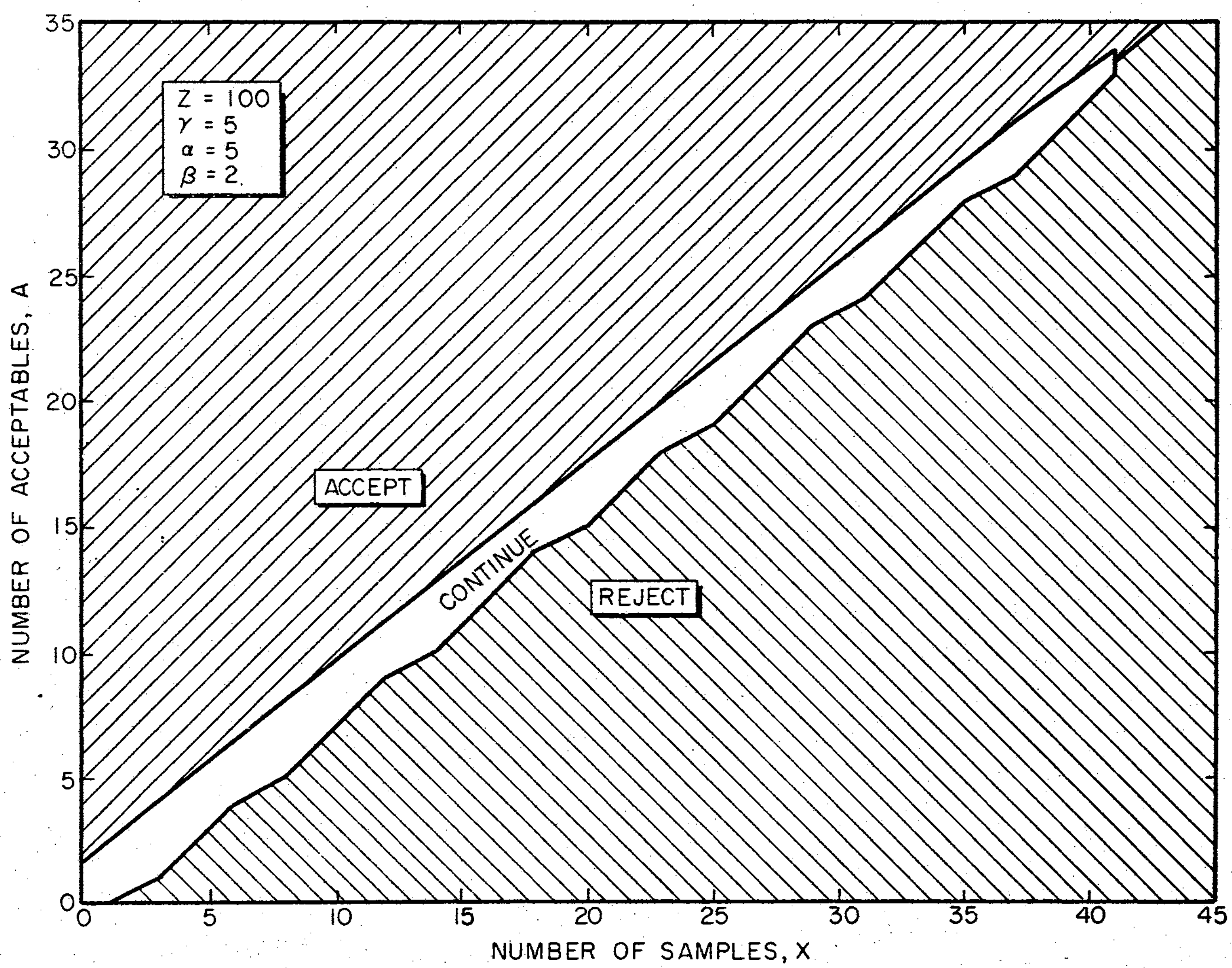

Figure 13. Sequential Sampling Policy (Z = 100, γ = 5)

A comparison of the results presented in the figures with sequential sampling graphs produced by conventional methods using the Wald (4) technique reveals several differences. The conventional technique, using Neyman confidence limits, produces a pair of lines of the same slope separating the accept, continue, and reject regions. Thus, the maximum number of samples to be taken cannot be predetermined. The statistical decision-dynamic programming approach used here eliminates this undesirable characteristic. As shown in the figures, each example produces a definite maximum number of samples (in these cases, always less than half the lot size) which will be taken under any sampling circumstances.

CHAPTER IV

THE TIME VARYING PARAMETER

We now consider the case where P describes a stochastic process. It will be assumed that the coefficients producing this change are also random variables with some a priori distribution. This development will only consider equally spaced sampling intervals, i.e.: samples will be taken at times t + m with m = 0, 1, 2, 3, ... Further, while P is subject to change from t to t+1, it is considered time invariant during the time sampling is being done. This, in effect, means that the time taken to accomplish sampling at time t is very small compared to the time interval between t and t+1.

The A Priori Beta Distribution

We must first review the previously developed forms of the distributions on the random variables A, the number of acceptable samples, and P. Consider the a priori distribution of P as Beta, that is

$$f_{p|\mathcal{E}_p}(p;\lambda,\psi) = \frac{\Gamma(\lambda+2)}{\Gamma(\psi+1)\Gamma(\lambda-\psi+1)}\, p^{\psi}(1-p)^{\lambda-\psi} \qquad (4.1)$$

where $\mathcal{E}_p$ indicates the pre-sampling a priori estimate with parameters λ and ψ. From Chapter II, it is recalled that, when the distribution of A given P is binomial, application of Bayes Theorem gives, upon carrying out the integration of equation 2.5,

$$f_{p|\mathcal{E}_1}(p|a;x,\psi,\lambda) = \frac{\Gamma(x+\lambda+2)}{\Gamma(a+\psi+1)\Gamma(x+\lambda-a-\psi+1)}\, p^{a+\psi}(1-p)^{x+\lambda-a-\psi} \qquad (4.2)$$

with $\mathcal{E}_1$ denoting the sampling experience, a favorables of x samples, at time one. The restrictions stated in Chapter II pertaining to the range of equation 4.2, (i.e.: valid for P in the closed interval [0,1] and zero elsewhere), to the permissable values of λ and ψ, (i.e.: $\psi > -1$, $\lambda > \psi - 1$), and to x and a being non-negative integers, still hold.

When considering a time-varying P, equation 4.2 can be written

$$f_{P(t)|\mathcal{E}_t}(p|a_t;x_t,\lambda_t,\psi_t)$$
$$= \frac{\Gamma(x_t+\lambda_t+2)}{\Gamma(a_t+\psi_t+1)\Gamma(x_t+\lambda_t-a_t-\psi_t+1)}\, p^{a_t+\psi_t}(1-p)^{x_t+\lambda_t-a_t-\psi_t} \qquad (4.3)$$

where the t subscript indicates the value of the variable at time t and $\mathcal{E}_t$ denotes the pre-sampling estimate, $\mathcal{E}_p$, and all sampling experiences thru time t. This Beta density give the following moments:

$$E\,[P(t)] = \frac{a_t+\psi_t+1}{x_t+\lambda_t+2} \qquad (4.4)$$

$$E\left[\left(P(t) - E\,[P(t)]\right)^2\right] = \frac{(a_t+\psi_t+1)(x_t+\lambda_t-a_t-\psi_t+1)}{(x_t+\lambda_t+2)^2\,(x_t+\lambda_t+3)} \qquad (4.5)$$

To further simplify notation, moments will henceforth be subscripted with only the variable concerned followed by the integer time. As an example, μ_{M2}, would, in this notation, represent the mean of M at time 2. Thus, equation 4.4 becomes μ_{Pt} and 4.5 is σ^2_{Pt}.

The Difference Equation Model

We now consider the problem of predicting the distribution of P at time (t+1). The following difference equation is established:

$$P(t+1) = C(t)P(t). \tag{4.6}$$

In this equation, the distribution of P(t+1) and P(t) are assumed to be Beta and C(t) is a sample at time t of random variable C, independent of P, with a priori mean, μ_C, and variance, σ_C^2. The sample values of C, C(t), are also considered to be independent so that no learning of C is possible.

To determine the mean of P(t+1), we can write

$$E\,[P(t+1)] = E\,[C(t)P(t)]$$

which, due to independence, is

$$\hat{\mu}_{P(t+1)} = \mu_C\,\mu_{Pt} \tag{4.7}$$

where the "hat" indicates an estimate made prior to time t+1.

A similar procedure gives

$$\hat{\sigma}^2_{P(t+1)} = \sigma^2_{Pt}\,(\sigma^2_{Ct} + \mu^2_C) + \sigma^2_C\,\mu^2_{Pt}. \tag{4.8}$$

We now take advantage of the fact that the distribtuion of P at time t+1 is assumed to be Beta when P at t is Beta and A at t+1 is Binomial. Thus, our a priori of P for time t+1 is of the form

$$f_{P(t+1)|\varepsilon_t}(p|\lambda_{t+1},\psi_{t+1}) = \frac{\Gamma\left(\lambda_{t+1}+2\right)}{\Gamma(\psi_{t+1}+1)\Gamma(\lambda_{t+1}-\psi_{t+1}+1)}\,p^{\psi_{t+1}}\left(1-p\right)^{\lambda_{t+1}-\psi_{t+1}} . \tag{4.9}$$

From this, proceeding as for equations 4.4 and 4.5, we can determine

$$\hat{\mu}_{P(t+1)} = \frac{\psi_{t+1}+1}{\lambda_{t+1}+2} \qquad (4.10)$$

and

$$\hat{\sigma}^2_{P(t+1)} = \frac{\left(\psi_{t+1}+1\right)\left(\lambda_{t+1}-\psi_{t+1}+1\right)}{\left(\lambda_{t+1}+2\right)^2\left(\lambda_{t+1}+3\right)}. \qquad (4.11)$$

As shown in Appendix D, $\hat{\mu}_{P(t+1)}$ and $\hat{\sigma}^2_{P(t+1)}$ are sufficient to determine unique values for λ_{t+1} and ψ_{t+1} as

$$\lambda_{t+1} = \frac{\hat{\mu}_{P(t+1)}\left(1-\hat{\mu}_{P(t+1)}\right)}{\hat{\sigma}^2_{P(t+1)}} -3 \qquad (4.12)$$

and

$$\begin{aligned} \psi_{t+1} &= \frac{\hat{\mu}^2_{P(t+1)}\left(1-\hat{\mu}_{P(t+1)}\right)}{\hat{\sigma}^2_{P(t+1)}} - \hat{\mu}_{P(t+1)} -1 \\ &= \hat{\mu}_{P(t+1)}\left(\lambda_{t+1}+2\right) -1. \end{aligned} \qquad (4.13)$$

Using the results of equations 4.7 and 4.8 in 4.12 and 4.13 will thus give the parameters of the a priori distribution of P for time t+1 in terms of the means and variances of P and C at time t. It should be remembered that this was done prior to sampling at time t+1. This is indicated by the notation $\mathcal{E}_t$ which indicates all experience, including a priori, thru time t.

After sampling at time t+1, obtaining a_{t+1} favorables from x_{t+1} samples, we find, from equation 4.3

$$f_{P(t+1)|\mathcal{E}_{t+1}}\left(p\,|a_{t+1};x_{t+1},\lambda_{t+1},\psi_{t+1}\right)$$

$$= \frac{\Gamma\left(x_{t+1}+\lambda_{t+1}+2\right)}{\Gamma\left(a_{t+1}+\psi_{t+1}+1\right)\Gamma\left(x_{t+1}+\lambda_{t+1}-a_{t+1}-\psi_{t+1}+1\right)} \left[p^{a_{t+1}+\psi_{t+1}}(1-p)^{x_{t+1}+\lambda_{t+1}-a_{t+1}-\psi_{t+1}}\right]. \tag{4.14}$$

From equation 4.4, the mean of this distribution is

$$\mu_{P(t+1)} = \frac{a_{t+1}+\psi_{t+1}+1}{x_{t+1}+\lambda_{t+1}+2} \tag{4.15}$$

and its variance, from equations 4.4 and 4.5,

$$\sigma^2_{P(t+1)} = \frac{\mu_{P(t+1)}\left(1-\mu_{P(t+1)}\right)}{x_{t+1}+\lambda_{t+1}+3} \tag{4.16}$$

Expanding equations 4.15 and 4.16 from the results of equations 4.12 and 4.13 gives

$$\mu_{P(t+1)} = \frac{a_{t+1}\hat{\sigma}^2_{P(t+1)} + \hat{\mu}^2_{P(t+1)}\left(1-\hat{\mu}_{P(t+1)}\right) - \hat{\mu}_{P(t+1)}\hat{\sigma}^2_{P(t+1)}}{x_{t+1}\hat{\sigma}^2_{P(t+1)} + \hat{\mu}_{P(t+1)}\left(1-\hat{\mu}_{P(t+1)}\right) - \hat{\sigma}^2_{P(t+1)}} \tag{4.17}$$

and

$$\sigma^2_{P(t+1)} = \frac{\mu_{P(t+1)}\left(1-\mu_{P(t+1)}\right)\hat{\sigma}^2_{P(t+1)}}{\hat{\sigma}^2_{P(t+1)}x_{t+1} + \hat{\mu}_{P(t+1)}\left(1-\hat{\mu}_{P(t+1)}\right)} \tag{4.18}$$

Further substitution from equations 4.7 and 4.8 yields

$$\mu_{P(t+1)} = \frac{a_{t+1}\left[\sigma^2_{Ct}\sigma^2_{Pt} + \mu^2_{Ct}\sigma^2_{Pt}\right] + \sigma^2_{Ct}\mu^2_{Pt} + \mu^2_{Ct}\mu^2_{Pt}\left(1-\mu_{Ct}\mu_{Pt}\right)}{x_{t+1}\left[\sigma^2_{Ct}\sigma^2_{Pt} + \mu^2_{Ct}\sigma^2_{Pt}\right] + \sigma^2_{Ct}\mu^2_{Pt} + \mu_{Ct}\mu_{Pt}\left(1-\mu_{Ct}\mu_{Pt}\right)} \tag{4.19}$$

$$\frac{-\mu_{Ct}\mu_{Pt}\left[\sigma^2_{Ct}\sigma^2_{Pt} + \mu^2_{Ct}\sigma^2_{Pt} + \sigma^2_{Ct}\mu^2_{Pt}\right]}{-\left[\sigma^2_{Ct}\sigma^2_{Pt} + \mu^2_{Ct}\sigma^2_{Pt} + \sigma^2_{Ct}\mu^2_{Pt}\right]}$$

and

$$\sigma^2_{P(t+1)} = \frac{\mu_{P(t+1)}\left(1-\mu_{P(t+1)}\right)}{x_{t+1}\left[\sigma^2_{Ct}\sigma^2_{Pt} + \sigma^2_{Pt}\mu^2_{Ct} + \sigma^2_{Ct}\mu^2_{Pt}\right]} \tag{4.20}$$

$$\frac{\left[\sigma^2_{Ct}\sigma^2_{Pt} + \mu^2_{Ct}\sigma^2_{Pt} + \sigma^2_{Ct}\mu^2_{Pt}\right]}{+\mu_{Ct}\mu_{Pt}\left(1-\mu_{Ct}\mu_{Pt}\right)} .$$

While these expressions seem extremely unwieldy, calculation of $\mu_{P(t+1)}$ and $\sigma^2_{P(t+1)}$ is relatively simple if carried out step-by-step. First, calculate $\hat{\mu}_{(t+1)}$ and $\hat{\sigma}^2_{P(t+1)}$ from equations 4.7 and 4.8. Next, calculate λ_{t+1} and ψ_{t+1} from equations 4.12 and 4.13. Finally, $\mu_{P(t+1)}$ and $\sigma^2_{P(t+1)}$ are computed using 4.15 and 4.16.

At this point, the relative weights, wt, implicitly assigned to the estimate, $\hat{\mu}_{Pt}$, and to the sample result at time t can be calculated as follows:

$$\mu_P = \frac{a_t+\psi_t+1}{x_t+\lambda_t+2} = \frac{(\psi_t+1)}{(\lambda_t+2)}\left[\frac{\lambda_t+2}{x_t+\lambda_t+2}\right] + \frac{a_t}{x_t}\left[\frac{x_t}{x_t+\lambda_t+2}\right] \tag{4.21}$$

$$= \hat{\mu}_{Pt}\left[\frac{\lambda_t+2}{x_t+\lambda_t+2}\right] + \frac{a_t}{x_t}\,wt_t$$

Thus

$$wt_t = \frac{x_t}{x_t+\lambda_t+2} \tag{4.22}$$

and

$$\sum_{i=o}^{t-1} wt_i = \frac{\lambda_t+2}{x_t+\lambda_t+2} \tag{4.23}$$

With these equations, it is possible to find the weight of any prior sampling experience, say at time s, by

$$wt_{s|t} = wt_s \prod_{i=t}^{s-1} (1-wt_i) \tag{4.24}$$

where s < t. To determine the weight of the a priori estimate of P after sampling through time t,

$$wt_{\hat{\mu}o|t} = \prod_{i=t}^{o} (1-wt_i) \tag{4.25}$$

Summary of the Procedure

The entire preceding development for a time-varying P is summarized in the following set of equations:

$$\hat{\mu}_{P(t+1)} = \mu_C\mu_{Pt} \tag{4.26}$$

$$\hat{\sigma}^2_{P(t+1)} = \sigma^2_{Pt}(\sigma^2_C + \mu^2_C) + \sigma^2_C\mu^2_{Pt} \tag{4.27}$$

$$\lambda_{t+1} = \begin{cases} \dfrac{\hat{\mu}_{P(t+1)}\left(1-\hat{\mu}_{P(t+1)}\right)}{\sigma^2_{P(t+1)}} -3, & t \geq o \\ \\ \lambda_o, & t = -1 \end{cases} \tag{4.28}$$

$$\psi_{t+1} = \begin{cases} \hat{\mu}_{P(t+1)}\left(\lambda_{t+1}+2\right) -1, & t \geq o \\ \psi_o, & t = -1 \end{cases} \tag{4.29}$$

$$\mu_{P(t+1)} = \frac{a_{t+1}+\psi_{t+1}+1}{x_{t+1}+\lambda_{t+1}+2} \tag{4.30}$$

$$\sigma^2_{P(t+1)} = \frac{\mu_{P(t+1)}\left(1-\mu_{P(t+1)}\right)}{x_{t+1}+\lambda_{t+1}+3}. \tag{4.31}$$

These equations establish a method for predicting the distribution of a random variable, P(t+1), when the applicable model is P(t+1) = C(t)P(t), and binomial sampling is done. It should be noted that this is possible without considering the actual distribution of the random variable C but merely its a priori mean and variance.

Computer Simulation

A computer simulation program using the foregoing development was written and appears in Appendix E. For the simulation, the Monte Carlo method was used which requires an assumption of the density of C. The form $f_C(c) = (d+1)c^d$ was arbitrarily chosen. With this density, $P(C \leq c) = c^{d+1}$ so that, given the probability, the value of C is the (d+1) root of the probability. Probabilities are obtained using a random number generator. The number of favorable samples, a, at this sampling time is similarly randomly generated using the binomial probability

$$P\ (A=a) = \binom{x}{a} p_t'^{\,a}\ (1-p_t')^{x-a}$$

with $p_t' = C_t\mu_{Pt}$, C_t being the previously described randomly obtained value of C, and μ_{Pt} being calculated by equation 4.27.

Results of this simulation using a constant sample size, x, of 100, a priori λ_o and ψ_o of 98, and d of 10, appear in Appendix F. The estimated and calculated means of P(t) together with the calculated variance of P(t) are shown in Figure 14. As expected, the figure indicates a decreasing variance and generally more accurate estimates as time, and thus the number of samples taken, increases. The relatively large difference between the estimated and calculated mean at time 8 was caused by the low value of 'CRAN' randomly generated at that time. It should be noted that this rather large perturbation had only minor effects on the subsequent estimates, the error of which approximated the magnitude of the errors in the estimates immediately preceeding time 8.

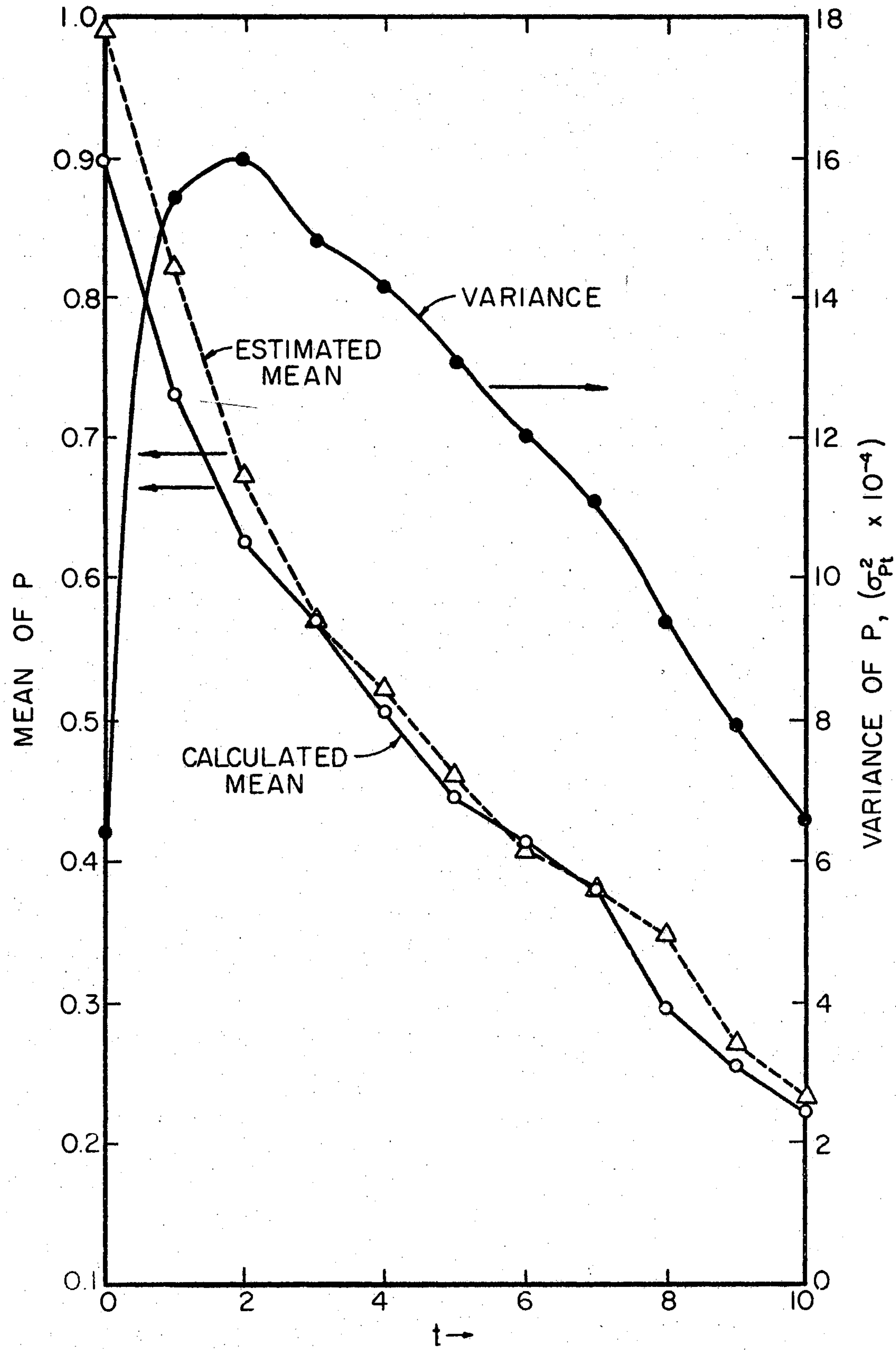

Figure 14. Simulated Mean and Variance of P(t)

CHAPTER V

SAMPLING IN THE STOCHASTIC CASE

Having developed the methods of Chapter IV for a time-varying binomial variable, we can consider optimum sampling of a stochastic process when the sampling is to be done periodically. As in Chapter IV, we will consider the sampling time to be small in relation to the time between samples so that P is time-invariant during any one sampling period. A posteriori sampling results will also be considered available prior to subsequent sample size decisions. If this were not the case, all decisions would be made on the a priori information, reducing the problem to essentially that considered in Chapters II and III.

Sequential Block Sampling

The decision tree involved in the sequential block sampling situation is shown in Figure 15 for a two-stage problem. This modified tree incorporates the result of the test, the action, and the outcome into one of two portions of a stage. As explained in Chapter II, this is permissible when the outcome is governed by nature (as described by $f_{P|\mathcal{E}}$) and the decision resulting in minimum risk is selected with probability one. In the figure, the sampling results designated a are those which would result in acceptance while the b's are those which choose rejection as the best decision. That is

$$b_t \leq g(x_t)$$

$$a_t > g(x_t)$$

where $g(x_t)$ is the decision boundary described in equation 2.12. D_A and D_B are the accept and reject decisions respectively. The expected risks are as follows:

$$E\left[R|a_t,x_t,\&_{t-1},D_A\right] = \left[\prod_{i=o}^{t} P\ (A_i = a_i|x_i;\lambda_i,\psi_i,)\right] R_{D_A} \tag{5.1}$$

$$E\left[R|b_t,x_t,\&_{t-1},D_B\right]$$

$$= P\left(A_t = b_t|x_t,\lambda_t,\psi_t\right)\left[\prod_{i=o}^{t-1} P\left(A_i = a_i|x_i;\lambda_i,\psi_i\right)\right] R_{D_B}. \tag{5.2}$$

Where the a_t and x_t are the number of acceptables and the sample size at time t, $\&_t$ indicates a priori and sampling experience through time t. As before, $\&_t$ implies λ_{t+1} and ψ_{t+1}, the a priori parameters for the Beta density of P for time t+1, calculated as shown in Chapter IV. Each of these risks is a function of the random variable P which exists at that stage. The probabilities of A are calculated using the expected mean of P calculated by equation 4.26 for the appropriate stage. Thus,

$$P\left(A_t = a_t|x_t,\lambda_t,\psi_t\right) = \binom{x_t}{a_t}\hat{\mu}_{Pt}^{a_t}\left(1-\hat{\mu}_{Pt}\right)^{x_t-a_t} \tag{5.3}$$

where

$$\hat{\mu}_{Pt} = \frac{\psi_t+1}{\lambda_t+2} \tag{5.4}$$

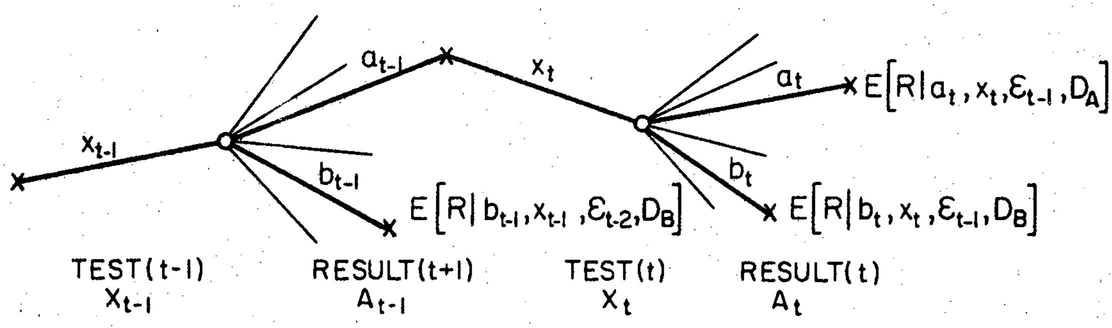

Figure 15. Stochastic Sequential Sampling Tree

To perform these risk calculations, it is first necessary to calculate the λ_i and ψ_i pairs that result for each path thru the tree up to the final stage, n. This can be done by the methods of Chapter IV. When λ_n and ψ_n for a path have been determined, the appropriate risks for each possible A_n for a given X_n can then be calculated. When this has been done, the X_n for which the summation of all A_n risks is less than or equal to that sum for every other X_n becomes the optimum value for X_n, x_n. This is the optimum sequential block sample size for stage n, that is, the sample size which would be selected if non-sequential sampling was to be done. For this last stage only, the calculation of optimum x_n can be considerably simplified by adapting the method of Chapter II for optimum single sample size determination to this problem. The equations necessary are developed later in this chapter in the "Successive Block Sampling" section. When the expected risk, $\overline{R}(x_n)$ from this calculation is found, it must be modified by pre-multiplication by the appropriate probabilities, i.e.

$$\overline{R}\left(a_o, x_o, a_1, x_1, \ldots, a_{n-1}, x_{n-1}\right) = \left[\prod_{i=0}^{n-1} P\left(A_i = a_i | x_i, \lambda_i, \psi_i\right)\right] \overline{R}(x_n) \quad (5.5)$$

Whichever method is used, a numerical value for the risk and an associated optimum x_n result for each value of A_{n-1} for which the decision is accept. When A_{n-1} is a value resulting in rejection, $A_{n-1} \leq g(x_{n-1})$, then the lot is rejected and the sequential testing ends. This is also true of all preceding stages.

The determination of the optimum x_{n-1} requires the summation of the risk values for all the A_{n-1} pertaining to that value of x_{n-1}. This is done for every x_{n-1} and the one with the lowest expected risk is chosen as the optimum. This is then x_{n-1} for the given value of A_{n-2} and its concomitant expected risk is the expected risk if the results of testing at time n-2 give that value of A_{n-2}.

This procedure is repeated down through the tree until an optimum value of x_o is determined. The result is an optimum sequential policy when blocks of samples are to be taken at discrete time intervals and the single sample size at time t must be determined after time t-1 but before any sampling at time t.

While the above procedure for the sequential block sampling case is straightforward and the calculations simple, the number of individual calculations required is enormous. If every possible combination is investigated for an n-stage problem with a lot size of z, as many as $\left(\frac{z}{2}\right)^{2(n-1)}$ sets of calculations could be necessary. As each set of calculations involves solution of six equations for determination of λ_n and ψ_n, plus the probability and risk determination, this means $(4z)^{2(n-1)}$ possible computations.

Fortunately, considerable reduction in this number is possible. Most importantly, the maximum number of samples which will be taken at any one stage will not exceed the optimum value of x_{tot} calculated for

z_{tot} = nz at the greatest P. When C is restricted to the range zero to one, this is the a priori P. For most realistic problems, this value of x_{tot} will be on the order of the square root of z_{tot}. This reduces the calculations to approximately $\left(\frac{nz}{2}\right)^{n-1}$.

The next reduction is possible by considering the fact that the problem terminates when a reject decision is made. Thus, no further calculations are necessary after a reject decision. While the exact number of computations eliminated by this is completely dependent on the problem, for this general consideration it will be assumed that the number of acceptable samples must be greater than one-half of the number sampled for an "accept" decision. This makes no more than $\left(\frac{nz}{4}\right)^{n-1}$ sets of computations necessary. For a problem involving a lot size of 100 and ten stages, this reduces the number of computations required from $5x10^{19}$ to $2.5x10^{11}$. Further reductions are possible when the particular problem at hand is carefully examined and unnecessary computations eliminated but reduction past one more order of magnitude than already achieved would probably not be possible.

Since each set of computations involves at least 15 multiplications and additions, the 100 item, 10 stage problem would require more than three years to solve on the latest commercially available digital computers such as the IBM 360 series. A state-of-the-art computer designed especially for this problem would still require approximately 50 days to perform the necessary calculations.

Thus, while the sequential block sampling problem can be easily solved theoretically, practical considerations make this approach impractical. We are therefore forced to consider some sub-optimization scheme for solution of the problem. While the sequential block problem can be sub-optimized in many ways, such as considering only two or three stages at a time and applying the above development, there is generally little to be gained by any sequential sub-optimization vis-a-vis optimization at each stage. The following section discusses stage by stage optimization when P is time varying.

Successive Block Sampling

To optimize the sample size to be drawn at time t considering only the experience prior to t requires the modification of the development of Chapter II in accordance with the method of a priori parameter determination of Chapter IV.

Assuming we have values for the parameters λ and ψ for time t based on experience through time t-1,

$$f_{P(t)|\mathcal{E}_{t-1}}(p;\lambda_t,\psi_t) = \frac{\Gamma(\lambda_t+2)}{\Gamma(\psi_t+1)\Gamma(\lambda_t-\psi_t+1)} p^{\psi_t}(1-p)^{\lambda_t-\psi_t} \qquad (5.6)$$

with the previously described restrictions on p_t, λ_t and ψ_t obtaining. $f_{P(t)|\mathcal{E}_{t-1}}$ is our present a priori for P_t. After sampling at t, observing a_t favorable items from a total of x_t, our a posteriori density is, from equation 4.3,

$$f_{P(t)|\mathcal{E}_t}(p|a_t;x_t,\lambda_t,\psi_t) = \frac{\Gamma(x_t+\lambda_t+2)}{\Gamma(a_t+\psi_t+1)\Gamma(x_t+\lambda_t-a_t-\psi_t+1)} \left[p^{a_t+\psi_t}(1-p)^{x_t+\lambda_t-a_t-\psi_t}\right] \qquad (5.7)$$

We use this latter density with R_{D_A} and R_{D_B} from equation 2.11 to determine the decision boundary, $g(x_t)$, of equation 2.12, again choosing $a_t \leq g(x_t)$ as the criteria for choosing decision B, reject, and $a_t > g(x_t)$

for the accept decision, A.

Equation 2.14 becomes

$$\overline{R}\ (x_t, p | \mathcal{E}_{t-1}) = R_T + R_{D_A} + \left(R_{D_B} - R_{D_A}\right) \sum_{a_t=o}^{w_t} \binom{x_t}{a_t} p^{a_t}(1-p)^{x_t-a_t} \quad (5.8)$$

This equation implies certain assumptions. First, the risks involved are considered time-invariant. Secondly, the sampling at time t is statistically independent of sampling at past or future times. While this latter restriction will remain, the former will be slightly relaxed in the examples. For this general development, to avoid the confusion of additional subscripting, the time invariant form will be used.

With equations 5.7 and 5.8, we can find the expected risk as a function of x_t:

$$\overline{R}\ (x_t | p, \mathcal{E}_{t-1}) = \int_{-\infty}^{\infty} \overline{R}\ (x_t, \theta | \mathcal{E}_{t-1})\ f_{P(t)|\mathcal{E}_{t-1}}\ (\theta; \lambda_t, \psi_t)\ d\theta$$

$$= \int_{o}^{1} \left(R_T + R_{D_A}\right) \frac{\Gamma(\lambda_t+2)}{\Gamma(\psi_t+1)\Gamma(\lambda_t-\psi_t+1)}\ \theta^{\psi_t}(1-\theta)^{\lambda_t-\psi_t}\ d\theta$$

$$(5.9)$$

$$+ \sum_{a_t=o}^{w_t} \int_{o}^{1} \left(R_{D_B} - R_{D_A}\right) \left[\frac{\Gamma(\lambda_t+2)\ x_t!}{\Gamma(\psi_t+1)\Gamma(\lambda_t-\psi_t+1)\ a_t!(x_t-a_t)!}\right]$$

$$\left[\theta^{a_t+\psi_t}(1-\theta)^{x_t+\lambda_t-a_t-\psi_t}\ d\theta\right]$$

When $\overline{R}(x_t | p_t, \mathcal{E}_t)$ has been determined by solution of equations 5.9, the integer value of x_t which minimizes this risk must be found. Because of the comparative complexity of the factorial expressions when λ_t and ψ_t are non-zero, an approximation of the optimum x_t by differential calculus is usually not feasible. The most efficient method of solution depends upon the form of the risks, R_{D_A} and R_{D_B}. Selection of a method of determination must consider that many reasonable forms of these risks produce an expected risk which is not unimodal, such as in the example which follows. As a last resort, when a digital computer is available, risk values for all x_t's can be calculated and that which produces a minimum chosen. Whenever a computer is used, whatever the solution method, care must be exercised in calculation of the factorials to insure that the machine capacity is not exceeded. As an example, 34! will exceed the capacity of an IBM 7040, while 70! will exceed that of the IBM 1620. This limitation can be circumvented by taking advantage of the division by and of factorials in equation 5.9.

The optimum sample size thus determined becomes x_t. After observation of a_t acceptable items from the x_t samples, the procedures of Chapter IV can be utilized to determine the a priori distribution of P_{t+1}. With λ_{t+1} and ψ_{t+1}, the above equations can again be utilized for determination of the optimum value of x_{t+1}.

This procedure should be successively applied until the sampling results indicate the reject decision or, in the case where the same items are sampled, the lot depleted.

Successive Sequential Sampling

If we now consider the case where the sampling at time t is to be sequential (as described in Chapter III) rather than that described above, we eliminate much of the computational difficulty previously encountered. Here, the future of the problem past the current sampling time, t, does not in general, determine the sample size at t. Rather, the expected risks and the sampling results at time t dictate how many samples will be drawn. A possible exception to this would be when the risk at time t was a function of the future sampling results. This would require a special formulation, depending on the problem, beyond the scope of this study. For our purposes, we need merely modify the sequential sampling policy determination of Chapter III to accommodate the a priori parameters of Chapter IV.

The determinations of expected risks under each of the final decisions, A and B, as functions of a_t and x_t, can be accomplished by use of equation 3.1. Again, the density of P_t given a_t and x_t from equation 5.7 should be used. Thus,

$$\overline{R}\,(a_t,x_t|D_A\&_t) = \frac{\Gamma(x_t+\lambda_t+2)}{\Gamma(a_t+\psi_t+1)\Gamma(x_t+\lambda_t-a_t-\psi_t+1)} \int_0^1 \left(R_T - R_{D_A}\right)\theta^{a_t+\psi_t}(1-\theta)^{x_t+\lambda_t-a_t-\psi_t}\,d\theta \tag{5.10}$$

and

$$\overline{R}\,(a_t,x_t|D_A\&_t) = \frac{\Gamma(x_t+\lambda_t+2)}{\Gamma(a_t+\psi_t+1)\Gamma(x_t+\lambda_t-a_t-\psi_t+1)} \int_0^1 \left(R_T - R_{D_B}\right)\theta^{a_t+\psi_t}(1-\theta)^{x_t+\lambda_t-a_t-\psi_t}\,d\theta \tag{5.11}$$

With these risks, the $\overline{R}(a_t,x_t)$ can be calculated as in equation 3.4.

$$\overline{R}\ (a_t,x_t) = \begin{cases} \overline{R}(a_t,x_t|D_B,\mathcal{E}_t), & o \leq a_t \leq g(x_t) \\ \overline{R}(a_t,x_t|D_A,\mathcal{E}_t), & g(x_t) < a_t \leq x_t \end{cases} \tag{5.12}$$

where $g(x_t)$ is as described above for successive block sampling.

The probability of the next sample being acceptable give a_t of x_t becomes the expected value of P_t when P_t has the density of equation 5.2.

$$P\ (\Omega=1|\mathcal{E}_t) = \frac{a_t+\psi_t+1}{x_t+\lambda_t+2} \tag{5.13}$$

and

$$P\ (\Omega=0|\mathcal{E}_t) = \frac{x_t+\lambda_t-a_t-\psi_t+1}{x_t+\lambda_t+2} \tag{5.14}$$

The expected risk incurred if sampling is continued is

$$R\ (a_t,x_t|\text{continue}) = \frac{a_t+\psi_t+1}{x_t+\lambda_t+2} R\ (a_t+1,x_t+1) + \frac{x_t+\lambda_t-a_t-\psi_t+1}{x_t+\lambda_t+2} R\ (a_t,x_t+1) \tag{5.15}$$

where

$$R\ (a_t,x_t) = \min \left\{ \overline{R}\ (a_t,x_t);\ R\ (a_t,x_t|\text{continue}) \right\} \tag{5.16}$$

The policy for time t can be completely determined with these equations and the dynamic programming techniques described in Chapter III.

After sampling at t, the results must be observed, and a λ_{t+1} and ψ_{t+1} calculated by the method of Chapter IV. At this point, it should again be observed that, if the sample results at time t result in a "reject" decision, no further sampling is necessary and the problem for this process or lot is terminated.

If, sampling indicates an accept decision at time t, the sequential policy determination is repeated for time t+1 and continued for t+2, t+3, etc. until a reject decision is made.

Successive Block Sampling Example

The example problem of Chapters II and III can be readily modified according to the procedures described above for successive block sampling. Recalling from Chapter II

$$R_T = C(1+\beta)x_t$$

$$R_{D_A} = Cy_t\ [\gamma(1+\alpha)-\alpha-\gamma(1+\alpha)p]$$

$$R_{D_B} = Cy_t$$

we have, with equations 5.6 through 5.9, all that is necessary for solution. For this example and for the successive sequential sampling example which follows we will consider that we are sampling the same lot of items, as, for instance, items subject to deterioration which are held in storage. This as opposed to items that are being produced by a process where the process itself is deteriorating. In the latter case, the lot size is not affected by the number of previous samples taken. In the "storage" case, the number of items remaining after sampling at time t is $z_t - x_t$, which becomes z_{t+1}. The effect of this is time modification of the risk functions.

The equations of this example are as follows:

$$\bar{R}\ (x_t,P|\mathcal{E}_{t-1}) = C\left\{z_t+\beta x_t+y_t(1+\alpha)(\gamma-1-\gamma p)\sum_{a_t=w_t+1}^{x_t}\binom{x_t}{a_t}p^{a_t}(1-p)^{x_t-a_t}\right. \quad (5.17)$$

$$\bar{R}\ (x_t|p,\mathcal{E}_{t-1}) = C\left\{z_t+\beta x_t+y_t(1+\alpha)\right. \quad (5.18)$$

$$\left.\sum_{a_t=w_t+1}^{x_t} \frac{x_t!\Gamma(\lambda_t+2)\Gamma(a_t+\psi_t+1)\Gamma(x_t+\lambda_t-a_t-\psi_t+1)}{a_t!\Gamma(\psi_t+1)(x_t-a_t)!\Gamma(\lambda_t-\psi_t+1)\ (x_t+\lambda_t+2)}\left[\gamma-1-\gamma\mu_{Pt}\right]\right\}.$$

$$g(x_t) = \frac{(\gamma-1)(x_t+\lambda_t)+\gamma(1-\psi_t)-2}{\gamma}. \quad (5.19)$$

$$w_t = [g(x_t)]. \quad (5.20)$$

The value of x_t which minimizes equation 5.18 is the optimum block sample size for this stage of sequence.

Successive Sequential Sampling Example

Using the same assumptions as in the above examples, equations for the successive sequential sampling can be written. These are as follows:

$$\bar{R}\ (a_t,x_t|D_A,\mathcal{E}_t) = C\left\{z_t+\beta x_t+y_t(1+\alpha)\left[\gamma-1-\frac{\gamma(a_t+\psi_t+1)}{x_t+\lambda_t+2}\right]\right\}. \quad (5.21)$$

$$\bar{R}\ (a_t,x_t|D_B,\mathcal{E}_t) = C\ (z_t+\beta x_t). \quad (5.22)$$

$$P\ (\Omega=1|\mathcal{E}) = \frac{a_t+\psi_t+1}{x_t+\lambda_t+2}. \quad (5.23)$$

With these equations, equation 5.19 for $g(x_t)$ and the equations of Chapter IV, an optimum successive sequential sampling policy can be found using the techniques of Chapter III.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Summary

The problem of minimizing the expected risk of a dichotomous process capable of being binomially sampled has been examined. Both the time-invariant and stochastic cases have been considered and the methods of determining optimum sampling policies developed.

Chapter II considered a time-invariant process when an optimum single sample size was to be determined in advance of sampling. The statistical decision theory method of solution as it applied to this problem was explained by first considering the distributions of the random variables involved and then by use of them in formulating the expected risk functions. The Bayesian method of determining probability densities was used to quantify the available prior experience and to formulate the after-sampling density of the binomial parameter, P. Use of the Beta distribution as the a priori of P was proposed because it met the parameter criteria and is the Bayesian conjugate of the distribution which applies to the samples, the binomial. The equally likely form of the Beta was chosen for this initial development.

The expected risk involved as a function of the sample size was developed and a parallel drawn between the method of this paper and a second Bayesian based method which considers the distribution and risk determination simultaneously. An example problem was introduced to illustrate an application of the preceding development and the effects of varying the parameters of the example investigated.

Chapter III introduced the sequential sampling problem and the dynamic programming approach to its solution in the time-invariant case. The required form for the expected risks was shown and the necessary recursion relation developed. The example was considered in the sequential case, the equations for it and a digital computer program incorporating them written, and certain results from the program presented in graphical form.

The method of determining probability densities when the random variable possessed certain stochastic qualities was considered in Chapter IV. The distribution was assumed to remain Beta. Also assumed was a difference equation model for describing its time variation. A method for determining the time-modified Beta parameters for successive a priori densities was devised and a computer simulation program written and run.

Finally, the stochastic developments were incorporated into the risk determinations in both the single sample and sequential sampling cases. The dynamic programming method of determining the optimum sequential single sample sizes was outlined. Sub-optimization in this case was also considered and formulated. The sequential sampling method of Chapter III was modified to accommodate the stochastic case. Both the successive block sampling and stochastic sequential sampling developments were applied to the example problem and all pertinent equations developed.

Conclusions

A Bayesian approach to the optimum sampling problem when the samples are discrete, independent and binomially distributed and the assumed P distributions are Beta yields mathematically feasible and intuitively satisfactory results for both the time-invariant and the stochastic case.

Dynamic programming can be easily adapted to the problem of sequential sampling of a binomial variable. When used in conjunction with statistical decision theory techniques, it produces a sampling policy which, when used with finite lot sizes, results in a decision prior to exhaustion of the lot due to sampling. Further, the expected risks in the sequential sampling case are less than those in the single sample size case. A digital computer is required to feasibly produce a sequential policy by the dynamic programming method.

The stochastic case is relatively easy to solve conceptually when a difference equation for uniform time intervals is the appropriate model and the random variable concerned is Beta distributed. Part of the ease of this determination is due to the fact that the Beta distribution is uniquely determined by its first two moments.

The dynamic programming approach to the sequential block sampling problem, while not difficult to formulate, results in computations too time consuming to be feasible. Sub-optimization is feasible and easily accomplished.

Successive sequential sampling produces optimum results in the stochastic case as future change in the random variable does not effect present policy determinations. The improvement in expected risk when sequential sampling is used in both the time-invariant and stochastic situations argues strongly for its adoption whenever possible.

Suggestions for Further Study

The example of this study could be made nearly universal if modified to include non-destructive testing. The major problem would arise in the stochastic case of testing the same lot where the return of only the favorable survivors to the population would bias the subsequent sampling.

The methods developed herein should be investigated for applicability when other distributions govern. For the stochastic case, the Bayesian conjugate property and that of unique distribution determination by a finite number of determinable moments are desirable.

The problem of learning the distribution of the stochastic modification variable, C, perhaps on the basis of the learned distributions of P, should be investigated.

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APPENDIX A

CALCULATION OF $\overline{R}(x|\mathcal{E})$ FOR THE EXAMPLE

$$\overline{R}(x|\mathcal{E}) = \int_{-\infty}^{\infty} \overline{R}_{P|\mathcal{E}}(p|a,x)\ f_{P|\mathcal{E}}(p)\ dp$$

$$= \int_{0}^{1} C\Bigg\{(1+\beta)x + y\ [\gamma(1+\alpha) - \alpha - \gamma(1+\alpha)p]$$

$$+ y\ (1+\alpha)[(1-\gamma+\gamma p)] \sum_{a=o}^{w} \binom{x}{a} p^{a}\ (1-p)^{x-a}\Bigg\}(1)\ dp$$

$$= C\Bigg\{x + \beta x + y + y\gamma(1+\alpha) - y - \alpha y - \frac{\gamma(1+\alpha)}{2}$$

$$+ y(1+\alpha)\sum_{a=o}^{w}\left[(1-\gamma)\binom{x}{a}\int_{0}^{1} p^{a}(1-p)^{x-a}dp + \gamma\binom{x}{a}\int_{0}^{1} p^{a+1}(1-p)^{x-a}\ dp\right]\Bigg\}$$

$$= C\Bigg\{z + \beta x + \frac{y(1+\alpha)(\gamma-2)}{2} + y(1+\alpha)\sum_{a=o}^{w}\left[\frac{(1-\gamma)\ x!\ a!\ (x-a)!}{(x+1)!\ a!\ (x-a)!}\right.$$

$$\left. + \frac{(x!)(a+1)!\ (x-a)!}{(x+2)!\ a!\ (x-a)!}\right]\Bigg\}$$

$$= C\Bigg\{z + \beta x + \frac{y(1+\alpha)(\gamma-2)}{2} + y(1+\alpha)\sum_{a=o}^{w}\left[\frac{(1-\gamma)}{x+1} + \frac{\gamma(a+1)}{(x+2)(x+1)}\right]\Bigg\}$$

$$= C\left\{z + \beta x + \frac{y(1+\alpha)(\gamma-2)}{2} + y(1+\alpha)\sum_{a=o}^{w} \frac{x+2-\gamma x-\gamma+\gamma a}{(x+1)(x+2)}\right\}$$

$$= C\left\{\beta z + \ \ x + \frac{y(1+\alpha)(\gamma-2)}{2} + \frac{y(1+\alpha)(w+1)}{(x+1)(x+2)}\left[x+2-\gamma x-\gamma+\frac{\gamma w}{2}\right]\right\}$$

$$= C\left\{z + \beta x + \frac{y(1+\alpha)}{2}\left[\gamma-2+ \frac{w+1}{(x+1)(x+2)}(2x+4-2\gamma x-2\gamma+\gamma w)\right]\right\}.$$

APPENDIX B

```
C
C                   FORTRAN PROGRAM
C
C     SEQUENTIAL SAMPLING POLICY FOR AN OPERATIONS RESEARCH PROBLEM
C     USING DYNAMIC PROGRAMMING TECHNIQUES - W. C. MCCORMICK,JR.
C
      DIMENSION RSKEP (200), RSK(200)
    1 FORMAT (5F10.4)
    2 FORMAT(/31HSEQUENTIAL SAMPLING POLICY FOR F5.0,13H TOTAL ITEMS.)
    3 FORMAT(/51H NR OF   NR OF                          RISK UNDER)
    4 FORMAT(53HSAMPLES  ONES    RISK        DECISION  OTHER DECISION/)
    5 FORMAT(I5,I8,F11.4,15H  STOP - REJECTF12.4)
    6 FORMAT(I5,I8,F11.4,13H     CONTINUEF14.4)
    7 FORMAT(/)
    8 FORMAT(I5,I8,F11.4,15H  STOP - ACCEPTF12.4,/)
    9 FORMAT(9H ALPHA = F7.4,9H, BETA = F7.4)
   91 FORMAT(8HGAMMA = F7.4,14H, MFG. COST = F8.4)
   10 READ 1, C, Z, ALPHA, BETA, GAMMA
C
C     C IS MFG COST, Z IS LOT SIZE, ALPHA IS MARK-UP, BETA IS SAMPLING COST
C     GAMMA IS PENALTY FACTOR
C
      IF (Z) 500, 500, 12
   12 PUNCH 2, Z
      PUNCH 9, ALPHA, BETA
      PUNCH 91, GAMMA, C
      G1 = GAMMA - 1.0
      G2 = G1 - 1.0
      EN = (((Z*G1) + G2)/GAMMA) + 1.0
      NREJ = EN
      L = Z
      RSINT = C*(1.0+BETA)*Z
C
C     INITIAL VALUES OF RSKEP (KEPT RISK) RESULT FROM 100 PER CENT SAMPLE
C
      DO 11 I = 1,NREJ
   11 RSKEP(I) = RSINT
      NPUN = NREJ - 1
      PUNCH 5, L, NPUN, RSINT, RSINT
      DO 15 IAC = NREJ,L
      IDX = IAC + 1
   15 RSKEP(IDX) = RSINT
      PUNCH 8, L, NREJ, RSINT, RSINT
      PUNCH 7
      DO 50 J = 1,L
      Y=J
      X = Z-Y
C
C     X IS NR OF SAMPLED ITEMS, Y IS NR OF NON-SAMPLED ITEMS.
C
      XREJ = (((X*G1) + G2)/GAMMA) + 1.0
      KREJ = XREJ
C
C     KREJ IS UPPER BOUND FOR REJECT DECISION PLUS ONE - F(X) + 1
C
      C1 = C*(Z+(BETA*X))
      C2 = C*(ALPHA+1.0)*Y/(X+2.0)
      KX = X
      IF (KREJ) 27, 27, 16
   16 JIX = 0.0
      DO 25 KR = 1, KREJ
```

```
      KR1 = KR - 1
      KR2 = KR1 - 1
      A = KR1
C
C     A IS NR OF ACCEPTABLE SAMPLES
C
      K1R = KR + 1
      RSKST = C1
C
C     RISK INCURRED IF SAMPLING IS STOPPED W/ A ACCEPTABLE OF X SAMPLES
C
      P1AX = (A+1.0)/(X+2.0)
C
C     PROBABILITY OF NEXT SAMPLE BEING OK GIVEN A OK'S OF X SAMPLES
C
      RSKCN = RSKEP(KR) +P1AX*(RSKEP(K1R) - RSKEP(KR))
C
C      RISK INCURRED IF SAMPLING IS CONTINUED
C
      IF (RSKST - RSKCN) 20,20,18
   18 RSK(KR) = RSKCN
      JIX = JIX + 1
      IF (JIX - 1) 20, 21, 19
   19 PUNCH 6, KX, KR1, RSKCN, RSKST
      GO TO 25
   20 RSK(KR) = RSKST
      RSKO = RSKCN
      GO TO 25
   21 IF (KR1) 23, 23, 22
   22 PUNCH 5, KX, KR2, RSK(KR1), RSKO
   23 PUNCH 6, KX, KR1, RSKCN, RSKST
   25 CONTINUE
      IF (JIX) 27, 26, 27
   26 PUNCH 5, KX, KR1, RSKST, RSKCN
   27 KAX = 0.0
      IF (KX - KREJ) 35, 270, 270
  270 DO 35 KA = KREJ, KX
      A = KA
      KA1 = KA + 1
      KA2 = KA + 2
      RSKST = C1 + C2*((GAMMA*(X-A+1.0)) - (X + 2.0))
      P1AX = (A+1.0)/(X+2.0)
      RSKCN = RSKEP(KA1) + P1AX*(RSKEP(KA2) - RSKEP(KA1))
      IF (RSKST-RSKCN) 30,30,28
   28 RSK(KA1) = RSKCN
      PUNCH 6, KX, KA, RSKCN, RSKST
      GO TO 35
   30 RSK(KA1) = RSKST
      KAX = KAX + 1
      IF (KAX - 1) 28, 31, 35
   31 PUNCH 8, KX, KA, RSKST, RSKCN
   35 CONTINUE
      MIND = KX + 1
      DO 40 M = 1, MIND
   40 RSKEP(M) = RSK(M)
      PUNCH 7
   50 CONTINUE
      GO TO 10
  500 STOP
      END
```

APPENDIX C

SEQUENTIAL SAMPLING POLICY FOR 50. TOTAL ITEMS.
ALPHA = 5.0000, BETA = 2.0000
GAMMA = 5.0000, MFG. COST = 10.0000

NR OF SAMPLES	NR OF ONES	RISK	DECISION	RISK UNDER OTHER DECISION
50	40	1500.0000	STOP - REJECT	1500.0000
50	41	1500.0000	STOP - ACCEPT	1500.0000
49	39	1480.0000	STOP - REJECT	1500.0000
49	40	1478.8236	STOP - ACCEPT	1500.0000
48	39	1460.0000	STOP - REJECT	1479.0589
48	40	1448.0000	STOP - ACCEPT	1474.0001
47	38	1440.0000	STOP - REJECT	1460.0000
47	39	1425.3062	STOP - ACCEPT	1450.2041
46	37	1420.0000	STOP - REJECT	1440.0000
46	38	1405.0000	STOP - ACCEPT	1428.0613
45	36	1400.0000	STOP - REJECT	1420.0000
45	37	1387.2341	STOP - ACCEPT	1407.8724
44	35	1380.0000	STOP - REJECT	1400.0000
44	36	1372.1740	STOP - ACCEPT	1389.7318
43	35	1360.0000	STOP - REJECT	1373.7392
43	36	1313.3334	STOP - ACCEPT	1340.0001
42	34	1340.0000	STOP - REJECT	1360.0000
42	35	1296.3637	STOP - ACCEPT	1321.8183
41	33	1320.0000	STOP - REJECT	1340.0000
41	34	1282.3256	STOP - ACCEPT	1304.4821
40	32	1300.0000	STOP - REJECT	1320.0000
40	33	1271.4286	STOP - ACCEPT	1289.5017
39	31	1280.0000	STOP - REJECT	1300.0000
39	32	1263.9025	STOP - ACCEPT	1277.0036
38	31	1260.0000	STOP - REJECT	1267.1220
38	32	1170.0000	STOP - ACCEPT	1197.5001
37	30	1240.0000	STOP - REJECT	1260.0000

37	31	1160.0000	STOP - ACCEPT	1186.1539
36	29	1220.0000	STOP - REJECT	1240.0000
36	30	1153.6843	STOP - ACCEPT	1174.7369
35	28	1200.0000	STOP - REJECT	1220.0000
35	29	1151.3514	STOP - ACCEPT	1166.2306
34	27	1180.0000	STOP - REJECT	1200.0000
34	28	1153.3334	STOP - ACCEPT	1160.8109
33	25	1160.0000	STOP - REJECT	1180.0000
33	27	1158.6668	CONTINUE	1160.0000
33	28	1014.2858	STOP - ACCEPT	1042.8573
32	26	1140.0000	STOP - REJECT	1158.9413
32	27	1012.9412	STOP - ACCEPT	1039.7649
31	25	1120.0000	STOP - REJECT	1140.0000
31	26	1016.3637	STOP - ACCEPT	1036.0429
30	24	1100.0000	STOP - REJECT	1120.0000
30	25	1025.0000	STOP - ACCEPT	1035.7956
29	22	1080.0000	STOP - REJECT	1100.0000
29	23	1080.0000	STOP - REJECT	1100.0000
28	22	1060.0000	STOP - REJECT	1080.0000
28	23	1047.4840	CONTINUE	1060.0000
28	24	840.0000	STOP - ACCEPT	870.0001
27	22	1040.0000	STOP - REJECT	1050.0736
27	23	849.6552	STOP - ACCEPT	875.7732
26	21	1020.0000	STOP - REJECT	1040.0000
26	22	865.7143	STOP - ACCEPT	883.6454
25	20	1000.0000	STOP - REJECT	1020.0000
25	21	888.8889	STOP - ACCEPT	894.2858
24	19	980.0000	STOP - REJECT	1000.0000
24	20	910.2565	CONTINUE	920.0000
24	21	620.0000	STOP - ACCEPT	653.8462
23	18	960.0000	STOP - REJECT	980.0000
23	19	924.2052	CONTINUE	960.0000
23	20	636.0000	STOP - ACCEPT	666.4410

22	17	940.0000	STOP - REJECT	960.0000
22	18	931.6624	CONTINUE	940.0000
22	19	660.0000	STOP - ACCEPT	684.0342
21	17	920.0000	STOP - REJECT	933.4749
21	18	693.0434	STOP - ACCEPT	707.2456
20	16	900.0000	STOP - REJECT	920.0000
20	17	734.3083	CONTINUE	736.3636
20	18	327.2727	STOP - ACCEPT	366.3636
19	15	880.0000	STOP - REJECT	900.0000
19	16	765.8686	CONTINUE	791.4285
19	17	348.5714	STOP - ACCEPT	385.4206
18	14	860.0000	STOP - REJECT	880.0000
18	15	788.6949	CONTINUE	860.0000
18	16	380.0000	STOP - ACCEPT	411.1660
17	13	840.0000	STOP - REJECT	860.0000
17	14	803.7065	CONTINUE	840.0000
17	15	423.1579	STOP - ACCEPT	444.5307
16	12	820.0000	STOP - REJECT	840.0000
16	13	811.7717	CONTINUE	820.0000
16	14	480.0000	STOP - ACCEPT	486.5826
15	12	800.0000	STOP - REJECT	813.7078
15	13	538.5479	CONTINUE	552.9411
15	14	-64.7058	STOP - ACCEPT	-19.9999
14	11	780.0000	STOP - REJECT	800.0000
14	12	587.5702	CONTINUE	645.0000
14	13	-30.0000	STOP - ACCEPT	10.7008
13	10	760.0000	STOP - REJECT	780.0000
13	11	626.0561	CONTINUE	760.0000
13	12	20.0000	STOP - ACCEPT	52.3427
12	9	740.0000	STOP - REJECT	760.0000
12	10	654.7584	CONTINUE	740.0000
12	11	88.5714	STOP - ACCEPT	106.5794
11	8	720.0000	STOP - REJECT	740.0000
11	9	674.4295	CONTINUE	720.0000
11	10	175.6771	CONTINUE	180.0000
11	11	-720.0000	STOP - ACCEPT	-663.0768
10	7	700.0000	STOP - REJECT	720.0000

10	8	685.8221	CONTINUE	700.0000
10	9	258.8025	CONTINUE	300.0000
10	10	-700.0000	STOP - ACCEPT	-645.3602
9	7	680.0000	STOP - REJECT	689.6888
9	8	336.4424	CONTINUE	456.3636
9	9	-661.8181	STOP - ACCEPT	-612.8361
8	6	660.0000	STOP - REJECT	680.0000
8	7	405.1540	CONTINUE	660.0000
8	8	-600.0000	STOP - ACCEPT	-561.9920
7	5	640.0000	STOP - REJECT	660.0000
7	6	461.7864	CONTINUE	640.0000
7	7	-506.6666	STOP - ACCEPT	-488.3162
6	4	620.0000	STOP - REJECT	640.0000
6	5	506.3398	CONTINUE	620.0000
6	6	-385.6099	CONTINUE	-370.0000
5	3	600.0000	STOP - REJECT	620.0000
5	4	538.8141	CONTINUE	600.0000
5	5	-258.1885	CONTINUE	-171.4285
4	2	580.0000	STOP - REJECT	600.0000
4	3	559.2094	CONTINUE	580.0000
4	4	-125.3547	CONTINUE	120.0000
3	2	560.0000	STOP - REJECT	567.5256
3	3	11.5581	CONTINUE	560.0000
2	1	540.0000	STOP - REJECT	560.0000
2	2	148.6685	CONTINUE	540.0000
1	0	520.0000	STOP - REJECT	540.0000
1	1	279.1123	CONTINUE	520.0000
0	0	399.5562	CONTINUE	500.0000

SEQUENTIAL SAMPLING POLICY FOR 50. TOTAL ITEMS.
ALPHA = 5.0000, BETA = 2.0000
GAMMA = 2.0000, MFG. COST = 10.0000

NR OF SAMPLES	NR OF ONES	RISK	DECISION	RISK UNDER OTHER DECISION
50	25	1500.0000	STOP - REJECT	1500.0000
50	26	1500.0000	STOP - ACCEPT	1500.0000
49	24	1480.0000	STOP - REJECT	1500.0000
49	25	1478.8236	STOP - ACCEPT	1500.0000
48	24	1460.0000	STOP - REJECT	1479.4118
48	25	1455.2000	STOP - ACCEPT	1477.6001
47	23	1440.0000	STOP - REJECT	1460.0000
47	24	1436.3266	STOP - ACCEPT	1457.5511
46	23	1420.0000	STOP - REJECT	1438.1633
46	24	1410.0000	STOP - ACCEPT	1432.5001
45	22	1400.0000	STOP - REJECT	1420.0000
45	23	1393.6171	STOP - ACCEPT	1414.8937
44	22	1380.0000	STOP - REJECT	1396.8086
44	23	1364.3479	STOP - ACCEPT	1386.9566
43	21	1360.0000	STOP - REJECT	1380.0000
43	22	1350.6667	STOP - ACCEPT	1372.0001
42	21	1340.0000	STOP - REJECT	1355.3334
42	22	1318.1819	STOP - ACCEPT	1340.9092
41	20	1320.0000	STOP - REJECT	1340.0000
41	21	1307.4419	STOP - ACCEPT	1328.8373
40	20	1300.0000	STOP - REJECT	1313.7210
40	21	1271.4286	STOP - ACCEPT	1294.2858
39	19	1280.0000	STOP - REJECT	1300.0000
39	20	1263.9025	STOP - ACCEPT	1285.3659
38	19	1260.0000	STOP - REJECT	1271.9513
38	20	1224.0000	STOP - ACCEPT	1247.0001
37	18	1240.0000	STOP - REJECT	1260.0000

37	19	1220.0000	STOP - ACCEPT	1241.5385
36	18	1220.0000	STOP - REJECT	1230.0000
36	19	1175.7895	STOP - ACCEPT	1198.9474
35	17	1200.0000	STOP - REJECT	1220.0000
35	18	1175.6757	STOP - ACCEPT	1197.2974
34	17	1180.0000	STOP - REJECT	1187.8379
34	18	1126.6667	STOP - ACCEPT	1150.0001
33	16	1160.0000	STOP - REJECT	1180.0000
33	17	1130.8572	STOP - ACCEPT	1152.5715
32	16	1140.0000	STOP - REJECT	1145.4286
32	17	1076.4706	STOP - ACCEPT	1100.0001
31	15	1120.0000	STOP - REJECT	1140.0000
31	16	1085.4546	STOP - ACCEPT	1107.2728
30	15	1100.0000	STOP - REJECT	1102.7273
30	16	1025.0000	STOP - ACCEPT	1048.7501
29	14	1080.0000	STOP - REJECT	1100.0000
29	15	1039.3549	STOP - ACCEPT	1061.2904
28	13	1060.0000	STOP - REJECT	1080.0000
28	14	1059.6775	CONTINUE	1060.0000
28	15	972.0000	STOP - ACCEPT	996.0001
27	13	1040.0000	STOP - REJECT	1059.8444
27	14	992.4138	STOP - ACCEPT	1014.3271
26	12	1020.0000	STOP - REJECT	1040.0000
26	13	1016.2069	CONTINUE	1020.0000
26	14	917.1429	STOP - ACCEPT	941.4285
25	12	1000.0000	STOP - REJECT	1018.1737
25	13	944.4445	STOP - ACCEPT	964.8404
24	11	980.0000	STOP - REJECT	1000.0000
24	12	972.2223	CONTINUE	980.0000
24	13	860.0000	STOP - ACCEPT	884.6154
23	11	960.0000	STOP - REJECT	976.2667
23	12	895.2000	STOP - ACCEPT	913.8667

22	10	940.0000	STOP - REJECT	960.0000
22	11	927.6000	CONTINUE	940.0000
22	12	800.0000	STOP - ACCEPT	825.0000
21	10	920.0000	STOP - REJECT	934.0695
21	11	844.3478	STOP - ACCEPT	861.0260
20	9	900.0000	STOP - REJECT	920.0000
20	10	882.1739	CONTINUE	900.0000
20	11	736.3636	STOP - ACCEPT	761.8182
19	9	880.0000	STOP - REJECT	891.5114
19	10	791.4285	STOP - ACCEPT	805.7971
18	8	860.0000	STOP - REJECT	880.0000
18	9	835.7142	CONTINUE	860.0000
18	10	668.0000	STOP - ACCEPT	694.0000
17	8	840.0000	STOP - REJECT	848.4962
17	9	735.7894	STOP - ACCEPT	747.4436
16	7	820.0000	STOP - REJECT	840.0000
16	8	787.8947	CONTINUE	820.0000
16	9	593.3333	STOP - ACCEPT	620.0000
15	7	800.0000	STOP - REJECT	804.8916
15	8	676.4705	STOP - ACCEPT	684.8916
14	6	780.0000	STOP - REJECT	800.0000
14	7	738.2353	CONTINUE	780.0000
14	8	510.0000	STOP - ACCEPT	537.5000
13	6	760.0000	STOP - REJECT	760.5098
13	7	612.0000	STOP - ACCEPT	616.5098
12	5	740.0000	STOP - REJECT	760.0000
12	6	686.0000	CONTINUE	740.0000
12	7	414.2857	STOP - ACCEPT	442.8571
11	4	720.0000	STOP - REJECT	740.0000
11	5	715.0769	CONTINUE	720.0000
11	6	539.6923	CONTINUE	540.0000
11	7	180.0000	STOP - ACCEPT	213.8461
10	4	700.0000	STOP - REJECT	717.9487
10	5	627.3846	CONTINUE	700.0000
10	6	300.0000	STOP - ACCEPT	329.8718
9	3	680.0000	STOP - REJECT	700.0000

9	4	666.9930	CONTINUE	680.0000
9	5	448.8112	CONTINUE	456.3636
9	6	9.0909	STOP - ACCEPT	45.4545
8	3	660.0000	STOP - REJECT	674.7972
8	4	557.9021	CONTINUE	660.0000
8	5	156.0000	STOP - ACCEPT	184.9790
7	2	640.0000	STOP - REJECT	660.0000
7	3	614.6231	CONTINUE	640.0000
7	4	334.6231	CONTINUE	353.3333
7	5	-219.9999	STOP - ACCEPT	-179.9999
6	2	620.0000	STOP - REJECT	630.4836
6	3	474.6231	CONTINUE	620.0000
6	4	-40.0000	STOP - ACCEPT	-12.0162
5	1	600.0000	STOP - REJECT	620.0000
5	2	557.6956	CONTINUE	600.0000
5	3	180.5527	CONTINUE	214.2857
5	4	-557.1428	STOP - ACCEPT	-511.4285
4	1	580.0000	STOP - REJECT	585.8985
4	2	369.1242	CONTINUE	580.0000
4	3	-340.0000	STOP - ACCEPT	-311.2442
3	0	560.0000	STOP - REJECT	580.0000
3	1	495.6496	CONTINUE	560.0000
3	2	-56.3503	CONTINUE	-4.0000
3	3	-1132.0000	STOP - ACCEPT	-1076.0000
2	0	540.0000	STOP - REJECT	543.9124
2	2	-900.0000	STOP - ACCEPT	-863.0875
2	1	219.6496	CONTINUE	540.0000
1	0	433.2156	CONTINUE	520.0000
1	1	-526.7833	CONTINUE	-460.0000
0	0	-46.7833	CONTINUE	500.0000

SEQUENTIAL SAMPLING POLICY FOR 100. TOTAL ITEMS.
ALPHA = 5.0000, BETA = 2.0000
GAMMA = 2.0000, MFG. COST = 10.0000

NR OF SAMPLES	NR OF ONES	RISK	DECISION	RISK UNDER OTHER DECISION
100	50	3000.0000	STOP - REJECT	3000.0000
100	51	3000.0000	STOP - ACCEPT	3000.0000
99	49	2980.0000	STOP - REJECT	3000.0000
99	50	2979.4060	STOP - ACCEPT	3000.0000
98	49	2960.0000	STOP - REJECT	2979.7030
98	50	2957.6000	STOP - ACCEPT	2978.8001
97	48	2940.0000	STOP - REJECT	2960.0000
97	49	2938.1819	STOP - ACCEPT	2958.7879
96	48	2920.0000	STOP - REJECT	2939.0910
96	49	2915.1021	STOP - ACCEPT	2936.3266
95	47	2900.0000	STOP - REJECT	2920.0000
95	48	2896.9073	STOP - ACCEPT	2917.5259
94	47	2880.0000	STOP - REJECT	2898.4537
94	48	2872.5000	STOP - ACCEPT	2893.7501
93	46	2860.0000	STOP - REJECT	2880.0000
93	47	2855.5790	STOP - ACCEPT	2876.2106
92	46	2840.0000	STOP - REJECT	2857.7895
92	47	2829.7873	STOP - ACCEPT	2851.0639
91	45	2820.0000	STOP - REJECT	2840.0000
91	46	2814.1936	STOP - ACCEPT	2834.8388
90	45	2800.0000	STOP - REJECT	2817.0968
90	46	2786.9566	STOP - ACCEPT	2808.2610
89	44	2780.0000	STOP - REJECT	2800.0000
89	45	2772.7473	STOP - ACCEPT	2793.4067
88	44	2760.0000	STOP - REJECT	2776.3737
88	45	2744.0000	STOP - ACCEPT	2765.3334
87	43	2740.0000	STOP - REJECT	2760.0000

87	44	2731.2360	STOP - ACCEPT	2751.9102
86	43	2720.0000	STOP - REJECT	2735.6180
86	44	2700.9091	STOP - ACCEPT	2722.2728
85	42	2700.0000	STOP - REJECT	2720.0000
85	43	2689.6552	STOP - ACCEPT	2710.3449
84	42	2680.0000	STOP - REJECT	2694.8276
84	43	2657.6745	STOP - ACCEPT	2679.0699
83	41	2660.0000	STOP - REJECT	2680.0000
83	42	2648.0000	STOP - ACCEPT	2668.7060
82	41	2640.0000	STOP - REJECT	2654.0000
82	42	2614.2858	STOP - ACCEPT	2635.7143
81	40	2620.0000	STOP - REJECT	2640.0000
81	41	2606.2651	STOP - ACCEPT	2626.9880
80	40	2600.0000	STOP - REJECT	2613.1326
80	41	2570.7318	STOP - ACCEPT	2592.1952
79	39	2580.0000	STOP - REJECT	2600.0000
79	40	2564.4445	STOP - ACCEPT	2585.1853
78	39	2560.0000	STOP - REJECT	2572.2223
78	40	2527.0000	STOP - ACCEPT	2548.5001
77	38	2540.0000	STOP - REJECT	2560.0000
77	39	2522.5317	STOP - ACCEPT	2543.2912
76	38	2520.0000	STOP - REJECT	2531.2659
76	39	2483.0770	STOP - ACCEPT	2504.6155
75	37	2500.0000	STOP - REJECT	2520.0000
75	38	2480.5195	STOP - ACCEPT	2501.2988
74	37	2480.0000	STOP - REJECT	2490.2598
74	38	2438.9474	STOP - ACCEPT	2460.5264
73	36	2460.0000	STOP - REJECT	2480.0000
73	37	2438.4000	STOP - ACCEPT	2459.2001
72	36	2440.0000	STOP - ACCEPT	2449.2000
72	37	2394.5946	STOP - ACCEPT	2416.2163

71	35	2420.0000	STOP - REJECT	2440.0000
71	36	2396.1644	STOP - ACCEPT	2416.9864
70	35	2400.0000	STOP - REJECT	2408.0822
70	36	2350.0000	STOP - ACCEPT	2371.6668
69	34	2380.0000	STOP - REJECT	2400.0000
69	35	2353.8029	STOP - ACCEPT	2374.6479
68	34	2360.0000	STOP - REJECT	2366.9015
68	35	2305.1429	STOP - ACCEPT	2326.8573
67	33	2340.0000	STOP - REJECT	2360.0000
67	34	2311.3044	STOP - ACCEPT	2332.1740
66	33	2320.0000	STOP - REJECT	2325.6522
66	34	2260.0000	STOP - ACCEPT	2281.7648
65	32	2300.0000	STOP - REJECT	2320.0000
65	33	2268.6568	STOP - ACCEPT	2289.5523
64	32	2280.0000	STOP - REJECT	2284.3284
64	33	2214.5455	STOP - ACCEPT	2236.3638
63	31	2260.0000	STOP - REJECT	2280.0000
63	32	2225.8462	STOP - ACCEPT	2246.7693
62	31	2240.0000	STOP - REJECT	2242.9231
62	32	2168.7500	STOP - ACCEPT	2190.6251
61	30	2220.0000	STOP - REJECT	2240.0000
61	31	2182.8572	STOP - ACCEPT	2203.8096
60	30	2200.0000	STOP - REJECT	2201.4286
60	31	2122.5807	STOP - ACCEPT	2144.5162
59	29	2180.0000	STOP - REJECT	2200.0000
59	30	2139.6722	STOP - ACCEPT	2160.6558
58	28	2160.0000	STOP - REJECT	2180.0000
58	29	2159.8361	CONTINUE	2160.0000
58	30	2076.0000	STOP - ACCEPT	2098.0001
57	28	2140.0000	STOP - REJECT	2159.9195
57	29	2096.2712	STOP - ACCEPT	2117.2076

56	27	2120.0000	STOP - REJECT	2140.0000
56	28	2118.1356	CONTINUE	2120.0000
56	29	2028.9656	STOP - ACCEPT	2051.0346
55	27	2100.0000	STOP - REJECT	2119.0842
55	28	2052.6316	STOP - ACCEPT	2072.7685
54	26	2080.0000	STOP - REJECT	2100.0000
54	27	2076.3158	CONTINUE	2080.0000
54	28	1981.4286	STOP - ACCEPT	2003.5715
53	26	2060.0000	STOP - REJECT	2078.1914
53	27	2008.7273	STOP - ACCEPT	2028.0096
52	25	2040.0000	STOP - REJECT	2060.0000
52	26	2034.3637	CONTINUE	2040.0000
52	27	1933.3334	STOP - ACCEPT	1955.5557
51	25	2020.0000	STOP - REJECT	2037.2351
51	26	1964.5284	STOP - ACCEPT	1982.8955
50	24	2000.0000	STOP - REJECT	2020.0000
50	25	1992.2642	CONTINUE	2000.0000
50	26	1884.6154	STOP - ACCEPT	1906.9232
49	24	1980.0000	STOP - REJECT	1996.2080
49	25	1920.0000	STOP - ACCEPT	1937.3845
48	23	1960.0000	STOP - REJECT	1980.0000
48	24	1950.0000	CONTINUE	1960.0000
48	25	1835.2000	STOP - ACCEPT	1857.6000
47	23	1940.0000	STOP - REJECT	1955.1021
47	24	1875.1021	STOP - ACCEPT	1891.4286
46	22	1920.0000	STOP - REJECT	1940.0000
46	23	1907.5511	CONTINUE	1920.0000
46	24	1785.0000	STOP - ACCEPT	1807.5001
45	22	1900.0000	STOP - REJECT	1913.9080
45	23	1829.7873	STOP - ACCEPT	1844.9719
44	21	1880.0000	STOP - REJECT	1900.0000
44	22	1864.8937	CONTINUE	1880.0000
44	23	1733.9131	STOP - ACCEPT	1756.5219
43	21	1860.0000	STOP - REJECT	1872.6147
43	22	1784.0000	STOP - ACCEPT	1797.9481

42	20	1840.0000	STOP - REJECT	1860.0000
42	21	1822.0000	CONTINUE	1840.0000
42	22	1681.8182	STOP - ACCEPT	1704.5455
41	20	1820.0000	STOP - REJECT	1831.2094
41	21	1737.6745	STOP - ACCEPT	1750.2791
40	19	1800.0000	STOP - REJECT	1820.0000
40	20	1778.8373	CONTINUE	1800.0000
40	21	1628.5715	STOP - ACCEPT	1651.4287
39	19	1780.0000	STOP - REJECT	1789.6768
39	20	1690.7318	STOP - ACCEPT	1701.8719
38	18	1760.0000	STOP - REJECT	1780.0000
38	19	1735.3659	CONTINUE	1760.0000
38	20	1574.0000	STOP - ACCEPT	1597.0001
37	18	1740.0000	STOP - REJECT	1747.9988
37	19	1643.0770	STOP - ACCEPT	1652.6142
36	17	1720.0000	STOP - REJECT	1740.0000
36	18	1691.5385	CONTINUE	1720.0000
36	19	1517.8948	STOP - ACCEPT	1541.0527
35	17	1700.0000	STOP - REJECT	1706.1539
35	18	1594.5946	STOP - ACCEPT	1602.3702
34	16	1680.0000	STOP - REJECT	1700.0000
34	17	1647.2973	CONTINUE	1680.0000
34	18	1460.0000	STOP - ACCEPT	1483.3334
33	16	1660.0000	STOP - REJECT	1664.1159
33	17	1545.1429	STOP - ACCEPT	1550.9730
32	15	1640.0000	STOP - REJECT	1660.0000
32	16	1602.5715	CONTINUE	1640.0000
32	17	1400.0000	STOP - ACCEPT	1423.5295
31	15	1620.0000	STOP - REJECT	1621.8529
31	16	1494.5455	STOP - ACCEPT	1498.2165
30	14	1600.0000	STOP - REJECT	1620.0000
30	15	1557.2728	CONTINUE	1600.0000
30	16	1337.5000	STOP - ACCEPT	1361.2501
29	13	1580.0000	STOP - REJECT	1600.0000
29	14	1579.3256	CONTINUE	1580.0000

29	15	1442.5807	STOP - ACCEPT	1443.8417
28	13	1560.0000	STOP - REJECT	1579.6853
28	14	1510.9532	CONTINUE	1560.0000
28	15	1272.0000	STOP - ACCEPT	1296.0001
27	12	1540.0000	STOP - REJECT	1560.0000
27	13	1536.3223	CONTINUE	1540.0000
27	14	1387.3568	CONTINUE	1388.9656
27	15	1086.8966	STOP - ACCEPT	1113.1035
26	12	1520.0000	STOP - REJECT	1538.2925
26	13	1461.8396	CONTINUE	1520.0000
26	14	1202.8572	STOP - ACCEPT	1226.3960
25	11	1500.0000	STOP - REJECT	1520.0000
25	12	1491.9969	CONTINUE	1500.0000
25	13	1327.5525	CONTINUE	1333.3334
25	14	1000.0001	STOP - ACCEPT	1026.6668
24	11	1480.0000	STOP - REJECT	1496.3063
24	12	1409.7747	CONTINUE	1480.0000
24	13	1129.2308	STOP - ACCEPT	1151.1782
23	10	1460.0000	STOP - REJECT	1480.0000
23	11	1446.2919	CONTINUE	1460.0000
23	12	1263.8919	CONTINUE	1275.2000
23	13	905.6000	STOP - ACCEPT	932.8001
22	10	1440.0000	STOP - REJECT	1453.7172
22	11	1355.0919	CONTINUE	1440.0000
22	12	1050.0000	STOP - ACCEPT	1069.8172
21	9	1420.0000	STOP - REJECT	1440.0000
21	10	1399.3918	CONTINUE	1420.0000
21	11	1195.9136	CONTINUE	1213.9131
21	12	801.7392	STOP - ACCEPT	829.5653
20	9	1400.0000	STOP - REJECT	1410.6327
20	10	1297.6527	CONTINUE	1400.0000
20	11	963.6364	STOP - ACCEPT	980.9094
19	8	1380.0000	STOP - REJECT	1400.0000
19	9	1351.2632	CONTINUE	1380.0000
19	10	1122.6918	CONTINUE	1148.5715
19	11	685.7143	STOP - ACCEPT	714.2857
18	8	1360.0000	STOP - REJECT	1367.0685
18	9	1236.9775	CONTINUE	1360.0000
18	10	868.0000	STOP - ACCEPT	882.3542

17	7	1340.0000	STOP - REJECT	1360.0000
17	8	1301.7262	CONTINUE	1340.0000
17	9	1042.7789	CONTINUE	1077.8948
17	10	553.6843	STOP - ACCEPT	583.1579
16	7	1320.0000	STOP - REJECT	1322.9895
16	8	1172.2526	CONTINUE	1320.0000
16	9	760.0000	STOP - ACCEPT	771.0597
15	6	1300.0000	STOP - REJECT	1320.0000
15	7	1250.4719	CONTINUE	1300.0000
15	8	954.0013	CONTINUE	1000.0000
15	9	400.0000	STOP - ACCEPT	430.5882
14	5	1280.0000	STOP - REJECT	1300.0000
14	6	1278.3315	CONTINUE	1280.0000
14	7	1102.2366	CONTINUE	1280.0000
14	8	635.0000	STOP - ACCEPT	642.3755
13	5	1260.0000	STOP - REJECT	1279.3326
13	6	1196.1539	CONTINUE	1260.0000
13	7	853.0438	CONTINUE	912.0000
13	8	216.0000	STOP - ACCEPT	248.0000
12	4	1240.0000	STOP - REJECT	1260.0000
12	5	1232.6374	CONTINUE	1240.0000
12	6	1024.5989	CONTINUE	1240.0000
12	7	485.7143	STOP - ACCEPT	489.0187
11	4	1220.0000	STOP - REJECT	1237.1683
11	5	1136.6197	CONTINUE	1220.0000
11	6	734.4303	CONTINUE	809.2308
11	7	-12.3076	STOP - ACCEPT	21.5384
10	3	1200.0000	STOP - REJECT	1220.0000
10	4	1185.2583	CONTINUE	1200.0000
10	5	935.5250	CONTINUE	1200.0000
10	6	298.8332	CONTINUE	300.0000
10	7	-600.0000	STOP - ACCEPT	-559.9999
9	3	1180.0000	STOP - REJECT	1194.6394
9	4	1071.7432	CONTINUE	1180.0000
9	5	588.2385	CONTINUE	683.6364
9	6	-309.0908	STOP - ACCEPT	-273.1515
8	2	1160.0000	STOP - REJECT	1180.0000
8	3	1136.6973	CONTINUE	1160.0000
8	4	829.9909	CONTINUE	1160.0000
8	5	49.8409	CONTINUE	56.0000
8	6	-1048.0000	STOP - ACCEPT	-1003.9999

7	2	1140.0000	STOP - REJECT	1152.2325
7	3	1000.3834	CONTINUE	1140.0000
7	4	396.5742	CONTINUE	520.0000
7	5	-720.0000	STOP - ACCEPT	-682.0529
6	1	1120.0000	STOP - REJECT	1140.0000
6	2	1087.6438	CONTINUE	1120.0000
6	3	698.4788	CONTINUE	1120.0000
6	4	-301.2846	CONTINUE	-290.0000
6	5	-1700.0000	STOP - ACCEPT	-1650.0000
5	1	1100.0000	STOP - REJECT	1110.7554
5	2	920.8589	CONTINUE	1100.0000
5	3	127.1854	CONTINUE	285.7143
5	4	-1342.8571	STOP - ACCEPT	-1300.3670
4	0	1080.0000	STOP - REJECT	1100.0000
4	1	1040.2864	CONTINUE	1080.0000
4	2	524.0221	CONTINUE	1080.0000
4	3	-852.8428	CONTINUE	-840.0000
4	4	-2760.0000	STOP - ACCEPT	-2699.9999
3	0	1060.0000	STOP - REJECT	1072.0573
3	1	833.7807	CONTINUE	1060.0000
3	2	-302.0968	CONTINUE	-104.0000
3	3	-2432.0000	STOP - ACCEPT	-2378.5685
2	0	1003.4452	CONTINUE	1040.0000
2	1	265.8419	CONTINUE	1040.0000
2	2	-1900.0000	STOP - ACCEPT	-1899.5242
1	0	757.5775	CONTINUE	1020.0000
1	1	-1178.0526	CONTINUE	-960.0000
0	0	-210.2375	CONTINUE	1000.0000

SEQUENTIAL SAMPLING POLICY FOR 100. TOTAL ITEMS.
ALPHA = 5.0000, BETA = 2.0000
GAMMA = 5.0000, MFG. COST = 10.0000

NR OF SAMPLES	NR OF ONES	RISK	DECISION	RISK UNDER OTHER DECISION
100	80	3000.0000	STOP - REJECT	3000.0000
100	81	3000.0000	STOP - ACCEPT	3000.0000
99	79	2980.0000	STOP - REJECT	3000.0000
99	80	2979.4060	STOP - ACCEPT	3000.0000
98	79	2960.0000	STOP - REJECT	2979.5248
98	80	2954.0000	STOP - ACCEPT	2977.0001
97	78	2940.0000	STOP - REJECT	2960.0000
97	79	2932.7273	STOP - ACCEPT	2955.1516
96	77	2920.0000	STOP - REJECT	2940.0000
96	78	2912.6531	STOP - ACCEPT	2934.1374
95	76	2900.0000	STOP - REJECT	2920.0000
95	77	2893.8145	STOP - ACCEPT	2914.0922
94	75	2880.0000	STOP - REJECT	2900.0000
94	76	2876.2500	STOP - ACCEPT	2895.0388
93	75	2860.0000	STOP - REJECT	2877.0000
93	76	2837.8948	STOP - ACCEPT	2861.0527
92	74	2840.0000	STOP - REJECT	2860.0000
92	75	2819.5745	STOP - ACCEPT	2842.1278
91	73	2820.0000	STOP - REJECT	2840.0000
91	74	2802.5807	STOP - ACCEPT	2823.5279
90	72	2800.0000	STOP - REJECT	2820.0000
90	73	2786.9566	STOP - ACCEPT	2805.9889
89	71	2780.0000	STOP - REJECT	2800.0000
89	72	2772.7473	STOP - ACCEPT	2789.5367
88	71	2760.0000	STOP - REJECT	2774.1979
88	72	2720.0000	STOP - ACCEPT	2743.3335

87	70	2740.0000	STOP - REJECT	2760.0000
87	71	2704.9439	STOP - ACCEPT	2727.6405
86	69	2720.0000	STOP - REJECT	2740.0000
86	70	2691.3637	STOP - ACCEPT	2711.7162
85	68	2700.0000	STOP - REJECT	2720.0000
85	69	2679.3104	STOP - ACCEPT	2696.9593
84	67	2680.0000	STOP - REJECT	2700.0000
84	68	2668.8373	STOP - ACCEPT	2683.4003
83	67	2660.0000	STOP - REJECT	2671.0699
83	68	2600.0000	STOP - ACCEPT	2623.5295
82	66	2640.0000	STOP - REJECT	2660.0000
82	67	2588.5715	STOP - ACCEPT	2611.4286
81	65	2620.0000	STOP - REJECT	2640.0000
81	66	2578.7952	STOP - ACCEPT	2598.4855
80	64	2600.0000	STOP - REJECT	2620.0000
80	65	2570.7318	STOP - ACCEPT	2586.8352
79	63	2580.0000	STOP - REJECT	2600.0000
79	64	2564.4445	STOP - ACCEPT	2576.5132
78	63	2560.0000	STOP - REJECT	2567.5556
78	64	2477.5000	STOP - ACCEPT	2501.2501
77	62	2540.0000	STOP - REJECT	2560.0000
77	63	2470.1266	STOP - ACCEPT	2493.1646
76	61	2520.0000	STOP - REJECT	2540.0000
76	62	2464.6154	STOP - ACCEPT	2483.5638
75	60	2500.0000	STOP - REJECT	2520.0000
75	61	2461.0390	STOP - ACCEPT	2475.4047
74	59	2480.0000	STOP - REJECT	2500.0000
74	60	2459.4737	STOP - ACCEPT	2468.7287
73	59	2460.0000	STOP - REJECT	2463.5790
73	60	2352.0000	STOP - ACCEPT	2376.0001
72	58	2440.0000	STOP - REJECT	2460.0000
72	59	2349.1892	STOP - ACCEPT	2372.4325

71	57	2420.0000	STOP - REJECT	2440.0000
71	58	2348.4932	STOP - ACCEPT	2366.6050
70	56	2400.0000	STOP - REJECT	2420.0000
70	57	2350.0000	STOP - ACCEPT	2362.3974
69	55	2380.0000	STOP - REJECT	2400.0000
69	56	2353.8029	STOP - ACCEPT	2359.8592
68	54	2360.0000	STOP - REJECT	2380.0000
68	55	2359.0424	CONTINUE	2360.0000
68	56	2222.8572	STOP - ACCEPT	2247.1430
67	54	2340.0000	STOP - REJECT	2359.2367
67	55	2225.2174	STOP - ACCEPT	2248.5153
66	53	2320.0000	STOP - REJECT	2340.0000
66	54	2230.0000	STOP - ACCEPT	2247.1612
65	52	2300.0000	STOP - REJECT	2320.0000
65	53	2237.3135	STOP - ACCEPT	2247.4627
64	51	2280.0000	STOP - REJECT	2300.0000
64	52	2247.2728	STOP - ACCEPT	2249.6609
63	50	2260.0000	STOP - REJECT	2280.0000
63	51	2253.8183	CONTINUE	2260.0000
63	52	2089.2308	STOP - ACCEPT	2113.8462
62	50	2240.0000	STOP - REJECT	2255.0740
62	51	2097.5000	STOP - ACCEPT	2120.0910
61	49	2220.0000	STOP - REJECT	2240.0000
61	50	2108.5715	STOP - ACCEPT	2124.6429
60	48	2200.0000	STOP - REJECT	2220.0000
60	49	2122.5807	STOP - ACCEPT	2130.1384
59	47	2180.0000	STOP - REJECT	2200.0000
59	48	2137.8108	CONTINUE	2139.6722
59	49	1938.0328	STOP - ACCEPT	1963.9345
58	46	2160.0000	STOP - REJECT	2180.0000
58	47	2146.2487	CONTINUE	2160.0000
58	48	1950.0000	STOP - ACCEPT	1974.6588

57	46	2140.0000	STOP - REJECT	2149.0456
57	47	1965.0848	STOP - ACCEPT	1986.5888
56	45	2120.0000	STOP - REJECT	2140.0000
56	46	1983.4483	STOP - ACCEPT	1998.2584
55	44	2100.0000	STOP - REJECT	2120.0000
55	45	2005.2632	STOP - ACCEPT	2009.8004
54	43	2080.0000	STOP - REJECT	2100.0000
54	44	2023.8723	CONTINUE	2030.7143
54	45	1784.2858	STOP - ACCEPT	1810.7144
53	42	2060.0000	STOP - REJECT	2080.0000
53	43	2035.0979	CONTINUE	2060.0000
53	44	1803.6364	STOP - ACCEPT	1827.8470
52	42	2040.0000	STOP - REJECT	2040.1706
52	43	1826.6667	STOP - ACCEPT	1846.4997
51	41	2020.0000	STOP - REJECT	2040.0000
51	42	1853.5850	STOP - ACCEPT	1866.9183
50	40	2000.0000	STOP - REJECT	2020.0000
50	41	1884.6154	STOP - ACCEPT	1885.5879
49	39	1980.0000	STOP - REJECT	2000.0000
49	40	1907.2399	CONTINUE	1920.0000
49	41	1620.0000	STOP - ACCEPT	1647.0589
48	38	1960.0000	STOP - REJECT	1980.0000
48	39	1921.7920	CONTINUE	1960.0000
48	40	1648.0000	STOP - ACCEPT	1671.7032
47	37	1940.0000	STOP - REJECT	1960.0000
47	38	1929.5896	CONTINUE	1940.0000
47	39	1680.4082	STOP - ACCEPT	1698.2884
46	37	1920.0000	STOP - REJECT	1931.7585
46	38	1717.5000	STOP - ACCEPT	1727.1298
45	36	1900.0000	STOP - REJECT	1920.0000
45	37	1756.2766	CONTINUE	1759.5745
45	38	1408.5107	STOP - ACCEPT	1437.4469
44	35	1880.0000	STOP - REJECT	1900.0000
44	36	1784.3964	CONTINUE	1806.9566
44	37	1441.7392	STOP - ACCEPT	1468.9918

43	34	1860.0000	STOP - REJECT	1880.0000
43	35	1803.5172	CONTINUE	1860.0000
43	36	1480.0000	STOP - ACCEPT	1502.6561
42	33	1840.0000	STOP - REJECT	1860.0000
42	34	1815.0706	CONTINUE	1840.0000
42	35	1523.6364	STOP - ACCEPT	1538.8214
41	33	1820.0000	STOP - REJECT	1820.2884
41	34	1573.0233	STOP - ACCEPT	1577.8568
40	32	1800.0000	STOP - REJECT	1820.0000
40	33	1620.0665	CONTINUE	1628.5715
40	34	1200.0001	STOP - ACCEPT	1230.0001
39	31	1780.0000	STOP - REJECT	1800.0000
39	32	1655.1755	CONTINUE	1690.7318
39	33	1244.3903	STOP - ACCEPT	1271.7188
38	30	1760.0000	STOP - REJECT	1780.0000
38	31	1680.1404	CONTINUE	1760.0000
38	32	1295.0000	STOP - ACCEPT	1316.2778
37	29	1740.0000	STOP - REJECT	1760.0000
37	30	1696.5219	CONTINUE	1740.0000
37	31	1352.3077	STOP - ACCEPT	1364.1278
36	28	1720.0000	STOP - REJECT	1740.0000
36	29	1705.6752	CONTINUE	1720.0000
36	30	1415.7156	CONTINUE	1416.8422
36	31	911.5790	STOP - ACCEPT	944.2107
35	28	1700.0000	STOP - REJECT	1708.7725
35	29	1470.5729	CONTINUE	1489.1892
35	30	962.1622	STOP - ACCEPT	993.3309
34	27	1680.0000	STOP - REJECT	1700.0000
34	28	1515.1838	CONTINUE	1570.0000
34	29	1020.0000	STOP - ACCEPT	1046.8974
33	26	1660.0000	STOP - REJECT	1680.0000
33	27	1548.1471	CONTINUE	1660.0000
33	28	1085.7143	STOP - ACCEPT	1104.8887
32	25	1640.0000	STOP - REJECT	1660.0000
32	26	1571.1757	CONTINUE	1640.0000
32	27	1160.0000	STOP - ACCEPT	1167.3201
31	24	1620.0000	STOP - REJECT	1640.0000

31	25	1585.7748	CONTINUE	1620.0000
31	26	1234.7593	CONTINUE	1243.6364
31	27	616.3637	STOP - ACCEPT	650.9091
30	23	1600.0000	STOP - REJECT	1620.0000
30	24	1593.2616	CONTINUE	1600.0000
30	25	1300.5748	CONTINUE	1337.5000
30	26	681.2500	STOP - ACCEPT	712.9881
29	23	1580.0000	STOP - REJECT	1594.7832
29	24	1357.2239	CONTINUE	1442.5807
29	25	755.4839	STOP - ACCEPT	781.1412
28	22	1560.0000	STOP - REJECT	1580.0000
28	23	1401.7792	CONTINUE	1560.0000
28	24	840.0000	STOP - ACCEPT	855.7740
27	21	1540.0000	STOP - REJECT	1560.0000
27	22	1434.5146	CONTINUE	1540.0000
27	23	935.8621	STOP - ACCEPT	936.8585
26	20	1520.0000	STOP - REJECT	1540.0000
26	21	1457.1187	CONTINUE	1520.0000
26	22	1024.9072	CONTINUE	1044.2858
26	23	251.4287	STOP - ACCEPT	288.5714
25	19	1500.0000	STOP - REJECT	1520.0000
25	20	1471.0924	CONTINUE	1500.0000
25	21	1104.9464	CONTINUE	1166.6667
25	22	333.3334	STOP - ACCEPT	366.0182
24	18	1480.0000	STOP - REJECT	1500.0000
24	19	1477.7634	CONTINUE	1480.0000
24	20	1175.3591	CONTINUE	1304.6154
24	21	427.6924	STOP - ACCEPT	452.0431
23	18	1460.0000	STOP - REJECT	1478.3002
23	19	1235.8400	CONTINUE	1460.0000
23	20	536.0000	STOP - ACCEPT	547.3191
22	17	1440.0000	STOP - REJECT	1460.0000
22	18	1282.5401	CONTINUE	1440.0000
22	19	652.6401	CONTINUE	660.0000
22	20	-315.0000	STOP - ACCEPT	-272.5000
21	16	1420.0000	STOP - REJECT	1440.0000
21	17	1316.7706	CONTINUE	1420.0000
21	18	762.1880	CONTINUE	801.7392
21	19	-228.6956	STOP - ACCEPT	-188.7860
20	15	1400.0000	STOP - REJECT	1420.0000

20	16	1340.2319	CONTINUE	1400.0000
20	17	863.0213	CONTINUE	963.6364
20	18	-127.2726	STOP - ACCEPT	-93.5751
19	14	1380.0000	STOP - REJECT	1400.0000
19	15	1354.4625	CONTINUE	1380.0000
19	16	953.9186	CONTINUE	1148.5715
19	17	-8.5714	STOP - ACCEPT	14.1979
18	14	1360.0000	STOP - REJECT	1360.8469
18	15	1034.0274	CONTINUE	1360.0000
18	16	130.0000	STOP - ACCEPT	135.8021
17	13	1340.0000	STOP - REJECT	1360.0000
17	14	1102.6533	CONTINUE	1340.0000
17	15	272.7412	CONTINUE	291.5790
17	16	-1018.9473	STOP - ACCEPT	-970.5263
16	12	1320.0000	STOP - REJECT	1340.0000
16	13	1155.3971	CONTINUE	1320.0000
16	14	411.0599	CONTINUE	480.0000
16	15	-920.0000	STOP - ACCEPT	-875.4263
15	11	1300.0000	STOP - REJECT	1320.0000
15	12	1194.1272	CONTINUE	1300.0000
15	13	542.4136	CONTINUE	700.0000
15	14	-800.0000	STOP - ACCEPT	-763.4047
14	10	1280.0000	STOP - REJECT	1300.0000
14	11	1220.5954	CONTINUE	1280.0000
14	12	664.6099	CONTINUE	957.5000
14	13	-655.0000	STOP - ACCEPT	-632.1983
13	9	1260.0000	STOP - REJECT	1280.0000
13	10	1236.4367	CONTINUE	1260.0000
13	11	775.8070	CONTINUE	1260.0000
13	12	-480.0000	STOP - ACCEPT	-479.0520
12	9	1240.0000	STOP - REJECT	1243.1691
12	10	874.5134	CONTINUE	1240.0000
12	11	-300.5989	CONTINUE	-268.5714
12	12	-2154.2856	STOP - ACCEPT	-2095.7142
11	8	1220.0000	STOP - REJECT	1240.0000
11	9	958.8565	CONTINUE	1220.0000
11	10	-119.8123	CONTINUE	-12.3076
11	11	-2066.1538	STOP - ACCEPT	-2011.6943
10	7	1200.0000	STOP - REJECT	1220.0000
10	8	1024.1424	CONTINUE	1200.0000
10	9	59.9658	CONTINUE	300.0000
10	10	-1950.0000	STOP - ACCEPT	-1903.9586

9	6	1180.0000	STOP - REJECT	1200.0000
9	7	1072.1036	CONTINUE	1180.0000
9	8	235.2707	CONTINUE	683.6364
9	9	-1798.1817	STOP - ACCEPT	-1767.2758
8	5	1160.0000	STOP - REJECT	1180.0000
8	6	1104.4726	CONTINUE	1160.0000
8	7	402.6373	CONTINUE	1160.0000
8	8	-1600.0000	STOP - ACCEPT	-1594.8364
7	4	1140.0000	STOP - REJECT	1160.0000
7	5	1122.9818	CONTINUE	1140.0000
7	6	558.6008	CONTINUE	1140.0000
7	7	-1377.4847	CONTINUE	-1340.0000
6	4	1120.0000	STOP - REJECT	1129.3637
6	5	699.6961	CONTINUE	1120.0000
6	6	-1135.4740	CONTINUE	-995.0000
5	3	1100.0000	STOP - REJECT	1120.0000
5	4	819.7830	CONTINUE	1100.0000
5	5	-873.3068	CONTINUE	-528.5714
4	2	1080.0000	STOP - REJECT	1100.0000
4	3	913.1887	CONTINUE	1080.0000
4	4	-591.1251	CONTINUE	120.0000
3	1	1060.0000	STOP - REJECT	1080.0000
3	2	979.9133	CONTINUE	1060.0000
3	3	-290.2623	CONTINUE	1060.0000
2	0	1040.0000	STOP - REJECT	1060.0000
2	1	1019.9567	CONTINUE	1040.0000
2	2	27.2816	CONTINUE	1040.0000
1	0	1020.0000	STOP - REJECT	1033.3190
1	1	358.1734	CONTINUE	1020.0000
0	0	689.0867	CONTINUE	1000.0000

APPENDIX D

UNIQUE BETA DETERMINATION BY FIRST TWO MOMENTS

Proof that the mean and variance of a Beta distribution are sufficient to uniquely determine the parameters of the density.

1. $f_{P|A,B} = \frac{(A+1)!}{A!\ (B-A)!}\ P^{A}(1-p)^{B-A}$, $A > -1$, $B-A > -1$, $0 \le p \le 1$.

2. $\mu = \frac{A+1}{B+2}$

3. $\sigma^2 = \frac{(A+1)(B-A+1)}{(B+2)^2(B+3)}$

4. From 2, $A = \mu(B+2) -1$ for every $A > -1$.

5. From 3 and 4, $\sigma^2 = \frac{\mu(B+2)[B-\mu(B+2)+2)}{(B+2)^2\ (B+3)} = \frac{\mu(B+2)[(B+2)(1-\mu)]}{(B+2)^2\ (B+3)}$

6. From 1, $B - A > -1$ and $A > -1$ imply $B > -2$ and $\mu \neq 0$, so that 5 becomes $\sigma^2 = \frac{\mu(1-\mu)}{B+3}$

7. Since $(B+3) > 1$, i.e.: unequal 0, $B+3 = \frac{\mu(1-\mu)}{\sigma^2}$

8. $B = \frac{\mu(1-\mu)}{\sigma^2} -3$

9. From 4 and 8, $A = \mu(B+2) -1$

10. Since, from 1 and 2, $\mu \neq 0, 1$, A and B are uniquely determined by μ and σ^2 as shown in 8 and 9.

APPENDIX E

FORTRAN SIMULATION FOR STOCHASTIC 'P'

```
C
C                   FORTRAN PROGRAM
C                   PROGRAM NAME - STOSIM
C
C     SIMULATION OF A STOCHASTIC PROCESS IN WHICH THE PARAMETER 'P' IS
C     DIMINISHING WITH TIME ACCORDING TO THE MODEL P(T+1) = C*P(T). 'C'
C     IS ASSUMED TO HAVE THE DENSITY FUNCTION (D+1)C**D. AN A PRIORI
C     BETA DISTRIBUTION OF 'P' IS ASSUMED.  - W. C. MCCORMICK, JR.
C
    1 FORMAT (6F10.4)
    2 FORMAT(70H     T         A          PSI        PMU        CMU        EPMU
     1      CRAN    )
    4 FORMAT(70H     WT        X          AMDA       PVAR       CVAR       EPVAR
     1      PRAN    /)
    5 FORMAT (I6, I9, F12.4, 4F10.4)
    6 FORMAT (F8.5, I7, F12.4, 3F10.7, F10.4,/)
   10 READ 1, X, AMDA, PSI, D, TMAX, AAA
C
C     'X' IS THE NUMBER OF SAMPLES TO BE DRAWN EACH TIME. AMDA AND PSI
C     ARE THE PARAMETERS OF THE A PRIORI DISTRIBUTION OF 'P'. 'D' IS THE
C     COEFFICIENT OF THE ASSUMED DENSITY OF 'C'. TMAX IS THE NUMBER OF
C     TIMES SAMPLING IS TO BE DONE. AAA IS ANY TEN DIGIT NUMBER USED FOR
C     RANDOM NUMBER GENERATION.
C
      IF (X) 500, 500, 11
   11 CMU = (D + 1.0)/(D + 2.0)
      CVAR = CMU/((D + 2.0) * (D + 3.0))
      EPMU = (PSI + 1.0)/(AMDA + 2.0)
      EPVAR= (PMU*(1.0 - PMU))/(AMDA + 3.0)
      JMAX = TMAX
      L = X
      PUNCH 2
      PUNCH 4
   20 DO 50 J = 1, JMAX
      J1 = J - 1
      T = J1
      RANC = RANDOM(AAA)
      XPON = 1.0/(D + 1.0)
      CRAN = RANC**XPON
C
C     'CRAN' IS THE RANDOMLY GENERATED VALUE OF 'C' THAT EXISTS NOW.
C
      PPRIME = CRAN*EPMU
C
C      'PPRIME' IS THE 'P' WHICH EXISTS AT THIS TIME.
C
      QPRIME = (1.0 - PPRIME)
      FAC = 1.0
      PPROB = (1.0 - PPRIME)**X
      PRAN = RANDOM (AAA)
C
C     'PRAN' IS THE RANDOM NUMBER USED TO DETERMINE THIS 'A'.
C
      PTOA = 1.0
      QTOB = PPROB
      DO 35 I = 1, L
      A = I
      PTOA = PTOA*PPRIME
      QTOB = QTOB/QPRIME
      FAC = (FAC*(X - A + 1.0))/A
      PROBA = PPROB + (FAC*PTOA*QTOB)
```

```
      IF (PROBA - PRAN) 35, 38, 36
   35 PPROB = PROBA
   36 IF((PROBA - PRAN) - (PRAN - PPROB)) 38, 38, 37
   37 A = A - 1.0
   38 WTMU = X/(X+AMDA + 2.0)
      PMU = (A + PSI + 1.0)/(X + AMDA + 2.0)
      PVAR = (PMU*(1.0 - PMU))/(X + AMDA + 3.0)
      EPMU = PMU*CMU
      EPVAR = PVAR*(CVAR + (CMU**2)) + (CVAR*(PMU**2))
      PUNCH 5, J1, A, PSI, PMU, CMU, EPMU, CRAN
      PUNCH 6, WTMU, X, AMDA, PVAR, CVAR, EPVAR, PRAN
C
C     AMDA AND PSI ARE THE PARAMETERS OF THE 'P' DENSITY YIELDING THIS
C     RESULT.  CMU AND CVAR ARE THE MEAN AND VARIANCE OF THE A PRIORI
C     OF 'C'. PMU AND EPVAR ARE THE ESTIMATES OF MEAN AND VARIANCE FOR
C     THE NEXT 'P'. WTMU AND WTVAR ARE THE RELATIVE WEIGHTS OF THE MEAN
C     AND VARIANCE OF THIS SAMPLE ONLY IN RELATION TO THE PREVIOUS EX-
C     PERIENCE.
C
      AMDA = ((EPMU*(1.0 - EPMU))/EPVAR) - 3.0
   50 PSI = (EPMU*(AMDA + 2.0)) - 1.0
      GO TO 10
  500 STOP
      END
```

APPENDIX F

SIMULATION RESULTS

T WT	A X	PSI AMDA	PMU PVAR	CMU CVAR	EPMU EPVAR	CRAN PRAN
0 .5000C	81 100	98.0000 98.0000	.9000 .0004477	.9166 .0058760	.8249 .0051384	.8385 .3183
1 .78680	71 100	21.3548 25.0967	.7345 .0015222	.9166 .0058760	.6733 .0044583	.9758 .0137
2 .67413	60 100	31.5462 46.3378	.6238 .0015712	.9166 .0058760	.5718 .0036167	.9234 .3589
3 .59990	57 100	37.1421 64.6940	.5707 .0014609	.9166 .0058760	.5231 .0031504	.9599 .7311
4 .56121	49 100	39.9054 76.1838	.5045 .0013950	.9166 .0058760	.4625 .0026764	.8541 .8235
5 .52115	43 100	41.4975 89.8828	.4455 .0012807	.9166 .0058760	.4084 .0022503	.9867 .3445
6 .48456	42 100	42.4457 104.3698	.4140 .0011699	.9166 .0058760	.3795 .0019972	.9948 .6721
7 .46103	38 100	43.3696 114.9042	.3797 .0010809	.9166 .0058760	.3481 .0017620	.9424 .6939
8 .43900	23 100	43.4835 125.7877	.2962 .0009112	.9166 .0058760	.2715 .0012868	.7480 .2647
9 .39568	23 100	40.4762 150.7283	.2551 .0007489	.9166 .0058760	.2338 .0010161	.9895 .2193
10 .36322	20 100	39.9991 173.3146	.2215 .0006241	.9166 .0058760	.2030 .0008166	.9524 .3613

VITA

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Doctor of Philosophy

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