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AN ALGORITHM FOR THE SOLUTION OF THE DISTRIBUTION PROBLEM OF PROBABILISTIC LINEAR PROGRAMMING

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ABSTRACT

Although a considerable amount of work has been done in the area of probabilistic linear programming, a method does not exist which can compute the distribution function of the optimal value in a practical computational manner. This is particularly true for those problems involving more than a few random variables.

This study proposes an algorithm based upon a "best" choice criterion for entering and leaving variables as a method for computing the distribution function. These criteria are similar to those that apply to regular deterministic linear programming except that they apply to the cases where the coefficients of the objective function or the restrictions of the constraints are random variables.

Additionally, a modification to the algorithm is developed which will yield an approximation for the distribution function without the requirement of a complete investigation of all possible bases.

Two computer programs based on the algorithm have been developed that compute the information that is required for solution of the problem.

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INTRODUCTION

Linear programs with some of their coefficients subject to random variation have been considered in several forms and under several different names. Among these are probabilistic linear programming, stochastic linear programming, chance constrained programming, linear programming under uncertainty, and recourse programming. In effect, these are different problems selected from the general class of linear programming problems that are non-deterministic. This general class of non-deterministic linear programs (i.e., those having random variables for some of their coefficients) will hereafter be referred to in this paper as probabilistic linear programming.

The initial interest in probabilistic linear programs arose from a problem dealing with the allocation of aircraft to routes when the demand for their service was unknown. This problem was considered by G. B. Dantzig and A. R. Ferguson (8) under the conditions that the demand distribution was discrete, and later by S. E. Elmaghraby (11) for the case where the demand distribution is continuous.

The next efforts in this area consisted of attempts by several authors, notably Dantzig, Madansky, and Charnes and Cooper, to eliminate the effect of the random variables by optimizing the expected value or the variance of the objective function or selecting alternatives such that the constraints would be violated with only a small probability.

Later developments have fallen into one of two categories which, according to Dempster (9), comprise the entire class of linear programming problems that have random variables for their parameters. These two categories are defined essentially by the timing of the decision making process relative to the realization of the random variables.

If the decision is to be made before the behavior of the random variables is known, the category is known as the "Here-and-Now" approach in the terminology of Madansky (13). This category has been investigated under different names by various authors, notably Dempster (9, 10), Walkup and Wets (21), Wets (22), and Williams (24). Their general approach has been to select some criterion, usually optimizing the expected value of the objective function, and including a penalty function that represents the cost associated with making an incorrect decision. This leads to the development of equivalent convex programs which in general are non-linear and serve as approximations to the original probabilistic linear programming problem.

If the decision is to be made after the behavior of the random variables is known, we have the type of problem that Madansky (13) referred to as the "Wait-and-See" problem. This problem was classified by Tintner (19) as the "Distribution Problem" and subsequently was investigated by Tintner (20, 21) and Sengupta and Tintner (17, 18). This is the kind of problem with which this paper will be concerned. It consists of determining the distribution of the optimum value of the objective function when the distribution of the random variables in the problem is known.

Tintner and Sengupta's work was based mainly upon problems related to agricultural economics. Their basic approach was to take all of the possible combinations of the values of the random variables and compute the optimum value of the deterministic linear program that is defined by each combination. They then used the method of sample moments to fit a probability distribution function to these values. The following two problems exist with this method: (1) the number of linear programs to be solved increases rapidly as the number of possible values for these random variables increase, and (2) the distributions so derived are approximations.

Bereanu (1, 2) developed a method for determining the distribution of the optimal value of the objective function when the coefficients of the objective function or the constraint restrictions are random variables. Bereanu assumes that the random variables have finite lower and upper bounds and proceeds by setting each random variable at its lower bound and solving the resulting deterministic linear program. He determines the range over which the optimal solution remains feasible by using the sensitivity analysis technique of parametric linear programming. He changes bases and applies the sensitivity analysis technique to the new basis and repeats this process until all optimal solutions have been investigated. Using the information so obtained, Bereanu computes the distribution of the optimum value of the objective function.

The objective of this research will be to develop an algorithm to determine the distribution of the optimum value of the objective function that does not depend upon solving a sequence of linear programs. This algorithm will be based upon Bereanu's results, but will offer improvements in the method of changing bases. Additionally, a modification to the algorithm will be developed which will allow an approximation of the optimum value of the objective function without the complete enumeration of all of the possible bases.

The algorithm is based upon methods that are similar to the simplex technique of deterministic linear programming and enables the investigator to determine the value of the objective function and the value of the basic variables at each iteration.

CHAPTER II

AN ALGORITHM FOR LINEAR PROGRAMS THAT ARE FUNCTIONS OF RANDOM VARIABLES

Consider the general maximization problem of linear programming represented in the following form

$$\text{Maximize } X_o = CX$$
$$\text{subject to } AX \leq b$$
$$X \geq 0$$

where A is an (m×n) matrix; b is (m×1), X is (n×1), and C is (1×n). Adding slack variables to the constraints, the problem can be represented by

$$\text{Maximize } X_o = CX$$
$$\text{subject to } AX = b$$
$$X \geq 0$$

where A is [m×(m+n)]; b is (m×1), x is [(m+n)×1]; C is [1×(m+n)].

When some (or all) of the parameters of this problem (i.e., C, b, or A) are random variables, the problem becomes a Stochastic (or Probabilistic) Linear Programming Problem.

Two cases of this general problem will be considered in this paper. Case I will deal with the problem when the vector C is a vector of random variables and all other parameters are known deterministically. Case II will deal with the problem when b is a vector of random variables and all other parameters are known deterministically.

Case I: C-vector is a Vector of Random Variables

When the C vector is the only parameter that is a random variable, the problem becomes essentially one of determining the probability that a feasible basis is optimal.

Since the number of feasible bases is finite, the problem of determining the distribution of the optimum value of the objective function becomes one of determining the distribution of the optimal value of the objective function for a particular basis, the probability that this particular basis is optimal, and summing the product of these two values over all possible bases to obtain the distribution of the optimal value of the objective function.

One might question the statement that the optimal value of the objective function occurs at an extreme point (a basis) when the coefficients of the objective function are random variables. To see that this is true, consider the following theorem:

Theorem 1: The optimum solution of the linear programming problem,

$$\text{Maximize } X_o = CX$$
$$\text{subject to } (A,I)X = P_o$$
$$X \geq 0$$

where C is a vector of independent random variables whose distributions are known in advance, when it is finite must occur at an extreme point (a basis) of the feasible space defined by the constraint set.

Proof of the theorem:

1. Let X_i, i = 1, ..., k be the extreme points of the feasible space and let $X_o^* = \underset{i}{\text{Max}}\ CX^{(i)} = CX^{(m)}$ where $X^{(m)}$ is the extreme point at which the value of objective function is a maximum. Note: Since this is a "Wait-and-See" Problem, let the random vector C assume a particular value denoted by C'.
2. Suppose there exists a point $X^{(1)}$ which is not an extreme point but which can yield a better value of the objective function, i.e., $X_o^{(1)} = C'X^{(1)} \geq X_o^* = C'X^{(m)}$.
3. Since $X^{(1)}$ is not an extreme point, it can be expressed as a convex combination of the extreme points of the feasible space.

$$X^{(1)} = \sum_{i=1}^{k} \lambda_i X^{(i)},\ \lambda_i \geq 0;\quad \sum_{i=1}^{k} \lambda_i = 1$$

4. Therefore: $X_o^{(1)} = C'X^{(1)} = C' \sum_{i=1}^{k} \lambda_i X^{(i)}$

$$X_o^{(1)} = \sum_{i=1}^{k} C_i' \lambda_i X^{(i)} = \sum_{i=1}^{k} \lambda_i (C_i' X^{(i)})$$

5. By hypothesis: $CX^{(m)} = \underset{i}{Max}\ CX^{(i)}$, so $CX^{(m)} \geq C'X^{(i)}$.

6. $X_o^{(1)} = \sum_{i=1}^{k} \lambda_i (C_i'X^{(i)}) \leq \sum_{i=1}^{k} \lambda_i C\ X^{(m)}$, $X_o^{(1)} \leq CX^{(m)} \sum_{i=1}^{n} \lambda_i$

7. $X_o^{(1)} \leq CX^{(m)} = X_o^*$.

8. This contradicts the assumption that a better value for the solution can occur at a point which is not an extreme point and the proof of the theorem is complete.

Let $f_\ell(X_o)$ be the distribution of the objective function for the ℓth basis and P_ℓ be the probability that the ℓth basis is optimal. The distribution of the objective function will then be given by

$$f(X_o) = \sum_{\ell=1}^{q} P_\ell f_\ell(X_o)$$

Since the number of bases is finite, this sum exists and is finite.

In order to determine P_ℓ, let us examine the conditions required for a given basis to be optimal. For the maximization problem this condition is satisfied for a particular basis when all non-basic variables are such that their coefficients in the objective function row are non-negative. Let C_B be the coefficient corresponding to the current basic solution, let B be the basis matrix from the matrix A corresponding to the current basic solution, and let P_j be the column from A corresponding to a particular non-basic variable. The non-negativity condition stated above becomes: for any X_j

that is a non-basic variable, its coefficient $(C_B B^{-1} P_j - C_j)$ must be non-negative [i.e., $(C_B B^{-1} P_j - C_j) \geq 0$]. Let $S_\ell = \{C | C_B B^{-1} P_j - C_j) \geq 0,\ c \in C\}$. The set S_ℓ defines the space over which a particular basis ℓ is optimal. Thus

$$P_\ell = \int_{S_\ell} f(c)\ dc$$

$$f(X_o) = \sum_{\ell=1}^{q} P_\ell f_\ell(X_o)$$

These are the conditions stated by Bereanu (1) expressed in the notation used by Taha (20). They form the basis of the solution algorithm.

The algorithm depends upon two criteria for choosing the entering and leaving variables as a means of changing bases. These criteria are respectively the optimality and feasibility criteria. Since for this problem, the C-vector is the only random variable in the problem, the feasibility criterion remains the same as for deterministic linear programming.

To reiterate, the feasibility criterion is based upon selecting the leaving variable so that all remaining basic variables remain non-negative (≥ 0). Using matrix notation, let X_B be a vector corresponding to a basic feasible solution, and let the vectors of the A matrix corresponding to X_B be denoted by $P_1, P_2, \ldots, P_m$. Let $B = (P_1, P_2, \ldots, P_m)$ denote the basis matrix where B is square (m×m) and non-singular. Let P_o denote the right hand side of the contraint equations. The linear programming problem can now be expressed as

$$\text{Maximize } X_o = CX$$
$$\text{subject to } (A,I)X = P_o$$
$$X \geq 0$$

Since X_B is a basic feasible solution and B is its corresponding matrix as defined above, we have that

$$BX_B = P_o$$

and thus $$X_B = B^{-1}P_o.$$

Also, $$BX_B = P_o = \sum_{k=1}^{m} X_k P_k$$

so that $$X_B = B^{-1}P_o$$

and $$x_k = (B^{-1}P_o)_k$$

where the subscript k denotes the kth element of the vector X_B.

Let P_j be a vector from the remaining n non-basic vectors of (A,I), and let x_j be its corresponding variable. Then

$$\sum_{k=1}^{m} \alpha_k^{\ j} P_k = P_j$$

where $\alpha_k^{\ j}$ is a scalar with a least one non-zero value. It follows that

$$B\alpha^j = P_j$$

or $$\alpha^j = B^{-1}P_j.$$

Let θ be any real number. Thus $\theta B\alpha^j = \theta P_j$. Since $BX_B = P_o$, then by subtraction, $B(X_B - \theta\alpha^j) + \theta P_j = P_o$.

The new vector, X', where

$$X' = \begin{pmatrix} X_B - \theta\alpha^j \\ \theta \end{pmatrix} = \begin{pmatrix} B^{-1}P_o - \theta\alpha^j \\ \theta \end{pmatrix}$$

is a solution to the linear program with $X_j = \theta$. However, it contains (m+l) variables and is non-basic, so θ must be selected so that one of the former basic variables is set to zero. Additionally, all of the elements of X' must remain non-negative. These conditions may be expressed mathematically as,

$$(X_k - \theta\alpha_k{}^j) \geq 0, \; k = 1, 2, \ldots, m$$

and

$$X_j = \theta \geq 0$$

This yields the following selection criteria: Select

$$\theta = \min_k \{\frac{X_k}{\alpha_k{}^j}, \; \alpha_k{}^j > 0\}$$

and call this value θ*. Therefore

$$\theta^* = \min_k \; \frac{(B^{-1}P_o)_k}{\alpha_k{}^j}, \; \alpha_k{}^j > 0$$

Since this criteria for selecting the leaving variable depends upon B, B^{-1}, P_o and $\alpha_k{}^j$, and none of these depends upon C, then the feasibility criterion for the case where C is the only random variable remains the same as for ordinary deterministic linear programming.

In deterministic linear programming, the criterion for selecting the entering variable, the optimality criterion, is based upon selecting the most promising non-basic variable X_j which when introduced into solution will cause improvement in the objective function. When the coefficients of the objective function of the linear program are random variables, the optimality criterion of ordinary linear programming does not apply. A modified version of the optimality criterion suitable for use with the probabilistic case will now be developed.

Consider the linear program stated earlier,

$$\text{Maximize } X_o = CX$$
$$\text{subject to } AX \leq b$$
$$X \geq 0$$

where C is a vector of random variables. Adding slack variables and renaming the vector b as P_o, the problem can be written as

$$\text{Maximize } X_o = CX$$
$$\text{subject to } (A,I)X = P_o$$
$$X \geq 0.$$

Let X_B be a basic feasible solution and $C_B = (C_1, C_2, \ldots, C_m)$ be the corresponding coefficients of the objective function. For the current basis, $X_o = C_B X_B$ and $BX_B = P_o \rightarrow X_B = B^{-1}P_o$. It follows that, $X_o = C_B X_B = C_B B^{-1} P_o$. Now if X_j is the entering variable, then

$$X_B' = \begin{pmatrix} X_B - \theta\alpha^j \\ \theta \end{pmatrix}$$

and

$$\alpha^j = B^{-1}P_j$$

Let X_o' be the new value of the objective function corresponding to X_B'. The only reason to introduce the variable X_j into the solution would be to improve the solution (i.e., $X_o'>X_o$). We have that

$$X_o = C_B X_B$$

$$X_o' = (C_B, C_j)\begin{pmatrix} X_B - \theta\alpha^j \\ \theta \end{pmatrix}$$

$$X_o' = C_B X_B - \theta C_B \alpha^j + \theta C_j$$

$$X_o' = C_B X_B - \theta(C_B\alpha^j - C_j)$$

$$X_o' = X_o - \theta(C_B\alpha^j - C_j)$$

Therefore, the only way for $X_o' > X_o$ is for $\theta(C_B\alpha^j - C_j)$ to be negative, and since $\theta > 0$ this implies that $(C_B\alpha^j - C_j) < 0$.

When the C vector is a vector of random variables, this condition, $(C_B\alpha^j - C_j) < 0$, does not have a meaning in the absolute sense, so a probability statement will be developed to represent this condition.

The first condition will be that a variable will be considered as a candidate for an entering variable as long as the following probability statement holds:

$$P[(C_B B^{-1} P_j - C_j) < 0] > 0$$

The selection criterion thus becomes: from among those non-basic variables whose $P[(C_B B^{-1} P_j - C_j) < 0] > 0$, select as the candidate for the entering variable that variable X_j corresponding to the greatest probability value.

This modified optimality criterion and the regular feasibility criterion function to change bases until all feasible bases of the linear program have been investigated.

At each feasible basis the linear program will have a probability of being optimal. This is the probability P_ℓ mentioned earlier in this chapter.

The conditions that determine P_ℓ are contained in the optimality criterion. A particular basis will be optimal as long as there are no non-basic variables whose coefficients are negative. This is the condition $(C_B B^{-1} P_j - C_j) \geq 0$, and, since C is a vector of random variables, this condition defines the set

$$S_\ell = \{C | (C_B B^{-1} P_j - C_j) \geq 0\}$$

The probability of a particular basis α being optimal is now given by

$$P_\ell = \int_{S_\ell} f(c)dc$$

<u>The Case I Algorithm</u>

Step 1: a. Select a starting basic feasible solution. The normal condition will be to select the slack variables as the starting solution.

b. Set the solution index to correspond to the starting solution.

Step 2: Select the candidate for the entering variable using the modified optimality criterion.

Step 3: Select the candidate for the leaving variable using the regular feasibility criterion.

Step 4: a. Set the solution index to correspond to the new solution.

b. Check the solution list to determine if this solution has been investigated before.

1. If solution has been investigated before, return to Step 2 and select the next most promising candidate.

2. If solution has not been investigated before, proceed to Step 5.

Step 5: Carry out a primal simplex iteration in order to update the tableau.

Step 6: Use the information contained in the objective function row to compute P_ℓ and the probabilities required to evaluate the new non-basic variables. Return to Step 2.

Step 7: The algorithm terminates when all feasible solutions have been investigated.

The solution index mentioned in Steps 1b, 4a, and 4b of the algorithm is a reference system that is used to prevent cycling in the algorithm. The modified optimality criterion which selects the entering variable at each iteration can produce as the new basis a basis that has previously been considered. This is an undesirable feature since no new information is obtained by returning to a previous basis, and this may cause the algorithm to cycle and not consider all of the feasible bases.

The solution index prevents the algorithm from cycling by creating a reference number for each basis when it is determined by the selection of the entering and leaving variables. This reference number is compared to the list of reference numbers corresponding to previously considered bases and if the current basis has been investigated before, it will not be checked again. The algorithm proceeds by selecting the next most promising candidate for an entering variable using the modified optimality criterion and repeating the process of assigning a solution index and checking the list of reference numbers.

The reference system uses a number that consists of as many digits as there are problem variables. The digits in the number are set to a value of one if the corresponding variable is a member of the basis or to a value of zero if the variable is not a member. Consider an example with four variables and two constraints. The solution index for the basis that consists of variables X_2 and X_4 would be the number 0101.

The complete procedure is shown by a flowchart in Figure 1.

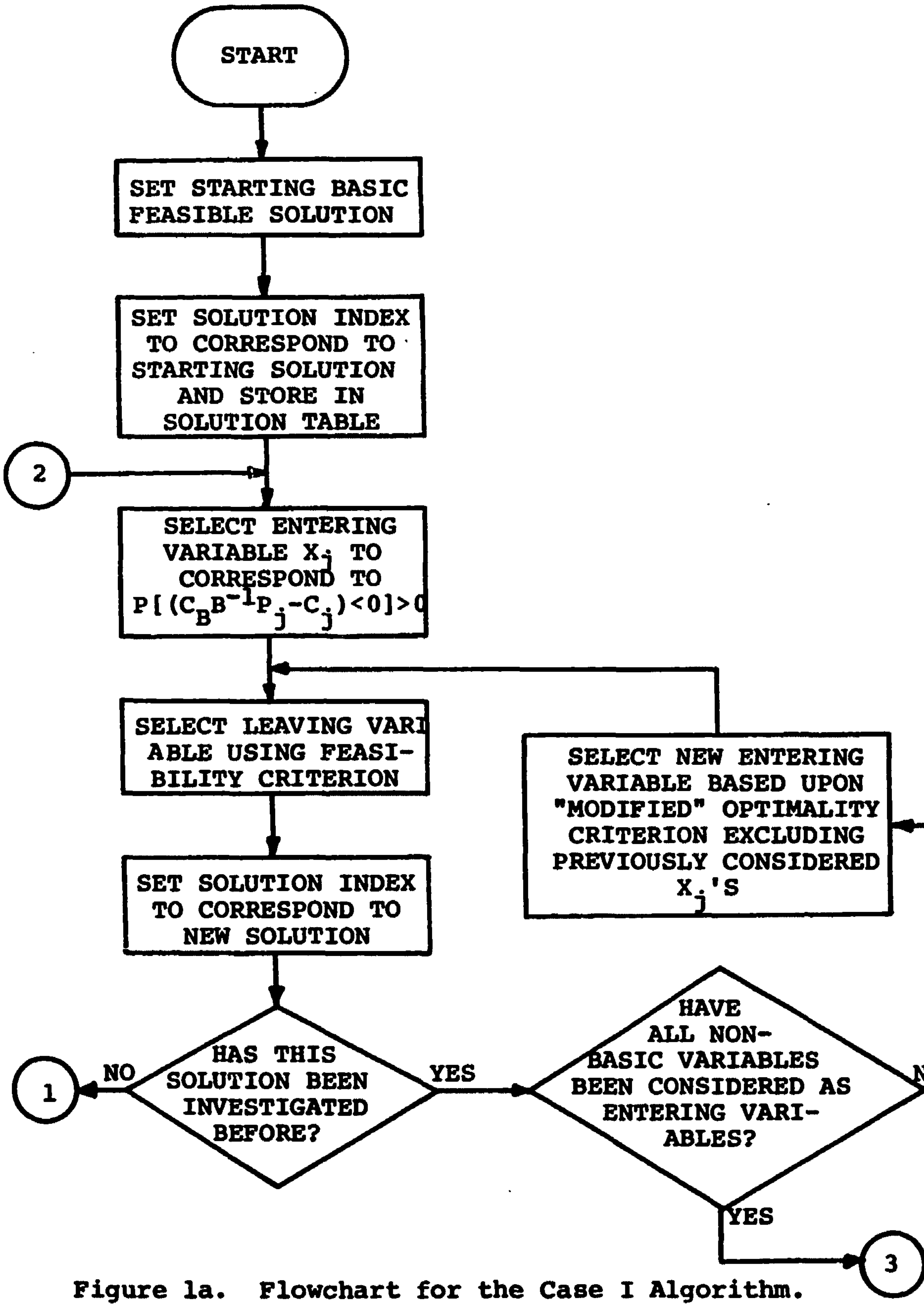
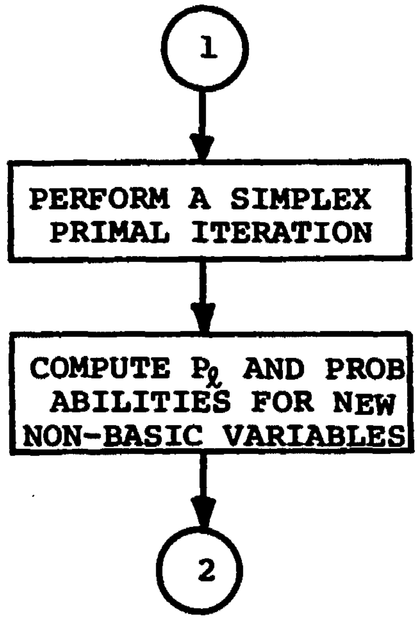


Figure 1a. Flowchart for the Case I Algorithm.



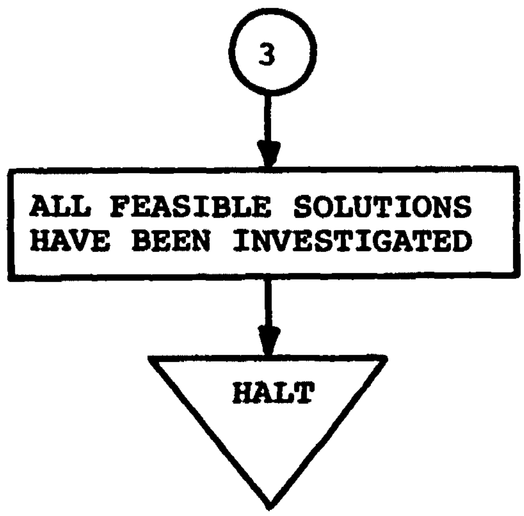


Figure 1b. Case I Flowchart Continued.

Example Problem

$$\text{Maximize } X_o = C_1X_1 + C_2X_2$$

$$\text{subject to } X_1 + 2X_2 \leq 10$$

$$2X_1 + X_2 \leq 10$$

$$X_1, X_2 \geq 0$$

where C_1 and C_2 are independent random variables.

Adding slack variables and rewriting,

$$\text{Maximize } X_o = C_1X_1 + C_2X_2 + 0\ X_3 + 0\ X_4$$

$$\text{subject to } X_1 + 2X_2 + X_3 = 10$$

$$2X_1 + X_2 + X_4 = 10$$

$$X_1, X_2, X_3, X_4 \geq 0$$

Assume that the distributions of C_1 and C_2 are distributed exponentially with parameters $\lambda_1 = 1/10$, $\lambda_2 = 1/10$, respectively. That is,

$$f(C_1) = \lambda_1 e^{-\lambda_1 C_1} = \frac{1}{10} e^{-\frac{1}{10} C_1},\ 0 \leq C_1 \leq \infty$$

$$f(C_2) = \lambda_2 e^{-\lambda_2 C_2} = \frac{1}{10} e^{-\frac{1}{10} C_2},\ 0 \leq C_2 \leq \infty$$

Tableau 1:

X_o	X_1	X_2	X_3	X_4	
1	$-C_1$	$-C_2$	0	0	0
	1	2	1	0	10
	2	1	0	1	10

$X_o^1 = 0$

$P_1 = 0$

Solution Index: 0011

Current Basic Variables: (X_3, X_4)

Selection of Entering Variable:

$$P[-C_1 < 0] = P[C_1 > 0] = \int_0^{\infty} f(C_1)dC_1 = 1$$

$$P[-C_2 < 0] = P[C_2 > 0] = \int_0^{\infty} f(C_2)dC_2 = 1$$

The probability values are equal, so the selection at this point is arbitrary. Select X_2 as the entering variable.

Selection of Leaving Variable:

Ratios: $10/2 = 5 \rightarrow$ select X_3 as being $10/1 = 10$

New Solution Index: 0101

Tableau 2:

X_o	X_1	X_2	X_3	X_4	
	$C_2/2 - C_1$	0	$C_2/2$	0	$5C_2$
	1/2	1	1/2	0	5
	3/2	0	-1/2	1	5

$X_o^2 = 5C_2$

$$P_2 = \iint_{S_2} f(C_1, C_2)dC_1dC_2 = \iint_{S_2} f(C_1)f(C_2)dC_1dC_2$$

$S_2 = \{C | C_2/2 \geq 0 \text{ and } C_2/2 - C_1 \geq 0\}$

$$P_2 = \int_0^{\infty} f(C_2) \int_0^{C_2/2} f(C_1)dC_1dC_2$$

$$P_2 = \int_0^\infty \frac{1}{10} e^{-\frac{1}{10}C_2} \int_0^{C_2/2} \frac{1}{10} e^{-\frac{1}{10}C_1} dC_1 dC_2$$

$$P_2 = \int_0^\infty \frac{1}{10} e^{-\frac{1}{10}C_2} \left[-e^{-\frac{1}{10}C_1}\right]_0^{C_2/2} dC_2$$

$$P_2 = \int_0^\infty \frac{1}{10} e^{-\frac{1}{10}C_2} \left[-e^{-C_2/20} + 1\right] dC_2$$

$$P_2 = \int_0^\infty \left[\frac{1}{10} e^{-\frac{1}{10}C_2} - \frac{1}{10} e^{-\frac{3}{20}C_2}\right] dC_2$$

$$P_2 = \int_0^\infty \frac{1}{10} e^{-\frac{1}{10}C_2} dC_2 - \frac{2}{3} \int_0^\infty \frac{3}{20} e^{-\frac{3}{20}C_2} dC_2$$

$$P_2 = \left[-e^{-\frac{1}{10}C_2}\right]_0^\infty + \frac{2}{3}\left[e^{-\frac{3}{20}C_2}\right]_0^\infty$$

$$P_2 = [0 + 1] + 2/3[0 - 1] = 1 - 2/3 = 1/3$$

Selection of Entering Variable:

$P[X_3 \text{ will enter}] = P[C_2/2 < 0] = 0$

$P[X_1 \text{ will enter}] = P[C_2/2 - C_1 < 0]$

$P[C_2/2 - C_1 < 0] = \int_0^\infty f(C_2) \int_{C_2/2}^\infty f(C_1) dC_1 dC_2$

$P[C_2/2 - C_1 < 0] = 1 - P[C_2/2 - C_1 \geq 0] = 1-1/3 = 2/3$

Select X_1 as the entering variable.

Selection of Leaving Variable:

Ratios: $\frac{5}{1/2} = 10$

$\frac{5}{3/2} = 10/3 \rightarrow$ select X_4 as leaving

New Solution Index: 1100

Tableau 3:

X_o	X_1	X_2	X_3	X_4	
	0	0	$\frac{2}{3}C_2-\frac{1}{3}C_1$	$\frac{2}{3}C_1-\frac{1}{3}C_2$	$10/3(C_1 + C_2)$
	0	1	2/3	-1/3	10/3
	1	0	-1/3	2/3	10/3

$$X_o^{\,3} = 10/3 \;(C_1 + C_2)$$

$$P_3 = \iint_{S_3} f(C_1, C_2)\; dC_1 dC_2$$

$$S_3 = \{C \,|\, (2/3\; C_2 - 1/3\; C_1 \geq 0) \text{ and } (2/3\; C_1 - 1/3\; C_2 \geq 0)\}$$

$$P_3 = \int_0^\infty f(C_1) \int_{C_1/2}^{2C_1} \frac{1}{10}\, e^{-\frac{1}{10}C_2}\; dC_2$$

$$= \int_0^\infty \frac{1}{10}\, e^{-\frac{1}{10}C_1} \Big[-e^{-\frac{1}{10}C_2}\Big]_{C_1/2}^{2C_1}$$

$$P_3 = \int_0^\infty (-\frac{1}{10}\, e^{-\frac{3}{10}C_1} + \frac{1}{10}\, e^{-\frac{3}{20}C_1})\; dC_1 = 1/3$$

Selection of Entering Variable:

$$P[X_3 \text{ will enter}] = P[2/3\; C_2 - 1/3\; C_1 < 0]$$

$$P[X_4 \text{ will enter}] = P[2/3\; C_1 - 1/3\; C_2 < 0]$$

These two probabilities are both equal to 1/3 so the choice is again arbitrary. However, if X_4 is chosen as the entering variable, the leaving variable will be X_1. This would cause the new basis to be (X_2, X_4) which has already been investigated in Tableau 2. The Solution

Index would be set to 0101, and since this basis has been considered previously, the algorithm would proceed by selecting X_3 as the entering variable.

Selection of Leaving Variable:

Ratios: $3.33/0.667 = 5 \rightarrow$ leaving variable is X_2.

New Solution Index: 1010

Tableau 4:

X_o	X_1	X_2	X_3	X_4	
	0	$C_1/2-C_2$	0	$C_1/2$	$5C_1$
	0	3/2	1	-1/2	5
	1	1/2	0	1/2	5

$$X_o^4 = 5C_1$$

$$P_4 = \iint_{S_4} f(C_1, C_2)\, dC_1 dC_2$$

$$S_4 = \{C \mid (C_1/2 - C_2) \geq 0 \text{ and } C_1/2 \geq 0\}$$

$$P_4 = \int_0^{\infty} f(C_1) dC_1 \int_0^{C_1/2} f(C_2) dC_2$$

$$P_4 = 1/3$$

The algorithm terminates at this point because all feasible bases have been investigated. For this problem, the only non-basic variable in Tableau 4 with a probability of improving the objective function is X_2. The feasibility criterion would select X_3 as the leaving variable yielding the new basis (X_1, X_2) which has been investigated previously.

The distribution of the optimal value of the objective function may now be calculated.

Case II. b-Vector is a Vector of Random Variables

Again consider the linear programming problem in the following form:

$$\text{Maximize } X_o = CX$$
$$\text{subject to } (A,I)X = b$$
$$X \geq 0$$

and let $P_o = b$. This yields the form

$$\text{Maximize } X_o = CX$$
$$\text{subject to } (A,I)X = P_o$$
$$X \geq 0$$

In order to develop the algorithm for this case, it is necessary to consider the optimality and feasibility criteria. The optimality criterion remains the same as for deterministic linear programming. Let C_B be the coefficient vector corresponding to the current basis that is defined by the basis matrix B. It follows that

$$X_o = C_B B^{-1} P_o$$

and the only reason for changing basis is if the new basis will cause an improvement in the value of the objective function. Letting X_o' denote the value of the objective function for the new basis, this condition yields,

$$X_o' > X_o$$

$$X_o' = (C_B, C_j)\begin{pmatrix} X_B - \theta\alpha^j \\ \theta \end{pmatrix}$$

$$X_o' = C_B X_B - C_B\alpha^j + \theta C_j$$

$$X_o' = X_o - \theta(C_B\alpha^j - C_j)$$

Since θ is positive the condition $X_o' > X_o$ requires that $(C_B\alpha^j - C_j) < 0$, $C_B B^{-1} P_j < C_j$. The vector P_o is the only random variable in this problem, so the optimality criterion remains the same as that for deterministic linear programming.

Considering the feasibility criterion, let the vectors of current basis B be denoted by $(P_1, P_2, \ldots, P_m)$. Then,

$$BX_B = P_o$$

$$\sum_{k=1}^{m} x_k P_k = P_o, \text{ and}$$

$$X_B = B^{-1} P_o$$

Let P_j be a new vector from the remaining non-basic vectors in (A,I). It follows that,

$$\sum_{k=1}^{m} \alpha_k^j P_k = P_j$$

and letting $\alpha^j = (\alpha_1^j, \alpha_2^j, \ldots, \alpha_m^j)$

$$B\alpha^j = P_j$$

Multiplying both sides by θ we obtain $\theta B\alpha^j = \theta P_j$, and subtracting $BX_B = P_o$ yields

$$B(X_B - \theta\alpha^j) + \theta P_j = P_o$$

$$BX' = P_o - \theta P_j$$

$$X' = B^{-1}(P_o - \theta P_j) = B^{-1}P_o - \theta B^{-1}P_j$$

$$X' = \begin{pmatrix} B^{-1}P_o - \theta\alpha^j \\ \theta \end{pmatrix}$$

The feasibility criterion would dictate that the new basis be selected so that its components will be non-negative. This means that $x_k - \theta\alpha^j \geq 0$ for $k = 1, \ldots, m$ and $x_j = \theta \geq 0$. Since

$$X_B - \theta\alpha^j = B^{-1}P_o - \theta\alpha^j \geq 0$$

then $$\theta \leq \frac{B^{-1}P_o}{\alpha^j} \text{ and } \alpha^j = B^{-1}P_j$$

which yields $$\theta \leq \frac{B^{-1}P_o}{B^{-1}P_j}$$

The selection of θ is therefore dependent upon B^{-1}, P_o, and P_j. P_o is the vector of random variables, so the choice of θ must be accomplished by some mechanism that accounts for this factor. This consideration suggests selecting as the leaving variable that variable which has the lowest probability of creating an infeasibility if it is selected as the leaving variable.

For this case where the b vector is a vector of random variables, the distribution of the optimum value of the objective function is determined by considering the value of the objective function at each optimal basis and taking into account the likelihood of feasibility of each basis.

For the purpose of considering the feasibility condition, consider the matrix form of the linear programming problem. At any iteration the current values of the problem are given in the following matrix-tableau form:

$$\left(\begin{array}{c|c|c} 1 & C_B B^{-1}A - C_I & C_B B^{-1} - C_{II} \\ \hline 0 & B^{-1}A & B^{-1} \end{array}\right) \begin{pmatrix} X_o \\ X_I \\ X_{II} \end{pmatrix} = \begin{pmatrix} C_B B^{-1} P_o \\ B^{-1} P_o \end{pmatrix}$$

where C_B is an defined previously and C_I and C_{II} represent a division of the C vector with C_{II} representing the coefficients corresponding to the starting solution of the problem.

At any iteration the value of the basic variable is given by

$$X_B = B^{-1}P_o$$

In the case where P_o is a vector of random variables, the problem becomes one of determining if a basis is feasible. In order to determine the distribution of the optimal value of the objective function, one must find an optimal basis and then determine the probability that the basis is feasible. The feasibility condition is satisfied as long as all basic variables are non-negative. This means that $X_B \geq 0$ and since $X_B = B^{-1}P_o$, then $B^{-1}P_o \geq 0$.

Let $Q_\ell = \{b | B^{-1}P_o \geq 0\}$. The set Q_ℓ is the set of all values of the random variables for which the problem is feasible. The set of inequalities so defined determines the space over which a particular optimal basis is feasible. Letting q_ℓ denote the probability that an optimal basis is feasible, this yields

$$q_\ell = \int_{Q_\ell} f(b)db$$

This analysis forms the basis of the algorithm for the case where the b vector is a vector of random variables.

The Case II Algorithm

Step 1: a. Place the problem in tableau form and select a starting solution.

b. Check the optimality of the problem using the regular Optimality Criterion.

c. If the problem is optimal, go to Step 3; otherwise go to Step 2.

Step 2: a. Select the candidate for the entering variable using the optimality criterion.

b. Select the candidate for the leaving variable based upon the modified Feasibility Criterion, i.e., select the variable j with the least value for the probability statement $P[B^{-1}P_o/B^{-1}P_j < 0]$.

c. Perform a Primal Simplex iteration and go to Step 1, part b.

Step 3: a. Record the current value of the objective function.

b. Set the solution index to correspond to the solution.

c. Compute the probability that the current optimal basis is feasible. That is,

$$q_\ell = \int_{Q_\ell} f(b)\,db$$

d. Compute the probability that the basic variables are negative and proceed to Step 4.

Step 4: a. Select the leaving variable to be that variable corresponding to the variable that has the greatest value for the probability statement, $P[(B^{-1}P_o)_i < 0]$.

b. Select the entering variable from among the nonbasic variables according to the Optimality Criterion of the Dual Simplex Method.

c. Set the solution index to correspond to the new solution.

d. If all optimal feasible solutions have been investigated, go to Step 5. If not, go to Step 4e.

e. Check the solution list to determine if this solution has been investigated before.

1. If the solution has been investigated before, return to Step 4a and select the next most promising candidate.

2. If the solution has not been checked before, update the tableau by performing a Dual Simplex iteration. Return to Step 3.

Step 5: If all optimal feasible basis have been considered, the algorithm terminates. (This will be detected when optimality has been achieved and the processes of changing basis to consider feasibility causes the problem to become non-optimal.)

The complete procedure is shown by a flow chart in Figure 2.

Example Problem

Consider the dual problem of the example that was presented for Case I.

$$\text{Minimize } Y_o = 10Y_1 + 10Y_2$$

$$\text{subject to } Y_1 + 2Y_2 \geq C_2$$

$$2Y_1 + Y_2 \geq C_2$$

$$Y_1, Y_2 \geq 0$$

Changing the sense of the inequalities and adding slack variables gives

$$\text{Minimize } Y_o = 10Y_1 + 10Y_2 + 0 \cdot Y_3 + 0 \cdot Y_4$$

$$\text{subject to } -Y_1 - 2Y_2 + Y_3 = -C_1$$

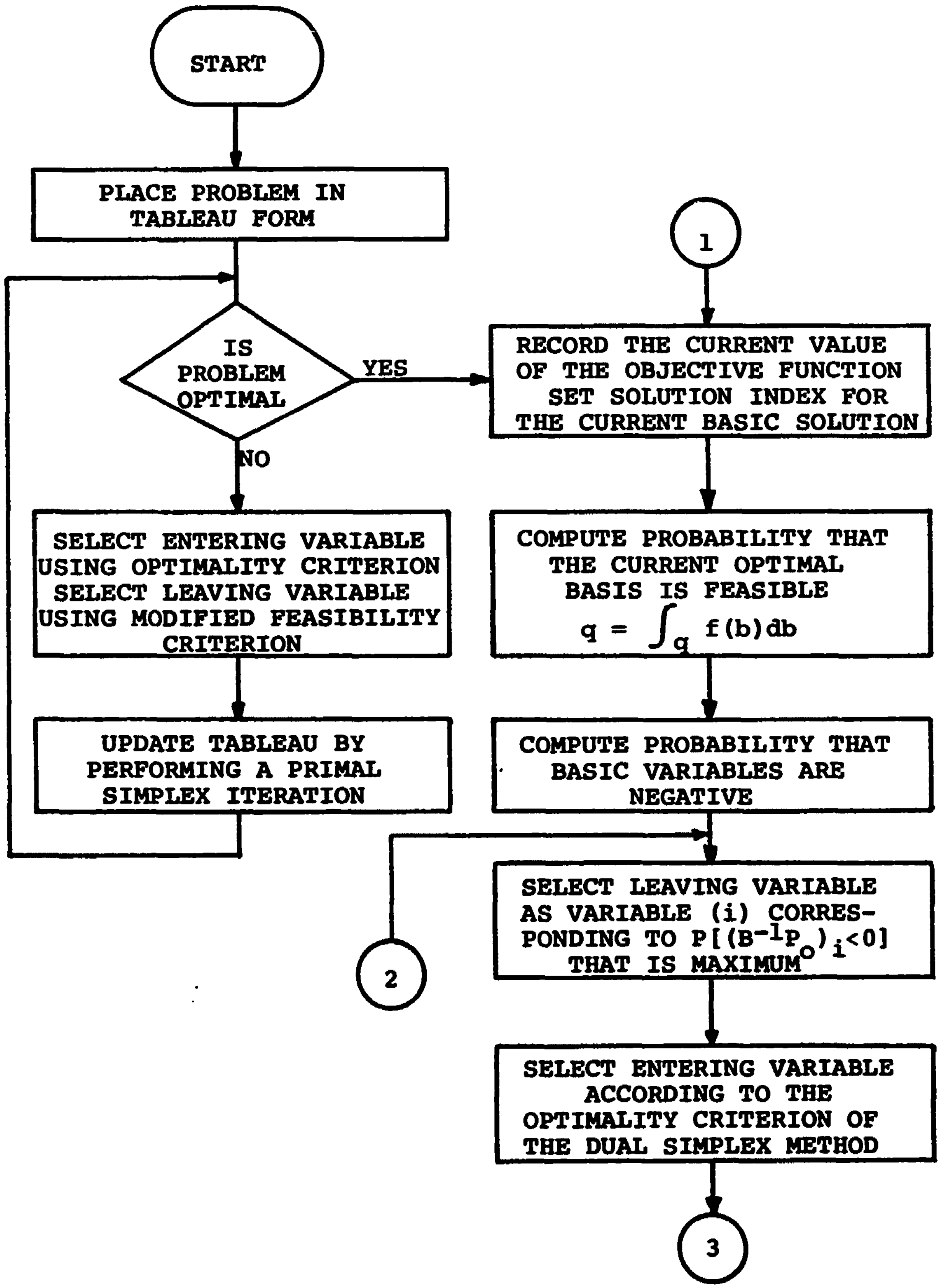


Figure 2a. Flowchart for the Case II Algorithm.

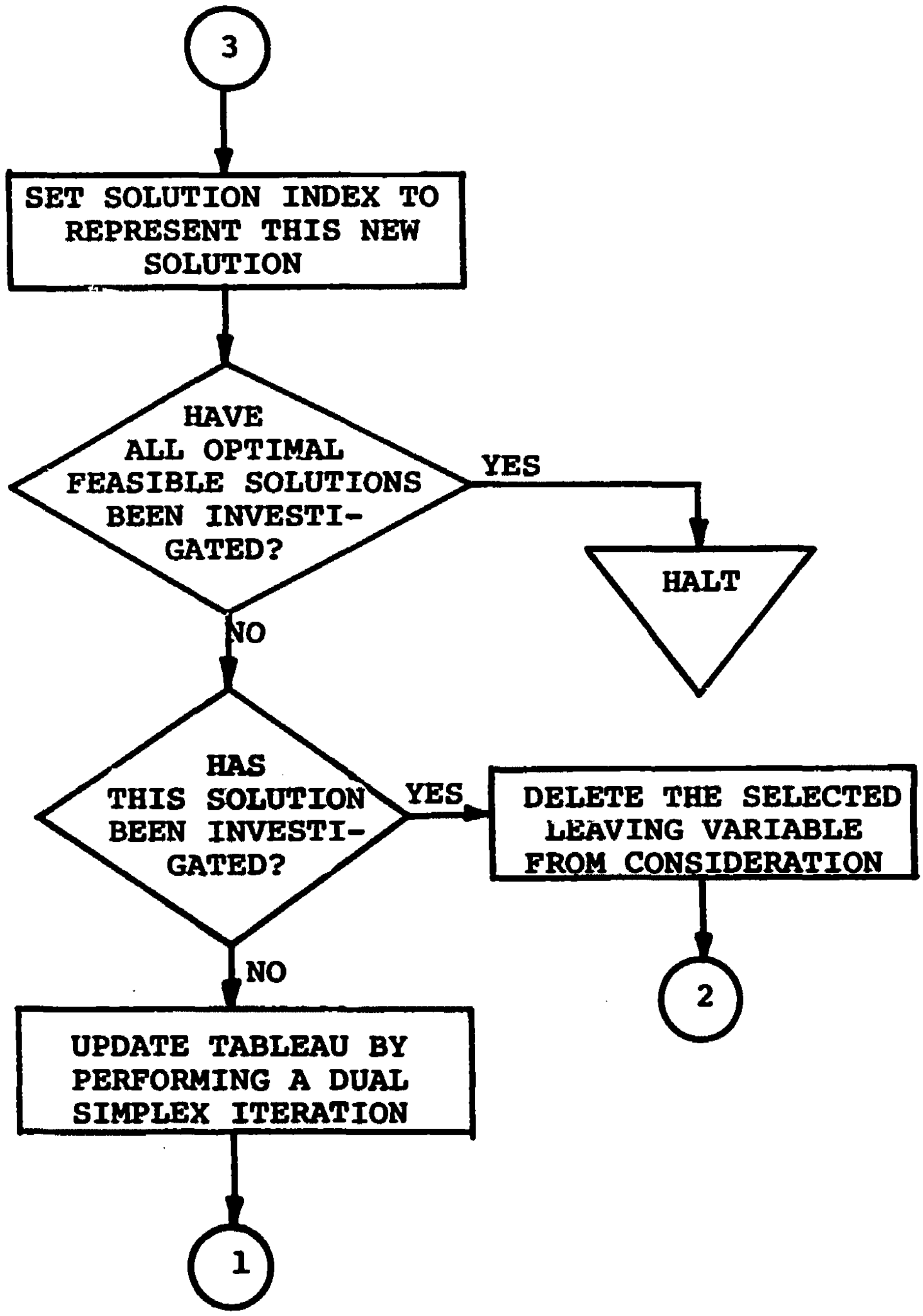


Figure 2b. Case II Flowchart Continued.

$$-2Y_1 - Y_2 + Y_4 = -C_2$$

$$Y_1, Y_2, Y_3, Y_4 \geq 0$$

Tableau 1:

Y_o	Y_1	Y_2	Y_3	Y_4	
1	-10	-10	0	0	0
	-1	-2	1	0	$-C_1$
	-2	-1	0	1	$-C_2$

Since the problem is a minimization problem, it is optimal. The current basis is (Y_3, Y_4). The set Q_1 defining the space over which the basis is feasible is given by

$$Q_1 = \{C| -C_1 \text{ and } -C_2 \geq 0\}$$

Thus
$$q_1 = \iint_{Q_1} f(C_1, C_2)\, dC_1 dC_2$$

$$q_1 = \int_{-\infty}^{0} f(C_2) \int_{-\infty}^{0} f(C_1) dC_1 dC_2 = 0$$

$Y_o^1 = 0$

$q_1 = 0$

Solution Index: 0011

Selection of Leaving Variable:

$$P[-C_1 < 0] = P[C_1 > 0] = 1$$

$$P[-C_2 < 0] = P[C_2 > 0] = 1$$

The probability values are equal, so the selection at this point is arbitrary. Select Y_4 as the leaving variable.

Selection of Entering Variable:

Ratios: $-10/-1 = 10$

$-10/-2 = 5 \rightarrow$ select Y_1 as entering.

New Solution Index: 1010

Tableau 2:

Y_o	Y_1	Y_2	Y_3	Y_4	
1	0	-5	0	-5	$5C_2$
	0	-3/2	1	-1/2	$C_2/2 - C_1$
	1	1/2	0	-1/2	$C_2/2$

$$Y_o^2 = 5C_2$$

$$Q_2 = \{C|(C_2/2 - C_1) \text{ and } C_2/2 \geq 0\}$$

$$q_2 = \iint_{Q_2} f(C_1,C_2)\, dC_1 dC_2$$

$$q_2 = \int_0^\infty f(C_2) \int_0^{C_2/2} f(C_1) dC_1 dC_2$$

$$q_2 = \int_0^\infty f(C_2)\left[-e^{-\frac{1}{10}C_1}\right]_0^{C_2/2} dC_2$$

$$q_2 = \left[-e^{-\frac{1}{10}C_2}\right]_0^\infty + 2/3\left[e^{-\frac{3}{20}C_2}\right]_0^\infty$$

$$q_2 = 1 - 2/3 = 1/3$$

Selection of Leaving Variable:

$P[Y_1 \text{ will leave}] = P[C_2/2 < 0] = 0$

$P[Y_3 \text{ will leave}] = P[(C_2/2 - C_1) < 0]$

$$P[(C_2/2 - C_1) < 0] = \int_0^{\infty} f(C_1) \int_{C_2/2}^{\infty} f(C_1)dC_1dC_2 = 2/3$$

Select Y_3 as the leaving variable.

Selection of Entering Variable:

Ratios: $-5/-\frac{3}{2} = 10/3 \rightarrow$ select Y_2 as entering

$-5/-\frac{1}{2} = 10$

New Solution Index: 1100

Tableau 3:

Y_o	Y_1	Y_2	Y_3	Y_4	
1	0	0	-10/3	-10/3	$10/3(C_1 + C_2)$
	0	1	-2/3	1/3	$2/3C_1 - 1/3C_2$
	1	0	1/3	-2/3	$2/3C_2 - 1/3C_1$

$Y_o^3 = 10/3\ C_1 + 10/3\ C_2$

$Q_3 = \{C|\ (2/3\ C_1 - 1/3\ C_2) \text{ and } (2/3\ C_2 - 1/3\ C_1) \geq 0\}$

$$q_3 = \iint_{Q_3} f(C_1,C_2)\ dC_1dC_2$$

$$q_3 = \int_0^{\infty} f(C_2) \int_{C_2/2}^{2C_2} f(C_1)dC_1dC_2$$

$q_3 = 1/3$ (Note calculation is the same as the previous example.)

Selection of Leaving Variable:

$$P[Y_1 \text{ will leave}] = P[(2/3\ C_2 - 1/3\ C_1) < 0]$$

$$P[Y_2 \text{ will leave}] = P[(2/3\ C_1 - 1/3\ C_2) < 0]$$

$$P[Y_1 \text{ will leave}] = \int_0^\infty f(C_1) \int_0^{C_2/2} f(C_2)dC_1dC_2 = 1/3$$

$$P[Y_2 \text{ will leave}] = \int_0^\infty f(C_2) \int_0^{C_2/2} f(C_1)dC_1dC_2 = 1/3$$

These values are equal, so the selection is arbitrary; however, if Y_2 is selected as the leaving variable the entering variable becomes Y_3, and this will produce a new basis of (Y_1, Y_3) which has been considered before. The solution index feature of the algorithm will detect this and choose Y_1 as the leaving variable. The entering variable becomes Y_4 and this will produce a new basis of (Y_2, Y_4), which has not been investigated previously.

New Solution Index: 0101

Tableau 4:

Y_o	Y_1	Y_2	Y_3	Y_4	
1	-5	0	-5	0	$5C_1$
	1/2	1	-1/2	0	$C_1/2$
	-3/2	0	-1/2	1	$C_1/2 - C_2$

$$Y_o^4 = 5C_1$$

$$Q_4 = \{C | C_1/2 \text{ and } (C_1/2 - C_2) \geq 0\}$$

$$q_4 = \iint_{Q_4} f(C_1, C_2)\ dC_1dC_2$$

$$q_4 = \int_0^{\infty} f(C_1) \int_0^{C_1/2} f(C_2)\, dC_2 dC_1$$

$$q_4 = 1/3$$

Any further attempt to change bases causes the problem to become non-optimal or to try to return to a previously investigated basis. Thus, the algorithm terminates with the following results:

$$Y_o^{\ 1} = 0,\ Y_o^{\ 2} = 5C_2,\ Y_o^{\ 3} = \frac{10}{3}(C_1 + C_2),\ Y_o^{\ 4} = 5C_1$$

$$q_1 = 0,\ q_2 = 1/3,\ q_3 = 1/3,\ q_4 = 1/3$$

These results are the same as those that were obtained when the primal problem was solved using the Case I Algorithm.

In order to determine the distribution of optimal Y_o we use the following relationship

$$f(Y_o) = \sum_{i=1}^{q} q_i f(Y_o^{\ i})$$

that was developed earlier in this chapter. Since $q_1 = 0$ and $Y_o^{\ 1} = 0$ with probability one the product of $q_1 f(Y_o^{\ 1})$ equals zero and does not contribute to the distribution of Y_o. The remaining terms are $[q_2 f(Y_o^{\ 2}) + q_3 f(Y_o^{\ 3}) + q_4 f(Y_o^{\ 4})]$. Since for this example $q_2 = q_3 = q_4$, then

$$f(Y_o) = 1/3\ [f(Y_o^{\ 2}) + f(Y_o^{\ 3}) + f(Y_o^{\ 4})]$$

Let us now examine the nature of Y_o^2 and Y_o^4. Notice that $Y_o^2 = 5C_2$ and that C_2 is distributed exponentially with $\lambda = 1/10$. Since Y_o^2 is the product of a constant and a random variable, its distribution is a function of the random variable. This condition yields the distribution of Y_o^2 which is exponential with $\lambda = 1/50$. Notice also, that $Y_o^4 = 5C_1$ and since C_1 is distributed exponentially with $\lambda = 1/10$, we have that Y_o^4 is distributed exponentially with $\lambda = 1/50$.

The random variable Y_o^3 is a function of both C_1 and C_2 and is in particular the product of a constant times the sum of C_1 and C_2. Since C_1 and C_2 are exponentially distributed with equal parameters the distribution of their sum is a gamma distribution with parameters $r = 2$ and $\lambda = 1/10$. Thus Y_o^3 is distributed as a gamma with $r = 2$ and $\lambda = 3/100$.

The distribution of optimal Y_o is given by the algebraic sum of these distributions. Therefore

$$f(Y_o) = 1/3\ [f(Y_o^2) + f(Y_o^3) + f(Y_o^4)]$$

where $f(Y_o^2)$ is gamma with $r = 1$, $\lambda = 1/50$,

$f(Y_o^3)$ is gamma with $r = 2$, $\lambda = 3/100$,

$f(Y_o^4)$ is gamma with $r = 1$, $\lambda = 1/50$.

The range of values over which these distributions are valid is given by the conditions that define the range of the random variables that define each Y_o^i. Since $Y_o^2 = 5C_2$ and range of C_2 is 0 to ∞ the range on Y_o^2 is 0 to ∞ and similarly the range of Y_o^3 and Y_o^4 is 0 to ∞. This yields the following:

$$f(Y_o^2) = \frac{1}{50} e^{-\frac{1}{50}u}, \; 0 \leq u \leq \infty$$

$$f(Y_o^3) = \frac{3}{100 \quad (2)} \cdot \frac{3}{100}v \, e^{-\frac{3}{100}v}, \; 0 \leq v \leq \infty$$

$$f(Y_o^4) = \frac{1}{50} e^{-\frac{1}{50}w}, \; 0 \leq w \leq \infty$$

In order to check that $f(Y_o)$ is a probability distribution function consider

$$\int_{\text{All } Y_o} f(Y_o) dY_o$$

This integral should be equal to a value of one if $f(Y_o)$ is a probability distribution function. This is verified as follows:

$$f(Y_o)dY_o = \int 1/3 \; f(Y_o^2) \; dY_o^2 + \int 1/3 \; f(Y_o^3) \; dY_o^3 + \int 1/3 \; f(Y_o^4) \; dY_o^4$$

$$f(Y_o)dY_o = 1/3 \quad \int_0^\infty f(Y_o^2) \; dY_o^2 + \int_0^\infty f(Y_o^3) \; dY_o^3 + \int_0^\infty f(Y_o^4) \; dY_o^4$$

$$f(Y_o)dY_o = 1/3 \; [1 + 1 + 1] = 1$$

and this demonstrates that the function $f(Y_o)$ is a probability distribution function.

CHAPTER III

MODIFICATION OF THE ALGORITHM FOR CALCULATION OF AN APPROXIMATION OF THE DISTRIBUTION OF THE OPTIMAL VALUE OF THE OBJECTIVE FUNCTION

The algorithm developed in Chapter II for Case I considers all of the feasible bases of a problem. In order to develop an approximation algorithm let us investigate how the algorithm functions to cover the probability space defined by the random variables.

Notice that the probability space over which the bases are optimal is a subset of R_n. The algorithm developed in Chapter II divides this space into subspaces by cutting the original space with hyperplanes that yield convex subspaces. This is easily seen since the space R_n is convex and the hyperplanes are convex and it follows that the intersection of convex sets is convex. The only common points are the boundaries of the subspace which are the hyperplanes and these are of measure zero as far as the probability functions are concerned.

To show that the algorithm proceeds contiguously around the probability space it is necessary to show that one of the hyperplanes that defines the subspace over which the i^{th} basis is optimal also is one of the hyperplanes that defines the subspace over which the $(i + 1)^{st}$ basis is optimal. This is equivalent to showing that these two subspaces have a common boundary. These boundaries are defined by the coefficients of the non-basic variables at each basis.

Let X_j and X_r be the entering and leaving variables respectively at the i^{th} iteration of the simplex method. By definition X_j is a non-basic variable at the i^{th} iteration and a basic variable at the $(i + 1)^{st}$ iteration and X_r is basic at the i^{th} iteration and non-basic at the $(i + 1)^{st}$ iteration. Consider the general linear programming problem in the form:

$$\text{Maximize } X_o = CX$$
$$\text{subject to } (A,I)X = P_o$$
$$X \geq 0$$

Let $C = (C_I, C_{II})$, where C_{II} represents the vector corresponding to the coefficients of the starting solution X_{II}. Let

$$M = \begin{pmatrix} 1 & -C_B \\ 0 & B \end{pmatrix}$$
$$M^{-1} = \begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

Then, according to Taha (20), it follows that the following matrix tableau is equivalent to the tableau form of the simplex method:

$$\left(\begin{array}{c|c|c|c} 1 & C_B B^{-1} A - C_I & C_B B^{-1} - C_{II} & C_B B^{-1} P_o \\ \hline 0 & B^{-1} A & B^{-1} & B^{-1} P_o \end{array} \right)$$

The coefficients of X_j and X_r can be found by producing this tableau at any iteration.

Let the coefficients of X_j and X_r be denoted by $(Z_j - C_j)$ and $(Z_r - C_r)$ respectively. Since $Z_j = C_B B^{-1} P_j$ and $Z_r = C_B B^{-1} P_r$, these coefficients are found in the top row of the tableau as indicated by the general form of the coefficients. Since the simplex pivot method is equivalent to the above method, consider the argument based upon the simplex technique.

Theorem 2: The Case I algorithm of Chapter II investigates the probability space of optimal bases in a contiguous manner.

The coefficient of X_j at the i^{th} iteration is $(Z_j - C_j)$ and the coefficient of X_r at the i^{th} iteration is equal to zero since it is a basic variable (see Appendix A). At the $(i + 1)^{st}$ iteration, the coefficient of X_r is $(Z_r - C_r)$, and the coefficient of X_j is equal to zero. According to the simplex technique, the pivot element selected for the i^{th} iteration is determined by the selection of X_j and X_r and is denoted by α_r^j. The simplex pivot method then creates the $(i + 1)^{st}$ iteration by replacing each element $a_{k\ell}$ at the i^{th} iteration that is not in the row and column of the pivot by

$$a_{k\ell}^{(i+1)} = \frac{a_{k\ell}^{(i)} \cdot \alpha_r^j - a_{kj}^{(i)} \cdot a_{r\ell}^{(i)}}{\alpha_r^j}$$

The coefficient of X_r at the $(i+1)^{st}$ iteration is thus given by

$$(Z_r - C_r)^{(i+1)} = \frac{(Z_r - C_r)^{(i)} \cdot \alpha_r^{\,j} - (Z_j - C_j)^{(i)} \cdot 1}{{}^{j}_{r}}$$

$$(Z_r - C_r)^{(i+1)} = \frac{0 \cdot \alpha_r^{\,j} - (Z_j - C_j)^i}{\alpha_r^{\,j}}$$

$$(Z_r - C_r)^{(i+1)} = -\frac{1}{\alpha_r^{\,j}} (Z_j - C_j)^i$$

Therefore, the coefficient of the non-basic variable X_r at the $(i + 1)^{st}$ iteration is equal to a constant K times the coefficient of the non-basic variable X_j at the i^{th} iteration where $K = -1/\alpha_r^{\,j}$.

Since the hyperplanes that form the boundaries of the optimal space for each basis are defined by the condition $(Z_k - C_k) \geq 0$ for the non-basic variables k, the condition $(Z_j - C_j) \geq 0$ at the i^{th} iteration and $(Z_r - C_r) = K(Z_j - C_j) \geq 0$ at the $(i + 1)^{st}$ iteration define the same hyperplane. This shows that the probability space over which the bases at the i^{th} and $(i + 1)^{st}$ iteration are optimal have a common boundary and that the algorithm presented in Chapter II proceeds contiguously around the probability space, and Theorem 2 is proved.

Letting M denote the number of feasible bases and noticing that

$$\sum_{i=1}^{M} P_i = 1$$

we have a method readily available for modifying the algorithm to give an approximation algorithm. Instead of continuing to iterate until all feasible bases have been considered, the accumulated probability of feasible bases being optimal will be used to cause the iterative process to stop.

Let γ denote a variable whose value is to be selected by the decision maker and reflects his evaluation of the worth of complete information. The criterion for continuing iterations will be to continue selecting entering and leaving variables to compute the $(s + 1)^{st}$ iteration until

$$(1 - \sum_{i=1}^{s} P_i) \leq \gamma, \text{ where } s \leq M$$

Since this modified algorithm will operate to produce $f^*(X_o)$ which is an approximation of $f(X_o)$, it is only necessary to consider what occurs as γ approaches zero. This means that

$$(1 - \sum_{i=1}^{s} P_i) \leq \gamma$$

as γ approaches zero gives

$$(1 - \sum_{i=1}^{s} P_i) \leq 0$$

or $$1 \leq \sum_{i=1}^{s} P_i$$

but since $$\sum_{i=1}^{M} P_i = 1$$

it follows

$$\sum_{i=1}^{s} p_i = 1$$

and therefore s equals M which means that

$$f^*(X_o) = \sum_{\ell=1}^{s=M} p_\ell f_\ell (X_o^{\ell}) = f(X_o)$$

The Case I algorithm modified for the approximation algorithm becomes:

Step 1: a. Select a value for γ and select a starting basic feasible solution.

b. Same as before.

Step 2: a. If $(1 - \sum_{i=1}^{s} p_i) \leq \gamma$, proceed to Step 7.

b. If not, select the candidate for the entering variable using the modified optimality criterion.

Step 3: Same as before.

Step 4: a. Same as before

b. Same as before except for

1. If solution has been investigated before return to Step 2b and select the next most promising candidate.

Step 5: Same as before.

Step 6: Use the information contained in the objective function row to compute p_ℓ, $\sum_{\ell=1}^{s} p_\ell$, and the probabilities required to evaluate the new non-basic variables. Return to Step 2a.

Step 7: The algorithm terminates when the γ limit is violated.

The algorithm in Chapter II for Case I and the modified version just presented selects entering variables based upon their probability of improving the objective function. As shown previously, this causes the probability space over which feasible bases are optimal to be swept out in a contiguous fashion. This does not imply that the bases are considered in decreasing order of probability of being optimal.

To see that the bases are not necessarily considered in this manner consider the following example problem:

$$\begin{aligned} \text{Maximize } X_o &= C_1X_1 + C_2X_2 \\ \text{subject to } \quad X_1 + 2X_2 &\le 10 \\ 2X_1 + X_2 &\le 10 \\ X_1 + X_2 &\le 6 \\ X_1, X_2 &\ge 0 \end{aligned}$$

Figure 3 is a graph of the feasible space for this problem and shows that it has five feasible bases, namely ($X_1 = 5$, $X_2 = 0$), ($X_1 = 0$, $X_2 = 5$), ($X_1 = 4$, $X_2 = 2$), ($X_1 = 2$, $X_2 = 4$), and ($X_1 = 0$, $X_2 = 0$).

Assume C_1 and C_2 are exponentially distributed independent random variables with parameters $\lambda_1 = 1/10$ and $\lambda_2 = 1/10$ respectively, and add slack variables to make the constraints equalities. Select γ to be equal to 0.4.

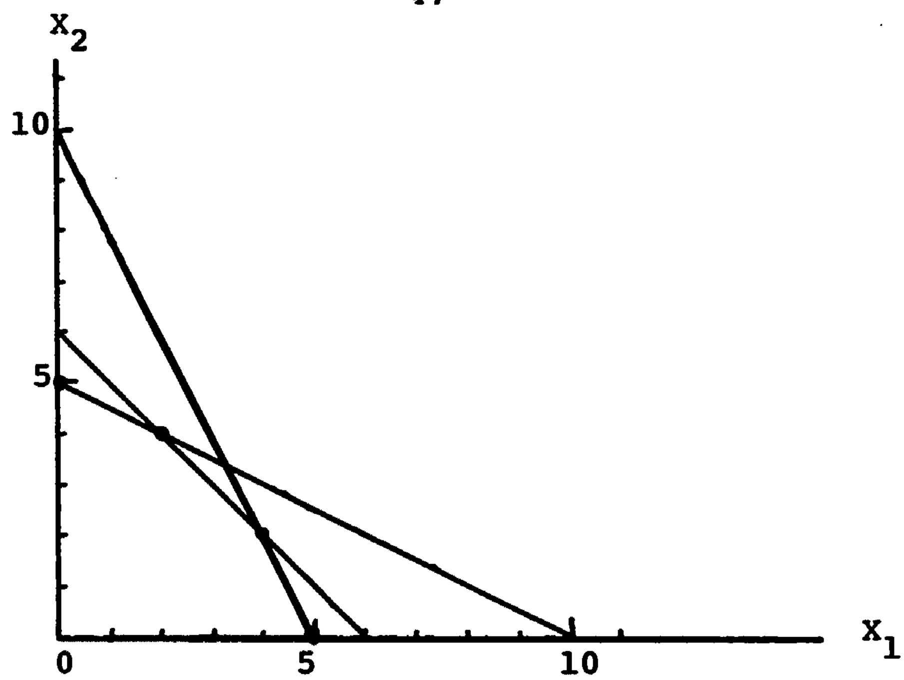


Figure 3. Graph of Feasible Space for Example Problem.

Tableau 1:

X_o	X_1	X_2	X_3	X_4	X_5	
1	$-C_1$	$-C_2$	0	0	0	0
	1	2	1	0	0	10
	2	1	0	1	0	10
	1	1	0	0	1	6

$X_o^1 = 0$

$p_1 = P(-C_1 \text{ and } -C_2 \geq 0) = 0$

Solution Index: 00111

Current Basic Variables: (X_3, X_4, X_5)

$$(1 - \sum_{i=1}^{s} p_i) = 1 - p_1 = 1 - 0 = 1 \not\leq \gamma$$

Selection of Entering Variable:

$$P[-C_1 < 0] = P[C_1 > 0] = 1, \; P[-C_2 < 0] = P[C_2 > 0] = 1$$

The probability values are equal, so the selection is arbitrary. Select X_1 as the entering variable.

Selection of Leaving Variable:

Ratios: 10/1 = 10

10/2 = 5 → select X_4 as leaving variable

6/1 = 6

New Solution Index: 10101

Tableau 2:

X_o	X_1	X_2	X_3	X_4	X_5	
1	0	$C_1/2-C_2$	0	$C_1/2$	0	$5C_1$
	0	3/2	1	-1/2	0	5
	1	1/2	0	1/2	0	5
	0	1/2	0	-1/2	1	1

$$X_o^2 = 5C_1$$

$$p_2 = P[C_1/2 - C_2 \text{ and } C_1/2 \geq 0]$$

$$p_2 = \int_0^\infty f(C_2) \int_{2C_2}^\infty f(C_1)\, dC_1 dC_2 = 1/3$$

Current Basic Variables: (X_1, X_3, X_5)

$$(1 - \sum_{i=1}^{2} p_i) = 1 - (p_1 + p_2) = 1 - (0 + 1/3) = 2/3 \not\leq \gamma$$

Selection of Entering Variable:

$$P[C_1/2 < 0] = 0$$

$$P[C_1/2 - C_2 < 0] = 1 - P[C_1/2 - C_2 \geq 0] = 1 - p_1 = 2/3$$

Selection of Leaving Variable:

Ratios: $\frac{5}{3/2} = 10/3$

$$\frac{5}{1/2} = 10$$

$$\frac{1}{1/2} = 2 \rightarrow \text{select } X_5 \text{ as leaving.}$$

New Solution Index: 11100

Tableau 3:

X_o	X_1	X_2	X_3	X_4	X_5	
1	0	0	0	C_1-C_2	$2C_2-C_1$	$4C_1+2C_2$
	0	0	1	1	-3	2
	1	0	0	1	-1	4
	0	1	0	-1	2	2

$$X_o^{\,3} = 4C_1 + 2C_2$$

$$p_3 = P[(C_1 - C_2) \text{ and } (2C_2 - C_1) \geq 0]$$

$$p_3 = \int_0^{\infty} f(C_2) \int_{C_2}^{2C_2} f(C_1)\, dC_1 dC_2 = 1/6$$

Current Basic Variables: (X_1, X_2, X_3)

$$(1 - \sum_{i=1}^{3} p_i) = 1 - (0 + 1/3 + 1/6) = 1 - 1/2 = 0.5 \nleq \gamma$$

Using the entering and leaving variables selection criteria as before we obtain

New Solution Index: 11010

Tableau 4:

X_o	X_1	X_2	X_3	X_4	X_5	
1	0	0	C_2-C_1	0	$2C_1-C_2$	$2C_1+4C_2$
	0	0	1	1	3	2
	1	0	-1	0	2	2
	0	1	1	0	-1	4

$$X_o^{\ 4} = 2C_1 + 4C_2$$

$$p_4 = P[(C_2 - C_1) \text{ and } (2C_1 - C_2) \geq 0]$$

$$p_4 = \int_0^{\infty} f(C_1) \int_{C_1}^{2C_1} f(C_2)\, dC_2 dC_1 = 1/6$$

Current Basic Variables: (X_1, X_2, X_4)

$$(1 - \sum_{i=1}^{4} p_i) = 1 - (0 + 1/3 + 1/6 + 1/6) = 1 - 2/3 = 1/3 \leq \gamma$$

At this point the approximation algorithm would stop leaving one feasible basis uninvestigated (see Figure 3). Continuing to the next iteration produces

New Solution Index: 01011

Tablesu 5:

X_o	X_1	X_2	X_3	X_4	X_5	
1	$C_2/2-C_1$	0	$C_2/2$	0	0	$5C_2$
	3/2	0	-1/2	1	0	5
	1/2	0	-1/2	0	1	1
	1/2	1	1/2	0	0	5

$$X_o^{\ 5} = 5C_1$$

$$p_5 = P[(C_2/2 - C_1) \text{ and } C_2/2 \geq 0]$$

$$p_5 = \int_0^{\infty} f(C_1) \int_{2C_1}^{\infty} f(C_2)\, dC_2 dC_1 = 1/3$$

Since $p_5 > p_4$ the bases are not considered in decreasing order of probability of being optimal and this example serves as a counter example to the premise that bases are considered in decreasing order.

Figure 4 is a graph of the space of the random variables over which the feasible bases are optimal and shows how this space is swept out by the algorithm.

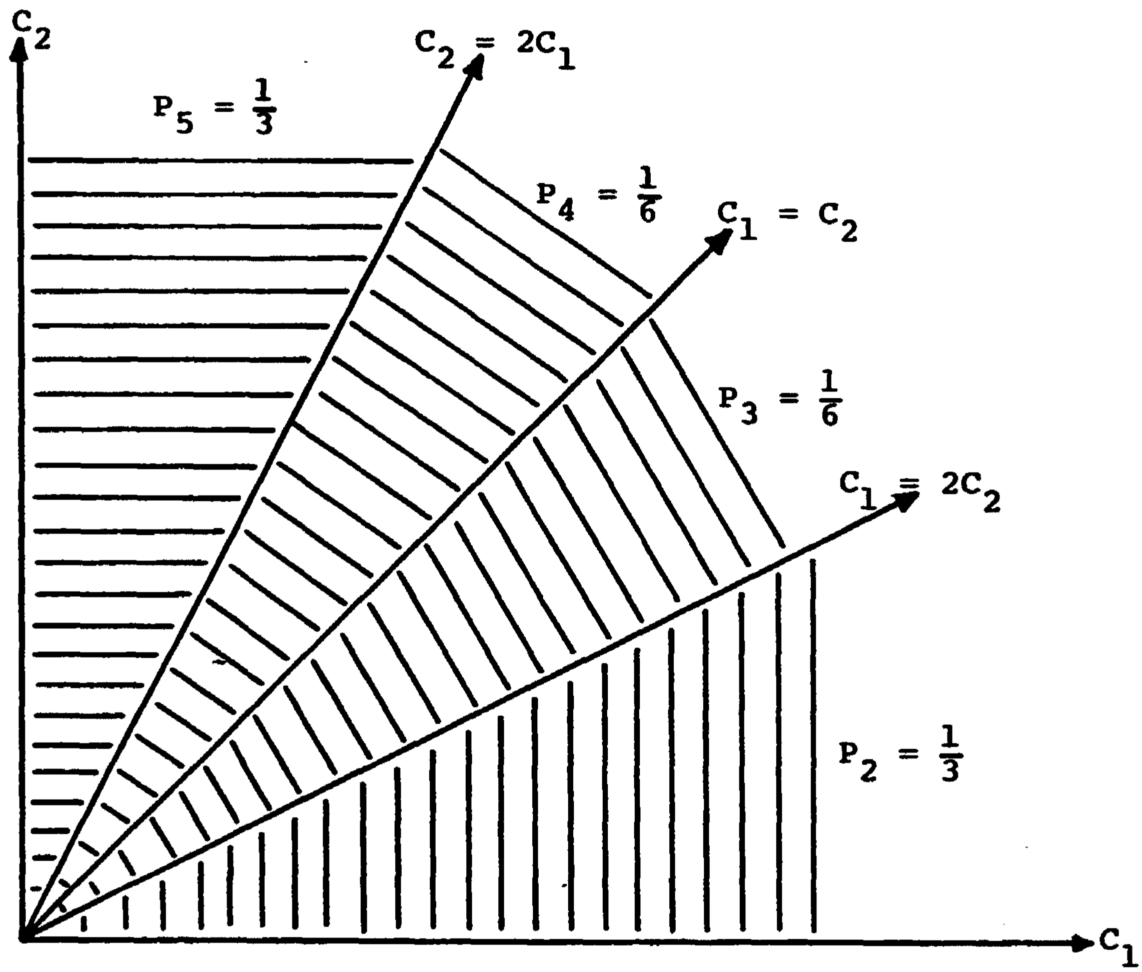
For the case where the b-vector is a vector of random variables (Case II of Chapter II), we are concerned with the probability of feasibility of optimal bases. The algorithm for this case considers all of the optimal bases of a problem of this type. An argument similar to that given for Case I provides the basis for the construction of an approximation algorithm for the case where the b-vector is a vector of random variables.

Let N denote the number of optimal bases of a problem and q_ℓ the probability that the ℓ^{th} optimal basis is feasible. Then,

$$\sum_{\ell=1}^{N} q_\ell = 1$$

and again we have a method for modifying the Case II algorithm to give an approximation algorithm. Instead of continuing to iterate until all optimal bases have been considered, the accumulated probability of optimal bases being feasible will be used to cause the iterative process to stop.

Let α denote a variable whose value is to be selected by the decision maker and reflects his evaluation of the worth of complete information. The criterion for continuing iterations will be to continue selecting leaving and entering variables to compute the $(s + 1)^{st}$ iteration until



C_1 is distributed $f(C_1) = \frac{1}{10} e^{-\frac{1}{10}u}$, $0 \le u \le \infty$

C_2 is distributed $f(C_2) = \frac{1}{10} e^{-\frac{1}{10}v}$, $0 \le v \le \infty$

Figure 4. Graph of Probability Space.

$$(1 - \sum_{\ell=1}^{s} q_\ell) \leq \alpha, \text{ where } s \leq N$$

The Case II algorithm modified for this α limit approximation now becomes:

Step 1: Same.

Step 2: Same.

Step 3: a. Same.

b. Same.

c. Same.

d. If $(1 - \sum_{\ell=1}^{s} q_\ell) \leq \alpha$ proceed to Step 5. If not, go to Step 3e.

e. Compute the probability that the basic variables are negative and proceed to Step 4.

Step 4: Same.

Step 5: The algorithm terminates when the α limit is violated.

Example Problem

As an example of the use of the modified Case II algorithm consider the following problem that was presented by Bereanu (2).

$$\text{Minimize } X_o = 3X_1 + 4X_2 + 5X_3 + 2X_4 + 2Y_1 + 3Y_2 + 5Y_3 + 4Y_4 + 6Z_1 + 5Z_2 + 2Z_3 + 3Z_4$$

subject to

$$X_1 + X_2 + X_3 + X_4 \leq 11{,}000$$

$$Y_1 + Y_2 + Y_3 + Y_4 \leq 13{,}000$$

$$Z_1 + Z_2 + Z_3 + Z_4 \leq 8{,}000$$
$$X_1 + Y_1 + Z_1 = 7{,}000+u$$
$$X_2 + Y_2 + Z_2 = 10{,}000$$
$$X_3 + Y_3 + Z_3 = 5{,}000$$
$$X_4 + Y_4 + Z_4 = 1{,}000+v$$
$$X_i, Y_i, Z_i \geq 0, i = 1, \ldots, 4$$

where u and v are exponentially distributed random variables with $\lambda_u = 2/1000$ and $\lambda_v = 3/1000$. Select α to be 0.01.

Replacing each equality constraint by two inequality constraints and adding slack variables puts the problem in the form found in Tableau 1.

The random variables in this problem appear in the restrictions of the constraints so the problem will be in determining the feasibility of optimal solutions. Tableau 1 is not feasible so S_7 is selected as the leaving variable, Y_2 is selected as the entering variable and a Dual Simplex iteration is performed to produce Tableau 2.

Tableau 2 is not feasible so S_5 is selected as the leaving variable, Y_1 is selected as the entering variable and a Dual Simplex iteration is performed to produce Tableau 3.

The former process continues until Tableau 6 is obtained.

Examination of Tableau 6 reveals that the problem is optimal (minimization problem) and the basis has a probability

TABLEAU 1

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Z_1	Z_2	Z_3	Z_4	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	
-3	-4	-5	-2	-2	-3	-5	-4	-6	-5	-2	-3	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	11,000
0	0	0	0	1	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	13,000
0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	0	0	0	0	0	0	0	0	8,000
1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	7,000+u
-1	0	0	0	-1	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	-7,000-u
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	10,000
0	-1	0	0	0	-1*	0	0	0	-1	0	0	0	0	0	0	0	0	1	0	0	0	0	-10,000
0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	5,000
0	0	-1	0	0	0	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	1	0	0	-5,000
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1,000+v
0	0	0	-1	0	0	0	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	-1,000-v

*Indicates pivot element determined by optimality and feasibility criteria.

TABLEAU 2

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Z_1	Z_2	Z_3	Z_4	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	
-3	-1	-5	-2	-2	0	-5	-4	-6	-2	-2	-3	0	0	0	0	0	0	-3	0	0	0	0	30,000
1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	11,000
0	-1	0	0	1	0	1	1	0	-1	0	0	0	1	0	0	0	0	1	0	0	0	0	3,000
0	0	0	0	0	0	0	0	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	8,000
1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	7,000+u
-1	0	0	0	-1*	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	-7,000-u
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	-1	0	0	0	0	10,000
0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	5,000
0	0	-1	0	0	0	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	1	0	0	-5,000
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1,000+v
0	0	0	-1	0	0	0	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	-1,000-v

*Indicates pivot element determined by optimality and feasibility criteria.

TABLEAU 3

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Z_1	Z_2	Z_3	Z_4	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	
-1	-1	-5	-2	0	0	-5	-4	-4	-2	-2	-3	0	0	0	0	-2	0	-3	0	0	0	0	44,000+2u
1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	11,000
-1	-1	0	0	0	0	1	1	-1	-1	0	0	0	1	0	0	0	0	0	0	0	0	0	-4,000-u
0	0	0	0	0	0	0	0	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	8,000
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	7,000+u
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	-1	0	0	0	0	10,000
0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	5,000
0	0	-1	0	0	0	-1	0	0	0	-1*	0	0	0	0	0	0	0	0	0	1	0	0	-5,000
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1,000+v
0	0	0	-1	0	0	0	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	-1,000-v

*Indicates pivot element determined by optimality and feasibility criteria.

TABLEAU 4

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Z_1	Z_2	Z_3	Z_4	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	
-1	-1	-3	-2	0	0	-3	-4	-4	-2	0	-3	0	0	0	0	-2	0	-3	0	-2	0	0	54,000+2u
1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	11,000
-1*	-1	0	0	0	0	1	1	-1	-1	0	0	0	1	0	0	1	0	1	0	0	0	0	-4,000-u
0	0	0	0	0	0	0	0	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	8,000
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	7,000+u
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	-1	0	0	0	0	10,000
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	-1	0	0	5,000
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1,000+v
0	0	0	-1	0	0	0	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	-1,000-v

*Indicates pivot element determined by optimality and feasibility criteria.

TABLEAU 5

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Z_1	Z_2	Z_3	Z_4	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	
0	0	-3	-2	0	0	-4	-5	-3	-1	0	-3	0	-1	0	0	-3	0	-3	0	-2	0	0	58,000+3u
0	0	1	1	0	0	1	1	-1	-1	0	0	1	1	0	0	1	0	1	0	0	0	0	7,000-u
1	1	0	0	0	0	-1	-1	1	1	0	0	0	-1	0	0	-1	0	-1	0	0	0	0	4,000+u
0	0	0	0	0	0	0	0	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	8,000
0	-1	0	0	0	0	1	1	-1	-1	0	0	0	1	0	1	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	3,000
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	-1	0	0	0	0	10,000
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	-1	0	0	5,000
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1,000+v
0	0	0	-1*	0	0	0	-1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	1	-1,000-v

*Indicates pivot element determined by optimality and feasibility criteria.

TABLEAU 6

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Z_1	Z_2	Z_3	Z_4	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	
0	0	-3	0	0	0	-4	-3	-3	-1	0	-1	0	-1	0	0	-3	0	-3	0	-2	0	-3	60,000+3u+2v
0	0	1	0	0	0	1	0	-1	-1*	0	-1	1	1	0	0	1	0	1	0	0	0	1	6,000-u-v
1	1	0	0	0	0	-1	-1	1	1	0	0	0	-1	0	0	-1	0	-1	0	0	0	0	4,000+u
0	0	0	0	0	0	0	0	1	1	0	1	0	0	1	0	0	0	0	0	0	0	0	8,000
0	-1	0	0	0	0	1	1	-1	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	3,000
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	-1	0	0	0	0	10,000
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	-1	0	0	5,000
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	1,000+v

*Indicates pivot element determined by optimality and feasibility criteria.

of being feasible. This condition is $(6000 - u - v) \geq 0$ which defines the set $Q_1 = \{(u,v) : (u+v) \leq 6000\}$. The current value of X_o is $60{,}000 + 3u + 2v$ and at this point the distribution of X_o is given by

$$f(X_o) = q_1 f_1(X_o)$$

The probability that this optimal basis is feasible is given by

$$q_1 = P[6000 - u - v \geq 0] = P[u + v \leq 6000]$$

$$q_1 = \int_0^{\infty} f(v) \int_0^{6000-v} f(u)\, dudv$$

$$q_1 = \int_0^{\infty} 3/1000\, e^{-3/1000\, v} \left[-e^{-2/1000\, u}\right]_0^{6000-v} dv$$

$$q_1 = \int_0^{\infty} 3/1000\, e^{-3/1000\, v}\, dv$$

$$- 3e^{-12} \int_0^{\infty} 1/1000\, e^{-1/1000\, v}\, dv$$

$$q_1 = 1 - 3e^{-12} = 1 - 0.00001875$$

$$q_1 = 0.99998125$$

Now $\quad 1 - \sum_{\ell=1}^{s=1} q_\ell = 1 - 0.99998125 = 0.00001875 \leq \alpha$

so the algorithm terminates.

The only reason for changing bases (i.e., investigating another solution) would be if an infeasibility could exist in the current solution. This condition would be satisfied if (6000 - u - v) were negative, but the probability of this is extremely small (approximately 0.00001875). The approximation algorithm with α set at any value above this small probability would terminate and yield $X_o = X_o^1 = 60{,}000 + 3u + 2v$ with probability q_1 and $f(X_o^*) = q_1 f_1(X_o^1)$. Bereanu's technique would be to use sensitivity analysis to find the set that defines q_1 and then would require a change of basis using parametric programming methods to investigate the other basis which has a probability of optimality of $q_2 = 0.00001875$.

An estimate of the amount of information that is lost by using the approximation algorithm is desirable. Of course, the total amount of the probability space that is left out is less than or equal to α by the nature of the algorithm. Another interesting measure is the change in expected value that would occur if more bases were investigated. Let s denote the last basis that is checked and consider the contribution of the $(s + 1)^{st}$ basis. We know that

$$\left(1 - \sum_{\ell=1}^{s} q_\ell\right) \le \alpha$$

and that
$$\sum_{\ell=s+1}^{N} q_\ell = \left(1 - \sum_{\ell=1}^{s} q_\ell\right) \le \alpha$$

so
$$\sum_{\ell=s+1}^{N} q_\ell \le \alpha$$

Also since $q_\ell \ge 0$ for each ℓ, we have that

$$q_{(s+1)} \le \sum_{\ell=s+1}^{N} q_\ell \le \alpha$$

so
$$q_{(s+1)} \le \alpha$$

TABLEAU 7

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Z_1	Z_2	Z_3	Z_4	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	
0	0	-4	0	0	0	-5	-3	-2	0	0	0	-1	-2	0	0	-4	0	-4	0	-2	0	-4	54,000+4u+3v
0	0	-1	0	0	0	-1	0	1	1	-	1	-1	-1	0	0	-1	0	-1	0	0	0	-1	u+v-6,000
																							10,000-v
																							14,000-u-v
																							0
																							3,000
																							0
																							16,000-u-v
																							0
																							5,000
																							0
																							1,000+v

The difference between $f^*(X_o)$ at the s^{th} iteration and $f^*(X_o)$ at the $(s + 1)^{st}$ iteration is given by

$$f^*(X_o)_{(s+1)} - f^*(X_o)_s = q_{(s+1)} \; f_{(s+1)}(X_o)$$

Therefore, $f^*(X_o)_{(s+1)} - f^*(X_o)_s \leq \alpha \; f_{(s+1)}(X_o)$

At any iteration, the value of the basic variables is given by

$$X_B = B^{-1}P_o$$

For the case where all of the constraint relationships are sums, it follows that

$$B^{-1}P_o \leq P_o$$

Also, at any iteration

$$X_o = C_B B^{-1} P_o$$

so

$$X_o = C_B B^{-1} P_o \leq C_B P_o$$

The largest possible value for C_B would be the m largest coefficients in the objective function. Let $C_B^{\,1}$ denote these; this gives

$$X_o \leq C_B P_o \leq C_B^{\,1} \; P_o$$

This yields

$$f^*(X_o)_{s+1} - f^*(X_o)_s \leq \alpha \; f_{s+1}(X_o) \leq \alpha \; f(C_B^{\,1} \; P_o)$$

and the difference in the expected value is less than

$$\alpha\ E[X_{o_{(s+1)}}] = \alpha \int X_{o_{(s+1)}}\ f(C_B^1\ P_o)\ dX_{o_{(s+1)}}$$

These values for Bereanu's example problem are

$$P_o = \begin{pmatrix} 11,000 \\ 13,000 \\ 8,000 \\ 7,000+u \\ 10,000 \\ 5,000 \\ 1,000+v \end{pmatrix}$$

$$C_B^1 = (5,\ 5,\ 4,\ 6,\ 5,\ 3,\ 4)$$

This gives

$$C_B^1\ P_o = (55,000 + 65,000 + 32,000 + 28,000 + 6u + 50,000 + 15,000 + 4,000 + 4v)$$

$$C_B^1\ P_o = 249,000 + 6u + 4v$$

For the selected α value (i.e., $\alpha = 0.01$) we have that

$$\alpha\ f(C_B^1\ P_o) = 0.01\ f(249,000\ 6u + 4v)$$

The random variables u and v are independently distributed according to exponential probability distribution functions with parameters $\lambda_u = 2/1000$ and $\lambda_v = 3/1000$. Let $y_1 = 6u$ and $y_2 = 4v$. The distribution of these random variables is exponential with parameters $\lambda_1 = 2/6000$ and $\lambda_2 = 3/4000$ respectively. Let $f_{y_1+y_2}(X)$ denote the distribution of the sum of these two random variables. This distribution must be approximated since it is the distribution of the sum of two gamma distributions each with parameter $r = 1$ but with unequal λ's.

The characteristic function of y_1 is given by

$$\phi_{y_1}(u) = (1 - \frac{iu}{\lambda_1})^{-1}$$

and for y_2 by

$$\phi_{y_2}(u) = (1 - \frac{iu}{\lambda_2})^{-1}$$

If these characteristic functions are approximately equal, then their probability laws are approximately equal.

In order to construct an approximation function, let $\lambda^* = (\lambda_1 + \lambda_2)/2$ and let $f_{y_1}^*(X)$ be an approximation of $f_{y_1}(X)$ and $f_{y_2}^*(X)$ be an approximation of $f_{y_2}(X)$ where $f_{y_1}^*(X)$ and $f_{y_2}^*(X)$ are given by

$$f(X) = \lambda^* e^{-\lambda^* X} dX, \ 0 \le X \le \infty$$

Now,

$$\phi_{y_1+y_2}(u) = \phi_{y_1}(u)\ \phi_{y_2}(u)$$

$$\phi_{y_1+y_2}(u) = (1 - \frac{iu}{\lambda^*})^2$$

and this indicates that $f_{y_1+y_2}(X)$ is a gamma distribution with parameters $r = 2$, and $\lambda^* = 13/24{,}000$.

The difference in expected values when the $(s+1)^{st}$ basis is not investigated is less than or equal to $\alpha(249{,}000) + \alpha E[X_o^{s+1}]$ or $0.01(249{,}000) + 0.01(48{,}000/13) = 2527$.

Examination of Tableau 7 gives the value of $X_o^2 = 54{,}000 + 4u + 3v$ and $q_2 = 0.00001875$. This means that for this problem the actual difference in the expected value of the objective function is approximately

$$1.875 \times 10^{-5} (54 \times 10^{3}) + 1.875 \times 10^{-5} E[X_o^{2}]$$

As before, the distribution of X_o^2 is the distribution of the sum of the random variables $y_1 = 4u$ and $y_2 = 3v$ which are exponentially distributed with λ parameters 1/2000 and 1/1000 respectively. Using the same type of approximation as before, this means that the actual difference is approximately

$$1.875 \times 10^{-5} [54,000 + 8000/3] = 1.06$$

This value represents a very small difference in the expected value of the optimum of the objective function and indicates that the approximation algorithm eliminates only a small amount of information when it does not investigate this basis.

The choice rule for entering variables used by the algorithms of Chapter II and the approximation algorithms ensures that if a basis has a zero probability of being optimal, it will never be considered. This occurs because the selection condition for the entering variable represents the amount of the probability space that has not yet been considered.

The selection rule selects a non-basic variable to enter only if it has a positive probability of contributing to the objective function. This means that for some non-basic variable, the following statement is true:

$$P[(C_B B^{-1} P_j - C_j) < 0] > 0$$

The actual value of this probability statement is the amount of the probability space that has not been previously considered. Therefore, the entering variable choice rule will eliminate from consideration those bases that have a zero probability of being optimal. This feature offers a further savings in computational effort since it is not necessary to consider these bases in the computation of the distribution of the optimal value of the objective function.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

Algorithms have been presented for the solution of the distribution problem of probabilistic linear programming under the conditions that either the coefficients of the objective function or the restrictions of the constraints are random variables. These algorithms are based upon the simplex technique with appropriate modification of the rules for selecting entering and leaving variables to account for the presence of random variables in the problem.

The method developed by Tintner and Sengupta (18, 19) for finding the distribution of the optimal value of the objective function requires the solution of as many linear programs as there are combinations of random variables and the possible values that they may assume. For problems involving more than a few random variables, the number of computations required becomes quite burdensome. The algorithms presented in this paper offer the advantage of producing the desired distribution of the optimal value of the objective function without requiring the solution of more than one linear program.

Another feature of Tintner and Sengupta's method is that it is dependent upon sampling techniques for selecting values for the random variables and for fitting a function to the optimal values that are obtained. Because of this, the method actually produces an approximate distribution and not an exact distribution. Tintner and Sengupta do not give any bounds upon the error that may be produced by this technique.

The algorithm presented in Chapter II of this paper will produce the exact distribution of the optimal value of the objective function. The approximation algorithm of Chapter III produces an approximate distribution and gives a bound on the error in the mean and variance of the distribution of X_o that is created by the approximation.

The algorithms are not dependent upon placing upper and lower bounds on the range of the random variables as in Bereanu's computational method. Additionally, Bereanu's method does not give a specific procedure for changing bases, but relies on parametric programming methods. The algorithms developed in this paper are based upon the simplex method and contain specific rules for changing bases that prevent consideration of bases that have a probability of being optimal equal to zero. This condition is not necessarily guaranteed by Bereanu's computational technique.

The method of selecting entering and leaving variables is important since it selects candidates that are best according to a probabilistic measure of their ability to improve the solution. As stated in Chapter III, the algorithms investigate the probability space in a contiguous manner and this leads to the concept of producing the approximation algorithms.

Bereanu's technique requires an investigation of all of the feasible bases of the problem and in this sense it is totally enumerative. He does not present an approximation algorithm based upon his technique.

The modified approximation algorithms make possible a determination of an approximation of the distribution of the optimal value of the objective function. Bounds are given for the amount of probability of either optimality or feasibility that is omitted and the difference in the expected value of the distribution function that is caused by using the approximation algorithms. These algorithms are important because of the savings in computational effort that they produce. This reduction of computations becomes even more significant as the number of random variables in a problem increases.

Two computer programs are given in Appendix B. The first of these programs is designed to produce the information required for determination of the distribution of the optimal value of the objective function when the C-vector is a random vector whose elements are described by discrete probability distribution functions. The second program handles the same case with the exception that the elements of the C-vector are described by continuous normal distribution functions.

The algorithms developed in this dissertation do not consider the case where both the C-vector and the b-vector are simultaneously vectors of random variables. This condition offers an area for further research and a possible extension of this work. Additionally, different rules for selecting entering and leaving variables may be developed and their effect upon the investigation of the probability space will need careful consideration.

The integrations that are necessary for the determination of the probability that either a feasible basis is optimal or that an optimal basis is feasible present some difficulty. In general, these are conditional integrals, and in the case where the original variables are normally distributed, this integration must be accomplished by numerical methods. This process has not been included in the computer program and provides an area for further work in terms of reducing the effort required in solving a problem of this type. A similar comment holds for the discrete program since the summations required are conditional sums, and again this process has not been included in the current computer program.

Another area for further work is the development of computer programs that will consider probability distributions other than the discrete or continuous normal cases.

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APPENDIX A

PROOF THAT THE COEFFICIENTS OF THE BASIC VARIABLES ARE ZERO WHEN THE C-VECTOR IS A VECTOR OF RANDOM VARIABLES

Theorem: Let the C-vector be a vector whose elements are independent random variables and consider the linear program

$$\text{Maximize } X_o = CX$$
$$\text{subject to } AX = P_o$$
$$X \geq 0$$

The coefficients of the basic variables of this problem are equal to zero.

Proof: 1. Consider the general tableau form of the problem which represents the solution at any iteration.

$$\left(\begin{array}{c|c|c} 1 & C_B B^{-1}A - C_I & C_B B^{-1} - C_{II} \\ \hline 0 & B^{-1}A & B^{-1} \end{array}\right) \begin{pmatrix} X_o \\ X_I \\ X_{II} \end{pmatrix} = \begin{pmatrix} C_B B^{-1} P_o \\ B^{-1} P_o \end{pmatrix}$$

C_B - Coefficients of basic variables

B - portion of the A matrix associated with the basic variables

(C_I, C_{II}) - a partition of C-vector where C_{II} represents the coefficients of the starting solution.

2. The coefficients of all of the variables are given by either $(C_B B^{-1} A - C_I)$ or $(C_B B^{-1} - C_{II})$
3. Consider a variable X_j and assume that it is a basic variable.
4. The coefficient of X_j is given by $(C_B B^{-1} P_j - C_j)$ or $(C_B B^{-1} - C_j)$.
5. Since X_j is a basic variable, its column vector P_j is an element of B.
6. Therefore, B^{-1} contains the inverse of this column vector and thus $C_B B^{-1} P_j$ for the jth variable which is basic becomes $C_j I = C_j$.
7. Therefore, $(C_B B^{-1} P_j - C_j) = (C_j - C_j)$.
8. Now, C_j is a random variable so let C_j assume any of its possible values, say C_j^*.
9. Therefore, $(C_B B^{-1} P_j - C_j) = (C_j^* - C_j^*) = 0$ and since $(C_B B^{-1} P_j - C_j)$ is the coefficient of a basic variable, the theorem is proved.

APPENDIX B

COMPUTER PROGRAMS

The two computer programs presented in this appendix entitled respectively PROGRAM 'DISCRETE' and PROGRAM 'CONTINUOUS' are written in Fortran IV. They are designed to use the Case I Algorithm when the coefficients of the objective function are either discretely distributed random variables or continuously distributed normal random variables.

Each of these programs produces as its output a modified tableau that contains the information needed for the construction of the inequalities that define the set S_ℓ. This set is necessary for the computation of the probability that a feasible basis is optimal, i.e., P_ℓ.

In addition, the tableaux contain the basic variables and their values at each iteration. This information is presented in a form that is similar to the regular simplex method tableau.

The program user must supply the data that defines the problem that he wishes to solve. This includes the number of original variables in the problem, the number of constraint equations, the probability distributions of the random variables, the coefficients of the constraint equation (the A matrix), the constraint restrictions (the b vector), and the starting solution. The programs use this data and the Case I Algorithm to produce the output described above.

PROGRAM 'DISCRETE' User Information

Program Limitations

As currently written this program will handle a linear program with a maximum of nine random coefficients in the objective function and a maximum of nine constraint equations. This limitation may be removed by changing the Dimension Statements in the program. Let n be the number of variables and m be the number of constraints and k be the number of possible values of the random variables. The Dimension Statements should be modified as follows: A[m, (m+n)], CONTAB [(m+n), (m+n+1)], VRV [(m+n), k], PRU [(m+n), k], PR (m+n), KV (m+n). At the beginning of the program the statement MAX = 9 should be changed to MAX = (m+n). These changes will handle any size linear program up to the limitations of the computer being used.

Input Required

Card 1.

Columns 1-5 : The number of constraint equations in Format I5

Columns 6-10: The number of original variables (not including slacks) in Format I5

Columns 11-80: Blank

Card 2.

Column 1 : Blank

Columns 2-16: Value of the random variable in Format F15.6

Columns 17-32: Probability of the random variable having the value in Col. 2-16 in Format F15.6

Columns 33-80: Blank

Note: The second card is repeated until all values of the random variable have been read in and the last card of this set for each random variables is followed by Card 3 which signals the end of this random variable.

Card 3.

Column 1 : Blank

Columns 2-16: Zero in Format F15.6

Columns 17-32: Zero in Format F15.6

Columns 33-80: Blank

Note: Slack variables are indicated by a Card 2 with zero in Columns 2-16 and 1.0 in Columns 17-32 and again each of these is followed by a Card 3.

Card 4. Beginning in Column 2 in a F7.3 Format, this card contains the coefficients of the A matrix and the b vector with each card representing one equation. This card is repeated as required to read in all of the constraint equations.

Card 5. This card contains the information concerning the starting solution.

Columns 1-5 : The column of the basic variable in the starting solution in Format I5

Columns 6-10: The row of the basic variable in the starting solution in Format I5

Columns 11-80: Blank

Note: There will be as many of these cards as there are constraint equations (i.e., m of these).

A listing of this program and an example problem follow at the end of this appendix.

PROGRAM 'CONTINUOUS' User Information

Program Limitations

The same general comments given for program limitations for PROGRAM 'DISCRETE' apply to this program. The Dimension Statement for PROGRAM 'CONTINUOUS' should be modified as follows: A[(m, (m+n)], CONTAB [(m+n), (m+n+1)], RVMV (m+n), RVAR (m+n), PR (m+n) in order to handle program with more than nine total variables. Also, for this program the statement MAX = 9 should be changed to MAX = (m+n).

Input Required

Card 1. Same as for PROGRAM 'DISCRETE.'

Card 2.

Column 1 : Blank

Columns 2-8 : Mean value of the random variable in Format F15.6

Columns 9-15: Variance of the random variable in Format F15.6

Columns 16-80: Blank

Note: There will be (m+n) of these cards with the slack variables having zero mean and zero variance.

<u>Card 3</u>. Same as Card 4 for PROGRAM 'DISCRETE.'

<u>Card 4</u>. Same as Card 5 for PROGRAM 'DISCRETE.'

A listing of this program and an example problem follow at the end of this appendix.

```
C PROGRAM DISCRETE
C PROGRAM DISCRETE COMPUTES THE INFORMATION NECESSARY FOR THE
C CALCULATION OF THE PROBABILITY THAT A FEASIBLE BASIS IS
C OPTIMAL WHEN THE COEFFICIENTS OF THE OBJECTIVE FUNCTION ARE
C DISCRETELY DISTRIBUTED RANDOM VARIABLES
      INTEGER*2 IHEAD,IXO
      DIMENSION IHEAD(9)
      DIMENSION A(9,9), CONTAB(9,10), VRV(9,9), PRV(9,9)
      DIMENSION TVRV(1000), TPRV(1000), KV(9), IBASIS(100)
      DIMENSION PR(9)
      DATA IHEAD/'C1','C2','C3','C4','C5','C6','C7','C8','C9'/
      DATA IXO/'XO'/
      DATA VRV,PRV,CONTAB/81*.0,81*.0,90*.0/
      IN=5
      IOUT=6
      MAX=9
      NBASE=0
      IB=0
       READ(IN,5000) M,N,B
      WRITE(IOUT,4900) M,N,B
      NV=M+N
      NN=NV+1
      MM=M
      DO 300 J=1,NV
      K=0
      DO 100 L=1,MAX
      READ(IN,5600) VRV(L,J),PRV(L,J)
      IF(PRV(L,J).EQ.0.0) GO TO 200
      K=K+1
  100 CONTINUE
  200 CONTINUE
      KV(J)=K
  300 CONTINUE
      DO 400 J=1,NV
      KVN=KV(J)
      DO 400 L=1,KVN
       WRITE(IOUT,5200) VRV(L,J),PRV(L,J)
  400 CONTINUE
      DO 450 I=1,NV
      CONTAB(I,I)=-1.
  450 CONTINUE
      DO 500 I=1,MM
      READ(IN,5250) (A(I,J),J=1,NN)
  500 CONTINUE
      DO 540 K=1,MM
      READ(IN,5000) JJ,II
      DO 520 J=1,NN
      IF(J.EQ.JJ) GO TO 520
```

```
      DO 520 I=1,NV
      CONTAB(I,J)=CONTAB(I,J)-A(II,J)*CONTAB(I,JJ)
  520 CONTINUE
      DO 530 I=1,NV
      CONTAB(I,JJ)=0.
  530 CONTINUE
  540 CONTINUE
  550 CONTINUE
      WRITE(IOUT,5300)
      WRITE(IOUT,5800) (IHEAD(J),J=1,NV),IXO
      DO 600 I=1,NV
      WRITE(IOUT,5900) IHEAD(I),(CONTAB(I,J),J=1,NN)
  600 CONTINUE
       WRITE(IOUT,5700)
      DO 650 I=1,MM
      WRITE(IOUT,5200) (A(I,J),J=1,NN)
  650 CONTINUE
      JJ=0
      IBASE=0
      PT=1.0
      PMAX=0.0
      DO 900 J=1,NV
      ICT=0
      DO 800 I=1,NV
      IF(CONTAB (I,J).EQ.0.) GO TO 800
      IF(ICT.NE.0) GO TO 700
      ICT=KV(I)
      CON=CONTAB(I,J)
      DO 675 K=1,ICT
      TVRV(K)=VRV(K,I)*CON
      TPRV(K)=PRV(K,I)
  675 CONTINUE
      GO TO 800
  700 CONTINUE
      LMT=KV(I)
      KM=LMT*ICT
      CON=CONTAB(I,J)
      DO 750 K=1,LMT
      KK=KM-K*ICT+1
      DO 750 L=1,ICT
      TVRV(KK)=TVRV(L)+CON*VRV(K,I)
      TPRV(KK)=TPRV(L)*PRV(K,I)
      KK=KK+1
  750 CONTINUE
      ICT=ICT*LMT
  800 CONTINUE
      IF(ICT.NE.0) GO TO 850
      IBASE=IBASE+10**(NV-J)
```

```
      PR(J)=0.0
      GO TO 900
  850 CONTINUE
      PRVLZ=0.0
      DO 875 K=1,ICT
      IF(TVRV(K).GT..0) GO TO 875
      IF(TVRV(K).NE..0) GO TO 860
      PRVLZ=PRVLZ+TPRV(K)/2.
      GO TO 875
  860 CONTINUE
      PRVLZ=PRVLZ+TPRV(K)
  875 CONTINUE
      PT=PT*(1.0-PRVLZ)
      IF(PMAX.GT.PRVLZ) GO TO 890
      PMAX=PRVLZ
      JJ=J
  890 CONTINUE
      PR(J)=PRVLZ
  900 CONTINUE
      IB=IB+1
      IBASIS(IB)=IBASE
      IF(JJ.EQ.0) GO TO 1600
  950 CONTINUE
      NBASE=IBASE+10**(NV-JJ)
      RTEST=9999.
      II=0
      DO 1000 I=1,MM
      IF(A(I,JJ).LE.0.) GO TO 1000
      R=A(I,NN)/A(I,JJ)
      IF(R.GE.RTEST) GO TO 1000
      II=I
      RTEST=R
 1000 CONTINUE
      IF(II.EQ.0) GO TO 1060
      DO 1020 J=1,NV
      IF(A(II,J).NE.1) GO TO 1020
      DO 1010 I=1,M
      IF(II.EQ.I) GO TO 1010
      IF(A(I,J).NE.0) GO TO 1020
 1010 CONTINUE
      DO 1015 I=1,NV
      IF(CONTAB(I,J).NE.0.) GO TO 1020
 1015 CONTINUE
      NBASE=NBASE-10**(NV-J)
      GO TO 1040
 1020 CONTINUE
      WRITE(IOUT,5500)
      GO TO 1600
```

```
1040 CONTINUE
     DO 1050 J=1,IB
     IF(NBASE.EQ.IBASIS(J)) GO TO 1060
1050 CONTINUE
     GO TO 1090
1060 CONTINUE
     PR(JJ)=.0
     JJ=0
     PMAX=0.0
     DO 1070 J=1,NV
     IF(PMAX.GE.PR(J)) GO TO 1070
     PMAX=PR(J)
     JJ=J
1070 CONTINUE
     IF(JJ.EQ.0) GO TO 1600
     GO TO 950
1090 CONTINUE
     PIVOT=A(II,JJ)
     DO 1100 I=1,MM
     DO 1100 J=1,NN
     IF(I.EQ.II) GO TO 1100
     IF(J.EQ.JJ) GO TO 1100
     A(I,J)=(A(I,J)*PIVOT-A(I,JJ)*A(II,J))/PIVOT
1100 CONTINUE
     DO 1200 J=1,NN
     A(II,J)=A(II,J)/PIVOT
1200 CONTINUE
     DO 1300 I=1,MM
     A(I,JJ)=0.
1300 CONTINUE
     A(II,JJ)=1.
     DO 1400 J=1,NN
     DO 1400 I=1,NV
     IF(J.EQ.JJ) GO TO 1400
     CONTAB(I,J)=CONTAB(I,J)-A(II,J)*CONTAB(I,JJ)
1400 CONTINUE
     DO 1500 I=1,NV
     CONTAB(I,JJ)=0.0
1500 CONTINUE
     GO TO 550
1600 CONTINUE
     CALL EXIT
4900 FORMAT(' ',2I5,F8.3)
5000 FORMAT(2I5,2F4.2)
5100 FORMAT(1H1)
5200 FORMAT(5H     ,10F7.3)
5250 FORMAT(1X,10F7.3)
5300 FORMAT(12H1NEW TABLEAU)
```

```
5400 FORMAT(1H ,F7.3,2I5)
5500 FORMAT(' ','ERROR 1')
5600 FORMAT(1H ,3F15.6)
5700 FORMAT(1H0)
5800 FORMAT ('0',4X,10(2X,A2,3X))
5900 FORMAT (1H ,A2,2X,10F7.3)
     END
```

2	2	0.400
	0.0	0.100
	1.000	0.200
	2.000	0.400
	3.000	0.200
	4.000	0.100
	1.000	0.100
	2.000	0.200
	3.000	0.400
	4.000	0.200
	5.000	0.100
	0.0	1.000
	0.0	1.000

NEW TABLEAU

	C1	C2	C3	C4	X0
C1	-1.000	0.0	0.0	0.0	0.0
C2	0.0	-1.000	0.0	0.0	0.0
C3	1.000	2.000	0.0	0.0	10.000
C4	2.000	1.000	0.0	0.0	10.000

1.000	2.000	1.000	0.0	10.000
2.000	1.000	0.0	1.000	10.000

NEW TABLEAU

	C1	C2	C3	C4	X0
C1	-1.000	0.0	0.0	0.0	0.0
C2	0.500	0.0	0.500	0.0	5.000
C3	0.0	0.0	-1.000	0.0	0.0
C4	1.500	0.0	-0.500	0.0	5.000
	0.500	1.000	0.500	0.0	5.000
	1.500	0.0	-0.500	1.000	5.000

NEW TABLEAU

	C1	C2	C3	C4	XC
C1	0.0	0.0	-0.333	0.667	3.333
C2	0.0	0.0	0.667	-0.333	3.333
C3	0.0	0.0	-1.000	0.0	0.0
C4	0.0	0.0	-0.000	-1.000	0.000
	0.0	1.000	0.667	-0.333	3.333
	1.000	0.0	-0.333	0.667	3.333

NEW TABLEAU

	C1	C2	C3	C4	XO
C1	0.0	0.500	0.0	0.500	5.000
C2	0.0	-1.000	0.0	0.0	0.000
C3	0.0	1.500	0.0	-0.500	5.000
C4	0.0	0.000	0.0	-1.000	0.000
	0.0	1.500	1.000	-0.500	5.000
	1.000	0.500	0.0	0.500	5.000

```
C PROGRAM CONTINUOUS
C PROGRAM CONTINUOUS COMPUTES THE INFORMATION NECESSARY FOR
C THE CALCULATION OF THE PROBABILITY THAT A FEASIBLE BASIS IS
C OPTIMAL WHEN THE COEFFICIENTS OF THE OBJECTIVE FUNCTION ARE
C INDEPENDENT NORMAL RANDOM VARIABLES
      INTEGER*2 IHEAD,IXO
      DIMENSION IHEAD(9)
      DIMENSION A(9,9), CONTAB(9,10),RVMV(9),RVAR(9)
      DIMENSION IBASIS(100),PR(9)
      DATA CONTAB/90*0.0/
      DATA IHEAD/'C1','C2','C3','C4','C5','C6','C7','C8','C9'/
      DATA IXO/'XO'/
      IN=5
      IOUT=6
      MAX=9
      NBASE=0
      IB=0
       READ(IN,5000) M,N,B
      WRITE(IOUT,4900) M,N,B
      NV=M+N
      NN=NV+1
      MM=M
      K=0
      DO 200 J=1,NV
      READ(IN,5500) RVMV(J),RVAR(J)
      WRITE(IOUT,5200) RVMV(J), RVAR(J)
  200 CONTINUE
  400 CONTINUE
      DO 450 I=1,NV
      CONTAB(I,I)=-1.
  450 CONTINUE
      DO 500 I=1,MM
      READ(IN,5250) (A(I,J),J=1,NN)
  500 CONTINUE
      DO 540 K=1,MM
      READ(IN,5000) JJ,II
      DO 520 J=1,NN
      DO 520 I=1,NV
      IF(J.EQ.JJ) GO TO 520
      CONTAB(I,J)=CONTAB(I,J)-A(II,J)*CONTAB(I,JJ)
  520 CONTINUE
      DO 530 I=1,NV
      CONTAB(I,JJ)=0.
  530 CONTINUE
  540 CONTINUE
  550 CONTINUE
      WRITE(IOUT,5300)
      WRITE(IOUT,5800) (IHEAD(J),J=1,NV),IXO
```

```
      DO 600 I=1,NV
      WRITE(IOUT,5900) IHEAD(I),(CONTAB(I,J),J=1,NN)
  600 CONTINUE
      WRITE(IOUT,5100)
      DO 650 I=1,MM
      WRITE(IOUT,5200) (A(I,J),J=1,NN)
  650 CONTINUE
      JJ=0
      IBASE=0
      PT=1.0
      PMAX=0.0
      DO 900 J=1,NV
      DMEAN=0.0
      DVAR=0.0
      ICT=0
      DO 800 I=1,NV
      IF(CONTAB (I,J).EQ.0.) GO TO 800
      ICT=ICT+1
      DMEAN=DMEAN+RVMV(I)*CONTAB(I,J)
      DVAR=DVAR+RVAR(I)*CONTAB(I,J)*CONTAB(I,J)
  800 CONTINUE
      IF( ICT.NE.0) GO TO 850
      IBASE=IBASE+10**(NV-J)
      PR(J)=0.0
      GO TO 900
  850 CONTINUE
      PRVLZ=PROB(DMEAN,DVAR)
      WRITE(IOUT,5100)
      PT=PT*(1.0-PRVLZ)
      IF(PMAX.GT.PRVLZ) GO TO 890
      PMAX=PRVLZ
      JJ=J
  890 CONTINUE
      PR(J)=PRVLZ
  900 CONTINUE
      IB=IB+1
      IBASIS(IB)=IBASE
      IF(JJ.EQ.0) GO TO 1600
  950 CONTINUE
      NBASE=IBASE+10**(NV-JJ)
      RTEST=9999.
      II=0
      DO 1000 I=1,MM
      IF(A(I,JJ).LE.0.) GO TO 1000
      R=A(I,NN)/A(I,JJ)
      IF(R.GE.RTEST) GO TO 1000
      II=I
      RTEST=R
```

```
1000 CONTINUE
     IF(II.EQ.0) GO TO 1060
     DO 1020 J=1,NV
     IF(A(II,J).NE.1) GO TO 1020
     DO 1010 I=1,M
     IF(II.EQ.I) GO TO 1010
     IF(A(I,J).NE.0) GO TO 1020
1010 CONTINUE
     DO 1015 I=1,NV
     IF(CONTAB(I,J).NE.0.) GO TO 1020
1015 CONTINUE
     NBASE=NBASE-10**(NV-J)
     GO TO 1040
1020 CONTINUE
     WRITE(IOUT,5500)
     GO TO 1600
1040 CONTINUE
     DO 1050 J=1,IB
     IF(NBASE.EQ.IBASIS(J)) GO TO 1060
1050 CONTINUE
     GO TO 1000
1060 CONTINUE
     PR(JJ)=.0
     JJ=0
     PMAX=0.0
     DO 1070 J=1,NV
     IF(PMAX.GE.PR(J)) GO TO 1070
     PMAX=PR(J)
     JJ=J
1070 CONTINUE
     IF(JJ.EQ.0) GO TO 1600
     GO TO 950
1090 CONTINUE
     PIVOT=A(II,JJ)
     DO 1100 I=1,MM
     DO 1100 J=1,NN
     IF(I.EQ.II) GO TO 1100
     IF(J.EQ.JJ) GO TO 1100
     A(I,J)=(A(I,J)*PIVOT-A(I,JJ)*A(II,J))/PIVOT
1100 CONTINUE
     DO 1200 J=1,NN
     A(II,J)=A(II,J)/PIVOT
1200 CONTINUE
     DO 1300 I=1,MM
     A(I,JJ)=0.
1300 CONTINUE
     A(II,JJ)=1.
     DO 1400 J=1,NN
```

```
      DO 1400 I=1,NV
      IF(J.EQ.JJ) GO TO 1400
      CONTAB(I,J)=CONTAB(I,J)-A(II,J)*CONTAB(I,JJ)
 1400 CONTINUE
      DO 1500 I=1,NV
      CONTAB(I,JJ)=0.0
 1500 CONTINUE
      GO TO 550
 1600 CONTINUE
      CALL EXIT
 4900 FORMAT(' ',2I5,F8.3)
 5000 FORMAT(2I5,2F4.2)
 5100 FORMAT(1H0)
 5200 FORMAT(5H     ,10F7.3)
 5250 FORMAT(1X,10F7.3)
 5300 FORMAT(12H1NEW TABLEAU)
 5400 FORMAT(1H ,F7.3,2I5)
 5500 FORMAT(' ','ERROR 1')
 5600 FORMAT(1H ,3F15.6)
 5700 FORMAT(1H1)
 5800 FORMAT ('0',4X,10(2X,A2,3X))
 5900 FORMAT (1H ,A2,2X,10F7.3)
      END
C THIS IS A SUBPROGRAM THAT CALCULATES THE PROBABILITY OF A
C NORMALLY DISTRIBUTED RANDOM BEING NONNEGATIVE GIVEN
C ITS MEAN AND VARIANCE
      FUNCTION PROB(FM,SM)
      DIMENSION A(73)
      DATA A/.5000,.5199,.5398,.5596,.5793,.5987,.6179,.5368
     1        ,.6554,.6736,.6915,.7088,.7257,.7422,.7580,.7734
     2        ,.7381,.3023,.8159,.8289,.8413,.8531,.8643,.8749
     3        ,.8849,.8944,.9032,.9115,.9192,.9265,.9332,.9394
     4        ,.9452,.9505,.9554,.9599,.9641,.9678,.9713,.9744
     5        ,.9772,.9798,.9821,.9842,.9861,.9878,.9893,.9906
     6        ,.9918,.9929,.9938,.9946,.9953,.9960,.9965,.9970
     7        ,.9974,.9978,.9981,.9984,.9987,.9989,.9990,.9992
     8        ,.9993,.9994,.9995,.9996,.9997,.9997,.9998,.9999
     9        ,1.000/
      Y=-FM/SQRT(SM)
      IF(Y.NE.0.0) GO TO 100
      PROB=0.5
      RETURN
  100 CONTINUE
      YY=ABS(Y)
      IF(YY.LT. 3.6) GO TO 200
      PROB=1.0
      GO TO 300
  200 CONTINUE
```

```
      I=YY/.05
      OI=I*.05
      I=I+1
      OJ=I*.05
      AI=A(I)
      AJ=A(I+1)
      PROB=AI+(YY-OI)/.05*(AJ-AI)
  300 CONTINUE
      IF(Y.GT. .0) RETURN
      PROB=1.-PROB
      RETURN
      END
```

```
3     2    0.100
 10.000100.000
 10.000100.000
  0.0      0.0
  0.0      0.0
  0.0      0.0
```

NEW TABLEAU

	C1	C2	C3	C4	C5	XC
C1	-1.000	0.0	0.0	0.0	0.0	0.0
C2	0.0	-1.000	0.0	0.0	0.0	0.0
C3	1.000	2.000	0.0	0.0	0.0	10.000
C4	2.000	1.000	0.0	0.0	0.0	10.000
C5	1.000	1.000	0.0	0.0	0.0	6.000

1.000	2.000	1.000	0.0	0.0	10.000
2.000	1.000	0.0	1.000	0.0	10.000
1.000	1.000	0.0	0.0	1.000	6.000

NEW TABLEAU

	C1	C2	C3	C4	C5	XC
C1	-1.000	0.0	0.0	0.0	0.0	0.0
C2	0.500	0.0	0.500	0.0	0.0	5.000
C3	0.0	0.0	-1.000	0.0	0.0	0.0
C4	1.500	0.0	-0.500	0.0	0.0	5.000
C5	0.500	0.0	-0.500	0.0	0.0	1.000

0.500	1.000	0.500	0.0	0.0	5.000
1.500	0.0	-0.500	1.000	0.0	5.000
0.500	0.0	-0.500	0.0	1.000	1.000

NEW TABLEAU

	C1	C2	C3	C4	C5	X0
C1	0.0	0.0	-1.000	0.0	2.000	2.000
C2	0.0	0.0	1.000	0.0	-1.000	4.000
C3	0.0	0.0	-1.000	0.0	0.0	0.0
C4	0.0	0.0	1.000	0.0	-3.000	2.000
C5	0.0	0.0	0.0	0.0	-1.000	0.0

0.0	1.000	1.000	0.0	-1.000	4.000
0.0	0.0	1.000	1.000	-3.000	2.000
1.000	0.0	-1.000	0.0	2.000	2.000

NEW TABLEAU

	C1	C2	C3	C4	C5	X0
C1	0.0	0.0	0.0	1.000	-1.000	4.000
C2	0.0	0.0	0.0	-1.000	2.000	2.000
C3	0.0	0.0	0.0	1.000	-3.000	2.000
C4	0.0	0.0	0.0	-1.000	0.0	0.0
C5	0.0	0.0	0.0	0.0	-1.000	0.0

0.0	1.000	0.0	-1.000	2.000	2.000
0.0	0.0	1.000	1.000	-3.000	2.000
1.000	0.0	0.0	1.000	-1.000	4.000

NEW TABLEAU

	C1	C2	C3	C4	C5	XC
C1	0.0	0.500	0.0	0.500	0.0	5.000
C2	0.0	-1.000	0.0	0.0	0.0	0.0
C3	0.0	1.500	0.0	-0.500	0.0	5.000
C4	0.0	0.0	0.0	-1.000	0.0	0.0
C5	0.0	0.500	0.0	-0.500	0.0	1.000

0.0	0.500	0.0	-0.500	1.000	1.000
0.0	1.500	1.000	-0.500	0.0	5.000
1.000	0.500	0.0	0.500	0.0	5.000

APPENDIX C

PROOF THAT THE PROBABILITY OF THE INTERSECTION OF THE SETS THAT DEFINE THE PROBABILITY SPACE OVER WHICH FEASIBLE BASES ARE OPTIMAL IS EQUAL TO ZERO

Theorem: Let $S_i = \{C | (C_B B^{-1} P_j - C_j) \geq 0\}$ define the set over which the i^{th} basis is optimal, and let $S_k = \{C | (C_B B^{-1} P_r - C_r) \geq 0\}$ define the set over which the k^{th} basis is optimal. Then $P[S_i \cap S_k] = 0$ when $i \neq k$.

Proof:

1. Theorem 2 of Chapter III proved that the bases were investigated contiguously, so to prove this theorem, it is only necessary to prove it for two adjacent bases.
2. S_i and S_k are convex sets since they were formed by the intersection of convex sets.
3. Assume that $P(S_i \cap S_k) \neq 0$ when $i \neq k$, i.e., $P(S_i \cap S_k) = P_i > 0$.
4. For 3 to be true, then $(S_i \cap S_k)$ contains at least one element that has a positive probability with respect to the probability distribution functions.
5. This requires that there exist some element of S_i and S_k that is in the interior of both of these sets.

6. Let $K = (i + 1)$, so S_i and S_k are sets associated with adjacent bases. Let X_j be the entering variable at the i^{th} basis and X_r be the leaving variable at the i^{th} basis.
7. Then, one of the conditions that defines the set S_k is $(Z_r - C_r) \geq 0$ in the $(i + 1)^{st}$ tableau, and one of the conditions that defines the set S_i is $(Z_j - C_j) \geq 0$ in the i^{th} tableau. Since the sets are adjacent, $(Z_r - C_r) = K\ (Z_j - C_j)$ where $K = -\ 1/\alpha_r{}^j$.
8. Therefore, at the $(i + 1)^{st}$ tableau

$$(Z_r - C_r) = -\frac{1}{\alpha_r{}^j}\,(Z_j - C_j)$$

and since $\quad (Z_j - C_j) = (C_B B^{-1} P_j - C_j)$

we have $\quad (Z_r - C_r) = -\dfrac{1}{\alpha_r{}^j}\,(C_B B^{-1} P_j - C_j)$

9. Therefore $(Z_r - C_r) \geq 0$ gives $(-\ 1/\alpha_r{}^j)\ (C_B B^{-1} P_j - C_j) \geq 0$ and since $\alpha_r{}^j > 0$, $(C_B B^{-1} P_j - C_j) \leq 0$.
10. The set of C's that are elements of S_i must satisfy the condition $(C_B B^{-1} P_j - C_j) \geq 0$ and the set of C's that are elements of S_k must satisfy the condition $(C_B B^{-1} P_j - C_j) \leq 0$.
11. The only set of C's that will satisfy these conditions is the set of C's that satisfy the equality $(C_B B^{-1} P_j - C_j) = 0$.
12. Therefore, $(S_i \cap S_k) = \{C \mid (C_B B^{-1} P_j - C_j) = 0\}$, and this defines the boundary between the sets.

13. This boundary is of probability measure zero with respect to the joint probability distribution functions, so $P[S_i \cap S_k] \neq 0$ and this condradicts the assumption of Step 3.

14. Therefore, $P[S_i \cap S_k] = 0$ and the theorem is proved.