THE FUNDAMENTAL THEOREM OF ALGEBRA:

A SURVEY OF HISTORY

AND PROOFS

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CHAPTER I

INTRODUCTION

The fundamental theorem of algebra states that a polynomial with complex coefficients of degree n where n is at least one has at least one complex root. We will examine the history of this theorem and investigate some proofs that have been devised in different areas of mathematics.

> If by algebra we mean the science which allows us to solve the equation $ax^2 + bx + c = 0$, expressed in these symbols, then the history begins in the 17th century; if we remove the restriction as to these particular signs, and allow for other and less convenient symbols, we might properly begin the history in the 3d century; if we allow for the solution of the above equation by geometric methods, without algebraic symbols of any kind we might say that algebra begins with the Alexandrian School or a little earlier; and if we say that we should class as algebra any problem that we should now solve by algebra (even though it was first solved by mere guessing or by some cumbersome

> arithmetic process), then the science was known about 1800 BC, and probably still earlier. (Smith, History II 378)

Our survey of the history of the fundamental theorem of algebra begins with the Babylonians. They were able to solve linear and quadratic equations. We will work our way through Egypt, India, Arabia, and Greece examining their ability and methods for solving linear and quadratic (and even some cubic) equations. Continuing on through Christian Europe, we will arrive in Italy in the 1500's.

The algebraists of the sixteenth century discovered general methods to solve cubic and quartic equations and began the search for a method to solve the general quintic equation by radicals.

With the onset of the seventeenth century, mathematicians' interests lay in infinitesimal analysis and analytic geometry. The search for a method to solve the quintic equation continued, while approximation methods for solving polynomial equations began to appear. Albert Girard is generally credited as being "the first to assert that [for an equation of the nth degree] there are always n solutions" in 1629 (Remmert 99). René Descartes' La Géométrie appeared in 1637. He gave a brief summary of all that was known of equations at this time. In this paper we can find what is known as "Descartes' Rule of Signs" which described the signs and number of real roots that a real polynomial equation may have.

> The fundamental theorem of algebra is of outstanding significance in the history of the theory of complex numbers because it was the possibility of proving this theorem in the complex domain that, more than anything else, paved the way for a general recognition of complex numbers. (Remmert 97)

By the eighteenth century, complex numbers were known but not completely understood or accepted. Even the great mathematicians of the time, Newton and Leibniz, were unclear about their exact meaning.

D'Alembert, a French mathematician, published an essay in 1746 in which the algebra of complex numbers was analyzed. He gave the first "proof" of the fundamental theorem of algebra, and for this reason, it is known as d'Alembert's Theorem in France. However, he assumed that all complex numbers were of the form a + bi, a fact which was yet to be proven, and in his proof of the fundamental theorem, he made some assumptions that were unfounded.

The reason for the sudden interest in the proof of the fundamental theorem of algebra was a claim made by Bernoulli that every polynomial could be factored into a product of linear and quadratic factors. This is just a restatement of the fundamental theorem of algebra that Bernoulli and others needed for their work on integrating rational functions.

Other "proofs" followed d'Alembert's, including proofs by Euler, Lagrange, and Laplace, the leading mathematicians of the day. Each of these proofs was found to be faulty, and Carl Gauss pointed the faults out in his dissertation of 1799 on the fundamental theorem of algebra. His proof is considered to be the first logically sound proof, and in his lifetime he found three other proofs.

A few years later it was found that the general quintic equation was not solvable by radicals, a fact that mathematicians had suspected for some time. Galois was able to "give a simple necessary condition for the solvability by radicals of a polynomial equation" (Hadlock 5). Thus the question of which equations could be solved was hereafter answered.

A blossoming of mathematics occurred, and many branches of mathematics sprouted. As the branches began to mature, different methods of proof of the fundamental theorem of algebra appeared.

We will look at Gauss' four proofs, proofs from Complex Analysis, Algebra, Analysis, and Topology. Then we will consider some historically significant approximation methods for finding roots of polynomials.

There are many other proofs of the fundamental theorem of algebra. The Appendix contains a bibliographical listing of some of these other proofs.

CHAPTER II

PREHISTORY

The Babylonians

To begin the history of the fundamental theorem of algebra, we look to the time of King Hammurabi of Babylon, around 1800 BC. Two tablets from that period were found in 1854 at Senkerah on the Euphrates by a British geologist, W. K. Loftus. These tablets contain the squares of numbers from one to sixty and the cubes of numbers from one to thirty-two. Since this important discovery, 50,000 tablets have been unearthed. Many relate to mathematics. These include multiplication and division tables, tables of squares and square roots, geometric progressions, a few computations, and some work on mensuration. The study of a large number of tablets shows that the Sumerians and Babylonians could solve linear, quadratic, and cubic equations and had some knowledge of negative numbers (Smith 40).

As early as 3000 BC the Babylonians had begun cuneiform writing on clay tablets, and the sexagesimal (base 60) number system was already in use. King Hammurabi's time was a time of "flowering of algebra and geometry" in Babylonia (van der Waerden 62).

Although the city of Babylon was not the center of the culture associated with the two rivers, the Tigris and the Euphrates, the series of people who occupied that area are now referred to as Babylonians by historians of mathematics. This region, then known as Mesopotamia, is now part of Iraq. This area was settled by the Sumerians about 4000 BC.

Much information about the lives of the Babylonians can be found in the tablets that were unearthed. Carl Boyer writes: "Laws, tax accounts, stories, school lessons, personal letters - these and many other records were impressed on soft clay tablets with a stylus, and the tablets then were baked in the hot sun or in ovens" (27). These tablets have weathered well, and we know a great deal more about Babylonia than some other ancient civilizations because of it.

The great majority of the texts are Old Babylonian - that is, contemporary with Hammurabi - and the rest are from the Seleucid period, which took place during the last three centuries BC. It is not known why there have been no tablets found from between the Old Babylonian and Seleucid periods. The language and symbolism had changed from the first tablets to those of later date, but says Otto Neugebauer in <u>The Exact Sciences in Antiquity</u>,

> So far as the contents are concerned, little change can be observed from one group to the other. The only essential progress which was

> made consists in the use of the zero sign in the Seleucid texts (29).

The texts fall into two classes, besides the tables mentioned earlier. One kind of text formulates the problem and then shows the method of solution. The second kind consists of lists of problems arranged from the simplest cases to problems with very "elaborate polynomials." Neugebauer found that sometimes the solutions to all the problems on a certain tablet were the same, and he believed that these tablets were for students.

> It was no concern to the teacher that the result must have been known to the pupil. What he obviously had to learn was the method of transforming such horrible expressions into simpler ones and to arrive finally at the correct solutions. (42)

The problems found on these tablets were stated and solved verbally.

> The words us (length), sag (breadth), and asa, (area) were often used for the unknowns, not because the unknowns necessarily represented the geometric quantities, but probably because many algebraic problems came from geometric situations and the geometric terminology became standard. (Kline 9)

There now exist translations of some of the mathematical tablets. The one that contained the following example originally had twenty-two problems solvable as linear equations in one unknown.

> I found a stone, (but) did not weigh it; (after) I weighed (out) 6 times (its weight), [added] 2 gin, (and) added one-third of one-seventh multiplied by 24, I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? The origin(al weight) of the stone was 4 1/3 gin.

In our symbols:

$(6x + 2) + (1/3)\cdot(1/7)\cdot 24\cdot(6x + 2) = 60$. Note that 1 ma-na is 60 gin (Fauvel 26).

A tablet which can now be found in the British Museum gives this problem:

> I have added the area and two thirds of the side of my square and it is 0;35. You take 1, the coefficient. Two thirds of 1, the coefficient, is 0;40. Half of this, 0;20, you multiply by 0;20 (and the result) 0;6,40 you add to 0;35, and (the result) 0;41,40 has 0;50 as its square root. 0;20, which you multiplied by itself, you subtract from 0;50 and 0;30 is the (side of) the square. (Aaboe 23)

This example states and solves the quadratic equation $x^2 + (2/3)x = 0;35$. Recall that this problem is in base 60 and that 0;35 means 35/60. So 2/3 of 1 is 2/3 of 60/60 which is 40/60 or 0;40. 0;6,40 is $(6\cdot 60 + 40)/3600$ or

400/3600 and 0;41,40 is (41·60 + 40)/3600 or 2500/3600. We should note also that words like the coefficient and the result are used, perhaps indicating that this shows a general method for solving all quadratic equations, not just this specific example.

Many Old Babylonian texts show solutions of quadratic equations. "[The Babylonians] could transpose terms in an equation by adding equals to equals, and they could multiply both sides by like quantities to remove fractions or to eliminate factors" (Boyer 33). One problem asks for a number which added to its reciprocal yields a given number. In our symbols:

Find x and y such that $xy = 1$ and $x + y = b$. By solving for y in the second equation, and substituting into the first equation, we arrive at a quadratic equation in x, $x^2 - bx + 1 = 0$. As the example above shows, the Babylonians formed $(b/2)^2$: then $\sqrt{(b/2)^2 - 1}$; and then $(b/2) + \sqrt{(b/2)^2 - 1}$ and $(b/2) - \sqrt{(b/2)^2 + 1}$. Other problems were reduced to the above problem and then solved. The solution to each specific problem was found in this general way, without explicitly stating this formula; the Babylonians emphasized procedure (Kline 8).

The following problem is taken from a text from Senkereh where many of the tablets were found and was written during the Hammurabi dynasty.

> Length, width. I have multiplied length and width, thus obtaining the area. Then I added to

> the area, the excess of the length over the width: 183. Moreover, I have added length and width: 27. Required length, width, and area.

This problem also leads to a quadratic equation. The complete Babylonian solution is given in Science Awakening (van der Waerden 63).

We should note that the Babylonians added lengths to areas. Both Aaboe and Boyer point out that these problems could not be practical mensuration problems (Aaboe 25; Boyer 33).

The Babylonians also solved some cubic equations. Pure cubics were solved by using tables of cubes and cube roots. Linear interpolation was used when the value was not listed in the tables. Similarly, tables were used to solve cubics of the form $x^3 + x^2 = a$ (Boyer 36).

With regard to the Babylonians and the fundamental theorem of algebra Boyer writes, "questions about the solvability or unsolvability of a problem do not seem to be raised" (44).

The Egyptians

Next, we look to the fertile Nile valley where as early as 3000 BC the Egyptians were using hieroglyphics and had symbols for numbers up to 100,000. The pyramids of Egypt were built in the period now referred to as the Old Kingdom. During the Middle Kingdom, from 2000 BC to 1800 BC, many papyri were written by the scribes of that time. The most famous of the mathematical papyri, now called the Rhind Papyrus, was copied sometime around 1700 BC. This papyrus was discovered in some ruins of the Rameseum at Thebes, and was acquired in 1858 by A. H. Rhind, a Scottish Egyptologist who specialized in tombs (Gillings 89).

The Rhind Papyrus, which is now in the British Museum, was deciphered by Eisenlohr in 1877 and found to be a "mathematical manual" (Cajori 9). The papyrus is one foot high and eighteen feet long. The papyrus is not written in hieroglyphics but in a more cursive script, called hieratic, which, according to Boyer, was "better adapted to the use of pen and ink on prepared papyrus leaves" (12). It was copied by the scribe Ah-mose or Ahmes and is sometimes referred to as the Ahmes Papyrus. The Ahmes Papyrus is contemporary with the Babylonian tables of squares and cubes found at Senkerah (Karpinski 3).

The prologue promises much: "Complete and thorough study of all things, insight into all that exists, knowledge of all secrets." Van der Waerden says,

> It soon becomes evident that we shall not witness the revelation of the origin of things, but that we shall merely be initiated into the secrets of numbers and into the art of calculating with fractions, in order to apply these to various practical problems with which

> the officials of the great state had to deal. (16)

There seems to be differing opinions on the purpose of the Ahmes Papyrus. Smith says,

> It is not a textbook, but is rather a practical handbook. It contains material on linear equations of such types as $x + 1/7x = 19$; it treats extensively of unit fractions; it has a considerable amount of work on mensuration, and it includes problems in elementary series. (History I 48)

Boyer, on the other hand, believes that the calculations are "practice exercises for young students." He says, "Although a large proportion of them are of a practical nature, in some places the scribe seems to have had puzzles or mathematical recreations in mind" (17). The papyrus contains eighty-five problems and solutions. Kline remarks, "Presumably such problems occurred in the work of the scribes and they were expected to know how to solve them" (16).

The problems found on the Ahmes papyrus are equivalent to linear equations of the form $x + ax = b$ or $x + ax + bx = c$, where a, b, and c are known and x is unknown. The unknown quantity was referred to as aha or heap. The following problem, number 24 from the Rhind Papyrus, is solved by the method of false position or the rule of false. In this method, a value is assumed for aha, usually a false one, the operations are performed on this value, and then the result is compared to the result desired. Then by the use of proportions the correct value is found.

Problem 24: A quantity and its 1/7 added together become 19. What is the quantity? Assume 7.

\1	7
\1/7	1
Total	8.

As many times as 8 must be multiplied to give 19, so many times 7 must be multiplied to give the required number.

1	8
\2	16
1/2	4
\1/4	2
\1/8	1

Total 2 1/4 1/8.

\1	2 1/4 1/8
\2	4 1/2 1/4
\4	9 1/2

Do it thus: The quantity is 16 1/2 1/8,

1/7	2 1/4 1/8
Total	19. (Chace 36)

The problem is solved like this: Assume 7 is the answer. Find 1/7 of 7 which is 1. 7 and 1 are 8, so we will find how many times 8 will go into 19, and then multiply 7 by this to arrive at the correct answer. The Egyptians did not divide 19 by 8, but calculated parts of 8, like twice 8, half 8, and so on. The sum of 16 and 2 and 1 is 19, and you will see the parts marked in the column on the left: 2, 1/4, and 1/8. And so, 8 will divide into 19, 2 1/4 1/8 times. (The Egyptians almost always worked with unit fractions, except for the fraction 2/3.) Then since 7 is 1 and 2 and 4, Ahmes finds once, twice, and four times 2 1/4 1/8 and adds to get the result, 16 1/2 1/8.

These problems employed practically no symbolism. However, addition and subtraction are represented by the legs of a man coming and going, and , and the symbol was used to denote square roots (Kline 19).

Other mathematical papyri have been found including the Moscow papyrus which is 18 feet long and 3 inches wide.

> It was written, less carefully than the work of Ahmes, by an unknown scribe of the twelfth dynasty (1890 BC). It contains twenty-five examples, mostly from practical life and not differing greatly from those of Ahmes . . . (Boyer 20)

Although we know that the Babylonians could solve quadratic and cubic equations, the Egyptians could get no further than linear equations and simple quadratic equations which we would write in the form $ax^2 = b$. In

Science Awakening, van der Waerden says,

> the aha calculations constitute the climax of Egyptian arithmetic. The Egyptians could not possibly get beyond linear equations and pure quadratics with one unknown, with their primitive and laborious computing technique. (29)

The Hindus

There was a highly cultured civilization in India as far back as the days of the pyramid builders in Egypt. The religious leader Buddha was active about the same time Pythagoras was supposed to have visited India (Boyer 229). Trade was carried on between India and Greece, and India and Rome via Alexandria. The Indian society was one of castes, and only the religous caste and the war and government caste had the privelege and leisure to think about mathematics (Cajori 83).

We know little of the development of Hindu mathematics. Cajori mentions the discovery of an anonymous arithmetic written on birch bark whose probable date was the eighth century. The Indians put all mathematical results into verse, "clothing them in obscure and mystic languages" (Cajori 83).

Hindu mathematics may be resolved into two periods: the S'ulvasùtra period which lasted until AD 200 and the Siddhàntas period, which lasted until AD 1200. The word

S'ulvasùtra means the rules of the cord. The main purpose of the book was religious and the mathematical portion dealt with the construction of squares and rectangles (Cajori 85).

Siddhàntas means a system of astronomy. Aryabhata, born in AD 476, was a noted Hindu astronomer of this period. His celebrity rests on a work entitled Aryabhatiya which was a summary of earlier developments in astronomy and mensuration. Most of the work of Hindu mathematicians was motivated by astronomy and astrology. There were no separate mathematical texts. Mathematics is presented in chapters of works on astronomy (Kline 184).

Brahmagupta was born in AD 598; in 628 he wrote Brahma-sphuta-Siddhànta which translates as The Revised System of Brahma. Two chapters deal with mathematics. He applies algebra to astronomical calculations and solves indeterminate equations (Smith 158). The arithmetic of negative numbers and zero are first found in his work. Brahmagupta knew the general solution to quadratic equations and found two roots, even when one was a negative number (Boyer 242). His recognition of negative roots enabled him to bring the three forms of the quadratic equation previously studied under one general case, $px^2 + qx + r = 0$ (Cajori 94).

Mahavira lived in the ninth century. According to C. N. Srinivasiengar in The History of Ancient Indian Mathematics, Mahavira was a mathematician only, not an astronomer. He wrote Ganita Sara Sangraha in AD 850. There are no new discoveries and the problems are often long and complicated.

> Out of a certain number of Sarasa birds, one-fourth the number are moving about in lotus plants; one-ninth coupled with one-fourth as well as seven times the square root of the number move on a hill; 56 birds remain in Vakula trees. What is the total number of birds?

If x is the number of birds, this problem leads to the equation $x = x/4 + x/9 + x/4 + 7\sqrt{x}$ [+ 56], whose solution is $x = 576$ (71).

Three centuries later we find Bhaskara filling gaps in Brahmagupta's work. Boyer says "There is a striking lack of continuity of tradition in the mathematics of India; significant contributions are episodic events separated by intervals without achievement" (229). Bhaskara wrote Lilavati which contained problems from Brahmagupta and others, adding new observations of his own. He discussed linear and quadratic equations, both determinate and indeterminate (Boyer 245). Bhaskara says

> the square of a positive as also of a negative number, is positive; that the square root of a positive number is twofold, positive and negative. There is no square root of a negative number, for it is not a square.

An example of the type of problem found in the Lilavati follows. Note, as Cajori calls it, "the pleasing poetic garb in which all arithmetical problems are clothed" (92).

> Out of a swarm of bees a number equal to the square root of half their number went to the Malati flowers; 8/9th of the total number also went to the same place. A male bee enticed by the fragrance of the lotus got into it. But when it was inside it, night fell, the lotus closed, and the bee was caught inside. To its buzz, its consort was replying from outside. What is the number of bees? (Srinivasiengar 86)

If x is the total number of bees, this problem leads to the equation $\sqrt{(x/2)} + (8/9)x + 2 = x$ and the solution is $x = 72$. Quadratic equations were solved by both completing the square and by using some version of the quadratic formula.

The Indians "greatly aided the progress of mathematics" by "never discerning the dividing line between numbers and magnitude" as the Greeks had. They advanced beyond Diophantus in observing that a quadratic equation always has two roots. "But," says Bhaskara, "the [negative] value is not to be taken, for it is inadequate; people do not approve of negative roots" (Cajori 93). The Indian arithmetic and algebra was completely independent of geometry (Kline 186).

The most important contribution of the Hindu mathematicians was the development of our system of notation for integers. They could solve linear and quadratic equations and were not hampered by negative solutions. Of equations of degree three and higher, they could solve special cases in which both sides of the equation could be made perfect powers by the addition of some terms (Cajori 94).

The Greeks

> . . . we now have reason to believe, on the basis of the <u>Iliad</u> and the <u>Odyssey</u> of Homer, the decipherment of ancient languages and scripts, and archeological investigations, that the Greek civilization dates back to 2800 BC. (Kline 24)

The Greeks lived in Asia Minor and on the mainland of Europe, in southern Italy, Sicily, Crete, Rhodes, Delos, and in North Africa. During the time period 1000 BC to AD 600, most of the Mediterranean world, as far as mathematical achievements are concerned, can be considered to be under Hellenic influence.

Some authors find it desirable to distinguish two periods in the history of Greek civilizaton, the Classical Period, which lasted from 600 BC to 300 BC, and the Alexandrian or Hellenistic Period, 300 BC to AD 600. "The adoption of the alphabet and the fact that papyrus became available in Greece during the seventh century BC may account for the blossoming of cultural activity about 600 BC" (Kline 25). Schools were formed in which knowledge was passed from one person to the next, with the first, the Ionian school, founded by Thales in Miletus around 585 BC. Pythagoras is supposed to have learned from Thales and then formed his own school in Southern Italy around 550 BC. The most celebrated school was the Academy of Plato in Athens. Plato was taught by two Pythagoreans, Theodorus of Cyrene and Archytas of Tarentum. Plato founded his school in 385 BC (Kline 27).

Our knowledge of the mathematics of ancient Greece comes from Byzantine Greek codices or manuscript books that were written 500 to 1500 years after the original Greek works were composed. Kline writes "These codices are not literal reproductions but critical editions, so that we cannot be sure what changes may have been made by the editors" (25). We also have some Arabic translations of the works of the Greeks. Neugebauer points out another important fact:

> Any attempt to reconstruct the origin of Hellenistic mathematics and astronomy must face the fact that Euclid's Elements and Ptolemy's Almagest reduced all their predecessors to objects of mere historical interest with little chance of survival. As Hilbert once expressed it, the importance of a scientific work can be measured by the number of previous publications it makes superfluous to read. (145)

Eudemus, who lived in the fourth century BC, wrote a history of arithmetic, a history of geometry, and a history of astronomy. These histories are lost except for fragments quoted by later writers (Kline 26).

The intellectuals or scholars of ancient Greece did not "concern themselves with practical problems. They confined themselves to philosophical and scientific activities and took no hand in commerce or the trades" (Kline 49). Thus we find that they did not concern themselves with arithmetic nor algebra, but focused instead on geometry. Kline says: "It is clear . . . that Plato and other Greeks for whom he speaks valued abstract ideas and preferred mathematical ideas as a preparation for philosophy" (44). He also points out that the Greeks were the first to consciously recognize that "mathematical entities, numbers, and geometrical figures are abstractions, ideas entertained by the mind and sharply distinguished from physical objects or pictures" (29).

We have established that the Babylonians could solve quadratic equations. Neugebauer raises the question of the "specific way in which such knowledge found its way to Greece. Here we are left to mere speculation" (150). Boyer describes the creation of geometrical algebra:

> The dichotomy between number and continuous magnitude required a new approach to the Babylonian algebra that the Pythagoreans had inherited. The old problems in which, given the sum and product of the sides of a rectangle, the

> dimensions were required, had to be dealt with differently from the numerical algorithms of the Babylonians. A geometrical algebra had to take the place of the older arithmetical algebra, and in this new algebra there could be no adding of lines to areas or of areas to volumes. From now on there had to be a strict homogeneity of terms in equations, and the Mesopotamian normal forms, $xy = A$, $x \pm y = b$, were to be interpreted geometrically. (85)

Aaboe points out that the "irrationality of $\sqrt{2}$ has serious consequences for algebra, for it showed that the simple problem of finding x such that $x^2 = 2$ which could easily be stated, had no exact solution in numbers, for numbers meant rational numbers." He felt this is what gave rise to geometrical algebra. (44)

Algebra was reformulated in geometric terms. The phrase "the rectangle of sides a and b" was used instead of "a times b." Even today we say x squared and x cubed for x^2 and x^3. Throughout Greek mathematics, there are numerous applications of this algebra.

> The line of thought is always algebraic, the formulation geometric. The greater part of the theory of polygons and polyhedra is based on this method; the entire theory of conic sections depend on it. (van der Waerden 119).

The arguments of van der Waerden make a good case for geometrical algebra although some mathematics historians are in total disagreement with him.

We have seen that the Babylonian treatment of problems of second degree consist in reducing them to normal form where two quantities, x and y, should be found from their given product and their sum or difference. It seems significant that the geometric formulation of this problem leads precisely to the central problem of the geometrical algebra, a problem which is otherwise rather difficult to motivate. This problem is known as the application of area, which consists, in its simplest form, in the following: Given an area A and a line segment b; construct a rectangle of area A such that one of its sides falls on b but in such a way that the rectangle of equal height and of length b is either larger or smaller by a square than the rectangle of area A. The identity of this strange geometrical problem with the Babylonian normal form is at once evident when we formulate it algebraically. Let us call, in both cases, x and y the sides of the rectangle. Then we are given $xy = A$. In the first case a square should remain free; its sides are y and we must require $x + y = b$. In the second case, a square should exceed the rectangle of side b; thus we should

> have $x - y = b$. These are indeed the normal forms. Attempts have been made to motivate the problem of application of area independently of this algebraic background. There is no doubt, however, that the above assumption of a direct geometrical interpretation of the normal form of quadratic equations is by far the most simple and direct explanation. (149)

However, not all historians are in agreement about geometrical algebra. Sabetai Unguru refutes the arguments of van der Waerden and others by saying

> the view that Greek mathematics, especially after the discovery of the irrational by the Pythagorean school, is algebra dressed up primarily for the sake of rigor, in geometrical garb . . . I believe such a view is offensive, naive, and untenable. It is certainly indefensible on the basis of the historical record . . . (85)

Discussion of geometric algebra leads us to Euclid and <u>Elements</u>. Euclid is believed to have lived around 300 BC and it is

> . . . most probable that Euclid received his mathematical training in Athens from the pupils of Plato . . . and it was in Athens that the older writers of elements and the other

> mathematicians on whose works Euclid's <u>Elements</u> depend, had lived and taught. (Heath 2)

Not much else is known of Euclid's life.

<u>Elements</u> is written in thirteen parts or books. Book I covers congruence, parallels, and the Pythagorean Theorem; Book II discusses identities which we would now treat algebraically, like $(a + b)^2 = a^2 + 2ab + b^2$ but which were then treated geometrically, application of area problems, and the Golden Ratio. In Book III we find circles, in IV, inscribed and circumscribed polygons. Book V treats proportion geometrically, as Smith says in <u>History of Mathematics</u>, "a geometric way of solving fractional algebraic equations" (105). Book VI is on similarity of polygons, VII-IX are on arithmetic (the ancient theory of numbers) treated geometrically, Book X is about incommensurable magnitudes and the rest of the books are on solid geometry.

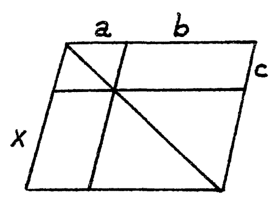
By Euclid's time, geometrical algebra had "reached such a stage of development that it could solve the same problems as our algebra so far as they do not involve the manipulation of expressions of a degree higher than the second" (Heath 372). The theory of proportions was necessary to make the geometric algebra effective. Eudoxus, who lived from 408 BC to 355 BC, is credited with the discovery of the theory of proportions.

> What Eudoxus accomplished was to avoid irrational numbers as numbers . . . Eudoxus'

> theory enabled the Greek mathematicians to make tremendous progress in geometry . . . it forced a sharp separation between number and geometry, for only geometry could handle incommensurable ratios. (Kline 48)

Theorem 4 in Book I of Elements is the first example of a geometric solution of an equation, according to Aaboe.

> Theorem 4: In any parallelogram the complements of parallelograms about the diagonal are equal. (58)

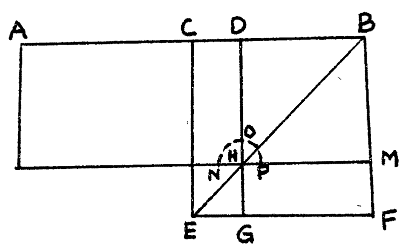


This translates to a construction problem, the task is to find an x such that $x \cdot a = b \cdot c$ where a, b, and c are given line segments.

The application of area was used to solve simple linear equations such as $ax = F$; and to solve $x^2 = F$ which amounts to the transformation of a given area into a square; and to solve the pure cubic $x^3 = V$ which poses the problem of constructing a cube of a given volume. Other quadratics were solved with this method after first being reduced to one of the forms $x(x + a) = F$, $x(a - x) = F$, or $x(x - a) = F$ (van der Waerden 124).

The geometrical solution of a quadratic equation can be found in Proposition 5 of Book II.

> Proposition 5: If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.



Heath translates this into algebraic symbols in this way:

> Suppose that AB = a, DB = x then
>
> $ax - x^2$ = the rectangle AH
>
> = the gnomon NOP.
>
> Thus if the area of the gnomon is given ($= b^2$, say), and if a is given (= AB), the problem of solving the equation $ax - x^2 = b^2$ is, in the language of geometry, to a given straight line (a) to apply a rectangle which shall be equal to

> a given square (b^2) and shall fall short by a square figure. (383)

Heath defines a gnomon as the figure which remains of a square when a smaller square is cut out of one corner.

The geometrical solution of the quadratic equation derived from Euclid is a geometric version of our practice of completing the square on the side containing the terms in x^2 and x.

In Science Awakening, van der Waerden wonders about geometric algebra:

> Why did the Greeks not simply adopt Babylonian algebra as it was, why did they put it in geometric form? . . . Would these worshippers of numbers have solved quadratic equations, not in terms of numbers, but by means of segments and areas, purely for the delight in the visible? This is hard to believe; there must have been another push towards the geometrisation of algebra.

He discusses the discovery of the irrational, which originated in the Pythagorean school. He then says,

> For the Babylonians, every segment and every area simply represented a number. They had no scruples in adding the area of a rectangle to its base. When they could not determine a square root exactly, they calmly accepted an approximation. . . . But the Greeks were

> concerned with exact knowledge, with the diagonal itself, as Plato expresses it, not with an acceptable approximation. In the domain of numbers, the equation $x^2 = 2$ can not be solved, not even in that of ratios of numbers. But it is solvable in the domain of segments: indeed the diagonal of the unit square is a solution. Consequently, in order to obtain exact solutions of quadratic equations, we have to pass from the domain of numbers to that of geometric magnitudes. . . . It is therefore logical necessity, not the mere delight in the visible, which compelled the Pythagoreans to transmute their algebra into a geometric form. (125)

Archimedes, around 250 BC, was able to solve cubic equations which we would now write as $x^3 \pm ax^2 \pm b^2c = 0$ by the intersection of conics. In <u>On the Sphere and Cylinder</u>, we find this problem:

> Proposition 4: To cut a given sphere by a plane so that the volumes of the segments are to one another in a given ratio.

This problem is equivalent to solving the equation $x^2(a - x) = b^2c$ (Works 70). Archimedes solved it by intersecting the parabola, $x^2 = (a^2/c)y$ and the hyperbola, $y(c - x) = bc$ (Kasir 13).

Hero, or Heron of Alexandria, who lived in AD 60 was able to solve $ax^2 + bx = c$. But his method of solving resembled the Babylonian method of solving and he felt free to add areas and line segments. Heath discusses a problem from Hero's Geometrica that leads to a quadratic equation.

> Given a square such that the sum of its area and perimeter is 896 feet: to separate the area from the perimeter.

Hero solves this by the method we now call completing the square. Neugebauer says this return to the Babylonian methods should no longer be "viewed as a novel sign of the rapid degeneration of the so called Greek spirit, but simply reflects the algebraic or arithmetic tradition of Mesopotamia" (146). We will find whole sections of these works again in the famous Algebra of al-Khowârizmi, who lived in AD 800.

Diophantus of Alexandria lived around AD 250. He was the author of Arithmetica, a book containing 189 problems and solutions. He reduces all of his problems to equations in one unknown. He apparently knew how to solve the general quadratic equation of the form $ax^2 = 2bx + c$ by using what we now know as the quadratic formula. Many of the problems and solutions are closely related to Babylonian problems, and as van der Waerden expresses it:

> It is probable that the tradition of these algebraic methods was never interrupted so that, along with the scholarly tradition of Greek geometry, there has always existed a more

> popular tradition of small algebraic problems and methods of solution . . . " (280)

Arithmetica covers much that is now included in algebra. Diophantus introduced some symbolism, including a symbol for the unknown quantity, and special names for powers of the unknown (van der Waerden 281). Cajori notes,

> If we except the Ahmes Papyrus, which contains the first suggestions of algebraic notation and of the solution of equations, then Diophantus' *Arithmetica* is the earliest treatise on algebra now extant. (60)

"The *Arithmetica* is not," according to Boyer, "a systematic exposition of the algebraic operations or of algebraic functions or of the solution of algebraic equations" (202). It is a collection of problems solved with specific numerical examples and Diophantus makes no effort to find all solutions. Even for indeterminate equations he is satisfied with one solution.

The following problem and solution are from Book VI of *Arithmetica*.

> Problem 6: To find a right angled triangle such that the area added to one of the perpendiculars makes a given number.
>
> Given number 7, triangle $(3x,4x,5x)$.
>
> Therefore $6x^2 + 3x = 7$.
>
> In order that this might be solved, it would be

necessary that

(half coefficient of $x)^2$ + product of

coefficient of x^2 and absolute term should be a

square;

but $(3/2)^2 + 6 \cdot 7$ is not a square.

Hence we must find, to replace (3,4,5), a right

angled triangle such that

(1/2 one perpendicular)^2 + 7 times area

= a square.

Let one perpendicular be m, the other 1.

Therefore $(7/2)m + 1/4$ = a square,

or $14m + 1$ = a square.

Also, since the triangle is rational,

$m^2 + 1$ = a square.

The difference $m^2 - 14m = m(m - 14)$;

and putting, as usual, $7^2 = 14m + 1$,

we have $m = 24/7$.

The auxiliary triangle is therefore

(24/7,1,25/7) or (24,7,25).

Starting afresh, we take as the triangle

(24x,7x,25x).

Therefore $84x^2 + 7x = 7$, and $x = 1/4$.

We have then (6,7/4,25/4) as the solution.

(Heath, Diophantes 228)

Note that no instructions are given to solve the quadratic equation $84x^2 + 7x = 7$.

Diophantus accepted only positive roots. If a quadratic equation led to two negative roots or imaginary roots, he rejected the equation as not solvable. Until negative and imaginary roots are accepted, the fundamental theorem of algebra can not be formulated.

The Arabs

Around AD 600 the peninsula of Arabia was inhabited by desert nomads called Bedouins. These Bedouins could neither read nor write. Mohammed was a Bedouin and he became a military as well as a religious leader; he established an Islamic state whose center was Mecca. The expansion of the state was not slowed by his death, and soon the whole Mesopotamian valley as well as Alexandria fell to the Arabic conquerors. Legend has it that the leader of the victorious troops was told to burn the books of the Alexandrian library; "for if they were in agreement with the Koran they were superfluous, if they were in disagreement they were worse than superfluous." The story continues that the baths of the city were long heated from the flames of the burning books (Boyer 249).

By 750, these conquerors became eager to absorb the learnings of the civilizations they had overrun. An astronomical-mathematical work known by the Arabic name <u>Sindhind</u> was brought to Baghdad from India about 766. It is generally believed to be the <u>Siddhànta</u> of Brahmagupta (Boyer 250). Greek physicians and scholars were called to Baghdad and translations of other works began.

> The translation of mathematical works must have been very deficient at first, as it was evidently difficult to obtain translators who were masters of both Greek and Arabic and at the same time proficient in mathematics. The translations had to be revised again and again before they were satisfactory. (Cajori 101)

Important authors that were translated into Arabic were Euclid, Ptolemy, Appolonius, Archimedes, Heron, and Diophantus.

One of the translators was Mohammed ibn Musa al-Khowarizmi who lived during the ninth century. The Book of Chronicles, a collection of biographies of learned men of all nations, was completed in AD 987. We find this entry:

> Mohammed ibn Musa, born in Khowarizm, worked in the library of the caliphs under Al-Mamun. During his lifetime and afterward, where observations were made, people were accustomed to rely upon his tables, which were known by the name Sind-Hind. He wrote: The Book of Astronomical Tables, On the Sundial, On the Use of the Astrolabe, The Book of Chronology.

The bibliography does not mention four other works by al-Khowarizmi, including The Book of Algebra and On the Hindu Art of Reckoning, but we find them listed mistakenly in the biography that follows his (Karpinski 14).

Al-Khowarizmi was the first Islamic author to write on the solutions of problems by al-jabr and al-muqabala. The nearest English translation of these words is "restoration and reduction. By restoration was meant the transposing of negative terms to the other side of the equation; by reduction the uniting of similar terms" (Cajori 103). It is believed that the term "algebra" originates from "al-jabr" and the term "algorithm" is from the name "al-Khowarizmi." This work explains the elementary operations and the solutions of linear and quadratic equations. Kline says,

> The algebra of Al-Khowarizmi is founded on Brahmagupta's work but also shows Babylonian and Greek influences . . . He uses special names for the powers of the unknown. The unknown he refers to as the 'thing' or the root of a plant, whence our term root . . . He recognizes that there can be two roots of quadratics, but gives only the real positive roots, which can be irrational. (192)

He presented demonstrations of his solutions to quadratic equations by drawing a square and adding rectangles to it. The following is from The Algebra, Chapter IV: Concerning squares and roots equal to numbers.

> The following is an example of squares and roots equal to numbers: a square and 10 roots are equal to 39 units. The question is therefore in

> this type of equation is about as follows: what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore, take 5, which multiplied by itself gives 25, an amount which you add to 39, giving 64. Having taken then the square root of this which is 8, subtract from it the half of the roots, 5, leaving 3. The number 3 therefore represents one root of this square, which itself, of course, is 9. Nine therefore gives that square. (Karpinski 71)

The geometric solution is discussed on page 77 of Robert of Chester's Latin Translation of the Algebra of al-Khowarizmi.

At the beginning of the eleventh century, a work on the theory of numbers and algebra was done by Al-Karkhi of Baghdad. This work, drawn largely from Hindu sources, includes algebraic operations, finding roots, solving equations of the first and second degree, and indeterminate analysis. Al-Karkhi gives both arithmetical and geometrical proofs for the solutions of quadratic equations (Kasir 17).

Abu Ja'far Alchazin was the first Arab to solve cubic equations by using conic sections. The one who did most to "elevate to a method the solution of algebraic equations by intersecting conics, was the poet Omar Khayyam." Omar Khayyam lived in the eleventh century and was a poet, philosopher, mathematician, and astronomer. His algebra was mainly geometric, solving linear and quadratic equations with the methods of Euclid. He solved six types of trinomials that had a cubic term. To solve a problem that leads to "a cube and a number equal to sides", which we would write as $x^3 + a = bx$, Omar Khayyam used a parabola and a hyperbola. He showed methods to solve equations of the form $x^3 + cx^2 + bx = a$ and $x^3 + cx^2 = bx + a$ (Kasir 75). In The Algebra of Omar Khayyam by Daoud S. Kasir, Omar Khayyam states that if the numerical solution is not supplemented by geometric construction, or vise versa, "the art of algebra could not be verified" (21). He rejected negative roots of equations and sometimes failed to discover all the positive ones.

With Al-Kharki and Omar Khayyam, mathematics among the Arabs of the East reached a high mark and then began to fade. In the West, in Spain, there lived an astronomer named Jabir ibn Aflah. It was formerly believed that he was the inventer of algebra and that the word algebra came from the name, Geber, by which Aflah was frequently known. He was a very good astronomer, but "like so many of his contemporaries, his writings contain a great deal of mysticism" (Cajori 109).

We have seen that the Arabs could solve linear, quadratic, and some cubic equations. It was not clear to them that all quadratic equations should have two solutions unless both those solutions were positive.

Christian Europe

> With the third century after Christ begins an era of migration of nations in Europe. The powerful Goths quit their swamps and forests in the North . . . crossing the Roman territory, and stopping and recoiling only when reaching the shores of the Mediterranean. . . . wild hordes sweep down on the Danube. The Roman Empire falls to pieces, and the Dark Ages begin. (Cajori 113)

During this period of the Dark Ages and onward, to about AD 1000, we find, quoting Smith in History of Mathematics,

> the slow civilizing of the northern races, the development of monastic schools, the work of Charlemagne, and the contact with Oriental civilization, chiefly through the Moors of Spain. In mathematics it was the era of the development of the Christian calendar in the West, and little else. (177)

Scattered groups of people who had been part of the Roman Empire had acquired some learning. Before the collapse of the Empire, the Catholic Church was organized and powerful. The Church converted the "Germanic and Gothic barbarians" and began to found schools, attaching them to already existing monasteries.

> As the Church extended its influence it imposed the culture it favored. Latin was the official language of the Church and so Latin became the international language of Europe and the language of mathematics and science. (Kline 201)

As the Europeans began to seek knowledge, they naturally turned to books written in Latin. As Roman mathematics was practically insignificant, only a few facts of arithmetic and a primitive number system were available to the people.

Boethius lived at the opening of the Dark Ages and was a "statesman, philosopher, mathematician, man of letters, and founder of medieval scholasticism" (Smith 178). His mathematical works included arithmetic, geometry, and music. The <u>Geometry</u> consisted of the statements of the propositions of Book I, III, and IV of Euclid's <u>Elements</u>, and the <u>Arithmetic</u> is a translation of Nichomachus' <u>Introduction to Arithmetic</u>, a book concerned with the properties of numbers. Boethius understood the subject sufficiently so as not to leave out anything essential (Gibson 138). "The importance of Boethius", says Eves,

> is the fact that his writings remained standard

> texts in the monastic schools for many centuries. These very meager works came to be considered the height of mathematical achievement, and thus well illustrate the poverty of the subject in Christian Europe during the Dark Ages. (206)

"The rise of Christianity had unfortunate consequences," says Kline. The Christian leaders opposed "pagan learning and ridiculed mathematics, astronomy, and physical science" (180). Few of the first Christian scholars had an interest in mathematics or science because "their religious faith was too intense, their persecutors too real, and their lives too precarious" (Smith 179).

Bede the Venerable, who lived from 672 to 735, was considered to be the most learned man of his time. "His works contain treatises on the Computus, or the computation of Easter-time, and on finger reckoning" (Cajori 114). It seems that the determination of Easter and other holidays was a problem, and so each monastery needed a monk who was able to compute the calendar. Some mathematics was taught in the early medieval schools for this purpose and others: "finding heights and distance, [as] good training for theological reasoning, and [for] astrology (Kline 202).

Alcuin, who lived from 735 to 804, was called to the court of Charlemagne to direct the progress of education in the empire. A book, <u>Problems for Quickening the Mind</u>, is attributed to Alcuin by some scholars, although there is some doubt about the authorship. According to Cajori

> The solutions require no further knowledge than the recollection of some few formulas used in surveying, the ability to solve linear equations and to perform the four fundamental operations with integers. Extraction of roots was nowhere demanded, fractions hardly ever occur. (114)

The type of problem found in this book can be illustrated by the following example:

> A wolf, goat, and some cabbage need to be rowed across a river in a boat, holding only one besides the ferry-man. Query: How must he carry them across so that the goat shall not eat the cabbage, nor the wolf the goat?
> (Cajori 114)

The greatest mathematician of the tenth century was Gerbert, who was born in Auvergne. He received a monastic education and then studied mathematics. In 999, he was made pope and reigned under the name of Sylvester II. He wrote A Small Book on the Division of Numbers and Rule of Computation on the Abacus (Cajori 115).

By 1100, the civilization of Europe had begun to stabilize. The Europeans had come into contact with the Arabs of the Mediterranean area and with the people of the Eastern Roman Empire. The Crusades, which were military campaigns to conquer territory, brought Europeans into

Arab lands. The Europeans began to learn about Greek works. This awareness created great excitement; "Europeans energetically sought out copies of Greek works, their Arabic versions, and texts written by Arabs" (Kline 205). Translating Arabic manuscripts into Latin began about 1100. Two of the earliest scholars-turned-translators were Athelard of Bath, and Gerard of Cremona. They translated the Elements of Euclid, the Almagest of Ptolemy, works of al-Khowarizmi and Jabir ibn Aflah (Cajori 118). "The twelfth century was to Christian Europe what the ninth century was to the eastern Mohammedan world, a period of translations" (Smith 200).

This "influx of Arabic learning" led to the establishment of universities in Europe (Cajori 129). The University of Paris received a charter from the state in 1200 and its degrees were recognized by the Pope in 1283, Oxford's degrees were recognized in 1296 and Cambridge's in 1318 (Smith 212). Very little mathematics was taught at these universities.

The first great mathematician of the thirteenth century and the most productive mathematician of the Middle Ages, was Leonardo Fibonacci, known also as Leonardo Pisano, or Leonardo of Pisa (Smith 214). Leonardo traveled about the Mediterranean and collected all the knowledge he could on mathematics. In 1202, he published Liber Abaci which introduced Hindu-Arabic numerals, methods of calculation with integers and fractions, and methods of solving certain problems of concern to merchants. He also explained square and cube roots, and both determinate and indeterminate equations of the first and second degree. He recognized that the quadratic $x^2 + c = bx$ may be satisfied by two values of x. But, says Cajori, "he took no cognizance of negative and imaginary roots." In 1220, he published Practica Geometria which contained all the knowledge of geometry and trigonometry transmitted to him (Cajori 123). According to Eves in An Introduction to the History of Mathematics, Fibonacci was invited to a mathematical tournament by Emperor Frederick II. One of his problems was to find a soluton to $x^3 + 2x^2 + 10x = 20$.

> Fibonacci attempted a proof that no root of the equation can be expressed by means of irrationalities of the form $\sqrt{a} + \sqrt{b}$, or in other words, that no root can be constructed with straightedge and compass. He then obtained an approximate answer correct to nine decimal places. This answer appears, without accompanying discussion in a work by Fibonacci entitled Flos (blossom or flower). It seems very probable that the approximation was found by the Arabian method of double false position. (Eves 210)

The Black Death swept across Europe during the fourteenth century taking more than a third of the population to their deaths. The Hundred Years War began and absorbed the energies of the people.

> Mathematical science was almost stationary . . . The growth of science was retarded not only by war, but also by the injurious influence of scholastic philosophy. The intellectual leaders of those times quarrelled over subtle subjects in metaphysics and theology. Frivolous questions, such as 'How many angels can stand on the point of a needle?' were discussed with great interest. Indistinctness of ideas characterized the reasoning during this period. (Cajori 124)

Van der Waerden mentions Master Dardi of Pisa who lived in the fourteenth century. He wrote Aliabraa Argibraa, an algebra textbook that presented a list of 198 different types of equations and their rules of solution. Benedeeto of Florence wrote a literal Italian translation of a Latin translation of the algebra of al-Khowarizmi. This book also contained a long list of equations (42).

At the universities, the study of mathematics was gaining ground. At the University of Paris, no student was allowed a degree without attending lectures in mathematics. At the University of Prague, the six books of Euclid were studied along with applied mathematics. By the middle of the fifteenth century, the first two books of Euclid were read at Oxford (Cajori 129).

No great mathematician appeared and significant mathematical progress did not occur. "Fortunately, forces of revolutionary strength did begin to exert their effects on the European intellectual, political, and social scene" (Kline 214). The invention of the printing press helped disseminate ideas throughout Europe. The first comprehensive algebra was printed in 1494. Written in Italian by Luca Pacioli, Summa de Arithmetica, Geometria, Proportioni e Proportionalita was very influential. It contained a simpler notation than that of Fibonacci (van der Waerden 47). It was comprised of all the knowledge of the day on arithmetic, such as devices for multiplication and for finding square roots, algebra, including standard solutions of linear and quadratic equations, very elementary Euclidean geometry, and double-entry bookkeeping (Boyer 307). "[Pacioli] closes his book by saying that the solution of the equations $x^3 + mx = n$, $x^3 + n = mx$ is as impossible at the present state of science as the quadrature of the circle. This remark doubtless stimulated thought" (Cajori 133).

CHAPTER III

HISTORY

1500 to 1650

> The Italian algebraists of the sixteenth century tacitly assumed that every rational integral equation has a root. The later ones of that century were also aware that a quadratic equation has two roots, a cubic equation three roots, and a biquadratic equation four roots. (Smith, History I 473)

As we pass from the Middle Ages to the sixteenth century and beyond, the fundamental theorem of algebra begins to take form, and is stated and restated and proofs are attempted. As we shall see, the proof was not successfully completed until 1799, but many interesting events led to the improvement of the state of mathematics and hence to a correct proof of the theorem.

In 1515, Scipione del Ferro, a professor at the University of Bologna, succeeded in solving $x^3 + mx = n$. He did not publish his results, but he imparted the solution to his student, Antonio Maria Fiore, also called Floridas. Nicolo of Brescia, also known as Tartaglia, found an imperfect method for solving $x^3 + px^2 = q$, but also kept his result a secret. Cajori says:

> [Tartaglia] spoke about his secret in public and thus caused Floridas to proclaim his knowledge of the form $x^3 + mx = n$. Tartaglia, believing him to be a mediocrist and braggart, challenged him to a public discussion. Hearing meanwhile that his rival had gotten the method from a deceased master, and fearing that he would be beaten in the contest, Tartaglia put in all the zeal, industry, and skill to find the rule for the equations, and he succeeded in it ten days before the appointed date, as he himself modestly states. (133)

These challenges, often for a great deal of money, were a normal form of competition in the learned world. Tartaglia was very successful in these contests and had won several prizes. This particular contest was to have thirty questions and the loser would have to pay for thirty banquets. Tartaglia prepared a variety of problems, but Fiore's problems all led to equations of the form $x^3 + mx = n$. Tartaglia, having discovered the method of solution, solved all of his problems in a few hours while Fiore was unable to solve most problems presented to him. Fiore was declared the loser. "The honour alone was satisfaction enough for Tartaglia, and he renounced the thirty banquets" (van der Waerden 55).

He was asked to reveal his solution but he would not, saying that later he would publish a large algebra containing his method. Enter Girolamo Cardano, a famous medical doctor, astrologer, philosopher, and mathematician from Milan. Cardano invited Tartaglia to Milan. Cardano swore an oath, "the most solemn and sacred promises of secrecy," that he would never publish Tartaglia's discovery. Tartaglia then divulged his secret to Cardano. Right after Tartaglia's visit, Cardano extended the method of solution to other types of cubics: $x^3 = mx + n$ and $x^3 + n = mx$. Cardano was writing his Ars Magna and Cajori remarks "he knew no better way to crown his work than by inserting the much sought for rules for solving cubics" (134). He went to Bologna to examine the papers of del Ferro after hearing a rumor that del Ferro had previously solved the cubic. He decided to publish the results and stated that the equation had been solved by del Ferro, the solution rediscovered by Tartaglia, and that he had extended it to other cases (Cardano 8). Thus Cardano broke his vows, and published in 1545 in his Ars Magna Tartaglia's solution to cubics. Tartaglia now was without his secret and was not able to write his own "immortal work." He wrote a history of his invention including Cardano's promises and then challenged Cardano and his pupil to a contest. Tartaglia succeeded in solving all problems offered him in a few days whereas Cardano and his pupil, Ferrari, were only able to solve one of their problems correctly. Following this, Tartaglia proceeded to write the algebra he had been determined to write for so long, but he died before he reached the consideration of cubic equations (Cajori 134).

The method for solving cubics is explained by Cardano in this fashion:

> Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate this and to one of the two you add one-half the number you have already squared and from the other you subtract one half the same. You will then have a binomium and its apotome. Then, subtracting the cube root of the apotome from the cube root of the binomium, the remainder [or] that which is left is the value of x. (98)

In other words, if you have a cubic of the form $x^3 + mx = n$, form $t = \sqrt{[(m/3)^3 + (n/2)^2]} + (n/2)$

and $u = \sqrt{[(m/3)^3 + (n/2)^2]} - (n/2)$.

Your solution will be $x = \sqrt[3]{t} - \sqrt[3]{u}$. To arrive at this solution, let $n = t - u$ and $[(1/3)m]^3 = tu$ in the original equation.

In the first chapter of Cardano's <u>Ars Magna</u> or <u>The Great Art</u>, he presents a discussion of the number of positive and negative roots of equations. He says "If, therefore, an even power is equal to a number, its root has two values, one plus, the other minus, which are equal to each other." He described a case of the fourth degree equation that had four roots, and cases that he declared had only two roots. He then went case by case through cubic equations explaining the number of positive and negative roots each particular type has (Cardano 10). In explaining the solutions of cubic equations, he avoided the casus irreducibilis (van der Waerden 56). In the irreducible case, t and u in the above formulas are complex numbers, but the solutions to the cubic equation are three real and distinct values. Kline remarks: "One would think that the fact that real numbers can be expressed as combinations of complex numbers would have caused Cardan to take complex numbers seriously, but it did not" (266).

Lodovico Ferrari was born in 1522 and was a servant in Cardano's household. He developed into a very good mathematician, discovering that the general equation of degree four can be reduced to a cubic equation and then solved by means of square roots and cube roots. Cardano published Ferrari's method in Ars Magna and says, "There is another rule, more noble than the preceding (on cubic equations). It is Lodovico Ferrari's, who gave it to me on my request" (237).

Rafael Bombelli published L'Algebra parte maggiore dele'aritmetica divisa in tre libri in 1572. He discussed solutions of equations up to degree four following the methods of Cardano. He fully discusses the casus irreducibilis. "Bombelli introduced a notation for what we call ±i and he presented rules of calculation" (van der Waerden 59). "This book contained the most teachable and the most systematic treatment of algebra that had appeared in Italy up to that time" (Smith 301).

After the solution to fourth degree equations had been found, mathematicians' interests naturally turned to equations of fifth degree and higher. Attempts at solutions proved fruitless and so they began to devise processes to approximate real roots of equations. Francois Viète, or Vieta, authored De numerosa protestatum purarum atque adfectarum ad exegesin resolutione tractatus in which he demonstrated a method of approximating roots that resembled the rule of ordinary root extraction. Viète, who lived from 1540 to 1603 was French. He solved an equation of the forty-fifth degree by using a trigonometric substitution. He only found twenty-three roots however, since the remaining ones involved negative sines which he did not understand (Cajori 138).

Cajori, in History of Mathematics, describes Viète's approximation method.

> In $x^5 - 5x^3 + 500x = 7905504$, he takes $r = 20$, then computes $7905504 - r^5 + 5r^3 - 500r$ and divides the result by a value which in our modern notation takes the form $|(f(r + s_1) - f(r)| - s_1{}^n$, where n is the degree of the equation and s_1 is a unit of the

> denomination of the digit next to be found. Thus if the required root is 243, and r has been taken to be 200, then s_1 is 10; but if r is taken as 240, then s_1 is 1. In our example, where r = 20, the divisor is 878295, and the quotient yields the next digit of the root equal to four. We obtain x = 20 + 4 = 24, the required root. (137)

Viète's most famous work is <u>In artem analyticem isagoge</u>, published in 1591. He used general letters instead of numbers in equations. "He studied $ax^2 + bx + c = 0$ instead of $x^2 + 5x + 6 = 0$" (Struik, Source Book 74). Viète

> considered the possibility of resolving the polynomial f(x) in an algebraic equation f(x) = 0 into linear factors. Anything approaching completeness or proof in this direction was far beyond the algebra of that time . . . (Bell 119)

Another notable algebraist of this time was Robert Recorde who wrote the first English treatise on algebra, <u>The Whetstone of Witte</u>. He is credited with the modern symbol for equality, =.

> I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: =, bicause noe.2. thynges, can be more equalle. (N. pag.)

In "The Art of Cossike Nombers", a chapter of The Whetstone of Witte, Recorde gave this rule:

> The Somme of the rule of equation:
> When any question is proponded, apperteinyng to this rule, you shall imagin a name for the number, that is to bee soughte, as you remember, that you learned in the rule of false position. And with that nomber shall you procede, accordyng to the question, until you find a Cossike nomber, equalle to that nomber, that the question expresseth, whiche you shal reduce evermore to the leaste nombers. And then divide the nomber of the lesser denomination, and the quotient doeth aunswere to the question. Except the greater denominatiō, doe beare the signe of some rooted nomber. For then must you extract the roote of that quotiente, accordyng to that sign of denomination. (N. pag.)

He then discussed simplifying equations by adding "equalle portions to thynges that bee equal."

Simon Stevin lived in Belgium from 1548 to 1620. He published several books on mathematics and mechanics. He introduced several simplifications of algebraic notation, including +, −, M, D, and √ (van der Waerden 58). He was the first person to discuss the theory and arithmetic of decimal numbers. From his book De Thiende (or in French, La Disme):

> Dîme is a kind of arithmetic, invented by the tenth progression, consisting in characters of ciphers, whereby a certain number is described and by which also all accounts which happen in human affairs are dispatched by whole numbers, without fractions or broken numbers. . . . As 3 (1) 7 (2) 5 (3) 9 (4), that is to say: 3 primes, 7 seconds, 5 thirds, 9 fourths and so proceeding infinitely, but to speak of their value, you may note that according to this definition the said numbers are 3/10, 7/100, 5/1000, 9/10000, together 3759/10000. . . (405)

This general notion of real numbers was accepted by all later scientists. Stevin accepted negative numbers, but not imaginary solutions to equations (van der Waerden 58).

Thomas Harriot, born in 1560 and died in 1621, was the founder of the English school of algebraists. He accompanied the first colony sent out by Sir Walter Raleigh to Virginia (Cajori 156). No mathematical work of Harriot's was published in his lifetime and "his reputation has oscillated as his papers were studied or forgotten" (Fauvel 291). His most famous work, Artis analyticae praxis discusses equations of the first, second, third, and fourth degrees.

John Wallis wrote Treatise of Algebra in 1685. In it, he discusses Harriot and his work.

> [Harriot] hath also made a strange improvement of Algebra, by discovering the true construction of Compound Equations, and how they be raised by a Multiplication of Simple Equations, and may therefore be resolved into such. By this means he shews the number of Roots (real or imaginary) in every Equation, and the Ingredients of all the Coefficients, in each degree of Affection . . . And amongst other things, teacheth (thereby) to resolve, not only Quadraticks, but all Cubick Equations; even those whose roots have, by others, been thought Inexplicable, and but Imaginary. (Fauvel 294)

The following is from the second section of Artis analyticae praxis.

> Propositio Prima
>
> Aequatio canonica $\begin{array}{l} aa - ba \\ \quad + ca == + bc \end{array}$
>
> ab originali $\begin{array}{c} a - b \\ \underline{a + c} \end{array}$ $== ==$ $\begin{array}{l} aa - ba \\ \quad + ca - bc \end{array}$
>
> posito b. ipsi a. aequali deducitur.
>
> Nam fi ponatur $a == b$ erit $a - b == 0$.
>
> Posito igitur $a == b$ erit $\left.\begin{array}{c} a - b \\ \underline{a + c} \end{array}\right| == 0$.
>
> Est autem ex genesi $\left.\begin{array}{c} a - b \\ \underline{a + c} \end{array}\right| ==$ $\begin{array}{l} aa - ba \\ \quad + ca - bc \end{array}$
>
> que est aequatio originalis hic designata.

> Ergo . . . aa - ba
>
> + ca - bc == 0.
>
> Ergo . . . aa - ba
>
> + ca == + bc quae est aequatio
>
> proposita. Aequatio igitur canonica proposita ab originali delignata, posito b. ipsi a. aequatio deducitur. Ut est enunciatum. (16)

Harriot is showing that in an equation such as aa - ba + ca = bc, the solution would be a = b. There is example after example showing this type of equation and solution.

"Harriot is erroneously credited with the statement that any polynomial of degree n has n roots and Descartes' rule of signs" (Eves 250). Since Harriot did not recognize imaginary and even negative roots, he could not have known the fundamental theorem of algebra (Cajori 157).

Another mathematician to consider is William Oughtred, an Englishman who "contributed vastly to the propagation of mathematical knowledge in England by his treatises, the Clavis mathematicae, Circles of Proportion, and Trigonometrie . . . " He invented the circular and rectilinear slide rules (Cajori 157).

John Wallis writes on Oughtred's Clavis mathematicae, which translates as Key of Mathematics:

> Mr. Oughtred contents himself (for the most part) with the solution of Quadratick Equations

> . . . in Resolving Equations, to take notice of the Affirmative or Positive Roots; omitting the Negative or Ablative Roots, and such as are called Imaginary or Impossible Roots. And of those which he calls Ambiguous Equations, (as having more Affirmative Roots than one,) he doth not (that I remember) any where take notice of more than Two Affirmative Roots; (Because in Quadratick Equations there are indeed no more.) Whereas yet in Cubick Equations, there may be Three, and in those of Higher Powers, yet more. Which Vieta was well aware of, and mentioneth in some of his Writings; and of which Mr. Oughtred could not be ignorant. (Fauvel 303)

The last mathematician we will consider here is Albert Girard who lived from 1595 to 1632 in Holland. He was responsible for the earliest use of the abbreviations sin, tan, and sec for sine, tangent, and secant. He edited the works of Simon Stevin (Eves 301). In L'Invention nouvelle en l'algebra, Girard stated:

> Theorem II. Every algebraic equation except the incomplete ones admits of as many solutions as the denominations of the highest quantity indicates. Explication. Let there be a complete equation 1(4) equal to 4(3) + 7(2) - 34(1) - 24. Then the denomination of the highest quantity is (4), which signifies that there are four

> determinate solutions, neither more nor less . . . As for the incomplete equation, they do not always have as many solutions. . . .

An incomplete equation is a "mixed equation that does not have all its quantities", that is, an equation where some power of x less than the highest degree is missing. The notation 1(4) equal to 4(3) + 7(2) - 34(1) - 24 represents in our notation the equation $x^4 = 4x^3 + 7x^2 - 34x - 24$, and denomination means the exponent. Girard continues:

> In the same way, if 1(4) is equal to 4(1) - 3 . . . the four solutions will be 1, 1, $-1 + \sqrt{-2}$, $-1 - \sqrt{-2}$ . . . Someone could ask what good these impossible solutions are. I would answer that they are good for three things: for the certainty of the general rule, for being sure that there are no other solutions, and for its utility. (Girard 139)

Struik says that "many authors seem to be willing to give Girard priority in the formulation of the fundamental theorem of algebra" (85).

By the middle of the 1600's, we see the fundamental theorem of algebra taking shape just as we see the beginnings of the necessary acceptance and understanding of complex numbers taking place.

1650 to 1700

Most mathematicians of the second half of the seventeenth century were interested in analytic geometry and infinitesimal analysis. "It is likely that it was the very success in these branches that made men of the time relatively oblivious to other aspects of mathematics" (Boyer 391). And so, although many consider this era to be the greatest in mathematical achievement since the time of the Greeks, we find very little work on algebra and the fundamental theorem of algebra. However, we should consider what transpired during this century to better understand later developments.

No professional mathematical organizations existed, but there were loosely organized scientific groups which helped to spread knowledge and discoveries from one mathematician to another. A Minimite friar, Marin Mersenne, who lived from 1588 to 1648, was a friend of Descartes, Fermat, and other mathematicians of the time, and "through correspondence [Mersenne] served as a clearinghouse for mathematical information" (Boyer 367).

René Descartes wrote <u>La Géométrie</u> as an appendix to his philosophical work <u>Discours de la Methode</u> in 1637. He is credited with the introduction of the modern exponential notation, and the practice of using the first letters of the alphabet as known quantities and the last letters as unknown quantities (Eves 281). In his <u>La Geometrie</u>, Descartes discussed what is now called

Descartes' Rule of Signs. In this section, he came very close to stating the fundamental theorem of algebra.

> Every equation can have as many distinct roots (values of the unknown quantity) as the number of dimensions of the unknown quantity in the equation. Suppose, for example, $x = 2$ or $x - 2 = 0$, and again, $x = 3$, or $x - 3 = 0$. Multiplying together the two equations $x - 2 = 0$ and $x - 3 = 0$, we have $x^2 - 5x + 6 = 0$, or $x^2 = 5x - 6$. This is an equation in which x has the value 2 and at the same time x has the value 3. If we next make $x - 4 = 0$ and multiply this by $x^2 - 5x + 6 = 0$, we have $x^3 - 9x^2 + 26x - 24 = 0$ another equation, in which x, having three dimensions, has also three values, namely, 2, 3, and 4.
>
> It often happens, however, that some of the roots are false or less than nothing. Thus, if we suppose x to represent the defect of a quantity 5, we have $x + 5 = 0$ which, multiplied by $x^3 - 9x^2 + 26x - 24 = 0$, yields $x^4 - 4x^3 - 19x^2 + 106x - 120 = 0$, an equation having four roots, namely three true roots, 2, 3, and 4, and one false root, 5. . . .
>
> On the other hand, if the sum of the terms of an equation is not divisible by a binomial consisting of the unknown quantity plus or minus

some other quantity then this latter quantity is not a root of the equation. Thus the above equation $x^4 - 4x^3 - 19x^2 + 106x - 120 = 0$ is divisible by $x - 2$, $x - 3$, $x - 4$, and $x + 5$, but is not divisible by x plus or minus any other quantity. Therefore the equation can have only the four roots, 2, 3, 4, and 5. We can determine also the number of true and false roots that any equation can have, as follows: An equation can have as many true roots as it contains changes of sign, from + to - or from - to +; and as many false roots as the number of times two + signs or two - signs are found in succession. (159)

Descartes is frequently criticized for his lack of completeness in stating this rule. "J. Wallis claimed that Descartes failed to notice that the rule breaks down in case of imaginary roots, but Descartes does not say that the equations always has but that it may have so many roots" (Cajori 179). Descartes went on to describe methods used to increase or decrease the value of the roots. He then said:

Neither the true nor the false roots are always real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite

> quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation $x^3 - 6x^2 + 13x - 10 = 0$ as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remain always imaginary. (175)

D. E. Smith and M. L. Latham, the translators of this edition of La Géométrie, remark in a footnote to this passage, "This seems to indicate that Descartes realized the fact that an equation of the nth degree has exactly n roots" (175).

> The merits of [Descartes' La Géométrie] according to the commonly accepted point of view, consist mainly in the creation of so-called analytic geometry. . . . La Géométrie itself can hardly be considered a first textbook on the subject. There are no explicit 'Cartesian' axes, and no equations of the straight line or of conic sections are derived . . . (Struik 96)

Boyer remarks, "the goal [of La Géométrie] is generally a geometric constuction, and not necessarily the reduction of geometry to algebra" (370). So it appears that although the creation of analytic geometry was not Descartes' goal, this was the eventual result of La Géométrie.

Descartes was very thorough in his symbolic algebra, and La Géométrie is the earliest mathematical manuscript that a "present-day student of algebra can follow without encountering difficulties in notation" (Boyer 371).

Pierre Fermat is credited by some as discovering analytic geometry nearly simultaneously and independently of Descartes (van der Waerden 69). In Introduction to Loci, Fermat was applying Renaissance algebra to problems from ancient geometry. He emphasized the sketching of indeterminate equations and was aware that every quadratic equation in x and y represented a line or a conic. Of course, Fermat is most famous for what is now called Fermat's Last Theorem. In the margins of Claude Gaspard Bachet's edition of Diophantus' Arithmetica, Fermat wrote notes and ideas. Next to the equation $x^n + y^n = z^n$, he wrote

> To divide a cube into two other cubes, a fourth power, or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it. (Smith, Source Book 213)

This conjecture is still unproven.

Fermat is also credited, together with Blaise Pascal, with the creation of the theory of probability. Pascal "came remarkably close to a discovery of the calculus - so close that Leibniz later wrote that it was upon reading this work by Pascal that a light suddenly burst upon him . . . " (Boyer 400)

Probably the most important geometric problems of the day concerned finding tangents to curves and areas under curves. Algebraic analysis and geometric intuition were used to try to find solutions to these problems, but the success of this endeavor rested on the invention of the calculus by Newton and Leibniz.

Isaac Newton had just earned his A.B. degree from Trinity College when it was closed for a year because of the plague. He went home to

> live and think. The result was the most productive period of mathematical discovery ever reported, for it was during these months, Newton later averred, that he had made four of his chief discoveries: (1) the binomial theorem, (2) the calculus, (3) the law of gravitation, and (4) the nature of colors. (Boyer 430)

In 1687 he published Philosophiae Naturalis Principia Mathematica. In Newton's lectures of 1673 to 1683, published as Arithmetica Universalis, we find that Newton has stated that imaginary roots of real polynomials occur in pairs and he gives rules for finding upper bounds to the roots of polynomials (Eves 335). In his book Method of Fluxions, we find Newton's method, a method used to find approximate solutions to equations. Joseph Raphson published a tract in 1690, Analysis aequationum universalis, in which he also discussed a method of approximating solutions to equations. In fact the method taught in modern texts is not really Newton's method but Raphson's modification of it.

Newton worked with imaginary numbers in 1685. This work was "confined to the question of the number of roots of an equation" (Smith, History II 264).

Gottfried Wilhelm Leibniz was born at Leipzig in 1646. He studied theology, law, philosophy, and mathematics at the university and "he sometimes is regarded as the last scholar to achieve universal knowledge" (Boyer 438). Fauvel credits Leibniz' interest in logic and language as "the key ingredient in the invention of calculus" (424).

> Leibniz realized, in about 1673, that the determination of the tangent to a curve depended on the ratio of the differences in the ordinates and abscissas, as these became infinitely small, and that quadratures depended on the sum of the ordinates or infinitely thin rectangles making up the area. (Boyer 441)

Leibniz developed an "appropriate language and notation" for this calculus much of which we still use today.

Many arguments have taken place over who was the first to discover calculus. It has been established, however, that both men deserve equal credit as they found their methods independently of one another. Newton is credited with discovering it first (in 1665) while Leibniz' work was done in 1673. But Leibniz published it first in 1684, while Newton did not publish his until 1704.

The Bernoullis were brothers who lived in Switzerland. Jakob Bernoulli began correspondence with Leibniz about the new calculus in 1687 (Struik, History 118). Jakob Bernoulli invented the method of partial fractions to integrate rational functions in 1699. Leibniz had also discovered the method independently and published it in 1702. Johann Bernoulli, Jakob's younger brother,began working on $\int \frac{dx}{ax^2 + bx + c}$. Having succeeded in integrating some rational functions by the method of partial fracions, Johann

> asserted in the <u>Acta Eruditum</u> of 1702 that the integral of any rational function need not involve any other transcendental functions than trigonometric and logarithmic functions. (Kline 411)

Since the denominator of a rational function is a polynomial of degree n, this statement could only be true if a polynomial with real coefficients can be factored into the product of linear and quadratic factors with real coefficients. If this is true, than it implies that every polynomial equation with real coefficients has a root of the form a + bi, in other words, the fundamental theorem of algebra. Serious attempts at proofs would now be made.

1700 to 1799

By 1700 all the numbers we are now familiar with were known. Mathematicians used whole numbers, fractions, irrationals, and negative numbers with ease. Complex numbers were used but not completely understood. Cardano had discussed them in Ars Magna but found them to be useless. Girard discussed the need for them in l'Invention nouvelle en l'algebra, but his "advanced views were not influential." Descartes rejected complex roots and "coined the term imaginaries." Since thay had no physical meaning, Newton regarded them as insignificant and Leibniz was less than clear about their meaning. He says,

> The Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, that amphibian between being and not-being, which we call the imaginary root of negative unity. (Kline 253)

Despite the trouble with imaginary numbers, mathematicians continued to search for methods of solving all polynomial equations. Many thought that perhaps the way to solve quintic or higher degree polynomial equations was to use substitutions. Count Ehrenfried Walter von Tschirnhaus (or Tschirnhausen) believed that the secret was to reduce an equation of the nth degree to a pure equation of the nth degree, that is, an equation containing only the terms of degree n and degree zero.

The substitutions that he used are still known as "Tschirnhaus transformations." Although this method seemed promising, it proved inadequate for solving the quintic, as the best that could be done was to eliminate the x^4 and the x^3 term (Boyer 473).

Another method of solving equations, that of using infinite series, had caught the attention of several mathematicians. Daniel Bernoulli solved a quartic using recurring series but was aware that there may not always be convergence to the root. Others to successfully employ series were Brook Taylor in 1717, Francois Nicole in 1738, and A. C. Clairaut in 1746. Later Leonard Euler used similar methods (Cajori 227).

Jean le Rond D'Alembert was born in France in 1717. He was abandoned by his parents at birth, but his father provided for his education. He showed remarkable intelligence at an early age and was broadly educated: in law, in medicine, in science, and in mathematics. Later in life he became secretary of the French Academy and was "the most influential man of science in France" (Struik, HIstory 128). He collaborated with Denis Diderot to write the twenty-eight volume <u>Encyclopedie</u>. D'Alembert was very interested in proving the fundamental theorem of algebra. He published a prize-winning essay in <u>Memoirs</u> of the Berlin Academy entitled "The General Cause of the Winds" in 1746. This essay, according to Boyer, is the reason why the fundamental theorem is known as d'Alembert's

Theorem in France (490). In his essay, d'Alembert showed that any "algebraic quantity, composed of as many imaginaries as one wishes, can always be reduced to a + b√-1, for a and b real numbers." He showed that addition, subtraction, multiplication, and division of two complex numbers led to a complex number. An imaginary quantity raised to an imaginary quantity was more difficult, and no strictly algebraic method would give the necessary result. He used logarithms, differentiation, integration, exponentials, and trigonometric methods to simplify his expression (Rider 47). Robin Rider, in her doctoral dissertation, <u>Mathematics in the Enlightenment</u>, points out that d'Alembert had made an assumption that could have made all of his work useless. Like others before and after him, he had assumed that all imaginary quantities were of the form a + b√-1. They assumed that these imaginaries would follow the same algebraic rules as real numbers, except for the fact that √-1·√-1 = -1 and not √[(-1)·(-1)]. As we now know, thay had not made a critical mistake but this was a reasonable complaint about d'Alembert's and other works on imaginaries (50).

D'Alembert published a proof of the fundamental theorem of algebra in <u>Memoirs</u> in a paper entitled "Recherches sur le calcul integral". His goal was to demonstrate that any polynomial had a real root or a root of the form a + bi. This, in turn, would prove Bernoulli's claim that any polynomial could be factored into a product of linear and quadratic factors with real coefficients since it was commonly known that imaginary roots appeared in conjugate pairs.

For a polynomial, P(z), d'Alembert wanted to find a z = a + bi such that P(z) = 0. He considered the curve u = P(z). He then supposed that $z = \sum_k a_k u^{r_k}$ where u was very small and r_k was an increasing sequence of rational numbers. (Rider says this is "consonant with much eighteenth-century practice.") He assumed that u does not equal zero here; this is necessary since 0^{r_k} does not make sense for all rational numbers r_k. However, if u was positive, then u^{r_k} would be a sequence of real numbers and hence z would be real. If u was negative, then each u^{r_k} would have the form p + qi. (D'Alembert had shown this previously.) The sum of the quantities of the form p + qi would itself be of this form. He claimed the expression for z would converge to a value satisfying the equation of the curve. Next, he considered the behavior of u = P(z) in a small neighborhood of zero. He assumed that u attained a "minimal" value, that is, a value closest to zero but not equal to it, in this neighborhood. He chose this u_0 infinitesimally small. Then d'Alembert considered a u_1 between u_0 and 0. He then claimed he would find a corresponding series development for z using u_1. But u_1 was closer to zero than the closest value, u_0. Therefore, by contradiction, u = P(z) actually attained a zero value for some z (Rider 55).

Rider remarks that d'Alembert's proof was "composed more of assertions than demonstrated conclusions. Even d'Alembert recognized that the scrupulous reader might question the convergence of the series for z." Other criticisms of the proof were made. Mathematicians had doubts that every u near u_0 was the image of P(z) for some z; there was skepticism that there existed a value z such that P(z) lay between zero and the supposed "closest" value, and even that P(z) had a minimum at all (Rider 58).

One of the most prolific writers of all time was from Basel, Switzerland. Leonard Euler lived from 1707 to 1783 and published more than 500 books and papers during his lifetime. He made "significant contributions in every field of mathematics which existed in his day" (Struik, History 120). His textbooks were very influential and his notation became standard notation. Some of the symbolism credited to Euler is the letter e for the base of the natural logarithm, π, i for the square root of -1, a, b, c for the lengths of the sides of triangles and A, B, C for the corresponding angles, lx for the logarithm of x, Σ for summation, and f(x), our standard function notation (Boyer 484). Later in life Euler became blind but his blindness did not stop his work. Euler dictated an elementary algebra text, Anleitung zur Algebra, to his servant, which was "meritorious as one of the earliest attempts to put the fundamental processes on a sound basis" (233).

Euler searched for a general solution to algebraic equations. He substituted $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$ into a biquadratic equation. This led him to a system of linear equations which he solved by the method of elimination. He is credited with the invention of the method of elimination as is Etienne Bezout, who discovered it independently of Euler (Cajori 235).

Another of Euler's interests was logarithms. Much mathematical discussion had taken place on the logarithms of negative numbers. In 1747, he disproved d'Alembert's claim that $\log(-1) = 0$. He believed that $\log(n)$ had an infinite number of imaginary values, except when n was a positive number, in which case one of the infinite number of values is real. In 1751 he published a paper in the Berlin Memoirs entitled "Recherches sur les racines imaginaries des equations." In this paper, we find Euler's proof of the fundamental theorem of algebra.

To begin his work, Euler showed that complex roots occur in conjugate pairs. He then used geometric arguments to prove three theorems: (1) an equation of odd degree has at least one root, (2) an equation of even degree has either no real roots or pairs of such roots, and (3) an equation of even degree with a positive leading term and a negative constant has at least one positive and one negative real root.

The initial goal was to show the truth of

Theorem 4. Every equation of the fourth degree, as

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0,$$

> can always be decomposed into two real factors of the second degree.

Euler first eliminated the x^3 term by use of the substitution $x = y - \frac{1}{4}A$. The equation was simplified to one of the form

$$x^4 + Bx^2 + Cx + D = 0.$$

Struik then translates: "It is clear that these factors will be of the form $(xx + ux + \alpha)(xx - ux + \beta) = 0$." If u, α, and β can be determined to be real, the proof will be complete.

The next step was to multiply the factors given above and equate the coefficients with the coefficients of $x^4 + Bx^2 + Cx + D$. This gives three equations which Euler solved for α and β. Finding $2\alpha = uu + B - (C/u)$ and $2\beta = uu + B + (C/u)$, he then eliminated α and β to get an equation in u, B, C, and D:

$$u^6 + 2Bu^4 + (B^2 - 4D)u^2 - C^2 = 0.$$

Now since the constant term is $-C^2$, by Theorem 3, this equation has one negative and one positive root.

> When we take one of them as u, then the values of α and β will also be real and hence the supposed factors of the second degree $xx + ux + \alpha$ and $xx + ux + \beta$ will be real. Q. E. D. (Struik, Source Book 100)

Rider says:

> Euler observed that the force of this demonstration derived from the degree of the

> equation in u and the sign of its last term. . . . He therefore wanted to establish the degree and sign a priori . . . (61)

So now Euler supposed that the roots of the fourth degree equation were a, b, c, and d. Since the equation had no x^3 term, $a + b + c + d = 0$, and u would be the sum of two roots of the fourth degree equation. (This is true since the quadratic factors are of the form $x^2 + ux + \alpha$, so u must be the sum of two factors of α, hence the sum of two solutions of the fourth degree equation.) Thus, u could assume six values, $(a + b)$, $(a + c)$, $(a + d)$, $(b + c)$, $(b + d)$, and $(c + d)$. Since $a + b + c + d = 0$, these six values can be reduced to $\pm p$, $\pm q$, and $\pm r$, and so the sixth degree equation in u can be written

$$(u^2 - p^2)(u^2 - q^2)(u^2 - r^2) = 0.$$

To guarantee that u has a real value, $(-p^2)(-q^2)(-r^2) = -(pqr)^2$ had to be negative (from Theorem 3.) This would be true if (pqr) was real. Since

$$x^4 + Bx^2 + Cx + D = (x - a)(x - b)(x - c)(x - d),$$

Euler concluded that a, b, c, and d were dependent on B, C, and D. Since $pqr = (a + b)(a + c)(a + d)$, (pqr) was also determined by B, C, and D. Hence (pqr) was real, implying $-(pqr)^2$ was negative and therefore u was real.

Euler used the same reasoning to prove similar results for equations of degree 8 and degree 16, and then in general for equations of degree 2^n. Struik notes:

> Euler believes that the proof is solid, but to strengthen the argument he gives extra proofs for degree 6, 4n + 2, 8n + 4, . . . , and $2^n p$ for p prime. (Sourcebook 102)

Gauss (in his dissertation) pointed out that Euler used relationships among roots and coefficients that were known to be true for real numbers. Euler had not established "the nature of their roots and their mathematical behavior" (Rider 64).

Joseph-Louis Lagrange was born in Italy in 1736. He too attempted a proof of the fundamental theorem of algebra. He began his task by analyzing the methods of solving third and fourth degree equations. By doing this he hoped to find why these methods worked and to look for adaptations of these methods that might help him solve higher degree equations (Kline 601). He published "Reflexions sur la resolution algebrique des equations" in the Memoirs of the Berlin Academy in 1770. This paper dealt with the question of why the methods used to solve equations of degree four or less were not useful for solving equations of higher degree. "This led Lagrange to rational functions of the roots and their behavior under the permutations of the roots " (Struik, History 133). Van der Waerden notes that several "fundamental ideas of Galois theory" can be found in this paper (79).

Lagrange also studied the use of continued fractions to find approximations to the irrational roots of equations (Kline 460). He continued to research the theory of equations and in 1772 he published "Sur la forme des racines imaginaires des equations". In this paper, he attempted to complete Euler's proof of the fundamental theorem of algebra.

Lagrange's proof is similar to Euler's. He also "wanted to discover a priori the degree and character of the auxiliary equation in u." But again Gauss pointed out that Lagrange also "freely manipulat[ed] functions of imaginary roots without ascertaining the patterns of their mathematical behavior" (Rider 66).

Daviet de Foncenex, a student of Lagrange's, tried a similar proof that suffered from similar difficulties. Pierre Laplace's proof sought real quadratric factors instead of linear imaginary ones, but his too left some doubts in people's minds (Rider 70).

Other mathematicians continued to work on solving general polynomial equations. Edward Waring found a process for approximating the value of imaginary roots of an equation in 1757. In 1762, he published Miscellanea Analytica in which he discussed polynomial equations of degree n that have n roots. Van der Waerden says, " . . . it is well known since the time of Vieta that the coefficients of the equation are all equal to the elementary symmetric functions of the roots" (76). Waring showed that all rational symmetric functions of the roots can be expressed as rational functions of the coefficients of the equations.

Waring and others investigated the cyclotomic equation $x^n - 1 = 0$. Alexandre-Theophile Vandermonde presented a paper to the Paris Academy, "Sur la resolution des equations". He claimed that if n is prime, this equation is solvable by radicals. But he could only verify it up to 11 (Kline 600). The important work on the cyclotomic equation was done by Gauss.

And so, as the turn of the century approached, mathematicians realized that the fundamental theorem of algebra was true and that the issue of imaginaries had not been completely settled. The theory of equations became a branch of mathematics, separate from other studies. Notation had become standard and mathematicians agreed on a certain amount of rigor in their proofs. In 1799, a proof of the fundamental theorem of algebra was accepted as correct.

CHAPTER IV

GAUSS AND HIS PROOFS

Carl Friedrich Gauss

Johann Friedrich Carl Gauss was born the only child of a bricklayer on April 30, 1777 in the city of Brunswick (Braunschweig), Germany. By the age of three, he was able to perform long computations in his head (Gallian 257). Gauss said of himself that

> he "could count before he spoke." The earliest mathematical legend about him claims that at the age of three he followed his father's calculations with a bricklayer, unexpectedly corrected him, and turned out to be right. (Gindikin 116)

In 1784, he entered elementary school, the Collegium Catharineum. Once, a taskmaster asked a group of students to sum the integers from 1 to 100. Gauss finished his calculation almost as soon as the teacher had finished dictating the problem. His answer was correct; he had figured out the formula for the sum of an arithmetic progression (Gindikin 117). This and other incidents brought him to the attention of the Duke of Brunswick who secured an education for Gauss.

In 1788, he entered the gymnasium. There he studied classical languages and "acquired a solid knowledge of Latin, the indispensible prerequisite for the pursuit of higher learning and an academic career" (Bühler 6).

Felix Klein says:

> [Gauss] calculated continually, with overpowering industry and untiring perserverance. By this incessant exercise in manipulating numbers (for example, calculating decimals to an unbelievable number of places) he acquired not only the astounding virtuosity in computational techniques that marked him throughout his life, but also an immense memory stock of definite numerical values, and thereby an appreciation and overview of the realm of numbers such as probably no one, before or after him, has possessed. (29)

At eighteen, Gauss left Brunswick to study at the University of Göttingen where he invented the method of least squares for handling statistical data. At nineteen, he began his diary which had entries dated up until 1814. His was no regular diary, but a list of important mathematical discoveries that he had made. The first entry is dated March 30, 1796, and refers to his discovery of a method of inscribing a regular 17-sided polygon into a circle (Cajori 435). The question of the constructibility of regular n-sided polygons using only a compass and straightedge had been open for more than 2000 years. It is often said that this discovery convinced Gauss to devote his time and energy to mathematics instead of the study of languages (Bühler 10).

While at Göttingen, he began to write Disquisitiones Arithmeticae, which was not published until 1801. In this treatise, the full proof of the solvability of $x^m - 1 = 0$ by radicals was given. An equation $x^m - 1 = 0$ is called a cyclotomic equation because its solution is closely connected with the construction of a regular polygon of n sides inscribed in a given circle. The equation $x^{17} - 1 = 0$ was treated as a special case (van der Waerden 91). He also discussed the law of quadratic reciprocity, "a law which involves the whole theory of quadratic residues" (Cajori 435). This book had an enormous impact on the development of number theory.

In the fall of 1798, Carl Gauss left the University of Göttingen without a diploma. He submitted his doctoral dissertation to the University of Helmstadt in 1799. "The degree was awarded in absentia, without the usual oral examination" (Bühler 17).

In a letter, Gauss described the contents of his dissertation:

> The title describes the main objective of the paper quite well though I devote to it only a third of the space. The rest mainly contains history and criticisms of the works of other

> mathematicians (namely d'Alembert, Bouganville, Euler, de Foncenex, Lagrange, and the authors of compendia - the latter will presumably not be too happy) about the subject, together with diverse remarks about the shallowness of contemporary mathematics. (Bühler 41)

In the following sections of this chapter, we will look at the proof of the fundamental theorem of algebra given in the dissertation and the three others that Gauss provided during his lifetime.

In 1801, Gauss was able to calculate the orbit of a newly found planet, Ceres, by using only three observations. "He showed that the variation inherent in experimentally derived data follows a bell shaped curve, now called the Gaussian distribution" (Gallian 257). Astronomers managed to locate Ceres at positions very close to where Gauss had predicted it would be. "The result made Gauss a European celebrity" (Bühler 44). Gauss was then offered a position as professor of astronomy at Göttingen. Bühler lists several reasons Gauss decided to take the position:

> . . . the firm commitment of the administration to erect a new observatory, the presence of the experienced and skillful observer C. L. Harding as Gauss's assistant, and the fact that Gauss would only be loosely connected with the university. This gave him relative freedom from

> lecturing and from participation in the administrative affairs of the university. (53)

Most authors remark on Gauss's dislike for teaching. "He typified much more the research-oriented eighteenth century scholar than the educator and teacher." But he seems to have been willing to advise any interested student who asked for his help or for explanations. (Bühler 70).

Gauss married Johanna Osterhoff in October of 1805 Following the birth of their third child, both Johanna and the baby died. Shortly after her death, Gauss remarried. His second wife, Minna Waldeck, bore him three children.

Gauss held the post of professor of astronomy at Göttingen from 1807 until his death in 1855 (Smith, Source Book 292). Astronomy "absorbed him most" but he was actively interested in many other areas of science. He "earned great distinction in his physical research on theoretical and experimental magnetism" and, although he did not invent the telegraph, he and Wilhelm Weber improved earlier inventions. Gauss also studied optics (Kline 870). The kingdom of Hanover was surveyed from 1818 to 1832 and Gauss was the director of the initial phase of this vast project. "Geodesists list Gauss as one of the greatest geodesists, a man who introduced new standards of observational and theoretical accuracy" (Bühler 110).

In 1849, the University of Göttingen celebrated the fiftieth anniversary of Gauss's doctorate. (The University of Helmstadt no longer existed.) Gauss submitted his fourth and final proof of the fundamental theorem of algebra, an improved version of his original proof. Gauss died in February of 1855 and was buried in Göttingen (Bühler 155).

Gauss was one of the first mathematicians to use Argand's "visualization of the complex numbers in the two-dimensional plane" (Bühler 43). Gauss coined the term complex numbers (Gallian 258). His interest in geodesy, surveying, and map-making led to a now famous work on differential geometry, Disquisitiones generales circe superficies curvas. He made many other contributions to algebra, complex functions, and potential theory (Kline 870). Struik says:

> His publications, however, do not give an adequate picture of his full greatness. The appearance of his diaries and some of his letters has shown that he kept some of his most penetrating thoughts to himself. We now know that Gauss, as early as 1800, had discovered elliptic functions and around 1816 was in possession of non-Euclidean geometry. (History 145)

As most of Gauss's contemporaries were specialized in one field or another, Gauss's universal activities are "all the more remarkable." Kline says he was not "so much an innovator as a transitional figure from the eighteenth to the nineteenth century" (871).

In closing, it was Gauss who remarked that "mathematics is the queen of the sciences and arithmetic is the queen of mathematics" (Boyer 553). Gauss is sometimes called the Prince of Mathematics. And Bühler says, "Gauss's brain with its exceptionally deep and numerous convolutions has been incorporated in the anatomical collection of the University of Göttingen" (155).

Gauss's First Proof

In 1799, Gauss published his doctoral dissertation under the title <u>Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse</u> or <u>New proof of the theorem that every integral rational algebraic function of one variable can be decomposed into real factors of the first or second degree</u>. Before giving his proof, Gauss criticized the proofs of others, including d'Alembert, Euler, de Foncenex, and Lagrange.

He first discussed d'Alembert's <u>Recherches sur le Calcul Integral</u> which was published in 1746. He summarized d'Alembert's proof and then he discussed "the basic points which seem able to be brought against d'Alembert's demonstration" (Fauvel and Gray 490).

> 1. Ill. d'A. nullum dubium mouet de "existentia" valorem ipsius x quibus valores datisius X respondent, sed illam supponit, solamque "formam" istorum valorum inuestigat. . . . (Gauss 11)
>
> D'Alembert raises no doubts about the <u>existence</u> of values of x, to which given values of X may correspond, but supposes their existence, and investigates only the form of the values. (Fauvel and Gray 490) 2.)
>
> Assertio, w per talem seriem qualem ponit semper exprimi posse, certo est falsa, si X etiam functionem quamlibet transcendentem designare debet (uti d'A. pluribus locis innuit). Hoc e. g. manifestum est, si ponitur $x = e^{1/x}$, siue $x = 1/(\log X)$. (Gauss 11)
>
> The assertion, that w can always be expressed through such a series as he proposes, is certainly false, if X is meant to designate any transcendental function (as d'Alembert indicates in several places). This is clear if, for example, $X = e^{1/x}$, or $x = 1/(\log X)$. (Fauvel and Gray 490)

Gauss also discussed d'Alembert's free use of infinitely small quantities, which was not "consistent with the rigours of geometry, at least in our age . . . " (Fauvel and Gray 491)

He then discussed Euler's "Recherches sur les racines imaginaires des equations" published in 1749. Gauss remarked:

> Euler tacitly supposes that the equation X = 0 has 2m roots, of which he determines the sum to be = 0 because the second term in X is missing. What I think of this licence I have already declared in art. 3. The proposition that the sum of all the roots of an equation is equal to the first coefficient with the sign changed does not seem applicable to other equations unless they have roots; now although it ought to be proved by this same demonstration that the equation X = 0 really does have roots, it does not seem permissible to suppose the existence of these. . . . (Fauvel and Gray 491)

After fully describing Euler's attempt to prove the fundamental theorem, Gauss discussed the proofs of de Foncenex and Lagrange. Of Lagrange's work he remarked:

> Magnus hic geometra imprimis operam dedit, defectus in Euleri demonstratione prima supplere et reu era praesertim ea, quae supra . . . (26)
> This great geometer handed his work to the printers when he was worn out with completing Euler's first demonstration. . . (Fauvel and Gray 491)

Gauss begins his proof with a polynomial $X = x^m + Ax^{m-1} + Bx^{m-2} + \ldots + Lx + M$ where $A, B, \ldots, M$ are real numbers. His goal was to show that a linear or quadratic factor of X existed. A real linear factor of X would imply the existence of a root $\pm r$. The existence of a quadratic factor implied a pair of complex roots, z and z, defined by $r(\cos \phi \pm i \sin \phi)$ for $r > 0$. Thus the quadratic factor $(x - z)(x - z)$ could be written $x^2 - 2xr\cos\phi + r^2$. He first stated and proved this lemma:

> Lemma. Denotante m numerum integrum positimum quemeunque functio
> $\sin\phi \cdot x^m - \sin m\phi \cdot r^{m-1}x + \sin(m - 1)\phi \cdot r^m$
> divisibilis erit per $xx - 2\cos\phi \cdot rx + rr$. (26)
> Lemma. If m is an arbitrary positive integer, then the function
> $\sin\phi \cdot x^m - \sin m\phi \cdot r^{m-1}x + \sin(m - 1)\phi \cdot r^m$ is divisible by $x^2 - 2\cos\phi \cdot rx + r^2$ (Struik, Source Book 115).

Next Gauss considered this:

> Lemma. If the quantity r and the angle ϕ are so determined that the equations

$$r^m\cos m\phi + Ar^{m-1}\cos(m - 1)\phi + Br^{m-2}\cos(m - 2)\phi + \text{etc.} + Krr\cos 2\phi + Lr\cos\phi + M = 0, \quad (1)$$

$$r^m\sin m\phi + Ar^{m-1}\sin(m - 1)\phi + Br^{m-2}\sin(m - 2)\phi + \text{etc.} + Krr\sin 2\phi + Lr\sin\phi = 0 \quad (2)$$

> exist, then the function

$x^m + Ax^{m-1} + Bx^{m-2} + \text{etc.} + Kx^2 + Lx + M = X$ will be divisible by the quadratic factor $x^2 - 2\cos r\phi + r^2$, unless $r\sin\phi = 0$. If $r\sin\phi = 0$, then the same function is divisible by $x - r\cos\phi$.

This may be proved by examining the sum of the following functions, each of which is divisible by $x^2 - 2\cos\phi\cdot xr + r^2$ (from the first lemma).

$$\begin{array}{lclcl}
\sin\phi\cdot rx^m & - & \sin m\phi\cdot r^m x & + & \sin(m-1)\phi\cdot r^{m+1} \\
A\sin\phi\cdot rx^{m-1} & - & A\sin(m-1)\phi\cdot r^{m-1}x & + & A\sin(m-2)\phi\cdot r^m \\
B\sin\phi\cdot rx^{m-2} & - & B\sin(m-2)\phi\cdot r^{m-2}x & + & B\sin(m-3)\phi\cdot r^{m-1} \\
\cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot \\
K\sin\phi\cdot rx^2 & - & K\sin 2\phi\cdot r^2 x & + & K\sin\phi\cdot r^3 \\
L\sin\phi\cdot rx & - & L\sin\phi\cdot rx & & \\
M\sin\phi\cdot r & & & + & M\sin(-\phi)\cdot r.
\end{array}$$

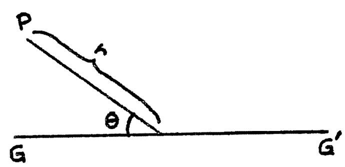
The sum is $\sin\phi\cdot rX + 0 + 0$. (The second and third column add up to zero by the hypothesis of the theorem.) If $\sin\phi = 0$, then $\cos\phi = \pm 1$, $\cos 2\phi = \pm 1$, $\cos 3\phi = \pm 1$, etc. and X becomes zero for $x = r\cos\phi$.

In the proof, Gauss substituted $a + bi = r(\cos\phi + i\sin\phi)$ in for x in X. After substitution, he separated the result into two equations, the real and the imaginary parts. These two pieces, represented above by equations (1) and (2), Gauss consider as curves, $U = 0$ and $T = 0$, in the Cartesian plane. Next, he showed that U and T must intersect somewhere.

> . . . if the two curves intersected, then there necessarily existed a pair of points a_1, b_1 such that $U(a_1,b_1) = 0$ and $T(a_1,b_1) = 0$. But then $X(a_1 + \sqrt{-1}b_1) = 0$; that is, there was at least one imaginary value of the form $a + b\sqrt{-1}$ satisfying the polynomial equation. (Rider 72)

To do this, Gauss must describe the behavior of T and U and convince his reader that they do in fact intersect. He said:

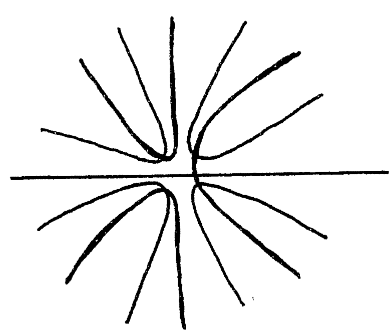
> We consider a fixed infinite plane (the plane of our Fig. 1) and in it a fixed infinite straight line GG' passing through the fixed point C.



> In order to express all line segments by numbers we take an arbitrary segment as unit, and erect at an arbitrary point P of the plane, with distance r from center C and with angle GCP = μ, a perpendicular equal to the value of the expression $r^m \sin m\phi + Ar^{m-1}\sin(m - 1)\phi$ + etc. + $Lr\sin\phi$. I shall denote this expression by T. (Struik, Source Book 116)

He next described how the endpoints of this perpendicular formed a "continuous, curved surface, infinite in all directions" which is above the plane when T is positive, below the plane when T is negative, and vanishes when T = 0. The expression for U forms a similar surface using the same plane, center, and axis. The intersection of the first surface and the plane form the curve T = 0, while the intersection of the second surface and the plane form the curve U = 0. Each curve, T = 0 and U = 0, could have multiple branches but "each by itself forms a continuous curve (Struik, Source Book 117).

Rider, in Mathematics in the Enlightenment, says Gauss offered as an example the polynomial $X = x^4 - 2x^2 + 3x + 10$. The sketch of U = 0 and T = 0 would look like this:



> If a circle were drawn about the center C with radius R, intersecting the curves T = 0 and U = 0, Gauss indicated that, consonant with the example and our intuition, there would be 2m points on the circle where T = 0 (and 2m where U = 0). (Rider 74)

Gauss remarked that these points "are situated in such a way that each point of the second kind lies between two of the first kind" (Struik, Source Book 118).

Then he said:

> It is now possible to deduce from the relative position of the branches which enter into the circle that inside the circle there must be an intersection of a branch of the first curve with a branch of the second curve, and this can be done in so many ways that I hardly know which method is to be preferred to another. . . . Now it is known from higher geometry that every algebraic curve (or the single parts of an algebraic curve when it happens to consist of several parts) either runs into itself or runs out to infinity in both directions and that therefore, if a branch of an algebraic curve enters into a limited space, it necessarily has to leave it again. (Struik, Source Book 121)

Since $U = 0$ and $T = 0$ intersect in the circle, then there exists a point where $X = 0$. Thus he had proved the fundamental theorem of algebra.

Gauss's Second Proof

The second proof uses no geometrical arguments. The following is from a translation of the second proof by C. Raymond Adams and can be found in A Source Book in

Mathematics by D.E. Smith. Smith says "The term fundamental theorem of algebra appears to have been introduced by Gauss." This proof first appeared in 1816 in Commentationes Societatis regiae scientiarum Gottingensis recentiores under the title "Demonstratio nova altera theorematis omnem functionem algrebraicam rationalem integram unius variabilis in factores reales primi vel secondi gradus resolvi posse" (292).

The solution of Y = 0 is made to depend on the solution of the auxiliary equation F(u,X) = 0 provided the discriminant of Y is not zero.

To determine the discriminant of Y, suppose

$$v \equiv (x - a)(x - b)(x - c) \ . \ . \ .$$
$$\equiv x^m - \lambda' x^{m-1} + \lambda'' x^{m-2} - \ . \ . \ . \ .$$

Note that each λ or function of λ is a symmetric function of a, b, c, . . . Form the product

$$\pi = (a - b)(a - c)(a - d)\ldots \text{ X } (b - a)(b - c)(b - d)\ldots$$
$$\text{X } (c - a)(c - b)(c - d)\ldots \text{ X } \ldots$$

This product is an integral function of the . Denote the same function of l', l", . . . by p. Then p is defined to be the discriminant of

$$y = x^m - l' x^{m-1} + l'' x^{m-2} - \ . \ . \ .$$

Now let Y be a particular, arbitrary function of the same type with constant coefficients L', L", . . .

Then Y has the form

$Y = x^m - L' x^{m-1} + L'' x^{m-2} + \ . \ . \ .$ Then P, which is the same as p with the l's replaced by L's, is the discriminant of Y.

Gauss proved the following two theorems.

I. If P is zero, then Y and Y' = dY/dx have a common factor.

II. If P is not zero, then Y and Y' have no common factors.

He described how to create the auxiliary equation.

> We will now consider the product of all $u - (a + b)x + ab$ without repetitions, where u and x indicate unknowns, and denote the same by ζ. Then ζ will be the product of the following $\frac{1}{2}m(m - 1)$ factors:
>
> $u - (a + b)x + ab$, $u - (a + c)x + ac$,
> $u - (a + d)x + ad$, . . . ;
> $u - (b + c)x + bc$, $u - (b + d)x + bd$, . . . ;
> $u - (c + d)x + cd$, . . . ; . . .
>
> Since this function involves the unknowns a, b, c, . . . symmetrically, it determines an integral function of the unknowns u, x, l', l", . . . , which shall be denoted by z, with the property that it goes over into ζ if the unknowns l', l", . . . are replaced by λ', λ'', ... Finally we will denote by Z the function of the unknowns u and x alone to which z reduces if we assign to the unknowns l', l", . . . the particular values L', L", . . . (299)

Gauss then proved the following theorem:

> Theorem. Whenever P is not zero, the discriminant of the function Z certainly cannot vanish identically.

Rename Z as $Z = F(u,x)$. If P is not zero, the discriminant of Z is a function of x that is not identically zero. So the number of particular values of x for which the discriminant of Z can vanish is finite. Thus there exists an infinite number of x's which make the discriminant of Z nonzero.

Let X be a real value of x such that the discriminant of Z is not zero. Then by the second theorem $F(u,X)$ and $dF(u,X)/du$ have no common divisors. Using this information, Gauss generated a zero for $F(u,X)$ of the form $g + h\sqrt{-1}$. From this, it follows that for the same $g + h\sqrt{-1}$, Y will also be zero.

If the discriminant of Y is zero and the degree of Y is $m = 2^{\mu}k$ where k is an odd number, he showed that it is possible to find another function whose degree is $2^{\gamma} k$, where $\gamma < \mu$. This new function is a divisor of Y and its discriminant is not zero. Every solution of the new equation is a solution of $Y = 0$, and the solution is

> made to depend upon the solution of another equation whose degree is expressed by a number of the form $2^{\gamma-1}k$.
>
> From this we conclude that in general the solution of every equation whose degree is

expressed by an even number of the form $2^{\mu}k$ can be made to depend on the solution of another equation whose degree is expressed by a number of the form $2^{\mu'}k$ with $\mu' < \mu$. In case this number is also even - i. e., if μ' is not zero, - this method can be applied again, and so we proceed until we come to an equation whose degree is expressed by an odd number; the coefficients of this equation are all real if all the coefficients of the original equation are real. It is known, however, that such an equation of odd degree is surely solvable and indeed has a real root. Hence each of the preceding equations is solvable, having either real roots or roots of the form $g + h\sqrt{-1}$. (306)

Gauss's Third Proof

Gauss's third proof also appeared in 1816 in Commentationes Societis regia scientiarum Göttingensis recentiores with the title "Theorematis de resolubilitate omnem functionem algebraicam rationalem integram unius variablilis in factores reales primi vel secundi gradus resolvi posse" (Smith, Source Book 292).

This explanation of the proof is from an article in Bulletin of the American Mathematical Society by Maxime Bocher published in 1895. He said:

> It is hoped that the following note may be of interest to some readers of the Bulletin as indicating the connection between Gauss's third proof that every algebraic equation has a root and those branches of mathematics which have since been developed under the names of the Theory of Equations and the Theory of the Potential. . . . [This] is essentially Gauss's proof. (205)

It should be noted that in Gauss's original third proof the coefficients of the polynomial were real and that he only used complex coefficients in his fourth proof.

Let $f(z) = z^n + (a_1 + b_1 i)z^{n-1} + \ldots$

$+ (a_{n-1} + b_{n-1}i)z + a_n + b_n i = \sigma + \tau i.$

Let $zf'(z) = \sigma' + \tau' i$. Now substitute $z = r(\cos\phi + i\sin\phi)$ and solve for σ, τ, σ', and τ'.

Define $$F(z) = \frac{\sigma' + \tau' i}{\sigma + \tau i}$$

$$= \frac{\sigma\sigma' + \tau\tau'}{\sigma^2 + \tau^2} + \frac{\sigma\tau' - \tau\sigma'}{\sigma^2 + \tau^2} i$$

$$= u + vi.$$

Next, find formulas for the derivative of u and v with respect to r and ϕ. We find

$$\frac{\delta u}{\delta r} = \frac{1}{r}\frac{\delta v}{\delta \phi}$$

$$= \frac{(\sigma^2 + \tau^2)(\sigma\sigma'' + \tau\tau'') + (\sigma\tau' - \tau\sigma')^2 - (\sigma\sigma' + \tau\tau')^2}{r(\sigma^2 + \tau^2)^2}$$

$$= T.$$

Form the double integral $\Omega = \int_0^\alpha \int_0^{2\pi} T \, d\phi \, dr.$

First, integrate with respect to ϕ and then r, and we find $\Omega = 0$. Now integrate in the opposite order.

Remembering that u vanishes at the origin,

$\Omega = \int_0^{2\pi} u \, d\phi$, the integral being taken around the circumference of a circle with radius a and centre at the origin, so that Ω will be positive if a is sufficiently large. The fact that we get different values for Ω according to the order of integration shows that T cannot be everywhere finite, continuous, and single valued, and this can be explained only by the vanishing of $\sigma^2 + \tau^2$. (208)

If $\sigma^2 + \tau^2$ vanishes at a point, this point must be a root of $f(z) = 0$. Hence, the fundamental theorem of algebra has been proved.

Gauss's Fourth Proof

Gauss submitted his fourth proof in honor of the fiftieth anniversary of the awarding of his doctorate. The title of his fourth proof was "Beitrage zur Theorie der algebraichen Gleichungen" and was published in Abbandlungen der Könilgiden Gesellshaft der Wissenschaften zu Göttingen (Smith, Source Book 292).

This version of his proof is from Appendix I of Theory of Equations by J. V. Uspensky. He says:

> Among the many existing proofs perhaps the first and the fourth (which is only another presentation of the first) proofs by Gauss show

> in the clearest intuitive way why any equation should have a root, and although partisans of extreme rigor may insist on the necessity of various additions, we shall present here the fourth Gaussian proof as the most suitable for the purposes of this book. (293)

Let $f(x) = x^n + ax^{n-1} + bx^{n-2} + . . . + 1$ be a polynomial with complex coeficcients. Substitute $x = r(\cos \phi + i\sin \phi)$. Separate the real and imaginary parts to get two equations, T and U. The goal will be to show the existence of a point where both T and U vanish.

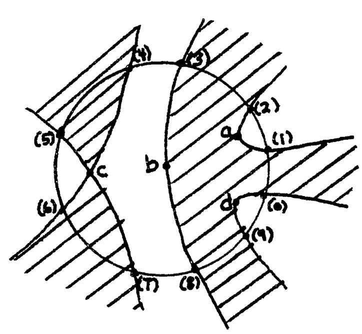
A circle Γ of a certain radius is selected and divided into 2n arcs. On each of the arcs, the sign of T is shown to be alternately negative and positive.

It is shown that U has positive values at the even endpoints of the arcs and negative values at the odd endpoints. Uspensky says:

> To describe the situation more intuitively we shall call those n regions outside of Γ, where $T < 0$, "seas", and the other n regions where $T > 0$, "lands." Lines on which $T = 0$ will be then "seashores." Now the n seas and the n lands existing in the interior of Γ extend themselves into the interior of Γ across the arcs (0)(1), (1)(2); etc. Starting from the endpoint (1) of the arc (0)(1), through which a sea penetrates into the interior of Γ, we

imagine that we walk along the seashore so that the land is always on our right, heading inward. We must eventually come out of Γ, and when we cross it again, heading outward, the land must still be on our right. If the circumference is followed in a counterclockwise direction, lands and seas alternate, whence it follows that we cross Γ, heading outward, at a point (k) with k even, that is either (2), which is the simplest case, or at (4), (6), etc. Thus, there is a continuous line L leading from (1) to some point (k) with even k. On the line L constantly $T = 0$, and at the point (1), $U < 0$, whereas $U > 0$ at the point (k). Since U varies continuously, at some point of L it must take the value 0, so that there is a point within Γ at which both $T = 0$ and $U = 0$, which proves the existence of a root. (297)

Uspensky said the following diagram was borrowed from Gauss. The shaded area represents "seas" and the white area "lands." It represents an equation of the fifth degree.



CHAPTER V

COMPLEX VARIABLES

Augustin-Louis Cauchy

> In a very important sense, it may be said that Cauchy brought ancient and modern mathematics together. He cast his rigorous calculus in the deductive mould characteristic of ancient geometry . . . he not only gave his work a Euclidean form but presented definitions that generally are adequate to support the desired results, proofs that basically are valid, and methods that were fruitful sources for later mathematical work. . . . There is little in nineteenth century analysis that was not marked, directly or indirectly, by his ideas. (Grabiner 164)

Augustin-Louis Cauchy was born in Paris in 1789. His father educated him and his five brothers and sisters when they were young. The elder Cauchy was a barrister and a police lieutenant (Bell, Men 272). At sixteen, Cauchy began to study engineering at l'Ecole Polytechnique and from there went to l'Ecole des Pontes et Chaussées (Calinger 548). In 1813, he became an instructor at l'Ecole Polytechnique.

At forty, he left France and taught in Switzerland and Italy, and then became the private tutor of the heir of the deposed king of France, Charles X. Cauchy continued his research, and following the Revolution of 1848, he returned to France. Cauchy resumed the chair of celestial mechanics at the Sorbonne, where he remained the rest of his career (Calinger 549).

Cauchy published rapidly and is second only to Euler in volume of output (Calinger 549). "Cauchy's productivity was so prodigious that he had to found a sort of journal of his own, the Exercises de Mathématiques, continued in a second series as Exercises d'Analyse Mathématique et de Physique" (Bell, Men 287). His numerous contributions include researches in convergence and divergence of infinite series, real and complex function theory, differential equations, determinants, probability, and mathematical physics. Fauvel credits him with "the ϵ and δ beloved of analysts" in his definitions (563). His importance in the world of mathematics is acknowledged by the number of concepts and theorems named for him; more than any other mathematician (Calinger 549).

Definitions

As with most branches of mathematics, the theory of functions of a complex variable has its own terminology and definitions. To prove the fundamental theorem of algebra using this theory, we will need to look at a few of these terms.

We begin with the notion of a simple closed curve. A simple closed curve is a curve which does not cross itself and whose initial and terminal points are the same. A useful example is a circle. The interior of a circle is an example of a simply connected domain. One way to think about simply connected domains is that they are domains such that every closed curve in them can be shrunk continuously to a point in the domain without bumping into the boundary of the domain (Boas 44).

If you were walking along a simple closed curve and the inside of the curve was to your left, then we would say that the curve is positively oriented.

A function that has a complex derivative at every point in a region is said to be analytic or holomorphic there. If a function is analytic in the whole finite plane, it is called entire. In some literature, especially British, entire functions are called integral functions.

The last term that we will need is line integral. In ordinary calculus, we integrate functions over intervals. In multivariable calculus, when we integrate functions over curves, we get these line integrals. In complex analysis, we integrate complex functions over curves to get contour integrals.

Cauchy Integral Theorem

Many theorems in Complex Variables are due to Cauchy and bear his name. This one is usually called the Cauchy Integral Theorem.

> Suppose R is a closed region that consists of points interior to and on a simple closed curve C in the xy plane. If f is an analytic function on R and f' is continuous there, then
>
> $$\int_C f(z)dz = 0.$$

To see this, let $f(z) = u(x,y) + iv(x,y)$ be analytic throughout R. The integral along C can be written as a contour integral which is equal to a line integral.

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy).$$

Since f is analytic in R, u and v are analytic there, and since f' is continuous, so are the first-order partial derivatives of u and v. Assume C is positively oriented. Then Green's Theorem from multivariable calculus then enables us to write

$$\int_C f(z)\ dz = \iint_R (-v_x - u_y)dxdy + i\iint_R (u_x - v_y)dxdy$$

But, the Cauchy-Riemann equations state $u_x = v_y$ and $u_y = -v_x$, so that the integrands of these two double integrals are zero throughout R. Thus $\int_C f(z)dz = 0.$

This theorem is sometimes called the Cauchy-Goursat Theorem, after E. Goursat, a French mathematician who lived from 1858 to 1936. He was the first to prove that the condition of continuity of f' could be omitted. However, since our functions are polynomials, we know that their derivatives are continuous and so we may use the weaker form of this theorem (Churchill 94).

Proof Using the Cauchy Integral Theorem

This proof of the fundamental theorem of algebra needs only the Cauchy Integral Theorem. Like most complex variables proofs, it is a proof by contradiction.

> If $p(z)$ is a nonconstant polynomial of degree n, then $p(z)$ has at least one root.

Suppose not. Let

$p(z) = a_0 + a_1 z + \dots + a_n z^n = a_0 + zq(z)$, then

$$\frac{1}{z} = \frac{p(z)}{zp(z)} = \frac{a_0 + zq(z)}{zp(z)} = \frac{a_0}{zp(z)} + \frac{q(z)}{p(z)}.$$

Integrating around the circle $|z| = r$, we find

$$2\pi i = \int_{|z|=r} \frac{a_0}{zp(z)}\,dz + 0.$$

The last term on the right integrates to zero by the Cauchy Integral Theorem since $q(z)/p(z)$ is analytic. As $|z| = r \to \infty$, $p(z)/a_n z^n \to 1$, and hence for large r,

$$\left|\frac{p(z)}{a_n z^n}\right| \geq \frac{1}{2} \quad \text{which implies} \quad |p(z)| \geq \frac{|a_n z^n|}{2}.$$

Thus

$$2\pi = \left|\int_{|z|=r} \frac{a_0}{zp(z)}\,dz\right|$$

$$\leq 2\pi r \cdot \frac{2|a_0|}{r|a_n||r|^n}$$

$$= \frac{4\pi|a_0|}{|a_n|r^n}$$

Now, as $r \to \infty$, this inequality gives $2\pi \leq 0$, which is a contradiction. Thus, $p(z)$ has a root.

Another Proof Using the Cauchy Integral Theorem

The following proof uses only the Cauchy Integral Theorem and is from Invitation to Complex Analysis by R. P. Boas. This proof is based on a proof by N. C. Ankeny of Stanford University. Ankeny's proof may be found in the October 1947 issue of The American Mathematical Monthly, Volume 54.

> Every polynomial of positive degree has at least one zero.

Assume not. Since every polynomial of degree one obviously has a zero, there is a polynomial $p(z)$ of degree greater than one with no zeros. We may assume that p is real on the real axis. Since $p(z) \neq 0$, $p(2\cos\theta) \neq 0$. Consider

$$I = \int_{-\pi}^{\pi} \frac{d\theta}{p(2\cos\theta)}.$$

Since $p(z)$ is real for real z, and never zero, we see that $p(2\cos\theta)$ is always of the same sign, and hence the integral is never zero. Now interpret I as an integral around the unit circle, parameterized by $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$:

$$I = \int_{-\pi}^{\pi} \frac{d\theta}{p(2\cos\theta)} = -i\int_{|z|=1} \frac{dz}{zp(z + 1/z)}.$$

If $p(z) = a_0 + a_1 z + . . . + a_n z^n$, then

$$p(z + 1/z) = a_0 + a_1(z + 1/z) + . . . + a_n(z + 1/z)^n$$
$$= z^{-n}q(z),$$

where $q(z)$ is a polynomial with $q(0) = a_n \neq 0$. Thus the integrand in I is $z^{n-1}/q(z)$ with $1/q(z)$ analytic, $n \geq 1$, and therefore $I = 0$ by the Cauchy Integral Theorem. This is a contradiction and so the assumption must be false.

Joseph Liouville

There are many mathematical journals now in existence. Some of these have a very long history, published for the first time in the first half of the nineteenth century.

> Foremost among these are the German journal entitled Journal für die reine und angewandte Mathematik, first published in 1826 by A. L. Crelle, and the French journal entitled Journal de mathématiques pures et appliquées, which appeared in 1836 under the editorship of J. Liouville. (Eves 303)

These journals are frequently known as Crelle's Journal and Liouville's Journal. This is the same Liouville of Liouville's Theorem which we will use to prove the fundamental theorem of algebra.

Joseph Liouville lived from 1809 to 1882. He was a professor at the College de France (Cajori 440). He showed, in his Journal of 1844, that neither e nor e^2

could be the root of a quadratic equation with integral coefficients (Boyer 602). He founded his Journal in 1836 (Kline 624).

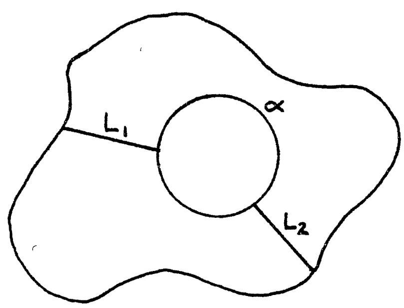
Although the theorem we are interested in is called Liouville's Theorem, some believe

> it is incorrectly attributed to Liouville by Borchardt (whom others copied), who heard it in Liouville's lectures in 1847. It is due to Cauchy, in Comptes Rendus, Volume 19 (1844), although it may have been known to Gauss earlier. (Marsden 171)

Borchardt was then the editor of Crelle's Journal.

Liouville's Theorem

For the next proofs, we will need Liouville's Theorem. We will need to consider α, a small circle around a point inside a simple closed curve, C. It can be shown that the contour integral of an analytic function f over C is equal to the contour integral of f over α. Suppose that C is in a simply connected domain where f is analytic. Connect C and α with lines L_1 and L_2 as in the following diagram.



Consider $\int f(z)dz$ separately along the upper and lower curves formed by C, α, L_1, and L_2. By Cauchy's Integral Theorem, each of those two integrals is zero and the integrals along L_1 and L_2 cancel, so

$$\int_C f(z)dz = \int_\alpha f(z)dz. \quad \text{(Boas 52)}$$

Another Cauchy theorem follows, this one called Cauchy's Integral Formula.

> Let C be a simple closed positively oriented curve in a simply connected domain, and z a point inside C. Then
>
> $$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\,dw.$$

To see this, notice that the function $\frac{f(w)}{w - z}$ is analytic on and inside C except at $w = z$. Let α be a small circle around z, so small that it is entirely in C. Then by the fact above we have

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\,dw = \frac{1}{2\pi i}\int_\alpha \frac{f(w)}{w - z}\,dw.$$

If we let r be the radius of the circle α whose center is z, then

$$\frac{1}{2\pi i}\int_\alpha \frac{f(w)}{w - z}dw = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})d\theta.$$

Since $f(z + re^{i\theta}) \to f(z)$ uniformly as $r \to 0$ with respect to θ, we obtain

$$\lim_{r\to 0}\int_0^{2\pi} f(z + re^{i\theta})d\theta = f(z).$$

From Cauchy's Integral Formula it is not hard to derive the following useful formula:

$$f'(z) = \frac{1}{2\pi i}\int_\alpha \frac{f(w)}{(w - z)^2}dw.$$

Cauchy's Estimate will give us a bound on the derivative of f at the center of a circle.

> Suppose f is analytic inside and on a circle of radius r, whose center is a. Suppose also that $|f(z)| \le M$ for all z inside the circle. Then
>
> $$|f'(a)| \le M/r.$$

This is easy to see using the formula above.

$$|f'(a)| = \left|\frac{1}{2\pi i}\int_\alpha \frac{f(w)}{(w - a)^2}dw\right|$$

$$\le \frac{1}{2\pi} \cdot \frac{M}{r^2} \cdot 2\pi r = \frac{M}{r}. \quad \text{(Conway 73)}$$

Now we can prove Liouville's Theorem.

> If f is entire and there is a constant M such that $|f(z)| \le M$ for all $z \in C$, then f is constant.

For any $z \in C$, by Cauchy's Estimate, $|f'(z)| \le M/r$. Letting r approach ∞, we may conclude that $|f'(z)| = 0$, and therefore $f'(z) = 0$, so f is constant.

Proof Using Liouville's Theorem

Most complex variables textbooks use a version of the following proof. Marsden and Hoffman, in their book Basic Complex Analysis, believe this proof is essentially due to Gauss.

If $p(z)$ is a nonconstant polynomial then there is a complex number z_0 such that $p(z_0) = 0$.

To see this, suppose $p(z) = a_0 + a_1 z + \ldots + a_n z^n = 0$ for all z. Let $f(z) = \frac{1}{p(z)}$. Then f is an entire function. We may write

$$p(z) = z^n[a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}],$$

from which it is easily seen that if $z \to \infty$, $p(z) \to \infty$ and so $\lim_{z\to\infty} f(z) = 0$. In particular, there is a number $R > 0$ such that $|f(z)| < 1$ if $|z| > R$. But f is continuous in and on the circle of radius R and center 0, so there is a constant $M > 0$ such that $|f(z)| \le M$ for $|z| \le R$. Hence f is bounded and, therefore, must be constant by Liouville's Theorem. It follows that p must be constant which is a contradiction. Hence, there exists a $z_0 \in C$ such that $p(z_0) = 0$. (Conway 77)

Proof Using Rouché's Theorem

Another complex variables proof uses a theorem known as Rouché's Theorem. This theorem was first published in 1862 in Journal de l'Ecole Impèriale Polytechnique by M. Eugène Rouché, a graduate of l'Ecole Polytechnique and a professor at Lycée Charlemagne. This is a proof of the alternate version of the fundamental theorem of algebra:

> Any polynomial $p(z) = a_0 + a_1z + . . . + a_nz^n$ of degree n, where $n \geq 1$, has precisely n zeros, counting multiplicities.

First, we will look at Rouché's Theorem. We will not prove it here. The proof is not difficult, but some powerful theorems are needed for it. The original proof may be found in the journal mentioned above, while easy-to-understand proofs may be found in <u>Invitation to Complex Analysis</u> by R. P. Boas.

> Let f and g be functions which are analytic in and on a positively oriented simple closed curve C. If $|f(z)| > |g(z)|$ at each point z on C, the functions f(z) and f(z) + g(z) have the same number of zeros, counting multiplicities, inside C.

Now we will prove the fundamental theorem of algebra. Without loss of generality, we may assume that $a_n = 1$ in p(z). Let $f(z) = a_nz^n = z^n$ and

$$g(z) = a_0 + a_1z + . . . + a_{n-1}z^{n-1},$$

$$\text{so } f(z) + g(z) = p(z).$$

Let $R > \max \{1, |a_0| + |a_1| + . . . + |a_{n-1}|\}$. Now f(z) has n zeros inside a circle C of radius R, so if we can show that $|f(z)| > |g(z)|$ at each point z of C, we will be done. We have $|f(z)| = R^n$ for z on C and

$$|g(z)| \leq (|a_0| + |a_1| + . . . + |a_{n-1}|)|z^{n-1}|$$
$$< R(R^{n-1}) = R^n.$$

Thus $|f(z)| > |g(z)|$. The theorem is proved (Churchill).

CHAPTER VI

ALGEBRA

Definitions

The proof of the fundamental theorem that we are about to consider uses the basic ideas of Gauss. This proof is based on a proof from Algebra by Serge Lang. He says, "The variation of the ideas which we have selected, making a particularly efficient use of the Sylow group, is due to Artin" (311). Other algebra textbooks give proofs similar to this one. Unlike the complex variables proofs, much background information is needed.

We will begin our discussion with some assumptions about the real numbers that have to be made in order for this and similar proofs to be valid.

We will assume that the real numbers, R, form an ordered field. An ordered field is a commutative field such that the property of positiveness ($r > 0$) is defined for its elements; and for every r in the field either $r = 0$, $r > 0$, or $r < 0$; and if $r_1 > 0$ and $r_2 > 0$, then $r_1 + r_2 > 0$ and $r_1 r_2 > 0$.

We must also assume that every positive $r \in R$ has a square root. This is easily seen by recalling that the function $f(x) = x^2$ restricted to the positive real numbers is one-to-one and onto. Therefore it has an inverse. Note also that every polynomial of odd degree in R[x] has a root in R and thus has a linear factor in R[x]. (R[x] is the polynomial ring in x over R; that is, the set of all polynomials in x whose coefficients are real numbers.) To see this, consider a monic polynomial f(x) of odd degree. As x approaches positive infinity, then f(x) approaches positive infinity. As x approaches negative infinity, f(x) approaches negative infinity. So the graph of f(x) must cross the real axis somewhere, thus f(r) = 0 for some real number r.

If we adjoin the roots of the polynomial $x^2 + 1$ to the real numbers, we get a new field denoted R(i), where ±i are the roots of $x^2 + 1$. R(i) is another name for C, the field of complex numbers. This field is a finite extension field of R. A field F is an extension field of K if K is a subfield of F. F is always a vector space over K. F is a finite extension field if the dimension of F over K is finite.

It is neccessary to establish that any element in R(i) has a square root. All elements in R(i) have the form a + bi, where a,b ϵ R. Our goal then is to show for each a + bi there exists x such that $x^2 = a + bi$ for some x ϵ R(i). So assume x has the form c + di, c, d ϵ R. Then $(c + di)^2 = a + bi$ implies $c^2 - d^2 = a$ and $2cd = b$. Solving simultaneously we arrive at

$c^2 = \dfrac{a \pm \sqrt{a^2 + b^2}}{2}$ and $d^2 = \dfrac{-a \pm \sqrt{a^2 + b^2}}{2}$.

Since choosing the plus sign in both right-hand expressions yields non-negative quantities, real values for c and d exist, therefore $x = c + di$ exists such that $x^2 = a + bi$.

A ring R has characteristic n if there exists a least positive integer n such that $na = 0$ for all $a \in R$. For the real numbers no such n exists: we say R has characteristic zero.

Let F be an extension field of K. An element $u \in F$ is algebraic over K if u is a root of some nonzero polynomial $f \in K[x]$. A polynomial $f(x) \in K[x]$ is irreducible if the degree of f is greater than or equal to one and f(x) can not be written as the product $f(x) = g(x)h(x)$ with $g, h \in K[x]$ and both $g, h \in K$. The irreducible polynomial over K of an element u of an extension field of K is the polynomial in K[x] of least degree satisfied by u. In general, and consistently with the notation for R(i), K(u) is a new field whose elements take the form g(u) for $g \in K[x]$ with the degree of g less than the degree of the minimal polynomial of u.

Let $f \in K[x]$ be a polynomial of positive degree. If f can be written as a product of linear factors in K[x], then f splits over K. An extension field F of K is a splitting field over K of the polynomial f if f splits in F[x] and $F = K(u_1, u_2, \ldots, u_n)$ where $u_1, u_2, \ldots, u_n$ are the roots of f in F. A splitting field for a polynomial over a base field can always be constructed abstractly.

Let's pause and see how these definitions relate to R and R(i). The element i is algebraic over R since i is the root of $x^2 + 1 \in R[x]$. Since it cannot be written as a product of linear factors whose coefficients are real, $x^2 + 1$ does not split in R[x]. R(i) is a splitting field over R of the polynomial $x^2 + 1$ since $x^2 + 1$ splits in C[x] where C = R(i). $x^2 + 1$ is irreducible in R[x].

In characteristic zero, F is a finite Galois extension of K if and only if F is the splitting field of a polynomial over K (Hungerford 257). R(i) is a finite Galois extension of R since R(i) is a splitting field of $x^2 + 1$ over R.

The proof of the fundamental theorem of algebra considers finite extensions of R(i). These extensions are contained in some extension K of R(i) that is finite and Galois over R. Our goal will be to show that K = R(i), that is, there is no finite extension of R(i). This would imply that every polynomial in C[x] has a root in C.

We will need to use theorems from Galois theory and one of the Sylow theorems. So let's look at the lives of Galois and Sylow, and at Emil Artin whose proof we are imitating.

Evariste Galois

> The life and death of Galois have long been a source of fascination and speculation for

> mathematics historians. One article argues convincingly that three of the most widely read accounts of Galois's life are highly fictitious. (Gallian 158)

The information in this section is gleaned from both the "highly fictitious" accounts and the article mentioned by Gallian in the quote above.

Evariste Galois was born on October 25, 1811 in a village just outside Paris. His father was the mayor of the village. Galois's mother taught him lessons at home until he reached the age of twelve. He received a "thorough classical and religious education." He then entered the Lyceé of Louis-le-Grand (Bell, Men 362). His first two years were marked by a number of successes including a prize in General Concourse and three mentions. Galois was asked to repeat his third year because of his poor work in rhetoric (Rothman 86).

After he had been demoted, Galois enrolled in his first mathematics course. "With the discovery of mathematics, Galois became absorbed and neglected his other courses" (Rothman 86). He quickly mastered the works of Legendre and Lagrange. His mathematics teacher "constantly implored Galois to work more systematically . . . Galois did not take the advice." At the age of 16, he attempted the entrance examination to l'Ecole Polytechnique a year early, without the special course in mathematics that was usually taken (Rothman 87). He did most of the work in his head but apparently did not know some basics (Gallian 158). Thus, he failed the examination and Bell says he "suspected his examiners of incompetence in their office" (366).

At 17, he was encouraged by his mathematics instructor and soon published his first small paper in Annales de Gergonne titled "Démonstration d'un théorème sur les fractions continues périodiques" which translates as "Proof of a Theorem on Periodic Continued Fractions". He then submitted a paper to the Academy on the solvability of equations of prime degree. Cauchy was appointed referee (Rothman 87).

That same year, Galois's father comitted suicide (Rothman 87). Just days after his father's death, Galois attempted the entrance exams at l'Ecole Polytechnique again. He failed them and returned to prepare for a teaching career (Bell, Men 369).

Cauchy did not present the paper Galois submitted to the Academy on the date that had been scheduled (Rothman 88). It was believed that he had lost the manuscript. Galois published a paper in June 1830, "Sur la théorie des nombres", in which the structure of finite fields was determined (van der Waerden 104). Galois then submitted a paper to the Academy of Science in competition for the Grand Prize in Mathematics. Fourier, who was the secretary of the Academy at the time, took the manuscript home to read. Soon after, Fourier died and no trace of the paper was ever found (Bell, Men 370).

"[Galois's] hatred grew," Bell says, "he flung himself into politics." He wrote a letter to the Gazette des Ecoles, complaining of inaction of the students and the school director during the beginning of the Revolution. He was expelled. He then joined the Artillery of the National Guard, a republican organization (Bell, Men 371). Since he was no longer a student, he attempted to organize a private class in mathematics. About forty students attended the first meeting, but the "endeavor did not last long, evidently because of Galois's political activities" (Rothman 90). Around the same time, the Academy received a revised version of Galois's memoir, "Mémoire sur les conditions de résolubilité des équations par radicaux". The Academy asked two of its members, Poisson and Lacroix, to read the manuscript. Poisson "examined it carefully, but he declared he could not understand it" (van der Waerden 104).

A banquet was held at a restaurant to celebrate the acquittal of nineteen republicans on conspiracy charges. Alexander Dumas wrote in his memoirs:

> A young man who had raised his glass and held an open dagger in the same hand was trying to make himself heard. . . .
>
> Evariste Galois was scarcely 23 or 24 at the time. He was one of the most ardent republicans. The noise was such that the very

> reason for this noise had become incomprehensible.
>
> All I could perceive was that there was a threat and that the name of Louis-Phillipe had been mentioned; the intention was made clear by the open knife. (Rothman 92)

Galois was arrested the following day and held in detention. He was acquitted at his trial on the charge of threatening the king's life (Rothman 93). Later, he was arrested again as a precautionary measure. He spent six months in jail and then was paroled (Bell, Men 373).

He was soon after challenged to a duel. The reasons for the challenge are not clear. Bell says that Galois spent the last hours before the "affair of honor feverishly dashing off his scientific last will and testament" (Men 375). This correspondence took the form of a letter to his friend Chevalier. Galois wrote:

> My Dear Friend,
>
> I have made some new discoveries in analysis. The first concern the theory of equations, the other integral functions.
>
> In the theory of equations I have researched the condition for the solvability of equations by radicals; this has given me the occasion to deepen this theory and describe all the transformations possible on an equation even

> though it is not solvable by radicals.
>
> All this will be found here in three memoirs. (Rothman 102)

Rothman says that Galois then went on to describe the memoir rejected by Poisson and other previous work. "During the course of the night he annotated and made corrections on some of his papers" (Rothman 102). His last work contained "no less than the theory of groups, the key to modern algebra and to modern geometry" (Struik, History 153).

Galois died at twenty-one, "shot through the intestines and left to die. He was buried in a common ditch. His enduring monument is his collected works. They fill sixty pages" (Bell M 377).

Galois's works were first published by Liouville in 1846 in his Journal. Cajori reports that "As a rule Galois did not fully prove his theorems. It was only with difficulty that Liouville was able to penetrate Galois's ideas." Galois was the first to use the word group in a technical sense (351).

> Modern algebra begins with Evariste Galois. With Galois, the character of algebra changed radically. Before Galois, the efforts of algebraists were mainly directed towards the solution of algebraic equations. . . . Galois was the first to investigate the structure of fields and groups, and he showed that these two

> structures are closely connected. If one wants to know if an equation can be solved by radicals, one has to analyze the Galois group. After Galois, the efforts of the leading algebraists were mainly directed towards the investigation of the structure of rings, fields, algebras, and the like. (van der Waerden 76)

In 1852, Enrico Betti published the first "rigorous exposition of Galois's theory of equations that made the theory intelligible to the general public." The theory is first found in a textbook in 1866, J. A. Serret's <u>Algebre</u> (Cajori 352).

Ludvig Sylow

Ludvig Sylow was born on December 12, 1832 in Christiania, Norway (now called Oslo). He attended Christiania University and while a student there, won a gold medal for competitive problem solving.

In 1855, he became a high school teacher but "found time to study the papers of Abel." Sylow received a temporary appointment at Christiania University in 1862 and gave lectures on Galois's theory and permutation groups (Gallian 341).

In the spring of 1872, Sylow presented a "paper of fundamental importance for the structure and theory of finite groups" to the <u>Mathematische Annalen</u>, entitled "Theorems sur les groupes de substitutions" (van der

Waerden 139). In it, Sylow extended a theorem given nearly thirty years earlier by Cauchy, which is now known as Sylow's Theorem (Cajori 354). "The result took on greater importance when the theory of abstract groups flowered in the late nineteenth and early twentietn centuries" (Gallian 341).

Retired from high school teaching, Sylow held a chair at Christiania University until his death in 1918. We note that the mathematician Sophus Lie, after whom Lie algebras and groups are named, was a student of Sylow's at the University.

Emil Artin

Emil Artin was born on March 3, 1898 in Vienna, Austria. He grew up in what is now known as Czechoslovokia. In 1921, he received his Ph.D. from the University of Leipzig. He was a professor at the University of Hamburg, Notre Dame, Indiana University, and Princeton.

Artin solved one of the twenty-three problems posed in 1900 by the mathematician David Hilbert (Gallian 285). Artin solved the ninth problem which "concern[ed] the most general reciprocity law in an arbitrary algebraic number field" (Browder 311).

We are most interested in Artin and his work with Galois theory. Joseph Rotman in his book Galois Theory explains:

> . . . for its first century, 1830-1930, the Galois group was a group of permutations. In the late 1920's, E. Artin, developing ideas of E. Noether going back at least to Dedekind, recognized that it is both more elegant and more fruitful to describe Galois groups in terms of field automorphisms (Artin's version is isomorphic to the original version). In 1930, van der Waerden incorporated much of Artin's viewpoint into his influential text Moderne Algebra, and a decade later Artin published his own lectures. (93)

Artin made contributions in many areas of mathematics. He invented the theory of braids; he did much work in ring theory, "in fact, there is a class of rings named after him." He also made advances in number theory, group theory, field theory, geometric algebra, and algebraic topology (Gallian 285).

Emil Artin died in 1962 at the age of 64.

More Definitions

The Galois group of F over K is the set of all automorphisms of F that fix K. That is, $G = G(F|K) = \{\sigma | \sigma$ is an automorphism of F and $\sigma|_K =$ identity$\}$. Recall that an automorphism is a one-to-one onto mapping from a set to itself that preserves the operations. So $\sigma(a + b) = \sigma(a) + \sigma(b)$ and $\sigma(ab) = \sigma(a)\sigma(b)$ for a and b $\in$ F.

If H is a subgroup of the Galois group $G = G(F|K)$, then the fixed field of H is the subset of elements of K that are fixed by every function in H. This subset forms a subfield of K. Keep in mind that G and H are sets of functions while F and the fixed field of H are sets of elements.

F is a Galois extension of K if the fixed field of $G(F|K)$ is K. This says F is a Galois extension of K if the only elements that are fixed by all of the automorphisms in G are the elements of K itself.

A p-group is a finite group whose order (or number of elements) is a power of p. H is a p-Sylow subgroup of a group G if the order of H is p^n and p^n is the highest power of p dividing the order of G.

We will need to use a theorem known as the First Sylow Theorem. It states that if G is a group of order $p^n m$, with $n \geq 1$, p prime and $(p,m) = 1$, then G contains a subgroup of order p^i for each $1 \leq i \leq n$. You may find the proof in Algebra by Hungerford (94).

The Fundamental Theorem of Galois Theory

Now we are ready to discuss the fundamental theorem of Galois Theory. This version is from Hungerford's Algebra (245).

> If F is a finite dimensional Galois extension of K, then there is a one-to-one correspondence between the set of all intermediate fields of

the extension and the set of all subgroups of the Galois group $G(F|K)$ such that:

i.) the relative dimension of two intermediate fields is equal to the relative index of the corresponding subgroups.

ii.) F is Galois over every intermediate field E, but E is Galois over K if and only if the corresponding subgroup $G(F|E)$ is normal in $G(F|K)$.

It may be helpful to consider the following diagrams. We are given a Galois extension F of K and two intermediate fields L and M.

$$\begin{array}{ccc} F & \longrightarrow & G(F|F) = 1 \\ \cup & & \cap \\ M & \longrightarrow & G(F|M) \\ \cup & & \cap \\ L & \longrightarrow & G(F|L) \\ \cup & & \cap \\ K & \longrightarrow & G(F|K) \end{array}$$

What the fundamental theorem says is that not only is there a one-to-one correspondence between the elements in the first and second columns but that the dimension of F over M is equal to the relative index of 1 and $G(F|M)$; that is,

$$\dim_M F = |G(F|M)|/|1|.$$

Further,

$$\dim_L M = |G(F|L)|/|G(F|M)|,$$

$$\dim_K L = |G(F|K)|/|G(F|L)|,$$

and F is Galois over M, L, and K but M (or L) is Galois over K only if G(F|M) (or G(F|L)) is normal in G(F|K).

We are given the Galois group G = G(F|K) and the subgroups J, H, and 1.

$$\begin{array}{lll} 1 & \longrightarrow & F \\ \cap & & \cup \\ H & \longrightarrow & \text{fixed field of } H \\ \cap & & \cup \\ J & \longrightarrow & \text{fixed field of } J \\ \cap & & \cup \\ G & \longrightarrow & K \end{array}$$

As before, the dimensions and indices match up.

Proof Using the Fundamental Theorem of Galois Theory

We now have all the definitions and theorems necessary to prove the fundamental theorem of algebra:

The complex numbers are algebraically closed.

Every finite extension of R(i) is contained in a finite Galois extension K over R. We need to show that K = R(i).

Let G be the Galois group of K over R.

(G = G(K|R))

Let H be a 2-Sylow subgroup of G.

Let L be the fixed field of H.

(The diagram looks like

$$\begin{array}{ccc} 1 & & K \\ \cap & & \cup \\ H & \dashrightarrow & L \\ \cap & & \cup \\ G & & R \end{array}$$

and we know from the fundamental theorem of Galois Theory that $\dim_L K = |H|$ and $\dim_R L = |G|/|H|$.) Since H is a 2-Sylow subgroup, $|H| = 2^m$ for some m and $|G|/|H|$ is odd. (Remember m is the highest power of 2 that will divide $|G|$.) Thus $\dim_R L$ is odd. In fact, we can show $\dim_R L = 1$, that is, R = L.

For let α be any element of L. Then $\dim_R L = \dim_R R(\alpha) \cdot \dim_{R(\alpha)} L$, so $\dim_R R(\alpha)$ is odd. This implies that α is the root of an irreducible real polynomial of odd degree. But the only irreducible real polynomials of odd degree are linear by one of our initial assumptions from calculus. So α is real. Hence $\dim_R L = 1$, so G = H and thus G is a 2-group.

From the fundamental theorem of Galois Theory we know that K is Galois over R(i).

Let G_1 be the Galois group of K over R(i). G_1 is a p-group with p = 2 (since G_1 G and G is a 2-group.)

If G_1 is not the trivial group, then G_1 has a subgroup G_2 of index 2 (from the first Sylow theorem).

Let F be the fixed field of G_2. (The diagram looks like

$$\begin{array}{lcl} 1 & \text{-->} & K \\ \cap & & \cup \\ G_2 & \text{-->} & F \\ \cap & & \cup \\ G_1 & \text{-->} & R(i) \\ \cap & & \cup \\ G & \text{-->} & R \end{array}$$

and we know that $\dim_{R(i)} F = |G_2|/|G_1|$.)
Thus $\dim_{R(i)} F = 2$. But we know that every element in R(i) has a square root. Hence the quadratic formula yields roots in R(i) for any quadratic polynomial in R(i)[x]. So there are no extensions of R(i) of degree 2. So G_1 must be the trivial group 1, which implies that K must be R(i), which was our goal. Thus, every polynomial in C[x] (or R(i)[x]) has a root in C.

Symmetric Polynomials

We will now look at another algebraic proof of the fundamental theorem of algebra that uses known relationships between coefficients and roots of polynomial equations. This proof is from <u>Modern Algebra: A Constructive Introduction</u> by Ian Connell. The basic relationship that we will use is:

$$\text{If } f = x^n - s_1x^{n-1} + s_2x^{n-2} - \cdots + (-1)^n s_n$$
$$= (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$
$$\text{then} \quad s_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

$$s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ . \ . \ . + \alpha_1\alpha_n$$
$$+ \alpha_2\alpha_3 + \ . \ . \ . + \alpha_2\alpha_n$$
$$+ \ . \ . \ . + \alpha_{n-1}\alpha_n$$

$$s_3 = \sum_{1\le i<j<k\le n} \alpha_i\alpha_j\alpha_k$$

.

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.

$$s_n = \alpha_1\alpha_2 \cdot \cdot \cdot \alpha_n$$

The history of the study of coefficients and roots of equations seems to begin with Viète. Although he failed to realize that coefficients and roots could be negative, he was aware of relationships between the roots and coefficients for some specific equations, for example $x^3 + b = 3ax$ (Boyer 336).

In Invention nouvelle en l'algebra, Girard stated clearly the basic relationships just given (Boyer 450).

In each s_i above, any permutation of the α's will not affect the value of the s_i. For this reason, each s_i is known as a symmetric polynomial.

There exist symmetric polynomials other than the s_i given above, for example, $1 + \alpha_1^4 + \alpha_2^4$ as a polynomial of only α_1 and α_2. (Note that if we exchange α_1 for α_2 and vice versa, we obtain $1 + \alpha_2^4 + \alpha_1^4$ which is equivalent to the original polynomial.)

It can be shown that every symmetric polynomial with integer coefficients can be written as a polynomial with integer coefficients in $s_1, s_2, \ . \ . \ . \ , s_n$. For example,

$$1 + \alpha_1^4 + \alpha_2^4 = 1 + s_1^4 - 4s_1^2s_2 + 2s_2^2.$$

Thus $s_1, s_2, \ldots, s_n$ are called the elementary symmetric polynomials. For a proof of the above proposition, see Connell page 293.

> The interest in symmetric functions arose when the seventeenth century algebraists noted and Newton proved that the various sums of the products of the roots of a polynomial equation can be expressed in terms of the coefficients. (Kline 600)

Vandermonde, in 1771, showed that any symmetric function of the roots can be expressed in terms of the coefficients of the equation, the proposition we just discussed.

Lagrange analyzed the methods of solution of third and fourth degree equations hoping to find a method to solve higher degree equations. Much work on symmetric functions can be found in his paper "Réflexions sur la résolution algébrique des équations" (Kline 600).

Proof Using Symmetric Polynomials

We will prove C is algebraically closed.

Let $f \in C[x]$ have degree $n > 0$.

We have shown that f has roots in C if either $n = 2$ or $f \in R[x]$ and n is odd. We will now consider $f \in R[x]$ for general n and show that such an f has a root in C. We will then show that any $f \in C[x]$ of general degree n has a root. Upon completion of these two tasks, we will have shown that C is algebraically closed.

Let $f = x^n + a_{n-1}x^{n-1} + . . . + a_0$, $a_j \in R$, and $n = 2^q m$ where 2 does not divide m. Since we know our theorem is true for $q = 0$, let's assume $q > 0$, and so n is even.

If we think of $f \in C[x]$, then f has a splitting field $k \subset C$. Over $k[x]$, factor f as $(x - \alpha_1) \cdot \cdot \cdot (x - \alpha_n)$, $\alpha_i \in k$.

Define $g = \pi(x - (\alpha_i + \alpha_j) - h\alpha_i\alpha_j) \in k[x]$ where h is an integer to be specified.

The coefficients of g are symmetric polynomials in the α_i, hence are polynomials with integer coefficients in the elementary symmetric polynomials $\pm a_i$ and are therefore real numbers. So $g \in R[x]$.

Calculating the degree of g we find $\deg g = [n(n - 1)]/2 = [2^q m(n - 1)]/2 = 2^{q-1}m(n - 1) = N$. (Note that $m(n - 1)$ is odd since n is even and 2 does not divide m.)

By induction on q, at least one of the roots $\alpha_i + \alpha_j + h\alpha_i\alpha_j$ is in C.

Now let $h = 0, 1, 2, . . . , N$.

This leads us to $N + 1$ different polynomials g.

Each of these g's has a root in C.

Each g is the product of N factors $(x - (\alpha_i + \alpha_j - h\alpha_i\alpha_j))$. Thus two of the roots guaranteed above are composed of the same pair of α_i and α_j. That is, there exist $h_1 = h_2$ such that

$\alpha_i + \alpha_j + h_1\alpha_i\alpha_j$ and $\alpha_i + \alpha_j + h_2\alpha_i\alpha_j$ are both in C. So $[(\alpha_i + \alpha_j + h_1\alpha_1\alpha_2) - (\alpha_i + \alpha_j + h_2\alpha_i\alpha_j)]/(h_1 - h_2)$ $= \alpha_i\alpha_j \in C$. And since $\alpha_i + \alpha_j + h_1\alpha_i\alpha_j \in C$, $\alpha_i + \alpha_j \in C$.

So $x^2 - (\alpha_i + \alpha_j)x + \alpha_i\alpha_j \in C[x]$. But

$$x^2 - (\alpha_i + \alpha_j)x + \alpha_i\alpha_j = (x - \alpha_i)(x - \alpha_j).$$

Thus α_i and α_j are in C, because quadratic polynomials have their roots in C.

Hence $f \in R[x]$ of even degree has roots in C.

Now we will consider $f \in C[x]$.

Recall if $f = \alpha_0 + \alpha_1 x + . . . + \alpha_n x^n$ then

$$\overline{f} = \overline{\alpha_0} + \overline{\alpha_1}x + . . . + \overline{\alpha_n}x^n.$$

Also $\overline{\overline{f}} = f$.

Let $g = f\cdot\overline{f}$. Then $\overline{g} = \overline{f\cdot\overline{f}} = \overline{f}\cdot\overline{\overline{f}} = \overline{f}\cdot f = g$. Thus $g \in R[x]$. Therefore, by the first part of the proof, g has a root in C, say α.

So $g(\alpha) = f(\alpha)\cdot\overline{f}(\alpha) = 0$.

If $f(\alpha) = 0$, we have found a root of f in C.

If $\overline{f}(\alpha) = 0$, then $\overline{\overline{f}(\alpha)} = \overline{0} = 0$, and $\overline{\overline{f}(\alpha)} = \overline{\overline{f}}(\overline{\alpha}) = f(\overline{\alpha})$ and thus $\overline{\alpha}$ is the required root.

Thus $f \in C[x]$ has a root in C (Connell 306).

CHAPTER VII

ANALYSIS

The Foundations of Analysis

> Up to the present . . . more concern has been given to enlarging the building than to illuminating the entrance, to raising it higher than giving proper strength to the foundations. (d'Alembert in Kline 619)

Some mathematicians of the eighteenth century were becoming increasingly alarmed over the "deepening crisis" in the foundation of analysis (Eves 367).

Geometrical methods were used during the first part of the century but mathematicians such as Euler and Lagrange had begun to realize the "greater effectiveness of analytic methods" (Kline 614). Intuition played a big role in the thinking of these mathematicians. Kline says:

> Any delicate question of analysis, such as the convergence of series and integrals, the interchange of the order of differentiation and integration, the use of differentials of higher order, and questions of existence of integrals and solutions of differential equations, were all but ignored. (617)

This mathematical era is sometimes called the heroic age because the mathematicians were willing to "plunge ahead so boldly without logical support" (Kline 617).

Rigorous analysis began in the early 1800's with the work of Bolzano, Cauchy, Abel, Direchlet, and Weierstrass (Kline 948). We will look at the lives and work of Bolzano, Weierstrass, and another analyst of a later date, Riemann. We have previously discussed Cauchy's life and some of his work, but let's look at his importance in the "rigorization of analysis".

Around 1820 Cauchy began to collect the lectures he had given on analysis at l'Ecole Polytechnique and other colleges to publish a book. The book, entitled Cours d'analyse de l'Ecole Polytechnique, was to become very famous (Grattan-Guinness 48). In his introduction, Cauchy explained that he "seeks to give rigor to analysis" (Kline 948).

> The legend surrounding this book and its companions is that they revolutionized the whole of analysis and created the standards of mathematical rigor to which we are now accustomed. (Grattan-Guinness 48)

But the books had many weaknesses although they "marked a great step forward from their predecessors" (Grattan-Guinness 48).

The work of Cauchy and others "freed the calculus and its extensions from all dependence upon geometrical notions, motion, and intuitive understanding" (Kline 972).

Bernhard Bolzano

Bernhard Bolzano lived as a priest, philosopher and mathematician in Bohemia. He was born in 1781 and died in 1848 (Kline 950). He held the post of professor of the philosophy of religion at the University of Prague (Cajori 367).

His mathematical interests were varied. He studied the foundations of real variable analysis, Euclidean geometry, number theory, and rational and irrational numbers. His papers were issued in the form of pamphlets and in journals, where he gave a proof: <u>Purely analytical proof of the theorem that between any two values which guarantee an opposing result [in sign] lies at least one real root of the equation</u> (Grattan-Guiness 52).

He also gave a proof of the binomial formula and "exhibited clear notions on the convergence of series. He held advanced views on variables, continuity and limits" (Cajori 367).

Karl Weierstrass

Karl Weierstrass was born in 1815 in Westphalia. He studied law at the University of Bonn, but did not complete his doctoral work. Instead he became a gymnasium (high school) teacher (Kline 643). He taught at several different gymnasia, instructing his pupils in subjects such as science, gymnastics, and writing. He taught at Braunsberg where he "entered upon the study of Abelian functions." After the publication of some scientific papers, he received an honorary doctorate from Königsberg (Cajori 424).

In 1856, he began teaching at the Industrial Institute in Berlin. Later that year he became an instructor at the University of Berlin, and then professor. He remained at this post until his death in 1897 (Kline 643).

Weierstrass' lectures were "meticulously prepared" and he became increasingly famous. "It is mainly through these lectures that Weierstrass' ideas have become the common property of mathematicians" (Struik 160). The number of successful research workers that he produced led Boyer to proclaim that Weierstrass was "the greatest mathematics teacher of the mid-nineteenth century" (609).

The "age of rigor" brought us the kind of analysis with which we are now familiar.

> But the age did not dawn before Weierstrass, for these levels of technique and subtlety of reasoning were introduced only in his analysis lectures at Berlin. Uniform and nonuniform convergence; noninfinitesimal analysis to avoid the difficulties of his infinitesimal predecessors; the "(ϵ, δ)" formulation of Bolzano's arithmetical approach to analysis; all

> these ideas were urged by Weierstrass on his students, who then began to use and develop them in their own research and teaching. (Grattan-Guinness 120)

In 1835, N. I. Lobachevsky had discussed the necessity of distinguishing between continuity and differentiability. "The mathematical world received a great shock" when Weierstrass brought forth his discovery of a function which was continuous over an interval but did not have a derivative at any point in the interval (Cajori 425). Mathematicians refused to take such functions seriously and called them "pathological functions" (Struik 158).

Weierstrass's fame rests on his "extremely careful reasoning" and the rigor which he used in all of his work. "He clarified the notions of minimum, of function, and of derivative," and this, claims Struik, "eliminated the remaining vagueness of expression in the fundamental concepts of the calculus" (160).

Georg F. B. Riemann

Georg Friedrich Bernhard Riemann was born in Hanover in 1826. He studied theology at the University of Göttingen and attended some mathematics lectures there (Smith 404). He was bashful and although his father wished for him a career in religion, "[Riemann] realized he would not be a preacher" (Bell, Men 485). He gave up theology and studied under Gauss and Stern. In 1847, he traveled to Berlin to study under the group of famous mathematicians that had gathered there, including Dirichlet, Jacobi, Steiner, and Eisenstein. He returned to Göttingen and studied physics under Weber (Smith 404).

In 1851 he wrote his doctoral thesis on the theory of complex functions. Riemann "clarified [his] definition of a complex function: its real and imaginary parts have to satisfy the Cauchy-Riemann equations (Struik, History 158). Soon after the completion of his thesis, Riemann began his career as a professor at Göttingen (Smith 404).

Riemann was a "many-sided mathematician with a fertile mind." He made contributions in analysis, geometry, and the theory of numbers. He is "recalled for his part in the refinement of the definition of the integral, for emphasis on the Cauchy-Riemann equations, and for Riemann surfaces" (Boyer 601).

Riemann was sickly as a child and as an adult. He died at the age of 40 in Italy. He published a relatively small number of papers but "each of them was - and is - important, and several have opened entirely new and productive fields" (Struik, History 158).

Proof Using the Cauchy-Riemann Equations

This proof uses the Cauchy-Riemann equations which were first stated by d'Alembert in 1752 in a paper on the resistance of fluids (Struik, History 151). Later, Cauchy published a paper in which he "concentrated on saving complex variable integration" by finding the conditions that made the integration valid. In this paper, written in 1814 and published in 1825, Cauchy states the equations for the first time (Grattan-Guinness 31). They are known as the Cauchy-Riemann equations "for the fundamental role that they play in Riemann's formulation of the theory of functions of a complex variable" (Grattan-Guiness 33).

A complex valued function $w(x,y) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations if its real and imaginary parts do, that is, if $u_x = v_y$ and $u_y = -v_x$.

We will need some theorems from calculus to complete this proof. The first result, if u is a sufficiently well-behaved function of x and y, then $u_{xy} = u_{yx}$, was stated in 1734 by Euler (Grattan-Guinness 3). It is very easy to prove if u is a polynomial.

We also need a theorem similar to the second derivative test for extrema of functions of one variable. This theorem states that at a minimum of the polynomial function $g(x,y)$, both $g_x = 0$ and $g_y = 0$, and $g_{xx} \geq 0$ and $g_{yy} \geq 0$. For a discussion of this, you may refer to Calculus and Analytic Geometry by Douglas Riddle, page 999.

The author of this proof of the fundamental theorem of algebra, Raymond Redheffer, says, "the only other deep theorem we need is that a continuous function in the closed circle attains a minimum in the closed circle.

This is not trivial even for polynomials" (582). See Advanced Calculus by John Olmsted, page 188, for more information on this theorem.

Before we begin the proof, we will look at some algebraic properties of the Cauchy-Riemann equations.

> Property 1: If w and W are complex functions which satisfy the Cauchy-Riemann equations, so are $w + W$ and wW.

Suppose $w = u + iv$ and $W = U + iV$. It is easy to show that the Cauchy-Riemann equations hold for $w + W$. Let's look at wW.

$$wW = (uU - vV) + i(uV + vU)$$

and
$$\begin{aligned}(uU - vV)_x &= (uU_x + Uu_x) - (vV_x + Vv_x)\\ &= uV_y + Uv_y + vU_y + Vu_y\\ &= (uV_y + Vu_y) + (Uv_y + vU_y)\\ &= (uV + vU)_y.\end{aligned}$$

Similarly, $(uU - vV)_y = -(uV + vU)_x$.

> Property 2: Let u and v be real and imaginary parts of a polynomial
> $u + iv = a_nz^n + a_{n-1}z^{n-1} + . . . + a_1z + a_0$
> where $z = x + iy$ and a_i is constant for all i. Then u and v are polynomials that satisfy the Cauchy-Riemann equations.

It is clear that $u + iv = x + iy$ satisfies the Cauchy-Riemann equations. Hence, $(x + iy)(x + iy) = (x + iy)^2$, $(x + iy)^3$, and consequently $(x + iy)^m$ satisfy the equations (by Property 1). Also, $u + iv = a + ib$ where a and b are constants satisfies the Cauchy-Riemann equations, and hence so does $a_m(x + iy)^m$. And so by Property 1, the sum of such terms satisfies the Cauchy-Riemann equations.

Next, suppose that u and v are the real and imaginary parts of a polynomial as in Property 2. Then we have

> Theorem 1: If $2f = u^2 + v^2$ and $m = u_x^2 + u_y^2$, then $f_x^2 + f_y^2 = m(u^2 + v^2)$ and $f_{xx} + f_{yy} = 2m$.

If $2f = u^2 + v^2$, then

$$f_x = uu_x + vv_x = uv_y - vu_y \text{ and}$$

$$f_y = uu_y + vv_y = -uv_x + vu_x.$$

Then
$$f_x^2 + f_y^2 = (uv_y - vu_y)^2 + (vu_x - uv_x)^2$$
$$= m(u^2 + u^2).$$

Now let's consider f_{xx} and f_{yy}.

$$f_{xx} = (uv_{yx} + v_y u_x) - (vu_{yx} + u_y v_x) \text{ and}$$

$$f_{yy} = -(uv_{xy} + v_x u_y) + (vu_{xy} + u_x v_y).$$

So
$$f_{xx} + f_{yy} = 2v_y u_x - 2v_x u_y$$
$$= 2u_x^2 + 2u_y^2$$
$$= 2m.$$

Now we will look at Theorem 2. Theorem 2 contains the fundamental theorem of algebra as a special case. So after proving theorem 2, we will have just one more step to finish our proof.

> Theorem 2: With f(x,y) as above, suppose $f(0,0) = c$ and $f(x,y) \geq c + 1$ on the circle $x^2 + y^2 = r^2$. Then $f = 0$ at some point inside this circle.

To prove this, define $g(x,y) = f(x,y) - hx^2$ where h is a small positive constant. Then $g(0,0) = c$, and on the circle $x^2 + y^2 = r^2$

$$g(x,y) \geq c + 1 - hx^2 \geq c + 1 - hr^2 > c$$

for sufficiently small h.

Since g is continuous, it reaches an absolute minimum in $x^2 + y^2 \leq r^2$. This minimum is not on the boundary since $g(0,0) < g(x,y)$ for all (x,y) on $x^2 + y^2 = r^2$. So the minimum is at an interior point. At the minimum,

$$g_x = f_x - 2hx = 0$$

$$g_y = f_y = 0$$

and

$$g_{xx} = f_{xx} - 2h \geq 0$$

$$g_{yy} = f_{yy} = 0.$$

So $m(u^2 + v^2) = f_x{}^2 + f_y{}^2 = (2hx)^2 + 0^2 = 4h^2x^2$ and $2m = f_{xx} + f_{yy} \geq 2h + 0 = 2h$. Dividing these two expressions gives us

$$(u^2 + v^2)/2 \leq (4h^2x^2)/(2h) = 2hx^2 \leq 2hr^2,$$

so $\quad (u^2 + v^2) \leq 4hr^2.$

Since h is arbitrary, $u^2 + v^2$ cannot have a positive lower bound in the circle, and so $u^2 + v^2 = 0$ at some point in the circle.

Now, the last step. Suppose

$u + iv = a_n z^n + \ldots + a_1 z + a_0$ where $z = x + iy$, $a_n = 0$, and $n \geq 1$. Then we know that the Cauchy-Riemann equations hold and that

$$\lim_{z \to \infty} \frac{u + iv}{a_n z^n} = 1.$$

Hence, for large z, $\frac{|u + iv|}{|a_n z^n|} > \frac{1}{2}$, or

$|u + iv| > (1/2)|a_n||z|^n$. So as $x^2 + y^2 \to \infty$, $u^2 + v^2 \to \infty$. Therefore, the hypothesis of Theorem 2 is satisfied if r is large enough, and so u + iv = 0 at some point inside the circle. The point (x,y) where u and v both vanish yields a complex number z = x + iy where the polynomial vanishes. Thus, the existence of a root is established. (Redheffer 582)

An Elementary Proof

This proof is from Principles of Mathematical Analysis by Walter Rudin. This proof uses the fact that if z is a complex number with $|z| = 1$, then there is a unique $t \in [0,2\pi)$ such that $e^{it} = z$.

Again, let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$. Without loss of generality, let $a_n = 1$. Let $L = \text{glb}|p(z)|$. If $|z| = r$, then

$$|p(z)| \geq r^n[1 - \frac{|a_{n-1}|}{r} - \ldots - \frac{|a_0|}{r^n}].$$

As $r \to \infty$, the right side of the inequality tends to ∞. Hence, there exists an r_0 such that $|p(z)| > L$ if $|z| > r_0$. Since p(z) is continuous in and on the circle of radius r_0 with center 0, then $|p(z_0)| = L$ for some z_0, $|z_0| \leq r$.

Now we want to show that L = 0. Suppose not. Let $q(z) = \frac{p(z + z_0)}{p(z_0)}$. Then q is a nonconstant polynomial, $q(0) = 1$ and $|q(z)| \geq 1$ for all z. There is a smallest integer k, $1 \le k \le n$, such that

$$q(z) = 1 + b_k z^k + \ldots + b_n z^n, \ b_k = 0.$$

By the fact above, there is a real $t \in [0,2\pi)$ such that $e^{ikt}b_k = -|b_k|$. Then if $r > 0$,

$$|q(re^{it})| \le 1 - r^k[|b_k| - r|b_{k+1}| - \ldots - r^{n-k}|b_n|]$$

For sufficiently small r, the expression in brackets is positive, hence $|q(re^{it})| < 1$, which is a contradiction. Thus, $L = 0$, that is, $p(z_0) = 0$. (Rudin 170)

A Precalculus Proof

The following proof was published in the April 1981 issue of the American Mathematical Monthly. The authors, J. L. Brenner and R. C. Lyndon remark:

> It is a truism that the [fundamental theorem of algebra] is not really a theorem of algebra but of analysis or topology. In the present note we present a proof that ought to be intelligible to a precalculus student. (253)

Since no "big" theorems are used, the proof seems very long. The proof of the fundamental theorem of algebra requires one brief step after we have proved the following theorem.

> Let $P(z)$ be a nonconstant polynomial with complex coefficients. Then there is a positive number S, depending only on P, with the following property: for every $\delta > 0$ there is a

complex number z such that $|z| \le S$ and $|P(z)| < \delta$.

The proof will use only the elementary algebra of the complex numbers and simple inequalities. What we will do is make a circle around the origin of a specific size. Around the circle we will put an equilateral triangle. This triangle will be divided into a mesh of smaller triangles. We will show that if z and z' are vertices of a triangle that are "close", then their corresponding P(z) and P(z') values will be "close". Then we will show that if P(z) and P(z') are in opposite quadrants, then each of $|P(z)|$ and $|P(z')|$ will be less than $|P(z') - P(z)|$. Since P(z) and P(z') are "close", this will imply that $|P(z)|$ is less than any arbitrary small positive number, from which we may conclude that $P(z) = 0$.

Define $P(z) = \sum_{j=0}^{n} a_j z^j$ and suppose $a_0 \neq 0$ and $a_n = 1$. Let $A = \sum_{j=0}^{n} |a_j|$ and $R = 2A$. (Note that $A > 1$.)

First we will need to prove the following three inequalities. If $|z| \ge R$ then

1.) $|P(z)| \ge A$

2.) $|P(z) - z^n| \le |z^n|/2$

3.) $|\arg[P(z)] - \arg[z^n]| \le \pi/6$

where arg[w] denotes the principal value of the angle of w.

So let's begin with the first inequality.

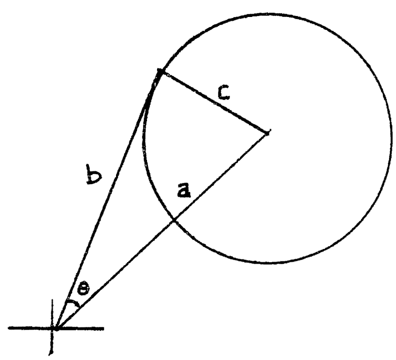
$$|P(z)| \ge |z^n| - \sum_{0}^{n-1} |a_j||z^j|$$
$$\ge |z^{n-1}|(|z| - \Sigma\, |a_j|)$$

$$= |z^{n-1}|(|z| - (A - a_n))$$
$$\geq R^{n-1}(R - A + 1)$$
$$\geq R^{n-1}(R - A)$$
$$= R^{n-1}A$$
$$\geq A.$$

Before we begin the next step we need to note that $|z| \geq R$ so $|z| \geq 2A$ and so $|z| \geq 2A - 2$.

$$|P(z) - z^n| \leq \sum_{j=0}^{n-1}|a_j||z|^j$$
$$\leq \sum_{j=0}^{n-1}|a_j||z|^{n-1}$$
$$= (A - 1)|z|^{n-1}$$
$$= (A - 1)|z|^{-1}|z|^n$$
$$\leq |z|^n(A - 1)/(2A - 2)$$
$$= |z|^n/2.$$

For the last inequality, we will use the previous inequality and the following diagram.



In the diagram, $a = |z^n|$ and $b = |P(z)|$.

$$c \leq |P(z) - z^n|$$
$$\leq |z|^n/2$$
$$= a/2.$$

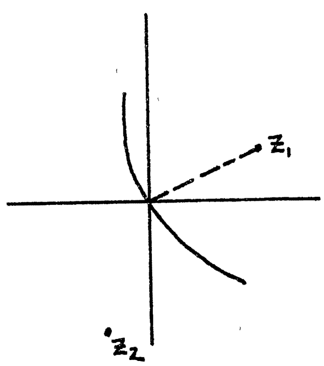
Let $\theta = |\arg[P(z)] - \arg[z^n]|$. Consider s, a perpendicular to b drawn from the endpoint of b to a, extended if necessary. Then $\tan\theta = s/b$

$$\begin{aligned} &\le c/b \\ &\le (\tfrac{1}{2}a)/b \\ &\le \tfrac{1}{2} \qquad \text{(since } |P(z)| \ge |z^n|.\text{)} \end{aligned}$$

Thus $\theta \le \pi/6$.

Now we need some notation. We say w lies in the kth quadrant (k = 1, 2, 3, or 4) when w = 0 and $(k - 1)\pi/2 \le \arg[w] < [k\pi]/2$. Let $Q(w) = k$ if w lies in the kth quadrant. If z_1 and z_2 lie in opposite quadrants, then $Q(z_1)$ and $Q(z_2)$ differ by two.

From the following diagram it is clear that if z_1 and z_2 lie in opposite quadrants then both $|z_1|$ and $|z_2|$ are less than or equal to $|z_2 - z_1|$.



We need to show that

> If $S > 0$, there exists a number $K > 0$, depending only on S and P, such that whenever $|z_1|$, $|z_2| \le S$, we have $|P(z_2) - P(z_1)| \le K|z_2 - z_1|$.
> Explicitly, we may choose $K = A \cdot \{\max(1,S)\}^n$.

Suppose that $S > 1$. Then

$$P(z_2) - P(z_1) = \sum_{j=0}^{n} a_j(z_2{}^j - z_1{}^j)$$

$$= (z_2 - z_1)\sum_{j=1}^{n} a_j \sum_{h+k=j-1} z_1{}^h z_2{}^k$$

So $|P(z_2) - P(z_1)| \le |z_2 - z_1| \sum_{j=1}^{n} |a_j| S^{j-1}$

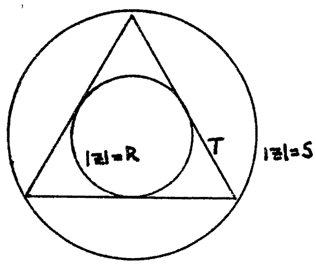
$$\le |z_2 - z_1| S^n A$$

$$= K|z_2 - z_1|.$$

Now we are ready to prove the theorem stated at the beginning.

Let δ be given such that $0 < \delta < 1$.

Choose an equilateral triangle T enclosing the circle $|z| = R$ and a number $S > 0$ such that T is enclosed in the circle $|z| = S$.



Let $\epsilon = \delta/K$ where K is as above.

Let be the closed two-dimensional set inside T. Then if z_1 and z_2 are in and $|z_2 - z_1| < \epsilon$,

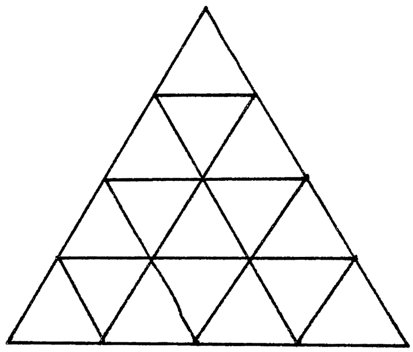
$$|P(z_2) - P(z_1)| \le K|z_2 - z_1|$$

$$= K \cdot \epsilon$$

$$= K \cdot (\delta/K)$$

$$= \delta.$$

Now divide into congruent equilateral triangles τ by equidistant lines parallel to the sides of T. We will make the mesh small enough so that each triangle has sides of length no greater than ϵ and so that if z and z' are adjacent vertices of the mesh lying on the sides of T, then $|\arg[z'] - \arg[z]| < \pi/(6n)$.



So if z and z' are vertices of the same small triangle Δ_T, then $|z' - z| < \epsilon$ and $|P(z') - P(z)| < \delta$.

Case I. Assume the two vertices z and z' are such that P(z) and P(z') lie in opposite quadrants.

Then $|P(z)| < |P(z') - P(z)| < \delta$.

Case II. Assume for a contradiction that none of the triangles Δ_T has two vertices z and z' such that P(z) and P(z') lie in opposite quadrants (and that there is no vertex z such that P(z) = 0).

Suppose that z and z' are two adjacent vertices of our mesh, so w = P(z) and w' = P(z') do not lie in opposite quadrants. Define $d(w,w') \equiv Q(w') - Q(w)$; then d(w,w') can take on only the values -1, 0, and 1. The following chart shows the values of $d(w,w')$ for w and w' in various quadrants.

$d(w',w)$		Q(w) 1	2	3	4
	1	0	-1	no	1
Q(w')	2	1	0	-1	no
	3	no	1	0	-1
	4	-1	no	1	0

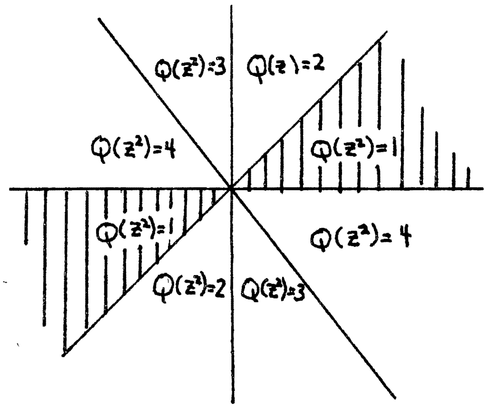
Let $z_1, \ldots, z_t$ be the vertices of the mesh that lie on the sides of T taken in counterclockwise order and let $w_j = P(z_j)$.

The sequence $Q(z_1^n), \ldots, Q(z_t^n)$ runs through the cycle 1, 2, 3, 4 exactly n times apart from repetitions.

Let's stop to examine an example here. Suppose $z = a + bi$ and $n = 2$. Then

$$z^2 = (a + bi)^2 = (a^2 - b^2) + 2abi.$$

We want to know what the sequence of Q's looks like as $z = a + bi$ goes through the various quadrants. We know that $Q(z^2) = 1$ if $a^2 - b^2 > 0$ and $ab > 0$. The shaded part of the diagram below indicates where z would have to be in order that $Q(z^2) = 1$. Going around counterclockwise we find where z would have to be for $Q(z^2) = 2$ and 3 and 4. Then the pattern repeats. So we see that 1, 2, 3, and 4 each appear twice in the sequence of Q's.



The sequence $Q(w_1), \ldots, Q(w_t)$ is almost the same as the previous sequence. The only difference is that certain subsequences of the form $h, \ldots, h, h', \ldots, h'$ where h and h' are numbers representing adjacent quadrants are replaced by subsequences of the form $h, h_1, \ldots, h_m, h'$ where the h_i's are either h or h' for all i. This is true since we know $|\arg[P(z)] - \arg[z^n]| < \pi/6$. For example, a subsequence of the form $1, 1, \ldots, 1, 2, 2, \ldots, 2$ might be replaced by $1, 2, 1, 2, 1, 1, 2, \ldots, 1, 2$.

Define $D(\Delta) = \Sigma d(w_{j+1}, w_j)$ summed over $j = 1, \ldots, t$ where $w_{t+1} = w_1$.

$$
\begin{aligned}
D(\Delta) &= \Sigma d(w_{j+1}, w_j) \\
&= \Sigma d(z_{j+1}{}^n, z_j{}^n) \\
&= 4n.
\end{aligned}
$$

Let's go back to our example to check on this calculation. Recall $Q(z_1{}^2), \ldots, Q(z_t{}^2)$ was $1, 2, 3, 4, 1, 2, 3, 4$ (without repetition.) Thus $D(\Delta)$ would be

$$D(\triangle) = \Sigma d(z_{z+1}{}^2, z_j{}^2)$$
$$= (2 - 1) + (3 - 2) + (4 - 3) + (1 - 4)$$
$$+ (2 - 1) + (3 - 2) + (4 - 3) + (1 - 4)$$
$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 8 = 4 \cdot 2.$$

(Refer to the chart to find these values.)

Let z_1, z_2, and z_3 be the vertices of some triangle $\triangle_\tau$, taken in order counterclockwise. We have assumed that no two vertices of $\triangle_\tau$ can be in opposite quadrants so $Q(P(z_1))$, $Q(P(z_2))$, and $Q(P(z_3))$ can assume at most two adjacent values. Therefore

$$D(\triangle_\tau) = d(w_2, w_1) + d(w_3, w_2) + d(w_1, w_3) = 0.$$

For example, if $\triangle_\tau$ was such that $Q(w_1) = 1$, $Q(w_2) = 1$, and $Q(w_3) = 2$, then

$$D(\triangle_\tau) = (1 - 1) + (2 - 1) + (1 - 2) = 0.$$

It follows then that $\Sigma D(\triangle_\tau) = 0$ where the summation is over all the triangles in the mesh.

Now we want to consider the relationship between $D(\triangle)$ and $\Sigma D(\triangle_\tau)$.

Let's look again at $\Sigma D(\triangle_\tau)$. Suppose that two small triangles, $\triangle_\alpha$ and $\triangle_\beta$, are joined at the vertices z and z'. Then $d(w',w)$ is in either $D(\triangle_\alpha)$ or $D(\triangle_\beta)$, and $d(w,w')$ is in the other. Now $d(w',w) = -d(w,w')$, so in the sum $\Sigma D(\triangle_\tau)$ all the interior vertices force similar pairs which cancel each other out. Thus $\Sigma D(\triangle_\tau)$ depends only on the vertices on the edges of T. So $\Sigma D(\triangle_\tau) = D(\triangle)$.

But $D(\triangle) = 4n$ and $\Sigma D(\triangle_\tau) = 0$, so we have a contradiction. Remember, we had assumed that no two vertices of a small triangle could lie in opposite quadrants and that there did not exist a vertex z such that $P(z) = 0$. Since we have contradicted this, either there does exist a vertex z such that $P(z) = 0$ in which case we are done, or two vertices z and z' of a triangle Δ_τ lie in opposite quadrants, which is Case I. We have proved the fundamental theorem of algebra. (253)

CHAPTER VIII

TOPOLOGY

History

> Topology is concerned with those properties of geometric figures that remain invariant when the figures are bent, stretched, shrunk, or deformed in any way that does not create new points or fuse existing points. The transformation presupposes, in other words, that there is a one-to-one correspondence between the points of the original figure and the points of the transformed figure, and that the transformation carries nearby points into nearby points. (Kline 1158)

Topology can be thought of as two separate and distinct areas: point-set topology and combinatorial or algebraic topology. Point-set topology is "concerned with geometrical figures regarded as collections of points with the entire collection often regarded as a space" (Kline 1158). According to Kline, the origins of point-set topology can be traced back to Maurice Frèchet's doctoral dissertation of 1906. In this paper, Frèchet treated functions as points of a space, a common practice in the study of the calculus of variations. He "launched the study of abstract spaces" and introduced the class of metric spaces (1159).

The subject of point-set topology has "continued to be enormously active. It's relatively easy to introduce variations, specializations, and generalizations of the axiomatic bases for the various types of spaces" (Kline 1162).

Combinatorial topology, or analysis situs as it was first called, is the

> study of intrinsic qualitative aspects of spatial configurations that remain invariant under one-to-one transformations. It is often referred to popularly as rubber-sheet geometry, for deformations of, say, a balloon, without puncturing or tearing it, are instances of topological transformations. (Boyer 652)

In 1679, Leibniz, in his Characteristica Geometrica, tried to "formulate basic geometric properties of geometrical figures." He called this study analysis situs or geometria situs. One geometric property, known to Euler and even Descartes before him, was that for a closed convex polyhedron, $V - E + F = 2$, where V is the number of vertices, E the number of edges, and F the number of faces. Euler used the property to classify polyhedra (Kline 1163).

The Koenigsberg bridge problem was solved in 1735 by

Euler. This is a problem "whose topological nature was later appreciated." In a river in Koenigsberg there exist two islands joined to the shore and each other by seven bridges. The townspeople amused themselves by trying to cross all seven bridges without recrossing any of them. Euler proved it was not possible by looking at a diagram of points and arcs. He also gave criteria to determine when such paths are possible for a given set of points and arcs (Kline 1163).

In 1848 Johann B. Listing published Vorstudien zur Topologie or Introductory Studies in Topology. Listing was a professor of physics at Göttingen and formerly was a student of Gauss. Topology, to him, was the "geometry of position." In 1858 he began a series of topological investigations seeking qualitative laws for geometrical figures (Kline 1164).

An assistant to Gauss, Augustus Ferdinand Möbius, is credited by Kline as being "the man who first formulated properly the nature of topological investigations" (Kline 1164). Möbius was a native of Prussia. He was a professor of astronomy at Leipzig from 1825 until his death in 1868 (Cajori 289). He classified the various geometrical properties, projective, affine, similarity, and congruence, and in 1863 in his "Theorie der elementaren Verwandschaft" (Theory of Elementary Relationships), he "proposed studying the relationships between two figures whose points are in one-to-one correspondence and such that neighboring points correspond to neighboring points" (Kline 1164). Möbius is best known for his discovery of a one-sided surface which is now known as the Möbius band or strip (Kline 1165).

The conjecture that four colors will always be sufficient to color all maps so that countries with at least one common border will be colored differently is known as the four-color problem or the map problem. This was first conjectured by Francis Guthrie, a professor of mathematics, whose brother communicated the problem to DeMorgan. The map problem is also considered to be topological in nature (Kline 1166). In 1977, Dr. Kenneth Appel and Dr. Wolfgang Haken of the University of Illinois published a paper proving that four colors "were indeed enough. But the 100-page proof relied on extensive computer calculations. . . . Many mathematicians found the proof difficult to swallow" (Kolata 4E).

Riemann's dissertation of 1851 on complex function theory also contained topological discussions. Riemann classified surfaces according to their connectivity, a topological property (Kline 1166). In 1882, Felix Klein introduced a two-dimensional closed figure now called the Klein bottle. This surface has "no edge, no inside, and no outside; it is one-sided and has a genus of one." Genus of one means that the bottle has one hole. The Klein bottle illustrates the complexity of figures that can be studied using topology (Kline 1168).

> As a date for the beginning of the subject [of topology] none is more appropriate than 1895, the year in which Poincaré published his Analysis situs. This book for the first time provided a systematic development. (Boyer 652)

Henri Poincaré was born at Nancy in 1854. He graduated from l'Ecole Polytechnique with a degree in mining engineering. In 1879 he earned a doctorate in science at the University of Paris where he held professorships in mathematics and science until his death in 1912 (Boyer 651). He wrote

> a vast number of research articles, texts, and popular articles, which concerned almost all the basic areas of mathematics and major areas of theoretical physics, electromagnetic theory, dynamics, fluid mechanics, and astronomy. (Kline 1170)

In his lectures at the Sorbonne he would lecture on different topics each year. The list of subjects included capillarity, elasticity, thermodynamics, optics, electricity, telegraphy, and cosmogony (Boyer 652).

Poincaré "decided that a systematic study of the analysis situs of general or n-dimensional figures was necessary." He published some notes in Comptes Rendus and articles in various journals (Kline 1170).

> Poincaré, like Riemann, was especially adept at handling problems of a topological nature, such

> as finding out the properties of a function without worrying about its formal representation in the classical sense. (Boyer 653)

Boyer says "others regard [Luitzen E. J.] Brouwer as the founder of topology." He published, in 1911, theorems on topological invariance. With Brouwer's "fusion of the methods of Cantor with those of analysis situs", there began a period of "intensive evolution of topology that has continued to the present day" (Boyer 668).

In 1913 Hermann Weyl lectured on Riemann surfaces at Göttingen. He "emphasized the abstract nature of a surface, or a two-dimensional manifold." In 1914, Felix Hausdorff, working on the same thing as Weyl, generalized the notion of a metric space. This led to a space having a neighborhood topology. In Grunzüge der Mengenlehre or Basic Features of Set Theory, Hausdorff gave a systematic exposition of set theory, where "the nature of elements is of no consequence; only the relations among the elements are important." The last half of the book was dedicated to the development of Hausdorff topological spaces from a set of axioms (Boyer 668).

> Topology has emerged in the twentieth century as a subject that unifies almost the whole of mathematics. . . . Because of its primitiveness, topology lies at the basis of a very large part of mathematics, providing it with unexpected cohesiveness. (Boyer 669)

Proof Using the Fundamental Group of a Circle

The first topological proof we will consider is based on the notion of the fundamental group of the circle. This proof is from Topology: A First Course by James Munkres.

We will need to define some terms, the first being path homotopy.

> Two paths f and f', mapping the interval $I = [0,1]$ into X, are path homotopic if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F: I \times I \to X$ such that
>
> $$F(s,0) = f(s) \quad \text{and} \quad F(s,1) = f'(s)$$
> $$F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1$$
>
> for each $s \in I$ and each $t \in I$. F is a path homotopy between f and f'.

We should think of F as representing a continuous way of deforming the path f to the path f' in such a way that the endpoints of the path remain fixed during the deformation (319).

We are interested in the unit circle, S^1, which we consider as a subspace of R^2 and we define as $S^1 = \{(x,y) | x^2 + y^2 = 1\}$.

Since we are going to discuss a group, we need a group operation, $*$. Munkres says:

If f is a path in X from x_0 to x_1, and g is a path in X from x_1 to x_2, the composition f * g is the path h defined by

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0,\frac{1}{2}] \\ g(2s - 1) & \text{for } s \in [\frac{1}{2},1] \end{cases}$$

We should note that h maps the interval [0,1] to X and it is a path from x_0 to x_2. We think of h as the path whose first half is the path f and whose second half is the path g (322).

> If b_0 is a point of S^1, and we have a path in S^1 that begins and ends at b_0, that path is called a loop based at b_0. The set of path homotopy classes of loops based at b_0, with the operation *, is called the fundamental group of S^1 relative to the base point b_0. It is denoted $\pi_1(S^1,b_0)$ (326).

Next we would like to define a covering space. We need some intermediate definitions.

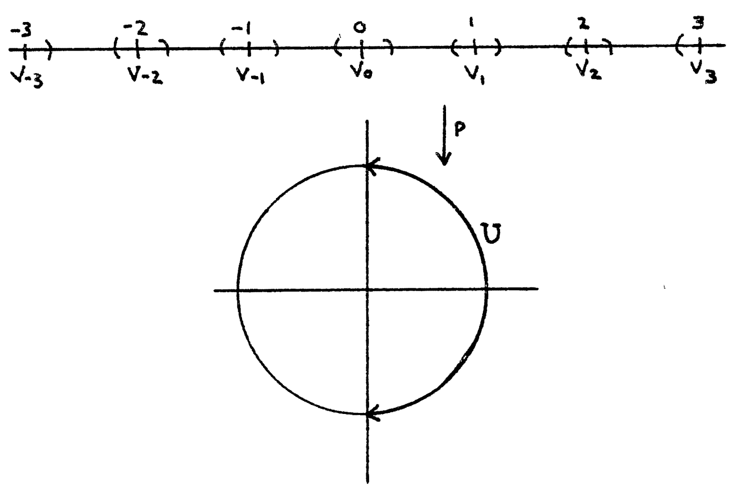
> Let p: R -> S^1 be the continuous onto map $p(x) = (\cos(2\pi x),\sin(2\pi x))$. Any open subset U of S^1 is evenly covered by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_n in R such that for each n, the restriction of p to V_n is a homeomorphism of V_n onto U (331).

Recall the definition of homeomorphism:

Let X and Y be topological spaces. Let f: X -> Y be a bijection. If both the function f and the inverse function f^{-1}: Y -> X are continuous, then f is called a homeomorphism (104).

If every point b of S^1 has a neighborhood U that is evenly covered by p, then p is called a covering map, and R is said to be a covering space of S^1 (331).

It's easiest to picture p as a function that wraps the real line R around S^1 and maps each interval [n,n + 1] onto S^1. In the following diagram from Topology: A First Course, Munkres considers the subset U of S^1 consisting of those points having positive first coordinates. Then the set $p^{-1}(U)$ consists of those points x for which $\cos(2\pi x)$ is positive; that is, it is the union of the intervals $V_n = (n - 1/4, n + 1/4)$, for all $n \in Z$ (332).



If f_n is a continuous mapping from I into S^1, a lifting of f_n is a map f from I to R such that $p \circ f = f_n$.

$$\begin{tikzcd} & R \arrow[d, "p"] \\ I \arrow[ur, "f"] \arrow[r, "f_n"'] & S^1 \end{tikzcd}$$

If f_n is a loop in $\pi_1(S^1,b_0)$ defined by $f_n(x) = (\cos(2\pi nx),\sin(2\pi nx))$, then f_n sends a point x_0 in $[0,1]$ to the point $(\cos(2\pi nx_0),\sin(2\pi nx_0))$ on S^1 that can be arrived at by traveling n times around S^1. The map f_n can be lifted to f, the map from $[0,1]$ to R defined by $f(x) = nx$. Clearly, $p \circ f = f_n$.

We define a map h to be inessential if h is homotopic to a constant map. Otherwise, h is essential.

This leads to a lemma that is important in our proof. It says:

Let $h: S^1 \to Y$. Then the following are equivalent:

1.) h is inessential

2.) h can be extended to a continuous map $g: B^2 \to Y$.

The unit ball, B^2 is defined by $B^2 = \{(x,y) \mid x^2 + y^2 \leq 1\}$. The proof of this lemma may be found in <u>Topology: A First Course</u> on page 358.

We need one more definition before we can prove our theorem.

> Consider $p: R \to S^1$ defined as above. If f is a loop on S^1 based at b_0, let f be the lifting of f to a path on R beginning at 0. The point f(1) must be a point of the set $p^{-1}(b_0)$; that is, f(1) must equal some integer n. Define $\phi: \pi_1(S^1,b_0) \to Z$ by letting $\phi(f)$ be this integer n. ϕ is called the standard isomorphism of $\pi_1(S^1,b_0)$ (340).

Now we are ready to prove the fundamental theorem of algebra.

> A polynomial equation of degree $n > 0$,
> $x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$,
> with real or complex coefficients has at least one (real or complex) root.

Step 1.

Consider the map $h: S^1 \to S^1$ given by $h(z) = z^n$, z a complex number. Our first goal is to show that the induced homomorphism

$$h_*: \pi_1(S^1,b_0) \to \pi_1(S^1,b_0),$$

defined by $h_*(f) = h \circ f$, carries a generator of the fundamental group to n times itself.

Let $p: R \to S^1$ be $p(s) = (\cos(2\pi s),\sin(2\pi s)) = e^{2\pi is}$. Then the image of p under h_* is the loop

$h(p(s)) = (e^{2\pi is})^n = e^{2\pi ins} = (\cos(2\pi ns),\sin(2\pi ns))$.

We know this loop lifts to the path $f(s) = ns$ in the covering space R. So the loop $h \circ p$ corresponds to the integer n under the standard isomorphism of $\pi_1(S^1,b_0)$.

Step 2. Given a polynomial equation

$$z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 = 0,$$

assume

$$|a_{n-1}| + \ldots + |a_1| + |a_0| < 1.$$

Our goal is to show that the equation has a root lying in the unit ball B^2.

Assume that it does not, that is, there is no root of the equation in B^2. Then we can define a map $g: B^2 \to R^2 - \{0\}$ by the equation

$$g(z) = z^n + a_{n-1}z^n + \ldots + a_1z + a_0.$$

Let $f: S^1 \to R^2 - \{0\}$ be the restriction of g to S^1. Because f is extendable to the map g of B^2 into $R^2 - \{0\}$, the map f is inessential by our lemma.

But f is homotopic to the map $k: S^1 \to R^2 - \{0\}$ defined by $k(z) = z^n$. To see how, define the homotopy $F: S^1 \times I \to R^2 - \{0\}$ by

$$F(z,t) = z^n + t(a_{n-1}z^{n-1} + \ldots + a_1z + a_0).$$

$F(z,t)$ is the required homotopy and $F(z,t)$ never vanishes since

$$\begin{aligned} |F(z,t)| &\geq |z^n| - |t(a_{n-1}z^{n-1} + \ldots + a_0)| \\ &\geq 1 - t(|a_{n-1}z^{n-1}| + \ldots + |a_0|) \\ &= 1 - t(|a_{n-1}| + \ldots + |a_0|) \\ &> 0. \end{aligned}$$

Furthermore, the map k is essential. To see this note the map k equals the composite of the map $h: S^1 \to S^1$ of

Step 1, given by $h(z) = z^n$, and the inclusion map $j: S^1 \to R^2 - \{0\}$. Since h_* is "multiplication by n" and j_* is an isomorphism, k_* is not the zero homomorphism. So k must be essential.

Since f is homotopic to k, f must also be essential. This contradicts the fact we just proved, that f is inessential. Therefore, the polynomial equation has a root in B^2.

Step 3. Given any polynomial equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0;$$

choose a real number $c > 0$ and substitute $x = cy$. This gives

$$(cy)^n + a_{n-1}(cy)^{n-1} + \ldots + a_1(cy) + a_0 = 0$$

or

$y^n + (a_{n-1}/c)y^{n-1} + \ldots + (a_1/c^{n-1})y + (a_0/c^n) = 0$. If this equation has the root $y = y_0$, then the original equation has the root $x_0 = cy_0$. So we need to choose c large enough so that $|a_{n-1}/c| + |a_{n-2}/c^2| + \ldots + |a_1/c^{n-1}| + |a_0/c^n| < 1$. Then this is the same as the special case that we considered in Step 2. Thus any polynomial equation of degree $n > 0$ with real or complex coefficients has at least one (real or complex) root (362).

Proof Using Brouwer's Fixed Point Theorem

The next topological proof is from the September 1949 issue of The American Mathematical Monthly. This proof was written by B. H. Arnold.

This proof is based on Brouwer's Fixed-point Theorem.

> If $g: B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such that $g(x) = x$.

For a proof, see Munkres' Topology: A First Course on page 365.

Now we will prove the fundamental theorem of algebra:

> If $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ is a complex polynomial with leading coefficient unity, $f(z)$ has at least one zero.

To begin, set $z = re^{i\theta}$, $0 \le \theta < 2\pi$ and $R = 2 + |a_1| + \ldots + |a_n|$. Define a function $g(z)$ by

$$g(z) = \begin{cases} z - f(z)/(Re^{i(n-1)\theta r}) & \text{for } |z| \le 1 \\ z - f(z)/(Rz^{n-1}) & \text{for } |z| > 1. \end{cases}$$

The function $g(z)$ is well-defined and continuous for all values of z. Here's why:

Each of the two expressions for $g(z)$ is continuous throughout the range specified since neither denominator becomes zero, and for $z = 0$, $r = 0$, so $i(n - 1)\theta r = 0$ for all θ. When $|z| = 1$, the two expressions are identical.

For $|z| \le R$, we have $|g(z)| \le R$. Here's why:

For $|z| \leq 1$,

$$
\begin{aligned}
|g(z)| &\leq |z| + |f(z)/R| \\
&\leq 1 + (1 + |a_1| + . . . + |a_n|)/R \\
&\leq 1 + 1 \\
&\leq R.
\end{aligned}
$$

For $1 \leq |z| \leq R$,

$$
\begin{aligned}
|g(z)| &\leq |z - (z/R) - [(a_1 + . . . + a_n z^{1-n})/R]| \\
&\leq |(R - 1)(z/R)| + (|a_1| + . . . + |a_n|)/R \\
&\leq R - 1 + (R - 2)/R \\
&\leq R.
\end{aligned}
$$

Now consider the correspondence $z \to g(z)$. We have shown that this is a continuous transformation which maps the circle $|z| \leq R$ into itself. Therefore, by Brouwer's fixed point theorem, there exists at least one value z_0 such that $g(z_0) = z_0$. This means, then, that $f(z_0) = 0$ and so $f(z)$ has at least one zero. (465)

CHAPTER IX

APPROXIMATION OF ROOTS

Methods of Approximations of Roots

The history of approximation methods of roots of equations may well begin with the Egyptians. Their Method of False Position starts with a first approximation to a root of an equation which is manipulated until the actual root is found.

Historians mention a Chinese method of approximating roots of quadratic and higher degree equations used in the thirteenth century. More discussion on this method follows.

Around the time of Vieta, other approximation methods began to appear in print. We will examine three methods: Horner's method, the Bisection Method, and Newton's Method, also called the Newton-Raphson Method.

Horner's Method

William George Horner was born in Bristol, England in 1786. He was educated at Kingswood School near Bristol, and became headmaster at the age of eighteen. In 1809, he established his own school at Bath where he remained until his death in 1837 (Dictionary 510).

Horner never received university training and was "not a noted mathematician" (Smith, Source Book 232). His only "significant contribution to mathematics lay in the method of solving algebraic equations which still bears his name." Horner submitted a paper to the Royal Society which was read by Davies Gilbert in July of 1819. It was entitled "A New Method of Solving Numerical Equations of All Orders by Continuous Approximations." Published first in the Philosophical Transactions of 1819, and then later in Ladies' Diary (1838) and Mathematician (1843), Horner

> found influential sponsors in J.R. Young of Belfast and Augustus de Morgan, who gave extracts and accounts of the method in their own publications. Horner's method spread rapidly in England but was little used elsewhere in Europe. (Dictionary 510)

Many English and American textbooks written in the nineteenth and early twentieth centuries that dealt with the theory of equations gave Horner's Method a prominent place. But with the increasing use of computers, Horner's Method has declined in importance, although some of his techniques can be found in courses on numerical analysis (Dictionary 510).

Horner's Method closely resembles a method used by the Chinese of the thirteenth century to approximate roots of equations. In 1247, Ch'in Chiu-shao wrote the Su-shu Chiu-chang which translates as Nine Sections of

Mathematics. In this treatise, he explained the process of solving numerical equations of all degrees.

Yoshio Mikami points out, in The Development of Mathematics in China and Japan, that although Genghis Khan terrorized the Asiatic countries at the beginning of the thirteenth century, mathematical progress was still made. He states "the dates of Ch'in's birth as well as his death are utterly unknown. Nor know we much about the particulars of his life" (64).

In Ch'in's work, eighty-one problems are found in nine sections (Mikami 65). Ch'in solves the problem $-x^4 + 763200x^2 - 40642560000 = 0$ by a method that is similar to Horner's Method. The computations were "probably carried out on a computing board, divided into columns, and by the use of computing rods" (Cajori 74). These rods were called sangis and the Chinese could extract square and cube roots with the sangi-board. For a complete discussion of Ch'in's method, see Mikami's The Development of Mathematics in China and Japan, page 73.

Another method similar to Horner's Method was discussed by Paolo Ruffini. The Italian scientific society offered a gold medal for improvements in the solutions of numerical equations. Ruffini was awarded the medal in 1804. Using calculus, he developed the theory of transforming one equation into another whose roots are diminished by a certain amount. His device "is simpler than Horner's scheme of 1819 and practically identical to what is now known as Horner's procedure." However, Ruffini's paper was neglected and forgotten (Cajori 271). Smith says:

> The probablity is that neither Horner nor Ruffini knew of the work of the other and that neither was aware of the ancient Chinese method. Apparently Horner knew very little of any previous work in approximation, as he did not mention in his article the contributions of Vieta, Harriot, Oughtred, or Wallis. (Source Book 232)

Horner's Method consists of synthetic division used to find a root of an algebraic equation, digit by digit. Horner stated in the introduction to "A New Method of Solving Numerical Equations of All Orders, by Continuous Approximations":

> The process which is the object of this Essay to establish, being nothing else than the leading theorem in the Calculus of Derivations, presented under a new aspect, may be regarded as a universal instrument of calculation, extending to the composition as well as analysis of functions of every kind. But it comes into most useful application in the numerical solutions of equations. (Smith, Source Book 233)

<u>Algebra</u>, the classic textbook by G. Chrystal, first published in 1886, has a complete discussion of Horner's

Method. Chrystal gives a few preliminary results.

Result 1: To deduce from the equation

$p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n = 0$ (*)

another equation each of whose roots is m times a corresponding root of (*).

To do this, let x be any root of (*); and let $\delta = mx$. Then $x = \delta/m$. Hence, from (*), we have

$p_0(\delta/m)^n + p_1(\delta/m)^{n-1} + \ldots + p_{n-1}(\delta/m) + p_n = 0.$

If we multiply by the constant m^n, we deduce the equivalent equation

$p_0\delta^n + p_1m\delta^{n-1} + \ldots + p_{n-1}m^{n-1}\delta + p_nm^n = 0,$

which is the equation desired.

Result 2: To deduce from the equation (*) another, each of whose roots is less by a than a corresponding root of (*).

To do this, let x denote any root of (*), δ the corresponding root of the required equation; so that $\delta = x - \alpha$, and $x = \delta + \alpha$. Then we deduce from (*), $p_0(\delta + \alpha)^n + \ldots + p_{n-1}(\delta + \alpha) + p_n = 0$. If we rearrange this equation according to powers of δ, we get $p_0\delta^n + q_1\delta^{n-1} + \ldots + q_{n-1}\delta + q_n = 0$ which is the equation desired.

It is important to have a systematic process for calculating the coefficients of the above equation. Comparing this and (*), we find

$$p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n$$
$$= p_0(x - \alpha)^n + q_1(x - \alpha)^{n-1} + \ldots + q_{n-1}(x - \alpha) + q_n.$$

The problem before us is to expand $p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n$ in powers of $(x - \alpha)$. When this polynomial is divided by $(x - \alpha)$, q_n is the remainder. If the integral quotient of the last division is divided by $(x - \alpha)$, the result is q_{n-1}, and so on. The calculation of these remainders will be carried out by means of synthetic division.

One last result is needed.

> Result 3: If one of the roots of the equation (*) be small then an approximate value of that root is $-p_n/p_{n-1}$.

This will give us approximate values to try in our synthetic division (Chrystal 338).

To perform Horner's Method, we first determine, by examining the sign of $f(x)$, an interval where a root is located. Then we diminish the roots by the value of the lower endpoint of the interval.

Now, Chrystal says, "to avoid the trouble and possible confusion arising from decimal points, we multiply the roots of [this and every following] subsidiary equation by 10" (343).

By Result 3, since the root we are now looking for is between 0 and 1, an approximate value of the root is $-p_n/p_{n-1}$. So we calculate this which suggests the next digit of our root. We diminish the roots of $f_1(x) = 0$ by this number. As long as the constant term continues to have the same sign as $f(0)$, we proceed to calculate the digits in this manner, stopping when we have reached the desired accuracy.

Let's look at the necessary calculations to find a root of $x^3 + x^2 + x - 100 = 0$, accurate to three decimal places, by Horner's Method.

If $f(x) = x^3 + x^2 + x - 100$, then $f(4) < 0$ and $f(5) > 0$. Thus, the first digit of our root is 4, and we must now diminish the roots of our equation by 4.

$$\begin{array}{ccccl}
1 & 1 & 1 & -100 & (4 \\
 & \underline{4} & \underline{20} & \underline{84} & \\
 & 5 & 21 & -16 & \\
 & \underline{4} & \underline{36} & & \\
 & 9 & 57 & & \\
 & \underline{4} & & & \\
 & 13 & & &
\end{array}$$

Our new equation is $x^3 + 13x^2 + 57x - 16 = 0$. Since we are now looking for a decimal root, we increase the value of the root by a factor of 10. Our equation becomes $x^3 + 130x^2 + 5700x - 16000 = 0$. Calculating $-p_n/p_{n-1}$, we find $16000/5700 = 2.8...$, so we try 2 as our next digit.

$$\begin{array}{ccccl}
1 & 130 & 5700 & -16000 & (2 \\
 & \underline{2} & \underline{264} & \underline{11928} & \\
 & 132 & 5964 & -4072 & \\
 & \underline{2} & \underline{268} & & \\
 & 134 & 6232 & & \\
 & \underline{2} & & & \\
 & 136 & & &
\end{array}$$

Our new equation (after multiplying the roots by 10) is $x^3 + 1360x^2 + 623200x - 4072000 = 0$. Calculating $-p_n/p_{n-1} = 4072000/623200 = 6.5...$, so we now try 6.

1	1360	623200	-4072000	(6
	6	8196	3788376	
	1366	631396	-283624	
	6	8232		
	1372	639628		
	6			
	1378			

Now we have $x^3 + 13780x^2 + 63962800x - 283624000 = 0$ and $283624000/63962800 = 4.4...$, so next we would try 4. If the last number in the right column is negative, then we will know that 4 is the next digit, and the root, accurate to three decimal places, is 4.264.

The advantages of Horner's Method over other methods of root calculations, according to J.V. Uspensky in Theory of Equations, is that "the necessary calculations are arranged in a very convenient manner, and the root can be computed to a greater number of decimals for a given expenditure of labor" (157).

The disadvantages are that this method applies only to algebraic equations and it is not an efficient method for a computer. For this last reason, Horner's Method is not usually found in up-to-date algebra or numerical analysis textbooks.

Here's how Horner's method would look in columnar form:

1	1	1	-100	(4.264
	4	20	84	
	5	21	-16000	
	4	36	11928	
	9	5700	-4072000	
	4	264	3788376	
	130	5964	-283624000	
	2	268	256071744	
	132	623200	-27552256	
	2	8196		
	134	631396		
	2	8232		
	1360	63962800		
	6	55136		
	1366	64017936		
	6			
	1372			
	6			
	13780			
	4			
	13784			

The Bisection Method

The bisection method is based on the Intermediate Value Theorem, a theorem from calculus. The Intermediate

Value Theorem states:

> Suppose f(x) is continuous on the closed interval [a,b] and f(a) = f(b). Then for any number k between f(a) and f(b), there exists c ϵ (a,b) such that f(c) = k.

So if we could find an interval [a,b] where f(a) < 0 and f(b) > 0 (or vice versa) then there is c ϵ (a,b) such that f(c) = 0. Horner's method used the Intermediate Value Theorem to find the first value.

The continuity of the function is essential for the Intermediate Value Theorem to work. The concept of continuity was an important idea in the development of Calculus and Analysis. Priestly, in Calculus: An Historical Approach quotes Leibniz:

> Nothing happens all at once, and it is one of my great maxims, and among the most completely verified, that nature never makes leaps: which I called the Law of Continuity . . . (117)

A precise formulation of the modern concept of continuity first appeared in a pamphlet published by Bolzano in 1817. In fact, in this pamphlet, "Purely analytical proof of the theorem, that between each two roots which guarantee an opposing result, at least one real root of the equation lies", Bolzano proves the Intermediate Value Theorem (Edwards 308).

The proof of the Intermediate Value Theorem depends on the definition for the least upper bound of a set S, lub S:

> If there exists a real number b such that $x \leq b$ for all $x \in S$, then S is bounded above and b is an upper bound of S. If β is an upper bound of S, but no number less than β is, then β is lub S. (Trench 5)

Now to prove the Intermediate Value Theorem, suppose $f(a) < k < f(b)$. The set $S = \{x | a \leq x \leq b \text{ and } f(x) \leq k\}$ is bounded and nonempty. Let β = lub S. If $f(\beta) > k$, then $\beta > a$ and since f is continuous at β, there exists an $\epsilon > 0$ such that $f(x) > k$ for $\beta - \epsilon < x \leq \beta$. Therefore $\beta - \epsilon$ is an upper bound for S, which is a contradiction of the definition of β as the least upper bound. So now suppose $f(\beta) < k$. Then $\beta < b$ and there exists an $\epsilon > 0$ such that $f(x) < k$ for $\beta \leq x < \beta - \epsilon$. This implies β is not an upper bound for S, another contradiction. Thus $f(\beta) = k$.

The other case, $f(b) < k < f(a)$, works similarly. (Trench 67)

Now with the Intermediate Value Theorem proved, we can consider the Bisection Method for calculating roots of equations.

Suppose we have an equation $f(x) = 0$, and $f(a)$ and $f(b)$ do not have the same sign. Calculate $p_1 = \frac{1}{2}(a + b)$. If $f(p_1) = 0$, we are done. If not, $f(p_1)$ has the same sign as either $f(a)$ or $f(b)$. Suppose $f(p_1)$ and $f(b)$ have the same sign. Then the root lies in the interval (a,p_1). Now calculate $p_2 = \frac{1}{2}(p_1 + a)$, and continue in the same manner until reaching the root or an appropriate approximation.

Let's look at the equation $8x^2 - 17x + 2 = 0$, and calculate one root by the bisection method. We find $f(0) > 0$ and $f(1) < 0$ so we know a root lies in the interval $(0,1)$. First we calculate $p_1 = \frac{1}{2}(0 + 1) = \frac{1}{2}$. Now $f(\frac{1}{2}) < 0$, so we know the root lies in $(0, \frac{1}{2})$. Next calculate $p_2 = \frac{1}{2}(0 + \frac{1}{2}) = 1/4$. We find $f(1/4) < 0$, so the root lies in $(0,1/4)$. We find $p_3 = \frac{1}{2}(0 + 1/4) = 1/8$. We see that $f(1/8) = 0$, thus the root we were looking for is 1/8.

This method is easily programmed even by beginners and it will always converge to a solution. But according to Richard Burden et al. in Numerical Analysis, it has "significant drawbacks. It is very slow in converging and, moreover, a good intermediate approximation may be inadvertantly discarded" (24).

Newton's Method or the Newton-Raphson Method

Newton's Method for approximating the roots of an equation was first published in Principia Mathematica (Goldstine 64). In his Papers, Volume II, Newton said this procedure "is essentially an improved version of the procedure expounded by Vieta and simplified by Oughtred."

Vieta's method first appeared in a work published in 1600 on solving equations. Like Horner's Method, this procedure yields one digit of the root at a time. Suppose the equation is $f(x) = N$ and the root is $x = a_0 \cdot 10^k + a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \ldots$, and let an approximation to that root be $x_p = a_0 \cdot 10^k + a_1 \cdot 10^{k-1} + \ldots + a_p \cdot 10^{k-p}$. To find the next digit a_{p+1}, Vieta formed the auxiliary value $g_k(x_p) = f(x_p + 10^{k-p-1}) - f(x_p) - 10^{(k-p-1)n}$, where n is the degree of the equation. Vieta then divided this quantity into $f(x_p) - N$ or perhaps $[f(x_p + 10^{k-p-1}) + f(x_p)]/2 - N$, and the integer part of the result gave the next digit a_{p+1}, and thus the next approximation, x_{p+1} (Goldstine 66).

Goldstine, in <u>A History of Numerical Analysis from the 16th through the 19th Century</u>, remarks:

> In passing it is worth noting that this method was very useful and, until Newton replaced it by his own, was much employed. There are instances of its use by Harriot, Oughtred, and Wallis. In fact Oughtred made simplifications of Vieta's method in his <u>Clavis Mathematicae</u> from 1647 onwards. (66)

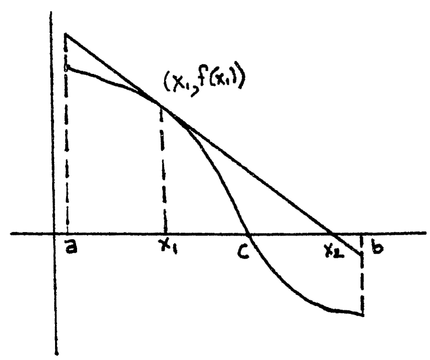
Kline discusses Newton's Method. He says:

> In his <u>De Analysi</u> and <u>Method of Fluxions</u>, [Newton] gave a general method of approximating the roots of $f(x) = 0$, which was published in Wallis's <u>Algebra</u> of 1685. In his tract <u>Analysis</u>

Aequationum Universalis (1690), Joseph Raphson (1648-1715) improved on this method; though he applied it only to polynomials, it is much more broadly useful. It is this modification that is now known as Newton's method or the Newton-Raphson method. (381)

To approximate roots of $f(x) = 0$ by the Newton-Raphson Method, we will have to insist that f is continuous on [a,b] and differentiable on (a,b), where f(a) and f(b) have different signs.

Estimate a zero, $x = x_0$. We will assume that the tangent line to f at $(x_0, f(x_0))$ crosses the x-axis at about the same point where f(x) crosses the x-axis. See the following diagram.



The equation of the tangent line is
$y - f(x_0) = f'(x_0)(x - x_0)$ since $f'(x_0)$ is the slope of

the tangent line to f at x_0. Solving for y we find $y = f'(x_0)(x - x_0) + f(x_0)$. On the x-axis $y = 0$. This gives $0 = f'(x_0)(x - x_0) + f(x_0)$. Solving for x, we arrive at $x = x_0 - f(x_0)/f'(x_0)$.

The Newton-Raphson Method generates a sequence $\{x_n\}$ defined by $x_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1})$. This is a sequence of successive approximations of the root of f(x) (Larson et al. 234).

Let's consider the equation $8x^2 - 17x + 2 = 0$ again, and let's approximate the root by the Newton-Raphson Method. We will use the following generating equation: $x_n = x_{n-1} - [8x_{n-1}^2 - 17x_{n-1} + 2]/[16x_{n-1} - 17]$. We know there exists a root near zero, so let $x_0 = 0$. Then $x_1 = 0 - [8\cdot 0^2 - 17\cdot 0 + 2]/[16\cdot 0 - 17] = 2/17$. Then

$$x_2 = 2/17 - [8\cdot(2/17)^2 - 17(2/17) + 2]/[16(2/17) - 17]$$
$$= 546/4369 \approx .1249713.$$

After only two iterations, we are already very close to the root $1/8 = .125$.

The Newton-Raphson Method is easily programmed and will always converge to a root provided a "sufficiently accurate initial approximation is chosen" (Burden et al. 38). Kline says:

> J. Raymond Mourraille showed in 1768 that [the first approximation] must be chosen so that the curve $y = f(x)$ is convex toward the axis of x in the interval between [the approximation] and the

root. Much later Fourier discovered this fact independently. (381)

Two disadvantages of the Newton-Raphson Method are that sometimes $f'(x)$ is difficult to find and sometimes $f'(x) = 0$, which leads to failure.

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APPENDIX

OTHER PROOFS

The following is a list of other proofs that were not used in the body of this paper. There are many other proofs that are not listed here.

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