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ALGEBRAIC LOGIC VIA ARENAS

APPROVED BY

Allen S Davis

Leonard R. Rubin

W. Reid

Bernard R. Dull

John C. Driggers

DISSERTATION COMMITTEE

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I would like to express by indebtedness to Professor A. S. Davis for his encouragement and direction of this thesis. It was in both his classroom lectures and private conversations that many of the ideas contained herein were initiated.

CHAPTER I

INTRODUCTION

The structure of many mathematical systems is much more complicated than just being an algebraic system. Many times two or more algebraic systems are involved where the action between certain of the algebraic systems is very significant.

For example, a module over a ring has both an Abelian group and a ring involved with the action having to do with the scalar multiplication. In topological groups there is a similar situation; there is a group G, a topology on G, and an action on the topological group having to do with the operations on G being continuous mappings with respect to the topology on G. We will not be concerned with this last situation but will restrict our discussion to where the systems involved are algebraic systems.

In fact, we will be interested in the case where there are only two algebraic systems involved. Such a system will be called a mixed algebra and will be studied in chapter II. There we will be concerned mainly with universal concepts, not with particular kinds of mixed algebras. It will be easy to see that the results contained there can be generalized to the case where more than just two algebraic systems are involved.

In 1966 Davis [10] first introduced the concept of an arena and studied some of its properties. An arena is a particular kind of mixed algebra. The concept of an arena was motivated by certain considerations in algebraic logic. In chapter III we will develop a theory of arenas which will be used in chapter IV to study some algebraic logic. We will look at several different kinds of arenas and several imbedding theorems for arenas.

The concept of a free mixed algebra is introduced in chapter II and in chapter III we show that there do not exist free arenas in this sense. It would be interesting to consider under what conditions a free mixed algebra would exist. We do not undertake this question here, but do show that a negative solution holds in the case of arenas.

Henkin, Monk, and Tarski [16] along with Halmos [15] have contributed much to the theory of algebraic logic. Their methods were somewhat different. Halmos uses the concept of a polyadic algebra whereas the others use the concept of a cylindric algebra. The arena concept gives a third approach to algebraic logic. It is quite different from the other two methods. In chapter IV we show, among other things, that certain mappings from a Boolean algebra into itself are quantifiers in the sense of Halmos.

CHAPTER II

A THEORY OF MIXED ALGEBRAS

In many mathematical contexts situations arise where two algebras of possibly different types are involved. For example, a vector space over a field has both an Abelian group and a field involved in which certain action, left and right scalar multiplication, hold satisfying certain prescribed conditions. Such a general system will be called a "mixed algebra". In this chapter we develop a beginning theory of such algebras.

Definition 2.1: By a mixed algebra we mean a system $\mathfrak{A} = (A, B, *_1, *_2)$, where

(1) A and B are algebraic systems, not necessarily of the same type,

(2) $*_1 : A \times B \to A : (a,b) \to ab$

$*_2 : B \times A \to A : (b,a) \to ba$

are mappings which satisfy a certain number of conditions.

The mappings $*_1$ and $*_2$ are called the action for the mixed algebra $\mathfrak{A}$.

Definition 2.2: Two mixed algebras $\mathfrak{A}_1 = (A_1, B_1, *_{11}, *_{12})$ and $\mathfrak{A}_2 = (A_2, B_2, *_{21}, *_{22})$ are of the same kind if A_1 and A_2 are algebras of the same kind and B_1 and B_2 are algebras of the same kind.

We assume throughout this chapter that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are mixed algebras of the same kind unless otherwise stated.

Example 2.3: Every module M over a ring R is a mixed algebra (M, R, $*_1$, $*_2$), where

(1) M is an Abelian group,

(2) R is a ring,

(3) $*_1 : M \times R \to M$

$*_2 : R \times M \to M$

are the right and left scalar multiplications defined on M satisfying the usual properties of a module.

Definition 2.4: A mixed algebra $\mathfrak{A}' = (A', B', *_1', *_2')$ is a mixed subalgebra of a mixed algebra $\mathfrak{A} = (A, B, *_1, *_2)$ if

(1) A' is a subalgebra of A,

(2) B' is a subalgebra of B,

(3) $*_1' = *_1\big|_{A' \times B'}$ and $*_2' = *_2\big|_{B' \times A'}$.

Thus $\mathfrak{A}'$ is a mixed subalgebra of $\mathfrak{A}$ if $\mathfrak{A}'$ is a subsystem of the mixed algebra $\mathfrak{A}$ which is also a mixed algebra. The proofs for the next two results are similar to those which will be done later for arenas and consequently are omitted here.

Theorem 2.5: The intersection of any family of mixed algebras $\mathfrak{A}_i = (A_i, B_i, *_{i1}, *_{i2})$ of a mixed algebra $\mathfrak{A} = (A, B, *_1, *_2)$ is a mixed subalgebra of $\mathfrak{A}$.

Corollary 2.6: The family of all mixed subalgebras of a mixed algebra forms a complete lattice.

Given a mixed algebra $\mathfrak{A} = (A, B, *_1, *_2)$ it follows that if S is a nonempty subset of the algebra A and T is a nonempty subset of the algebra B, then there exists a smallest mixed subalgebra $\mathfrak{A}' = (A', B', *_1', *_2')$ of $\mathfrak{A}$ so that $S \subseteq A'$ and $T \subseteq B'$.

Definition 2.7: The mixed subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ is called the mixed subalgebra of $\mathfrak{A}$ generated by the subsets S and T.

Definition 2.8: Let $\mathfrak{A} = (A, B, *_1, *_2)$ be a mixed algebra. By a mixed algebra congruence on $\mathfrak{A}$ we mean a pair $(\approx_1, \approx_2)$ so that

(1) $\approx_1$ is a congruence on A,

(2) $\approx_2$ is a congruence on B,

(3) if $a \approx_1 a_1$ and $b \approx_2 b_1$, then $ab \approx_1 a_1 b_1$ and $ba \approx_1 b_1 a_1$.

Theorem 2.9: Let $\mathfrak{A} = (A, B, *_1, *_2)$ be a mixed algebra and let $(\approx_{i1}, \approx_{i2})$ be congruences on $\mathfrak{A}$. Then $(\bigcap \approx_{i1}, \bigcap \approx_{i2})$ is a mixed algebra congruence on $\mathfrak{A}$.

Proof: Clearly $\bigcap \approx_{i1}$ is a congruence on A and $\bigcap \approx_{i2}$ is a congruence on B. If $p \, (\bigcap \approx_{i1}) \, q$ and $x \, (\bigcap \approx_{i2}) \, y$, then $p \approx_{i1} q$ and $x \approx_{i2} y$, for all i. Since each $(\approx_{i1}, \approx_{i2})$ is a congruence on $\mathfrak{A}$, then $px \approx_{i1} qy$ and $xp \approx_{i1} yq$, for all i. Thus $px \, (\bigcap \approx_{i1}) \, qy$ and $xp \, (\bigcap \approx_{i1}) \, yq$. Hence $(\bigcap \approx_{i1}, \bigcap \approx_{i2})$ is a mixed algebra congruence on $\mathfrak{A}$. //

Corollary 2.10: The family of all mixed algebra congruences on a mixed algebra $\mathfrak{A}$ forms a complete lattice.

A mapping from one mixed algebra into a second mixed algebra which preserves the action of the first mixed algebra and also preserves the structure of the algebras in the first mixed algebra will be called a mixed algebra homomorphism.

Definition 2.11: By a mixed algebra homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ we mean a pair (f,g) of mappings so that

(1) f is a homomorphism from A_1 to A_2,

(2) g is a homomorphism from B_1 to B_2,

(3) $f(ab) = f(a)g(b)$,

$f(ba) = g(b)f(a)$,

for all $a \in A_1$, for all $b \in B_1$.

Statement (3) written out explicitly would be

$$f(*_{11}(a,b)) = *_{21}(f(a),g(b)),$$

$$f(*_{12}(b,a)) = *_{22}(g(b),f(a)),$$

for all $a \in A_1$, for all $b \in B_1$. It simply states that (f,g) preserves the action of the mixed algebra $\mathfrak{A}_1$. We write

$$(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2.$$

Definition 2.12: A mixed algebra homomorphism (f,g) is one-to-one if both f and g are one-to-one. A mixed algebra homomorphism (f,g) is onto if both f and g are onto. If $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a mixed algebra homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ which is one-to-one and onto, then (f,g) is called a mixed algebra isomorphism and we say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic mixed algebras and write

$$\mathfrak{A}_1 \cong \mathfrak{A}_2.$$

By a mixed algebra monomorphism we mean a mixed algebra homomorphism (f,g) so that (f,g) is one-to-one. A mixed algebra monomorphism is also called a mixed algebra imbedding. By a mixed algebra epimorphism we mean a mixed algebra homomorphism (f,g) so that (f,g) is onto.

We now state a couple of trivial results for mixed algebras which have to do with mixed algebra homomorphisms.

Theorem 2.13: If $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a mixed algebra homomorphism, then $(f(A_1), g(B_1), *'_{21}, *'_{22})$ is a mixed subalgebra of $\mathfrak{A}_2$, where $*'_{21}$

and $*'_{22}$ are the natural restrictions of $*_{21}$ and $*_{22}$, respectively.

Theorem 2.14: If $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a mixed algebra homomorphism and $\mathfrak{A}'_2 = (A'_2, B'_2, *'_{21}, *'_{22})$ is a mixed subalgebra of $\mathfrak{A}_2$, then $(f^{-1}(A'_2), g^{-1}(B'_2), *'_{11}, *'_{12})$ is a mixed subalgebra of $\mathfrak{A}_1$, where $*'_{11}$ and $*'_{12}$ are the natural restrictions of $*_{11}$ and $*_{12}$, respectively.

Definition 2.15: If $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ and $(h,k) : \mathfrak{A}_2 \to \mathfrak{A}_3$ are mixed algebra homomorphisms, then by the composition of (f,g) and (h,k) we mean the mapping

$$(hf,kg) : \mathfrak{A}_1 \to \mathfrak{A}_3,$$

where hf and kg are the natural compositions.

Theorem 2.16: The composition of two mixed algebra homomorphisms is a mixed algebra homomorphism.

Theorem 2.17: Let $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ be a mixed algebra homomorphism and define

$$x\ (\approx_f, \approx_g)\ y \text{ if and only if } f(x) = f(y) \text{ or } g(x) = g(y).$$

Then $(\approx_f, \approx_g)$ is a mixed algebra congruence on $\mathfrak{A}_1$.

Proof: Clearly $\approx_f$ and $\approx_g$ are congruences on A_1 and B_1, respectively. Moreover, if $a \approx_f a_1$ and $b \approx_g b_1$, then $f(a) = f(a_1)$ and $g(b) = g(b_1)$. Thus

$$\begin{aligned} f(ab) &= f(a)g(b) \\ &= f(a_1)g(b_1) \\ &= f(a_1b_1), \end{aligned}$$

that is, $ab \approx_f a_1b_1$. Similarly $ba \approx_f b_1a_1$. Hence $(\approx_f, \approx_g)$ is a mixed algebra congruence on $\mathfrak{A}_1$. //

Definition 2.18: This congruence $(\approx_f, \approx_g)$ is called the mixed algebra congruence on $\mathfrak{A}_1$ induced by the mixed algebra homomorphism (f,g).

Let $(\approx_1, \approx_2)$ be a congruence on the mixed algebra $\mathfrak{A}$ and let

$$\bar{a} = \{ x \in A : x \approx_1 a \},$$

$$\bar{b} = \{ y \in B : y \approx_2 b \},$$

for all $a \in A$, for all $b \in B$. Then the quotient algebras $A/{\approx_1}$ and $B/{\approx_2}$ are defined. We wish to construct a mixed algebra

$$(A/{\approx_1}, B/{\approx_2}, \bar{*}_1, \bar{*}_2)$$

for some appropriate $\bar{*}_1$ and $\bar{*}_2$. Define

$$\bar{*}_1 : A/{\approx_1} \times B/{\approx_2} \to A/{\approx_1},$$

$$\bar{*}_2 : B/{\approx_2} \times A/{\approx_1} \to A/{\approx_1},$$

by

$$\bar{*}_1(\bar{a},\bar{b}) = \overline{ab},$$

$$\bar{*}_2(\bar{b},\bar{a}) = \overline{ba},$$

for all $\bar{a} \in A/{\approx_1}$, for all $\bar{b} \in B/{\approx_2}$. We must show that $\bar{*}_1$ and $\bar{*}_2$ are well-defined. If $\bar{a} = \bar{a}_1$ and $\bar{b} = \bar{b}_1$, then $a \approx_1 a_1$ and $b \approx_2 b_1$. Since $(\approx_1, \approx_2)$ is a congruence on the mixed algebra $\mathfrak{A}$, then $ab \approx_1 a_1b_1$ and $ba \approx_1 b_1a_1$, that is, $\overline{ab} = \overline{a_1b_1}$ and $\overline{ba} = \overline{b_1a_1}$. Hence $\bar{*}_1$ and $\bar{*}_2$ are well-defined.

<u>Theorem 2.19</u>: $(A/{\approx_1}, B/{\approx_2}, \bar{*}_1, \bar{*}_2)$ is a mixed algebra of the same kind as $(A, B, *_1, *_2)$.

<u>Proof</u>: Tedious but easy. //

<u>Definition 2.20</u>: This mixed algebra $(A/{\approx_1}, B/{\approx_2}, \bar{*}_1, \bar{*}_2)$ is called the <u>quotient mixed algebra</u> induced by the mixed algebra congruence $(\approx_1, \approx_2)$ and is denoted by $\mathfrak{A}/(\approx_1, \approx_2)$.

<u>Theorem 2.21</u>: (Fundamental Homomorphism Theorem) Let (f,g) be a homomorphism from the mixed algebra $\mathfrak{A}_1$ to the mixed algebra $\mathfrak{A}_2$ which is onto. Then

$$\mathfrak{A}_1/(\approx_f, \approx_g) \cong \mathfrak{A}_2,$$

where $(\approx_f, \approx_g)$ is the mixed algebra congruence on $\mathfrak{A}_1$ induced by the mixed algebra homomorphism (f,g).

Proof: Define

$$(h,k) : \mathfrak{A}_1/(\approx_f, \approx_g) \to \mathfrak{A}_2$$

by $\quad h(\bar{a}) = f(a), \quad \text{if } \bar{a} \in A_1/\approx_f,$

$$k(\bar{b}) = g(b), \quad \text{if } \bar{b} \in B_1/\approx_g.$$

We must show that (h,k) is well-defined. If $\bar{a} = \bar{a}_1$, then $f(a) = f(a_1)$ so that h is well-defined. If $\bar{b} = \bar{b}_1$, then $g(b) = g(b_1)$ so that k is well-defined. Thus (h,k) is well-defined. (h,k) is a mixed algebra homomorphism since

$$\begin{aligned} h(\bar{a}\bar{b}) &= h(\overline{ab}) \\ &= f(ab) \\ &= f(a)g(b) \\ &= h(\bar{a})k(\bar{b}), \end{aligned}$$

and

$$\begin{aligned} h(\bar{b}\bar{a}) &= h(\overline{ba}) \\ &= f(ba) \\ &= g(b)f(a) \\ &= k(\bar{b})h(\bar{a}). \end{aligned}$$

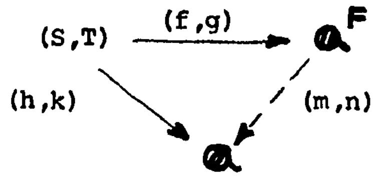
(h,k) is onto since both f and g are onto. If $h(\bar{a}) = h(\bar{a}_1)$, then $f(a) = f(a_1)$ so that $\bar{a} = \bar{a}_1$. Similarly, if $k(\bar{b}) = k(\bar{b}_1)$, then $g(b) = g(b_1)$ so that $\bar{b} = \bar{b}_1$. The last two sentences show that (h,k) is one-to-one. Hence

$$\mathfrak{A}_1/(\approx_f, \approx_g) \cong \mathfrak{A}_2. \quad //$$

We conclude this chapter with a discussion of a possible class of mixed algebras which we will call free mixed algebras. Throughout the remainder of this chapter we let $\mathfrak{C}$ be a class of mixed algebras of the same kind.

Definition 2.22: Let S and T be any sets. By a free mixed algebra on (S,T) with respect to the class $\mathcal{C}$ we mean a mixed algebra $\mathfrak{A}^F = (A^F, B^F, *_1^F, *_2^F)$ together with a mapping $(f,g) : (S,T) \to \mathfrak{A}^F$ so that (1) $\mathfrak{A}^F$ is a mixed algebra in $\mathcal{C}$,

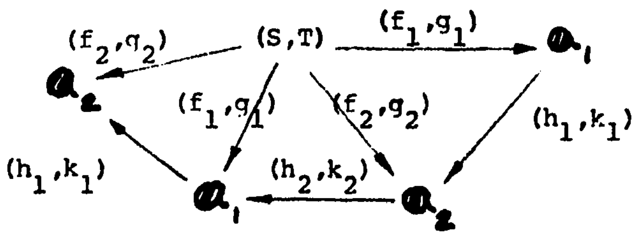
(2) If $\mathfrak{A}$ is any mixed algebra in $\mathcal{C}$ and (h,k) is any mapping from (S,T) to $\mathfrak{A}$, then there is a unique mixed algebra homomorphism $(m,n) : \mathfrak{A}^F \to \mathfrak{A}$ so that the following diagram is commutative.



It may or may not happen that a free mixed algebra on (S,T) exists; however, if one exists, then it is unique up to a mixed algebra isomorphism.

Theorem 2.23: If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are free mixed algebras on (S,T) with respect to the class $\mathcal{C}$, then $\mathfrak{A}_1 \approx \mathfrak{A}_2$.

Proof: Since $\mathfrak{A}_1$ is free, then there exists a mapping (f_1,g_1) from (S,T) to $\mathfrak{A}_1$ satisfying condition (2) of definition 2.22. Similarly, since $\mathfrak{A}_2$ is free, then there exists a mapping (f_2,g_2) from (S,T) to $\mathfrak{A}_2$ satisfying condition (2) of definition 2.22. Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are free mixed algebras, there exist unique mixed algebra homomorphisms (h_1,k_1) and (h_2,k_2) so that the following diagram is commutative.



By the uniqueness requirement, we have

$$h_1 h_2 = 1_{A_1}, \quad h_2 h_1 = 1_{A_2},$$

$$k_1 k_2 = 1_{B_1}, \quad k_2 k_1 = 1_{B_2},$$

where 1_Y is the identity mapping on the algebra Y. Hence (h_1, h_2) is a mixed algebra isomorphism and

$$\mathfrak{A}_1 \cong \mathfrak{A}_2. \quad //$$

If $\mathfrak{A} = (A, B, *_1, *_2)$ is a free mixed algebra with respect to the class $\mathcal{C}$, then the freedom of $\mathfrak{A}$ implies the freedom of A and B, but not conversely as we will see in the arena case.

<u>Theorem 2.24</u>: If $\mathfrak{A} = (A, B, *_1, *_2)$ is a free mixed algebra, then A and B are free algebras of their respective kinds.

<u>Proof</u>: This follows immediately from definition 2.22. //

CHAPTER III

ARENAS

This chapter deals with a particular kind of mixed algebra called an arena. The basic definitions and terminology in the beginning of this chapter are due to Davis [10].

Definition 3.1: By an arena we mean a mixed algebra

$$\mathfrak{A} = (P, X, *_1, *_2),$$

where (1) P is a lattice,

(2) X is a semigroup,

(3) $*_1 : P \times X \to P : (p,x) \to px$,

$*_2 : X \times P \to P : (x,p) \to xp$,

are mappings so that

(a) $p(xy) = (px)y$ and $(xy)p = x(yp)$,

(b) $p \leq q$ implies $px \leq qx$,

(c) $p \leq (xp)x$,

(d) $x(p \wedge qx) = xp \wedge q$,

for all $x,y \in X$, for all $p,q \in P$.

The mappings $*_1$ and $*_2$ are called the action of the arena $\mathfrak{A}$.

We now give some examples of arenas.

Example 3.2: Let S be a compact Hausdorff space. If we take P to be the class of compact subspaces of S and X to be the class of all continuous mappings of S into itself, then $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena, where px and xp are the inverse and direct images of p under x, respectively.

<u>Example 3.3</u>: Let S be any algebra. If we take P to be the class of all subsystems of S and X to be the class of all endomorphisms of S, then $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena, where px and xp are the inverse and direct images of p under x, respectively.

<u>Example 3.4</u>: Let S be a set. If P is a collection of subsets of S which forms a lattice and X is a collection of mappings from S into S which forms a semigroup so that $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena, where px and xp are the inverse and direct images of p under x, respectively, then $\mathfrak{A}$ is called a <u>set arena</u>.

For more examples, see Davis [10]. In this chapter we will be concerned with arenas where the lattice P is a Boolean algebra. Hence from now on we will always assume that given any arena, the lattice P is a Boolean Algebra.

An alternate definition of an arena is given by the following theorem.

<u>Theorem 3.5</u>: $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena if and only if P is a Boolean algebra, X is a semigroup, and the mappings $*_1$ and $*_2$ satisfy the following conditions:

(1) $p(xy) = (px)y$ and $(xy)p = x(yp)$,

(2) $p \leq q$ implies $px \leq qx$ and $xp \leq xq$,

(3) $x(px) \leq p \leq (xp)x$,

(4) if $p \leq xq$, then $p = xr$, for some $r \leq q$,

for all $x,y \in X$, for all $p,q \in P$.

Proof: Assume that $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena. Clearly (1) holds. Let $p \leq q$. If $xq \leq r$, then

$$\begin{aligned} p &\leq q \\ &\leq (xq)x \\ &\leq rx. \end{aligned}$$

Thus $p \wedge rx = p$. So

$$\begin{aligned} xp &= x(p \wedge rx) \\ &= xp \wedge r. \end{aligned}$$

Thus $xp \leq r$. Taking $r = xq$, we obtain $xp \leq xq$, and so (2) holds. If $xq \leq p$, then $q \leq (xq)x \leq px$. Thus $x(px) \leq p$ and (3) holds. Let $p \leq xq$. Then $xq \wedge p = p$. If $r = q \wedge px$, then

$$\begin{aligned} xr &= x(q \wedge px) \\ &= xq \wedge p \\ &= p \end{aligned}$$

and (4) holds. Conversely, if (1) - (4) hold, then we need only show that $x(p \wedge qx) = xp \wedge q$. Clearly $x(p \wedge qx) \leq xp$ and $x(p \wedge qx) \leq x(qx) \leq q$, so that $x(p \wedge qx) \leq xp \wedge q$. If $r \leq xp \wedge q$, then $r \leq xp$ and $r \leq q$. Since $r \leq xp$, then by (4) we have $r = xs$, where $s \leq p$. Since $xs = r \leq q$, then we have

$$\begin{aligned} s &\leq (xs)x \\ &\leq qx. \end{aligned}$$

Thus $s \leq p \wedge qx$ and $r = xs \leq x(p \wedge qx)$. Hence $x(p \wedge qx) = xp \wedge q$. //

Theorem 3.6: If $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena, then

(1) $x0 = 0, \quad 0 = x(ox),$

(2) $1x = 1, \quad 1 = (x1)x,$

(3) $x(qx) = x1 \wedge q,$

(4) $x(1x) = x1$,

for all $x \in X$, for all $q \in P$.

Proof: For any $q \in P$ we have

$$\begin{aligned} x0 &= x(0 \wedge qx) \\ &= x0 \wedge q. \end{aligned}$$

Thus $x0 = 0$. Since $x(px) \leq p$, for all $x \in X$, then $x(0x) = 0$. Hence (1) holds. Since $x1 \leq 1$, then $1 \leq (x1)x \leq 1x$. Thus $1x = 1$. Since $p \leq (xp)x$, for all $p \in P$, then $1 \leq (x1)x$. Hence $1 = (x1)x$ and (2) holds. For all $q \in P$ we have $x(1 \wedge qx) = x1 \wedge q$, that is, $x(qx) = x1 \wedge q$ and (3) holds. In particular for $q = 1$ we have

$$\begin{aligned} x(1x) &= x1 \wedge 1 \\ &= x1. \end{aligned}$$

This completes the proof of this theorem. //

Theorem 3.7: If $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena, then $xp \leq q$ if and only if $p \leq qx$.

Proof: If $xp \leq q$, then $p \leq (xp)x \leq qx$. Thus $p \leq qx$. Conversely, if $p \leq qx$, then $xp \leq x(qx) \leq q$. Thus $xp \leq q$. //

Theorem 3.8: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena. Then

$$x(p \vee q) = xp \vee xq,$$

for all $p,q \in P$, for all $x \in X$.

Proof: Since $p \leq p \vee q$ and $q \leq p \vee q$, then $xp \leq x(p \vee q)$ and $xq \leq x(p \vee q)$. Thus $xp \vee xq \leq x(p \vee q)$. If $xp \vee xq \leq r$, then $xp \leq r$ and $xq \leq r$. By theorem 3.7 we have $p \leq rx$ and $q \leq rx$. Thus $p \vee q \leq rx$ and $x(p \vee q) \leq x(rx) \leq r$. Taking $r = xp \vee xq$, we have $x(p \vee q) \leq xp \vee xq$. Thus $x(p \vee q) = xp \vee xq$, for all $p,q \in P$, for all $x \in X$. //

We can actually prove a more general result than the previous theorem.

Theorem 3.9: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena and let $x \in X$. Then the mapping $p \to px$ preserves all infima and the operation $p \to xp$ preserves all suprema.

Proof: Assume that $\bigwedge p_i$ exists. Since $\bigwedge p_i \leq p_j$, for all j, then $(\bigwedge p_i)x \leq p_j x$, for all j. If $q \leq p_j x$, for all j, then by theorem 3.7 we have $xq \leq p_j$, for all j. Thus $xq \leq \bigwedge p_i$. Again by theorem 3.7 we have $q \leq (\bigwedge p_i)x$. Hence $(\bigwedge p_i)x = \bigwedge (p_i x)$. The second part of the theorem can be shown by the dual argument of the first part. //

Corollary 3.10: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena and $\{p_i\} \subseteq P$. If $\bigwedge p_i$ exists, then $(\bigwedge p_i)x = \bigwedge (p_i x)$. If $\bigvee p_i$ exists, then $x(\bigvee p_i) = \bigvee (xp_i)$.

Proof: Immediate from theorem 3.9. //

Theorem 3.11: If $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena, $x \in X$, and $p \in P$, then

$$xp = \bigwedge \{q : p \leq qx\}$$

and

$$px = \bigvee \{q : xq \leq p\}.$$

Proof: We prove only the first half of the conclusion; the other half can be demonstrated by a similar argument. First note that $xp \in \{q : p \leq qx\}$. If $r \in \{q : p \leq qx\}$, then $p \leq rx$. Thus

$$\begin{aligned} xp &\leq x(rx) \\ &\leq r \end{aligned}$$

and xp is a lower bound for the set $\{q : p \leq qx\}$. If $r \leq q$, for all $q \in P$ so that $p \leq qx$, then $r \leq xp$, since $p \leq (xp)x$. Hence $xp = \bigwedge \{q : p \leq qx\}$. //

Corollary 3.12: In the context of theorem 3.11 given either $*_1$ or

$*_2$, the other is uniquely determined.

Proof: This is precisely the statement of the last theorem. //

One may expect that the action determined by $*_1$ and $*_2$ will completely determine P and X, given one or the other. In some cases this may be the case, but in general it is not.

If there are enough Boolean algebra elements to distinguish elements of the semigroup, then the arena $\mathfrak{A} = (P, X, *_1, *_2)$ is called operational.

Definition 3.13: By an operational arena we mean an arena $\mathfrak{A} = (P, X, *_1, *_2)$ so that if $xp = yp$, for all $p \in P$, implies that $x = y$, for all $x,y \in X$.

Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena and let $x \in X$. Define a mapping $T_x : P \to P$ by

$$T_x(p) = xp,$$

for all $p \in P$.

Theorem 3.14: An arena $\mathfrak{A} = (P, X, *_1, *_2)$ is operational if and only if $T_x = T_y$ implies $x = y$, for all $x,y \in X$.

Proof: Immediate from definition 3.13. //

In the non-operational arena case the action of the arena can be very general indeed.

Example 3.15: Let P be any Boolean algebra and X be any semigroup. Define the action by

$$px = xp = p,$$

for all $p \in P$, for all $x \in X$. Then $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena which is non-operational in the worst possible way, that is, the semigroup does not act on the Boolean algebra.

Theorem 3.16: An arena $\mathfrak{A} = (P, X, *_1, *_2)$ is operational if and only if $px = py$, for all $p \in P$, implies $x = y$, for all $x,y \in X$.

Proof: Assume that $\mathfrak{A} = (P, X, *_1, *_2)$ is an operational arena and let $px = py$, for all $p \in P$. If $q \in P$, then by theorem 3.11 we have

$$\begin{aligned} xq &= \bigwedge \{p : q \leq px\} \\ &= \bigwedge \{p : q \leq py\} \\ &= yq. \end{aligned}$$

Since $\mathfrak{A}$ is operational and $q \in P$ is arbitrary, then $x = y$. The converse can be shown by a dual argument. //

In an operational arena $\mathfrak{A} = (P, X, *_1, *_2)$ we will see that the number of elements in P and X are somewhat related.

Theorem 3.17: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an operational arena so that P is a finite Boolean algebra with n elements. Then the semigroup X has at most n^n elements.

Proof: If $\mathfrak{A}$ is an operational arena, then for each $x \in X$, there is a mapping $T_x : P \to P$ defined by

$$T_x(p) = xp,$$

for all $p \in P$. By theorem 3.14 $\mathfrak{A}$ is operational if and only if $T_x = T_y$ implies $x = y$, for all $x,y \in X$. There are n^n possible mappings of P into P, and hence only n^n possible mappings T_x. Hence X has at most n^n elements. //

Definition 3.18: An arena $\mathfrak{A}' = (P', X', *_1', *_2')$ is a subarena of the arena $\mathfrak{A} = (P, X, *_1, *_2)$ if

(1) P' is a Boolean subalgebra of P,

(2) X' is a subsemigroup of X,

(3) $*_1' = *_1\big|_{P' \times X'}$ and $*_2' = *_2\big|_{X' \times P'}$.

Thus $\mathfrak{A}_1$ is a subarena of $\mathfrak{A}$ if, in terms of mixed algebras, $\mathfrak{A}_1$ is a mixed subalgebra of $\mathfrak{A}$.

Theorem 3.19: The intersection of any family of subarenas $\mathfrak{A}_i = (P_i, X_i, *_{i1}, *_{i2})$ of an arena $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena if $\bigcap X_i \neq \emptyset$.

Proof: Let $\mathfrak{A}_i = (P_i, X_i, *_{i1}, *_{i2})$ be subarenas of the arena $\mathfrak{A} = (P, X, *_1, *_2)$. Then

$$\bigcap \mathfrak{A}_i = (\bigcap P_i, \bigcap X_i, \bigcap *_{i1}, \bigcap *_{i2})$$

is an arena, since $\bigcap P_i$ is a Boolean subalgebra of P, $\bigcap X_i$ is a subsemigroup of X, if $\bigcap X_i \neq \emptyset$, $*_{i1} = *_1 \big|_{P_i \times X_i}$, and $*_{i2} = *_2 \big|_{X_i \times P_i}$. //

Corollary 3.20: If the family of all subarenas $\mathfrak{A}_i$ of an arena $\mathfrak{A}$ is such that $\bigcap X_i \neq \emptyset$, then the family of all subarenas forms complete lattice.

Given an arena $\mathfrak{A} = (P, X, *_1, *_2)$ it follows that if S is a nonempty subset of the Boolean algebra P and T is a nonempty subset of the semigroup X, then there exists a smallest arena $\mathfrak{A}' = (P', X', *_1', *_2')$ so that $S \subseteq P'$ and $T \subseteq X'$.

Definition 3.21: The subarena $\mathfrak{A}'$ of $\mathfrak{A}$ is called the subarena of $\mathfrak{A}$ generated by the subsets S and T.

Definition 3.22: By an arena congruence on the arena $\mathfrak{A} = (P, X, *_1, *_2)$ we mean a mixed algebra congruence $(\approx_1, \approx_2)$ on $\mathfrak{A}$.

Theorem 3.23: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena and let $(\approx_{i1}, \approx_{i2})$ be congruences on $\mathfrak{A}$. Then $(\bigcap \approx_{i1}, \bigcap \approx_{i2})$ is a congruence on $\mathfrak{A}$.

Proof: Clearly $\bigcap \approx_{i1}$ is a congruence on P and $\bigcap \approx_{i2}$ is a congruence on X. If p $(\bigcap \approx_{i1})$ q and x $(\bigcap \approx_{i2})$ y, then p $\approx_{i1}$ q and x $\approx_{i2}$ y, for all i. Since each $(\approx_{i1}, \approx_{i2})$ is a congruence on $\mathfrak{A}$, then px $\approx_{i1}$ qy and xp $\approx_{i1}$ yq, for all i. Thus px $(\bigcap \approx_{i1})$ qy and xp $(\bigcap \approx_{i1})$ yq. Hence $(\bigcap \approx_{i1}, \bigcap \approx_{i2})$ is an arena congruence on $\mathfrak{A}$. //

Corollary 3.24: The family of all arena congruences on an arena $\mathfrak{A}$ forms a complete lattice.

A mapping from one arena into a second arena which preserves the action of the first arena and also preserves the structure of the Boolean algebra and semigroup of the first arena will be called an arena homomorphism.

Definition 3.25: By an arena homomorphism from an arena $\mathfrak{A} = (P, X, *_1, *_2)$ into an arena $\mathfrak{A}' = (P', X', *_1', *_2')$ we mean a pair (f,g) of mappings, where

(1) $f : P \to P'$ is a Boolean homomorphism,

(2) $g : X \to X'$ is a semigroup homomorphism,

(3) $f(px) = f(p)g(x)$,

$f(xp) = g(x)f(p)$,

for all $p \in P$, for all $x \in X$.

In the context of mixed algebras an arena homomorphism is simply a mixed algebra homomorphism for arenas. If (f,g) is an arena homomorphism from $\mathfrak{A}$ into $\mathfrak{A}'$, then we write

$$(f,g) : \mathfrak{A} \to \mathfrak{A}'.$$

Definition 3.26: By an arena monomorphism we mean an arena homomorphism (f,g) so that both f and g are one-to-one mappings. An arena monomorphism (f,g) is also called an imbedding. By an arena epimorphism we mean an arena homomorphism (f,g) so that both f and g are onto mappings. By an arena isomorphism we mean an arena homomorphism (f,g) so that both f and g are one-to-one and onto mappings. If there is an arena isomorphism from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$, then we say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic and write

$$\mathfrak{A}_1 \cong \mathfrak{A}_2.$$

We now state some trivial results for arenas.

Theorem 3.27: If $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ is an arena homomorphism, then $(f(P_1), g(X_1), *'_{21}, *'_{22})$ is a subarena of $\mathfrak{A}_2$, where $*'_{21}$ and $*'_{22}$ are the natural restrictions of $*_{21}$ and $*_{22}$, respectively.

Theorem 3.28: If $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ is an arena homomorphism and $\mathfrak{A}'_2 = (P'_2, X'_2, *'_{21}, *'_{22})$ is a subarena of $\mathfrak{A}_2$, then

$$(f^{-1}(P'_2), g^{-1}(X'_2), *'_{11}, *'_{12})$$

is a subarena of $\mathfrak{A}_1$, where $*'_{11}$ and $*'_{12}$ are the natural restrictions of $*_{11}$ and $*_{12}$, respectively.

Definition 3.29: If $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ and $(h,k) : \mathfrak{A}_2 \to \mathfrak{A}_3$ are arena homomorphisms, then by the composition of (f,g) and (h,k) we mean the mapping $(h \circ f, k \circ g) : \mathfrak{A}_1 \to \mathfrak{A}_3$, where $h \circ f$ and $k \circ g$ are the natural compositions.

Theorem 3.30: The composition of two arena homomorphisms is an arena homomorphism.

Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena and $(\approx_1, \approx_2)$ be an arena congruence on $\mathfrak{A}$. Then $P/\approx_1$ and $X/\approx_2$ are well-known quotient structures. We would like to define mappings $\bar{*}_1$ and $\bar{*}_2$ so that $(P/\approx_1, X/\approx_2, \bar{*}_1, \bar{*}_2)$ is an arena. Define

$$\bar{*}_1 : P/\approx_1 \times X/\approx_2 \to P/\approx_1,$$

$$\bar{*}_2 : X/\approx_2 \times P/\approx_1 \to P/\approx_1,$$

by $\qquad \bar{p}\bar{x} = \overline{px},$

$$\bar{x}\bar{p} = \overline{xp},$$

for all $\bar{p} \in P/\approx_1$, for all $\bar{x} \in X/\approx_2$. This construction gives precisely the quotient mixed algebra in the case of arenas. As a consequence of the theory of mixed algebras we obtain the following result.

Theorem 3.31: $(P/\approx_1, X/\approx_2, \bar{*}_1, \bar{*}_2)$ is an arena.

Proof: This is a quotient mixed algebra in the particular case of arenas. //

Definition 3.32: This arena is called the quotient arena of $(P, X, *_1, *_2)$ determined by the congruence $(\approx_1, \approx_2)$ and is denoted by $\qquad \mathfrak{A}/(\approx_1, \approx_2)$.

Let $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ be an arena homomorphism from an arena $\mathfrak{A}_1 = (P_1, X_1, *_{11}, *_{12})$ onto the arena $\mathfrak{A}_2 = (P_2, X_2, *_{21}, *_{22})$. Define $(\approx_f, \approx_g)$ on $\mathfrak{A}_1$ by

$$p \approx_f q \text{ if and only if } f(p) = f(q),$$

$$x \approx_g y \text{ if and only if } g(x) = g(y),$$

for all $p,q \in P_1$, for all $x,y \in X$.

Theorem 3.33: $(\approx_f, \approx_g)$ is an arena congruence on $\mathfrak{A}_1$.

Proof: This is a mixed algebra congruence in the particular case of arenas. //

Definition 3.34: This congruence $(\approx_f, \approx_g)$ is called the arena congruence on $\mathfrak{A}_1$ induced by the arena homomorphism (f,g).

Theorem 3.35: (Fundamental Homomorphism Theorem) If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are arenas and $(f,g) : \mathfrak{A}_1 \to \mathfrak{A}_2$ is an arena homomorphism from $\mathfrak{A}_1$

onto $\mathfrak{A}_2$, then

$$\mathfrak{A}_1 / (\approx_f, \approx_g) \cong \mathfrak{A}_2,$$

where $(\approx_f, \approx_g)$ is the congruence on $\mathfrak{A}_1$ induced by the arena homomorphism (f,g).

Proof: This is a corollary of the fundamental homomorphism theorem for mixed algebras in the arena case. //

There are two natural types of arenas: those which are similar to a set arena in which the join of two elements of the Boolean algebra is given by set-union and those which are not. The first kind of arena is distributive in the sense that the inverse image of the join of two elements is the join of the inverse images and the inverse image of the zero element is itself.

Definition 3.36: By a distributive arena we mean an arena $\mathfrak{A} = (P, X, *_1, *_2)$ so that

(1) $(p \vee q)x = px \vee qx$,

(2) $0x = 0$,

for all $p,q \in P$, for all $x \in X$.

At first glance there seems to be no properties of an arena which have anything to do with complements. We will show, however, that this is far from the case.

Theorem 3.37: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be a distributive arena. Then the following statements are equivalent:

(1) $(px)' = p'x$,

(2) $(p \wedge q)x = px \wedge qx$,

for all $p,q \in P$, for all $x \in X$.

Proof: Assume that $(px)' = p'x$, for all $p \in P$, for all $x \in X$.

Then $(p \wedge q)x = (p' \vee q')'x$

$$= ((p' \vee q')x)'$$

$$= (p'x \vee q'x)'$$

$$= (p'x)' \wedge (q'x)'$$

$$= p''x \wedge q''x$$

$$= px \wedge qx,$$

for all $p,q \in P$, for all $x \in X$. Conversely, assume that $(p \wedge q)x = px \wedge qx$, for all $p,q \in P$, for all $x \in X$. Then

$$px \vee p'x = (p \vee p')x$$

$$= 1x$$

$$= 1,$$

and $px \wedge p'x = (p \wedge p')x$

$$= 0x$$

$$= 0.$$

Hence $(px)' = p'x$, for all $p \in P$, for all $x \in X$. //

<u>Theorem 3.38</u>: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be a distributive arena. Then $(px)' = p'x$, for all $p \in P$, for all $x \in X$.

<u>Proof</u>: Let $p \in P$ and $x \in X$. Since $(p \wedge q)x = px \wedge qx$ holds in every arena, then by theorem 3.37 the result is immediate. //

<u>Theorem 3.39</u>: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena so that $(px)' = p'x$, for all $p \in P$, for all $x \in X$. Then $\mathfrak{A}$ is a distributive arena.

<u>Proof</u>: Let $p,q \in P$ and $x \in X$. Then

$$(p \vee q)x = (p' \wedge q')'x$$

$$= ((p' \wedge q')x)'$$

$$= (p'x \wedge q'x)'$$

$= (p'x)' \vee (q'x)'$

$= (px)'' \vee (qx)''$

$= px \vee qx.$

Also $\quad 0x = 1'x$

$= (1x)'$

$= 1'$

$= 0.$

Hence $\mathfrak{A}$ is a distributive arena. //

Corollary 3.40: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena. Then $\mathfrak{A}$ is a distributive arena if and only if $(px)' = p'x$, for all $p \in P$, for all $x \in X$.

This last corollary gives us an alternate definition for a distributive arena.

Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena and let $x \in X$. Define a mapping $R_x : P \to P$ by

$$R_x(p) = px,$$

for all $p \in P$.

Theorem 3.41: If $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena and $x \in X$, then the mapping R_x is a lower semi-lattice homomorphism.

Proof: This is the statement of corollary 3.10. //

If $\mathfrak{A}$ is a distributive arena, then much more can be said.

Theorem 3.42: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a distributive arena and $x \in X$, then the mapping R_x is a Boolean algebra homomorphism from P into P.

Proof: Let $p,q \in P$. Then by theorem 3.41 we have

$$R_x(p \wedge q) = R_x(p) \wedge R_x(q).$$

Since $\mathfrak{A}$ is a distributive arena, then

$$R_x(p \vee q) = (p \vee q)x$$
$$= px \vee qx$$
$$= R_x(p) \vee R_x(q),$$

and

$$R_x(p') = p'x$$
$$= (px)'$$
$$= (R_x(p))'.$$

Hence R_x is a Boolean algebra homomorphism from P into P. //

It is natural now to inquire whether or not the converse of theorem 3.42 is also true.

Theorem 3.43: If $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena so that the mapping R_x is a Boolean algebra homomorphism for all $x \in X$, then $\mathfrak{A}$ is a distributive arena.

Proof: If R_x is a Boolean algebra homomorphism, then

$$R_x(p') = (R_x(p))',$$

for all $p \in P$; that is, $p'x = (px)'$, for all $x \in X$, for all $p \in P$. By theorem 3.39 we have $\mathfrak{A}$ is a distributive arena. //

Corollary 3.44: Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena. Then $\mathfrak{A}$ is a distributive arena if and only if R_x is a Boolean algebra homomorphism, for all $x \in X$.

Combining corollary 3.40 and corollary 3.44 we have the following result.

Theorem 3.45: If $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena, then the following statements are equivalent:

(1) $\mathfrak{A}$ is a distributive arena,

(2) $(px)' = p'x$, for all $p \in P$, for all $x \in X$,

(3) R_x is a Boolean algebra homomorphism for all $x \in X$.

Theorem 3.46: A subarena of a distributive arena is a distributive arena.

Proof: Obvious. //

Theorem 3.47: The homomorphic image of a distributive arena is a distributive arena.

Proof: Obvious. //

Theorem 3.48: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a distributive arena and $(\approx_1, \approx_2)$ is an arena congruence on $\mathfrak{A}$, then the quotient arena $\mathfrak{A}/(\approx_1, \approx_2)$ is a distributive arena.

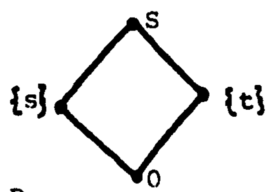
Proof: Let $\bar{p}, \bar{q} \in P/\approx_1$ and $\bar{x} \in X/\approx_2$. Then

$$\begin{aligned} (\bar{p} \vee \bar{q})\bar{x} &= (\overline{p \vee q})\bar{x} \\ &= \overline{(p \vee q)x} \\ &= \overline{px \vee qx} \\ &= \overline{px} \vee \overline{qx} \\ &= \bar{p}\bar{x} \vee \bar{q}\bar{x}. \end{aligned}$$

Hence $\mathfrak{A}/(\approx_1, \approx_2)$ is a distributive arena. //

Note that a subarena of an operational arena is not necessarily an operational arena.

Example 3.49: Let $S = \{s,t\}$ and $\mathfrak{A} = (P, X, *_1, *_2)$ be the set arena determined by S.



P:

X:

	e	a	b	c
e	e	a	b	c
a	a	a	c	c
b	b	a	e	c
c	c	a	a	c

The above diagram gives the Boolean algebra P and the operation table gives the semigroup X, where

$$e = \begin{pmatrix} s & t \\ s & t \end{pmatrix}, \quad a = \begin{pmatrix} s & t \\ s & s \end{pmatrix}, \quad b = \begin{pmatrix} s & t \\ t & s \end{pmatrix}, \quad c = \begin{pmatrix} s & t \\ t & t \end{pmatrix}.$$

That $\mathfrak{A}$ is an operational arena is easily checked by considering the following action table:

p	ep	ap	bp	cp
0	0	0	0	0
$\{s\}$	$\{s\}$	$\{s\}$	$\{t\}$	$\{t\}$
$\{t\}$	$\{t\}$	$\{s\}$	$\{s\}$	$\{t\}$
S	S	$\{s\}$	S	$\{t\}$.

The subarena $\mathfrak{A}_1 = (P_1, X_1, *_{11}, *_{12})$ of $\mathfrak{A}$ determined by $P_1 = \{0,S\}$ and $X_1 = \{e,b\}$ is clearly not operational.

By corollary 3.10 we have

$$x(\bigvee p_i) = \bigvee (xp_i),$$

whenever $\bigvee p_i$ exists and

$$(\bigwedge p_i)x = \bigwedge (p_i x),$$

whenever $\bigwedge p_i$ exists, where $\{p_i\} \subseteq P$. We may inquire whether or not

$$(\bigvee p_i)x = \bigvee (p_i x)$$

or

$$x(\bigwedge p_i) = \bigwedge (xp_i)$$

hold. The first condition seems very natural and is what motivates consideration of the following definition.

<u>Definition 3.50</u>: By a <u>complete</u> <u>arena</u> we mean an arena $\mathfrak{A} = (P, X, *_1, *_2)$ so that

(1) P is a complete Boolean algebra,

(2) If $\{p_i\} \subseteq P$ and $x \in X$, then $(\bigvee p_i)x = \bigvee (p_i x)$.

We would like to characterize complete arenas. Without some restrictions we will see that this is not easy.

Theorem 3.51: If $\mathcal{A} = (P, X, *_1, *_2)$ is a distributive arena where P is a complete Boolean algebra, then $\mathcal{A}$ is a complete arena.

Proof: Let $\{p_i\} \subseteq P$ and $x \in X$. Then by theorem 3.38 we have

$$\begin{aligned}(\bigvee p_i)x &= (((\bigvee p_i)x)')' \\ &= ((\bigvee p_i)'x)' \\ &= ((\bigwedge p_i')x)' \\ &= (\bigwedge (p_i'x))' \\ &= (\bigwedge (p_i x)')' \\ &= ((\bigvee (p_i x))')' \\ &= \bigvee (p_i x).\end{aligned}$$

Hence $\mathcal{A}$ is a complete arena. //

The converse of theorem 3.51 is not true unless $0x = 0$, for all $x \in X$.

Theorem 3.52: If $\mathcal{A} = (P, X, *_1, *_2)$ is a complete arena and $0x = 0$, for all $x \in X$, then $\mathcal{A}$ is a distributive arena.

Proof: Obvious. //

Example 3.53: Let $\mathcal{A} = (P, X, *_1, *_2)$ be the arena where $P = \{0,1\}$, $X = \{x\}$, and the structure of $\mathcal{A}$ is given by the following action table:

P:

X:

	x
x	x

p	xp	px
0	0	1
1	0	1

Then $\mathcal{A}$ is a complete arena which is not distributive, since $0x \neq 0$.

Corollary 3.54: If $\mathcal{A} = (P, X, *_1, *_2)$ is an arena so that $0x = 0$, for all $x \in X$, and P is a complete Boolean algebra, then $\mathcal{A}$ is a complete arena if and only if $\mathfrak{A}$ is a distributive arena.

Proof: Immediate from theorem 3.51 and theorem 3.52. //

For some reasons, which will become clear, it is desirable for the semigroup of an arena to contain an identity; therefore, we will show that every arena can be imbedded in an arena whose semigroup contains an identity.

Let (X, *) be a semigroup and X^e denote the semigroup ($X \cup \{e\}$, *'), where

$$x *' y = x * y,$$

$$x *' e = e *' x = x,$$

$$e *' e = e,$$

for all $x,y \in X$. This is the usual imbedding of a semigroup X in a semigroup X^e with identity e.

Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena. We would like to make $(P, X^e, *_1^e, *_2^e)$ into an arena $\mathfrak{A}^e$, where $*_1^e$ and $*_2^e$ are some appropriate mappings which extend $*_1$ and $*_2$, respectively. Define

$$*_1^e : P \times X^e \to P$$

by

$$*_1^e(p,x) = px,$$

$$*_1^e(p,e) = p,$$

for all $p \in P$, for all $x \in X$. Also define

$$*_2^e : X^e \times P \to P$$

by

$$*_2^e(x,p) = xp,$$

$$*_2^e(e,p) = p,$$

for all $p \in P$, for all $x \in X$.

Theorem 3.55: $\mathfrak{A}^e = (P, X^e, *_1^e, *_2^e)$ is an arena with $\mathfrak{A}$ being a subarena of $\mathfrak{A}^e$.

Proof: Let $x \in X$ and $p \in P$. Then

$$(xe)p = xp = x(ep),$$

$$(ex)p = xp = e(xp),$$

$$(ee)p = ep = e(ep).$$

Thus $(xy)p = x(yp)$, for all $x,y \in X^e$, for all $p \in P$. Similarly, $p(xy) = (px)y$, for all $x,y \in X^e$, for all $p \in P$. If $p \leq q$, then $pe \leq qe$. Thus $p \leq q$ implies $px \leq qx$, for all $x \in X^e$. Since $e = (ep)e$, then $x \leq (xp)x$, for all $x \in X^e$, for all $p \in P$. Since

$$\begin{aligned} e(p \wedge qe) &= e(p \wedge q) \\ &= p \wedge q \\ &= ep \wedge q, \end{aligned}$$

for all $p,q \in P$, then $x(p \wedge qx) = xp \wedge q$, for all $p,q \in P$, for all $x \in X^e$. Hence $\mathfrak{A}^e = (P, X^e, *_1^e, *_2^e)$ is an arena. Clearly $\mathfrak{A}$ is a subarena of $\mathfrak{A}^e$. //

Corollary 3.56: Every arena $\mathfrak{A} = (P, X, *_1, *_2)$ can be imbedded in an arena $\mathfrak{A}^e = (P, X^e, *_1^e, *_2^e)$, where X^e is a semigroup with identity e.

As a consequence of this corollary we may always assume, if necessary, that the semigroup X of an arena $\mathfrak{A} = (P, X, *_1, *_2)$ has an identity e such that $pe = ep = p$, for all $p \in P$.

We next consider the idea of a simple arena and the interplay between the simplicity of the arena and the simplicity of its Boolean algebra and semigroup. Since an arena $\mathfrak{A} = (P, X, *_1, *_2)$ can always be imbedded in an arena $\mathfrak{A}^e = (P, X^e, *_1^e, *_2^e)$ as above, we will assume throughout this portion on simple arenas that the semigroup of every arena has an identity e and that $pe = ep = p$, for all p.

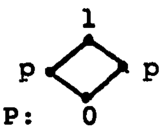
Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena and let Δ_P and Δ_X denote the equality relations on P and X, respectively. Similarly, let A_P and A_X denote the all relations on P and X, respectively. Then (Δ_P, Δ_X), (Δ_P, A_X), (A_P, Δ_X), and (A_P, A_X) are possible congruences on the arena $\mathfrak{A}$.

Theorem 3.57: If $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena, then (Δ_P, Δ_X), (A_P, Δ_X), and (A_P, A_X) are arena congruences on $\mathfrak{A}$.

Proof: It is immediate. //

As the next example shows it is not the case that (Δ_P, A_X) is always a congruence on $\mathfrak{A}$.

Example 3.58: Consider the arena $\mathfrak{A} = (P, X, *_1, *_2)$ given below:

P: 

X:

	e	u
e	e	u
u	u	u

q	eq	uq	qe	qu
0	0	0	0	0
p	p	p	p	1
p'	p'	0	p'	0
1	1	p	1	1

Note that $(1,1) \in \Delta_P$ and $(e,u) \in A_X$, but $(e1,u1) = (1,p) \notin \Delta_P$. Thus (Δ_P, A_X) is not a congruence on the arena $\mathfrak{A}$.

The following definition now seems very natural in view of theorem 3.57 and example 3.58.

Definition 3.59: The three congruences (Δ_P, Δ_X), (A_P, Δ_X), and (A_P, A_X) are called the trivial congruences on the arena $\mathfrak{A}$.

Definition 3.60: By a simple arena we mean an arena which has only the trivial congruences.

Recall that an algebra is simple if it has only the two trivial algebra congruences. In the case of semigroups a warning should be given. There are two common definitions of a simple semigroup. A simple semigroup in the sense of Clifford and Preston is not the same as a simple semigroup in the sense which we are using. Our definition is the universal algebraic definition. It is well-known that the only simple Boolean algebra is the two element Boolean algebra.

If P is a simple Boolean algebra and X is a simple semigroup, then there are just four possible congruences on the corresponding arena $\mathfrak{A}$, namely the three trivial congruences and (Δ_P, A_X). The next example will at first seem surprising.

Example 3.61: Consider the arena $\mathfrak{A}$ = (P, X, $*_1$, $*_2$) below:

P:

X:

	e	u
e	e	u
u	u	u

p	ep	up	pe	pu
0	0	0	0	0
1	1	1	1	1

Note that (Δ_P, A_X) is a congruence on $\mathfrak{A}$, for if $(p,p) \in \Delta_P$ and $(x,y) \in A_X$, then

$$(px,py) = (p,p) \in \Delta_P$$

and

$$(xp,yp) = (p,p) \in \Delta_P.$$

This last example shows that if P is a simple Boolean algebra and X is a simple semigroup, then the corresponding arena $\mathfrak{A}$ need not be a simple arena.

Note that this last example also shows given any Boolean algebra P and any semigroup X how to construct an arena $\mathfrak{A}$ having P and X as the Boolean algebra and semigroup, respectively. That is, if $p \in P$ and $x \in X$, then define

$$px = xp = p.$$

This gives an arena which essentially disregards the semigroup completely. It is clear that this type of arena is of little use.

Definition 3.62: An arena $\mathfrak{A}$ defined as above is called a null arena.

We use the term null arena to suggest that there is no action in the arena determined by the semigroup. We would now like to examine the simple arenas. As a first result we obtain the following theorem.

Theorem 3.63: If $\mathfrak{A} = (P, X, *_1, *_2)$ is not a null arena such that P is a simple Boolean algebra and X is a simple semigroup, then $\mathfrak{A}$ is a simple arena.

Proof: We need only show that (Δ_P, A_X) is not an arena congruence. Since $\mathfrak{A}$ is not a null arena, then $px \neq p$ or $xp \neq p$, for some $p \in P$, for some $x \in X$. For simplicity let us assume that $px \neq p$, for some $p \in P$, for some $x \in X$. The other possibility would be done in a similar manner. (Δ_P, A_X) is not an arena congruence since $(p,p) \in \Delta_P$ and $(x,e) \in A_X$, where e is the identity of X, but

$$(px,pe) = (px,p) \notin \Delta_P,$$

since $px \neq p$. Hence $\mathfrak{A}$ is a simple arena. //

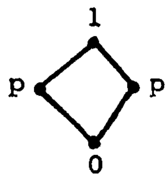
The natural question now is whether or not the converse of this last theorem is true.

Theorem 3.64: If $\mathfrak{A} = (P, X, *_1, *_2)$ is an arena where X is not a simple semigroup, then $\mathfrak{A}$ is not a simple arena.

Proof: If X is not a simple semigroup, then let R be a non-trivial congruence on X. It then easily follows that (A_P, R) is a non-trivial congruence on $\mathfrak{A}$. Hence $\mathfrak{A}$ is not a simple arena. //

It remains in doubt whether P not being a simple Boolean algebra implies that $\mathfrak{A}$ is not a simple arena.

Example 3.65: Consider the arena $\mathfrak{A}$ given below.

P:


X:

	e	v
e	e	v
v	v	e

q	eq	vq	qe	qv
0	0	0	0	0
p	p	p'	p	p'
p'	p'	p	p'	p
1	1	1	1	1

Clearly the semigroup X is simple but the Boolean algebra P is not simple. It is easily checked that the non-trivial congruences on P are

$$R_1 = \Delta_P \cup \{(p,0),\ (0,p),\ (p',1),\ (1,p')\}$$

and $$R_2 = \Delta_P \cup \{(p,1),\ (1,p),\ (p',0),\ (0,p')\}.$$

Hence the possible non-trivial congruences on $\mathfrak{A}$ are (Δ_P, A_X), (R_1, Δ_X), (R_1, A_X), (R_2, Δ_X), and (R_2, A_X). We consider each of these five possibilities:

(1) $(p,p) \in \Delta_P$ and $(e,v) \in A_X$ but $(pe,pv) = (p,p') \notin \Delta_P$.

(2) $(0,p) \in R_1$ and $(v,v) \in \Delta_X$ but $(0v,pv) = (0,p') \notin R_1$.

(3) $(0,p) \in R_1$ and $(v,v) \in A_X$ but $(0v,pv) = (0,p') \notin R_1$.

(4) $(0,p') \in R_2$ and $(v,v) \in \Delta_X$ but $(0v,p'v) = (0,p) \notin R_2$.

(5) $(0,p') \in R_2$ and $(v,v) \in A_X$ but $(0v,p'v) = (0,p) \notin R_2$.

Thus $\mathfrak{A}$ contains no non-trivial congruences and hence $\mathfrak{A}$ is a simple arena.

Thus given an arena $\mathfrak{A} = (P, X, *_1, *_2)$ with P not a simple Boolean algebra and with X a simple semigroup, it may well be the case that $\mathfrak{A}$ is still a simple arena.

Definition 3.66: Let $\mathfrak{A}_i = (P_i, X_i, *_{i1}, *_{i2})$ be a family of arenas. Consider the system $(P, X, *_1, *_2)$, where

(1) $P = \prod P_i$ is the direct product of the Boolean algebras P_i,

(2) $X = \prod X_i$ is the direct product of the semigroups X_i,

(3) $*_1 = \prod *_{i1}$ and $*_2 = \prod *_{i2}$ are the mappings $*_1 : P \times X \to P$ and $*_2 : X \times P \to P$ defined by

$*_1(p,x) = (p_i x_i)$,

$*_2(x,p) = (x_i p_i)$,

for all $p = (p_i) \in P$, for all $x = (x_i) \in X$.

We denote this system by $\prod \mathfrak{A}_i$.

Theorem 3.67: $\prod \mathfrak{A}_i$ is an arena.

Proof: Obvious. //

Definition 3.68: This arena $\prod \mathfrak{A}_i$ is called the direct product of the arenas $\mathfrak{A}_i$.

We will now consider a particular type of direct product of arenas. Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena. Then X^X is a semigroup under the natural pointwise operations and P^X is a Boolean algebra under the natural pointwise operations. Then

$$\mathfrak{A}^X = (P^X, X^X, *_1^X, *_2^X)$$

is an arena, the direct product of the arena $\mathfrak{A}$ with itself as many times as there are elements in the semigroup X. We consider conditions under which the arena $\mathfrak{A}$ can be imbedded in the arena $\mathfrak{A}^X$. Define

$$(f,g) : \mathfrak{A} \to \mathfrak{A}^X$$

by $f(p) = B_p$,

$g(x) = S_x$,

for all $p \in P$, for all $x \in X$, where $B_p : X \to P$ and $S_x : X \to X$ are defined by

$$B_p(y) = py,$$

$$S_x(y) = xy,$$

for all $y \in X$. If (f,g) is an arena homomorphism, then

(a) $f(p \vee q) = f(p) \vee f(q)$,

(b) $f(p') = (f(p))'$,

(c) $g(xz) = g(x)g(z)$,

(d) $f(px) = f(p)g(x)$,

(e) $f(xp) = g(x)f(p)$,

for all $p,q \in P$, for all $x,z \in X$. If $\mathfrak{A}$ is a distributive arena, then the mapping (f,g) satisfies conditions (a) - (d), since

(a) $$f(p \vee q) = B_{p \vee q} = B_p \vee B_q = f(p) \vee f(q),$$

(b) $$f(p') = B_{p'} = (B_p)' = (f(p))',$$

(c) $$g(xz) = S_{xz} = S_x S_z = g(x)g(z),$$

(d) $$f(px) = B_{px} = B_p S_x = f(p)g(x),$$

for all $p,q \in P$, for all $x,z \in X$. That $B_{p \vee q} = B_p \vee B_q$ follows since

$$B_{p \vee q}(y) = (p \vee q)y = py \vee qy = B_p(y) \vee B_q(y) = (B_p \vee B_q)(y),$$

for all $y \in X$. That $B_{p'} = (B_p)'$ follows since

$$B_{p'}(y) = p'y$$

$$= (py)'$$

$$= (B_p(y))'$$

$$= (B_p)'(y),$$

for all $y \in X$. That $S_{xz} = S_x S_z$ follows since

$$S_{xz}(y) = (xz)y$$

$$= x(zy)$$

$$= S_x(S_z(y)),$$

for all $y \in X$. That $B_p S_x = B_{px}$ follows since

$$(B_p S_x)(y) = B_p(S_x(y))$$

$$= B_p(xy)$$

$$= p(xy)$$

$$= (px)y$$

$$= B_{px}(y),$$

for all $y \in X$. It is not the case that (e) holds in general; for if it did, then

$$f(xp) = B_{xp}$$

$$= S_x B_p$$

$$= g(x)f(p).$$

But $B_{xp} = S_x B_p$ does not always hold. If $B_{xp} = S_x B_p$, then

$$(xp)y = B_{xp}(y)$$

$$= (S_x B_p)(y)$$

$$= S_x(B_p(y))$$

$$= S_x(py)$$

$$= x(py),$$

for all $y \in X$, and conversely. The following theorem has just been established.

Theorem 3.69: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a distributive arena so that $(xp)y = x(py)$, for all $p \in P$, for all $x,y \in X$, then the mapping $(f,g) : \mathfrak{A} \to \mathfrak{A}^{\mathfrak{X}}$ defined above is an arena homomorphism.

It may be useful to note that

$$\begin{aligned} f(p \wedge q) &= B_{p \wedge q} \\ &= B_p \wedge B_q \\ &= f(p) \wedge f(q), \end{aligned}$$

since

$$\begin{aligned} B_{p \wedge q}(y) &= (p \wedge q)y \\ &= py \wedge qy \\ &= B_p(y) \wedge B_q(y) \\ &= (B_p \wedge B_q)(y), \end{aligned}$$

for all $y \in X$. Under what conditions will the mapping (f,g) be an arena imbedding?

Definition 3.70: By a P-operational arena we mean an arena $\mathfrak{A} = (P, X, *_1, *_2)$ so that if $px = qx$, for all $x \in X$, then $p = q$.

A P-operational arena is simply an arena $\mathfrak{A} = (P, X, *_1, *_2)$ whose Boolean algebra is P and P has enough elements to distinguish the elements of the semigroup X.

Theorem 3.71: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a P-operational arena, then the mapping (f,g) defined above is one-to-one.

Proof: If $f(p) = f(q)$, then $B_p = B_q$. Thus $B_p(y) = B_q(y)$, for all $y \in X$. That is, $py = qy$, for all $y \in X$. Since $\mathfrak{A}$ is a P-operational arena, then $p = q$ and hence f is one-to-one. Also if $g(x) = g(y)$, then $S_x = S_y$. So

$$\begin{aligned} x &= xe \\ &= S_x(e) \end{aligned}$$

$= S_y(e)$

$= ye$

$= y.$

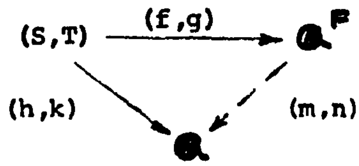
Thus g is a one-to-one mapping. Hence (f,g) is a one-to-one mapping of $\mathfrak{A}$ into $\mathfrak{A}^I$. //

Corollary 3.72: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a distributive P-operational arena so that $(xp)y = x(py)$, for all $p \in P$, for all $x,y \in X$, then the mapping (f,g) defined above is an arena imbedding.

Proof: Combine the results of theorem 3.69 and theorem 3.71. //

We consider next a possible collection of arenas which we will call free arenas.

Definition 3.73: Let S and T be sets. By a free arena on (S,T) we mean an arena $\mathfrak{A}^F = (P^F, X^F, *_1^F, *_2^F)$ together with a mapping $(f,g) : (S,T) \to \mathfrak{A}^F$ so that if $\mathfrak{A}$ is any arena and (h,k) is any mapping from (S,T) to $\mathfrak{A}$, then there is a unique arena homomorphism (m,n) from $\mathfrak{A}^F$ to $\mathfrak{A}$ so that the following diagram is commutative.



Thus a free arena on (S,T) is a free mixed algebra on (S,T) in the case of arenas. As an immediate consequence of this definition we can obtain the following theorem.

Theorem 3.74: If $\mathfrak{A}^F = (P^F, X^F, *_1^F, *_2^F)$ is a free arena on (S,T), then P^F is a free Boolean algebra on S and X^F is a free semigroup on T.

This concept of freedom contains most others. Consider the case of free R-modules. We take the mixed algebras to be of the form

(M, R, $*_1$, $*_2$), where M is an Abelian group, R is a (fixed) ring, and $*_1$ and $*_2$ are the right and left scalar multiplications. If we define an R-fixed homomorphism to be a mixed algebra homomorphism where the second mapping is restricted to be the identity mapping on the ring R and then define an R-free mixed algebra in the obvious manner, then an R-free mixed algebra is precisely a free R-module in the usual sense.

Theorem 3.75: There does not exist a free operational arena $\mathfrak{A}$ = (P, X, $*_1$, $*_2$) so that P is a finite Boolean algebra.

Proof: If $\mathfrak{A}$ = (P, X, $*_1$, $*_2$) is a free operational arena with P a finite Boolean algebra, then X must be finite by theorem 3.17. It is well known that every free semigroup is infinite. Hence $\mathfrak{A}$ can not be free after all. //

If $\mathfrak{A}^F$ is a free arena on (S,T), then by theorem 3.74 P^F and X^F are determined. The action is all that remains to be found. However, it may be, as the next example shows, that no action is definable so as to have a free arena.

Example 3.76: Consider the arena $\mathfrak{A}$ = (P, X, $*_1$, $*_2$), where P is the free Boolean algebra generated by a single element p and X is the free semigroup generated by a single element a. Then P is the four element Boolean algebra and X is a copy of the natural numbers. Let S = $\{x,y\}$, $P_1 = 2^S$, and $X_1 = S^S$, where $*_{11}$ and $*_{12}$ are the usual action. Let

$$e = \begin{pmatrix} x & y \\ x & y \end{pmatrix}, \quad u = \begin{pmatrix} x & y \\ x & x \end{pmatrix}, \quad w = \begin{pmatrix} x & y \\ y & y \end{pmatrix}, \quad v = \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

Then the operation table for X_1 is given below.

	e	u	w	v
e	e	u	w	v
u	u	u	w	w
w	w	u	w	u
v	v	u	w	e

Clearly $\mathfrak{A}_1$ is an arena. We wish to show that it is impossible for $\mathfrak{A}$ to be a free arena. Assume otherwise. Define $(h_1,k_1) : (\{p\}, \{a\}) \to \mathfrak{A}_1$ by

$$h_1(p) = \{x\},$$
$$k_1(a) = v.$$

If $\mathfrak{A}$ is free, then

$$\begin{aligned} h_1(pa) &= h_1(p)k_1(a) \\ &= \{x\}\,v \\ &= \{y\}. \end{aligned}$$

This implies that $pa = p'$. Define $(h_2,k_2) : (\{p\}, \{a\}) \to \mathfrak{A}_1$ by

$$h_2(p) = \{x\}$$
$$k_2(a) = u.$$

If $\mathfrak{A}$ is free, then

$$\begin{aligned} h_2(pa) &= h_2(p)k_2(a) \\ &= \{x\}\,u \\ &= \{x,y\}. \end{aligned}$$

But this implies that $pa = 1$. Hence we have a contradiction unless $\mathfrak{A}$ is not free.

The same argument as in this last example shows that there do not exist any free arenas.

<u>Theorem 3.77</u>: There do not exist any free arenas.

CHAPTER IV

ALGEBRAIC LOGIC VIA ARENAS

Now that we have established a theory of arenas we will consider algebraic logic by using this theory. Halmos [14] began a study of algebraic logic in the early fifties and about the same time Tarski and Thompson [18] were considering algebraic logic via cylindric algebras. Their points of view differ mainly in that Halmos's system doesn't have an equality relation whereas Tarski and Thompson's system does.

We will be concerned mainly with Halmos's formulization of algebraic logic. For sake of completeness we now give his definition of a quantifier in the monadic case.

<u>Definition 4.1</u>: Let P be a Boolean algebra. A mapping $\exists : P \to P$ is a <u>H-quantifier</u> on P if

(1) $\exists 0 = 0$,

(2) $p \leq \exists p$,

(3) $\exists (p \wedge \exists q) = \exists p \wedge \exists q$,

for all $p, q \in P$.

Let $\mathfrak{A} = (P, X, *_1, *_2)$ be an arena and let $x \in X$. Define a mapping $\exists_x : P \to P$ by

$$\exists_x p = (xp)x,$$

for all $p \in P$. We shall prove several theorems about this general setting.

Theorem 4.2: If $p \leq q$, then $\exists_x p \leq \exists_x q$.

Proof: Let $p \leq q$. Then $xp \leq xq$ and $(xp)x \leq (xq)x$. That is, $\exists_x p \leq \exists_x q$. //

Theorem 4.3: If $q \leq \exists_x p$, then $\exists_x q \leq \exists_x p$.

Proof: Let $q \leq \exists_x p$. Then $q \wedge \exists_x p = q$.

So
$$\begin{aligned} xq &= x(q \wedge \exists_x p) \\ &= x(q \wedge (xp)x) \\ &= xq \wedge xp. \end{aligned}$$

Thus $xq \leq xp$ and $(xq)x \leq (xp)x$. Hence $\exists_x q \leq \exists_x p$. //

Definition 4.4: By a closure operator on a Boolean algebra P we mean a mapping $C : P \to P$ so that

(1) $C0 = 0$,

(2) $p \leq Cp$,

(3) $CCp = Cp$,

(4) $C(p \vee q) = Cp \vee Cq$,

for all $p, q \in P$.

Theorem 4.5: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a distributive arena and $x \in X$, then $\exists_x$ is a closure operator on P.

Proof: (1)
$$\begin{aligned} \exists_x 0 &= (x0)x \\ &= 0x \\ &= 0, \end{aligned}$$

by theorem 3.6 and definition 3.36.

(2) $p \leq \exists_x p$, for all $p \in P$, by the definition of an arena.

(3) By (2) above we have $p \leq \exists_x p$. By theorem 4.2 we have $\exists_x p \leq \exists_x \exists_x p$. On the other hand, if $q \leq \exists_x p$, then by theorem 4.3

we have $\exists_x q \leq \exists_x p$. In particular, for $q = \exists_x p$, we obtain $\exists_x \exists_x p \leq \exists_x p$. Thus $\exists_x \exists_x p = \exists_x p$.

(4) $$\begin{aligned} \exists_x (p \vee q) &= (x(p \vee q))x \\ &= (xp \vee xq)x \\ &= (xp)x \vee (xq)x \\ &= \exists_x p \vee \exists_x q, \end{aligned}$$

by theorem 3.8 and definition 3.36. Hence $\exists_x$ is a closure operator on the Boolean algebra P. //

We can even say more about the mapping $\exists_x$ when $\mathfrak{A}$ is a distributive arena.

<u>Theorem 4.6</u>: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a distributive arena and $x \in X$, then $\exists_x$ is a H-quantifier on P.

<u>Proof</u>: We need only show that

$$\exists_x (p \wedge \exists_x q) = \exists_x p \wedge \exists_x q$$

for all $p,q \in P$. Let $p,q \in P$. Then

$$\begin{aligned} \exists_x (p \wedge \exists_x q) &= (x(p \wedge (xq)x))x \\ &= (xp \wedge xq)x \\ &= (xp)x \wedge (xq)x \\ &= \exists_x p \wedge \exists_x q. \end{aligned}$$

Hence $\exists_x$ is a H-quantifier on P. //

<u>Theorem 4.7</u>: If $r \in \exists_x P$, then $(xr)x = r$.

<u>Proof</u>: Since $r \leq (xr)x$, we need only show that $(xr)x \leq r$. Since $r \in \exists_x P$, then $r = (xp)x$, for some $p \in P$. Thus

$$\begin{aligned} (xr)x &= (x((xp)x))x \\ &= \exists_x \exists_x p \\ &= \exists_x p \end{aligned}$$

$= (xp)x$

$= r.$

Hence $(xr)x = r$. //

Corollary 4.8: $r \in \exists_x P$ if and only if $(xr)x = r$.

Theorem 4.9: If $\mathfrak{A} = (P, X, *_1, *_2)$ is any arena, then $\exists_x 1 = 1$.

Proof: Obvious. //

Theorem 4.10: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a distributive arena and $x \in X$, then $\exists_x(\exists_x p)' = (\exists_x p)'$, for all $p \in P$.

Proof: Since $\exists_x p \wedge (\exists_x p)' = 0$, then $x((\exists_x p)' \wedge (xp)x) = x0$. Thus $x(\exists_x p)' \wedge xp = 0$ and $(x(\exists_x p)' \wedge xp)x = 0$, since the arena $\mathfrak{A}$ is distributive. Thus $(x(\exists_x p)')x \wedge (xp)x = 0$, that is, we have $\exists_x(\exists_x p)' \wedge \exists_x p = 0$. Thus $\exists_x(\exists_x p)' \leq (\exists_x p)'$. By theorem 4.5 we have $(\exists_x p)' \leq \exists_x(\exists_x p)'$. Hence $\exists_x(\exists_x p)' = (\exists_x p)'$. //

Theorem 4.11: If $\mathfrak{A} = (P, X, *_1, *_2)$ is a distributive arena, then the range $\exists_x P$ of the quantifier $\exists_x$ is a Boolean subalgebra of P.

Proof: Clearly $0 \in \exists_x P$ and $1 \in \exists_x P$. If $p, q \in \exists_x P$, then $p = \exists_x p$ and $q = \exists_x q$. So

$$\begin{aligned} p \wedge q &= \exists_x p \wedge \exists_x q \\ &= (xp)x \wedge (xq)x \\ &= (xp \wedge xq)x \\ &= (x(p \wedge (xq)x))x \\ &= (x(p \wedge q))x \\ &= \exists_x (p \wedge q). \end{aligned}$$

This shows that $p \wedge q \in \exists_x P$. By theorem 4.10 we have

$$\begin{aligned} p' &= (\exists_x p)' \\ &= \exists_x(\exists_x p)'. \end{aligned}$$

This shows that $p' \in \exists_x P$. Hence $\exists_x P$ is a Boolean subalgebra of the Boolean algebra P. //

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