

ANALYSIS OF CONTINUOUS BENT BEAMS LOADED OUT OF
THE PLANE BY THE SIX MOMENT EQUATIONS

By

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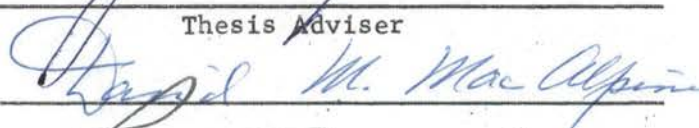
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PREFACE

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NOMENCLATURE

i, j, k	Letters designating supports of span.
ω	Angle between basic axis and principle axis of span.
\odot	Force acting up.
\bullet	Force acting down.
ΣP_z	Resultant of loads
x, x', u, u', v, v'	Cross section ordinates.
d_j, d_k	Length of spans.
x_j, y_j, x_k, y_k	Coordinates of supports.
$M_{ix'}, M_{kx}$	Support moments about the principal axis of the spans.
M_{ix}, M_{kx}	Support moments about basic axis.
SM_{ix}	Static moment of loads at i about the X-axis.
SM_y	Static moment of loads at a section in a span.
F_{ikxx}	Angular flexibility at i along the X-axis.
G_{ikxx}	Angular carry over flexibility at k along the X-axis.
τ_{ikxx}	Angular load function at i along the X-axis.
m_{ix}	Starting moment at i about the X-axis.
r_{ikxx}	Carry over value from m_{ix} to m_{kx} .
R_{iz}	Reaction at the support i parallel to Z-axis.
U	Strain Energy

- λ Elemental Angular Flexibility.
- EI Flexural Rigidity.
- GJ Torsional Rigidity.
- $BM_y^{(k)}$ Bending moment of loads at the section
with support k as origin.
- \sum Summation.
- δ First partial.

CHAPTER I

INTRODUCTION

1-1 General

This thesis presents the analysis of planar continuous bent beams, loaded out of the plane, by the six moment equations.

A carry over moment procedure of solving the six moment equations is introduced. The carry over moment method was developed by Tuma from the three moment equation and applied to the analysis of continuous beams (1), continuous trusses (2) and extended to numerous other problems in the plane. The analysis of beams and frames by means of this method was reported by Tuma in his lectures (3). The historical background of the problem investigated in this thesis has been discussed by Childress (4) and is not repeated here. The material presented in this thesis closely follows Tuma's lectures and reference to the lecture notes is made (3).

The analysis is based on the assumption of elastic deformations. The deformations due to bending and torsion are considered only and the deformations due to shear are considered small and are consequently neglected.

The symbols used in this thesis are rearranged under the heading of nomenclature at the beginning of the thesis. The signs of geometric quantities, loads, moments, forces and deformations are recorded in the appendix. The appendix material dealing with transformation matrices

was prepared on the basis of Tuma's lectures dealing with transformation matrices (3).

A list of a selected bibliography is presented at the end.

1-2 Statement of the Problem

A continuous bent member \overline{im} lying in the plane XY is acted upon by loads perpendicular to this plane. A system of reference axes XYZ is selected (Fig. 1-1), the co-ordinates being measured parallel to these axes and denoted by x , y , z , respectively.

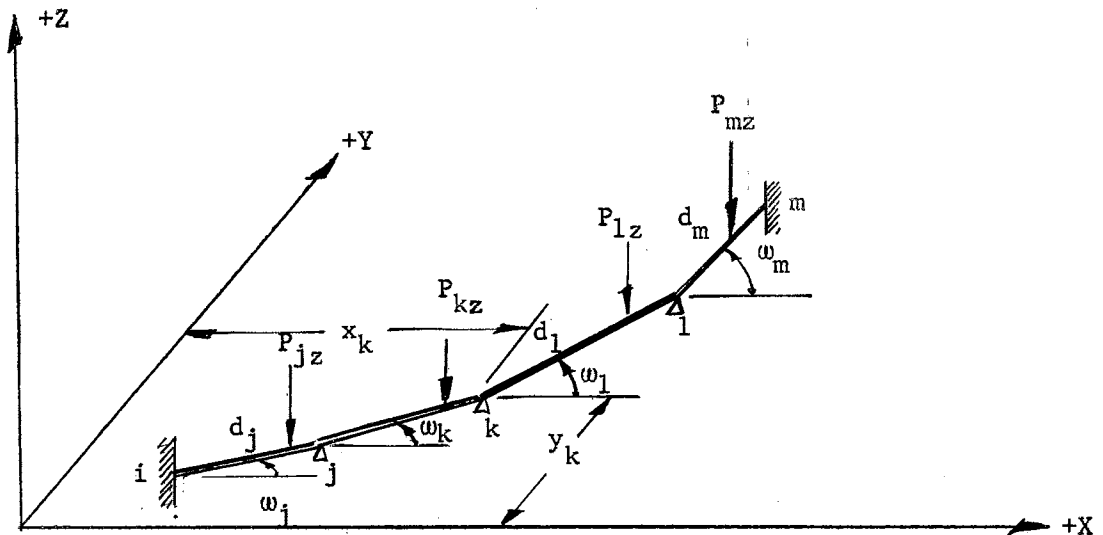


Figure 1-1

Continuous Bent Beam

The supports are designated as i , j , k , l , m and the span lengths are d_j , d_k , d_l , d_m . The slope of each span measured from a line parallel to the X -axis is represented by the symbol ω and the corresponding subscript as shown in Fig. 1-1. The exterior ends i , m are fixed and the interior supports are assumed to have spherical hinges. The foundation

under these supports is considered to be rigid and no displacement of supports is introduced.

The continuous bent beam shown in Fig. 1-1 has four reactive moments and five reactive forces. For the analysis of this beam, three equations of static equilibrium are available and six deformation conditions are essential. In general, for a continuous bent beam having the end supports fixed, and the number of spans equal to n , the number of reactive elements is $(n+5)$, three of which can be obtained from statics and $(n+2)$ deformation conditions are necessary. If the exterior ends of this beam are simply supported, the moments at the ends are equal to zero and the number of reactive elements decreases to $(n+1)$. In other words, a continuous bent beam with the end supports fixed is statically indeterminate to $(n+2)$ degree, and that with the exterior ends simply supported is statically indeterminate to $(n-2)$ degree. The degree of indeterminacy indicates the number of redundants which can be selected to the convenience of the analyst.

Basically, two systems of redundants are possible:

- (i) the case where moments acting at supports are selected as the redundants.
- (ii) the case where forces acting at supports are chosen as the redundants.

In this thesis, the support moments are taken as redundants. Two main types of continuous bent beams are considered:

- (a) The fixed-end continuous bent beam having even and odd number of spans. (Fig. 1-2a,b)
- (b) The continuous bent beam with simply-supported ends having even and odd number of spans. (Fig. 1-3a,b)

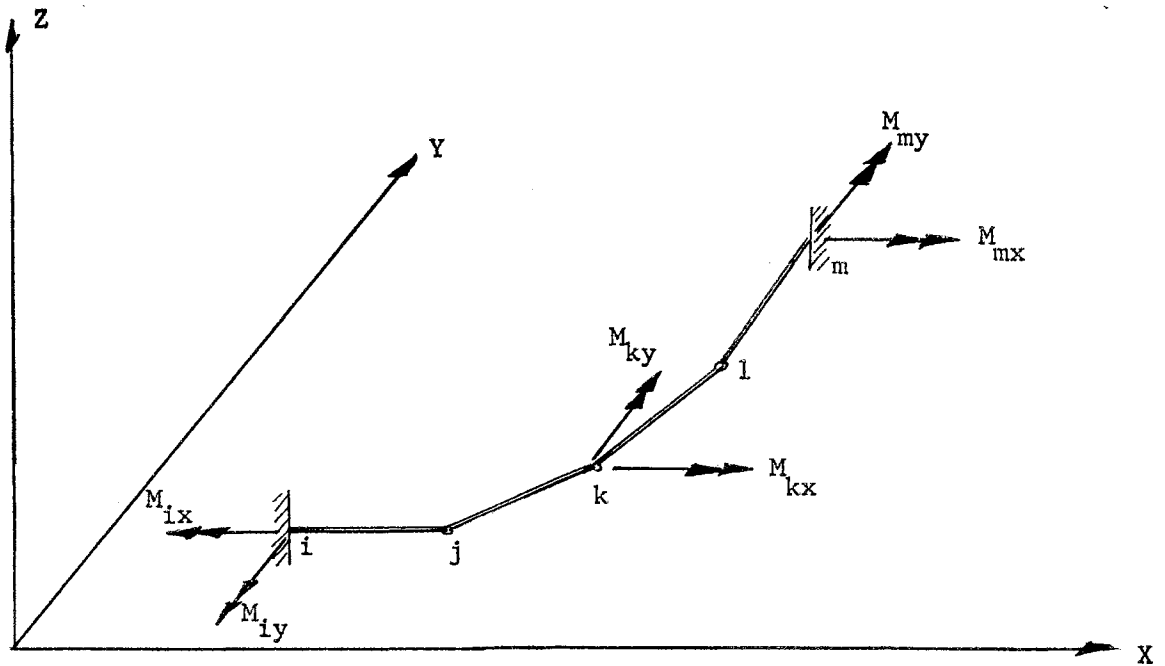


Figure 1-2a

Even span continuous bent beam - end supports fixed.

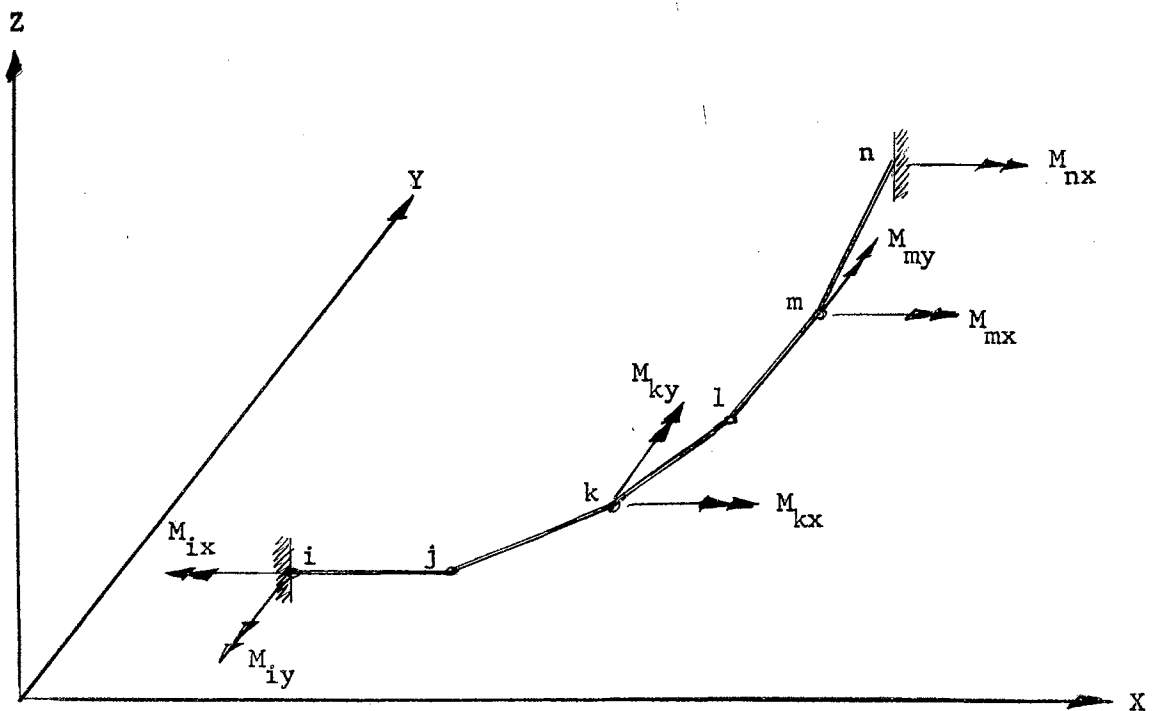


Figure 1-2b

Odd span continuous bent beam - end supports fixed.

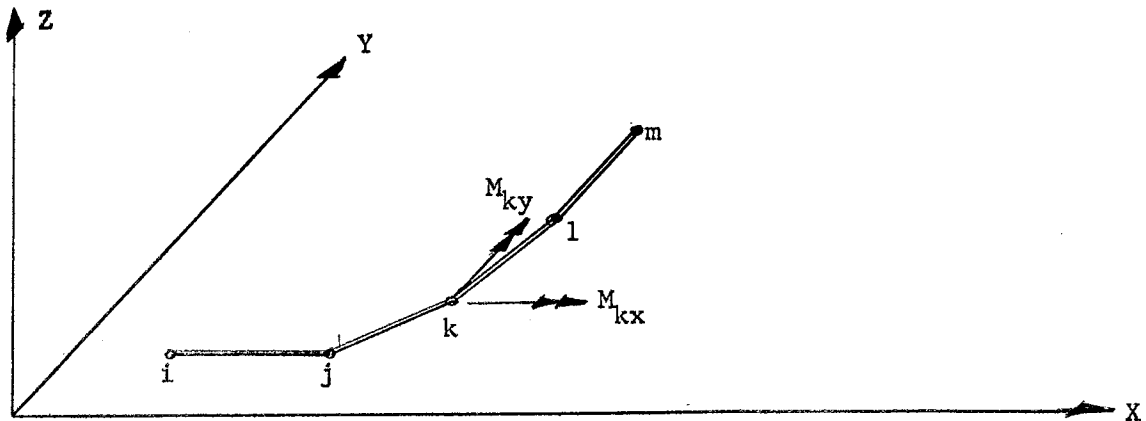


Figure 1-3a

Even span continuous bent beam - end supports simply-supported.

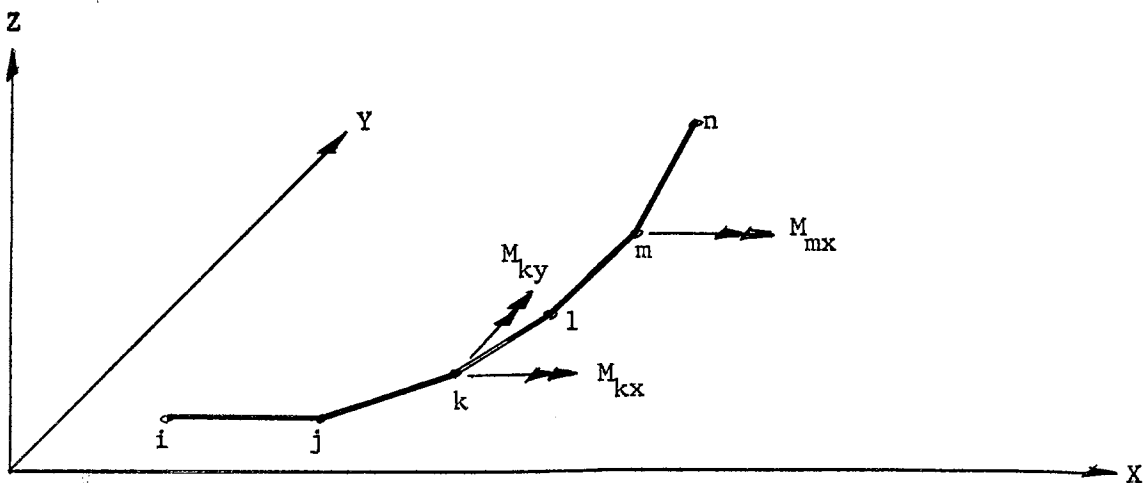


Figure 1-3b

Odd span continuous bent beam - end supports simply supported.

After selecting the moments as redundants as shown in Fig. (1-2a,b) and (1-3a,b), it is observed that three types of basic structures may be introduced.

- (1) A two span continuous bent bar \overline{ijk} simply supported at the exterior and the intermediate supports (Fig. 1-4a)
- (2) A single span bar \overline{mn} , simply supported at one end and fixed at the other end (Fig. 1-4b)

(3) A single span bar \overline{mn} , simply supported at both ends (Fig. 1-4c)

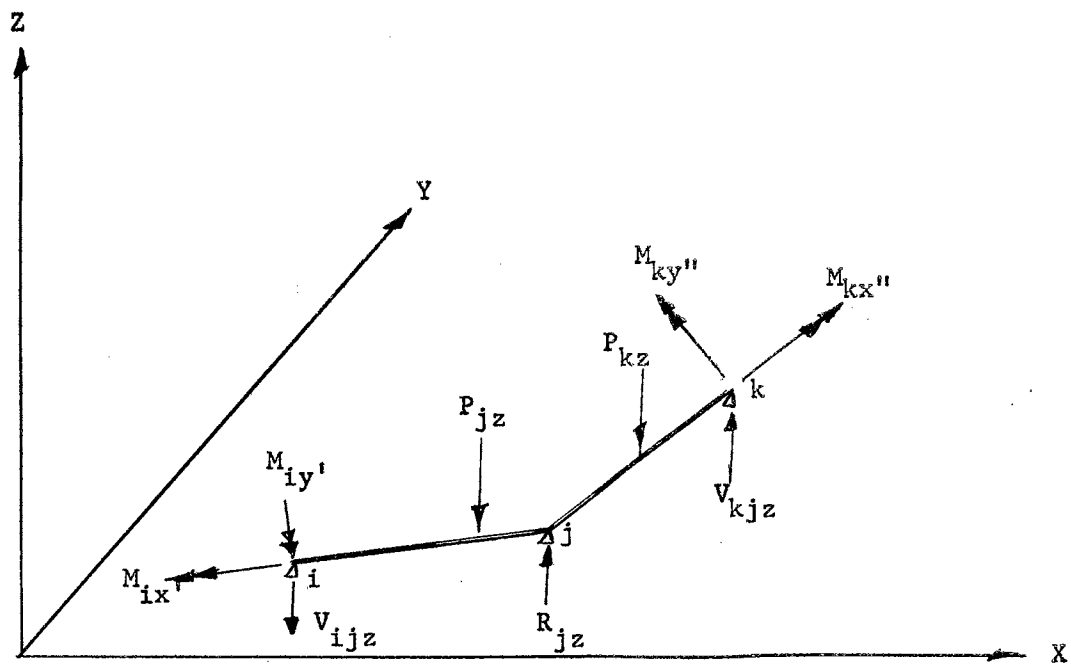


Figure 1-4a

Basic Structure No. 1

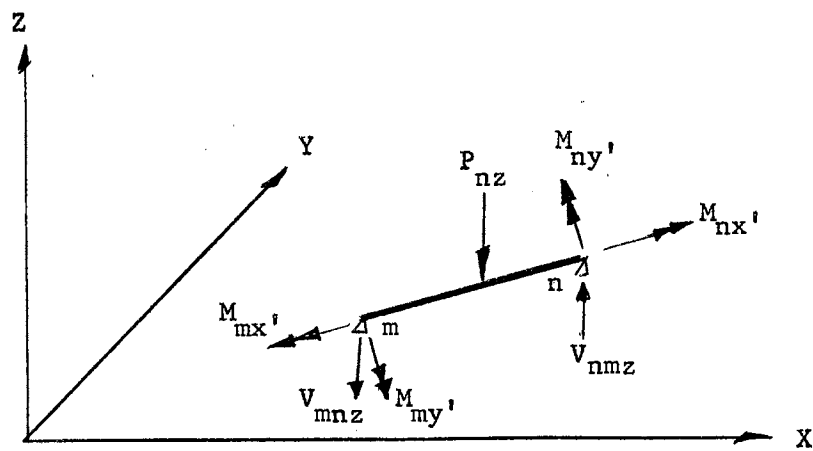


Figure 1-4b

Basic Structure No. 2

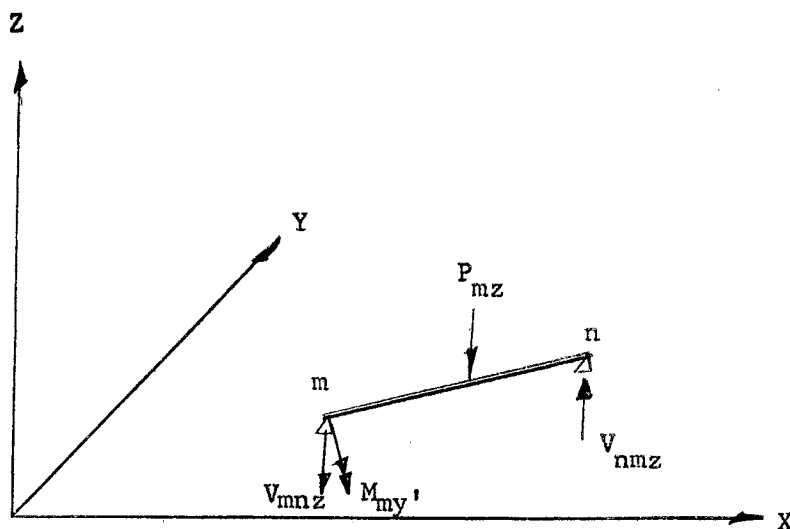


Figure 1-4c

Basic Structure No. 3

If the moments acting at the end of these bars are applied as unit moments, angular flexibilities may be derived and the compatibility equation may be expressed in terms of these angular flexibilities, load functions and the redundant moments. From this discussion, the necessity of studying four primary phases in the analysis of the problem becomes evident.

- (a) Geometry of basic structure.
- (b) Statics of basic structure.
- (c) Deformation of basic structure.
- (d) Compatibility Equations.

The geometry of the basic structure is discussed in this chapter and the next three topics mentioned above are treated in the subsequent chapters.

1-3 Geometry of the Problem.

(a) Basic Structure No. 1

The two span basic structure \overline{ijk} , lying in the plane XY and acted upon by loads perpendicular to this plane, is shown in Fig. 1-5. The bar rests on simple supports at i, j and k. The end moments and shears are also shown. As the structure is loaded perpendicular to its own plane, only the vertical shear V_z exists and the moment in the vertical direction, M_z , is zero.

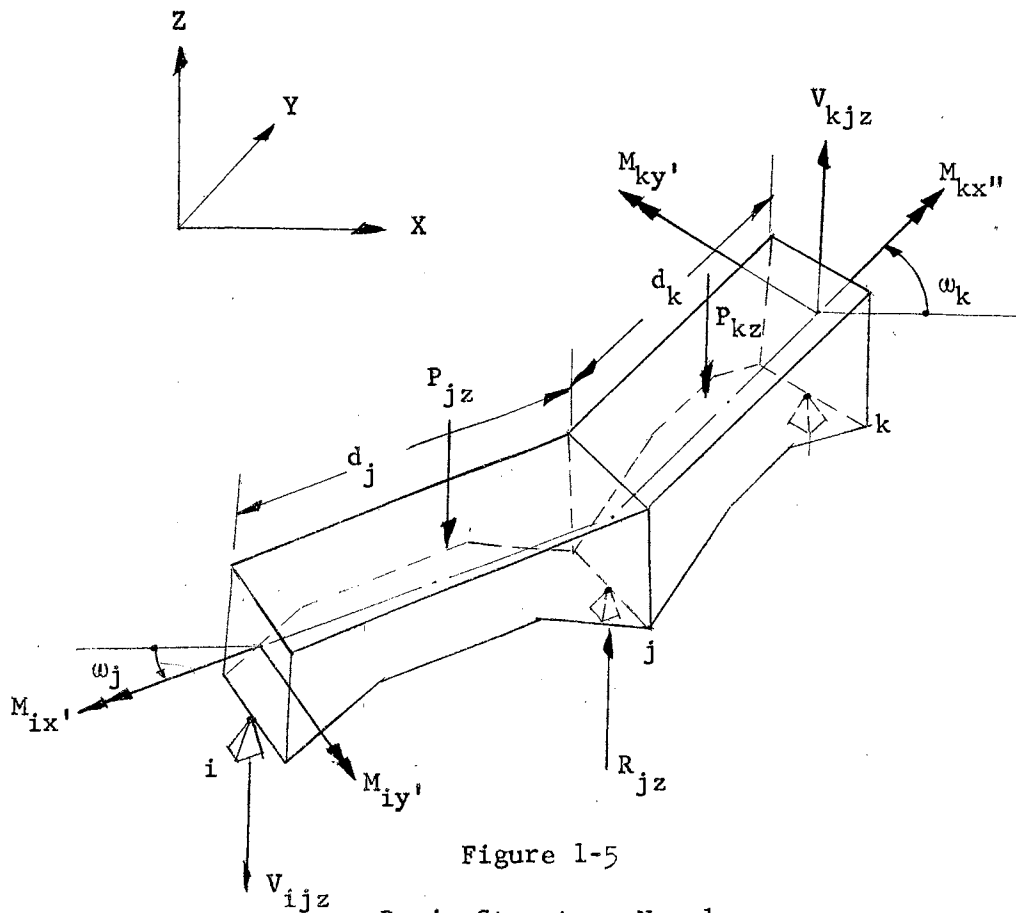


Figure 1-5

Basic Structure No. 1

The lengths of the members \overline{ij} and \overline{jk} are designated as d_j and d_k respectively. The slope of each member with reference to a line parallel to the X-axis is denoted by ω_j and ω_k respectively. The cross-section of the members varies along the length of the members.

Each span is related to its own co-ordinate system

(i) $X'Y'Z'$ for member \overline{ij} with origin at i

(ii) $X''Y''Z''$ for member \overline{jk} with origin at j

These particular co-ordinate systems can be related to the basic system of axes XYZ by making use of the Transformation Matrices shown in Tables 1-1 and 1-2.

	X'	Y'
X	α_{jx}	α_{jy}
Y	β_{jx}	β_{jy}

Table 1-1

Transformation Matrix
for span \overline{ij}

	X''	Y''
X	α_{kx}	α_{ky}
Y	β_{kx}	β_{ky}

Table 1-2

Transformation Matrix
for span \overline{jk}

The above Transformation Matrices are special cases of the Transformation Matrix derived for the case of a general space structure in the Appendix (Table No. 5). The " α " and the " β " terms appearing in the above tables are functions of the angles ω_j and ω_k respectively.

The Transformation Matrix provides for a systematic transformation of geometric quantities, moments, forces, slopes, elastic weights, etc. from one co-ordinate system to another. Thus it is seen that in having the Transformation Matrix, the analyst has a powerful tool in dealing

with problems in space structures.

(b) Basic Structure No. 2

In this case, a single span \overline{mn} with end "m" simply supported and end "n" fixed is considered. The basic structure lies in the plane XY and is loaded perpendicular to this plane (Fig. 1-6). The end moments and shears are shown. The member has a non-uniform cross-section, and its length is denoted by d_n . The principal axes of the member may be denoted by $X'Y'Z'$ with origin at m, and the slope of the member with a line parallel to the X axes of the basic system is designated as ω_n .

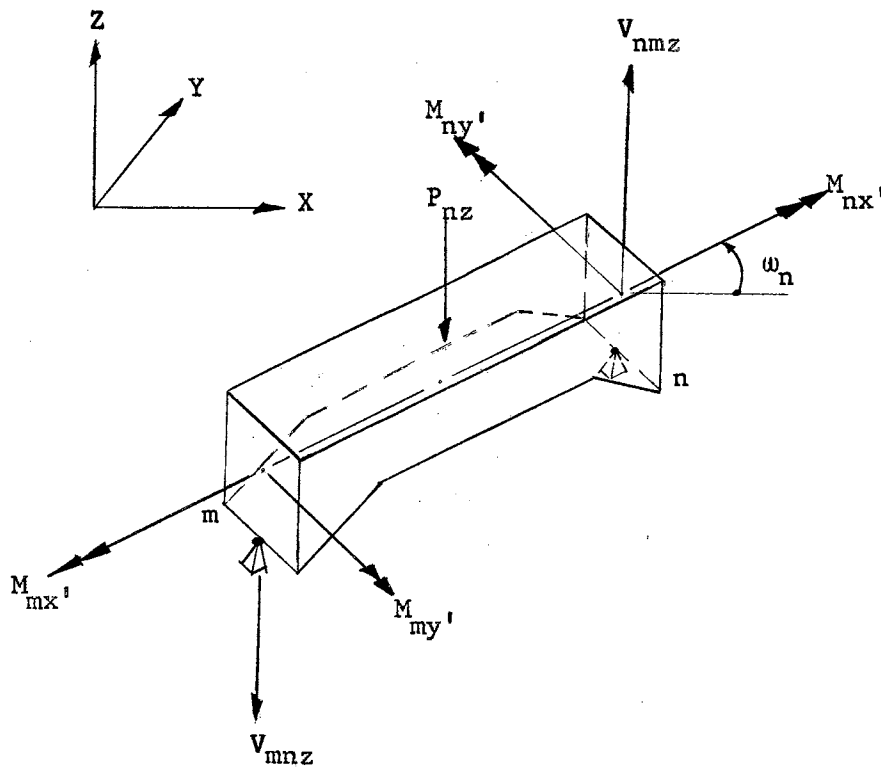


Figure 1-6

Basic Structure No. 2

The transformation angle is thus ω_n and the Transformation Matrix for the member is shown in Table 1-3.

	X'	Y'
X	α_{nx}	α_{ny}
Y	β_{nx}	β_{ny}

Table 1-3

Transformation Matrix
for span \overline{mn}

It is to be noted that the structure can be in stable equilibrium only if the end moments about the torsional axis of the span, M_{mx} , and M_{nx} , are equal.

(c) Basic Structure No. 3

A single span \overline{mn} with both ends simply-supported is considered. The basic structure is in the plane XY and is loaded out of the plane. The member has a length d_n and a cross-section varying along the length (Fig. 1-7).

The principal axes of the member may be designated as X'Y' and as in the case of the basic structure No. 2, ω_n represents the angle between a line parallel to the X-axis of the basic system and the X' axis and therefore the transformation angle. Thus the Transformation Matrix for this basic structure is the same as that shown in Table 1-3.

Since the end "n" is simply-supported, the end moments at n are equal to zero, and as such, the end moment at m in the X' direction will

also be zero to maintain equilibrium.

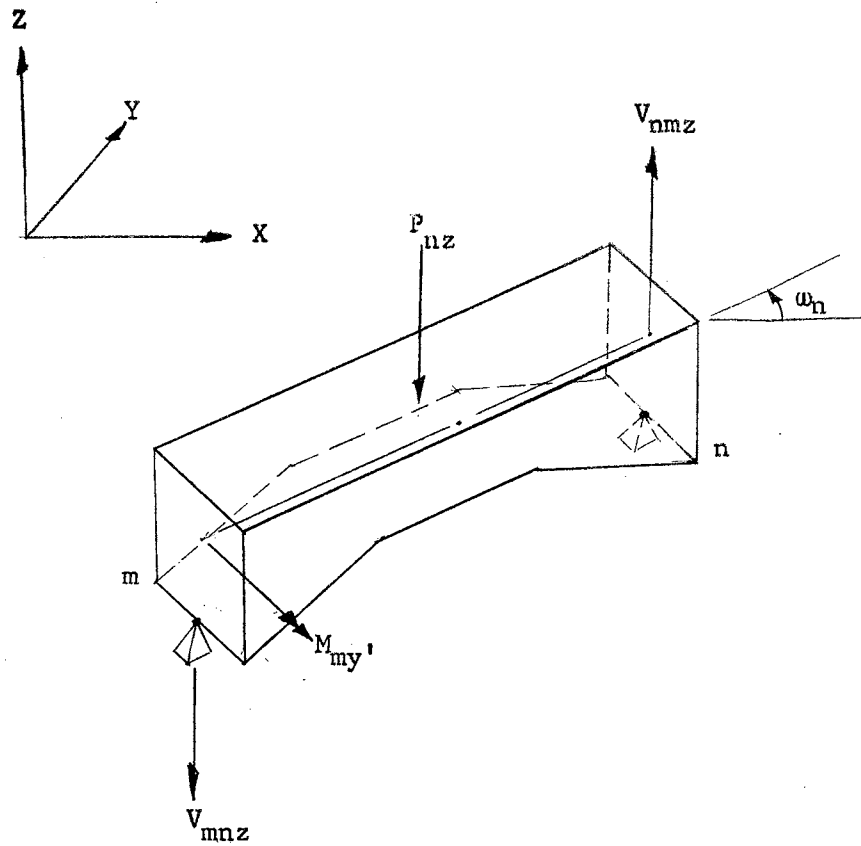


Figure 1-7

Basic Structure No. 3

CHAPTER II

STATICS OF BASIC STRUCTURE

The statics of the basic structures, described in the last chapter, is now studied.

2-1 Statics of Basic Structure No. 1

The basic structure \overline{ijk} , removed from the continuous bent beam \overline{im} (Fig. 1-2a) is shown in Fig. (2-1).

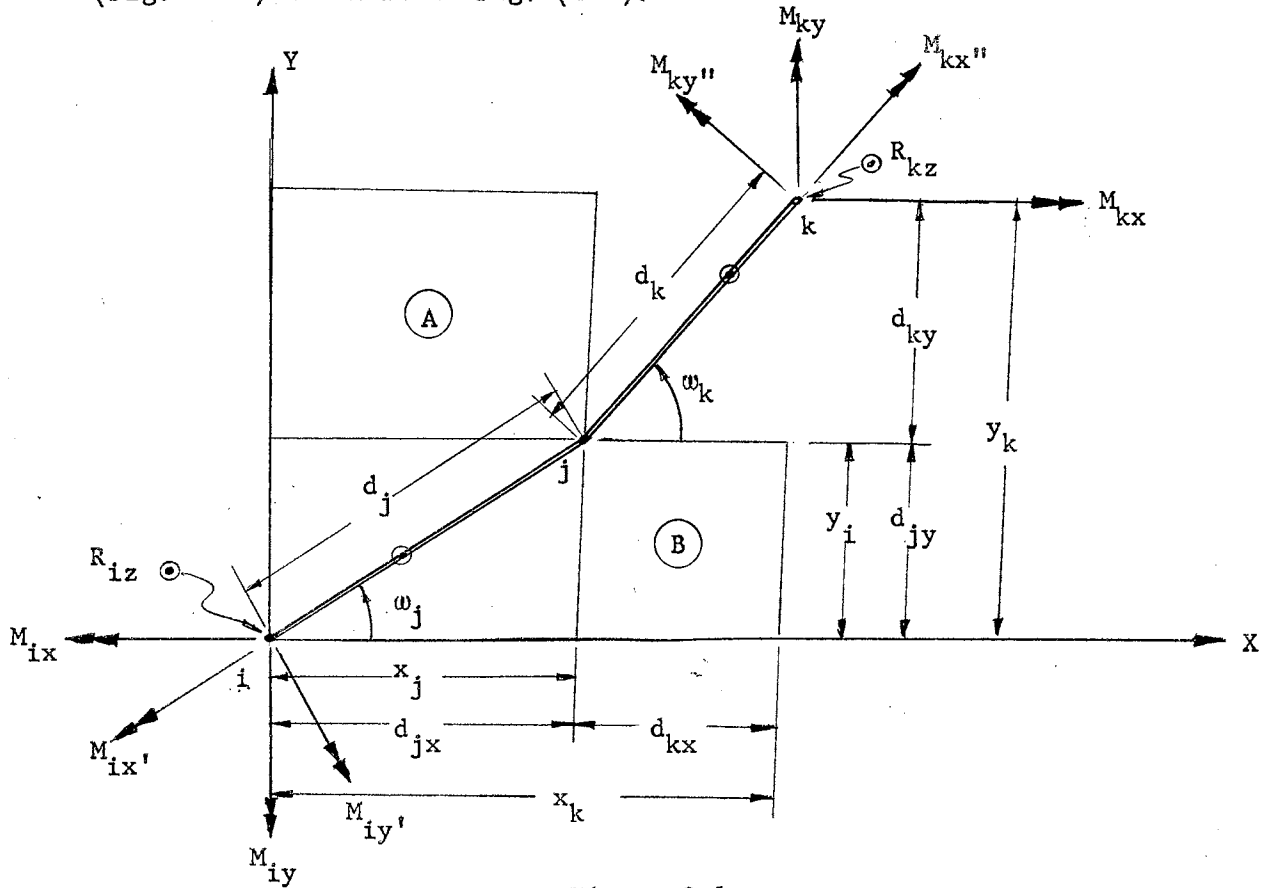


Figure 2-1

Basic Structure No. 1

The applied end moments at i and k about the principal axis of the member \overline{ij} and \overline{jk} are $M_{ix'}$, $M_{iy'}$ and $M_{kx''}$, $M_{ky''}$ respectively. $M_{ix'}$, $M_{iy'}$ and $M_{kx''}$, $M_{ky''}$ represent their transferred values about the basic system of reference XYZ.

Since the structure is in equilibrium under the action of the resultant of loads $\sum P_z$ assumed to act up, support reactions and basic end moments, the summation of moments at i about the X and Y axis is equal to zero.

Thus,

$$\sum M_x [@i] = 0$$

$$-M_{ix'} + M_{kx''} + SM_{ix'} + R_{jz} \cdot y_j + R_{kz} \cdot y_k = 0 \quad (2-1a)$$

$$\sum M_y [@i] = 0$$

$$-M_{iy'} + M_{ky''} - SM_{iy'} - R_{jz} x_j - R_{kz} x_k = 0 \quad (2-1b)$$

where $SM_{ix'}$ and $SM_{iy'}$ denote the static moments of loads at i about the X and Y axis respectively.

Rearranging equations (2-1a) and (2-1b)

$$R_{jz}(y_j) + R_{kz}(y_k) = M_{ix'} - M_{kx''} - SM_{ix'}$$

$$R_{jz}(-x_j) + R_{kz}(-x_k) = M_{iy'} - M_{ky''} + SM_{iy'}$$

Solving simultaneously,

$$R_{jz} = \frac{(M_{ix} - M_{kx} - SM_{ix})(-x_k) - (M_{iy} - M_{ky} + SM_{iy})(y_k)}{x_j y_k - x_k y_j} \quad (2-2a)$$

$$R_{kz} = \frac{(M_{ix} - M_{kx} - SM_{ix})(x_j) + (M_{iy} - M_{ky} + SM_{iy})(y_j)}{x_j y_k - x_k y_j} \quad (2-2b)$$

$$\text{Since } x_j y_k - x_k y_j = d_{jx}(d_{jy} + d_{ky}) - (d_{jx} + d_{kx})d_{jy}$$

$$= d_{jx} \cdot d_{ky} - d_{kx} \cdot d_{jy}$$

$$= \text{Area (A-B)} = C$$

(Fig. 2-1)

$$C = d_{jx} d_{ky} - d_{kx} d_{jy} \quad (2-2c)$$

$$R_{jz} = - \frac{(M_{ix} - M_{kx} - SM_{ix})(d_{jx} + d_{kx}) + (M_{iy} - M_{ky} + SM_{iy})(d_{jy} + d_{ky})}{C} \quad (2-3a)$$

$$R_{kz} = \frac{(M_{ix} - M_{kx} - SM_{ix})d_{jx} + (M_{iy} - M_{ky} + SM_{iy})d_{jy}}{C} \quad (2-3b)$$

Utilizing the third condition of static equilibrium in summing up the forces in the Z direction,

$$R_{iz} + R_{jz} + R_{kz} + \sum P_z = 0$$

$$R_{iz} = -R_{jz} - R_{kz} - \sum P_z$$

Substituting the values of R_{jz} and R_{kz} from Eq. (2-3a,b) and simplifying,

$$R_{iz} = (M_{ix'} - M_{kx'} - SM_{ix'}) \frac{d_{kx}}{C} + (M_{iy'} - M_{ky'} + SM_{iy'}) \frac{d_{ky}}{C} - \sum_z P_z \quad (2-3c)$$

To develop the expressions for the moments at a section in spans \overline{ij} and \overline{jk} of the basic structure, the free body diagrams shown in Fig. (2-2a) and (2-2b) are considered.

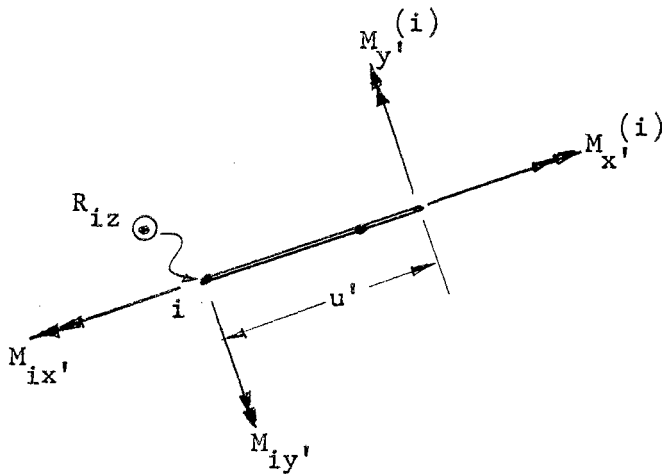


Figure 2-2a

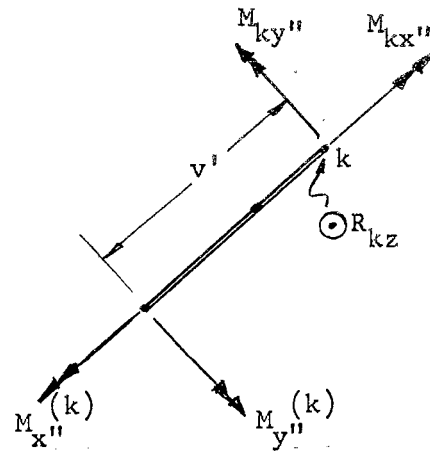


Figure 2-2b

Free-body diagrams.

Summing the moments at the section about the principal axes $X'Y'$ (Fig. 2-2a),

$$\sum M_{x'}^{(i)} = 0$$

$$M_{x'}^{(i)} = 0$$

$$M_{x'}^{(i)} = M_{ix'} \quad (2-4a)$$

where $M_{x'}^{(i)}$ denotes the twisting moment about the X' -axis at the section in the span \overline{ij} .

$$\sum M_{y'}^{(i)} = 0$$

$$-M_{iy'} + R_{iz} u^i + M_{y'}^{(i)} - SM_{y'} = 0$$

assuming that the resultant of loads between the support i and the section acts down.

Thus,

$$M_{y'}^{(i)} = M_{iy'} - R_{iz} \cdot u^i + SM_{y'} \quad (2-4b)$$

where $M_{y'}^{(i)}$ and $SM_{y'}$ represent the bending moment and the static moment due to loads alone, respectively, about the Y' -axis, at the section in the span \overline{ij} .

Similarly, considering the other free body (Fig. 2-2b)

$$\sum M_{x''}^{(k)} = 0$$

$$M_{x''}^{(k)} - M_{kx''} = 0$$

$$M_{x''}^{(k)} = M_{kx''} \quad (2-5a)$$

Also

$$\sum M_{y''}^{(k)} = 0$$

$$-M_{y''}^{(k)} + M_{ky''} - R_{kz} \cdot u' + SM_{y''} = 0$$

$$M_{y''}^{(k)} = M_{ky''} - R_{kz} \cdot u' + SM_{y''} \quad (2-5b)$$

where $M_{x''}^{(k)}$, $M_{y''}^{(k)}$ and $SM_{y''}$ have similar meanings as explained for the previous free body.

Equations (2-4a,b) and (2-5a,b) are expressed in terms of moments about the principal axes of members \bar{ij} and \bar{jk} . Using Tables 1-1 and 1-2 and Equations (2-3b,c), the above expressions can be put down in terms of basic end moments.

Thus

$$M_{x'}^{(i)} = M_{ix} \alpha_{jx} + M_{iy} \beta_{jx} \quad (2-6a)$$

$$M_{x''}^{(k)} = M_{kx} \alpha_{kx} + M_{ky} \beta_{kx} \quad (2-6b)$$

$$\begin{aligned}
M_y^{(i)} &= M_{ix} \cdot \alpha_{jy} + M_{iy} \cdot \beta_{jy} - \left[(M_{ix} - M_{kx} - SM_{ix}) \frac{d_{kx}}{C} + (M_{iy} - M_{ky} + SM_{iy}) \frac{d_{ky}}{C} + \sum P_z \right] (u') \\
&\quad + \left[+SM_x^{(i)} \cdot \alpha_{jy} + SM_y^{(i)} \cdot \beta_{jy} \right] \\
&= M_{ix} \cdot \alpha_{jy} + M_{iy} \cdot \beta_{jy} - \left[(M_{ix} - M_{kx}) d_{kx} + (M_{iy} - M_{ky}) d_{ky} \right] \left(\frac{u'}{C} \right) + BM_y^{(i)} \quad (2-6c)
\end{aligned}$$

$$\begin{aligned}
M_y^{(k)} &= M_{kx} \cdot \alpha_{ky} + M_{ky} \cdot \beta_{ky} - \left[(M_{ix} - M_{kx} - SM_{ix}) \frac{d_{ix}}{C} + (M_{iy} - M_{ky} + SM_{iy}) \frac{d_{iy}}{C} \right] (v') \\
&\quad + SM_x^{(k)} \alpha_{ky} + SM_y^{(k)} \beta_{ky} \\
&= M_{kx} \cdot \alpha_{ky} + M_{ky} \cdot \beta_{ky} - \left[(M_{ix} - M_{kx}) d_{jx} + (M_{iy} - M_{ky}) d_{jy} \right] \left(\frac{v'}{C} \right) + BM_y^{(k)} \quad (2-6d)
\end{aligned}$$

where $BM_y^{(i)}$ and $BM_y^{(k)}$ represent the bending moments at the section considered, in \bar{ij} and \bar{jk} respectively, due to loads alone.

$$BM_j^{(k)} = SM_x^{(k)} \cdot \alpha_{jy} + SM_y^{(i)} \cdot \beta_{jy} + \left[SM_{ix} \cdot d_{kx} - SM_{iy} \cdot d_{ky} + \sum P_z \right] \left(\frac{u'}{C} \right) \quad (2-7a)$$

$$BM_y^{(k)} = SM_x^{(k)} \cdot \alpha_{jy} + SM_y^{(k)} \cdot \beta_{jy} + \left[SM_{ix} \cdot d_{jx} - SM_{iy} \cdot d_{jy} \right] \cdot \left(\frac{v'}{C} \right) \quad (2-7b)$$

2-2 Statics of Basic Structure No. 2.

The basic structure \bar{mn} , isolated from the continuous bent beam in \bar{in}

(Fig. 1-2b) is shown in Fig. 2-3. This basic structure with end m simply supported and end n fixed is statically indeterminate to first degree and the end moment M_{nx} will be the selected unknown.

The structure is acted upon by the resultant of loads $\sum P_z$ acting up and the ends moments M_{mx}' , M_{my}' , and M_{nx}' , M_{ny}' , about the principle axis of the member at supports m and n respectively. The transferred values of these end moments about the basic system XYZ are denoted by M_{mx} , M_{my} and M_{nx} , M_{ny} respectively.

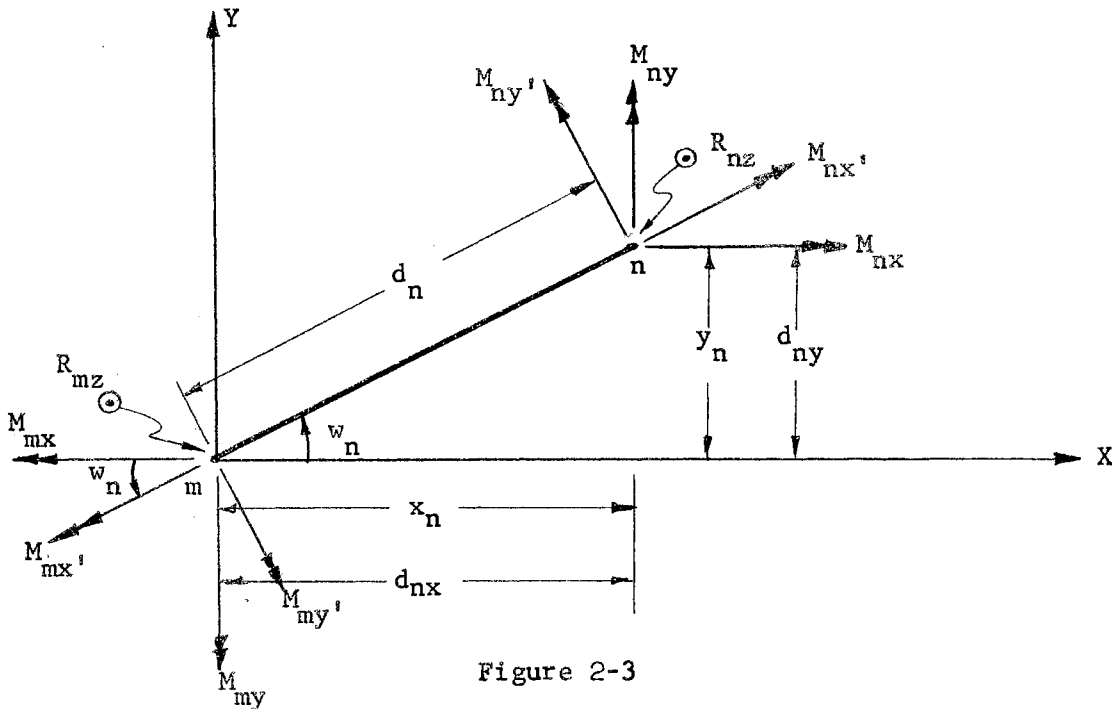


Figure 2-3

Basic Structure No. 2

Since

$$\sum M_x @ m = 0$$

$$-M_{mx} + M_{nx} + SM_{mx} + R_{nz} \cdot d_{ny} = 0$$

where SM_{mx} denotes the static moment of loads at m about the X-axis.

Solving for R_{nz} ,

$$R_{nz} = \frac{M_{mx} - M_{nx} - SM_{mx}}{d_{ny}} \quad (2-8a)$$

As the sum of forces in the Z-direction is zero,

$$R_{mz} + R_{nz} + \sum P_z = 0$$

$$R_{mz} = -R_{nz} - \sum P_z$$

$$= -\frac{M_{mx} - M_{nx} - SM_{mx}}{d_{ny}} - \sum P_z \quad (2-8b)$$

To calculate moments at a section in \overline{mn} , the free body diagram (Fig. 2-4) is considered.

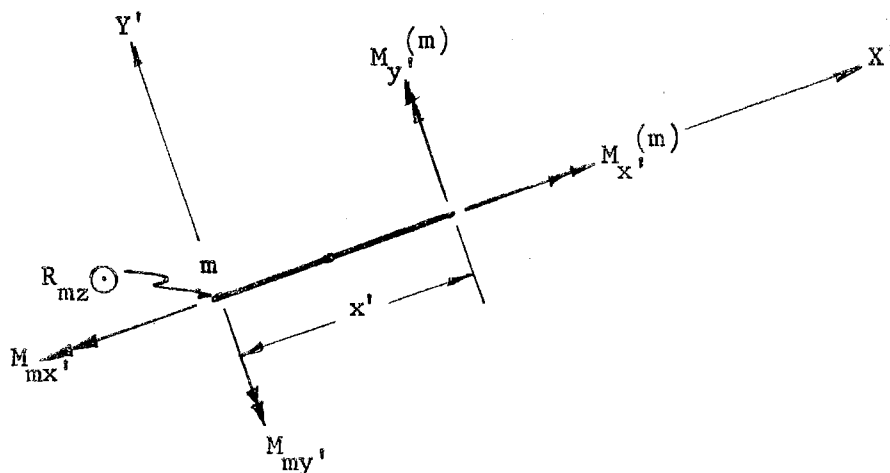


Figure 2-4

Free-body diagram.

Denoting by $M_{x'}^{(m)}$ and $M_{y'}^{(m)}$ the twisting moment and the bending moment respectively at the section considered, with reference to the support m ,

since

$$\sum M_{x'}^{(m)} = 0$$

$$M_{x'}^{(m)} - M_{mx'} = 0$$

$$M_{x'}^{(m)} = M_{mx'} \quad (2-9a)$$

Also

$$\sum M_{y'}^{(m)} = 0$$

$$M_{y'}^{(m)} - M_{my'} + R_{mz} \cdot x' - SM_{y'}^{(m)} = 0$$

assuming that the resultant of the loads between support m and the section acts down.

Thus,

$$M_{y'}^{(m)} = M_{my'} - R_{mz} \cdot x' + SM_{y'}^{(m)} \quad (2-9b)$$

where $SM_{y'}^{(m)}$ denotes the static moment due to loads at the section.

Using Eq. (2-8b) and transformation Table 1-3, Eq. (2-9a) and (2-9b) are expressed in terms of the basic end moments.

$$M_{x'}^{(m)} = M_{mx} \cdot \alpha_{nx} + M_{my} \cdot \beta_{nx} \quad (2-10a)$$

$$M_{y'}^{(m)} = M_{mx} \cdot \alpha_{ny} + M_{my} \cdot \beta_{ny} + \left[\frac{M_{mx} - M_{nx} - SM_{mx}}{d_{ny}} + \sum P_z \right] \cdot (x') \\ + SM_x^{(m)} \cdot \alpha_{ny} + SM_y^{(m)} \cdot \beta_{ny} \quad (2-10b)$$

Denoting

$$BM_y^{(m)} = \left[\sum P_z \frac{SM_{mx}}{d_{ny}} \right] \cdot x' + SM_x^{(m)} \cdot \alpha_{ny} + SM_y^{(m)} \cdot \beta_{ny} \quad (2-10c)$$

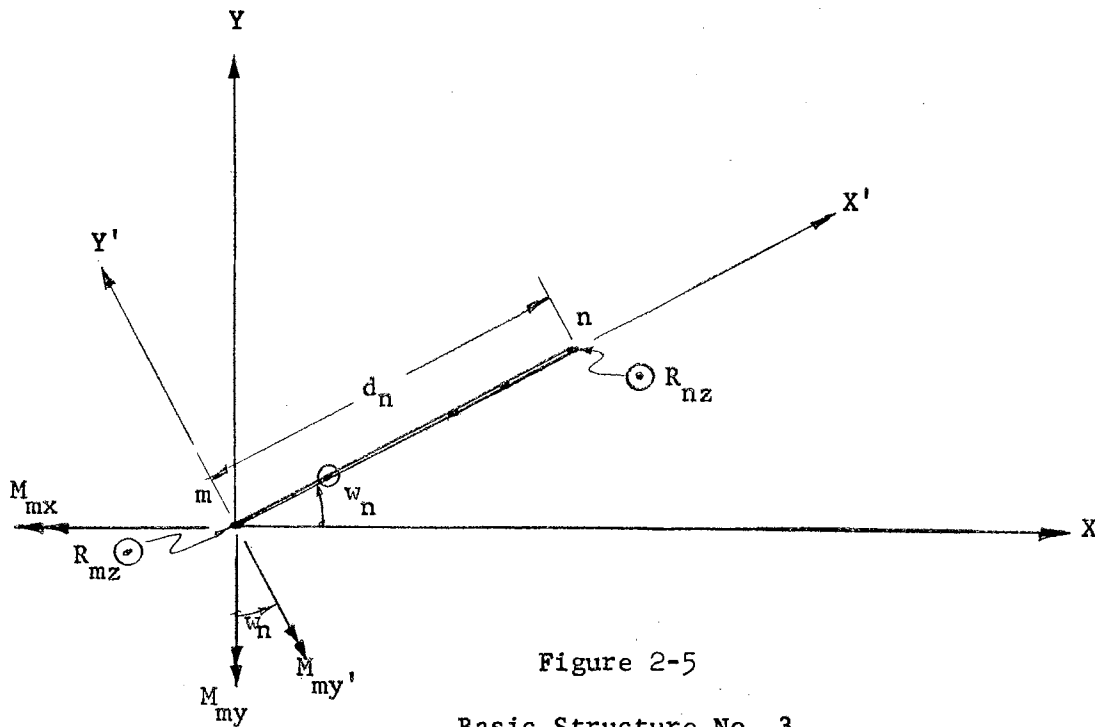
as the bending moment at the section due to loads alone, Eq. (2-10b) can be rewritten as

$$M_{y'}^{(m)} = M_{mx} \cdot \alpha_{ny} + M_{my} \cdot \beta_{ny} + (M_{mx} - M_{nx}) \cdot \frac{x'}{d_{ny}} + BM_y^{(m)} \quad (2-10d)$$

2-3 Statics of Basic Structure No. 3.

The single span \overline{mn} removed from the continuous bent beam in (Fig. 1-3b) is shown in Fig. 2-5. Since the basic structure is unable to resist any twisting moment as pointed out in discussing its geometry, it can be in equilibrium only under the action of:

- (i) the resultant of loads $\sum P_z$, assumed to act up.
- (ii) the end moment at m, M_{my} , about the Y' -axis.
- (iii) the vertical reactions, R_{mz} and R_{nz} .



The basic end moments M_{mx} and M_{my} denote the transferred values of the end moment M_{my}' .

Since

$$\sum M_y [\text{@n}] = 0$$

$$R_{nz} \cdot d_n + M_{my}' + SM_{my}' = 0$$

where SM_{my}' denotes the static moment about the Y' -axis at m due to loads.

Solving for R_{nz} ,

$$R_{nz} = - \frac{M_{my}' + SM_{my}'}{d_n} \quad (2-11a)$$

$$\sum F_z = 0$$

$$R_{mz} + R_{nz} + \sum P_z = 0$$

Utilizing Eq. (2-11a) and transposing,

$$R_{mz} = \frac{M_{my'} + SM_{my'}}{d_n} - \sum P_z \quad (2-11b)$$

To calculate moments at a section in \overline{mn} , the free-body diagram shown in Fig. 2-6 is considered.

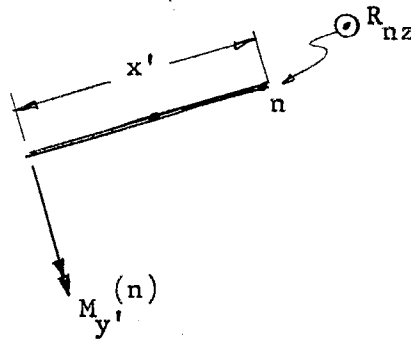


Figure 2-6

Free-body diagram

Since the end n is simply supported, only the bending moment, $M_{y'}^{(n)}$ at the section exists.

$$\sum M_{y'}^{(n)} = 0$$

$$M_{y'}^{(n)} + R_{nz} \cdot x' - SM_{y'}^{(n)} = 0$$

assuming that the resultant of loads between support n and the section acts down.

$$M_{y'}^{(n)} = -R_{nz} \cdot x' + SM_{y'}^{(n)} \quad (2-12a)$$

where $SM_{y'}^{(n)}$ refers to the static moment at the section due to loads.

Substituting for R_{nz} from Eq. (2-11a),

$$M_{y'}^{(n)} = \left[\frac{M_{my'} + SM_{my'}}{d_n} \right] \cdot x' + SM_{y'}^{(n)} \quad (2-12b)$$

Eq. (2-12b) is in terms of moments about the principal axis of the span \overline{mn} . Using transformation Table 1-3, it may be expressed in terms of moments related to the basic system XYZ.

Thus,

$$M_{y'}^{(n)} = (M_{mx} \cdot \alpha_{ny} + M_{ny} \cdot \beta_{ny} + SM_{mx} \cdot \alpha_{ny} + SM_{my} \cdot \beta_{ny}) \cdot \frac{x'}{d_n} + SM_x^{(n)} \cdot \alpha_{ny} + SM_y^{(n)} \cdot \beta_{ny} \quad (2-12c)$$

Denoting

$$BM_y^{(n)} = (SM_{mx} \cdot \alpha_{ny} + SM_{my} \cdot \beta_{ny}) \cdot \frac{x'}{d_n} + SM_x^{(n)} \cdot \alpha_{ny} + SM_y^{(n)} \cdot \beta_{ny} \quad (2-13)$$

as the bending moment at the section due to loads alone,

Eq. (2-12c) can be rewritten as

$$M_{y'}^{(n)} = (M_{mx} \cdot \alpha_{ny} + M_{my} \cdot \beta_{ny}) \frac{x'}{d_n} + BM_y^{(n)} \quad (2-14)$$

CHAPTER III

DEFORMATION OF BASIC STRUCTURE

The analytical expressions for the angular functions of the basic structures discussed earlier are now derived. These angular functions can be obtained by using several methods of analysis such as the Area-Moment method, the Elastic-Weights method, the String Polygon method, the Virtual Work method and the Castigliano's method. In the following derivation, the Castigliano's method is used.

3-1 Basic Structure No. 1.

The basic structure \overline{ijk} (Fig. 2-1) has four applied end moments M_{ix} , M_{iy} and M_{kx} , M_{ky} at its external supports i and k respectively. Thus, from Castigliano's theorem, it is possible to obtain four equations of end slopes in terms of the angular functions and these end moments. The following angular functions can be expected at the support k .

- (1) The angular flexibility F_{kixx}
- (2) The angular flexibility F_{kiyy}
- (3) The angular flexibility F_{kixy} ($=F_{kiyx}$)
- (4) The carry-over angular flexibility G_{ikxx}
- (5) The carry-over angular flexibility G_{ikyy}
- (6) The carry-over angular flexibility G_{ikyx}
- (7) The carry-over angular flexibility G_{ikxy}

(8) The angular load function τ_{kixx}

(9) The angular load function τ_{kiyy}

For constant cross-sections of elemental lengths in spans \overline{ij} and \overline{jk} , the elemental angular flexibilities are:

$$\lambda_{x'} = \frac{d_{x'}}{GJ_{x'}} \qquad \lambda_{y'} = \frac{d_{y'}}{EI_{y'}} \qquad (3-1)$$

$$\lambda_{x''} = \frac{d_{x''}}{GJ_{x''}} \qquad \lambda_{y''} = \frac{d_{y''}}{EI_{y''}}$$

The strain energy of the basic structure \overline{ijk} is :

$$\begin{aligned} U_{ijk} &= U_{ijx'} + U_{ijy'} + U_{jcx''} + U_{jky''} \\ &= \frac{1}{2} \left[\int_i^j [M_{x'}^{(i)}]^2 \cdot \lambda_{x'} + \int_i^j [M_{y'}^{(i)}]^2 \cdot \lambda_{y'} + \int_k^j [M_{x''}^{(k)}]^2 \cdot \lambda_{x''} + \int_k^j [M_{y''}^{(k)}]^2 \cdot \lambda_{y''} \right] \end{aligned} \qquad (3-2)$$

Partial differentiation of Eq. (3-2) with respect to M_{kx} , M_{ky} , M_{ix} and M_{iy} gives four deformation equations such as

$$\begin{aligned}
\frac{\delta U_{ijk}}{\delta M_{kx}} = & \int_i^j \left[M_{x'}^{(i)} \right] \left[\frac{\delta M_{x'}^{(i)}}{\delta M_{kx}} \right] \cdot \lambda_{x'} + \int_i^j \left[M_{y'}^{(i)} \right] \left[\frac{\delta M_{y'}^{(i)}}{\delta M_{kx}} \right] \cdot \lambda_{y'} \\
& + \int_k^j \left[M_{x''}^{(k)} \right] \left[\frac{\delta M_{x''}^{(k)}}{\delta M_{kx}} \right] \cdot \lambda_{x''} + \int_k^j \left[M_{y''}^{(k)} \right] \left[\frac{\delta M_{y''}^{(k)}}{\delta M_{kx}} \right] \cdot \lambda_{y''} \quad (3-3)
\end{aligned}$$

with correspondingly similar expressions for $\frac{\delta U_{ijk}}{\delta M_{ky}}$, $\frac{\delta U_{ijk}}{\delta M_{ix}}$ and $\frac{\delta U_{ijk}}{\delta M_{iy}}$.

Substituting in Eq. (3-3), the values of the moments at a section in spans \overline{ij} and \overline{jk} from Eq. (2-6a,b,c,d) and applying unit moments at ends i and k , the analytical expressions for the angular functions at the support k as denoted earlier and the counterpart expressions at the support i may be developed. This is done, and along with the load functions, recorded in Tables 3-2 and 3-3. Table 3-1 contains a list of the first partials required to obtain these angular functions.

Utilizing these angular functions (F's, G's, and τ 's), the deformation equations (Eq. 3-3) are now expressed as:

$$\frac{\delta U_{ijk}}{\delta M_{kx}} = \frac{M_{ix} G_{ikxx} + M_{kx} F_{kixx}}{M_{iy} G_{ikyx} + M_{ky} F_{kiyx}} + \tau_{kixx} \quad (3-4a)$$

$$\frac{\delta U_{ijk}}{\delta M_{ky}} = \frac{M_{ix} G_{ikxy} + M_{ky} F_{kixy}}{M_{iy} G_{iky y} + M_{ky} F_{kiyy}} + \tau_{kiyy} \quad (3-4b)$$

Partial Derivatives		Basic Structure No. 1
First Partial		First Partial Values
$\frac{\delta M_{x'}}{\delta M_{ix}}$		α_{jx}
$\frac{\delta M_{y'}}{\delta M_{ix}}$		$\alpha_{jy} - u' \cdot \frac{d_{kx}}{C}$
$\frac{\delta M_{x''}}{\delta M_{ix}} ; \frac{\delta M_{x'}}{\delta M_{kx}}$		0
$\frac{\delta M_{y''}}{\delta M_{ix}}$		$-v' \cdot \frac{d_{jx}}{C}$
$\frac{\delta M_{x'}}{\delta M_{iy}}$		β_{jx}
$\frac{\delta M_{y'}}{\delta M_{iy}}$		$\beta_{jy} - u' \cdot \frac{d_{ky}}{C}$
$\frac{\delta M_{x''}}{\delta M_{iy}} ; \frac{\delta M_{x'}}{\delta M_{ky}}$		0
$\frac{\delta M_{y''}}{\delta M_{iy}}$		$-v' \cdot \frac{d_{jy}}{C}$
$\frac{\delta M_{y'}}{\delta M_{kx}}$		$u' \cdot \frac{d_{kx}}{C}$
$\frac{\delta M_{x''}}{\delta M_{kx}}$		α_{kx}
$\frac{\delta M_{y''}}{\delta M_{kx}}$		$\alpha'_{ky} + v' \cdot \frac{d_{jx}}{C}$
$\frac{\delta M_{y'}}{\delta M_{ky}}$		$u' \cdot \frac{d_{ky}}{C}$
$\frac{\delta M_{y''}}{\delta M_{ky}}$		$\beta_{ky} + v' \cdot \frac{d_{jy}}{C}$

Table 3-1

F_{ikxx}	$\frac{\delta U_{ijk}}{\delta M_{ix}}$	$M_{ix}=+1.0; M_{iy}=M_{kx}=M_{ky}=Loads=0$	$\int_1^j \alpha_{jx}^2 \cdot \lambda_{x'} + \int_1^j (\alpha_{jy} - d_{kx} \cdot \frac{u'}{C})^2 \cdot \lambda_{y'} + \int_k^j (-d_{jx} \cdot \frac{v'}{C})^2 \cdot \lambda_{y''}$
F_{ikyy}	$\frac{\delta U_{ijk}}{\delta M_{iy}}$	$M_{iy}=+1.0; M_{ix}=M_{kx}=M_{ky}=Loads=0$	$\int_1^j \beta_{jx}^2 \cdot \lambda_{x'} + \int_1^j (\beta_{jy} - d_{ky} \cdot \frac{u'}{C})^2 \cdot \lambda_{y'} + \int_k^j (-d_{jy} \cdot \frac{v'}{C})^2 \cdot \lambda_{y''}$
F_{ikyx}	$\frac{\delta U_{ijk}}{\delta M_{ix}}$	$M_{iy}=+1.0; M_{ix}=M_{kx}=M_{ky}=Loads=0$	$\int_1^j \alpha_{jx} \cdot \beta_{jx} \cdot \lambda_{x'} + \int_1^j (\alpha_{jy} - d_{kx} \cdot \frac{u'}{C})(\beta_{ky} - d_{ky} \cdot \frac{u'}{C}) \lambda_{y'} + \int_k^j (-d_{jx} \cdot \frac{v'}{C})(-d_{jy} \cdot \frac{v'}{C}) \lambda_{y''}$
F_{ikxy}	$\frac{\delta U_{ijk}}{\delta M_{iy}}$	$M_{ix}=+1.0; M_{iy}=M_{kx}=M_{ky}=Loads=0$	
F_{kixx}	$\frac{\delta U_{ijk}}{\delta M_{kx}}$	$M_{kx}=+1.0; M_{ix}=M_{kx}=M_{ky}=Loads=0$	$\int_k^j \alpha_{kx}^2 \cdot \lambda_{x''} + \int_k^j (\alpha_{ky} + d_{jx} \cdot \frac{v'}{C})^2 \cdot \lambda_{y''} + \int_1^j (d_{kx} \cdot \frac{u'}{C})^2 \cdot \lambda_{y'}$
F_{kiyy}	$\frac{\delta U_{ijk}}{\delta M_{ky}}$	$M_{ky}=+1.0; M_{ix}=M_{iy}=M_{ky}=Loads=0$	$\int_k^j \beta_{kx}^2 \cdot \lambda_{x''} + \int_k^j (\beta_{ky} + d_{jy} \cdot \frac{v'}{C})^2 \cdot \lambda_{y''} + \int_1^j (d_{ky} \cdot \frac{u'}{C})^2 \cdot \lambda_{y'}$
F_{kiyx}	$\frac{\delta U_{ijk}}{\delta M_{kx}}$	$M_{ky}=+1.0; M_{ix}=M_{iy}=M_{kx}=Loads=0$	$\int_k^j \alpha_{kx} \cdot \beta_{kx} \cdot \lambda_{x''} + \int_k^j (\alpha_{ky} + d_{jx} \cdot \frac{v'}{C})(\beta_{ky} + d_{jy} \cdot \frac{v'}{C}) \lambda_{y''} + \int_1^j (d_{kx} \cdot \frac{u'}{C})(d_{ky} \cdot \frac{u'}{C}) \lambda_{y'}$
F_{kixy}	$\frac{\delta U_{ijk}}{\delta M_{ky}}$	$M_{kx}=+1.0; M_{ix}=M_{iy}=M_{ky}=Loads=0$	

Table 3-2

Angular Carry-Over Flexibilities and Load Functions

Basic Structure No. 1

G_{ikxx}	$\frac{\delta U_{ijk}}{\delta M_{kx}}$	$M_{ix}=+1.0; M_{iy}=M_{kx}=M_{ky}=Loads=0$	$\int_1^j (\alpha_{jy} - d_{kx} \frac{u^i}{C})(d_{kx} \frac{u^i}{C}) \lambda_y + \int_k^j (\alpha_{ky} + d_{jx} \frac{v^i}{C})(-d_{jx} \frac{v^i}{C}) \lambda_y$
G_{kixx}	$\frac{\delta U_{ijk}}{\delta M_{ix}}$	$M_{kx}=+1.0; M_{iy}=M_{kx}=M_{ky}=Loads=0$	
G_{ikyy}	$\frac{\delta U_{ijk}}{\delta M_{ky}}$	$M_{iy}=+1.0; M_{ix}=M_{kx}=M_{ky}=Loads=0$	$\int_1^j (\beta_{jy} - d_{ky} \frac{u^i}{C})(d_{ky} \frac{u^i}{C}) \lambda_y + \int_k^j (\beta_{ky} + d_{jy} \frac{v^i}{C})(-d_{jy} \frac{v^i}{C}) \lambda_y$
G_{kiyy}	$\frac{\delta U_{ijk}}{\delta M_{iy}}$	$M_{ky}=+1.0; M_{ix}=M_{kx}=M_{ky}=Loads=0$	
G_{ikxy}	$\frac{\delta U_{ijk}}{\delta M_{ky}}$	$M_{ix}=+1.0; M_{kx}=M_{ky}=M_{ix}=Loads=0$	$\int_1^j (\beta_{jy} - d_{ky} \frac{u^i}{C})(d_{kx} \frac{u^i}{C}) \lambda_y + \int_k^j (\alpha_{ky} + d_{jx} \frac{v^i}{C})(-d_{jy} \frac{v^i}{C}) \lambda_y$
G_{kiyx}	$\frac{\delta U_{ijk}}{\delta M_{ix}}$	$M_{ky}=+1.0; M_{ix}=M_{iy}=M_{kx}=Loads=0$	
G_{ikyx}	$\frac{\delta U_{ijk}}{\delta M_{kx}}$	$M_{iy}=+1.0; M_{ix}=M_{kx}=M_{ky}=Loads=0$	$\int_1^j (\alpha_{jy} - d_{kx} \frac{u^i}{C})(d_{ky} \frac{u^i}{C}) \lambda_y + \int_k^j (\beta_{ky} + d_{jy} \frac{v^i}{C})(-d_{jx} \frac{v^i}{C}) \lambda_y$
G_{kiyx}	$\frac{\delta U_{ijk}}{\delta M_{iy}}$	$M_{kx}=+1.0; M_{ix}=M_{iy}=M_{ky}=Loads=0$	
T_{ikxx}	$\frac{\delta U_{ijk}}{\delta M_{ix}}$	$M_{ix}=M_{iy}=M_{kx}=M_{ky}=0$	$\int_1^j BM_y^{(i)} (\alpha_{jy} - d_{kx} \frac{u^i}{C}) \cdot \lambda_y + \int_k^j BM_y^{(k)} (-d_{jx} \frac{v^i}{C}) \cdot \lambda_y$
T_{ikyy}	$\frac{\delta U_{ijk}}{\delta M_{iy}}$	$M_{ix}=M_{iy}=M_{kx}=M_{ky}=0$	$\int_1^j BM_y^{(i)} (\beta_{jy} - d_{ky} \frac{u^i}{C}) \cdot \lambda_y + \int_k^j BM_y^{(k)} (-d_{jy} \frac{v^i}{C}) \cdot \lambda_y$
T_{kixx}	$\frac{\delta U_{ijk}}{\delta M_{kx}}$	$M_{ix}=M_{iy}=M_{kx}=M_{ky}=0$	$\int_1^j BM_y^{(i)} (d_{kx} \frac{u^i}{C}) \cdot \lambda_y + \int_k^j BM_y^{(k)} (\alpha_{ky} + d_{kx} \frac{v^i}{C}) \cdot \lambda_y$
T_{kiyy}	$\frac{\delta U_{ijk}}{\delta M_{ky}}$	$M_{ix}=M_{iy}=M_{kx}=M_{ky}=0$	$\int_1^j BM_y^{(i)} (d_{ky} \frac{u^i}{C}) \cdot \lambda_y + \int_k^j BM_y^{(k)} (\beta_{ky} + d_{ky} \frac{v^i}{C}) \cdot \lambda_y$

Table 3-3

$$\frac{\delta U_{ijk}}{\delta M_{ix}} = \frac{M_{ix} F_{ikxx} + M_{kx} G_{kixx}}{M_{iy} F_{ikyx} + M_{ky} G_{kiyx}} + \tau_{ikxx} \quad (3-4c)$$

$$\frac{\delta U_{ijk}}{\delta M_{iy}} = \frac{M_{ix} F_{ikxy} + M_{kx} G_{kixy}}{M_{iy} F_{iky y} + M_{ky} G_{kiyy}} + \tau_{iky y} \quad (3-4d)$$

3-2 Basic Structure No. 2.

The basic structure \overline{mn} (Fig. 2-3) has four end moments M_{mx} , M_{my} and M_{nx} , M_{ny} at supports m and n respectively. As M_{ny} can be obtained from statics, only three deformation equations, two at support m and one at support n need to be derived. They are expressed, as before, in terms of the angular functions and the end moments M_{mx} , M_{my} and M_{nx} .

The following angular functions are expected.

(a) At support m in the x direction

- (i) The angular flexibility F_{mnxx}
- (ii) The angular flexibility F_{mnyx}
- (iii) The carry-over flexibility G_{nmxx}
- (iv) The angular load function τ_{mnxx}

(b) At support m in the y direction.

- (i) The angular flexibility F_{mnyy}
- (ii) The angular flexibility F_{mnxy} ($=F_{mnyx}$)
- (iii) The angular carry-over flexibility G_{mnxy}
- (iv) The angular load function τ_{mnyy}

(c) At support n in the x direction

- (i) The angular flexibility F_{nmxx}

- (ii) The angular carry-over value $G_{mnxx} (=G_{nmxx})$
- (iii) The angular carry-over value $G_{mnyx} (=G_{nmxy})$
- (iv) The angular load function τ_{nmxx}

The strain energy of the basic structure \bar{mn} is:

$$\begin{aligned}
 U_{mn} &= U_{mnx} + U_{mny} \\
 &= \frac{1}{2} \int_m^n [M_{x'}^{(m)}]^2 \cdot \lambda_{x'} + \frac{1}{2} \int_m^n [M_{y'}^{(m)}]^2 \cdot \lambda_{y'} \quad (3-5)
 \end{aligned}$$

where $\lambda_{x'}$ and $\lambda_{y'}$ have similar meanings as explained before (Eq. 3-2).

Partial differentiation of Eq. (3-5) with respect to M_{mx} , M_{my} and M_{nx} gives three deformation conditions

$$\frac{\delta U_{mn}}{\delta M_{mx}} = \int_m^n [M_{x'}^{(m)}] \left[\frac{\delta M_{x'}^{(m)}}{\delta M_{mx}} \right] \cdot \lambda_{x'} + \int_m^n [M_{y'}^{(m)}] \left[\frac{\delta M_{y'}^{(m)}}{\delta M_{mx}} \right] \cdot \lambda_{y'} \quad (3-6a)$$

$$\frac{\delta U_{mn}}{\delta M_{my}} = \int_m^n [M_{x'}^{(m)}] \left[\frac{\delta M_{x'}^{(m)}}{\delta M_{my}} \right] \cdot \lambda_{x'} + \int_m^n [M_{y'}^{(m)}] \left[\frac{\delta M_{y'}^{(m)}}{\delta M_{my}} \right] \cdot \lambda_{y'} \quad (3-6b)$$

$$\frac{\delta U_{mn}}{\delta M_{nx}} = \int_m^n [M_{x'}^{(m)}] \left[\frac{\delta M_{x'}^{(m)}}{\delta M_{nx}} \right] \cdot \lambda_{x'} + \int_m^n [M_{y'}^{(m)}] \left[\frac{\delta M_{y'}^{(m)}}{\delta M_{nx}} \right] \cdot \lambda_{y'} \quad (3-6c)$$

Substituting from Eq. (2-10a,d), the values of the moments at a section in span \bar{mn} , in Eq. (3-6a,b,c) and applying appropriate unit

moments at m and n, analytical expressions for angular functions as denoted earlier are obtained. These expressions (F's, G's and τ 's) are presented in Table 3-5 and a list of the first partials required for deriving them are recorded in Table 3-4.

Partial Derivatives	
Basic Structure No. 2	
First Partial	First Partial Value
$\frac{\delta M_{x'}}{\delta M_{mx}}$	α_{nx}
$\frac{\delta M_{y'}}{\delta M_{mx}}$	$(\alpha_{ny} + \frac{x'}{d_{ny}})$
$\frac{\delta M_{x'}}{\delta M_{my}}$	β_{nx}
$\frac{\delta M_{y'}}{\delta M_{my}}$	β_{ny}
$\frac{\delta M_{y'}}{\delta M_{nx}}$	$-\frac{x'}{d_{ny}}$

Table 3-4

In terms of these angular functions and the moments M_{mx} , M_{my} and M_{nx} , the deformation equations are:

$$\frac{\delta U_{mn}}{\delta M_{mx}} = \frac{M_{mx} F_{mxxx} + M_{nx} G_{nmxx}}{M_{my} F_{mnyx}} + \tau_{mxxx} \quad (3-7a)$$

Angular Flexibilities, Carry-Over Flexibilities and Load Functions

Basic Structure No. 2

F_{mnxx}	$\frac{\delta U_{mn}}{\delta M_{mx}}$	$M_{mx}=1.0; M_{my}=M_{nx}=Loads=0$	$\int_m^n \alpha_{nx}^2 \cdot \lambda_{x'} + \int_m^n (\alpha_{ny} + \frac{x'}{d_{ny}})^2 \cdot \lambda_{y'}$
F_{mnyx}	$\frac{\delta U_{mn}}{\delta M_{mx}}$	$M_{my}=1.0; M_{mx}=M_{nx}=Loads=0$	$\int_m^n \alpha_{nx} \cdot \beta_{nx} \cdot \lambda_{x'} + \int_m^n (\alpha_{ny} + \frac{x'}{d_{ny}}) \cdot \beta_{ny} \cdot \lambda_{y'}$
F_{mnxy}	$\frac{\delta U_{mn}}{\delta M_{my}}$	$M_{mx}=1.0; M_{my}=M_{nx}=Loads=0$	
F_{mnyy}	$\frac{\delta U_{mn}}{\delta M_{my}}$	$M_{my}=1.0; M_{mx}=M_{nx}=Loads=0$	$\int_m^n \beta_{nx}^2 \cdot \lambda_{x'} + \int_m^n \beta_{ny}^2 \cdot \lambda_{y'}$
F_{nmxx}	$\frac{\delta U_{mn}}{\delta M_{nx}}$	$M_{nx}=1.0; M_{mx}=M_{my}=Loads=0$	$\int_m^n (-\frac{x'}{d_{ny}})^2 \cdot \lambda_{y'}$
G_{nmxx}	$\frac{\delta U_{mn}}{\delta M_{mx}}$	$M_{nx}=1.0; M_{mx}=M_{my}=Loads=0$	$\int_m^n (\alpha_{ny} + \frac{x'}{d_{ny}})(-\frac{x'}{d_{ny}}) \cdot \lambda_{y'}$
G_{nmxy}	$\frac{\delta U_{mn}}{\delta M_{nx}}$	$M_{mx}=1.0; M_{nx}=M_{my}=Loads=0$	
G_{mnyx}	$\frac{\delta U_{mn}}{\delta M_{nx}}$	$M_{my}=1.0; M_{mx}=M_{nx}=Loads=0$	$\int_m^n (\beta_{ny})(-\frac{x'}{d_{ny}}) \cdot \lambda_{y'}$
G_{mnxy}	$\frac{\delta U_{mn}}{\delta M_{my}}$	$M_{nx}=1.0; M_{mx}=M_{my}=Loads=0$	
T_{mnxx}	$\frac{\delta U_{mn}}{\delta M_{mx}}$	$M_{nx}=M_{my}=M_{nx}=0$	$\int_m^n BM_y^{(m)} \cdot (\alpha_{ny} + \frac{x'}{d_{ny}}) \cdot \lambda_{y'}$
T_{mnyy}	$\frac{\delta U_{mn}}{\delta M_{my}}$	$M_{mx}=M_{my}=M_{nx}=0$	$\int_m^n BM_y^{(m)} \cdot (\beta_{ny}) \cdot \lambda_{y'}$
T_{nmxx}	$\frac{\delta U_{mn}}{\delta M_{nx}}$	$M_{mx}=M_{my}=M_{nx}=0$	$\int_m^n BM_y^{(m)} \cdot (-\frac{x'}{d_{ny}}) \cdot \lambda_{y'}$

Table 3-5

$$\frac{\delta U_{mn}}{\delta M_{my}} = \frac{M_{mx} F_{mnxy} + M_{nx} G_{nmxy}}{M_{my} F_{mnyy}} + \tau_{mnyy} \quad (3-7b)$$

$$\frac{\delta U_{mn}}{\delta M_{nx}} = \frac{M_{mx} G_{mnxx} + M_{nx} F_{nmxx}}{M_{my} G_{mnyx}} + \tau_{nmxx} \quad (3-7c)$$

3-3 Basic Structure No. 3.

In this case, the basic structure \overline{mn} has two end moments M_{mx} and M_{my} at support m. The moment M_{my} can be obtained from statics and as such, only one equation of slope at m along the X-direction is required to be derived. As done in previous cases, it is expressed in terms of angular functions and the end moments M_{mx} , M_{my} . Since there cannot be any moments at end support n, being the simply-supported end of a continuous structure \overline{in} (Fig. 1-3b), the expected angular functions at end m along the X-axis are:

- (i) The angular flexibility F_{mnxx}
- (ii) The angular flexibility F_{mnyx}
- (iii) The angular load function τ_{mnxx}

The strain energy (of bending) of the basic structure is:

$$U_{mny'} = \frac{1}{2} \int_m^n [M_{y'}^{(n)}]^2 \cdot \lambda_{y'} \quad (3-8)$$

where $\lambda_{y'}$ has been explained before (Eq. 3-2). By Castigliano's theorem, the deformation equation at m along the X-axis is

$$\frac{\delta U_{mn}}{\delta M_x} = \int_m^n [M_{y'}^{(n)}] \left[\frac{\delta M_{y'}^{(n)}}{\delta M_{mx}} \right] \cdot \lambda_{y'} \quad (3-9a)$$

Substituting for $M_{y'}^{(n)}$ from Eq. (2-14)

$$\frac{\delta U_{mn}}{\delta M_{mx}} = \int_m^n \left[(M_{mx} \cdot \alpha_{ny} + M_{my} \cdot \beta_{ny}) \left(\frac{x'}{d_n} \right) + \beta M_y^{(n)} \right] \left(\alpha_{ny} \cdot \frac{x'}{d_n} \right) \cdot \lambda_{y'} \quad (3-9b)$$

Eq. (3-9) is rewritten as

$$\begin{aligned} \frac{\delta U_{mn}}{\delta M_{mx}} = & M_{mx} \int_m^n \alpha_{ny}^2 \cdot \left(\frac{x'}{d_{ny}} \right)^2 \cdot \lambda_{y'} + M_{my} \int_m^n \alpha_{ny} \cdot \beta_{ny} \cdot \left(\frac{x'}{d_{ny}} \right)^2 \cdot \lambda_{y'} \\ & + \int_m^n \beta M_y^{(n)} \cdot \alpha_{ny} \cdot \left(\frac{x'}{d_{ny}} \right) \cdot \lambda_{y'} \quad (3-9c) \end{aligned}$$

From Eq. (3-9c), it is apparent that the expressions in the integrals are the three angular functions stated before.

Thus

$$\frac{\delta U_{mn}}{\delta M_{mx}} = M_{mx} \cdot F_{mnxx} + M_{my} \cdot F_{mnyx} + \tau_{mnxx} \quad (3-10)$$

where

$$F_{mnxx} = \int_m^n \alpha_{ny}^2 \cdot \left(\frac{x'}{d_{ny}}\right)^2 \cdot \lambda_{y'} \cdot \lambda_{y'} \cdot \lambda_{y'}$$

$$F_{mnyx} = \int_m^n \alpha_{ny} \cdot \beta_{ny} \cdot \left(\frac{x'}{d_{ny}}\right)^2 \cdot \lambda_{y'} \cdot \lambda_{y'} \cdot \lambda_{y'}$$

$$r_{mnxx} = \int_m^n BM_y^{(n)} \cdot \alpha_{ny} \cdot \left(\frac{x'}{d_{ny}}\right) \cdot \lambda_{y'} \cdot \lambda_{y'} \cdot \lambda_{y'}$$

(3-11)

CHAPTER IV

COMPATIBILITY EQUATIONS

The compatibility equations of deformation over several supports of a continuous bent beam are derived. For a planar continuous bent beam subjected to out of plane loading, the condition of consistent deformations over a support would result in a compatibility equation. As such, there will be as many compatibility equations as there are redundant support moments.

Such a set of compatibility equations is put in a matrix form. Carry over moment equations are derived from these compatibility equations. A neat and efficient carry over procedure is evolved, which can be used to solve for the redundant moments, as is illustrated algebraically and in the numerical example (Chap. V).

4-1 Derivation.

Consider the odd span continuous bent beams

- (i) \overline{gn} with end g simply supported and end n fixed (Fig. 4-1).
- (ii) \overline{gp} with end g fixed and end p simply supported (Fig. 4-2).

The isolated portion \overline{ikm} of these two structures can be separated into two basic structures \overline{ijk} and $\overline{k\overline{lm}}$, of the type classified earlier as basic structure No. 1. Consider the continuous support k . Using Castigliano's theorem, the first partial of the strain energy U_{ikm} with respect to the redundant moments M_{kx} and M_{ky} must vanish. In other words,

the summation of all slopes at k along the X-axis and the Y-axis must each be equal to zero.

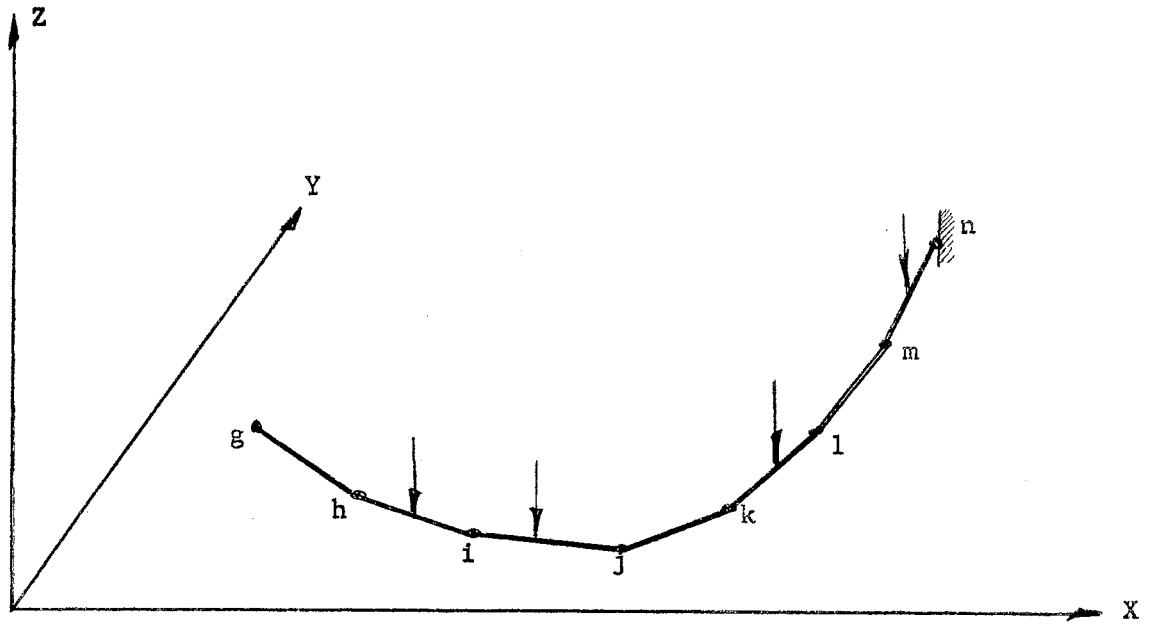


Figure 4-1

Continuous Bent Beam.

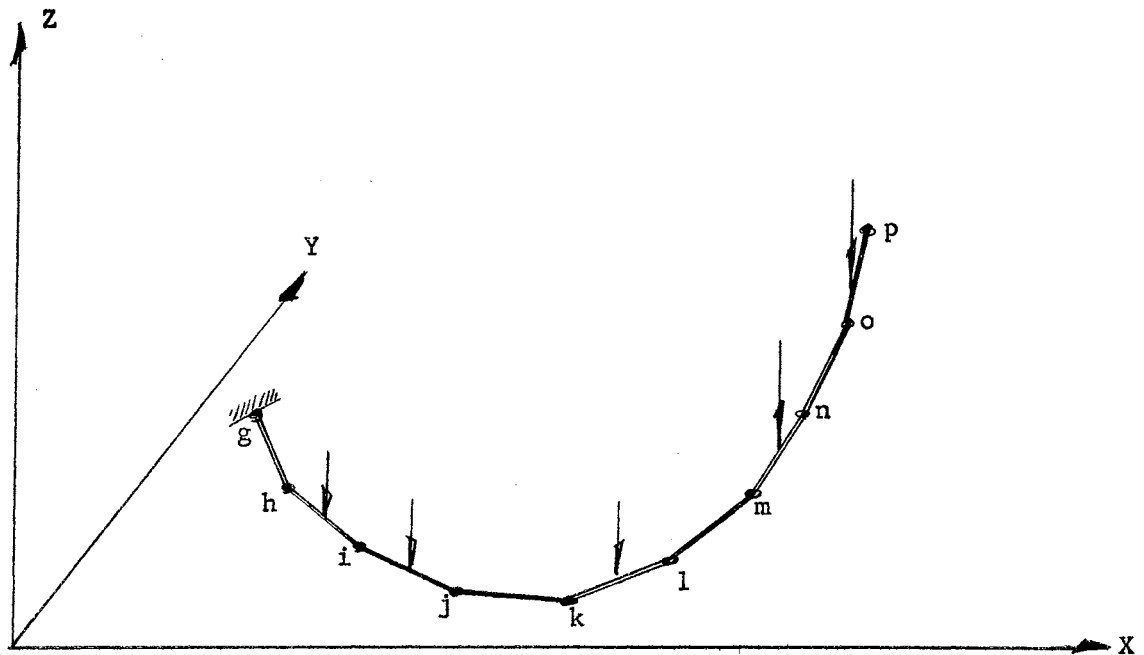


Figure 4-2

Continuous Bent Beam.

Thus

$$\frac{\delta U_{ikm}}{\delta M_{kx}} = \frac{\delta U_{ikm}}{\delta M_{ky}} = 0 \quad (4-1)$$

Using the deformation equations (3-4a,b) derived earlier for basic structures \overline{ijk} and adopting similar notation of symbols for basic structure \overline{klm} , equations (4-1) are expressed as:

$$M_{ix} G_{ikxx} + M_{kx} \sum F_{kxx} + M_{mx} G_{mkxx} + \sum \tau_{kxx} = 0 \quad (4-2a)$$

$$M_{iy} G_{ikyx} + M_{kx} \sum F_{kyx} + M_{my} G_{mkyx}$$

$$M_{ix} G_{ikxy} + M_{kx} \sum F_{kxy} + M_{mx} G_{mkxy} + \sum \tau_{kyy} = 0 \quad (4-2b)$$

$$M_{iy} G_{iky y} + M_{kx} \sum F_{kyy} + M_{my} G_{mkyy}$$

where

$$\begin{aligned} \sum F_{kxx} &= F_{kixx} + F_{kmxx}; & \sum F_{kyx} &= F_{kiyx} + F_{kmyx} \\ \sum F_{kxy} &= F_{kixy} + F_{kmxy}; & \sum F_{kyy} &= F_{kiyy} + F_{kmyy} \end{aligned} \quad (4-3)$$

$$\sum \tau_{kxx} = \tau_{kixx} + \tau_{kmxx}; \quad \sum \tau_{kyy} = \tau_{kiyy} + \tau_{kmyy}$$

Exactly similar six-moment compatibility equations apply to the support i of the continuous structure \overline{gp} (Fig. 4-2). As the end moments M_{gx} and M_{gy} for the continuous bent beam gn are zero, Eq. (4-2a) and (4-2b) may be applied to the support i with this modification (Fig. 4-1).

Next, consider the isolated portion \overline{kmn} of the continuous structure \overline{gn} . This can be separated into two parts, basic structure (No. 1) \overline{klm}

and basic structure (No. 2) \overline{mn} . At the support m, since M_{mx} and M_{my} are the redundant moments.

$$\frac{\delta U_{kmn}}{\delta M_{mx}} = \frac{\delta U_{kmn}}{\delta M_{my}} = 0 \quad (4-4)$$

From Eq. (3-4a,b) and (3-7a,b), the deformation conditions at m for basic structures \overline{klm} and \overline{mn} , Eq. (4-3) can be expressed as

$$M_{kx} G_{kmxx} + M_{mx} \sum F_{mxx} + M_{nx} G_{nmxx} + \sum \tau_{mxx} = 0 \quad (4-5a)$$

$$M_{ky} G_{kmyx} + M_{my} \sum F_{myx}$$

$$M_{kx} G_{kmxy} + M_{my} \sum F_{mxy} + M_{nx} G_{nmxy} + \sum \tau_{myy} = 0 \quad (4-5b)$$

$$M_{ky} G_{kmyy} + M_{my} \sum F_{myy}$$

where

$$\begin{aligned} \sum F_{mxx} &= F_{mkxx} + F_{mnxx}; & \sum F_{myx} &= F_{mkyx} + F_{mnyx} \\ \sum F_{mxy} &= F_{mkxy} + F_{mnxy}; & \sum F_{myy} &= F_{mkyy} + F_{mnyy} \\ \sum \tau_{mxx} &= \tau_{mkxx} + \tau_{mnxx}; & \sum \tau_{myy} &= \tau_{mkyy} + \tau_{mnyy} \end{aligned} \quad (4-6)$$

Eq. (4-5a,b) are the compatibility equations for the support m of the continuous structure \overline{gn} (Fig. 4-1).

Finally, the isolated portion \overline{mop} of the continuous bent beam \overline{gp} (Fig. 4-2) is considered. This can be separated into two parts, basic structure (No. 1) \overline{mno} and basic structure (No. 3) \overline{op} . At the support o, since M_{ox} is the only redundant moment,

$$\frac{\delta U_{\text{mop}}}{\delta M_{\text{ox}}} = 0 \quad (4-7)$$

Using the deformation equations along the X direction derived earlier (Eq. 3-4a,b, 3-10), the compatibility equation (Eq. 4-7) is expressed as

$$\begin{aligned} M_{\text{mx}} G_{\text{moxx}} + M_{\text{ox}} \Sigma F_{\text{oxx}} \\ + \Sigma \tau_{\text{oxx}} = 0 \end{aligned} \quad (4-8)$$

$$M_{\text{my}} G_{\text{moyx}} + M_{\text{oy}} \Sigma F_{\text{oyx}}$$

where

$$\begin{aligned} \Sigma F_{\text{oxx}} &= F_{\text{omxx}} + F_{\text{opxx}} \\ \Sigma F_{\text{oyx}} &= F_{\text{omyx}} + F_{\text{opyx}} \\ \Sigma \tau_{\text{oxx}} &= \tau_{\text{omxx}} + \tau_{\text{opxx}} \end{aligned} \quad (4-9)$$

Since M_{oy} is statically determinate, the three-moment compatibility equation in its final form is

$$\begin{aligned} M_{\text{mx}} \cdot G_{\text{moxx}} + M_{\text{ox}} \cdot \Sigma F_{\text{oxx}} \\ + \Sigma \tau_{\text{oxx}}^* = 0 \end{aligned} \quad (4-10)$$

$$M_{\text{my}} \cdot G_{\text{moyx}}$$

denoting

$$\Sigma \tau_{\text{oxx}}^* = \Sigma \tau_{\text{oxx}} + M_{\text{oy}} \Sigma F_{\text{oyx}} \quad (4-11)$$

It would be monotonous to derive compatibility equations for every support in the continuous structures \overline{gn} and \overline{gp} when the procedure of their derivation is similar to those obtained in Eq. (4-2), (4-5), and (4-10).

Instead, it is desirable to put these compatibility equations in matrix forms. The above treatment is general, and continuous bent beams having any number of spans can be analyzed similarly.

4-2 Matrix Forms.

(a) In the continuous structure \overline{gn} (Fig. 4-1), there are seven redundant support moments. Thus there are seven compatibility equations. The redundant matrix can be expressed as shown:

$$\begin{bmatrix}
 \Sigma F_{ixx} & \Sigma F_{iyx} & G_{kixx} & G_{kiyx} & - & - & - \\
 \Sigma F_{ixy} & \Sigma F_{iyy} & G_{kixy} & G_{kiyy} & - & - & - \\
 G_{ikxx} & G_{ikyxx} & \Sigma F_{kxx} & \Sigma F_{kyx} & G_{mkxx} & G_{mkyx} & - \\
 G_{ikxy} & G_{ikyxy} & \Sigma F_{kxy} & \Sigma F_{kyy} & G_{mkxy} & G_{mkyy} & - \\
 - & - & G_{kmxx} & G_{kmyx} & \Sigma F_{mxx} & \Sigma F_{myx} & G_{nmxx} \\
 - & - & G_{kmxy} & G_{kmyy} & \Sigma F_{mxy} & \Sigma F_{myy} & G_{nmxy} \\
 - & - & - & - & G_{mnxx} & G_{mnyx} & F_{nmxx}
 \end{bmatrix}
 \begin{bmatrix}
 M_{ix} \\
 M_{iy} \\
 M_{kx} \\
 M_{ky} \\
 M_{mx} \\
 M_{my} \\
 M_{nx}
 \end{bmatrix}
 = -
 \begin{bmatrix}
 \Sigma \tau_{ixx} \\
 \Sigma \tau_{iyy} \\
 \Sigma \tau_{kxx} \\
 \Sigma \tau_{kyy} \\
 \Sigma \tau_{mxx} \\
 \Sigma \tau_{myy} \\
 \tau_{nmxx}
 \end{bmatrix}$$

(b) There are nine redundant moments in the continuous bent beam \overline{gp} (Fig. 4-2). As such there will be nine compatibility equations. The redundant matrix is:

F_{gixx}	F_{giyx}	G_{igxx}	G_{igyx}	-	-	-	-	-	M_{gx}	T_{gixx}
F_{gixy}	F_{giyy}	G_{igxy}	$G_{igy y}$	-	-	-	-	-	M_{gy}	T_{giyy}
G_{gixx}	G_{giyx}	ΣF_{ixx}	ΣF_{iyx}	G_{kixx}	G_{kiyx}	-	-	-	M_{ix}	ΣT_{ixx}
G_{gixy}	G_{giyy}	ΣF_{ixy}	ΣF_{iyy}	G_{kixy}	G_{kiyy}	-	-	-	M_{iy}	ΣT_{iyy}
-	-	G_{ikxx}	$G_{iky x}$	ΣF_{kxx}	ΣF_{kyx}	G_{mkxx}	G_{mkyx}	-	M_{kx}	ΣT_{kxx}
-	-	G_{ikxy}	$G_{iky y}$	ΣF_{kxy}	ΣF_{kyy}	G_{mkxy}	G_{mkyy}	-	M_{ky}	ΣT_{kyy}
-	-	-	-	G_{kmxx}	$G_{km yx}$	ΣF_{mxx}	ΣF_{myx}	G_{omxx}	M_{mx}	ΣT_{mxx}
-	-	-	-	G_{kmxy}	$G_{km y y}$	ΣF_{mxy}	ΣF_{myy}	G_{omxy}	M_{my}	ΣT_{myy}
-	-	-	-	-	-	G_{moxx}	G_{moyx}	ΣF_{omx}	M_{ox}	ΣT_{omx}^*

4-3 Carry-Over Form.

The compatibility equations derived for the continuous support k (Eq. 4-2a,b) can be expressed in a neat carry over form by solving for the redundant moments.

Denoting

$$\begin{aligned}
 r_{ikxx} &= -\frac{G_{ikxx}}{\Sigma F_{kxx}} & r_{mkxx} &= -\frac{G_{mkxx}}{\Sigma F_{kxx}} \\
 r_{ikyx} &= -\frac{G_{ikyx}}{\Sigma F_{kxx}} & r_{mkyx} &= -\frac{G_{mkyx}}{\Sigma F_{kxx}} \\
 r_{kkyx} &= -\frac{\Sigma F_{kyx}}{\Sigma F_{kxx}} & r_{kkxy} &= -\frac{\Sigma F_{kxy}}{\Sigma F_{kyy}} \\
 r_{ikxy} &= -\frac{G_{ikxy}}{\Sigma F_{kyy}} & r_{mkxy} &= -\frac{G_{mkxy}}{\Sigma F_{kyy}} \\
 r_{ikyy} &= -\frac{G_{ikyy}}{\Sigma F_{kyy}} & r_{mkyy} &= -\frac{G_{mkyy}}{\Sigma F_{kyy}} \\
 m_{kx} &= -\frac{\Sigma \tau_{kxx}}{\Sigma F_{kxx}} & m_{ky} &= -\frac{\Sigma \tau_{kyy}}{\Sigma F_{kyy}}
 \end{aligned}
 \tag{4-12}$$

where the r-values represent the carry over values and the m-values are the starting moments, the carry-over forms of the compatibility equations (Eq. 4-2) are:

$$M_{kx} = m_{kx} + \frac{r_{ikxx} \cdot M_{ix} + r_{mkxx} \cdot M_{kx} + r_{kkyx} \cdot M_{ky}}{r_{ikyx} \cdot M_{iy} + r_{mkyx} \cdot M_{ky}} \tag{4-13a}$$

$$M_{ky} = m_{ky} + \frac{r_{ikxy} \cdot M_{ix} + r_{mkxy} \cdot M_{kx} + r_{kkxy} \cdot M_{kx}}{r_{ikyy} \cdot M_{iy} + r_{mkyy} \cdot M_{ky}} \tag{4-13b}$$

CHAPTER V

NUMERICAL APPLICATION

The application of the compatibility equations derived in the last chapter to a numerical example is illustrated. A systematic procedure for analysis is outlined in the first part of the chapter. The numerical example is then analyzed following the outline procedure in the second part of the chapter.

5-1 Outline for Numerical Procedure

(a) Transformation Matrices.

A reference system is selected and the transformation matrix for each span is established.

(b) Basic Structures.

After selecting the support moments as the unknowns, the continuous structure is isolated into appropriate basic structures.

(c) Angular Flexibilities and Carry Over Flexibilities.

The flexibility values (F's and G's) for the basic structures are calculated (Table 5-7).

(d) Angular Load Functions.

For the given loading, the angular load functions are computed (Table 3-3).

(e) Compatibility Equations.

The compatibility equations, in terms of these F, G, and τ

values and the redundant support moments are now obtained. The solution of this set of simultaneous equations can be achieved by several methods.

- (i) by the carry over procedure derived earlier.
- (ii) by synthetic elimination.
- (iii) by inverting the flexibility matrix on an electronic computer, if one is available or by Choleski's scheme.

The method, best suited to the example to be analysed, may be used.

(f) Shear Force, Twisting Moment and Bending Moment Diagrams

After obtaining the redundant support moments, the shear force, twisting moment and the bending moment diagrams are drawn, by considering the statics of the basic structures.

5-2 Numerical Example

A planar hexagonal beam ijklmn, of uniform square section is considered (Fig. 5-1). Each span is forty feet long and rests on simple supports. The structure is acted upon by loads perpendicular to the planar beam (Fig. 5-2). Unless stated otherwise, all values given are in units of feet, kips or kip-feet.

It is assumed that the Poisson's Ratio of the material of the structure is equal to 0.25, so that its flexural and torsional rigidity are equal.

Thus,

$$GJ_{x'} = EI_{y'} = GJ_{x''} = EI_{y''} = EI$$

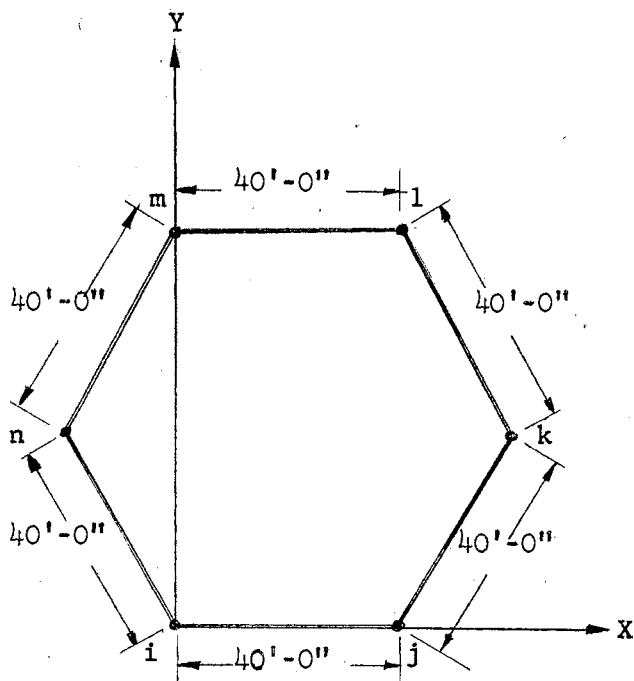


Figure 5-1

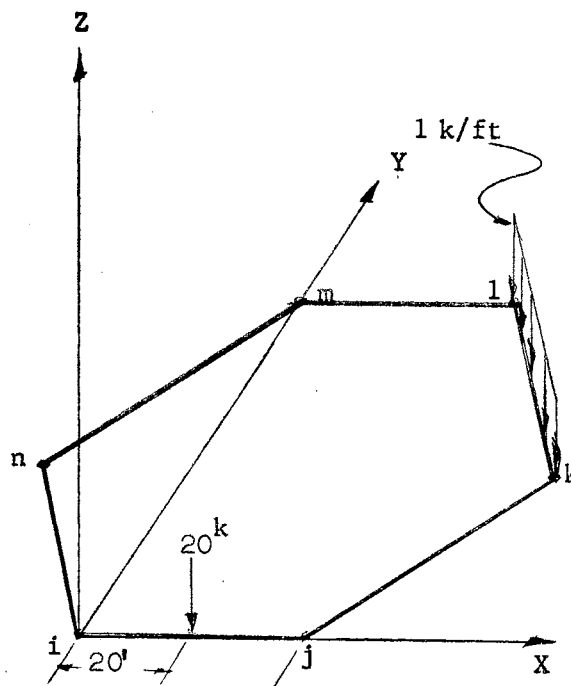


Figure 5-2

Hexagonal Beam

(a) Transformation Matrices

The transformation matrix for each span is calculated. As the selected reference system XYZ coincides with the principle axes of the span \overline{ij} , the transformation angle ω_j for the span \overline{ij} is equal to zero. Proceeding in a counter-clockwise direction on the structure from i to n, the subsequent transformation angles are:

$$\omega_k = 60^\circ$$

$$\omega_n = 240^\circ$$

$$\omega_l = 120^\circ$$

$$\omega_i = 300^\circ$$

$$\omega_m = 180^\circ$$

The transformation matrices for the particular spans are shown in the following tables.

	X'	Y'
X	$\alpha_{jx} = +1.000$	$\alpha_{jy} = 0.000$
Y	$\beta_{jx} = 0.000$	$\beta_{jy} = +1.000$

Table 5-1

Transformation Matrix
for Span \bar{ij} .

	X''	Y''
X	$\alpha_{kx} = +0.500$	$\alpha_{ky} = -0.866$
Y	$\beta_{kx} = +0.866$	$\beta_{ky} = +0.500$

Table 5-2

Transformation Matrix
for Span \bar{jk}

	X'	Y'
X	$\alpha_{lx} = -0.500$	$\alpha_{ly} = -0.866$
Y	$\beta_{lx} = +0.866$	$\beta_{ly} = -0.500$

Table 5-3

Transformation Matrix
for Span \bar{kl} .

	X''	Y''
X	$\alpha_{mx} = -1.000$	$\alpha_{my} = 0.000$
Y	$\beta_{mx} = 0.000$	$\beta_{my} = -1.000$

Table 5-4

Transformation Matrix
for Span \bar{lm}

	X'	Y'
X	$\alpha_{nx} = -0.500$	$\alpha_{ny} = +0.866$
Y	$\beta_{nx} = -0.866$	$\beta_{ny} = -0.500$

Table 5-5

Transformation Matrix
for Span \bar{mn}

	X''	Y''
X	$\alpha_{ix} = +0.500$	$\alpha_{iy} = +0.866$
Y	$\beta_{ix} = -0.866$	$\beta_{iy} = +0.500$

Table 5-6

Transformation Matrix
for Span \bar{ni}

(b) Basic Structures

As the structure is a closed even span bent beam continuous over the supports, it is isolated into three basic structures, of the type classified earlier as basic structure No. 1 (Fig. 5-3). The redundant moments, acting at supports i , k and m are shown.

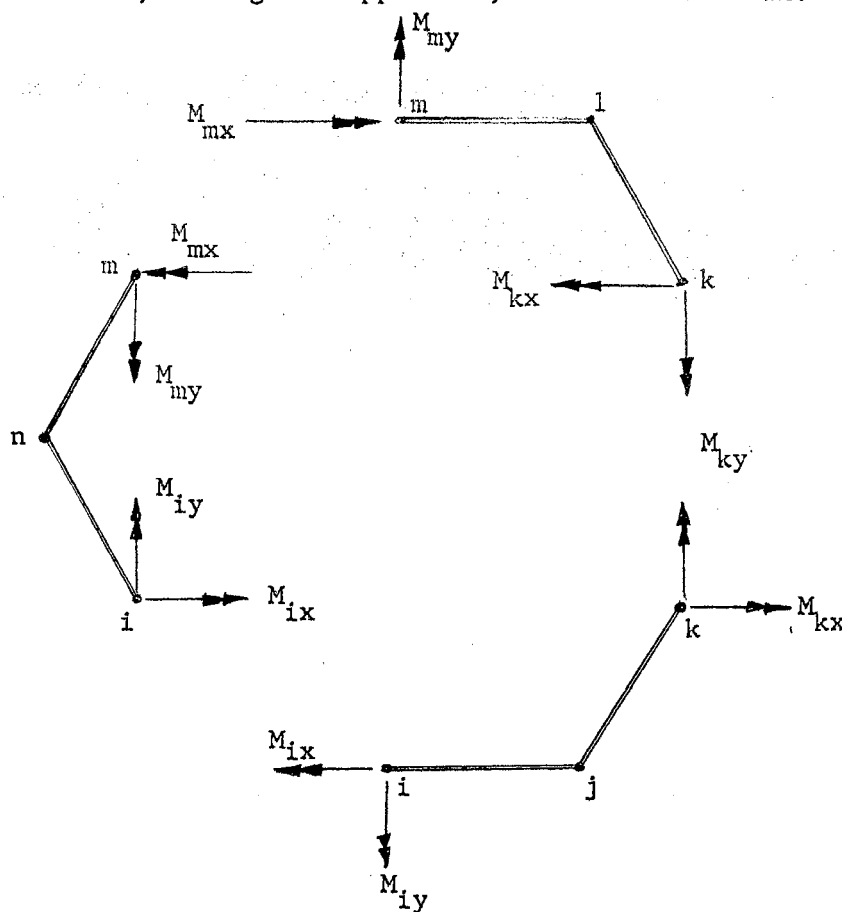


Figure 5-3

Basic Structures

(c) Angular Flexibilities and Carry Over Flexibilities

The angular flexibilities and the carry over flexibilities recorded in Tables 3-2 and 3-3 are for a non-uniform section. By integrating over the correct limits, they are simplified to apply to the case of a basic structure having a constant cross-section (Table 5-7).

Flexibility and Carry Over Flexibilities for Constant Section

Basic Structure No. 1

F_{ikxx}	$\frac{\alpha_{ix}^2 \cdot d_1}{GJ_{x'}} + \frac{\alpha_{iy}^2 \cdot d_1}{EI_{y'}} - \frac{\alpha_{iy} \cdot d_{kx} \cdot d_1^2}{CEI_{y'}} + \frac{d_{kx}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}} + \frac{d_{ix}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}}$
F_{ikyy}	$\frac{\beta_{ix} \cdot d_1}{GJ_{x'}} + \frac{\beta_{iy}^2 \cdot d_1}{EI_{y'}} - \frac{\beta_{iy} \cdot d_{ky} \cdot d_1^2}{CEI_{y'}} + \frac{d_{ky}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}} + \frac{d_{iy}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}}$
$F_{ikxy} = F_{ikyx}$	$\frac{\alpha_{ix} \cdot \beta_{ix} \cdot d_1}{GJ_{x'}} + \frac{\alpha_{iy} \cdot \beta_{iy} \cdot d_1}{EI_{y'}} - \frac{(\alpha_{iy} \cdot d_{ky} + \beta_{iy} \cdot d_{kx}) \cdot d_1^2}{2CEI_{y'}} - \frac{d_{kx} \cdot d_{ky} \cdot d_1^3}{3C^2 \cdot EI_{y'}} + \frac{d_{ix} \cdot d_{iy} \cdot d_k^3}{3C^2 \cdot EI_{y''}}$
F_{kixx}	$\frac{\alpha_{kx}^2 \cdot d_k}{GJ_{x''}} + \frac{\alpha_{ky}^2 \cdot d_k}{EI_{y''}} + \frac{\alpha_{ky} \cdot d_{ix} \cdot d_k^2}{CEI_{y''}} + \frac{d_{ix}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}} + \frac{d_{kx}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}}$
F_{kiyy}	$\frac{\beta_{kx} \cdot d_k}{GJ_{x''}} + \frac{\beta_{ky}^2 \cdot d_k}{EI_{y''}} + \frac{\beta_{ky} \cdot d_{iy} \cdot d_k^2}{CEI_{y''}} + \frac{d_{iy}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}} + \frac{d_{ky}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}}$
$F_{kixy} = F_{kiyx}$	$\frac{\alpha_{kx} \cdot \beta_{kx} \cdot d_k}{GJ_{x''}} + \frac{\alpha_{ky} \cdot \beta_{ky} \cdot d_k}{EI_{y''}} + \frac{(\alpha_{ky} \cdot d_{iy} + \beta_{ky} \cdot d_{ix}) \cdot d_k^2}{3CEI_{y''}} + \frac{d_{ix} \cdot d_{iy} \cdot d_k^3}{3C^2 \cdot EI_{y''}} + \frac{d_{kx} \cdot d_{ky} \cdot d_1^3}{3C^2 \cdot EI_{y'}}$
$G_{ikxx} = G_{kixx}$	$\frac{\alpha_{iy} \cdot d_{kx} \cdot d_1^2}{2CEI_{y'}} - \frac{d_{kx}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}} - \frac{\alpha_{ky} \cdot d_{ix} \cdot d_k^2}{2CEI_{y''}} - \frac{d_{ix}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}}$
$G_{ikyy} = G_{kiyy}$	$\frac{\beta_{iy} \cdot d_{ky} \cdot d_1^2}{2CEI_{y'}} - \frac{d_{ky}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}} - \frac{\beta_{ky} \cdot d_{iy} \cdot d_k^2}{2CEI_{y''}} - \frac{d_{iy}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}}$
$G_{ikxy} = G_{kiyx}$	$\frac{\beta_{iy} \cdot d_{kx} \cdot d_1^2}{2CEI_{y'}} - \frac{d_{kx} \cdot d_{ky} \cdot d_1^3}{3C^2 \cdot EI_{y'}} - \frac{\alpha_{ky} \cdot d_{iy} \cdot d_k^2}{2CEI_{y''}} - \frac{d_{ix} \cdot d_{iy} \cdot d_k^3}{3C^2 \cdot EI_{y''}}$
$G_{ikyx} = G_{kixy}$	$\frac{\alpha_{iy} \cdot d_{ky} \cdot d_1^2}{2CEI_{y'}} - \frac{d_{kx} \cdot d_{ky} \cdot d_1^3}{3C^2 \cdot EI_{y'}} - \frac{\beta_{ky} \cdot d_{ix} \cdot d_k^2}{2CEI_{y''}} - \frac{d_{ix} \cdot d_{iy} \cdot d_k^3}{3C^2 \cdot EI_{y''}}$

Table 5-7

The expressions in Table 5-7 are completely general for any basic structure No. 1 of uniform cross-section. The transformation matrix for each span determines the values of the geometric quantities in these expressions.

Basic Structure \overline{ijk}

The angular flexibilities and the carry over flexibilities are calculated by using Table 5-7 for the basic structure \overline{ijk} (Fig. 5-4).

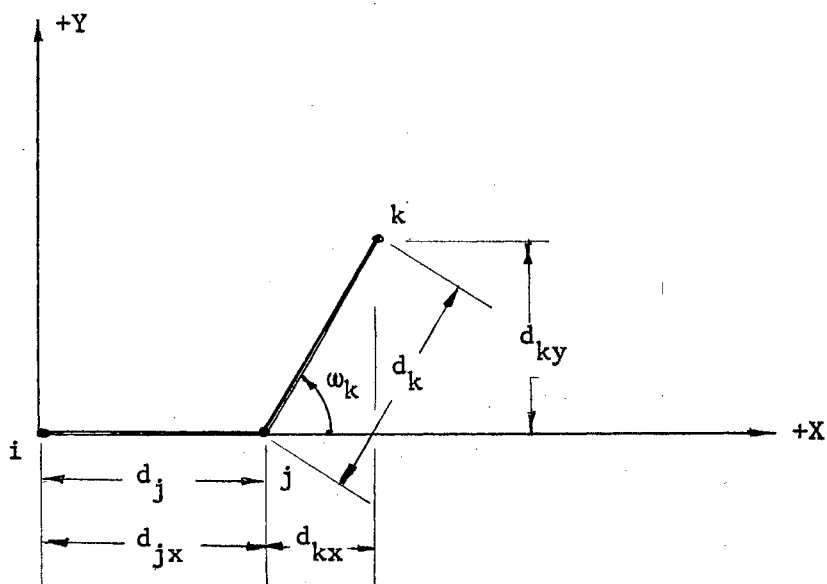


Figure 5-4

Basic Structure \overline{ijk}

$$d_j = d_k = 40'-0''$$

From the transformation matrix for the spans \overline{ij} and \overline{jk} (Tables 5-1, 5-2)

$$\alpha_{jx} = \beta_{jy} = +1.000$$

$$\alpha_{jy} = \beta_{jx} = 0.000$$

$$\alpha_{kx} = \beta_{ky} = +0.500$$

$$\alpha_{ky} = -\beta_{kx} = -0.866$$

$$d_{jx} = (40.000)(+1.000) = +40.000$$

$$d_{jy} = (40.000)(0.000) = 0.000$$

$$d_{kx} = (40.000)(+0.500) = +20.000$$

$$d_{ky} = (40.000)(+0.866) = +34.640$$

$$C = d_{jx} \cdot d_{ky} - d_{kx} \cdot d_{jy} \quad (\text{Eq. 2-2c})$$

$$= (+40.000)(+34.640)$$

$$= +1385.600$$

F values:

$$(i) \quad F_{ikxx} = \alpha_{jx}^2 \cdot \frac{d_j}{GJ_{x'}} + \frac{d_{kx} \cdot d_j^3}{3C^2 \cdot EI_{y'}} + \frac{d_{jx}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}}$$

$$= + \frac{62.25}{EI}$$

$$(ii) \quad F_{ikyy} = \beta_{jy}^2 \cdot \frac{d_j}{EI_{y'}} - \frac{\beta_{jy} \cdot d_{ky} \cdot d_j^2}{C \cdot EI_{y'}} + \frac{d_{ky}^2 \cdot d_j^3}{3C^2 \cdot EI_{y''}}$$

$$= + \frac{13.33}{EI}$$

$$(iii) \quad F_{ikxy} = F_{kixy} = - \frac{\beta_{jy} \cdot d_{kx} \cdot d_j^2}{2CEI_{y'}} + \frac{d_{kx} \cdot d_{ky} \cdot d_j^3}{3C^2 \cdot EI_{y'}}$$

$$= - \frac{3.85}{EI}$$

$$(iv) \quad F_{kixx} = \frac{\alpha_{kx}^2 \cdot d_k}{GJ_{x''}} + \frac{\alpha_{ky}^2 \cdot d_k}{EI_{y''}} + \frac{\alpha_{ky} \cdot d_{jx} \cdot d_k^2}{C \cdot EI_{y''}} + \frac{d_{jx}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}} + \frac{d_{kx}^2 \cdot d_j^3}{3C^2 \cdot EI_{y'}}$$

$$= + \frac{22.22}{EI}$$

$$\begin{aligned}
 \text{(v)} \quad F_{kiyy} &= \frac{\beta_{kx}^2 \cdot d_k}{GJ_{x''}} + \frac{\beta_{ky}^2 \cdot d_k}{EI_{y''}} + \frac{d_{ky}^2 \cdot d_j^3}{3C^2 \cdot EI_{y'}} \\
 &= + \frac{53.32}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad F_{kiyx} = F_{kixy} &= \frac{\alpha_{kx} \cdot \beta_{kx} \cdot d_k}{GJ_{x''}} + \frac{\alpha_{ky} \cdot \beta_{ky} \cdot d_k}{EI_{y''}} + \frac{\beta_{ky} \cdot d_{jx} \cdot d_k^2}{2CEI_{y''}} + \frac{d_{kx} \cdot d_{ky} \cdot d_j^3}{3C^2 EI_{y'}} \\
 &= + \frac{19.28}{EI}
 \end{aligned}$$

G values:

$$\begin{aligned}
 \text{(i)} \quad G_{ikxx} = G_{kixx} &= - \frac{d_{kx}^2 \cdot d_j^2}{3C^2 \cdot EI_{y'}} - \frac{\alpha_{ky} \cdot d_{jx} \cdot d_k^2}{2 \cdot CEI_{y''}} - \frac{d_{jx}^2 \cdot d_k^3}{3C^2 \cdot EI_{y''}} \\
 &= - \frac{2.22}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad G_{ikyy} = G_{kiyy} &= \frac{\beta_{jy} \cdot d_{ky} \cdot d_j^2}{2CEI_{y'}} - \frac{d_{ky}^2 \cdot d_j^3}{3C^2 \cdot EI_{y'}} \\
 &= + \frac{6.67}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad G_{ikxy} = G_{kiyx} &= \frac{\beta_{jy} \cdot d_{kx} \cdot d_j^2}{2CEI_{y'}} - \frac{d_{kx} \cdot d_{ky} \cdot d_j^3}{3C^2 \cdot EI_{y'}} \\
 &= + \frac{3.85}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv) } G_{iky x} = G_{kix y} &= -\frac{d_{kx} \cdot d_{ky} \cdot d_l^3}{3C^2 EI_{y'}} - \frac{\beta_{ky} \cdot d_{ix} \cdot d_k^2}{2CEI_{y''}} \\
 &= -\frac{19.28}{EI}
 \end{aligned}$$

Basic Structure \overline{klm}

The basic structure \overline{klm} with the origin of coordinates at k is shown in Fig. 5-5.

$$d_l = d_m = 40' - 0''$$

From Tables (5-3, 5-4)

$$\alpha_{lx} = \beta_{ly} = -0.500$$

$$\alpha_{ly} = -\beta_{lx} = -0.866$$

$$\alpha_{mx} = \beta_{my} = -1.000$$

$$\alpha_{my} = \beta_{mx} = 0.000$$

$$d_{lx} = (40.000)(-0.500) = -20.000$$

$$d_{ly} = (40.000)(+0.866) = +34.640$$

$$d_{mx} = (40.000)(-1.000) = -40.000$$

$$d_{my} = (40.000)(0.000) = 0.000$$

$$\begin{aligned}
 C &= d_{lx} \cdot d_{my} - d_{mx} \cdot d_{ly} \\
 &= -(-40.000)(+34.640) \\
 &= +1385.600
 \end{aligned}$$

F values:

$$\begin{aligned}
 \text{(i) } F_{kmxx} &= \frac{\alpha_{lx}^2 \cdot d_l}{GJ_x'} + \frac{\alpha_{ly} \cdot d_l}{EI_{y'}} - \frac{\alpha_{ly} \cdot d_{mx} \cdot d_l^2}{CEI_{y'}} + \frac{d_{mx}^2 \cdot d_l^3}{3C^2 \cdot EI_{y'}} + \frac{d_{lx}^2 \cdot d_m^3}{3C^2 \cdot EI_{y''}} \\
 &= +\frac{22.22}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad F_{kmyy} &= \frac{\beta_{1x}^2 \cdot d_1}{GJ_{x'}} + \frac{\beta_{1y}^2 \cdot d_1}{EI_{y'}} + \frac{d_{1y}^2 \cdot d_m^3}{3C^2 \cdot EI_{y''}} \\
 &= + \frac{53.32}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad F_{kmxy} &= F_{kmyx} = \frac{\alpha_{1x} \cdot \beta_{1x} \cdot d_1}{GJ_{x'}} + \frac{\alpha_{1y} \cdot \beta_{1y} \cdot d_1}{EI_{y'}} - \frac{\beta_{1y} \cdot d_{mx} \cdot d_1^2}{2CEI_{y'}} + \frac{d_{1x} \cdot d_{1y} \cdot d_m^3}{3C^2 \cdot EI_{y''}} \\
 &= - \frac{19.28}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad F_{mkxx} &= \frac{\alpha_{mx}^2 \cdot d_m}{GJ_{x''}} + \frac{d_{1x}^2 \cdot d_m^3}{3C^2 \cdot EI_{y''}} + \frac{d_{mx}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}} \\
 &= + \frac{62.25}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad F_{mkyy} &= \frac{\beta_{mx}^2 \cdot d_m}{GJ_{x''}} + \frac{\beta_{my}^2 \cdot d_m}{EI_{y''}} + \frac{\beta_{my} \cdot d_{1y} \cdot d_m^2}{CEI_{y''}} + \frac{d_{1y}^2 \cdot d_m^3}{3C^2 \cdot EI_{y''}} + \frac{d_{my}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}} \\
 &= + \frac{13.33}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad F_{mkxy} &= F_{kmyx} = \frac{\beta_{my} \cdot d_{1x} \cdot d_m^2}{2CEI_{y''}} + \frac{d_{1x} \cdot d_{1y} \cdot d_m^3}{3C^2 \cdot EI_{y''}} \\
 &= + \frac{3.85}{EI}
 \end{aligned}$$

G values:

$$(i) \quad G_{kmxx} = G_{mkxx} = \frac{\alpha_{ly} \cdot d_{mx} \cdot d_1^2}{2CEI_{y'}} - \frac{d_{mx}^2 \cdot d_1^3}{3C^2 \cdot EI_{y'}} - \frac{d_{lx}^2 \cdot d_m^3}{3C^2 \cdot EI_{y''}}$$

$$= - \frac{2.22}{EI}$$

$$(ii) \quad G_{kmyy} = G_{mkyy} = - \frac{\beta_{my} \cdot d_{ly} \cdot d_m^2}{2CEI_{y''}} - \frac{d_{ly}^2 \cdot d_m^3}{3C^2 \cdot EI_{y''}}$$

$$= + \frac{6.67}{EI}$$

$$(iii) \quad G_{kmxy} = G_{mkyx} = \frac{\beta_{ly} \cdot d_{mx} \cdot d_1^3}{2CEI_{y'}} - \frac{d_{lx} \cdot d_{ly} \cdot d_m^3}{3C^2 \cdot EI_{y''}}$$

$$= + \frac{19.28}{EI}$$

$$(iv) \quad G_{kmyx} = G_{mkxy} = - \frac{\beta_{my} \cdot d_{lx} \cdot d_m^2}{2CEI_{y''}} - \frac{d_{lx} \cdot d_{ly} \cdot d_m^3}{3C^2 \cdot EI_{y''}}$$

$$= - \frac{3.85}{EI}$$

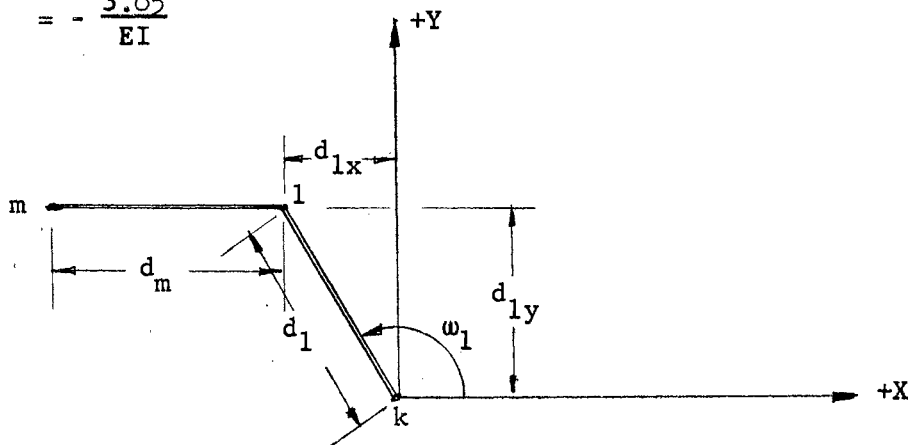


Figure 5-5

Basic Structure \overline{klm}

Basic Structure \overline{mni}

The basic structure \overline{mni} with the origin of co-ordinates at m is shown in Fig. 5-6.

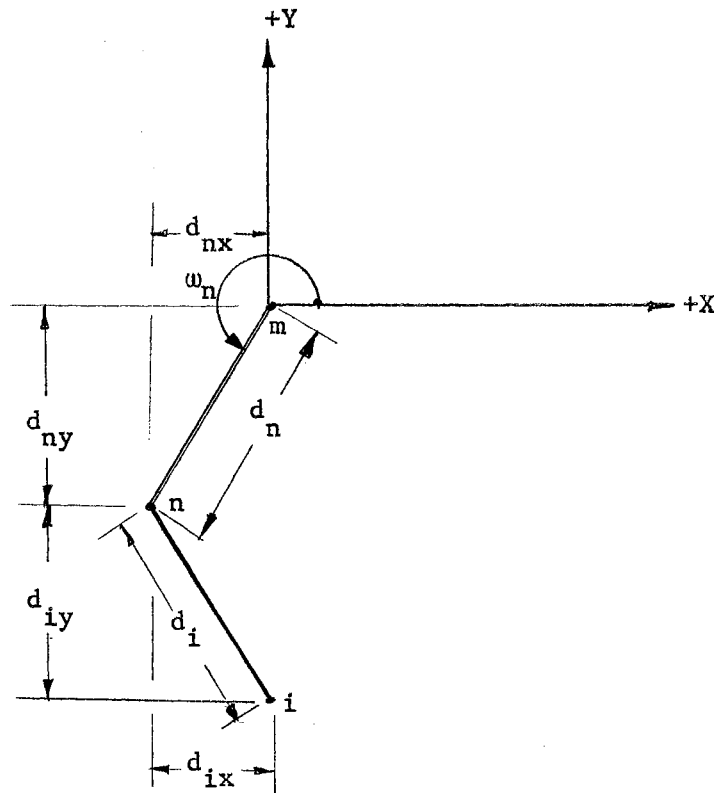


Figure 5-6

Basic Structure \overline{mni}

$$d_n = d_i = 40'-0''$$

From Tables 5-5 and 5-6

$$\alpha_{nx} = \beta_{ny} = -0.500$$

$$\alpha_{ny} = -\beta_{nx} = +0.866$$

$$\alpha_{ix} = \beta_{iy} = +0.500$$

$$\alpha_{iy} = -\beta_{ix} = +0.866$$

$$d_{nx} = (40.00)(-0.500) = -20.000$$

$$d_{ny} = (40.000)(-0.866) = -34.640$$

$$d_{ix} = (40.000)(+0.500) = +20.000$$

$$d_{iy} = (40.000)(-0.866) = -34.640$$

$$\begin{aligned} C &= d_{nx} \cdot d_{iy} - d_{ix} \cdot d_{ny} \\ &= (-20.000)(-34.640) - (+20.000)(-34.640) \\ &= +1385.600 \end{aligned}$$

F values:

$$\begin{aligned} \text{(i)} \quad F_{mixx} &= \frac{\alpha_{nx}^2 \cdot d_n^2}{GJ_{x'}} + \frac{\alpha_{ny}^2 \cdot d_n^2}{EI_{y'}} - \frac{\alpha_{ny} \cdot d_{ix} \cdot d_n^2}{CEI_{y'}} - \frac{d_{ix}^2 \cdot d_n^3}{3C^2 EI_{y'}} + \frac{d_{nx}^2 \cdot d_i^3}{3C^2 EI_{y''}} \\ &= + \frac{28.88}{EI} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad F_{miyy} &= \frac{\beta_{nx}^2 \cdot d_n^2}{GJ_{x'}} + \frac{\beta_{ny}^2 \cdot d_n^2}{EI_{y'}} - \frac{\beta_{ny} \cdot d_{iy} \cdot d_n^2}{CEI_{y'}} + \frac{d_{iy}^2 \cdot d_n^3}{3C^2 EI_{y'}} + \frac{d_{ny}^2 \cdot d_i^3}{3C^2 EI_{y''}} \\ &= + \frac{46.67}{EI} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad F_{mixy} = F_{miyx} &= \frac{\alpha_{nx} \cdot \beta_{nx} \cdot d_n^2}{GJ_{x'}} + \frac{\alpha_{ny} \cdot \beta_{ny} \cdot d_n^2}{EI_{y'}} - \frac{(\alpha_{ny} \cdot d_{iy} + \beta_{ny} \cdot d_{ix}) \cdot d_n^2}{2CEI_{y'}} \\ &\quad + \frac{d_{ix} \cdot d_{iy} \cdot d_n^3}{3C^2 EI_{y'}} + \frac{d_{nx} \cdot d_{ny} \cdot d_i^3}{3C^2 EI_{y''}} \\ &= + \frac{23.08}{EI} \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad F_{imxx} &= \frac{\alpha_{ix}^2 \cdot d_i^2}{GJ_{x''}} + \frac{\alpha_{iy}^2 \cdot d_i^2}{EI_{y''}} + \frac{\alpha_{iy} \cdot d_{nx} \cdot d_i^2}{CEI_{y''}} + \frac{d_{nx}^2 \cdot d_i^3}{3C^2 \cdot EI_{y''}} + \frac{d_{ix}^2 \cdot d_n^3}{3C^2 \cdot EI_{y'}} \\
 &= + \frac{28.88}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad F_{imyy} &= \frac{\alpha_{iy}^2 \cdot d_i^2}{GJ_{x''}} + \frac{\beta_{iy}^2 \cdot d_i^2}{EI_{y''}} + \frac{\beta_{iy} \cdot d_{ny} \cdot d_i^2}{CEI_{y''}} + \frac{d_{ny}^2 \cdot d_i^3}{3C^2 \cdot EI_{y''}} + \frac{d_{iy}^2 \cdot d_n^3}{3C^2 \cdot EI_{y'}} \\
 &= + \frac{46.67}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad F_{imxy} = F_{imyx} &= \frac{\alpha_{ix} \cdot \beta_{ix} \cdot d_i^2}{GJ_{x''}} + \frac{\alpha_{iy} \cdot \beta_{iy} \cdot d_i^2}{EI_{y''}} + \frac{(\beta_{iy} \cdot d_{nx} + \alpha_{iy} \cdot d_{ny}) d_i^2}{EI_{y''}} \\
 &\quad + \frac{d_{nx} \cdot d_{ny} \cdot d_i^3}{EI_{y''}} + \frac{d_{ix} \cdot d_{iy} \cdot d_n^3}{EI_{y'}} \\
 &= - \frac{23.08}{EI}
 \end{aligned}$$

G values:

$$\begin{aligned}
 \text{(i)} \quad G_{mixx} = G_{imxx} &= \frac{\alpha_{ny} \cdot d_{ix} \cdot d_n^2}{2CEI_{y'}} - \frac{d_{ix}^2 \cdot d_n^3}{3C^2 \cdot EI_{y'}} - \frac{\alpha_{iy} \cdot d_{nx} \cdot d_i^2}{2CEI_{y''}} - \frac{d_{nx}^2 \cdot d_i^3}{3C^2 \cdot EI_{y''}} \\
 &= + \frac{31.12}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad G_{miyy} = G_{imyy} &= \frac{\beta_{ny} \cdot d_{iy} \cdot d_n^2}{2CEI_{y'}} - \frac{d_{iy}^2 \cdot d_n^3}{3C^2 \cdot EI_{y'}} - \frac{\beta_{iy} \cdot d_{ny} \cdot d_i^2}{2CEI_{y''}} - \frac{d_{ny}^2 \cdot d_i^3}{3C^2 \cdot EI_{y''}} \\
 &= - \frac{6.67}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad G_{miyx} = G_{imxy} &= \frac{\alpha_{ny} \cdot d_{iy} \cdot d_n^2}{2CEI_{y'}} - \frac{d_{ix} \cdot d_{iy} \cdot d_n^3}{3C^2 \cdot EI_{y'}} - \frac{\beta_{iy} \cdot d_{nx} \cdot d_i^3}{2CEI_{y''}} \\
 &\quad - \frac{d_{nx} \cdot d_{ny} \cdot d_i^3}{3C^2 \cdot EI_{y''}} \\
 &= - \frac{11.56}{EI}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad G_{mixy} = G_{imyx} &= \frac{\beta_{ny} \cdot d_{ix} \cdot d_n^2}{2CEI_{y'}} - \frac{d_{ix} \cdot d_{iy} \cdot d_n^3}{3C^2 \cdot EI_{y'}} - \frac{\alpha_{iy} \cdot d_{ny} \cdot d_i^2}{2CEI_{y''}} \\
 &\quad - \frac{d_{nx} \cdot d_{ny} \cdot d_i^3}{3C^2 \cdot EI_{y''}} \\
 &= + \frac{11.56}{EI}
 \end{aligned}$$

(d) Angular Load Functions

For the given loads, the angular load functions (τ values) are expressed algebraically in terms of the end slopes of a simply supported beam loaded in the plane.

Basic Structure \overline{ijk}

A concentrated load of $P = 20$ kips acts at the center of the span \overline{ij} . (Fig. 5-2).

From Tables 3-1 and 3-3.

$$\begin{aligned}
 (i) \quad \tau_{ikyy} &= \int_0^{d_j} BM_y^{(i)} \frac{\delta M_{y'}^{(i)}}{\delta M_{iy}} \cdot \lambda_{y'} \\
 &= \int_0^{\frac{d_i}{2}} \left(-\frac{P}{2} \cdot u'\right) \left(\beta_{jy} - \frac{u' \cdot d_{ky}}{C}\right) \cdot \frac{d_{x'}}{EI} \\
 &\quad + \int_{\frac{d_i}{2}}^{d_j} \left(\frac{P}{2} \cdot u' - \frac{P}{2} \cdot d_j\right) \left(\beta_{jy} - \frac{u' \cdot d_{ky}}{C}\right) \cdot \frac{d_{x'}}{EI}
 \end{aligned}$$

On integrating and simplifying.

$$\tau_{ikyy} = -\frac{P \cdot d_i^2}{16EI} (2 \cdot \beta_{jy} - 1)$$

Since $\beta_{jy} = +1.000$ (Table 5-1)

$$\tau_{ikyy} = -\frac{P \cdot d_i^2}{16EI} = \tau_{ijy'} \quad (5-1)$$

Thus τ_{ikyy} is equal to the end slope at i of a simply supported span \overline{ij} loaded by the concentrated load P.

$$\begin{aligned}
 (ii) \quad \tau_{ikxx} &= \int_0^{d_j} BM_y^{(i)} \cdot \frac{\delta M_{y'}^{(i)}}{\delta M_{ix}} \cdot \lambda_{y'} \\
 &= \int_0^{\frac{d_i}{2}} \left(-\frac{P}{2} \cdot u'\right) \left(\alpha_{jy} - \frac{u' \cdot d_{kx}}{C}\right) \cdot \frac{d_{x'}}{EI} \\
 &\quad + \int_{\frac{d_i}{2}}^{d_j} \left(\frac{P}{2} \cdot u' - \frac{P}{2} \cdot d_j\right) \left(\alpha_{jy} - \frac{u' \cdot d_{kx}}{C}\right) \cdot \frac{d_{x'}}{EI}
 \end{aligned}$$

Integrating and simplifying

$$\tau_{ikxx} = \left[\frac{d_{kx}}{d_{ky}} \right] \frac{P \cdot d_j^2}{16EI}$$

From Fig. 5-4 and Table 5-2

$$\tau_{ikxx} = - \left[\frac{\alpha_{kx}}{\beta_{kx}} \right] \tau_{ijy'} \quad (5-2)$$

$$\begin{aligned} \text{(iii)} \quad \tau_{kiyy} &= \int_0^{d_j} BM_y^{(i)} \frac{\delta M_{y'}^{(i)}}{\delta M_{ky}} \cdot \lambda_{y'} \\ &= \int_0^{\frac{d_j}{2}} \left(-\frac{P}{2} \cdot u' \right) \left(\frac{u' \cdot d_{ky}}{C} \right) \cdot \frac{d_{x'}}{EI} + \\ &\quad + \int_{\frac{d_j}{2}}^{d_j} \left(\frac{P}{2} \cdot u' - \frac{P}{2} \cdot d_j \right) \left(\frac{u' \cdot d_{ky}}{C} \right) \cdot \frac{d_{x'}}{EI} \end{aligned}$$

Integration and simplification gives

$$\tau_{kiyy} = - \frac{P \cdot d_j^2}{16EI} = \tau_{ijy'} \quad (5-3)$$

$$\begin{aligned}
\text{(iv)} \quad \tau_{kixx} &= \int_0^{d_j} BM_y^{(i)} \cdot \frac{\delta M_{y'}^{(i)}}{\delta M_{kx}} \cdot \lambda_{y'} \\
&= \int_0^{\frac{d_j}{2}} \left(-\frac{P}{2} \cdot u'\right) \left(\frac{u' \cdot d_{kx}}{C}\right) \frac{d_{x'}}{EI} + \\
&\quad + \int_{\frac{d_j}{2}}^{d_j} \left(\frac{P}{2} \cdot u' - \frac{P}{2} \cdot d_j\right) \left(\frac{u' \cdot d_{kx}}{C}\right) \frac{d_{x'}}{EI} \\
&= - \left[\frac{d_{kx}}{d_{ky}} \right] \frac{P \cdot d_j^2}{16EI} \\
&= + \left[\frac{\alpha_{kx}}{\beta_{kx}} \right] \tau_{ijy'} \tag{5-4}
\end{aligned}$$

Substituting the numerical values in Eq. 5-1, 5-2, 5-3, and 5-4,

$$\tau_{iky} = \tau_{kiy} = -\frac{2000}{EI}$$

$$\tau_{ikx} = -\tau_{kix} = +\frac{1155}{EI}$$

Basic Structure \overline{klm}

A uniformly distributed load of $w = 1$ kip per foot acts on the span \overline{kl} .

From Tables 3-1 and 3-3.

$$\begin{aligned}
 \text{(i)} \quad \tau_{kmyy} &= \int_0^{d_1} BM_y^{(k)} \cdot \frac{\delta M_y^{(k)}}{\delta M_{ky}} \cdot \lambda_{y'} \\
 &= \int_0^{d_1} \left(-\frac{wd_1}{2} \cdot u' + \frac{w \cdot u'^2}{2} \right) \left(\beta_{1y} - \frac{u' \cdot d_{my}}{C} \right) \frac{d_{x'}}{EI}
 \end{aligned}$$

Integrating and simplifying,

$$\tau_{kmyy} = \beta_{1y} \left(-\frac{2}{24} \cdot \frac{wd_1^3}{EI} \right)$$

As $\beta_{1y} = -0.500$ (Table 5-3)

$$\tau_{kmyy} = + \frac{wd_1^3}{24EI} = - \tau_{kly'} \quad (5-5)$$

$$\begin{aligned}
 \text{(ii)} \quad \tau_{kmxx} &= \int_0^{d_1} BM_y^{(k)} \cdot \frac{\delta M_y^{(k)}}{\delta M_{kx}} \cdot \lambda_{y'} \\
 &= \int_0^{d_1} \left(-\frac{w \cdot d_1}{2} \cdot u' + \frac{w \cdot u'^2}{2} \right) \left(\alpha_{1y} - \frac{u' \cdot d_{mx}}{C} \right) \frac{d_{x'}}{EI}
 \end{aligned}$$

On integrating and simplifying

$$\tau_{kmxx} = \frac{wd_1^3}{24EI} \left(-2\alpha_{1y} - \frac{2}{3} \right)$$

Since $\alpha_{1y} = -0.866$

$$\tau_{kmxx} = + \frac{0.500}{0.866} \cdot \frac{wd_1^3}{24EI} = - \left[\frac{\alpha_{1x}}{\beta_{1x}} \right] \frac{wd_1^3}{24EI}$$

$$\tau_{kmxx} = \left[\frac{\alpha_{1x}}{\beta_{1x}} \right] \tau_{kly'} \quad (5-6)$$

$$\begin{aligned} \text{(iii)} \quad \tau_{mkyy} &= \int_0^{d_1} BM_y^{(k)} \cdot \frac{\delta M_{y'}^{(k)}}{\delta M_{my}} \cdot \lambda_{y'} \\ &= \int_0^{d_1} \left(-\frac{wd_1}{2} \cdot u' + \frac{w \cdot u'^2}{2} \right) \left(\frac{u' \cdot d_{my}}{C} \right) \cdot \frac{d_{x'}}{EI} \end{aligned}$$

Since $d_{my} = 0$

$$\tau_{mkyy} = 0 \quad (5-7)$$

$$\begin{aligned} \text{(iv)} \quad \tau_{mkxx} &= \int_0^{d_1} BM_y^{(k)} \cdot \frac{\delta M_{y'}^{(k)}}{\delta M_{mx}} \cdot \lambda_{y'} \\ &= \int_0^{d_1} \left(-\frac{wd_1}{2} \cdot u' + \frac{w \cdot u'^2}{2} \right) \left(\frac{u' \cdot d_{mx}}{C} \right) \cdot \frac{d_{x'}}{EI} \end{aligned}$$

Integrating,

$$\begin{aligned} \tau_{mkxx} &= + \left[\frac{1}{\beta_{1x}} \right] \frac{wd_1^3}{24EI} \\ &= - \left[\frac{1}{\beta_{1x}} \right] \tau_{kly}' \end{aligned} \quad (5-8)$$

Substituting the numerical values in Eq. 5-5, 5-6, 5-7 and 5-8

$$\tau_{kmyy} = + \frac{2667}{EI} \quad ; \quad \tau_{kmxx} = + \frac{1538}{EI}$$

$$\tau_{mkyy} = 0 \quad ; \quad \tau_{mkxx} = + \frac{3076}{EI}$$

Basic Structure \overline{mni}

As there are no loads on the spans \overline{mn} and \overline{ni} ,

$$\tau_{miyy} = \tau_{mixx} = \tau_{imyy} = \tau_{imxx} = 0$$

Summation of Flexibilities (F-values) at the supports k, m and i

$$\sum F_{kxx} = + \frac{44.44}{EI} \quad ; \quad \sum F_{mxy} = \sum F_{myx} = + \frac{26.93}{EI}$$

$$\sum F_{kyy} = + \frac{106.64}{EI} \quad ; \quad \sum F_{ixx} = + \frac{91.13}{EI}$$

$$\sum F_{kxy} = \sum F_{kyx} = 0 \quad ; \quad \sum F_{iyy} = + \frac{60.00}{EI}$$

$$\sum F_{mxx} = + \frac{91.13}{EI} \quad ; \quad \sum F_{ixy} = \sum F_{iyx} = - \frac{26.93}{EI}$$

$$\sum F_{myy} = + \frac{60.00}{EI} \quad ;$$

Summation of Load Functions (τ -values) at the supports k, m and i

$$\begin{aligned} \sum \tau_{kxx} &= + \frac{383}{EI} & ; & \quad \sum \tau_{myy} = 0 \\ \sum \tau_{kyy} &= + \frac{667}{EI} & ; & \quad \sum \tau_{ixx} = + \frac{1155}{GI} \\ \sum \tau_{mxx} &= + \frac{3076}{EI} & ; & \quad \sum \tau_{iyy} = - \frac{2000}{EI} \end{aligned}$$

(e) Compatibility Equations

The six moment compatibility equations are arranged in a matrix form.

$$\begin{bmatrix} 44.44 & 0.00 & -2.22 & 19.28 & -2.22 & -19.28 \\ 0.00 & 106.64 & -3.85 & 6.67 & 3.85 & 6.67 \\ -2.22 & -3.85 & 91.13 & 26.93 & 31.12 & 11.56 \\ 19.28 & 6.67 & 26.93 & 60.00 & -11.56 & -6.67 \\ -2.22 & 3.85 & 31.12 & -11.56 & 91.13 & -26.93 \\ -19.28 & 6.67 & 11.56 & -6.67 & -26.93 & 60.00 \end{bmatrix} \begin{bmatrix} M_{kx} \\ M_{ky} \\ M_{mx} \\ M_{my} \\ M_{ix} \\ M_{iy} \end{bmatrix} = \begin{bmatrix} 383 \\ 667 \\ 3076 \\ 0 \\ 1155 \\ -2000 \end{bmatrix}$$

The six moment equations are solved by synthetic elimination and by the carry over procedure derived earlier (Eq. 4-13a,b).

It is found that the carry-over procedure (Table 5-8) converges very slowly and consumes considerable labor and time as compared to the solution by synthetic elimination. In fact, this is apparent from observing that some of the carry over values vary from 0.25 to 0.50. As such, it turns out that the carry over solution is not an ideal solution for the analysis of a polygonal frame-work of continuous bent beams as treated

Carry Over Procedure

^{12}a	^{12}b	^{12}c	^{12}d	^{12}e	^{12}f
$r_{1km} = + 0.04995$	$r_{1km} = + 0.02636$	$r_{1km} = + 0.04995$	$r_{1ky} = - 0.06254$	$r_{1ky} = - 0.11136$	$r_{1ky} = - 0.06254$
$r_{1ms} = - 0.36149$	$r_{1ms} = + 0.02636$	$r_{1ms} = - 0.36149$	$r_{1my} = + 0.11136$	$r_{1my} = + 0.11136$	$r_{1my} = + 0.11136$
$r_{1ky} = + 0.03610$	$r_{1ky} = + 0.32133$	$r_{1ky} = + 0.03610$	$r_{1ys} = + 0.43306$	$r_{1ys} = - 0.06236$	$r_{1ys} = - 0.43306$
$r_{1my} = + 0.19266$	$r_{1my} = - 0.32133$	$r_{1my} = - 0.19266$	$r_{1ys} = - 0.12685$	$r_{1ys} = + 0.06236$	$r_{1ys} = + 0.12685$
$r_{1sy} = + 0.44883$	$r_{1sy} = 0.00000$	$r_{1sy} = + 0.44883$	$r_{1sy} = + 0.29251$	$r_{1sy} = 0.00000$	$r_{1sy} = - 0.29251$
-12.67420	- 0.61836	-32.75927	+32.32323	- 0.25468	0.00000
+11.52849	+ 1.68428		+ 6.50408	+ 1.21871	+15.15221
+11.77233	+17.28206	- 3.05227	+39.83741	- 2.48163	+ 4.42322
+ 2.68379	- 8.49482	+ 8.78636	+ 2.17627	+ 1.22636	+19.58033
+13.11043	+ 0.65483	- 4.47708	+ 5.88435	- 0.47328	+ 2.52586
+ 0.49263	0.00000	- 0.49263	+ 1.29642	-11.64266	+ 1.29642
+ 5.29872	- 0.78972	-12.80922	+ 2.06282	+ 0.37072	+ 7.09570
+ 2.46526	+ 1.38097	+ 1.57234	+12.40216	- 0.77269	+ 1.27874
+ 1.55983	+ 1.23490	- 3.43380	+ 1.36690	+ 0.76904	+12.29672
+11.11644	+ 0.25522	- 2.79615	+ 4.98929	- 0.40130	+ 2.14169
+ 3.07452	- 0.44971	- 9.00229	+ 1.73427	- 0.22502	+ 4.04094
+ 2.29092	+ 2.51014	- 1.02432	+ 8.09066	- 0.50600	+ 0.89920
+ 0.89812	- 3.07168	- 2.09227	+ 0.78704	- 0.44277	+ 7.08201
+ 4.36258	+ 0.21786	+ 2.17209	+ 2.85617	- 0.22922	+ 1.22600
+ 1.80722	- 0.26424	- 2.29219	+ 1.01929	- 0.19102	+ 2.27229
+ 1.27790	+ 1.02291	- 0.29148	+ 4.66280	- 0.29161	+ 0.51821
+ 0.19020	0.00000	- 0.19020	+ 0.50055	- 4.50052	+ 0.50055
+ 0.58607	+ 2.02281	+ 1.26230	+ 0.51258	+ 0.28896	+ 4.62015
+ 5.96160	+ 0.19287	- 1.35227	+ 1.77789	- 0.16261	+ 0.76220
+ 1.19152	- 0.17481	- 2.49972	+ 0.67426	+ 0.12629	+ 1.57080
+ 1.02422	+ 1.50285	- 0.42971	+ 2.46628	- 0.21679	+ 0.28222
+ 0.26492	- 1.17971	- 0.80256	+ 0.20227	- 0.17006	+ 2.71922
+ 2.26081	+ 0.12791	- 0.82449	+ 1.14927	- 0.09244	+ 0.49227
+ 0.72258	- 0.10269	- 2.11596	+ 0.40266	- 0.07629	+ 0.94971
+ 0.26494	+ 0.80662	- 0.22585	+ 1.82920	- 0.11628	+ 0.20668
+ 0.20927	- 0.71272	- 0.48752	+ 0.18229	+ 0.10218	+ 1.64976
+ 1.48129	+ 0.07299	- 0.28285	+ 0.66482	- 0.05247	+ 0.28229
+ 0.44709	- 0.06529	- 1.20922	+ 0.22222	- 0.06726	+ 0.58762
+ 0.22812	+ 0.48177	- 0.16086	+ 1.11047	- 0.06949	+ 0.12244
+ 0.12640	- 0.42220	- 0.29446	+ 0.11077	- 0.06222	+ 0.99642
+ 0.90164	+ 0.04206	- 0.20790	+ 0.40468	- 0.02252	+ 0.12720
+ 0.08722	0.00000	- 0.08722	+ 0.17770	- 1.29822	+ 0.17770
+ 0.27686	- 0.04049	- 0.81074	+ 0.12260	- 0.02927	+ 0.26882
+ 0.24992	+ 0.26692	- 0.10720	+ 0.84272	- 0.05289	+ 0.09401
+ 0.10229	- 0.25222	- 0.24062	+ 0.09051	- 0.42082	+ 0.81424
+ 0.69760	+ 0.02484	- 0.22822	+ 0.21210	- 0.02218	+ 0.12640
+ 0.20012	- 0.02927	- 0.28604	+ 0.11290	+ 0.02116	+ 0.26202
+ 0.12262	+ 0.22408	- 0.06222	+ 0.21620	- 0.02220	+ 0.02741
+ 0.02769	- 0.19722	- 0.12441	+ 0.02026	- 0.02844	+ 0.42484
+ 0.42062	+ 0.02100	+ 0.14228	+ 0.18871	- 0.02126	+ 0.08100
+ 0.11720	- 0.01216	- 0.24220	+ 0.06616	+ 0.01240	+ 0.12417
+ 0.09028	+ 0.12221	- 0.02874	+ 0.20242	- 0.01910	+ 0.02292
+ 0.02416	- 0.11672	+ 0.07922	+ 0.02891	- 0.01682	+ 0.26912
+ 0.26170	+ 0.01207	- 0.02222	+ 0.10848	- 0.00876	+ 0.04622
+ 0.04827	- 0.01002	- 0.20080	+ 0.02868	- 0.00722	+ 0.09012
+ 0.02228	+ 0.08116	+ 0.02222	+ 0.18707	+ 0.01189	+ 0.02079
+ 0.01928	- 0.06822	- 0.04624	+ 0.01721	- 0.00982	+ 0.12748
+ 0.14282	+ 0.00208	- 0.06912	+ 0.06422	- 0.00218	+ 0.02271
+ 0.04076	- 0.00296	- 0.11927	+ 0.02200	+ 0.00422	+ 0.02276
+ 0.04240	+ 0.04281	- 0.01240	+ 0.10260	- 0.00660	+ 0.01124
+ 0.01182	- 0.04044	- 0.02226	+ 0.01026	- 0.00282	+ 0.02221
+1.09912	- 0.40021	+72.89049	+72.40109	-18.12227	+59.72274

Table 5-8

in this example. The results obtained by these methods are recorded in Table 5-9.

Support Moments	Carry-Over Procedure	Synthetic Elimination
M_{ix}	+ 41.09915	+ 41.16508
M_{ly}	+ 73.40109	+ 73.40010
M_{kx}	- 0.40051	- 0.50143
M_{ky}	- 18.12727	- 18.10955
M_{mx}	- 72.894049	- 72.98405
M_{my}	+ 50.73374	+ 51.02418

Table 5-9

Support Moments

(f) Shear Force, Twisting Moment and Bending Moment Diagrams

The known support moments are now applied on the basic structures (Fig. 5-3), and the support reactions are computed from their statics.

The basic support moments at i, k and m are transformed to the principal axis of the spans. The shear force, twisting moment and the bending moment about the principal axes of the spans are shown on the unfolded structure (Fig. 5-7).

Basic Structure \overline{ijk}

$$\begin{aligned} \sum P_z &= +20.00000 \\ R_{iz} &= +12.88720 \\ R_{jz} &= +5.91478 \\ R_{kz} &= +1.19802 \\ M_{ijx}' = M_{jix}' &= +41.09915 \\ M_{kix}'' = M_{jix}'' &= +15.89837 \\ M_{ijy}' &= +73.40109 \\ M_{jiy}' &= -42.08691 \\ M_{jky}'' &= -56.63760 \\ M_{kly}'' &= -8.71680 \end{aligned}$$

Basic structure \overline{klm}

$$\begin{aligned} \sum P_z &= -40.00000 \\ R_{kz} &= +17.90773 \\ R_{lz} &= +24.86043 \\ R_{mz} &= -2.76785 \\ M_{klx}' = M_{lzx}' &= -15.49797 \\ M_{mlx}'' = M_{lmx}'' &= +72.89049 \\ M_{kly}' &= +9.41047 \\ M_{lky}' &= +93.11727 \\ M_{lmy}'' &= +59.98067 \\ M_{mly}'' &= -50.73374 \end{aligned}$$

Basic Structure \overline{mni}

$$\begin{aligned} \sum P_z &= 0.00000 \\ R_{mz} &= -1.07866 \\ R_{nz} &= -1.13336 \\ R_{iz} &= +2.21202 \\ M_{mnx}' = M_{nmx}' &= -7.49108 \\ M_{inx}'' = M_{nix}'' &= -43.01577 \\ M_{mny}' &= -88.49053 \\ M_{nmy}' &= -45.34413 \\ M_{niy}'' &= -16.19813 \\ M_{iny}'' &= +72.29240 \end{aligned}$$

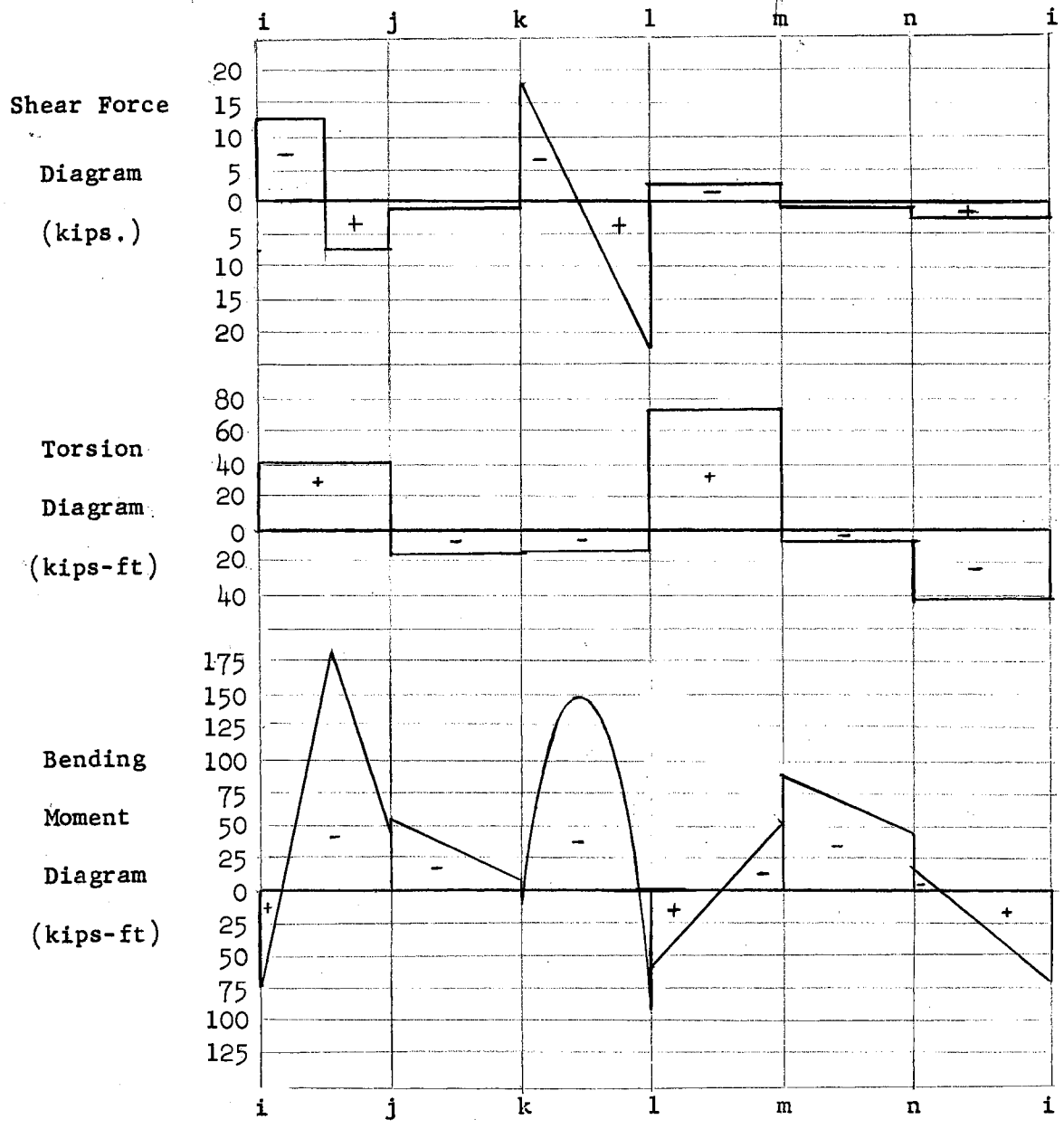


Figure 5-7

CHAPTER VI

SUMMARY AND CONCLUSIONS

6-1 Summary

The application of the six-moment equations to the analysis of planar continuous bent beams loaded laterally is presented in this thesis. The study is an extension of the analysis of continuous beams loaded in the plane by the three moment equation.

The continuous structure is isolated into appropriate basic structures, and the support moments, about the basic system of reference XYZ, are selected to be the unknowns. This is essential as the compatibility of deformations over a continuous support can only be achieved along a common reference axis. From the statics of the basic structures, the moments at a section are obtained and are expressed in terms of the selected unknowns by the transformation matrix. Angular constants of the basic structures are introduced and the deformation equations, in terms of these constants and the redundant moments are obtained by the Castigliano's method. Finally, by comparing the deformations over a continuous support, the six moment compatibility equations are obtained. The compatibility equations over the $(n-1)$ support of an odd span continuous bent beam having n supports are also derived. A carry over solution of the six moment equations is demonstrated. The procedure for the analysis of the problem is outlined and a numerical example is included.

6-2 Conclusions

The compatibility equations provide an adequate method in the analysis of planar continuous bent beams loaded laterally. However, it is observed, that, for the analysis of closed polygonal frame-works of continuous bent beams as considered in the numerical application, the carry over procedure is not an ideal method for solving the six moment equations. The convergency of the starting values is slow and the labor involved is more than solving by other methods such as Gauss's elimination, matrix inversion by a computer, and by Choleski's method.

The convergency of the carry over procedure is expected to improve in case of continuous bent beams, not as acutely inclined as analysed in the numerical example. The feasibility of applying the carry over procedure to such problems may then be considered. The study may be extended to continuous bent beams, not in one plane.

A SELECTED BIBLIOGRAPHY

1. Tuma, Jan J. "Analysis of Continuous Beams by Carry Over Moments," Journal of Structural Division, Proceedings, A.S.C.E., Vol. 84, No. 1762, September 1958.
2. Tuma, Jan J. "Analysis of Continuous Trusses by Carry Over Moments," Journal of Structural Division, Proceedings, A.S.C.E., Vol. 85, No. 2314, December 1959.
3. Tuma, Jan J. "Continuous Bent Members in Space," Lecture Notes, C.E. 5B4, Space Structures, Oklahoma State University, Summer 1961.
4. Childress, William H. "Analysis of Continuous Bent Members Loaded Out of Plane by the Carry Over Joint Moment Method," M.S. Thesis, Oklahoma State University, Spring 1962.
5. Tsui, Edward Y. "Analysis of Haunched Octagonal Girder Space Frames," Journal of Structural Division, Proceedings, A.S.C.E., January 1959.
6. Michalos, James. Theory of Structural Analysis and Design, New York: Ronald Press Co., 1958.

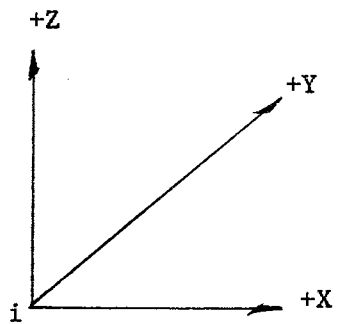
APPENDIX

In the first part, the signs of geometric quantities, loads, moments, and deformation are shown. Both the external and cross-sectional elements are considered.

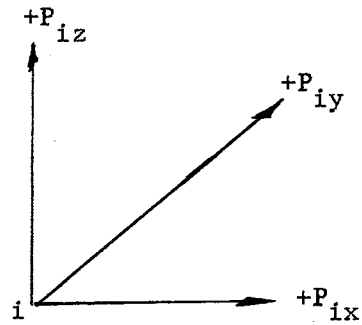
Several tables showing the application of the transformation matrix as discussed by Tuma (3) are presented in the second part of the appendix. The tables apply to the case of a general space structure (Table 5). For the planar problem treated in this thesis, ω_2 and ω_3 are zero so that the Z-terms vanish from the general transformation matrix.

Geometry and Forces

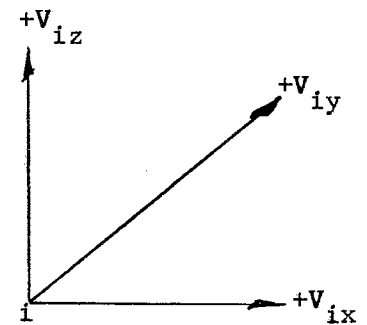
Part A - Table 1



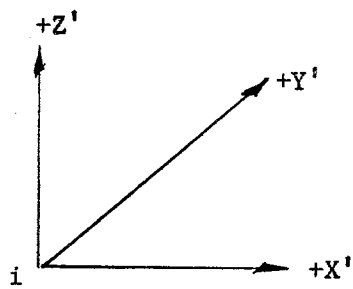
Basic Coordinates



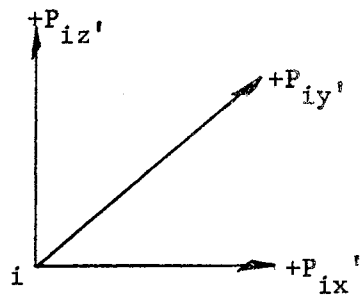
Basic Loads



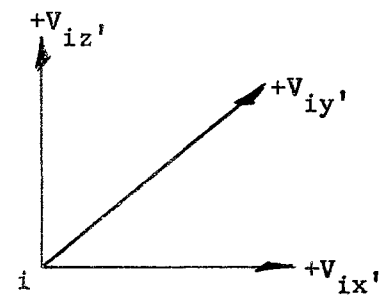
Basic Forces



Transformed Coordinates



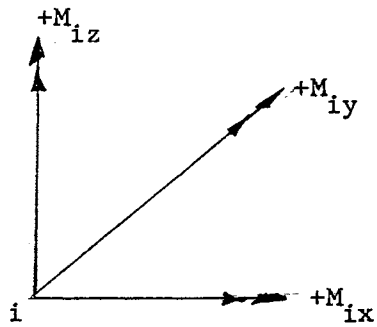
Transformed Loads



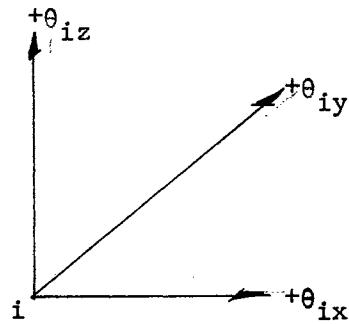
Transformed Forces

Moments, Rotations and Displacements

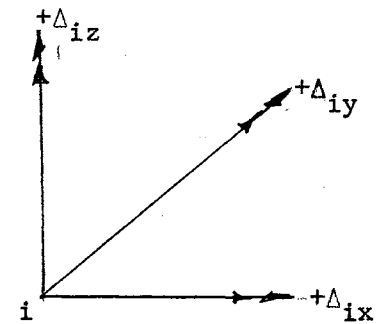
Part A - Table 2



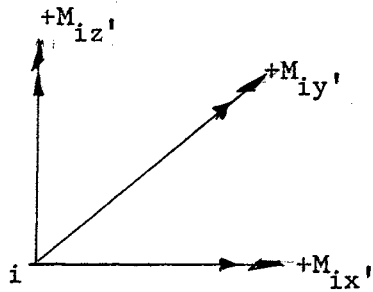
Basic Moments



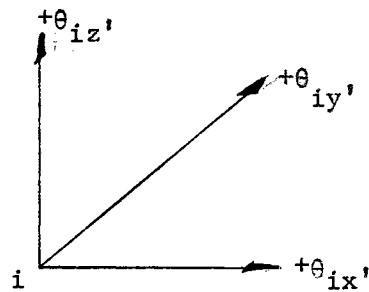
Basic Rotations



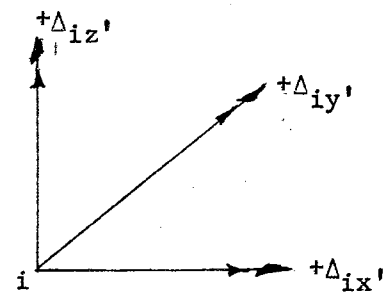
Basic Displacements



Transformed Moments



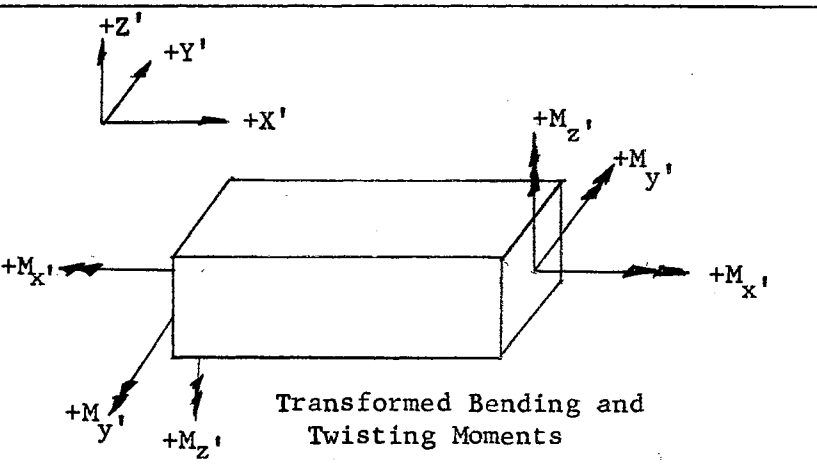
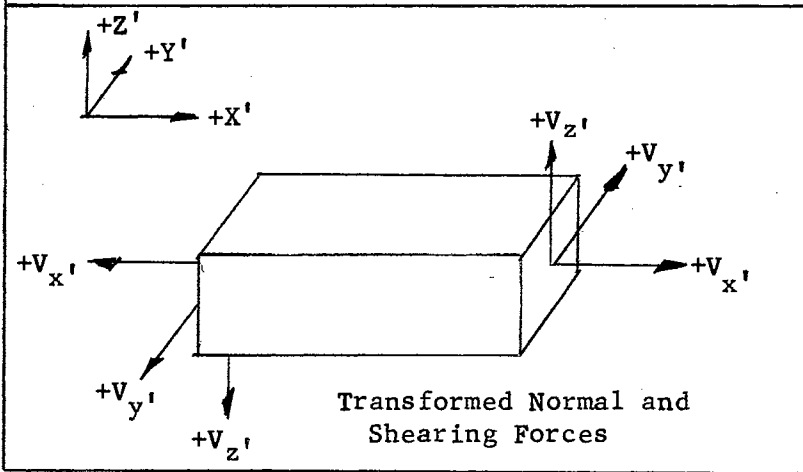
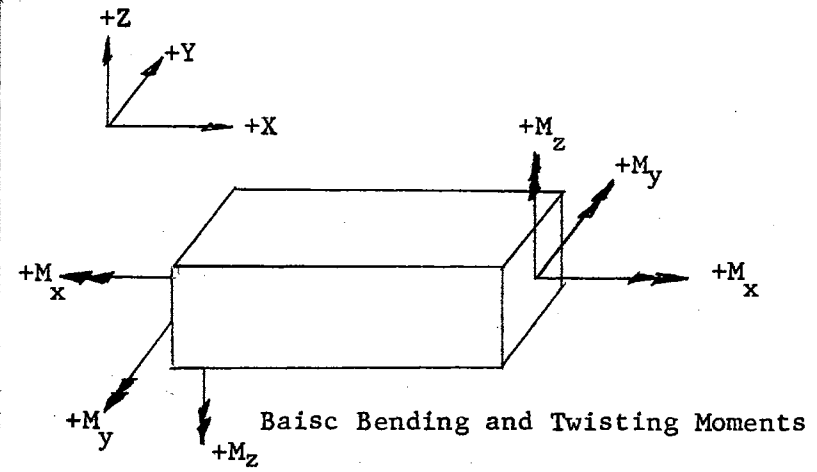
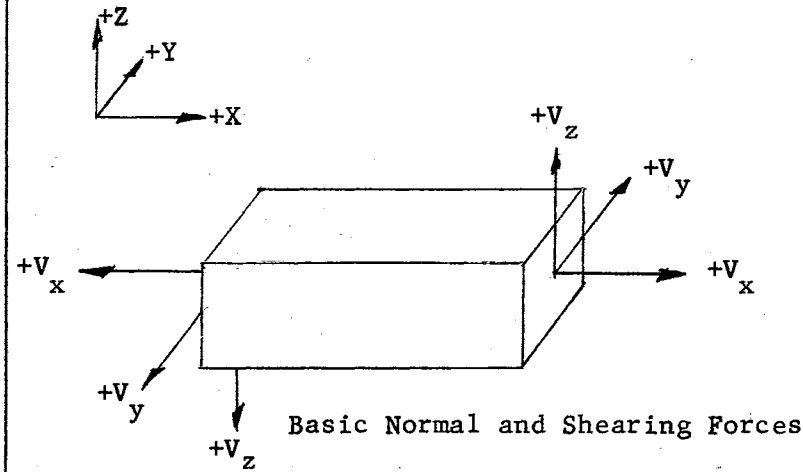
Transformed Rotations



Transformed Displacements

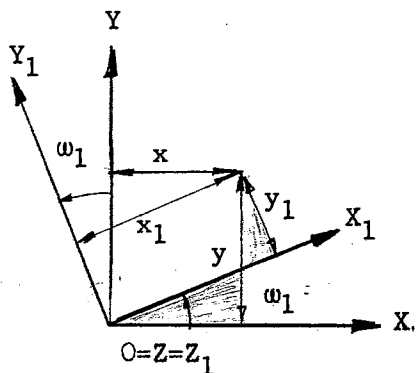
Cross Sectional Elements

Part A - Table 3



Transformation of Coordinates

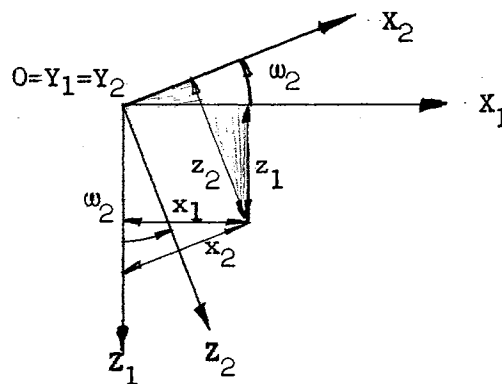
Part B - Table 4



Rotation ω_1

$$\begin{aligned} x &= x_1 \cos \omega_1 - y_1 \sin \omega_1 \\ y &= x_1 \sin \omega_1 + y_1 \cos \omega_1 \\ z &= z_1 \end{aligned}$$

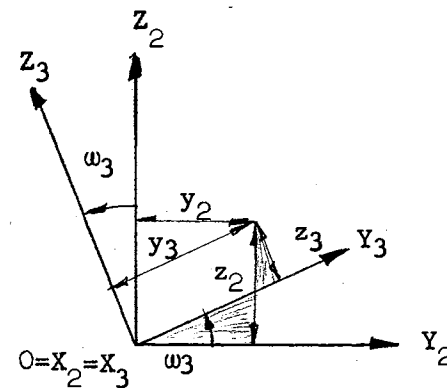
$$\begin{aligned} x_1 &= x \cos \omega_1 + y \sin \omega_1 \\ y_1 &= -x \sin \omega_1 + y \cos \omega_1 \\ z_1 &= z \end{aligned}$$



Rotation ω_2

$$\begin{aligned} x_1 &= x_2 \cos \omega_2 + z_2 \sin \omega_2 \\ y_1 &= y_2 \\ z_1 &= -x_2 \sin \omega_2 + z_2 \cos \omega_2 \end{aligned}$$

$$\begin{aligned} x_2 &= x_1 \cos \omega_2 - z_1 \sin \omega_2 \\ y_2 &= y_1 \\ z_2 &= x_1 \sin \omega_2 + z_1 \cos \omega_2 \end{aligned}$$



Rotation ω_3

$$\begin{aligned} x_2 &= x_3 \\ y_2 &= y_3 \cos \omega_3 - z_3 \sin \omega_3 \\ z_2 &= y_3 \sin \omega_3 + z_3 \cos \omega_3 \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 \\ y_3 &= y_2 \cos \omega_3 + z_2 \sin \omega_3 \\ z_3 &= -y_2 \sin \omega_3 + z_2 \cos \omega_3 \end{aligned}$$

Part B - Table 5

$$\alpha_x = \cos\omega_1 \cos\omega_2$$

$$\alpha_y = -\sin\omega_1 \cos\omega_3 + \cos\omega_1 \sin\omega_2 \sin\omega_3$$

$$\alpha_z = \sin\omega_1 \sin\omega_3 + \cos\omega_1 \sin\omega_2 \cos\omega_3$$

$$\beta_x = \sin\omega_1 \cos\omega_2$$

$$\beta_y = \cos\omega_1 \cos\omega_3 + \sin\omega_1 \sin\omega_2 \sin\omega_3$$

$$\beta_z = -\cos\omega_1 \sin\omega_3 + \sin\omega_1 \sin\omega_2 \cos\omega_3$$

$$\gamma_x = -\sin\omega_2$$

$$\gamma_y = \cos\omega_2 \sin\omega_3$$

$$\gamma_z = \cos\omega_2 \cos\omega_3$$

$$X = X'\alpha_x + Y'\alpha_y + Z'\alpha_z$$

$$Y = X'\beta_x + Y'\beta_y + Z'\beta_z$$

$$Z = X'\gamma_x + Y'\gamma_y + Z'\gamma_z$$

	X'	Y'	Z'
X	α_x	α_y	α_z
Y	β_x	β_y	β_z
Z	γ_x	γ_y	γ_z

Transformation Matrix

$$X' = X\alpha_x + Y\beta_x + Z\gamma_x$$

$$Y' = X\alpha_y + Y\beta_y + Z\gamma_y$$

$$Z' = X\alpha_z + Y\beta_z + Z\gamma_z$$

General Transformation Matrices

Part B - Table 6

	x'	y'	z'
x	α_x	α_y	α_z
y	β_x	β_y	β_z
z	γ_x	γ_y	γ_z

Coordinates

	$\Delta_{x'}$	$\Delta_{y'}$	$\Delta_{z'}$
Δ_x	α_x	α_y	α_z
Δ_y	β_x	β_y	β_z
Δ_z	γ_x	γ_y	γ_z

Displacements

	$\theta_{x'}$	$\theta_{y'}$	$\theta_{z'}$
θ_x	α_x	α_y	α_z
θ_y	β_x	β_y	β_z
θ_z	γ_x	γ_y	γ_z

Rotations

	$P_{x'}$	$P_{y'}$	$P_{z'}$
P_x	α_x	α_y	α_z
P_y	β_x	β_y	β_z
P_z	γ_x	γ_y	γ_z

Real Loads

	$V_{x'}$	$V_{y'}$	$V_{z'}$
V_x	α_x	α_y	α_z
V_y	β_x	β_y	β_z
V_z	γ_x	γ_y	γ_z

Real Forces

	$M_{x'}$	$M_{y'}$	$M_{z'}$
M_x	α_x	α_y	α_z
M_y	β_x	β_y	β_z
M_z	γ_x	γ_y	γ_z

Real Moments

	$\overline{P}_{x'}$	$\overline{P}_{y'}$	$\overline{P}_{z'}$
\overline{P}_x	α_x	α_y	α_z
\overline{P}_y	β_x	β_y	β_z
\overline{P}_z	γ_x	γ_y	γ_z

Conjugate Loads

	$\overline{V}_{x'}$	$\overline{V}_{y'}$	$\overline{V}_{z'}$
\overline{V}_x	α_x	α_y	α_z
\overline{V}_y	β_x	β_y	β_z
\overline{V}_z	γ_x	γ_y	γ_z

Conjugate Forces

	$\overline{M}_{x'}$	$\overline{M}_{y'}$	$\overline{M}_{z'}$
\overline{M}_x	α_x	α_y	α_z
\overline{M}_y	β_x	β_y	β_z
\overline{M}_z	γ_x	γ_y	γ_z

Conjugate Moments

VITA

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