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A SOLUTION OF THE BOLTZMANN'S EQUATION FOR THE ELECTRICAL CONDUCTIVITY OF A SLIGHTLY IONIZED GAS

(IN THE PRESENCE OF A WEAK MAGNETIC FIELD)

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A SOLUTION OF THE BOLTZMANN'S EQUATION FOR THE ELECTRICAL CONDUCTIVITY OF A SLIGHTLY IONIZED GAS

(IN THE PRESENCE OF A WEAK MAGNETIC FIELD)

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LIST OF SYMBOLS

$\overline{X}$	A bar over any quantity denotes that quantity as a vector.
X_y	A subscript is used to identify a quantity.
$(X)_1$	A quantity in parentheses with the subscript 1, 2, or 3 denotes the component of a vector in the direction of the subscript.
$\int$	Denotes the process of integration.
$\frac{\partial X}{\partial c}$	Denotes the partial derivative of the quantity x with respect to quantity c .
$X \cdot Y$	The dot between two quantities denotes the scalar product of the two quantities.
$X \times Y$	The cross between two quantities denotes the vector product of the two quantities.
$\frac{dx}{dc}$	Denotes the ordinary derivative of quantity x with respect to the quantity c .
$\log x$	Denotes the logarithm of the quantity x to the base "e".
$\exp x$	Denotes the exponential of quantity x .
$F(x)$ or $f(x)$	Denotes a function of the quantity x .
X^2	A superscript on a quantity denotes the power to which the quantity is taken.
X'	A prime is used to identify a quantity.

CHAPTER I

INTRODUCTION

The conductivity parameter with regard to electrical phenomena and with respect to ionized gases has for many years been a topic of more than casual interest by some of the foremost researchers and theorists in the fields of ionized-gas dynamics and plasma physics. The Bibliography reflects the names of many of those who have presented outstanding contributions.

Much of the ground work for the investigation of the phenomena of ionization was laid down in the early work of Maxwell (13) and Boltzmann (2), although the formulations of these two investigators were for the most part introduced in different contexts. Shortly following the turn of this century Lorentz (12) published a book dealing principally with the Theory of Electrons in metals,but which was to shed light on electron mobility in gases. Lorentz described a Boltzmann's distribution function to determine the conductivities of a "Lorentz gas". The "Lorentz gas" theory has remained ever since as a major jumping-off point in the discussion of the properties of ionized gases. The "Lorentz gas" was supposed to be a cloud of electrons whose mass was negligible compared to that of the atoms against which they collide.

Townsend's (19) book, Motion of Electrons in Gases was one of the earlier (1920's) publications dealing explicitly with particle gas dynamics. Townsend elaborated on the "mean free path" between electron-ion collisions and then deduced macroscopic properties for the gas. During this same period, Langmuir (10) was studying low pressure plasmas and generating his "space charge" theory of electron containment.

In the early and mid-1930's the now-famous and classical works of von Engle (21) were being published in the original German. The two-volume Elektrische Gasentladungen reviews the works of most major investigators up to that time and represents the most extensive work in the field during the 1930-1950 period.

Since 1950 a large number of significant publications has been forthcoming. Of special interest are the investigations by Alfven (1), Linhart (11), Vlasov (20), Spitzer (16), Delcroix (6), Montgomery (14), Chapman and Cowling (4), Green (5), Drummond (7), Druyvesteyn and Penning (8), Chandrasekhar (3), Rosenbluth (15), Jeans (9), and Taylor (18). The period since 1960 has been one of even greater effort with contributions principally covering the areas involving fully ionized gases, such as MHD generation and plasma propulsion.

The greatest problem unsolved in producing an exact theory of gas ionization is the most fundamental problem in any theory: where does one begin, or with what does one begin? As with any theory concerning the derivation of a quantity one expects to first agree upon a definition of that quantity. For the quantity of interest in this specific case, electrical conductivity of the ionized gas, a simple stated definition has seldom been agreed upon by investigators. Many authors do not specify in any sense what they intend the researcher to believe to be a common definitive agreement. Electrical conductivity in the case of classical conductors has been somewhat unanimously agreed to be an inherent property of the material, inherent, for example, as mass density is considered an inherent property. For some cases this is not a bad assumption; a functional relationship approximately correct for specific material, can be written which holds for that material over broad ranges of time, temperature, and any other parameter of interest, external or internal. The surprising thing is that electrical conductivity is seldom defined properly as a functional property, not of the material, but of the response of that material to an electrical field whether externally applied or internally generated or both. Any expression, therefore, for electrical conductivity must reflect the presence of an electric field either directly or indirectly. Electrical conductivity is explicitly defined in Chapter II.

The process of ionization is represented by the separation of electrical charges; it is the stripping of electrons from neutral gas atoms to produce positive ions or the impregnation of neutral gas atoms with electrons to produce negative ions. By either mechanism, if the gas were initially neutral, a separation of charge takes place. One cannot separate charges without developing an electric field, and the act of separation demands movement of charged particles. The movement establishes two other properties of the ionization process which must be considered, current density and magnetic flux density. Any theory proposing to derive an expression for electrical conductivity must have accounted for all three of the aforementioned properties of charge separation (ionization). The influence of ionization and ionization processes on the conductivity is treated in Chapter IV.

With the very beginning of the process of ionization and the separation of electrons from atoms, forces new to the classical kinetic theory of gases develop. In addition to the thermal-pressure forces, dispersion forces, and molecular momentum forces, the coulomb forces between unalike to alike charged particles become apparent, in fact become dominant. While the increase in the thermal pressure forces in classical kinetic theory accounts for the most significant part of the increase in molecular energy (or atomic energy) and is assumed to accompany a uniform temperature rise of the whole molecule (or atom), the thermal-pressure energy in an ionized gas is significantly found in the "free" electrons. (This is not precisely true; but so long as atoms remain intact, the assumption does not materially effect the results). For this reason and because the realm of influence of free electrons is wide in that interaction occurs at "coulomb distances" with many other particles or many different ions, classical gas-kinetic theory cannot be applied without modification. The thermal-pressure energy associated with a free electron is entirely contained in its momentum and, thence, is a function of its position, velocity magnitude, and velocity direction. The interchange of energy or the transfer of momentum during an interaction between particles depends in addition to particle momentum and position on interaction probability time. The classical gas-kinetic theory does not account in any proper way for this dependency. For this reason a new theory is justified and in fact has been developed in part, and for specialized models, for sometime. It is the development of the statistical-probability theory pioneered by such profound searchers as Maxwell and Lorentz that occupies the energy of modern investigators. Papers published since 1960, and indeed since 1950, have almost unanimously attempted to elaborate upon this theory. Chapter III discusses the significance of classical gas-dynamic theory to plasma theory, and the influence of the statistical model is treated in Chapter V.

Numerous theories have been suggested to define and explain conductivity. Theories take on a wide variety -- from extremely specific to very general, from apparently original to obviously extensional, and from complicated to over simplified. However, the major portion of these works can be broken into two groups which rely on greatly differing frames of approach. In the macroscopic approach the ionized gas is tested in the classical hydrodynamic manner, although the approach is more complicated than for ordinary fluids since the ionized gas is a mixture of several interacting constituents. In the microscopic approach the ionized gas is treated as a statistical interation of the atomic particles composing the gas. The two methods are generally complimentary, but the former has given away to the latter in recent years, as the more popular concept. In the end the usefulness of a theory can best be evaluated in light of observational proof from controlled experimentation. Unfortunately, experimentation under the most ideal of conceivable laboratory conditions does not completely bear out the validity of any one theory as yet proposed. This seems to be due to two very significant realities: first, the laboratory models (including all conditions of generation, measurement, and confinement) influence the net forces acting on the gas whether considered as macroscopic fluid or microscopic particles; second, no theory as yet proposed has been carried out without the involvement of a large number of simplifying assumptions, both mathematical and physical. In this regard a description of a laboratory model to be introduced into the theoretical derivation of the properties of the plasma presents a significant constraint upon the generality of the theory and has not induced the theorist to devote a great deal of time to considering specific model constraints. This has seemed reasonable in light of the fact that plasma theory is in its most energetic period of evolution. The years since 1960 have experienced a tremendous growth in both experimental and theoritical research into the understanding of ionized gases.

Beginning with the Maxwell's probability distribution (Appendix A) as the statistical frame of references, the Boltzmann's transport equation (Appendix B) is expanded in Chapter VI to generate three coupled equations relating the distribution function and its perturbations in velocity space. The solution of these ordinary differential equations allows the current density to be calculated and the conductivity to be determined in complex form. The theory discussed in Chapter VI deviates substantially from that pursued in the literature, and the assumptions necessary to produce a mathematical solution represents a significant extension to the techniques currently employed. It is possible, as shown in Chapter VI, to solve the Boltzmann's equation without neglecting cross-product terms or higher order terms.

CHAPTER II

THE DEFINITION OF ELECTRICAL CONDUCTIVITY WITH RESPECT TO A CONDUCTING GAS

Many definitions can and have been imposed on the general and rather broad term "conductivity", depending upon the context and the mutual agreement between investigators. The conductivity in any case will be an approximation, depending upon a variety of simplifying assumptions and upon the proposed theory. It is unfortunate that there is no universally agreed upon general definition for electrical conductivity; it is significant to note that even for specific cases a considerable variation in the definition exists.

Conductivity is often spoken of as though it were one of the simple properties of the material, much as mass and charge may be properties. This is obviously not true even as a functional relationship evolving time and temperature variables. Conductivity is a measure of the responsiveness of a material to external excitation. This responsiveness is normally a functional relationship incorporating many variables. The external excitation is always an applied electric field gradient. The response is always a net transfer of electric charge with respect to the gradient. Within a material, electric field intensities, electric current densities, and magnetic field intensities may be generated internally by whatever means. The relationships between an internally generated excitation and its response can often appear similar in every respect to conductivity; many authors refer to conductivity in this regard. It is not inconceivable that one would wish to study such an internal relationship, and it would not be deemed incorrect to speak of such a relationship as electrical conductivity. However, the term should carry an adjective which would identify the considerations involved. Such internal reactions as dispersion or diffusion, polarization, and ionization will soundly affect conductivity.

Although the necessary external excitation to the definition of conductivity is the presence of an electric field, this is not necessarily sufficient in a more general sense. Other external excitations cause major influences on the conductivity; therefore, the conductivity functional relationship should express these as independent variables. The influence of gas composition, time, temperature, density, gravitation, and magnetic intensity must be considered.

This paper presents a derivation of an expression which relates the current density, J, to the electric field intensity, E, in the following manner:

$$\bar{J} = \sigma \bar{E}. \tag{1}$$

σ is defined as the conductivity. $\bar{E}$ is a vector whose direction is specified; the current density, $\bar{J}$, is expected to be a vector quantity with quadrature components; σ is, therefore, a compound function involving at least three terms which allow $\sigma \bar{E}$ to have quadrature components.

Under these conditions equation (1) can be rewritten:

$$(J_1)_1 + (J_2)_2 + (J_3)_3 = (\sigma_1 E)_1 + (\sigma_2 E)_2 + (\sigma_3 E)_3 \tag{2}$$

From which the conductivity components with respect to $\overline{E}$ can be written:

$$\sigma_1 = J_1 / E, \quad \sigma_2 = J_2 / E, \quad \sigma_3 = J_3 / E \tag{3}$$

where direction 1, 2, and 3 are understood to be in quadrature. For the purpose of this paper conductivity is defined in its general sense to be the functional relationship, $\sigma(E,J)$, expressed by equation (1). In particular it is a complex quantity which relates the quadrature current density responsiveness of an ionized gas to an applied electric field.

CHAPTER III

CLASSICAL GAS KINETIC THEORY AND ITS SIGNIFICANCE TO THE STUDY OF IONIZED GASES

In this chapter it is proper to preview the task which the paper strives to accomplish and to review the contributions upon which it is set.

Two classical achievements are of particular significance. The first is due primarily to the early works of Boltzmann and Maxwell and later extended by Chapman, Cowling, Spitzer, et.al. The works of these men and others who followed their theory began with a phase-space distribution function called the Maxwell-Boltzmann equation or some small modification thereto and developed expressions for the electrical conductivity. The second achievement is primarily due to the work of Lorentz in studying optical dispersion. This theory begins with the premise that the system can be defined in terms of a classical oscillator. Extensions and modifications to this theory have likewise followed. In both cases mentioned, the results are much the same. This has not been because sufficient simplifying assumptions were necessary to relieve the mathematical burden that the results bespeak loudly of the influence of these assumptions. The assumptions were essentially the same; thus it is not startling to have observed similar results.

Having briefly summarized these achievements, this paper presents a derivation of an expression for the electrical conductivity which is founded in the well-established kinetic theory of gases based on a statistical model and extended by the application of the classical electrmagnetic theory proposed by Maxwell. In classical kinetic theories of gases the gas is generally visualized in one or the other of two different ways, macroscopically or microscopically. In the former the gas is considered as a uniform fluid, and the properties of the gas are considered as many individual but uniform particles. One is interested, for example, in particle velocity, particle forces, particle temperatures, etc. So far as a gas remains completely uniform, or on the average uniform, with respect to the particles that make up the gas, the macroscopic and the microscopic models produce much the same results. This is because the fluid acts precisely like the algebraic summation of its uniform constituents. It is not necessary to examine the gas in a model more microscopic than that of its particles at the point where uniformity of substance exists. In microscopic classical gas-kinetic theory the point of particle uniformity is the gas molecule; or if the gas is monatomic, the point of particle uniformity is the gas atom. The microscopic study of a gas involves the interaction of particles. Particles are known to "interact" in a variety of ways, and the manner of "interaction" depends upon the type of particle to a considerable extent. Thus the validity of an interaction theory for gas particles depends largely upon the number, type, and energy of the particles involved. Classical gas-kinetic theories, especially those of Boltzmann, et.al., appreciate the variation of energies of individual particles even under steady-state conditions; it appreciates the possible variation in type of particle to the extent that gas mixtures can be considered. Because particle type and energy are allowed to be variables, the theory becomes necessarily intangled with statistical mathematics, and a probability relationship is required to define the model. The theory, however, does not generally appreciate the type-variation of interaction. As long as a "classical" gas is the subject of interest, interaction is, for the substantial part, of a single kind, called collision. Particles are seen to "bump" into one another with insuing transfers of momentum and energy. Only when the subject particles are considered to take on properties which produce fields is there necessity to re-evaluate and redefine the interaction phenomena between particles. Gravitational and electromagnetic fields must be considered. The gravitational field effects do not drastically differ between particles and essentially are functions of the mass property of the particle. Gravitational field forces are generally neglected in kinetic theories as being small in comparison to other particle forces or as on-the-average tending to cancel out. This policy of neglecting inter-particle gravitational field forces is not altogether valid, but it seems justified in that its effect does not seriously disrupt the "collision" theory of interaction between particles with energies insufficient to influence nuclear reactions. In other words, at atomic distances and beyond inter-particle gravitational field effects are generally small. When particles take on net charge, electric and magnetic fields can result. The effects of these fields are appreciable; their interaction distances are atomically long, and their result is to require complete revision of the collision theory of interaction. This, more than any other, is the reason the classical kinetic theory of gases breaks down when the gas particles approach energies which produce ionization.

Ionization is here defined as the process of stripping neutral atoms or molecules of one or more electrons or as the process by which neutral atoms or molecules accept one or more electrons. Simply, ionization is the process by which atoms or molecules, originally neutral, take on charge. Therefore, the instant at which gas energies become sufficient to produce ionization is that time when particle interaction must deviate from that proposed in classical gas-kinetic theory. An additional aspect of classical theory which must be changed as "forces at a distance", or field forces, become appreciable is the proposition that interactions can be considered "binary" for anything but a first approximation. Binary, as the word implies, means two at a time, and a binary interaction means an interaction by only two particles at one time. As interaction distances become long, the collision theory becomes invalid; and with it the assumption that interactions are binary must be abandoned. Multiple simultaneous interactions between each and every particle must, then, be considered.

Because classical kinetic theories have been verified experimentally for the derivation of transport phenomena in gases with energies below ionization levels, and because the process of ionization has been observed to progress smoothly as a gas begins to be ionized, any theory which correctly produces an expression for conductivity of a slightly ionized gas must be soundly based on kinetic theories proven for gases up to ionization levels. The generation of a theory which properly produces an expression for electrical conductivity in ionized gases is expected, or so it would seem, to logically begin with the case of a simple gas under conditions of slight ionization. The theory could be extended to include states of partial and complete ionization as a natural evolution.

CHAPTER IV

IONIZATION PROCESSES AND THEIR INFLUENCE ON THE ELECTRICAL CONDUCTIVITY

Ionization, defined previously as the absorption or depletion of one or more electrons by an otherwise neutral gas atom or molecule, can be achieved in a variety of ways. In each case, however, the energy state of the electrons in the shell must be changed. Of interest here is the process of ionization by which a neutral atom loses an electron to produce a positive ion-electron pair. Macroscopically the gas is always neutral, there being exactly the same number of positive ions as electrons present at any time. Ionized gases of this type automatically exclude such gaseous states of ionization as electron clouds and gases of a single particle type. A gas which is initially neutral and ideally confined will remain electrically neutral, though ionized, outside a spherical cross section of radius d surrounding each particle of interest. This distance depends on several properties of the gas particle. However, for each particle type within the gas there is a distance, d, beyond which its coulombic field cannot be detected; all particles beyond this distance become shielded from the electrical presence of the subject particle. The distance is called the Debye length, the Debye radius, or the Debye shielding distance. Because of this coulombic shielding an external electric field applied to the gas causes no significant separation of charged particles within the gas, so long as the Debye radius is small with respect to the dimensions of the confinement. A gas of this type becomes very similar to a metallic conductor, in that no polarization or displacement currents are allowed, when the gas is in ionization equalibrium.

In order to produce a positive ion-electron pair a bound electron must be freed. Electrons bound to a nucleus may normally exist in a number of states, each particular to some minimum energy level. In order to reach a secondary state the electron must receive and hold a higher minimum level. These minimum energy-level bands are discrete, and an electron cannot exist in states between these discrete bands. The energy of an electron can be referred to as thermal energy for which a temperature can be designated, but the energy may equally well be described in terms of kinetic energy for which a velocity can be assigned. An electron in some discrete energy level may receive additional energy sufficient to raise its state to the next higher level. From this higher level two changes may occur; the energy of the electron may decay radiatively, allowing the energy state of the electron to reduce to the next stable and lower level, or, additional energy may be added in which case the electron may achieve a yet higher band and remain bound or be freed. If the energy is sufficient to allow the electron to escape the highest energy band, ionization is said to have occurred.

The simplest atom has a single electron bound to a nucleus composed of a single proton. When this electron is freed, the proton represents a completely ionized atom. A more complex atom may contain many proton-electron pairs. If energy is added in sufficient quantity to free the highest energy electron, the atom is said to be "singly ionized," and the amount of energy required is referred to as the "first ionization potential." Similarly, when energy equivalent to the "second ionization potential" is supplied and another electron is freed, the atom is "doubly ionized" and so forth. The energy necessary to produce ionization need not be added in lump form but may result due to many succeeding additions of lesser amounts; however, when the critical maximum energy for existence in any one band is reached, transition to the next allowable energy band or to freedom occurs immediately.

One of the simplest means of adding energy to a gas, and hence producing ionization, is to raise the gas temperature. Adding heat to a gas slowly, raises the temperature of the particles somewhat uniformly; thus, the kinetic temperature of both nucleus and electron remains approximately equal. Raising the temperature of the nucleus does not directly, in itself, produce ionization; the necessary ingredient is that electron temperature or energy becomes sufficiently large. Because the mass ratio of nucleus to electron is large, the majority of heat added is soaked up by the nucleus rather than the electron and does not appreciably assist ionization directly. Therefore, ionization generated by slow thermal heating, while simple, is an extremely inefficient process unless a thermally hot gas is desired. There are other ways to produce ionization through a rise in temperature; one of the most common methods utilizes the shock technique. By producing a pressure shock wave through the gas, pressure energy in the wave is applied to the gas particles as the wave passes. This energy is delivered very quickly and, therefore, is almost entirely absorbed by the electrons along the wave front. Shortly after the front has passed, the energy in the electrons dissipates to become absorbed by the nucleus mass, thus raising the mean gas temperature slightly. Ionization can readily occur along the shock wave, while the total energy delivered to the gas remains quite small. Ionization is transient; thermal efficiency is high; and the gas remains thermally cool.

Ionization can be induced in a gas by the external application of a strong electric field. The field accelerates a few random free electrons, raising their kinetic energies. These accelerated particles strike neutral atoms, deliver momentum to the atoms, and knock out some bound electrons. These are in turn accelerated to multiply the action. Hence, energy in the electric field is converted to produce ionization, but the process requires the initial presence of a few free electrons.

The derivations described in this paper are produced by considering the ionization process to be dynamically stable and the gas to maintain the property of electrical neutrality. The process of ionization is dynamic in that ionization and deionization are occurring simultaneously and continuously. It is considered stable if the rate of ionization equals the rate of deionization. Electrical neutrality requires that external electric force fields produce no significant charge separation across the gas and no net force on the gas; in other words the ionized gas acts like a normal conductor.

CHAPTER V

THE STATISTICAL MODEL

In classical gas-dynamic theory it is not necessary to consider the gas in terms of individual particles, but it is sufficient to investigate the gas in terms of the external forces acting on the gas. Properties of the gas can be determined in an average way which does not require a microscopic treatment. However, these same gas properties can be determined by an altogether different approach which observes the gas as individual atoms or molecules under thermal agitation and in collisions with one another.

Theories of ionized gases have been proposed utilizing both macroscopic and microscopic approaches; it appears, however, that especially where ionization must be contended with, the microscopic approach holds the most promise.

Where no ionization takes place and at normal pressure and temperature collisions between gas molecules occur at a rate of approximately one thousand per micro-second. Because of the large number of collisions, particles have a distribution of velocity with individual changes in velocity magnitude and direction occurring with each collision. Theoretically, the velocity distribution contains particles that at any one time have velocities varying from zero to very large in magnitude, and with no externally applied force the distribution in velocity direction contains particles with all directions represented. [Assumption — For a gas with no external forces applied and in thermal equilibrium the particle velocity directions for any volume element will be entirely random. Under these conditions the gas is assured isotropic.] Because for any volume element of the gas there is no net observable velocity direction, only particle velocity magnitudes will be of significance in the distribution function.

Let dn be the number of particles which occupy a given volume element of the isotropic gas that have random thermal velocities $c_1 + dc_1$, $c_2 + dc_2$, and $c_3 + dc_3$ in the quadrature velocity-space coordinate system. The volume element has position-space coordinates of s_1 , s_2 , and s_3 . The particle number, dn , is proportional to both the velocity-space element, dc , and the position space element, ds ; the proportionality term must be a function of not only position and velocity but also of time.

$$dn = f(s,c,t)\,ds\,dc \tag{4}$$

Since

$$ds = ds_1 ds_2 ds_3 \quad \text{and} \quad dc = dc_1 dc_2 dc_3 \ ,$$

$$dn = f(s,c,t)\,ds_1 ds_2 ds_3\, dc_1 dc_2 dc_3 \ . \tag{5}$$

The volume element consists of both position-space elements and velocity-space elements and is defined in the literature as a phase-space element. The function f(s,c,t) is referred to in the literature as a distribution function. [Assumption — Because of thermal equilibrium and no time-varying forces the function f(s,c,t) is constant with time and can be written f(s,c)]. If properties of the gas are at equilibrium with respect to time, the particle number becomes:

$$dn = f(s,c)dsdc \tag{6}$$

The function $f(s,c)$ is referred to as a phase-space distribution function. Each species of the gas will have its own distribution function; hence, for the r species

$$dn_r = f_r(s,c)ds_r dc_r \quad . \tag{7}$$

The number density of a given species r which has velocities between c and $c + dc$ is

$$dN_r = \frac{dn_r}{ds} = f_r dc \quad . \tag{8}$$

Thus,

$$N_r = \int dN_r = \iiint_{-\infty}^{\infty} f_r dc_1 dc_2 dc_3 = \int_{-\infty}^{\infty} f_r dc \tag{9}$$

where f_r approaches zero when c approaches ± infinity. The mass density for species r with particle mass m_r is

$$\rho_r = N_r m_r \quad . \tag{10}$$

The total mass density for all species present in the gas is

$$\rho = \sum_r \rho_r = \sum_r N_r m_r \quad . \tag{11}$$

The random kinetic energy associated with a particular particle of the r species is given by

$$KE_r = \frac{1}{2} m_r c_r^2 \quad . \tag{12}$$

The average random kinetic energy for all particles of the r species is

$$KE_r = \int_{-\infty}^{\infty} \frac{1}{2} m_r c_r^2 f_r \, dc_r \tag{13}$$

where $f_r dc_r$ is the number of particles of r species having velocities between c_r and $c_r + dc_r$, and where integration over the distribution function determines the process of averaging.

The average random kinetic energy of an r particle can be expressed as a function of the average thermal kinetic energy (assuming a Maxwellian energy distribution).

$$KE_r = \frac{3}{2} k T_r \tag{14}$$

where k is the Boltzmann's constant and T_r is the absolute temperature of the r particle. [Assumption — The number of particles of any species having energy in a given intremental range obeys the Maxwell's energy distribution function, namely:

$$\frac{dN_r}{dE_r} = 2\pi N_r E_r^{1/2} \left(\frac{1}{\pi k T_r}\right)^{3/2} e^{-KE_r/kT_r} \quad \Big]$$

The average random thermal energy per unit volume is

$$KE_r = \frac{3}{2} N_r k T_r \quad . \tag{15}$$

From equations (13) and (15) the average kinetic energy is equivalent to the average thermal energy.

$$\frac{3}{2} N_r k T_r = \int_{-\infty}^{\infty} \frac{1}{2} m_r c_r^2 f_r \, dc_r \tag{16}$$

Let w_r be the average velocity of the r species and w be the mass-average velocity for all species (to differentiate from the random thermal velocity c_r) . If w'_r is the particle velocity, then

$$w_r = \frac{1}{N_r}\int_{-\infty}^{\infty} w'_r f_r \, dw'_r \quad , \tag{17}$$

and

$$w = \sum_r \rho_r w_r \Big/ \sum_r \rho_r \quad . \tag{18}$$

Now the random thermal velocity can be expressed with respect to the mass-average velocity as

$$c_r = w'_r - w \tag{19}$$

or with respect to the average velocity as

$$c_r = w'_r - w_r \quad . \tag{20}$$

Since w_r was defined as the average value of w'_r , the average value of c_r is zero.

From equations (19) and (20) the average value of c_r is v_r .

$$v_r = w_r - w = \frac{1}{N_r}\int_{-\infty}^{\infty} c_r f_r dv_r \quad , \tag{21}$$

where v_r is referred to as the diffusion velocity of the r species and is equal to the difference between the average particle velocity and the mass-average particle velocity.

The electrical current density is defined by the relationship

$$J_r = N_r e Z_r w_r \tag{22}$$

where e is the unit electrical charge on the particle (equal to the magnitude of the charge on one electron), and Z_r is the number of net unit charges (negative or positive) on the r particle.

$$J_r = N_r e Z_r (v_r + w) \tag{23}$$

For all species

$$J = e\sum_r N_r Z_r v_r + ew\sum_r N_r Z_r \quad , \tag{24}$$

where the conduction current j is given by the first term on the right side of equation (24) and the convection current j' is given by the second term on the right.

$$j = \sum_r N_r e Z_r v_r \tag{25}$$

$$j' = \sum_r N_r e Z_r w \tag{26}$$

[Assumption — The gas under consideration will be electrically neutral in every volume element.]

If the gas is electrically neutral, the $\sum_r N_r e Z_r$ is exactly zero; hence,

$$J_r = j_r = N_r e Z_r v_r \quad , \tag{27}$$

and

$$J = j = \sum_r N_r e Z_r v_r \quad . \tag{28}$$

The conduction current density is, in the absence of external forces, equal to the diffusion (or dispersion) current density. In terms of the random thermal velocity equation (28) becomes:

$$J = \sum_r N_r e Z_r \frac{1}{N_r}\int_{-\infty}^{\infty} c_r f_r dc_r = \sum_r e Z_r \int_{-\infty}^{\infty} c_r f_r dc_r \qquad (29)$$

With the derivation of equation (29) the calculation of the current density is simple only when $f = f(s,c)$ is simple. This is never the case. In the succeeding chapter considerable effort is exhausted in the deduction of an approximate formulation for the velocity distribution function.

CHAPTER VI

SOLUTION OF THE BOLTZMANN'S EQUATION AND CALCULATION OF ELECTRICAL CONDUCTIVITY

The electrical current density in the units of charge per unit geometrical area can be defined for one species of charged particle as

$$\bar{j} = neZ\bar{v} \tag{30}$$

where n is the number density of particles in the units of per geometrical volume, e is the basic charge unit equivalent to the electron charge. Z is the number of unit electron charges on the particle, and $\bar{v}$ is the diffusion or mean drift velocity of the particles in the units of geometrical length per unit time.

$$\bar{J} = \sum_r \bar{j}_r = \sum_r n_r e Z_r \bar{v}_r \tag{31}$$

is the total current density due to the contribution of r species of particles composing the gas. From this definition of current it is obvious that the contribution directly to $\sum_r \bar{j}_r$ will be composed only of those species of the total gas species, r, which carry units of electrical charge $Z_r \neq 0$. Furthermore, the summation implies that the net contribution to $\bar{J}$ will depend not only upon the magnitude and sign of Z_r but also upon the magnitude and direction of the velocity drift $\bar{v}_r$.

Hence, for a common direction of particle drift the positive-ion contribution will reduce the electron and any negative-ion contribution.

In order that the equation (31) be solvable, it is necessary that the gas composition at all times be calculable; and the number density of all species must be known. [Assumption — The gas composition and number density of all particles are determinable at every point of the time-phase-space.] The diffusion velocity can be related to the random thermal kinetic velocity by the relationship,

$$\bar{v}_r = \int_{-\infty}^{\infty} \bar{c} f_r d\bar{c} \quad , \tag{32}$$

according to the definition for formulating the average. [Assumption — The kinetic velocity $\bar{c}$ contains only translational components. In other words, vibrational and rotational velocity components are entirely negligible for the particles under consideration. This assumption is generally invalid for heavy molecules. Assumption — The gas is composed of only constituents having simple atomic or molecular structures.] If $\bar{v}_r$ is to be calculable, the distribution function f_r must be known; it is necessary that an expression for $f_r(\bar{s},\bar{v},t)$ be available. The distribution function f_r can be determined by a solution of the Boltzmann's equation. A Boltzmann's equation for each species of the gas is necessary. The collision term,

$$\left(\frac{\partial f_r}{\partial t}\right)_c \quad ,$$

must be expressed in terms of the distribution functions, and r equations must be solved simultaneously. Because this is usually impossible, various simplifying assumptions are normally employed. For this reason an exact theory for conduction in ionized gases has never been determined.

Two theories are presently receiving major consideration. Both theories have received attention by many investigators; they are identified as the Lorentz method, or method of binary interaction, and the Fokker-Planck method, or the method of simultaneous-multiple interaction. The difference between the two theories arises from the manner in which the collision or interaction term is evaluated. Considering binary interaction, the interaction term is defined to be the time-rate-of-change of the distribution function. When multiple interactions are considered, the interaction term is proportional to the first and higher time-derivatives of the distribution function. Sutton and Sherman (17) have shown that under the approximations normally employed, the two theories produce identical results.

In the case of binary interactions the collision term in integral form is derived in the Appendix B and repeated here for convenience in the following form.

$$\left(\frac{\partial f}{\partial t}\right)_c = \int\left[(f'f_1' - f\ f_1)|\bar{v} - \bar{v}_1|\iint r\,dr\,d\phi\right]d\bar{v} \tag{33}$$

The complete Boltzmann's equation, repeated from the Appendix B, is

$$\frac{\partial f}{\partial t} + \bar{v} \cdot \frac{\partial f}{\partial \bar{s}} + \frac{\bar{F}}{m} \cdot \frac{\partial f}{\partial \bar{v}} = \left(\frac{\partial f}{\partial t}\right)_c = \int_{-\infty}^{\infty} \left[(f'f_1' - ff_1) \left| \bar{v} - \bar{v}_1 \right| \int_0^{2\pi} \int_0^{r_{max}} r dr d\phi \right] dv \tag{34}$$

$f(\bar{s},\bar{v},t)$ is the distribution function of the incident particle before interaction; it represents the number of particles in the phase-space volume-element per second in the units of per length-cubed velocity-cubed time. $f'(\bar{s}',\bar{v}',t)$ is the distribution of the incident particle after interaction. $f_1(\bar{s},\bar{v}_1^*,t)$ is the distribution of the scattering particle before interaction. $f'_1(\bar{s}',\bar{v}'_1,t)$ is the distribution of the scattering particle after interaction. m is the incident particle mass. r is the scattering impact parameter in units of length. ϕ is the angle between the normal to the scattering plane and an arbitrary reference direction in the plane normal to both the incident path and the plane of scattering. $\bar{F}$ is the superposition of all external forces in units of mass-length per time-squared. $\bar{v}$ is the incident particle average velocity in the laboratory frame of reference in the units of length per time.

External forces might include many different types, but for the purpose of this paper, the forces of interest are identified as follows: (a) For charged particles

$$\bar{F} = (eZ/m)(\bar{E} + \bar{v} \times \bar{B}) \quad . \tag{35a}$$

(b) For neutral particles

$$\bar{F} = 0 \ . \tag{35b}$$

[Assumption — For particles carrying electric charge the external force is given by equation (35a); for neutral particles this force is negligible.] Replacing the laboratory velocity $\bar{v}$ by the random velocity $\bar{c}$ and noting the assumption that $\bar{c}$ is translational, does not change the form of the equation (34); with this shift in coordinates the Boltzmann's equation becomes:

$$\frac{\partial f}{\partial t} + (\bar{c} + \bar{w}) \cdot \frac{\partial f}{\partial \bar{s}} + \frac{\bar{F}}{m} \cdot \frac{\partial f}{\partial \bar{c}}$$

$$= \int \left[\left(f' f_1' - f\, f_1 \right) \left| \bar{c} - \bar{c}_1 \right| \int \int r\,dr\,d\phi \right] d\bar{c} \tag{36}$$

If the gas is only slightly ionized, there will be few electrons or ions in a volume element; on the other hand, the number of neutrals will be correspondingly high. The thermal motion of the heavy particles may be neglected in comparison to the motion of electrons because of the large mass ratio; this is Lorentz's approximation. In the case of a slightly ionized gas the approximation is a valid one. Because the number of electrons and ions in a volume element is low, interactions between electron-electron and electron-ion will be few; and because the thermal motion of both ions and neutrals is relatively low, the interactions between ion-ion, neutral-ion, and neutral-neutral are neglected. [Assumptions —

Interactions of interest are those between electron and neutral. Electrical current is a function only of electron mobility. Neutrals are considered to be unaffected by electron impact. The gas is uniform in geometry space. Particle interactions are elastic. Ionization equalibrium has been achieved.] For these assumptions there is only one Boltzmann's equation to be solved. In equation (36)

$$\frac{\partial f}{\partial t} = 0 \ .$$

This term relates the change in the electron distribution function with respect to time while geometry and velocity components remain constant. For $\partial f/\partial t$ to be non-zero, electrons would have to be added to, or removed from, an element of phase-space. The only way this is possible for a contained gas is through an increase or decrease in the degree of ionization with time; this would invalidate the assumption that ionization equalibrium has been achieved. The second term on the left of equation (36) relates the change in the number of electrons in the distribution as a function of geometrical space while time and velocity components are held constant; for there to be such a change, the assumption that the gas is uniform would be invalidated. Hence,

$$(\bar{c} + \bar{w}) \cdot \frac{\partial f}{\partial \bar{s}} = 0 \ .$$

The inclusion of equation (35) in equation (36) and writing e and n subscripts for electron and neutral terms respectively gives the Boltzmann's equation for this case.

$$-\frac{e}{m_e}(\bar{E}' + \bar{c}_e \times \bar{B}) \cdot \frac{\partial f_e}{\partial \bar{c}_e}$$

$$= \int\left[\left(f_e' f_n' - f_e f_n\right) |\bar{c}_e - \bar{c}_n| \int\int r\,dr\,d\phi\right] d\bar{c}_n \qquad (37)$$

where

$$(\bar{E}' + \bar{c}_e \times \bar{B}) = (\bar{E} + \bar{v} \times \bar{B}) \quad ,$$

The neutral particle distribution function, f_n , is taken as Maxwellian in form prior to an interaction with an electron; and because of the large mass ratio m_n/m_e the distribution does not appreciably change after an interaction. Thus,

$$f_n = f_n' = a \exp(-Ac_n^2) \quad . \qquad (38)$$

Because interactions and external forces exert significant influence on the electron distribution, f_e cannot be Maxwellian. In the absence of an electric field $(E' = 0)$ there can be no drift velocity of electrons in the uniform gas. Under this condition the collision term is zero and the electron distribution is exactly Maxwellian. For a slightly ionized gas $\bar{E}'$ can be relatively small in magnitude. This allows the electron distribution to be perturbed from a Maxwellian function by an amount which is proportional to $\bar{E}'$. The electron distribution is approximated by adding to a Maxwellian term, $f_f(c)$, a term $F\left[E' f_g(c)\right]$ which is proportional to the perturbing force $\bar{E}'$. Thus, replacing $\bar{c}_e$ by $\bar{c}$, the existing distribution function for electrons can be written

$$f_e(c) = f_f(c) + F\left[\bar{E}'f_g(c)\right] \quad . \qquad (39)$$

[Assumption — The electron distribution function in a slightly ionized gas under the influence of an external force field can be approximated by a relationship of the form of equation (39)]. This approximation is in line with Coulomb's Law which states that the time-rate-of-change of velocity of a charged particle in an electric field is directly proportional to the field intensity, $\bar{E}$; in other words,

$$\bar{F} = m\bar{a} = m\frac{d\bar{v}}{dt} = q\bar{E}$$

and,

$$\frac{d\bar{v}}{dt} = \frac{q}{m}\,\bar{E} \ .$$

An alternate approximation to that proposed in equation (39) could be attained by the addition of succeeding terms of higher order in $\bar{E}'$.

Because of gas uniformity and ionization equalibrium the electron distribution function, like the distribution function for neutrals, is a function only of the particle velocity; and if electric currents are to result, then they must arise from the perturbation. Currents will not result unless the electron drift velocity, $\bar{v}$, is non-zero in equation (30); $\bar{v}$ will be non-zero only if the integral in equation (32) is taken for an even function. The Maxwellian function of the form of equation (38) is an even function of c ; however, the product $\bar{c}f_e(c)$ of equation (32) is an odd function of c . This requires that the perturbation term $F\left[\bar{E}'f_g(c)\right]$ in equation (39) be defined such that the contribution of this term in equation (30) does not disappear. No restriction is required at this time beyond that implied by designating F a function of the product $\bar{E}'$ and $f_g(c)$. The necessary restriction on $F\left[\bar{E}'f_g(c)\right]$ will become clear later.

The drift of electrons in the presence of electric and magnetic fields is known to have multi-component directions. In the presence of only an electric field the drift is in the direction $\bar{E}$. If only a magnetic field is present there is no resultant drift. When a magnetic field is added to an electric field already present, electrons feel accelerations toward $\bar{E}$ and transverse to $\bar{E}$ and $\bar{B}$. To account for electron velocity distribution perturbations in the directions 1 , 2 , and 3 (where 1 is in the direction of $\bar{E}$; where 1 , 2 , and 3 signifiy directions mutually perpendicular, but with the additional requirement that 2 is also perpendicular to $\bar{B}$) the second term on the right of equation (39) becomes:

$$\begin{aligned} F\left[\bar{E}'f_g(c)\right] &= E_1\left[c_1 f_{g_1}(c) + c_2 f_{g_2}(c) + c_3 f_{g_3}(c)\right] \\ &= \begin{bmatrix} \bar{c} \cdot \bar{E}' f_{g_1}(c) + \bar{c} \cdot \bar{E} \times \bar{B}\left(\frac{1}{B}\right) f_{g_2}(c) \\ + \bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}'\left(\frac{1}{B\underline{E}}\right) f_{g_3}(c) \end{bmatrix} \end{aligned} \tag{40}$$

Equation (40) defines the perturbation function which, when coupled to the non-perturbed Maxwellian distribution, generates the distribution of electrons in velocity-space. The subscripts on c in the equation identify the geometry-space coordinates. The subscript on $\bar{E}'$ indicates the component of $\bar{E}'$ in the direction 1 ; the fact that $\bar{E}'$ appears with no other subscript verifies that the direction 1 has been chosen for convenience to coincide with that of $\bar{E}'$. Further, the direction 2 is obviously that of $\bar{E}' \times \bar{B}$, and the direction 3 is that of $(\bar{E}' \times \bar{B}) \times \bar{E}'$. Since $f_e(c)$ is a scalar function, each term on the right of equation (40) is a scalar. $\bar{E}'$ is a vector; therefore, the dot product is necessary. $\bar{c} \cdot \bar{E}'$ can be written $cE' \mathrm{Cos}\, {}_c\theta_1$, where ${}_c\theta_1$ is the angle between $\bar{c}$ and $\bar{E}'$. The restriction mentioned earlier on $F\left[\bar{E}' f_g(c)\right]$ is in reality a restriction on ${}_c\theta_1$, and it can now be stated. If $\bar{c}$ is random and $\bar{E}'$ is not, then ${}_c\theta_1$ would appear to be random. Now, if $f_{g_1}(c)$ is a scalar function, the second term on the right of equation (40) will have no meaning. Therefore, some restriction on $F\left[\bar{E}' f_g(c)\right]$ is necessary. The required restriction is simple and intuitively obvious. Since the existence of any perturbation is due to the external forces, $\bar{E}$ and $\bar{B}$, and since it was previously supposed that the perturbation is proportional to the force, it is also reasonable to assume that the perturbation would not be negatively proportional to the force. In other words the angle ${}_c\theta_1$ is restricted to produce only positive cosines. A similar restriction is applied with respect to the other coordinates.

With the substitution of equation (40) equation (39) becomes:

$$f_e(c) = f_f(c) + (\bar{c} \cdot \bar{E}')f_{g_1}(c) + (\bar{c} \cdot \bar{E}' \times \bar{B})\frac{1}{B}f_{g_2}(c)$$

$$+ \bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}' \frac{1}{BE} f_{g_3}(c) \qquad (41)$$

c is the random electron velocity. f_f is the Maxwell's distribution function (equation A-20, Appendix A); $\bar{E}'$ is the electric field intensity, and $\bar{B}$ is the magnetic flux density. f_e is the perturbed electron distribution; f_{g_1} , f_{g_2} , and f_{g_3} are unknown perturbing functions. The existence of the f_g terms verifies the existence of quadrature current densities.

It has been observed previously, but the absence of an electric field ($E' = 0$) allows the electron distribution to be Maxwellian. If $\bar{B} = 0$ the third and fourth terms of equation (41) are zero; the current density is coincident with the electric intensity. If the applied $\bar{B}$ field is parallel to the $\bar{E}'$ field, the third and fourth terms disappear, and the situation duplicates the case where $\bar{B} = 0$. Should $\bar{B}$ and $\bar{E}'$ be normal, the third and fourth terms are maximum. The final case allows both $\bar{E}' \times \bar{B} \neq 0$ and $(\bar{E}' \times \bar{B}) \times \bar{E}' \neq 0$ to be non-maximal. Only when the electric field is absent will equation (32) be expected to fail to yield a net electron drift velocity and, therefore, no electrid current. This justifies the statement made earlier that electrical conductivity can only be discussed in terms of the existence both of average migration of charged particles and the presence of an electric force field. Migration of charged particles without the electric field gradient cannot lead to evaluation of finite electrical conductivity in the engineering sense.

Substitution of equation (41) into the Boltzmann's equation (37) yields a term on the left side involving the partial derivative of f_e with respect to c.

$$\frac{\partial}{\partial \bar{c}} f_e(c) = \frac{\partial}{\partial \bar{c}} f_f(c) + \frac{\partial}{\partial \bar{c}}\left[\bar{c} \cdot \bar{E}' f_{g_1}(c)\right]$$

$$+ \frac{\partial}{\partial \bar{c}}\left[\bar{c} \cdot \bar{E}' \times \bar{B} \, \frac{1}{B} f_{g_2}(c)\right]$$

$$+ \frac{\partial}{\partial \bar{c}}\left[\bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}' \frac{1}{BE'} f_{g_3}(c)\right] \qquad (42)$$

where

$$\text{a)} \quad \frac{\partial}{\partial \bar{c}} f_f(c) = \left[\frac{\partial}{\partial c} f_f(c)\right] \frac{\partial}{\partial \bar{c}} c$$

$$= \left[\frac{\partial}{\partial c} f_f\right]\left[\left(\frac{\partial}{\partial c_1}\right)_1 + \left(\frac{\partial}{\partial c_2}\right)_2 + \left(\frac{\partial}{\partial c_3}\right)_3\right] c$$

$$= \left[\frac{\partial}{\partial c} f_f\right]\left[\left(\frac{\partial}{\partial c_1}\right)_1 + \left(\frac{\partial}{\partial c_2}\right)_2 + \left(\frac{\partial}{\partial c_3}\right)_3\right]\left[c_1^2 + c_2^2 + c_3^2\right]$$

$$= \left[\frac{\partial}{\partial c} f_f\right]\left[(c_1)_1 + (c_2)_2 + (c_3)_3\right]\left[c_1^2 + c_2^2 + c_3^2\right]$$

$$= \left[\frac{\partial}{\partial c} f_f(c)\right] \frac{\bar{c}}{c} \qquad (43a)$$

$$\text{b)} \quad \frac{\partial}{\partial \bar{c}} \bar{c} \cdot \bar{E}' f_{g_1}(c) = \frac{\partial}{\partial \bar{c}} (\bar{c} \cdot \bar{E}') f_{g_1}(c)$$

$$+ (\bar{c} \cdot \bar{E}') \frac{\partial}{\partial \bar{c}} f_{g_1}(c) \qquad (43b)$$

From the results of the preceding work

$$(\bar{c}\cdot\bar{E}')\frac{\partial}{\partial\bar{c}}f_{g1}(c) = (\bar{c}\cdot\bar{E}')\frac{\bar{c}}{c}\frac{\partial}{\partial c}f_{g1}(c) ,$$

and

$$\frac{\partial}{\partial\bar{c}}(\bar{c}\cdot\bar{E}')f_{g1}(c) = f_{g1}(c)\frac{\partial}{\partial\bar{c}}\left[\bar{c}\cdot\bar{E}'\right]$$

$$= f_{g1}\left[\left(\frac{\partial}{\partial c_1}\right)_1 + \left(\frac{\partial}{\partial c_2}\right)_2 + \left(\frac{\partial}{\partial c_3}\right)_3\right]\left[E_1'c_1 + E_2'c_2 + E_3'c_3\right]$$

$$= f_{g1}\left[(E_1')_1 + (E_2')_2 + (E_3')_3\right]$$

$$= f_{g1}(c)\left[\bar{E}'\right] .$$

Therefore,

$$\frac{\partial}{\partial\bar{c}}\left[\bar{c}\cdot\bar{E}'f_{g1}(c)\right] = \bar{E}'f_{g1}(c) + \bar{c}\cdot\bar{E}'\frac{\bar{c}}{c}\frac{\partial}{\partial c}f_{g1}(c) . \quad (44)$$

c) From the results just obtained

$$\frac{\partial}{\partial\bar{c}}\left[(\bar{c}\cdot\bar{E}'\times\bar{B})\frac{1}{B}f_{g2}(c)\right]$$

$$= f_{g2}\left[(\bar{E}'\times\bar{B})\frac{1}{B}\right] + (\bar{c}\cdot\bar{E}'\times\bar{B})\frac{\bar{c}}{Bc}\frac{\partial}{\partial c}f_{g2}$$

$$= (\bar{E}'\times\bar{B})\frac{1}{B}f_{g2}(c) + (\bar{c}\cdot\bar{E}'\times\bar{B})\frac{\bar{c}}{Bc}\frac{\partial}{\partial c}f_{g2}(c) . \quad (45)$$

d) And,

$$\frac{\partial}{\partial \bar{c}}\left[\bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}' \frac{1}{BE'} f_{g3}(c)\right]$$

$$= f_{g3}\left[(\bar{E}' \times \bar{B}) \times \bar{E}' \frac{1}{BE'}\right] + \left[\bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}' \frac{\bar{c}}{BE'}\right] \frac{\partial}{\partial c} f_{g3}$$

$$= (\bar{E}' \times \bar{B}) \times \bar{E}' \frac{1}{BE'} f_{g3}(c) + \bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}' \frac{\bar{c}}{BE'} \frac{\partial}{\partial c} f_{g3}(c) . \qquad (46)$$

Substituting equations (43a), (44) (45), and (46) into equation (42) yields the value,

$$\frac{\partial}{\partial \bar{c}} f_e(c) = \frac{\bar{c}}{c} \frac{\partial}{\partial c} f_f(c) + \bar{E}' f_{g1}(c) + \frac{\bar{c}}{c}(\bar{c} \cdot \bar{E}') \frac{\partial}{\partial c} f_{g1}(c)$$

$$+ \frac{1}{B}(\bar{E}' \times \bar{B}) f_{g2}(c) + \frac{\bar{c}}{Bc}(\bar{c} \cdot \bar{E}' \times \bar{B}) \frac{\partial}{\partial c} f_{g2}(c)$$

$$+ \frac{1}{BE'}(\bar{E}' \times \bar{B}) \times \bar{E}' f_{g3}(c) + \frac{\bar{c}}{BE'c}\left[\bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}'\right] \frac{\partial}{\partial c} f_{g3}(c)$$

$$(47)$$

The substitution of equation (47) into equation (37) yields the left side of the Boltzmann's equation. The left side of equation (37) now becomes:

$$\frac{e}{m}(\bar{E}' + \bar{c} \times \bar{B}) \cdot \frac{\partial}{\partial \bar{c}} f_e(c)$$

$$
\begin{aligned}
= -\frac{e}{m} \Bigg\{ & \left[\frac{1}{c}(\bar{c} \cdot \bar{E}') + \frac{1}{c}(\bar{c} \cdot \bar{c} \times \bar{B}) \right] \frac{\partial}{\partial c} f_f(c) \\
& + \left[\bar{E}' \cdot \bar{E}' + \bar{E}' \cdot \bar{c} \times \bar{B} \right] f_{g_1}(c) \\
& + \left[\frac{\bar{c}}{c}(\bar{c} \cdot \bar{E}') \cdot \bar{E}' + \frac{\bar{c}}{c}(\bar{c} \cdot \bar{E}') \cdot (\bar{c} \times \bar{B}) \right] \frac{\partial}{\partial c} f_{g_1}(c) \\
& + \left[\frac{1}{B}(\bar{E}' \times \bar{B} \cdot \bar{E}') + \frac{1}{B}(\bar{E}' \times \bar{B}) \cdot (\bar{c} \times \bar{B}) \right] f_{g_2}(c) \\
& + \left[\frac{\bar{c}}{Bc}(\bar{c} \cdot \bar{E}' \times \bar{B}) \cdot \bar{E}' + \frac{\bar{c}}{Bc}(\bar{c} \cdot \bar{E}' \times \bar{B}) \cdot (\bar{c} \times \bar{B}) \right] \frac{\partial}{\partial c} f_{g_2}(c) \\
& + \left[\frac{1}{BE'}(\bar{E}' \times \bar{B}) \times \bar{E}' \cdot \bar{E}' + \frac{1}{BE'}(\bar{E}' \times \bar{B}) \times \bar{E}' \cdot (\bar{c} \times \bar{B}) \right] f_{g_3}(c) \\
& + \left[\frac{\bar{c}}{BE'c}\, \bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}' \cdot \bar{E}' + \frac{\bar{c}}{BE'c}\, \bar{c} \cdot (\bar{E}' \times \bar{B}) \times \bar{E}' \cdot (\bar{c} \times \bar{B}) \right] \frac{\partial}{\partial c} f_{g_3}(c) \Bigg\}
\end{aligned}
\qquad (48)
$$

The second, sixth, seventh, tenth, eleventh, and fourteenth terms in equation (48) are precisely zero, since a triple scalar product is zero if any two constituents are the same.

With respect to the third term

$$(\bar{E}' \cdot \bar{E}')\, f_{g_1}(c) \;=\; E'^2 f_{g_1}(c) \quad . \tag{49}$$

The fifth term becomes:

$$\left[\frac{\bar{c}}{c}(\bar{c} \cdot \bar{E}') \cdot \bar{E}'\right]\frac{\partial}{\partial c} f_{g_1} \;=\; \frac{1}{c}(\bar{c} \cdot \bar{E}')\;(\bar{c} \cdot \bar{E}')\frac{\partial}{\partial c} f_{g_1} \tag{50}$$

The fourth term may be written:

$$(\bar{E}' \circ \bar{c} \times \bar{B}) f_{g_1}(c) \;=\; -(\bar{c} \circ \bar{E}' \times \bar{B})\; f_{g_1}(c) \tag{51}$$

The eighth term can be rearranged to give

$$\frac{1}{B}(\bar{E}' \times \bar{B}) \cdot (\bar{c} \times \bar{B}) f_{g_2}(c) \;=\; \frac{1}{B}(\bar{E}' \times \bar{B}) \cdot \bar{c} \times \bar{B}\; f_{g_2}(c)$$

$$= \frac{1}{B}\,\vec{c} \cdot \left[-(\bar{B} \cdot \bar{E}')\bar{B} \;+\; (\bar{B} \cdot \bar{B})\bar{E}'\right] f_{g_2}(c)$$

$$= \left[\frac{-1}{B}(\bar{c} \cdot \bar{B})(\bar{B} \cdot \bar{E}) \;+\; B\bar{c} \cdot \bar{E}'\right] f_{g_2}(c) \quad . \tag{52}$$

The twelfth term can be rearranged as follows:

$$f_{g_3}(c)\frac{1}{BE'}\left[(\bar{E}' \times \bar{B}) \times \bar{E}'\right]\cdot(\bar{c}\times\bar{B})$$

$$= \frac{1}{BE'}\left[(\bar{E}'\cdot\bar{E}')\bar{B} - (\bar{B}\cdot\bar{E}')\bar{E}'\right]\cdot(\bar{c}\times\bar{B})f_{g_3}(c)$$

$$= \left[\frac{1}{BE}E'^2(\bar{B}\cdot\bar{c}\times\bar{B}) - \frac{1}{BE'}(\bar{B}\cdot\bar{E}')\bar{E}'\cdot\bar{c}\times\bar{B}\right]f_{g_3}(c)$$

$$= \left[-(\bar{c}\times\bar{B}\cdot\bar{E}')(\bar{B}\cdot\bar{E}')\frac{1}{BE'}\right]f_{g_3}(c)$$

$$= (\bar{c}\cdot\bar{E}'\times\bar{B})(\bar{B}\cdot\bar{E}')\frac{1}{BE'}f_{g_3}(c) \qquad (53)$$

The thirteenth term becomes:

$$\frac{\partial}{\partial c}f_{g_3}(c)\left[\frac{\bar{c}}{BE'c}\left[\bar{c}\cdot(\bar{E}'\times\bar{B})\times\bar{E}'\right]\cdot\bar{E}'\right]$$

$$= \left[\frac{1}{BE'c}(\bar{c}\cdot\bar{E}')\bar{c}\cdot(\bar{E}'\times\bar{B})\times\bar{E}'\right]\frac{\partial}{\partial c}f_{g_3}(c)$$

$$= \left[\frac{1}{BE'c}(\bar{c}\cdot\bar{E}')\bar{c}\right]\cdot\left[\bar{E}'^2\bar{B} - (\bar{B}\cdot\bar{E}')\bar{E}'\right]\frac{\partial}{\partial c}f_{g_3}(c)$$

$$= \frac{1}{BE'c}(\bar{c}\cdot\bar{E}')\left[(\bar{c}\cdot\bar{B})\bar{E}'^2 - (\bar{c}\cdot\bar{E}')(\bar{B}\cdot\bar{E}')\right]\frac{\partial}{\partial c}f_{g_3}(c)$$

$$= \left[\frac{E'}{Bc}(\bar{c}\cdot\bar{E}')(\bar{c}\cdot\bar{B}) - \frac{1}{BE'c}(\bar{c}\cdot\bar{E}')(\bar{c}\cdot\bar{E}')(\bar{B}\cdot\bar{E}')\right]\frac{\partial}{\partial c}f_{g_3}(c) \qquad (54)$$

The ninth and first term can remain unchanged. With the substitution of equations (49) through (54) into equation (48) the left side of the Boltzmann's equation (37) can be written as follows:

$$-\frac{e}{m}(\bar{E}' + \bar{c} \times \bar{B}) \cdot \frac{\partial}{\partial \bar{c}} f_e(c)$$

$$= -\frac{e}{m}\left\{ \begin{array}{l} \frac{1}{c}(\bar{c} \cdot \bar{E}')\frac{\partial}{\partial c} f_f(c) \\ \\ + (E'^2 - \bar{c} \cdot \bar{E}' \times \bar{B}) f_{g_1}(c) \\ \\ + \frac{1}{c}(\bar{c} \cdot \bar{E}')(\bar{c} \cdot \bar{E}')\frac{\partial}{\partial c} f_{g_1}(c) \\ \\ + \left[B(\bar{c} \cdot \bar{E}') - \frac{1}{B}(\bar{c} \cdot \bar{B})(\bar{B} \cdot \bar{E}')\right] f_{g_2}(c) \\ \\ + \frac{1}{Bc}(\bar{c} \cdot \bar{E}')(\bar{c} \cdot \bar{E}' \times \bar{B})\frac{\partial}{\partial c} f_{g_2}(c) \\ \\ + \frac{1}{BE'}(\bar{B} \cdot \bar{E}')(\bar{c} \cdot \bar{E}' \times \bar{B}) f_{g_3}(c) \\ \\ + \left[\begin{array}{l} \frac{E'}{Bc}(\bar{c} \cdot \bar{E}')(\bar{c} \cdot \bar{B}) \\ \\ - \frac{1}{BE'c}(\bar{c} \cdot \bar{E}')(\bar{c} \cdot \bar{E}')(\bar{B} \cdot \bar{E}') \end{array}\right]\frac{\partial}{\partial c} f_{g_3}(c) \end{array} \right\} \qquad (55)$$

The right hand side of the Boltzmann's equation (37) is now expanded by the substitution of equation (41). The right hand side of (37) becomes:

$$\left(\frac{\partial}{\partial t} f_e\right)_c = \iiint (f'_e\, f'_n - f_e\, f_n)\,|\bar{c} - \bar{c}_n|\, r\, dr\, d\phi\, d\bar{c}_n$$

$$\begin{aligned} = \iiint \Bigg\{ &\Big[f'_f + (\bar{c}' \cdot \bar{E}') f'_{g_1} + (\bar{c}' \cdot \bar{B} \times \bar{E}') \tfrac{1}{B} f'_{g_2} \\ &+ (\bar{c}' \cdot \bar{B})(\bar{B} \cdot \bar{E}') \tfrac{1}{B^2} f'_{g_3} \Big] f'_n\, |\bar{c}' - \bar{c}_n|\, r\, dr\, d\phi\, d\bar{c}'_n \\ &- \Big[f_f + (\bar{c} \cdot \bar{E}') f_{g_1} + (\bar{c} \cdot \bar{B} \times \bar{E}') \tfrac{1}{B} f_{g_2} \\ &- (\bar{c} \cdot \bar{B})(\bar{B} \cdot \bar{E}') \tfrac{1}{B^2} f_{g_3} \Big] f_n\, |\bar{c} - \bar{c}_n|\, r\, dr\, d\phi\, d\bar{c}_n \Bigg\} \end{aligned} \tag{56}$$

Equating equation (55) to (56) yields the complete Boltzmann's equation. Many techniques have been proposed to formulate an approximate solution for this equation, but a complete solution has not yet been achieved. A number of simplifications, however, can be suggested. The ratio of the electron mass to the neutral mass is very large. Electron-neutral interactions cause very small changes in the neutral particle velocity; for this reason

$$\bar{c}'_n \approx \bar{c}_n \; .$$

Because

$$m_e << m_n \; ,$$

the random velocity of the neutral particle is much smaller than that for the electron; in other words,

$$|\bar{c} - \bar{c}_n| \approx c \tag{57}$$

$$\left|\bar{c} - \bar{c}_n\right| \approx c \tag{57}$$

Since momentum is conserved under the presumption that the binary interaction is elastic,

$$mc = mc' ;$$

and the electron velocity magnitudes before and after interaction are equivalent

$$c \approx c' . \tag{58}$$

The term $\bar{c}'$ can be written with respect to $\bar{c}$ and the angle of scattering θ ,

$$\bar{c}' = \bar{c}\ (\mathrm{Cos}\ \theta - 1) ,$$

And

$$(\bar{c}' - \bar{c}) = \bar{c}(\mathrm{Cos}\ \theta - 1) , \tag{59}$$

An interaction between an electron and a neutral is of the type often referred to as "hard-sphere collision", where the interaction force is not a field function. The two particles collide when the impact parameter, r , is equal to the sum of the radii of the two particles, R. Note Figure No. 1.

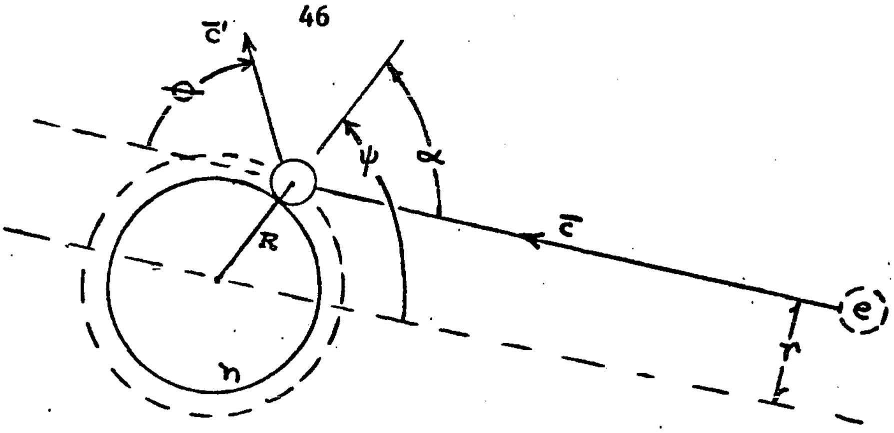

Figure No. 1

Scattering Angle for Hard-sphere Interaction

From Figure No. 1

$$r = R \operatorname{Sin} \psi$$

at the point of impact, and

$$dr = R \operatorname{Cos} \psi \, d\psi$$

Note that

$$\theta = \pi - 2\alpha$$

$$\operatorname{Cos} \theta = \operatorname{Cos}(\pi - 2\alpha)$$

$$= -\operatorname{Cos}(-2\alpha) = -\operatorname{Cos} 2\alpha$$

$$1 - \operatorname{Cos} \theta = 1 + \operatorname{Cos} 2\alpha = 2 \operatorname{Cos}^2 \alpha$$

$$\alpha = \psi$$

In the integrals of equation (56) r and ϕ are not functions of particle distribution or velocity.

$$\int_0^{2\pi}\int_0^{R}(1 - \mathrm{Cos}\,\theta)\; r\; d\phi\; dr$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 2\;\mathrm{Cos}^2\psi\,(R\;\mathrm{Sin}\,\psi d\psi\,)\; d\phi$$

$$= 2\;R^2\int_0^{2\pi}\int^{\pi/2}\mathrm{Cos}^3\,\psi\,(\mathrm{Sin}\psi\; d\psi\,)\; d\phi$$

$$= 2R^2\frac{1}{4}\int_0^{2\pi} -\mathrm{Cos}^4\psi\Big|_0^{\pi/2} d\phi \quad = \; -\frac{1}{2}R^2(-1)\int_0^{\pi/2} d\phi$$

$$= -\pi R^2 \qquad (60)$$

The third integration in equation (56) can be written quite simply upon noting that

$$d\bar{c}_n' \approx d\bar{c}_n \quad ,$$

since

$$\bar{c}_n' \approx \bar{c}_n \quad ,$$

and that

$$f_n' \;(c_n') \approx f_n \;(c_n) \quad .$$

The integral becomes:

$$\int_{-\infty}^{\infty} f_n(c_n)\, d\bar{c}_n = n_n \tag{61}$$

Under the conditions proposed and with the substitution of equations (58) through (61) into equation (56) equated to equation (55), the complete Boltzmann's equation becomes:

$$\begin{aligned}
-\frac{e}{m}\left\{ \begin{aligned}
&\frac{1}{c}(\bar{c}\cdot\bar{E}')\frac{\partial}{\partial c}f_f(c) \\
&+ (E'^2 - \bar{c}\cdot\bar{E}' \times \bar{B})f_{g_1}(c) \\
&+ \frac{1}{c}(\bar{c}\cdot\bar{E}')(\bar{c}\cdot\bar{E}')\frac{\partial}{\partial c}F_{g_1}(c) \\
&+ \left[B(\bar{c}\cdot\bar{E}') - \frac{1}{B}(\bar{c}\cdot\bar{B})(\bar{B}\cdot\bar{E}')\right]f_{g_2}(c) \\
&+ \frac{1}{Bc}(\bar{c}\cdot\bar{E}')(\bar{c}\cdot\bar{E}' \times \bar{B})\frac{\partial}{\partial c}f_{g_2}(c) \\
&+ \frac{1}{BE'}(\bar{B}\cdot\bar{E}')(\bar{c}\cdot\bar{E}' \times \bar{B})f_{g_3}(c) \\
&+ \left[\begin{aligned}&\frac{E'}{Bc}(\bar{c}\cdot\bar{E}')(\bar{c}\cdot\bar{B}) \\ &- \frac{1}{BE'c}(\bar{c}\cdot\bar{E}')(\bar{c}\cdot\bar{E}')(\bar{B}\cdot\bar{E}')\end{aligned}\right]\frac{\partial}{\partial c}f_{g_3}(c)
\end{aligned}\right\} \\
= \pi R^2 n_n c \left\{ \begin{aligned}
&(\bar{c}\cdot\bar{E}')f_{g_1}(c) + \frac{1}{B}(\bar{c}\cdot\bar{E}' \times \bar{B})f_{g_2}(c) \\
&+ \left[\frac{E'}{B}(\bar{c}\cdot\bar{B}) - \frac{1}{BE'}(\bar{c}\cdot\bar{E}')(\bar{B}\cdot\bar{E}')\right]f_{g_3}(c) \quad .
\end{aligned}\right\}
\end{aligned} \tag{62}$$

In the absence of external forces the velocity distribution of electrons is $f_f(c)$, a Maxwellian function. Appendix A develops this function. From Appendix A

$$f_f(c) = n\left(\frac{m}{2\pi K'T}\right)^{3/2} \exp\left[-mc^2/2kT\right] . \tag{63}$$

Equation (62) is a differo-integral relationship involving the four functions f_f, f_{g_1}, f_{g_2}, and f_{g_3}. The function, f_f, is known; this is the Maxwellian function given by equation (63). Equation (62) and the identity (63) can be combined such that a single equation in three unknowns results. A direct solution to equation (62) has not been formulated. One technique which has been used to find a solution to equations of the same type as (62) suggests that terms involving alike coefficients be equated. Hopefully, there are precisely the correct number of alike terms to allow a unique solution; otherwise, the technique fails or becomes much less precise. This technique is a valuable tool, even when approximations are required to adjust the coefficients, so long as any adjustment of the coefficients necessary to produce a solution can be justified as a reasonable approximation. An often used method to solve the Boltzmann's equation requires that the equation be separated into at least as many parts as there are unknowns. Each part is made to include an equality. The equations thus formed are then solved for the desired unknowns. With respect to equation (62), including identity (63) at least three equations would be required. The alike coefficients would hopefully be, for the first equation, simple multipliers; for the second equation, the scalar product, $\overline{c} \cdot \overline{E}'$; for the third equation, the triple scalar product, $\overline{c} \cdot \overline{E}' \times \overline{B}$; and for a fourth equation, the scalar triple-vector product, $\overline{c} \cdot (\overline{E}' \times \overline{B}) \times \overline{E}'$. From equating (62) and (63) no such set of coupled equations can be formulated without drastic approximations. The literature contains ample cases where approximations have been made and where useful results have followed, as proof of the utility of this technique as a tool. Because of the many varied approximations that would be required to solve equation (62) by this method, another technique is proposed.

Equation (62) was derived in a formal way, accounting for the fact that the result might produce vector components in the mutually perpendicular geometrical directions, 1 , 2 , and 3 , as defined previously. At the same time the functions, f_{g1} , f_{g2} , and f_{g3} , are unknown functions of the random velocity magnitude.

Equation (62) can be separated into three mutually coupled equations corresponding to three choices for the evaluation of $\overline{E}' \times \overline{B}$. The three cases of interest are:

$$(1) \quad \overline{E}' \times \overline{B} = 0 \; .$$

This is the case for $\overline{B}$ parallel to $\overline{E}'$. (It is also equivalent to $\overline{B} = 0$.)

$$(2) \quad \overline{E}' \cdot \overline{B} = 0 \; .$$

This is the case for $\bar{B}$ perpendicular to $\bar{E}'$.

(3) The general case where both

$$\bar{E}' \times \bar{B} \neq 0 ,$$

and

$$\bar{E}' \cdot \bar{B} \neq 0 .$$

In equation (62) the cross and dot product terms are expanded to eliminate the vector notation and to generate only scalar coefficients. Equation (62) becomes:

$$-\frac{e}{m}\left\{\begin{aligned}
&\frac{1}{c}cE'\operatorname{Cos}{}_{c}\theta_{1}\,\frac{\partial}{\partial c}f_{f}\\
&+\ (E'^{2}-cE'B\operatorname{Cos}{}_{c}\theta_{2}\ \operatorname{Sin}{}_{E}\theta_{B})f_{g_1}\\
&+\ \frac{1}{c}c^{2}E'^{2}\ \operatorname{Cos}^{2}{}_{c}\theta_{1}\,\frac{\partial}{\partial c}f_{g_1}\\
&+\ (cE'B\operatorname{Cos}{}_{c}\theta_{1}-cE'B\operatorname{Cos}{}_{c}\theta_{B}\operatorname{Cos}{}_{E}\theta_{B})f_{g_2}\\
&+\ cE'^{2}\ \operatorname{Cos}{}_{c}\theta_{1}\operatorname{Cos}{}_{c}\theta_{1}\ \operatorname{Sin}{}_{B}\theta_{E}\ \frac{\partial}{\partial c}f_{g_2}\\
&+\ (cE'^{2}\ \operatorname{Cos}{}_{c}\theta_{1}\operatorname{Cos}{}_{c}\theta_{B}-cE'^{2}\ \operatorname{Cos}^{2}{}_{c}\theta_{1}\operatorname{Cos}{}_{B}\theta_{E})\,\frac{\partial}{\partial c}f_{g_3}\\
&+\ cE'B\operatorname{Cos}{}_{B}\theta_{E}\operatorname{Cos}{}_{c}\theta_{2}\operatorname{Sin}{}_{B}\theta_{E}\,f_{g_3}
\end{aligned}\right\}$$

$$= R^2 \pi n_n \Big[(c^2 E' \mathrm{Cos}\,{}_c\theta_1) f_{g_1} + (c^2 E' \mathrm{Cos}\,{}_c\theta_2 \mathrm{Sin}\,{}_B\theta_E) f_{g_2} + (c^2 E' \mathrm{Cos}\,{}_c\theta_B$$

$$- c^2 E' \mathrm{Cos}\,{}_c\theta_1 \mathrm{Cos}\,{}_B\theta_E) f_{g_3} \Big] \qquad (64)$$

The three coupled equations can now be written.

I. For $\bar{B} \times \bar{E}' = 0$, and $\bar{B} \cdot \bar{E}' = BE$, i.e., $\bar{B} \parallel \bar{E}'$

$$-\frac{e}{m} \Big[E' \mathrm{Cos}\,{}_c\theta_1 \frac{\partial}{\partial c} f_f + E'^2 f_{g_1} + cE'^2 \mathrm{Cos}\,{}_c\theta_1 \frac{\partial}{\partial c} f_{g_1} \Big]$$

$$= \pi R^2 n_n \Big[c^2 E' \mathrm{Cos}\,{}_c\theta_1 f_{g_1} \Big] \qquad (65)$$

Rearranging

$$cE'^2 \mathrm{Cos}^2\,{}_c\theta_1 \frac{\partial}{\partial c} f_{g_1} + (E'^2 + \frac{m}{e} \pi R^2 n_n \, c^2 E' \mathrm{Cos}\,{}_c\theta_1) f_{g_1}$$

$$= -E' \mathrm{Cos}\,{}_c\theta_1 \frac{\partial}{\partial c} f_f \quad .$$

And,

$$\mathrm{Cos}^2\,{}_c\theta_1 \frac{\partial}{\partial c} f_{g_1} + (\frac{1}{c} + k_1 \mathrm{Cos}\,{}_c\theta_1 \; c) f_{g_1}$$

$$= -\frac{1}{E'} \mathrm{Cos}\,{}_c\theta_1 \frac{1}{c} \frac{\partial}{\partial c} f_f \quad , \qquad (66)$$

where,

$$k_1 = \frac{m}{eE'}\pi R^2 n_n$$

and ${}_c\theta_1$ is the angle between $\bar{c}$ and $\bar{E}'$. Because of the coefficients, $\mathrm{Cos}^2\,{}_c\theta_1$ and $\mathrm{Cos}\,{}_c\theta_1$, the function f_{g_1} would involve ${}_c\theta_1$, should an attempt be made to solve equation (66) directly. In fact, equation (66) can only be solved by a numerical process whereby values for $\mathrm{Cos}\,{}_c\theta_1$ are specified and then only with the assistance of tabulated functions. This process does not yield the solution in general terms.

It is recalled that the solution for the f_g functions will yield the drift velocity through integration of $\bar{c}f_e d\bar{c}$ over the complete range of $\bar{c}$. It was shown in the evaluation leading to equation (43) that

$$\frac{d}{d\bar{c}} = \frac{\bar{c}}{c}\frac{d}{dc} .$$

Therefore,

$$d\bar{c} = \frac{c}{\bar{c}}\ dc \quad , \tag{67}$$

and the integral

$$\int \bar{c}f_e d\bar{c} \quad (\text{over all } \bar{c}) \tag{68}$$

becomes

$$\int cf_e dc \quad . \tag{69}$$

This is true so long as f_e remains only a function of c . With the Cos $_c\theta_1$ coefficients of equation (66), integrals (68) and (69) would not be equivalent. For equivalence integral (69) must in some way include a summation over the range in $_c\theta_1$, in order that all $\bar{c}$ be included.

To overcome the predicament confronting a direct solution to equation (66), the Cos $_c\theta_1$, terms are treated as sources of weighting functions that can be evaluated and applied to equation (66) in terms of constant coefficients. If this process can be performed, f_{g_1} will then truly be a function only of c , and the partial derivative operating on f_{g_1} becomes an ordinary derivative. Since f_f is the Maxwell's distribution previewed earlier as a known function only of c , the partial operating on f_f can be replaced by the ordinary derivative and derivative taken forthwith. Thus, a first order, first degree, linear, ordinary differential equation replaces equation (66) if the weighting coefficients can be determined. Integral (68) demands that the complete range of $\bar{c}$ be included. Integral (69) allows for the complete range of $\bar{c}$ to be considered. The weighting coefficients must account for the complete range of the argument of $\bar{c}$, since equation (69) takes care of only the magnitude of $\bar{c}$. The complete range of $_c\theta_1$, can be stated:

$$0 \leq \theta_1 \leq \pi/2$$

Observing that

$$\frac{d}{dc}f_{g1} = \frac{\partial}{\partial c}f_{g1} + \frac{\partial}{\partial_c\theta_1}f_{g1}\frac{d}{dc}{}_c\theta_1 \qquad (70)$$

precisely and that the $\cos^2{}_c\theta_1$ term would appear preceding both particles on the right of equation (70), the weighting coefficient multiplying the $\frac{d}{dc}f_{g1}$ term should be double the value obtained by the integration process that follows.

To determine the weighting coefficients the modified equation (66) is multiplied term-wise by $d_c\theta_1$ and integrated over ${}_c\theta_1$ from 0 to $\pi/2$. [Assumption — The θ coefficients can be evaluated prior to determining the general relationship for the f_g function by a process of integration which ignores the contribution to the f_g's of θ.]

$$2\int_0^{\pi/2}\cos^2{}_c\theta_1\,d_c\theta_1\left[\frac{d}{dc}f_{g1}\right] + \int_0^{\pi/2}d_c\theta_1\left[\frac{1}{c}f_{g1}\right]$$

$$+\int_0^{\pi/2}\cos{}_c\theta_1\,d_c\theta_1\left[k_1 c\,f_{g1}\right]$$

$$= -\int_0^{\pi/2}\cos{}_c\theta_1\,d_c\theta_1\left[\frac{1}{E'c}A(1-\alpha c^2)c\,e^{-\alpha c^2}\right] \qquad (71)$$

where:

$$A = 8n\pi\left(\frac{m}{2\pi k'T}\right)^{3/2},$$

$$\alpha = m/2kT$$

Equation (71) becomes with integration:

$$\frac{\pi}{2}\frac{d}{dc}f_{g_1} + \left(\frac{\pi}{2}\frac{1}{c} + k_1 c\right)f_g = -\frac{A}{E}(1 - \alpha c^2)e^{-\alpha c^2} \tag{72}$$

Multiplying through by ,

$$\frac{d}{dc}f_{g_1} + \left(\frac{1}{c} + k_2 c\right)f_{g_1} = k_3(1 - \alpha c^2)e^{-\alpha c^2} \tag{73}$$

where $k_2 = 2k_1/\pi$ and where $k_3 = -2A/\pi E'$ /

II. For $\bar{B} \cdot \bar{E}' = 0$, $\bar{B} \times \bar{E}' \neq 0$, i.e. $\bar{B} \perp \bar{E}'$

$$-\frac{e}{m}\left\{\begin{aligned} &E'\mathrm{Cos}_c\theta_1\frac{\partial}{\partial c}f_f + (E'^2 - cE'B\mathrm{Cos}_c\theta_2)f_{g_1} \\ &+ cE'^2\mathrm{Cos}\ {}_c\theta_1\frac{\partial}{\partial c}f_{g_1} + cE'B\mathrm{Cos}_c\theta_1 f_{g_2} \\ &+ cE'^2\mathrm{Cos}_c\theta_1\,\mathrm{Cos}_c\theta_2\frac{\partial}{\partial c}f_{g_2} \\ &+ cE'^2\mathrm{Cos}_c\theta_1\,\mathrm{Cos}_c\theta_B\frac{\partial}{\partial c}f_{g_3} \end{aligned}\right\}$$

$$= \pi R^2 n_n\left\{\begin{aligned} &c^2E'\mathrm{Cos}_c\theta_1 f_{g_1} \\ &+ c^2E'\mathrm{Cos}_c\theta_2 f_{g_2} \\ &+ c^2E'\mathrm{Cos}_c\theta_B f_{g_3} \end{aligned}\right\} . \tag{74}$$

III. For $\bar{B} \times \bar{E}' \neq 0$, $\bar{B} \circ \bar{E}' \neq 0$, i.e. the third equation making up the three coupled sets is equation (64).

Since equations (65) and (74) represent two of the necessary equations coupling the f_g functions, the difference, equation (74) minue (65), must be an equality. This relationship becomes:

$$-\frac{e}{m}\left\{\begin{aligned} &-cE'B\mathrm{Cos}\,{}_c\theta_2\, f_{g_1} + cE'B\mathrm{Cos}\,{}_c\theta_1\, f_{g_2} \\ &+ cE'^2\mathrm{Cos}\,{}_c\theta\ \mathrm{Cos}\,{}_c\theta_2 \frac{\partial}{\partial c} f_{g_2} \\ &+ cE'^2\mathrm{Cos}\,{}_c\theta_1\, \mathrm{Cos}\,{}_c\theta_3 \frac{\partial}{\partial c} f_{g_3} \end{aligned}\right\}$$

$$= \pi R^2 n_n \left\{\begin{aligned} &c^2E'\mathrm{Cos}\,{}_c\theta_2\, f_{g_2} \\ &+ c^2E'\mathrm{Cos}\,{}_c\theta_3\, f_{g_3} \end{aligned}\right\}, \qquad (75)$$

where: $\mathrm{Cos}\,{}_c\theta_B = \mathrm{Cos}\,{}_c\theta_3$; ${}_c\theta_3$ is the angle between c and the third axis [B is along 3 in equation (74)] ; ${}_c\theta_2$ is the angle between c and 2 [2 is the E' x B axis] .

Equation (75) can be simplified and rearranged as follows:

$$\left\{\begin{aligned} &E'\frac{\partial}{\partial c} f_{g_2}\left[\mathrm{Cos}\,{}_c\theta_1\, \mathrm{Cos}\,{}_c\theta_2\right] - Bf_{g_2}\mathrm{Cos}\,{}_c\theta_1 \\ &+ \frac{m}{e}\pi R^2 n_n\ cf_{g_2}\ \mathrm{Cos}\,{}_c\theta_2 \\ &+ E'\frac{\partial}{\partial c} f_{g_3}\mathrm{Cos}\,{}_c\theta_1\, \mathrm{Cos}\,{}_c\theta_3 + \frac{m}{e}\pi R^2 n_n\ cf_{g_3}\mathrm{Cos}\,{}_c\theta_3 \end{aligned}\right\}$$

$$= Bf_{g_1}\ \mathrm{Cos}\,{}_c\theta_2 \tag{76}$$

Since equations (62) and (74) represent two of the necessary equations coupling the f_g functions, the difference, equation (62) minue (74), must be an equality. This relationship becomes:

$$-\frac{e}{m}\left\{\begin{array}{l} cE'B\ \mathrm{Cos}\,{}_c\theta_2(1 - \mathrm{Sin}\,{}_E\theta_B)f_{g_1} \\ \\ - cE'B\ \mathrm{Cos}\,{}_E\theta_B \mathrm{Cos}\,{}_c\theta_B f_{g_2} \\ \\ + cE'^2\,\mathrm{Cos}\,{}_c\theta_1\,\mathrm{Cos}\,{}_c\theta_2\,(\mathrm{Sin}\,{}_E\theta_B - 1)\frac{\partial}{\partial c}f_{g_2} \\ \\ + cE'B\ \mathrm{Cos}\,{}_E\theta_B \mathrm{Cos}\,{}_c\theta_2 \mathrm{Sin}\,{}_E\theta_B f_{g_3} \\ \\ - cE'^2\,\mathrm{Cos}^2\,{}_c\theta_1\,\mathrm{Cos}\,{}_E\theta_B \frac{\partial}{\partial c} f_{g_3} \end{array}\right\}$$

$$= \pi R^2 n_n \left\{\begin{array}{l} c^2E'\mathrm{Cos}\,{}_c\theta_2\,(\mathrm{Sin}\,{}_E\theta_B - 1)f_{g_2} \\ \\ - c^2E'\mathrm{Cos}\,{}_c\theta_1\,\mathrm{Cos}\,{}_E\theta_B f_{g_3} \end{array}\right\} \tag{77}$$

Equation (77) can be simplified and rearranged as follows:

$$\left(\begin{array}{l} E'\frac{\partial}{\partial c}f_{g_2}\mathrm{Cos}\,{}_c\theta_1\,\mathrm{Cos}\,{}_c\theta_2\,(\mathrm{Sin}\,{}_E\theta_B - 1) \\ \\ + \frac{m}{e}\pi R\ n_n\ cf_{g_2}\ \mathrm{Cos}\,{}_c\theta_2(\mathrm{Sin}\,{}_E\theta_B = 1) \end{array}\right)$$

$$\left\{\begin{aligned} &- Bf_{g_2}\, \mathrm{Cos}\,{}_{c}\theta_B \mathrm{Cos}\,{}_{E}\theta_B \quad + E'\frac{\partial}{\partial c} f_{g_3}\, \mathrm{Cos}^2\,{}_{c}\theta_1\, \mathrm{Cos}\,{}_{E}\theta_B \\ &- \frac{m}{e}\pi R^2 n_n\, cf_{g_3}\, \mathrm{Cos}\,{}_{c}\theta_1\, \mathrm{Cos}\,{}_{E}\theta_B \\ &+ Bf_{g_3}\, \mathrm{Cos}\,{}_{c}\theta_2 \mathrm{Cos}\,{}_{E}\theta_B \mathrm{Sin}\,{}_{E}\theta_B \end{aligned}\right\}$$

$$= Bf_{g_1}\left[\mathrm{Cos}\,{}_{c}\theta_2\,(\mathrm{Sin}\,{}_{E}\theta_B - 1)\right] \qquad (78)$$

Coefficients of equations (76) and (78), containing trigonometric functions of ${}_{c}\theta_1$, ${}_{c}\theta_2$, ${}_{c}\theta_3$, and ${}_{c}\theta_B$, can be evaluated by the process used in equation (71). Application of this process to equation (74) follows.

$$\left\{\begin{aligned} &E'\frac{d}{dc}f_{g_2}\,2\int_0^{\pi/2}\mathrm{Cos}\,{}_{c}\theta_1\, d_{c}\theta_1 \int_0^{\pi/2}\mathrm{Cos}\,{}_{c}\theta_2\, d_{c}\theta_2 \int_0^{\pi/2} d_{c}\theta_3 \\ &- Bf_{g_2}\int_0^{\pi/2}\mathrm{Cos}\,{}_{c}\theta_1\, d_{c}\theta_1 \int_0^{\pi/2} d_{c}\theta_2 \int_0^{\pi/2} d_{c}\theta_3 \\ &+ \frac{m}{e}\pi R^2 n_n\, cf_{g_2}\int_0^{\pi/2} d_{c}\theta_1 \int_0^{\pi/2}\mathrm{Cos}\,{}_{c}\theta_2\, d_{c}\theta_2 \int_0^{\pi/2} d_{c}\theta_3 \\ &+ E'\frac{d}{dc}f_{g_3}\,2\int_0^{\pi/2}\mathrm{Cos}\,{}_{c}\theta_1\, d_{c}\theta_1 \int_0^{\pi/2} d_{c}\theta_2 \int_0^{\pi/2}\mathrm{Cos}\,{}_{c}\theta_3\, d_{c}\theta_3 \\ &+ \frac{m}{e}\pi R^2 n_n\, cf_{g_3}\int_0^{\pi/2} d_{c}\theta_1 \int_0^{\pi/2} d_{c}\theta_2 \int_0^{\pi/2}\mathrm{Cos}\,{}_{c}\theta_3\, d_{c}\theta_3 \end{aligned}\right\}$$

$$= \left[Bf_{g_1}\int_0^{\pi/2} d_{c}\theta_1 \int_0^{\pi/2}\mathrm{Cos}\,{}_{c}\theta_2\, d_{c}\theta_2 \int_0^{\pi/2} d_{c}\theta_3\right] \qquad (79)$$

Evaluation of the integrals of equation (78) produces the following results.

$$\left(\begin{aligned} &E'\frac{d}{dc}f_{g2}\left[2(1)(1)\frac{\pi}{2}\right] - Bf_{g2}\left[(1)\frac{\pi}{2}(\frac{\pi}{2})\right] \\ &+ \frac{m}{e}\pi R^2 n_n \; cf_{g2}\left[(1)\frac{\pi}{2}(\frac{\pi}{2})\right] \\ &+ E'\frac{d}{dc}f_{g3}\left[2(1)\frac{\pi}{2}(1)\right] \\ &+ \frac{m}{e}\pi R^2 n_n \; cf_{g3}\left[(\frac{\pi}{2})\frac{\pi}{2}(1)\right] \end{aligned}\right)$$

$$= \quad Bf_{g1}\left[(1)\frac{\pi}{2}(\frac{\pi}{2})\right] \qquad (80)$$

Continuing the evaluation of the weighting coefficients,

$$\left\{\begin{aligned} &\pi E'\frac{d}{dc}f_{g2} - \frac{\pi^2}{4}(B - \frac{m}{e}\pi R^2 n_n \, c)f_{g2} \\ &+ \pi E'\frac{d}{dc}f_{g3} + \frac{\pi^2}{4}\frac{m}{e}\pi R \; n_n \; c \; f_{g3} \end{aligned}\right\}$$

$$= \left[\frac{\pi^2}{4}Bf_{g1}\right] \; . \qquad (81)$$

Equation (81) is multiplied termwise by .

$$\left(\frac{d}{dc}f_{g2} - \frac{\pi}{4E'}(B - \frac{m}{e}\pi R \; n_n \; c)f_{g2}\right.$$

$$\left(+ \frac{d}{dc} f_{g_3} \quad + \frac{\pi}{4E'} \cdot \frac{m}{e} \pi R \, n_n \, c \, f_{g_3} \right)$$

$$= \left[\frac{\pi B}{4E'} f_{g_1} \right] \qquad (82)$$

Application of the preceding process to equation (78) follows.

$$\left[\begin{array}{l} E'(\mathrm{Sin}\,_E\theta_B \; - 1)\frac{d}{dc} f_{g_2} \; 2\int_0^{\pi/2} \mathrm{Cos}_C\theta_1 \, d_C\theta_1 \int_0^{\pi/2} \mathrm{Cos}_C\theta_2 \, d_C\theta_2 \\ + \frac{m}{e}\pi R^2 n_n (\mathrm{Sin}\,_E\theta_B \; - 1) c f_{g_2} \int_0^{\pi/2} d_C\theta_1 \int_0^{\pi/2} \mathrm{Cos}_C\theta_2 \, d_C\theta_2 \\ - \; B f_{g_2} \int_0^{\pi/2} \mathrm{Cos}_C\theta_1 \; d_C\theta_1 \int_0^{\pi/2} d_C\theta_2 \\ + \; E'\mathrm{Cos}\,_E\theta_B \frac{d}{dc} f_{g_3} \; 2\int_0^{\pi/2} \mathrm{Cos}^2\,_C\theta_1 \, d_C\theta_1 \int_0^{\pi/2} d_C\theta_2 \\ - \frac{m}{e}\pi R^2 n_n \mathrm{Cos}\,_E\theta_B c f_{g_3} \int_0^{\pi/2} \mathrm{Cos}_C\theta_1 \, d_C\theta_1 \int_0^{\pi/2} d_C\theta_2 \\ + \; \frac{B}{2}\mathrm{Sin}2\,_E0_B \;\; f_{g_3} \int_0^{\pi/2} d_C\theta_1 \int_0^{\pi/2} \mathrm{Cos}_C\theta_2 \, d_C\theta_2 \end{array} \right]$$

$$= \left[B(\mathrm{Sin}\,_E\theta_B \; - 1) f_{g_1} \int_0^{\pi/2} d_C\theta_1 \int_0^{\pi/2} \mathrm{Cos}_C\theta_2 \, d_C\theta_2 \right] \qquad (83)$$

Where: $\mathrm{Cos}_C\theta_B = \mathrm{Cos}_C\theta_1 / \mathrm{Cos}_E\theta_B$, and $\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta = \frac{1}{2}\mathrm{Sin}\,2\theta$. Evaluation of the integrals of equation (83) produces the following results.

$$\left(E'(\mathrm{Sin}\,_E\theta_B \; - 1) \frac{d}{dc} f_{g_2}\left[2(1)(1)\right] \right.$$

$$\left[\begin{aligned} &+ \frac{m}{e}\pi R^2 n_n(\text{Sin}\,{}_E\theta_B - 1)cf_{g_2}\left[(1)\frac{\pi}{2}\right] \\ &- Bf_{g_2}\left[(1)\frac{\pi}{2}\right] + E'\text{Cos}\,{}_E\theta_B\frac{d}{dc}f_{g_3}\left[2\left(\frac{\pi}{2}\right)\frac{\pi}{2}\right] \\ &- \frac{m}{e}\pi R^2 n_n \text{Cos}\,{}_E\theta_B c f_{g_3}\left[(1)\frac{\pi}{2}\right] \\ &+ B\text{Sin}2\,{}_E\theta_B f_{g_3}\left[\left(\frac{1}{2}\right)\frac{\pi}{2}(1)\right] \end{aligned}\right]$$

$$= B(\text{Sin}\,{}_E\theta_B - 1)\ f_{g_1}\left[\frac{\pi}{2}(1)\right] \tag{84}$$

Continuing the evaluation of the weighting coefficients and dividing through by the coefficient of the first term, equation (84) becomes

$$\begin{aligned} &\frac{d}{dc}f_{g_2} + \left[\frac{\pi}{4}\frac{m}{eE'}\pi R^2 n_n\, c - \frac{\pi}{4}\frac{B}{E'}\left(\frac{1}{\text{Sin}\,{}_E\theta_B - 1}\right)\right]f_{g_2} \\ &+ \frac{\pi^2}{8}\left(\frac{\text{Cos}\,{}_E\theta_B}{\text{Sin}\,{}_E\theta_B - 1}\right)\frac{d}{dc}f_{g_3} \\ &+ \left[-\frac{\pi}{4}\frac{m}{e}\pi R^2 n_n\left(\frac{\text{Cos}\,{}_E\theta_B}{\text{Sin}\,{}_E\theta_B - 1}\right)c + \frac{\pi}{8}\frac{B}{E'}\left(\frac{\text{Sin}\,2\,{}_E\theta_B}{\text{Sin}\,{}_E\theta_B - 1}\right)\right]f_g \\ &= \left[\frac{\pi}{4}\frac{B}{E'}f_g\right] \end{aligned} \tag{85}$$

Equations (83) and (85) are two coupled equations relating the three f_g functions. The difference of equations (85) minus (83) is another coupled equation independent of either (83) or (85) but not both. Equation (85) minus equation (83) becomes:

$$\left\{\begin{aligned} &\frac{\pi}{4}\frac{B}{E'}\left[1 - \frac{1}{\text{Sin}\,{}_E\theta_B - 1}\right] f_{g2} + \left[\frac{\pi^2}{8}\left(\frac{\text{Cos}\,{}_E\theta_B}{\text{Sin}\,{}_E\theta_B - 1}\right) - 1\right]\frac{d}{dc}f_g \\ &+ \left[-\frac{\pi}{4}\frac{m}{eE'}\pi R^2 n_n\left(\frac{\text{Cos}\,{}_E\theta_B}{\text{Sin}\,{}_E\theta_B - 1} + 1\right)c + \frac{\pi}{8}\frac{B}{E'}\left(\frac{\text{Sin}\,2\,{}_E\theta_B}{\text{Sin}\,{}_E\theta_B - 1}\right)\right] f_g \end{aligned}\right\}$$

$$= 0 \qquad (86)$$

Equation (86) is simplified as follows:

$$\left[\frac{\pi}{4E'}(2 - \text{Sin}\,{}_E\theta_B)\, f_{g2}\right]$$

$$= \left\{\begin{aligned} &\left(\frac{\pi^2}{8}\text{Cos}\,{}_E\theta_B - \text{Sin}\,{}_E\theta_B + 1\right)\frac{d}{dc}f_{g3} \\ &+ - \frac{\pi m}{4E'e}\pi R^2 n_n(\text{Cos}\,{}_E\theta_B + \text{Sin}\,{}_E\theta_B - 1)c \\ &+ \frac{\pi}{8}\frac{B}{E'}\text{Sin}2\,{}_E\theta_B \end{aligned}\right\}, \qquad (87)$$

Equation (87) can be readily solved for f_{g2} in terms of f_{g3} .

$$f_{g2} = \left\{\begin{aligned} &\frac{4E'}{\pi}\left(\frac{1}{2 - \text{Sin}\,{}_E\theta_B}\right)\left[\left(\frac{\pi^2}{8}\text{Cos}\,{}_E\theta_B + 1 - \text{Sin}\,{}_E\theta_B\right)\frac{d}{dc}f_g\right. \\ &- \frac{\pi m}{4eE'}\pi R^2 n_n(\text{Cos}\,{}_E\theta_B + \text{Sin}\,{}_E\theta_B - 1)cf_{g3} \\ &\left. + \frac{\pi B}{8E'}\text{Sin}\ 2\,{}_E\theta_B f_{g3}\right] \end{aligned}\right]$$

$$= \left\{ \frac{E'}{2\pi}\left(\frac{1}{2 - \text{Sin}\,{}_E\theta_B}\right)\left[\pi^2\text{Cos}\,{}_E\theta_B + 8(1 - \text{Sin}\,{}_E\theta_B)\frac{d}{dc}f_{g3}\right] \right]$$

$$\left. + \left\{ \begin{array}{l} - \frac{m}{e} \pi R^2 n_n \left(\frac{\mathrm{Cos}\, {}_{E}\theta_{B} + \mathrm{Sin}\, {}_{E}\theta_{B} - 1}{2 - \mathrm{Sin}\, {}_{E}\theta_{B}} \right) c \\ \\ + \frac{B}{2} \left(\frac{\mathrm{Sin}\, 2\, {}_{E}\theta_{B}}{2 - \mathrm{Sin}\, {}_{E}\theta_{B}} \right) \end{array} \right\} f_{g_3} \right] \qquad (88)$$

For: $0 \leq {}_{E}\theta_{B} \leq \frac{\pi}{2}$

Equations (73) , (83) , and (88) are three independent equations in the functions f_{g_1}, f_{g_2}, and f_{g_3}. The equations are repeated below.

$$(1) \quad \left[\frac{d}{dc} f_{g_1} + \left(\frac{1}{c} + K_1 c \right) f_g = K_2 (1 - \alpha c^2) e^{-\alpha c^2} \right] \qquad (89)$$

$$(2) \quad \left[f_{g_2} = K_3 \frac{d}{dc} f_{g_3} + (K_4 + K_5 c) f_{g_3} \right] \qquad (90)$$

$$(3) \quad \left[\begin{array}{l} \frac{d}{dc} f_{g_2} + (K_6 + K_7 c) f_{g_2} + \frac{d}{dc} f_{g_3} + K_7 c\, f_{g_3} \\ \\ = K_8 f_{g_1} \end{array} \right] \qquad (91)$$

Where:

$$\left[\begin{array}{l} \alpha = m/2kT \\ \\ K_1 = + \frac{2}{eE'} m R^2 n_n \\ \\ K_2 = - \frac{16\, n}{E'} \left(\frac{m}{2 \pi k' T} \right)^{3/2} \\ \\ K_3 = \frac{E'}{2\pi} \left(\frac{1}{2 - \mathrm{Sin}\, {}_{E}\theta_{B}} \right) \left[\pi^2 \mathrm{Cos}\, {}_{E}\theta_{B} + 8 - 8\, \mathrm{Sin}\, {}_{E}\theta_{B} \right] \end{array} \right.$$

$$\left.\begin{aligned}
K_4 &= \frac{B}{2}\left(\frac{\sin 2\,{}_E\theta_B}{2 - \sin {}_E\theta_B}\right) \\
K_5 &= -\frac{m}{e}R^2 n_n \pi \left(\frac{\cos {}_E\theta_B + \sin {}_E\theta_B - 1}{2 - \sin {}_E\theta_B}\right) \\
K_6 &= -\frac{\pi B}{4E'} \\
K_7 &= +\frac{\pi m}{4eE'}\,\pi R^2 n_n \\
K_8 &= \frac{\pi B}{4E'}
\end{aligned}\right\} \qquad (92)$$

Equation (89) can be solved directly for f_{g_1}. Equations (90) and (91) can be combined to give f_{g_3} in terms of f_{g_1} by substitution for f_{g_2} in equation (91). Equation (90) specifies f_{g_2} in terms of f_{g_3}; from equation (91) the derivative of f_{g_2} in terms of f_{g_3} becomes:

$$\left.\begin{aligned}
\frac{d}{dc} f_{g_2} = K_3 \frac{d^2}{dc^2} f_{g_3} &+ K_4 \frac{d}{dc} f_{g_3} \\
&+ K_5 f_{g_3} + K_5 c \frac{d}{dc} f_{g_3}
\end{aligned}\right\} \qquad (93)$$

Equations (90) and (93) are substituted into equation (91) to generate equation (94); (94) is then simplified.

$$\left[\begin{aligned}
&K_3 \frac{d^2}{dc^2} f_{g_3} + K_4 \frac{d}{dc} f_{g_3} + K_5 f_{g_3} + K_5 c \frac{d}{dc} f_{g_3} \\
&+ K_3 K_6 \frac{d}{dc} f_{g_3} + K_4 K_6 f_{g_3} + K_5 K_6 c\, f_{g_3}
\end{aligned}\right.$$

$$\left. \begin{array}{l} + \; K_3K_7c\,\frac{d}{dc}f_{g3} \; + \; K_4K_7c\; f_{g3} \; + \; K_5K_7c^2\, f_{g3} \\ \\ + \; \frac{d}{dc}f_{g3} \; + \; K_7c\; f_{g3} \end{array} \right)$$

$$= \left[K_8 f_{g1} \right] \tag{94}$$

$$\left\{ \begin{array}{l} \frac{d^2}{dc^2}f_{g3} \; + \; \frac{1}{K_3}\left[(K_4 \; + \; K_3K_6 \; + \; 1) \; + \; (K_5 \; + \; K_3K_7)c\right]\frac{d}{dc}f_{g3} \\ \\ \qquad + \left[\begin{array}{l} \frac{1}{K_3}\;(K_5 \; + \; K_4K_6) \; + \; (K_5K_6 \; + \; K_4K_7 \; + \; K_7)\;c \\ \qquad\qquad + \; (K_5K_7)\;c^2 \end{array} \right] \end{array} \right\}$$

$$= \left[\left(\frac{K_8}{K_3}\right) f_{g1} \right] \tag{95}$$

$$\left[\begin{array}{l} \frac{d^2}{dc^2}f_{g3} \; + \; (K_9 \; + \; K_{10}\,c)\frac{d}{dc}f_{g3} \\ \\ \qquad + \; (K_{11} \; + \; K_{12}\,c \; + \; K_{13}\,c^2)\; f_{g3} \; = \; K_{14}\,f_{g1} \end{array} \right] \tag{96}$$

where:

$$\left[\begin{array}{l} K_9 \; = \; (K_4 \; + \; K_3K_6 \; + \; 1)/K_3 \\ \\ K_{10} \; = \; (K_5 \; + \; K_3K_7)/K_3 \\ \\ K_{11} \; = \; (K_5 \; + \; K_4K_6)/K_3 \\ \\ K_{12} \; = \; (K_5K_6 \; + \; K_4K_7 \; + \; K_7)/K_3 \end{array} \right.$$

$$\left.\begin{aligned} K_{13} &= (K_5K_7)/K_3 \\ K_{14} &= K_8/K_3 \end{aligned}\right\} \qquad (97)$$

The solution for equation (89) follows.

$$\frac{d}{dc}f_{g1} + \left(\frac{1}{c} + K_1c\right)f_{g1} = K_2(1-\alpha c^2)e^{-\alpha c^2} \qquad (89)$$

An integrating factor is

$$\left[e^{\int(\frac{1}{c}+K_1c)dc}\right] = e^{\ln c + \frac{K_1}{2}c^2} = ce^{\frac{K_1}{2}c^2}$$

$$f_{g1}\left[ce^{\frac{K_1}{2}c^2}\right] = \int K_2\,(1-\alpha c^2)c\,e^{(\frac{K_1}{2}-\alpha)c^2} + K'$$

$$= K_2\int c\,e^{(\frac{K_1}{2}-\alpha)c^2}dc - K_2\alpha\int c^3\,e^{(\frac{K_1}{2}-\alpha)c^2}dc + K' \qquad (98)$$

Let $c^2 = y$, a change of variable.

Then: $c = \sqrt{y}$.

$2cdc = dy$

$dc = dy/2\sqrt{y}$

With the change in variable the right side of equation (98) becomes

$$\frac{K_2}{2}\int e^{(\frac{K_1}{2}-\alpha)y}dy - \frac{K_2}{2}\alpha\int ye^{(\frac{K_1}{2}-\alpha)y}dy + K'$$

$$= \frac{K_2}{2}\left[\frac{1}{(\frac{K_1}{2}-\alpha)} - \frac{\alpha}{(\frac{K_1}{2}-\alpha)^2}\left[\left(\frac{K_1}{2}-\alpha\right)y - 1\right]\right]e^{(\frac{K_1}{2}-\alpha)y} + K'$$

$$= \frac{K_2}{K_1 - 2\alpha}\left[1 - \alpha y + \frac{2\alpha}{K_1 - 2\alpha}\right] e^{\left(\frac{K_1}{2} - \alpha\right)y} + K'$$

$$= \frac{K_2}{K_1 - 2\alpha}\left[\frac{K_1}{K_1 - 2\alpha} - \alpha y\right] e^{\left(\frac{K_1}{2} - \alpha\right)y} + K'$$

With the substitution back to c^2 and multiplying both sides of equation (98) by

$$\left[ce^{\frac{K_1}{2}c^2}\right]^{-1}$$

yield the following solution for f_g.

$$f_{g_1} = \frac{K_2}{K_1 - 2\alpha}\left[\frac{K_1}{K_1 - 2\alpha}\frac{1}{c} - \alpha c\right] e^{-\alpha c^2} + \frac{K'}{c}e^{-\frac{K_1}{2}c^2}$$

$$f_{g_1}(c) = \left(K_{15}\frac{1}{c} + K_{16}c\right)e^{-\alpha c^2} + \frac{K'}{c}e^{-kc^2} \qquad (99)$$

where:

$K_{15} = K_1K_2/(K_1 - 2\alpha)^2$

$K_{16} = -\alpha K_2/(K_1 - 2\alpha)$

$k = K_1/2$

K' is a constant of integration. (100)

The complete solution for $f_{g_1}(c)$ as displayed by equation (99) consists of a complimentary solution $f_{g_1}(c)_{comp.}$ plus a particular solution $f_{g_1}(c)_{part.}$

$$f_{g_1}(c)_{Comp.} = K' \frac{1}{c} e^{-kc^2}$$

$$f_{g_1}(c)_{Part.} = (K_{15} \frac{1}{c} + K_{16} c) e^{-\alpha c^2}$$

The arbitrary constant K' can be evaluated by the requirement that f_e be zero for c equal to zero; i.e.,

$$f_e(0) = f_f(0) + \bar{c} \cdot \bar{E}' f_{g_1}(0) = 0 .$$

$$f_f(0) = A(0) e^{0} = 0$$

$$f_{g_1}(0) = c \, E\mathrm{Cos}\, {}_c\theta_1 \left[\frac{K'}{c} e^{-kc^2} + (\frac{K_{15}}{c} + K_{16} c) e^{-\alpha c^2} \right]_{c=0} = 0$$

$$= E\mathrm{Cos}\, {}_c\theta_1 \left[K' e^{-kc^2} + (K_{15} + K_{16} c^2) e^{-\alpha c^2} \right]_{c=0} = 0$$

$$K'(1) = \left[K_{15} + K_{16}(0) \right] (1)$$

$$K' = -K_{15}$$

Equation (99) can now be written

$$f_{g_1}(c) = K_{15} \frac{1}{c} \left[e^{-\alpha c^2} - e^{-kc^2} \right] + K_{16} c \; e^{-\alpha c^2} \qquad (101)$$

With the solution for f_{g_1} accomplished in equation (101), attention can be turned to finding a solution for the linear, second order, non-homogeneous, ordinary differential equation with polynomial coefficients given by equation (96). No general method of solution for equations of this type is known. For certain special equations within this type, group methods have been worked out, usually involving an expedient change of variable, transformation, or operational factoring which allows the equation to reduce to a first order equation. For equation (96) no such technique has been discovered by the author. Differential equations of the form of equation (96) often yield to a series solution. This technique is attempted; the equation is repeated.

$$\frac{d^2}{dc^2} f_{g3} + (K_9 + K_{10} c) \frac{d}{dc} f_{g3} + (K_{11} + K_{12} c + K_{13} c) f_{g3}$$

$$= K_{14} f_{g1} = K_{14} K_{15} \frac{1}{c} \left[e^{-\alpha c^2} - e^{-kc^2} \right] + K_{14} K_{16} c \, e^{-\alpha c^2} \qquad (96)$$

The homogeneous case will be treated first; the homogeneous equation becomes:

$$\frac{d^2}{dc^2} f_{g3} + (K_9 + K_{10} c) \frac{d}{dc} f_{g3}$$

$$+ (K_{11} + K_{12} c + K_{13} c) f_{g3} = 0 \qquad (102)$$

A series solution is assumed of the type

$$f_{g3} = \sum_{n=0}^{\infty} a_n c^n \; . \qquad (103)$$

Where the a_n's are coefficients to be determined. From equation (102) the derivatives can be written

$$\frac{d}{dc} f_{g3} = \sum_{n=0}^{\infty} a_n (n) c^{n-1} \; , \qquad (104)$$

and

$$\frac{d^2}{dc^2} f_{g3} = \sum_{n=0}^{\infty} a_n (n)(n - 1) c^{n-2} \; .$$

Equation (74) now becomes :

$$\left[\begin{aligned} &\sum_{n=0}^{\infty} a_n(n)(n-1)c^{n-2} + K_9 \sum_{n=0}^{\infty} a_n(n)c^{n-1} \\ &+ K_{10} \sum_{n=0}^{\infty} a_n(n)c^{n} + K_{11} \sum_{n=0}^{\infty} a_n c^{n} \\ &+ K_{12} \sum_{n=0}^{\infty} a_n c^{n+1} + K_{13} \sum_{n=0}^{\infty} a_n c^{n+2} \end{aligned}\right\} \quad (105)$$

To solve equation (102) requires that the a_n coefficients of equation (105) be determined. This may be done by setting the coefficients of like powers of c equal to zero. Since equation (102) is of second order one expects two arbitrary constants. To begin, the indices on c in each of the summations of equations (105) are adjusted to be equivalent in the following way.

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n(n)(n-1)c^{n-2} + K_9 \sum_{n=1}^{\infty} a_{n-1}(n-1)c^{n-2} \\ &+ K_{10} \sum_{n=2}^{\infty} a_{n-2}(n-2)c^{n-2} + K_{11} \sum_{n=2}^{\infty} a_{n-2} c^{n-2} \\ &+ K_{12} \sum_{n=3}^{\infty} a_{n-3} c^{n-2} + K_{13} \sum_{n=4}^{\infty} a_{n-4} c^{n-2} = 0 \end{aligned} \quad (106)$$

For $n = 0$ only the first summation in equation (106) exists.

$a_o(0)(-1) = 0$ $\qquad a_o$ is arbitrary.

$n = 1$

$a_1(1)(0) + K_9 a_o(0) = 0$

a_1 is arbitrary.

$n = 2$

$$a_2(2)(1) + K_9 a_1(1) + K_{10} a_o(0) + K_{11} a_o = 0$$

$$2a_2 = K_9 a_1 + K_{11} a_o = 0, \quad a_2 = -\tfrac{1}{2}(K_9 a_1 + K_{11} a_o)$$

$n = 3$

$$a_3(3)(2) + K_9 a_2(2) + K_{10} a_1(1) + K_{11} a_1 + K_{12} a_o = 0$$

$$6a_3 + 2K_9 a_2 + (K_{10} + K_{11}) a_1 + K_{12} a_o = 0$$

$$a_3 = -\frac{1}{6}\left[2K_9 a_2 + (K_{10} + K_{11}) a_1 + K_{12} a_o\right]$$

$n = 4$

$$a_4(4)(3) + K_9 a_3(3) + K_{10} a_2(2) + K_{11} a_2 + K_{12} a_1 + K_{13} a_o$$

$$a_4 = -\frac{1}{12}\left[3K_9 a_3 + 2K_{10} a_2 + K_{11} a_2 + K_{12} a_1 + K_{13} a_o\right]$$

The recurrence relationship of a_n coefficients for $n > 1$ becomes:

$$a_n = -\frac{1}{n(n-1)}\Big[(n-1)K_9 a_{n-1} + (n-2)K_{10} a_{n-2} + K_{11} a_{n-2} + K_{12} a_{n-3} + K_{13} a_{n-4}\Big] \qquad (107)$$

where:

$$k < 0, \quad a_k = 0$$

The complimentary solution to equation (96) is, therefore, equation (103).

This solution will be truncated at $n = 4$, and the abbreviated form of the solution is taken to be an acceptable approximation to the complete solution. [Assumption — The complimentary solution to equation (96) can be approximated by a series of the form of equation (103) truncated at $n = 4$.] For this approximation $(f_{g3})_c$ becomes:

$$
\begin{aligned}
(f_{g3})_c &= a_0 + a_1 c + a_2 c^2 + a_3 c^3 + a_4 c^4 \qquad (108)\\
&= a_0 + a_1 c - \tfrac{1}{2}(K_9 a_1 + K_{11} a_0)c^2\\
&\quad + \tfrac{1}{6}\left[(K_9^2 - K_{10} - K_{11})a_1 + (K_9 K_{11} - K_{12})a_0\right]c^3\\
&\quad - \tfrac{1}{24}\Big[(K_9^3 - 3K_9K_{10} - 2K_9K_{11} + 2K_{12})\,a_1\\
&\qquad + (K_9^2 K_{11} - K_9K_{12} - 2K_{10}K_{11} - K_{11}^2\\
&\qquad + 2K_{13})a_0\Big]c^4
\end{aligned}
$$

$$
\begin{aligned}
(f_{g3})_c &= \Big[a_0(1 + K_{17}c^2 + K_{18}c^3 + K_{19}c^4)\\
&\quad + a_1(1 + K_{20}c + K_{21}c^2 + K_{22}c^3)c\Big] \qquad (109)
\end{aligned}
$$

where:

$$
\left(
\begin{aligned}
K_{17} &= -\tfrac{1}{2}K_{11}\\
K_{18} &= \tfrac{1}{6}(K_9K_{11} - K_{12})\\
K_{19} &= -\tfrac{1}{24}(K_9^2 K_{11} - K_9K_{12} - 2K_{10}K_{11} - K_{11}^2 + 2K_{13})\\
K_{20} &= -\tfrac{1}{2}K_9
\end{aligned}
\right.
$$

$$\left\{ \begin{aligned} K_{21} &= \tfrac{1}{6}(K_9^2 - K_{10} - K_{11}) \\ K_{22} &= -\tfrac{1}{24}(K_9^3 - 3K_9K_{10} - 2K_9K_{11} + 2K_{12}) \\ &\quad a_0 \text{ and } a_1 \text{ are arbitrary} \end{aligned} \right\} \qquad (110)$$

As a check on the validity of the foregoing assumption and approximation, equation (96) can be expanded through terms up to and including c^4. A check was made for all terms through c^2; the only terms not cancelling involved the a_0. These were

$$a_0(-\tfrac{1}{2}K_{11}K_{12} + \tfrac{1}{2}K_{11}^2)c^2 = 0 ,$$

or

$$K_{11} - K_{12} \neq 0 .$$

Each of the additional thirty-two terms cancelled.

A particular solution to equation (96) must be found. In order to generate terms on the left of (96) of the form $\frac{1}{c}e^{-c^2}$, the particular solution will require a term of the form $c(\ln c)e^{-c^2}$. In order to generate terms on the left of equation (96) of the form ce^{-c^2}, the particular solution must include a term on the form e^{-c^2}. Other than the information just given the form of the particular solution is vague. If a recurrence relationship for the a's involving only n and the constants K_9, K_{10}, K_{11}, K_{12}, and K_{13} can be specified, it appears that the best procedure for determining the particular solution is the method of "variation of parameters". The solution to the homogeneous equation (102) is an infinite series which was truncated at the

c^4 term as an approximation. It seems justified to seek a recurrance relation involving only n and the constant K's which holds through the a_5 as a reasonable approximation, from which an acceptable particular solution can be formulated. As a first step in this procedure, the values for the a's through a_7 are found in terms of the K's by referring to the recurrance relationship of equation (107).

$$a_0 = \text{arbitrary.}$$

$$a_1 = \text{arbitrary.}$$

$$a_2 = -\tfrac{1}{2}K_9 a_1 + K_{11} a_0$$

$$a_3 = +\tfrac{1}{6}\left[(K_9^2 - K_{10} - K_{11})A_1 + (K_9 K_{11} - K_{12})A_0\right]$$

$$a_4 = -\tfrac{1}{24}\left[\begin{aligned}&(K_9^3 - 3K_9K_{10} - 2K_9K_{11} + 2K_{12})A_1\\ &\quad + (K_9^2 K_{11} - K_9K_{12} - 2K_{10}K_{11} - K_{11}^2 + 2K_{13})A_0\end{aligned}\right]$$

$$a_5 = +\tfrac{1}{120}\left[\begin{aligned}&(K_9^4 - 6K_9^2K_{10} - 3K_9^2K_{11} + 5K_9K_{12} + 3K_{10}^2 + 4K_{10}K_{11}\\ &\quad + K_{11}^2 - 6K_{13})A_1\\ &+ (K_9^3K_{11} - K_9^2K_{12} - 5K_9K_{10}K_{11} - 2K_9K_{11}^2 + 2K_9K_{13}\\ &\quad + 3K_{10}K_{12} + 4K_{11}K_{12})A_0\end{aligned}\right]$$

$$a_6 = -\tfrac{1}{720}\left[\begin{aligned}&(K_9^5 - 10K_9^3K_{10} - 4K_9^3K_{11} + 9K_9^2 K_{12}\\ &\quad + 15K_9K_{10}^2 + 15K_9K_{10}K_{11} + 3K_9K_{11}^2\\ &\quad - 18K_9K_{13} - 12K_{10}K_{12} - 6K_{11}K_{12})A_1\end{aligned}\right.$$

$$\left[\begin{aligned}
&+ (K_9^4 K_{11} - K_9^3 K_{12} - 9K_9^2 K_{10} K_{11} = 3K_9^2 K_{11}^2 \\
&\quad + 2K_9^2 K_{13} + 7K_9 K_{10} K_{12} + 9K_9 K_{11} K_{12} \\
&\quad + 8K_{10}^2 K_{11} + 6K_{10} K_{11}^2 - 8K_{10} K_{13} \\
&\quad - 14K_{11} K_{13} + K_{11}^3 - 4K_{12}^2) A_0
\end{aligned}\right]$$

$$a_7 = -\frac{1}{5040}\left[\begin{aligned}
&(K_9^6 - 15K_9^4 K_{10} - 5K_9^4 K_{11} + 14K_9^3 K_{12} + 45K_9^2 K_{10}^2 \\
&\quad + 36K_9^2 K_{10} K_{11} + 6K_9^2 K_{11}^2 - 38K_9^2 K_{13} \\
&\quad - 52K_9 K_{10} K_{12} - 21K_9 K_{11} K_{12} - 15K_{10}^3 \\
&\quad - 23K_{10}^2 K_{11} - 9K_{10} K_{11}^2 + 50K_{10} K_{13} \\
&\quad - K_{11}^3 + 26K_{11} K_{13} + 10K_{12}^2) A_1 \\
&+ (K_9^5 K_{11} - K_9^4 K_{12} - 14K_9^3 K_{10} K_{11} - 4K_9^3 K_{11}^2 \\
&\quad + 2K_9^3 K_{13} + 12K_9^2 K_{10} K_{12} + 15K_9^2 K_{11} K_{12} \\
&\quad + 33K_9 K_{10}^2 K_{11} + 21K_9 K_{10} K_{11}^2 - 18K_9 K_{10} K_{13} \\
&\quad - 36K_9 K_{11} K_{13} + 3K_9 K_{11}^3 - 9K_9 K_{12}^2 \\
&\quad - 15K_{10}^2 K_{12} - 33K_{10} K_{11} K_{12} \\
&\quad - 9K_{11}^2 K_{12} + 30K_{12} K_{13}) A_0
\end{aligned}\right]$$

With a_2 through a_7 specified in terms of the arbitrary a_0 and a_1

a recurrence relationship that holds through a_7 can be written. This relationship is given in equation (111)

$$a_n = \frac{(-1)^{(n-1)}}{n!} \left\{ \begin{aligned} & K_9^{(n-1)} - \frac{(n-1)(n-2)}{2} K_9^{(n-3)} K_{10} - (n-2) K_9^{(n-3)} K_{11} \\ & + \frac{n(n-3)}{2} K_9^{(n-4)} K_{12} + \frac{(n-1)(n-2)(n-3)(n-4)}{8} K_9^{(n-5)} K_{10}^2 \\ & + \frac{(n-1)(n-3)(n-4)}{2} K_9^{(n-5)} K_{10} K_{11} + \frac{(n-3)(n-4)}{2} K_9^{(n-5)} K_{11}^2 \\ & - \frac{(n-4)(n^2-2n+3)}{3} K_9^{(n-5)} K_{13} - 2(n-5)(7n-36) K_9^{(n-6)} K_{10} K_{12} \\ & - \frac{(n-4)(n-5)n}{2} K_9^{(n-6)} K_{11} K_{12} - (n-2)(n-4)(n-6) K_9^{(n-7)} K_{10}^3 \\ & - \frac{(n-6)(n^2-n+4)}{2} K_9^{(n-7)} K_{10}^2 K_{11} - 3(n-4)(n-6) K_9^{(n-7)} K_{10} K_{11}^2 \\ & + \frac{5(n-2)(n-3)(n-6)}{2} K^{(n-7)} K_{10} K_{13} - (n-6) K_9^{(n-7)} K_{11}^3 \\ & + \frac{(n-6)(n^2+n-4)}{2} K_9^{(n-7)} K_{11} K_{13} + \frac{(n-2)(n-3)(n-6)}{2} K_9^{(n-7)} K_{12}^2 \end{aligned} \right\} a_1$$

$$\left[\begin{aligned} & K_9^{(n-2)} K_{11} - K_9^{(n-3)} K_{12} - \frac{n(n-3)}{2} K_9^{(n-4)} K_{10} K_{11} \\ & \qquad - (n-3) K_9^{(n-4)} K_{11}^2 \\ & + 2K_9^{(n-4)} K_{13} + \frac{(n-4)(n+1)}{2} K_9^{(n-5)} K_{10} K_{12} \\ & + \frac{(n-4)(n+3)}{2} K_9^{(n-5)} K_{11} K_{12} \\ & + \frac{(n-2)(n-4)(n-5)(n+4)}{10} K_9^{(n-6)} K_{10}^2 K_{11} \\ & + \frac{n(n-4)(n-5)}{2} K_9^{(n-6)} K_{10} K_{11}^2 \\ & - \frac{(n-4)(n-5)(n+2)}{2} K_9^{(n-6)} K_{10} K_{13} \\ & - \frac{(n-5)(n+1)(n+2)}{4} K_9^{(n-6)} K_{11} K_{13} \end{aligned} \right.$$

$$\left.\begin{array}{l}
+ \frac{(n-4)(n-5)}{2} K_9^{(n-6)} K_{11}^3 \\[2ex]
- \frac{(n-5)(n+2)}{2} K_9^{(n-6)} K_{12}^2 \\[2ex]
- \frac{(n-1)(n-2)(n-6)}{2} K_9^{(n-7)} K_{10}^2 K_{12} \\[2ex]
-(n-6)(n^2-2n-2) K_9^{(n-7} K_{10} K_{11} K_{12} \\[2ex]
-(n-6)(n+2) K_9^{(n-7)} K_{11}^2 K_{12} \\[2ex]
+ (n-1)(n-2)(n-6) K_9^{(n-7)} K_{12} K_{13}
\end{array}\right\} a_0 \quad (111)$$

Equation (111) holds for all a_n through $n = 7$; however, the last seven terms preceding a_1 and the last four terms preceding a_0 are not expected to be precisely correct. Generation of an a_8 and a_9 would be necessary to compute these terms with acceptable accuracy. The first eight terms preceding a_1 and the first seven terms preceding a_0 in equation (111) are precisely correct through $n = 7$; the fifteen terms are non-zero for $n = 5$. All other terms are zero for $n \leq 5$. Under the approximation that terms for $n > 5$ can be neglected, the recurrence relationship can be approximated by equation (112).

$$a_n = \frac{(-1)^{(n-1)}}{n!} \left\{ \left[\begin{array}{l}
K_9^{(n-1)} - \frac{1}{2}(n-1)(n-2) K_9^{(n-3)} K_{10} - (n-2) K_9^{(n-3)} K_{11} \\[2ex]
+ \frac{1}{2}(n-3)n K_9^{(n-4)} K_{12} + \frac{1}{8}(n-1)(n-2)(n-3)(n-4) K_9^{(n-5)} K_{10}^2 \\[2ex]
+ \frac{1}{2}(n-1)(n-3)(n-4) K_9^{(n-5)} K_{10} K_{11} \\[2ex]
+ \frac{1}{2}(n-3)(n-4) K_9^{(n-5)} K_{11}^2 \\[2ex]
- \frac{1}{3}(n-4)(n^2-2n+3) K_9^{(n-5)} K_{13}
\end{array} \right\} a_1 \right.$$

$$\left|\left[\begin{array}{l} K_9^{(n-2)} K_{11} - K_9^{(n-3)} K_{12} - \frac{1}{2}(n-3)n\, K_9^{(n-4)} K_{10} K_{11} \\ - (n-3)K_9^{(n-4)} K_{11}^2 + 2K_9^{(n-4)} K_{13} \\ + \frac{1}{2}(n-4)(n+1)K_9^{(n-5)} K_{10}K_{12} \\ + \frac{1}{2}(n-4)(n+3)K_9^{(n-5)} K_{11} K_{12} \end{array}\right\} a_0 \right| \Bigg\} \quad (112)$$

Equation (109) gives the complimentary solution to equation (96). The method of "variation of parameters" uses the complimentary solution to generate the particular solution by finding a function $X_1(c)$ and a function $X_2(c)$ such that

$$f_{g3\,\rho} = X_1 f_{g3c_1} + X_2 f_{g3c_2} \qquad (113)$$

where X_1 and X_2 are chosen to satisfy the equations

$$(1)\quad \frac{dX_1}{dc} f_{g3c_1} + \frac{dX_2}{dc} f_{g3c_2} = 0$$

$$\frac{dX_1}{dc}\left[a_0(1 + K_{17}c^2 + K_{18}c^3 + K_{19}c^4)\right] + \frac{dX_2}{dc}\left[a_1(c + K_{20}c^2 + K_{21}c^3 + K_{22}c^4)\right] = 0 . \qquad (114)$$

and

$$(2)\quad \frac{dX_1}{dc}\frac{df_{g3c_1}}{dc} + \frac{dX_2}{dc}\frac{df_{g3c_2}}{dc} = K_{14}K_{15}\frac{1}{c}\left[e^{-\alpha c^2} - e^{-kc^2}\right] + K_{14}K_{16}\,c e^{-\alpha c^2} .$$

$$\frac{dX_1}{dc}\left[a_0(2K_{17}c + 3K_{18}c^2 + 4K_{19}c^3)\right] + \frac{dX_2}{dc}\left[a_1(1 + 2K_{20}c + 3K_{21}c^2 + 4K_{22}c^3)\right]$$

$$= K_{14}K_{15}\frac{1}{c}\left[e^{-\alpha c^2} - e^{-kc^2}\right] + K_{14}K_{16}\,c e^{-\alpha c^2} \tag{115}$$

From equation (114)

$$\frac{dx_1}{dc} = -\frac{dx_2}{dc}\left(\frac{a_1}{a_0}\right)\left[\frac{c + K_{20}c^2 + K_{21}c^3 + K_{22}c^4}{1 + K_{17}c^2 + K_{18}c^3 + K_{19}c^4}\right] \tag{116}$$

Substituting equation (116) into equation (115) yields the following:

$$a_1\frac{dx_2}{dc}\left[\frac{(1 + 2K_{20}c + 3K_{21}c^2 + 4K_{22}c^3)(1 + K_{17}c^2 + K_{18}c^3 + K_{19}c^4)}{(1 + K_{17}c^2 + K_{18}c^3 + K_{19}c^4)} - \frac{(c + K_{20}c^2 + K_{21}c^3 + K_{22}c^4)(2K_{17}c + 3K_{18}c^2 + 4K_{19}c^3)}{(1 + K_{17}c^2 + K_{18}c^3 + K_{19}c^4)}\right]$$

$$= \left[K_{14}K_{15}\left[e^{-\alpha c^2} - e^{-kc^2}\right] + K_{14}K_{16}\,c\,e^{-\alpha c^2}\right]. \tag{117}$$

$$\frac{dx}{dc} = \frac{1}{a_1}\left[\frac{1 + K_{17}c^2 + K_{18}c^3 + K_{19}c^4}{1 + 2K_{20}c + (3K_{21} - K_{17})c^2 + (4K_{22} - 2K_{18})c^3 + (K_{17}K_{21} - K_{18}K_{20} - 3K_{19})c^4 + (2K_{17}K_{22} - 2K_{19}K_{20})c^5 + (K_{18}K_{22} - K_{19}K_{21})c^6}\right]\left[K_{14}K_{15}\frac{1}{c}\left[e^{-\alpha c^2} - e^{-kc^2}\right] + K_{14}K_{16}\,ce^{-\alpha c^2}\right] \tag{118}$$

The function X_2 can be determined by integration of equation (118) with respect to c. Formulation of this integration is extremely difficult if the equation is to remain general. From equations (116) and (118) $\frac{dx_1}{dc}$ can be written.

$$\frac{dx_1}{dc} = -\frac{1}{a_0}\left[\frac{c(1 + K_{20}c + K_{21}c^2 + K_{22}c^3)}{(\text{same denominator})}\right]\left[(\text{same exponientials})\right] \tag{119}$$

Equations (118) and (119) do not offer readily accessable general solutions. To procede any further requires numerous approximations because of the mathematical complexity required to maintain the solution general. Several approximations could be made and applied; however, those simplifications that allow a general integration of (118) and (119) seem to destroy the validity of the equations. For this reason an attempt to determine f_{g2} and f_{g3} explicitly is not advisable. It is likewise not necessary or desirable.

From the outset the purpose of this research has been to discover techniques which might permit the determination of electrical conductivity in an analytical manner more accurate than possible with methods presently employed. The term "accurate" is employed here in its engineering connotation. The accuracy of a theory and the usefulness of developed techniques are judged in an engineering sense, on the basis of the validity with which they may be used to verify and predict the natural behavior observed in scientific experimentation.

Determination of the perturbing functions, f_{g2} and f_{g3}, can readily be accomplished by numerical means from the equations derived in this chapter, when the experimental model is known and specified. The derivations through equation (119) involve few simplifying approximations comparatively. It will be noted that the cross product terms from equation (48) have not been neglected in the succeeding derivations; this marks a considerable deviation from the techniques employed in the literature. Coupled equations (89), (90), and (91) represent the extentional contribution of this paper. These ordinary linear differential equations are extremely difficult to solve except by numerical means.

Equation (101) specifies the solution for $f_{g_1}(c)$; it is repeated below for convenience.

$$f_{g_1}(c) = K_{15}\frac{1}{c}\left[e^{-\alpha c^2} - e^{-kc^2}\right] + K_{16}c\, e^{-\alpha c^2} \qquad (101)$$

For the case, $\bar{B} \| \bar{E}'$ or $\bar{B} = 0$, the perturbing function is $f_{g_1}(c)$, and the complete distribution function is given by equation (120).

$$f_e(c) = f_f(c) + \bar{c} \cdot \bar{E}' f_{g_1}(c) \qquad (120)$$

For this case a general solution for the conductivity can be calculated. The first step in this procedure requires that the average or drift velocity of electrons be calculated from equation (32).

$$\bar{v} = \int_{-\infty}^{\infty} \bar{c} f_e d\bar{c} \qquad (32)$$

where $\bar{v}$ is the electron drift velocity. The limits of integration are explicit for the function f_f; for the function f_{g_1} the limits are implicit. In both cases the integration is to be taken over all c. Since the function f_f is the unperturbed Maxwell's distribution, the limits $-\infty$ to ∞ on c hold. The perturbing function f_{g_1} was defined previously in such a way that the perturbation would be in the direction of the perturbing force. In the case of f_{g_1} the direction includes all possibilities in the half-space enclosing positive $\bar{E}$; i.e., all directions are possible in the positive half-space 1. In the integrations of equations (79) Cos ϑ_1 was integrated over limits on ϑ_1 from 0 to $\pi/2$. The integration on f_{g_1} in equation (32) must be for c between 0 and ∞ in order to avoid the negative half-space.

$$\bar{v} = \int_{-\infty}^{\infty} \bar{c} f_e d\bar{c} = \int_{-\infty}^{\infty} \bar{c} f_f d\bar{c} + (1) \int_{0}^{\infty} c f_{g_1} dc \qquad (121)$$

where (1) defines the direction of the drift for the f_{g_1} term. With

$$f_f = Kc^2 e^{-\alpha c^2}$$

the integration over c from $-\infty$ to ∞ will be zero for the first term in equation (121) because the unperturbed Maxwell's function is even while cf_f is odd. The contribution of the first integration on the right is zero. The second term gives $\bar{v}$; it becomes:

$$\begin{aligned}
\bar{v} = \int_0^{\infty} c f_{g_1} dc &= \int_0^{\infty} c \tfrac{1}{c} K_{15} \left[e^{-\alpha c^2} - e^{-kc^2} \right] dc + c\left[cK_{16} e^{-\alpha c^2} dc \right] \\
&= K_{15} \int_0^{\infty} e^{-\alpha c^2} dc - K_{15} \int_0^{\infty} e^{-kc^2} dc + K_{16} \int_0^{\infty} c^2 e^{-\alpha c^2} dc \\
&= K_{15} \left[\tfrac{1}{2} \sqrt{\tfrac{\pi}{\alpha}} - \tfrac{1}{2} \sqrt{\tfrac{\pi}{k}} + K_{16} \left[\tfrac{1}{4\alpha} \sqrt{\tfrac{\pi}{\alpha}} \right] \right. \\
&= \frac{K_{15} \sqrt{\pi}}{2} \frac{\sqrt{k} - \sqrt{\alpha}}{\sqrt{\alpha k}} + \frac{K_{16}}{4\alpha} \sqrt{\frac{\pi}{\alpha}} = K_{23} \qquad (122)
\end{aligned}$$

where:

$K_{15} = K_1 K_2 / (K_1 - 2\alpha)^2$

$K_{16} = -\alpha K_2 / (K_1 - 2\alpha)$

$K = 2mR^2 n_n / eE'$

$K = \frac{16 n}{E'} \left(\frac{m}{2\pi k T} \right)^{3/2}$

$\alpha = \bar{m} / 2k' \bar{T}$

$$k = mR^2 n_n / eE'$$

The current density $\bar{J}$ was given in equation (31), and for one species of particle (namely, electrons) it becomes:

$$\bar{J} = ne\bar{v} = K_{23}\, ne(1) \qquad (123)$$

$$\sigma_1 = J/E' = K_{23}\, ne/E'$$

where:

$$K_{23} = K_{15} \frac{\sqrt{k-\alpha}}{\sqrt{k}} + K_{16} \frac{1}{2\alpha}$$

n = Number density of electrons.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

A proven theory for the electrical conduction in ionized gases has never been developed. This is true for two significant reasons. First, the problem arises from a necessarily statistical probability phenomena, and the process is not entirely understood. Second, the mathematical burden in carrying out a solution in general terms requires simplifying approximations. These approximations cause major effective deviations from the theory as originally proposed. Thus, it is difficult to examine a theory without deviating from it. Indeed, a valid theory may already exist, but any proof of the theory has not been shown or even indicated.

All current treatment of the subject in the literature seems to begin with the Maxwell's probability statistics and to develop through application of the Boltzmann's transport equation. No theory greatly deviating from this general approach has received any substantial consideration. The hunt for an altogether new theory may be progressing among some theorists, principally theoretical physicists, but one would gather from the literature that the greatest majority of both physicists and engineers are concerned with the problems of applying the Maxwell-Boltzmann theory.

Two major conclusions can be drawn from the work in Chapter Six.

First, it is entirely possible to solve the Boltzmann's equation (62) without neglecting cross-product terms and higher order terms. Second, a complete solution of the Boltzmann equation can best be accomplished by numerical means after a laboratory model has been specified. Equations (89), (90), and (91) are three coupled equations, the solution to which produces the key to the calculation of electrical conductivity for a slightly ionized gas of simple atomic structure. The solution to these equations requires evaluation of integrals of the form of equation (118). Integrals of this type are normally solved by numerical techniques which are not applicable to equations involving unspecified constants.

An experimental program is recommended to verify the results obtained in this paper. The characteristics of the gas and conditions of containment should be made to duplicate the assumptions stated. With a laboratory model specified, the constants K_1 through K_{23} could be determined. Knowing these coefficients, a numerical calculation of drift velocities and conductivities could be made. The only proof of the value of the techniques used and the results obtained must come through experimental validation if practical engineering significance is of major concern.

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APPENDIX A

This Appendix discusses the derivation of the Maxwell's distribution function from the Maxwell's probability statistics.

MAXWELL'S DISTRIBUTION FUNCTION

The physical problem to be considered is that of the distribution of velocities within a gas. The gas is composed of a large number of particles in thermal agitation. The problem is to determine the probability that a particular particle has a specified velocity. At a random time a particle of the gas is observed, and its velocity is noted. It is necessary to determine the probability that the observed velocity components lie between c_1 and $c_1 + dc_1$, c_2 and $c_2 + dc_2$, and c_3 and $c_3 + dc_3$, respectively.

$p(c_1)$ is the probability that the component of the gas velocity in that directon lies within the interval c_1 and $c_1 + dc_1$; $p(c_2)$ and $p(c_3)$ are similarly defined. Now if the probability of component c_1 is completely independent of the probabilities of the other two components,

$$p(c_1, c_2, c_3) = p(c_1)p(c_2)p(c_3) \quad . \qquad \text{(A-1)}$$

[Assumption — The velocity probabilities in the velocity quadrature coordinate directions are mutually independent. (This assumption seems reasonable on the surface, but investigators from Maxwell's time hence have questioned, then accepted, it.) Other assumptions concerning the state of the gas are implied; namely, the gas is in thermal equalibrium, contained, and under the influence of no external forces.]

From equation (A-1)

$$\log p(c_1 , c_2, c_3) = \log p(c_1) + \log p(c_2) + \log p\ (c_3) \ . \qquad \text{(A-2)}$$

For a gas in thermal agitation with no external forces applied and in an equilibrium state, all directions of motion are equally likely. [Assumption — For any randomly selected time of observation all velocity directions of observed particle motion are equally likely.] In this case

$$p(c_1, c_2, c_3) = p(c) \ ,$$

where

$$c = \sqrt{c_1^2 + c_2^2 + c_3^2}\left[(1)_1 + (1)_2 + (1)_3\right] .$$

and where $\left[(1)_1 + (1)_2 + (1)_3\right]$ represents the direction coordinates only. Since the directional information is unnecessary $p(c)$ is a function of $(c_1^2 + c_2^2 + c_3^2)$ only; therefore,

$$p(c_1 , c_2, c_3) = f(c_1^2 + c_2^2 + c_3^2) \ , \qquad \text{(A-3)}$$

and one may write equation (A-2)

$$\log p(c_1 , c_2, c_3) = (c_1^2 + c_2^2 + c_3^2) \ . \qquad \text{(A-4)}$$

Suppose that the function $f(c_1^2 + c_2^2 + c_3^2)$ can be expanded in an infinite power series; then,

$$f(c_1^2 + c_2^2 + c_3^2) = a_0 + a_1(c_1^2 + c_2^2 + c_3^2)$$
$$+ a_2(c_1^2 + c_2^2 + c_3^2)^2$$
$$+ \ldots. + a_n(c_1^2 + c_2^2 + c_3^2)^n \; .$$
$$= \log p(c_1) + \log p(c_2) + \log p(c_3) \; . \qquad \text{(A-5)}$$

But the third term of the equality contains no cross product terms involving c_1, c_2, and c_3. Since cross products of this type appear for each term in the expansion except the first two, all coefficients beyond a_1, must be zero. [Assumption — The function $f(c_1^2 + c_2^2 + c_3^2)$ is of such a nature that it can be expanded in a power series. With equation (A-5) truncated at $n = 1$ and substituted into equation (A-5),

$$\log \; p(c_1, c_2, c_3) = a_0 + a_1(c_1^2 + c_2^2 + c_3^2) \; , \qquad \text{(A-6)}$$

and

$$p(c_1, c_2, c_3) = \exp\left[a_0 + a_1(c_1^2 + c_2^2 + c_3^2)\right]$$
$$= Ae^{a_1(c_1^2 + c_2^2 + c_3^2)} \qquad \text{(A-7)}$$

The probability of the particle having some velocity between the limits of positive and negative infinity is obviously one. This may be expressed functionally as

$$\iiint_{-\infty}^{\infty} p(c_1, c_2, c_3)dc_1dc_2dc_3 = 1 \; , \qquad \text{(A-8)}$$

or

$$\iiint_{-\infty}^{\infty} A e^{a_1(c_1^2 + c_2^2 + c_3^2)} dc_1 dc_2 dc_3 = A\left(\frac{\pi}{-a_1}\right)^{\frac{3}{2}} = 1 \quad . \tag{A-8}$$

Hence,

$$A = \left(\frac{-a_1}{\pi}\right)^{\frac{3}{2}} \quad . \tag{A-9}$$

By letting $-a_1 = a$, equation (A-7) becomes:

$$p(c_1, c_2, c_3) = F(c_1^2 + c_2^2 + c_3^2) = \left(\frac{a}{\pi}\right)^{\frac{3}{2}} e^{-a(c_1^2 + c_2^2 + c_3^2)} \tag{A-10}$$

Noting that

$$(c_1^2 + c_2^2 + c_3^2) = c^2 \quad ,$$

$$F(c_1^2 + c_2^2 + c_3^2) = F(c^2) \quad \text{or} \quad f(c) \quad .$$

Equation (A-10) now becomes:

$$f(\bar{c}) = \left(\frac{a}{\pi}\right)^{\frac{3}{2}} e^{-a c^2} \quad . \tag{A-11}$$

This is the Maxwell velocity distribution function; it can be transformed into an energy distribution function, readily, by noting that the kinetic energy is a function of the square of velocity. The function often appears in the literature in terms of the kinetic temperature, and the arbitrary constant has been shown to be a function only of the temperature.

In the Maxwell velocity distribution function of equation (A-11), the arbitrary constant, a , can be determined in an intuitive way by observing that the exponential term has no meaning unless it is nondemensional. This argues that the constant must have the dimensions of time-squared per length. The energy of a particle can be expressed directly in terms of its kinetic temperature by noting that the energy is directly proportional to the absolute kinetic temperature.

$$KE = K'kT \qquad \text{(A-12)}$$

K' is a dimensionless constant of proportionality, and k is the familiar Boltzmann's constant,

$$k = 1.380\ (10)^{-16}\ \text{erg}/{}^{\circ}\text{K} \qquad \text{(A-13)}$$

in the cgs system of units.

From equation (A-12) and the kinetic energy relationship,

$$KE = \tfrac{1}{2}mc^2 \ , \qquad \text{(A-14)}$$

$$KE = K'kT = \tfrac{1}{2}mc^2 \ . \qquad \text{(A-15)}$$

where m is the particle mass.

Solving for the square of velocity,

$$c^2 = K'\,2kT/m \ . \qquad \text{(A-16)}$$

Since $-ac^2$ must be dimensionless,

$$-a(2kT/m) = K \ .$$

where K is a dimensionless constant of proportionality.

$$a = -Km/2kT \ . \qquad \text{(A-17)}$$

Since only T is allowed to be a variable, it is obvious that a can be written as a function of T.

The Maxwell velocity distribution function can now be written in the following way:

$$f(c) = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-mc^2/2kT} . \qquad \text{(A-18)}$$

where n is the particle number-density per unit volume. Multiplying equation (A-18) by the volume element in velocity-space,

$$4\pi c^2 dc ,$$

produces the functional relationship for the number density of particles having speeds between c and $c + dc$; thus

$$f(c)dc = 4\pi c^2 n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-mc^2 2kT} dc . \qquad \text{(A-19)}$$

From which

$$f(c) = 4\pi nc^2\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-mc^2/2kT} \qquad \text{(A-20)}$$

is the distribution function for speeds between c and $c + dc$. Equation (A-20) is one of the most common forms of the Maxwell's velocity distribution function. Equation (A-20) expresses the number of particles in velocity space having velocities between c and $c + dc$.

The most probable speed for particles in $f(c)$ occurs when $f(c)$ is a maximum. Allowing a change of variable,

$$x = (m/2kT)^{\frac{1}{2}}c$$

to be substituted into equation (A-20) makes the equation easier to differentiate. This substitution gives

$$f(x) = 4n(\pi)^{-\frac{1}{2}}x^2e^{-x^2}. \qquad \text{(A-21)}$$

Further, letting

$$x^2 = u \quad ,$$

equation (A-21) becomes:

$$f(u) = 4n(\pi)^{-\frac{1}{2}}ue^{-u},$$

and

$$\frac{df(u)}{du} = 4n(\pi)^{-\frac{1}{2}}e^{-u}(1 - u) = 0 \quad .$$

Then, $u = 1$,

$x^2 = 1$,

and $x = 1$

for $x > 0$.

Hence, the most probable value of the particle speed is

$$c = \sqrt{2kT/m} = c_o \quad . \qquad \text{(A-22)}$$

Other speeds for the function are of interest.

The speed equivalent to the root-mean-square value of $f(c)$ is

$$\sqrt{3kT/m} = c_{rms} \quad . \qquad \text{(A-23)}$$

The speed equivalent to the average value of the function is

$$\sqrt{8kT/\pi m} = c_{av} \quad . \tag{A-24}$$

Some resultant speeds and velocities for the Maxwellian function can be summarized and discussed.

(a) The average value of the speed, c_{av} , is larger than the most probable value, c_o .

(b) The rms value is somewhat larger than both c_o and c_{av} .

Use of the rms value in expressing the particle effective kinetic energy gives the well-known result,

$$\frac{1}{2} mc^2_{rms} = (\frac{1}{2}) m(3kT/m) = 3kT/2 \quad . \tag{A-25}$$

Equation (A-25) is valid for translational particle speed, but does not account for particle vibration or rotation. [Assumption — For particles of interest rotation and vibration are negligible. (This assumption appears valid for single particles and simple atoms; it is much less valid for heavy molecules.)]

(c) The particle velocity direction is entirely random; hence, the average value of the velocity vector is zero.

$$\bar{c}_{av} = 0 \quad .$$

APPENDIX B

This Appendix discusses the formulation of the Boltzmann's transport equation. It interprets the physical meaning of the terms making up this equation, and adapts it to the particular case of interest in this paper.

THE BOLTZMANN'S EQUATION

The one-particle Maxwellian velocity distribution function was studied in Appendix A. A distribution function of this form, denoted by $f(\bar{s},\bar{v},t)$, is of interest where

$$\bar{s} = (s_1)_1 + (s_2)_2 + (s_3)_3$$

and

$$\bar{v} = (v_1)_1 + (v_2)_2 + (v_3)_3$$

in the laboratory coordinate system. t is the scalar time variable. The distribution function $f(\bar{s},\bar{v},t)$ is not necessarily known to be isotropic at this point, but it is assumed to hold for a gas in thermal equalibrium, electrically neutral, homogeneous in the particle species, and under the influence of external forces.

The function $f(\bar{s},\bar{v},t)$ is defined such that $f(\bar{s},\bar{v},t)d\bar{s}d\bar{v}$ is the probable number of particles per second in the phase-space volume element $d\bar{s}d\bar{c}$ at time t . In other words $f(\bar{s},\bar{v},t)d\bar{s}d\bar{v}$ is the number of particles in the geometrical volume between $\bar{s}$ and $\bar{s} + d\bar{s}$ having velocities between $\bar{v}$ and $\bar{v} + d\bar{v}$ at the time t . The probable number density of particles at a point in space determined by $\bar{s}$ at time t is

$$n(\bar{s},t) = \int_{-\infty}^{\infty} f(\bar{s},\bar{v},t)d\bar{v} \quad . \tag{B-1}$$

Now,

$$dn = f(\bar{s},\bar{v},t)d\bar{s}d\bar{v} \quad , \qquad \text{(B-2)}$$

is the number of particles in the differential phase-space volume $d\bar{s}d\bar{v}$ at time t , as previously stated. In a differential time Δt the particle geometric coordinates change to

$$\bar{s}' = \bar{s} + \bar{v}\Delta t \quad , \qquad \text{(B-3)}$$

and the velocity coordinates change to

$$\bar{v}' = \bar{v} + \bar{a}\Delta t \quad , \qquad \text{(B-4)}$$

where

$$\bar{a} = \bar{F}/m \qquad \text{(B-5)}$$

from Newton's force law. $\bar{F}$ is all external forces acting on the particle. Particle interactions are ignored for the present. Under these conditions the number density of particles does not change over Δt ; and

$$dn = dn' = f(\bar{s}',\bar{v}',t + \Delta t)d\bar{s}'d\bar{v}' \quad . \qquad \text{(B-6)}$$

If $\bar{F}$ is constant or slowly time varying, if

$$\bar{s}' = \bar{s} + d\bar{s} \quad ,$$

and if

$$\bar{v}' = \bar{v} + d\bar{v} \quad ,$$

then

$$d\bar{s}' \approx d\bar{s} \quad ,$$

and

$$d\bar{v}' \approx d\bar{v} \quad .$$

Equation (B-6) can be written as follows:

$$dn' = f(\bar{s} + \bar{v}\Delta t, \bar{v} + \frac{F}{m}\Delta t, t + \Delta t)d\bar{s}d\bar{v} \qquad \text{(B-7)}$$

A difference between equations (B-6) and (B-1) will occur only if interparticle interaction takes place. Since the same phase-space volume is involved and since the external forces $\bar{F}$ are presumed to remain constant over the time Δt , any difference in the number density of particles is a representation of the change in the distribution function during the time Δt .

$$dn' - dn = \left[\frac{\partial^3 f}{\partial \bar{s}\partial\bar{v}\partial t}\right] = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \bar{s}} \cdot \frac{d\bar{s}}{dt} + \frac{\partial f}{\partial \bar{v}} \cdot \frac{d\bar{v}}{dt}\right] = \left[\frac{\partial f}{\partial t}\right]_c \qquad \text{(B-8)}$$

where

$$\frac{\partial \bar{s}}{\partial t} = \frac{d\bar{s}}{dt} = \bar{v} \quad ,$$

and

$$\frac{\partial \bar{v}}{\partial t} = \frac{d\bar{v}}{dt} = \frac{\bar{F}}{m} \quad .$$

$$\frac{\partial f}{\partial t} + \bar{v} \cdot \frac{\partial f}{\partial \bar{s}} + \frac{\bar{F}}{m} \cdot \frac{\partial f}{\partial \bar{v}} = \left|\frac{\partial f}{\partial t}\right)_c \qquad \text{(B-9)}$$

where the subscript c identifies the collision term. Equation (B-9) is the well-known Boltzmann's equation; it represents little more than an informed guess as to the form of the collision term.

Two particles having initial velocities $\bar{v}$ and $\bar{v}_1$ and of masses m and m_1 are moving along paths whose proximity causes an interaction between the particles. After the interaction, due to whatever means, the particle velocities are $\bar{v}'$ and $\bar{v}'_1$ respectively. The interaction is a two-body encounter having functional relationships determined by the force governing the interaction. In the technique known as the "center-of-mass method," the encounter is equivalent to the interaction of a particle having reduced mass with a fixed center of scattering. The reduced mass of the auxiliary particle becomes $mm_1/(m + m_1)$, and the velocities before and after the interaction are $\bar{v} - \bar{v}_1$ and $\bar{v}' - \bar{v}'_1$ respectively. Figure No. 1 displays the mechanics of the encounter graphically.

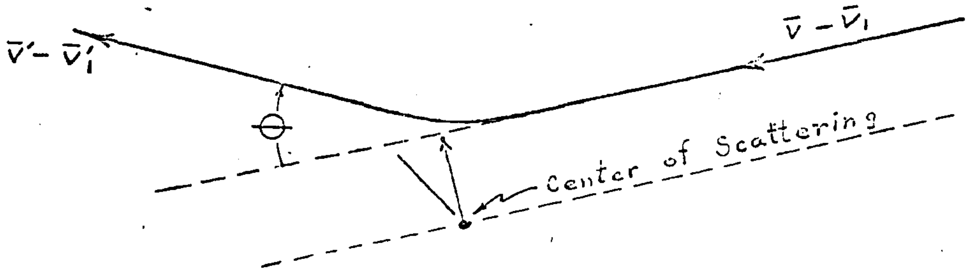

Figure No. B-1

Mechanics of Fixed Center Scattering

With no interaction the auxiliary particle would have equal initial and terminal velocities and would pass the center of scattering within some distance r, called the impact parameter. The path taken by the particle after the interaction lies in the plane determined by the path of incidence and the center of scattering. The scattering is planar with the scattering angle θ. The plane of scattering makes an angle ϕ with the normal to that plane. The number of particles passing through an element of geometric area $r d\phi dr$ and velocity area $|\bar{v} - \bar{v}_1| d\bar{v}$ is

$$dn_1 = f(\bar{s},\bar{v},t) r d\phi dr |\bar{v} - \bar{v}_1| dv_1 \quad . \tag{B-10}$$

The incident auxiliary particles are scattered through angles θ. The number of particles between $\bar{v}$ and $\bar{v} + d\bar{v}$ with which the incident particles react is

$$dn_c = f(\bar{s},\bar{v},t) d\bar{v} \quad . \tag{B-11}$$

The number of encounters per unit time, geometrically between r and $r + dr$ and ϕ and $\phi + d\phi$ is

$$dn_1 dn_c = f(\bar{s},\bar{v}_1,t) f(\bar{s},\bar{v},t) r d\phi dr |\bar{v} - \bar{v}_1| d\bar{v}_1 d\bar{v} \quad . \tag{B-12}$$

Integrating equation (B-12) over all $\bar{v}_1$, integrating r from 0 to r_{max} (where r_{max} is the maximum impact parameter for which there is an interaction), integrating θ from 0 to 2π, and dividing out $d\bar{v}$ gives the time-rate-of-change of particles due to collisions which scatter particles out of the velocity range $\bar{v}$ to $\bar{v} + d\bar{v}$; i.e.,

$$\left(\frac{\partial f}{\partial t}\right)_{out} = -\int_{-\infty}^{\infty} |\bar{v} - \bar{v}_1| f(\bar{s},\bar{v}_1,t) f(\bar{s},\bar{v},t) d\bar{v}_1 \int_0^{2\pi}\int_0^{r_{max}} r d\phi dr \quad . \tag{B-13}$$

By a similar argument the rate-of-change of particles into the range is

$$\frac{\partial f}{\partial t}_{in} = + \int_{-\infty}^{\infty} \left|\bar{v} - \bar{v}_1\right| f(\bar{s}',\bar{v}',t)\, f(\bar{s}',\bar{v}_1',t)\, d\bar{v}_1 \int_{0}^{2\pi}\int^{r_{max}} r\,d\phi\,dr \quad . \qquad \text{(B-14)}$$

Combining equations (B-13) and (B-14) for the collision term with equation (B-9), the complete Boltzmann's equation becomes:

$$\frac{\partial f}{\partial t} + \bar{v} \cdot \frac{\partial f}{\partial \bar{s}} + \frac{\bar{F}}{m} \cdot \frac{\partial f}{\partial \bar{v}} = \left(\frac{\partial f}{\partial t}\right)_c = \int \left|\bar{v} - \bar{v}_1\right| (f'f_1' - ff_1)\, d\bar{v}_1 \iint r\,d0\,dr \qquad \text{(B-15)}$$

where

$$f' = f(\bar{s}',\bar{v}',t)$$

$$f'_1 = f(\bar{s}',\bar{v}_1',t) \quad ,$$

$$f = f(\bar{s},\bar{v},t) \quad ,$$

and

$$f_1 = f(\bar{s},\bar{v}_1,t) \quad .$$

[Assumptions -- :

(a) $\bar{F}$ is assumed not to modify the process of interaction.

(b) Many-particle interactions are assumed negligible.

(c) The probability of one particle having velocity $\bar{v}$ and another having velocity $\bar{v}_1$ is $f(\bar{s},\bar{v},t)f(\bar{s},\bar{v}_1,t)$.

(d) The range of interaction is much less than the mean intraparticle separation.]

A brief discussion of the terms that make up the Boltzmann's equation and their physical significance with respect to the collision term follows. From Appendix A the Maxwellian distribution function, $f(\bar{s},\bar{v},t)$, is observed to involve the independent time-phase-space volume elements, $\bar{s}_1$, $\bar{v}_1$, and t. The Boltzmann's equation is repeated.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \bar{s}} \cdot \frac{d\bar{s}}{dt} + \frac{\partial f}{\partial \bar{c}} \cdot \frac{d\bar{c}}{dt} = \text{collision term} - F_c$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \bar{c} \cdot \frac{\partial f}{\partial \bar{s}} + \bar{a} \cdot \frac{\partial f}{\partial \bar{c}} = F_c$$

TERM	PHYSICAL EXPLANATION
$\frac{df}{dt} = F_c$	If there is a time-rate-of change of the distribution function, F_c, this equation states that it must be due to collisions.
$\left[\frac{\partial f}{\partial t} \longrightarrow \frac{df}{dt}\right]_{\substack{\bar{s}\text{ Const.} \\ \bar{c}\text{ Const.}}}$	This term asks the question, "Are there any new particles created at position $\bar{s}$ with velocity $\bar{c}$?" If no particles are added or subtracted externally, there is either no change or the change must be internally brought about. The only mechanism for the demise or creation of particles internally is certainly through collisions. Example: In the case of electrons the process of ionization or deionization could be the mechanism.

$\left[\frac{\partial f}{\partial s} \longrightarrow \frac{df}{d\bar{s}}\right]_{\substack{t\ \text{Const.} \\ \bar{c}\ \text{Const.}}}$	This term asks the question, "Is the geometrical density of the particles constant; in other words, is the gas geometrically uniform?" The gas would not be geometrically uniform for single-type particles if the particles possess charge. It would be non-uniform if gravitational or other external force gradients were present and unbalanced.
$\left[\frac{\partial f}{\partial c} \longrightarrow \frac{df}{d\bar{c}}\right]_{\substack{t\ \text{Const.} \\ \bar{s}\ \text{Const.}}}$	This term asks if thermal equalibrium of the particles has been achieved. In other words, "Is the particle number with velocity $\bar{c}$ increasing or decreasing?" In the absence of changing external influences, if internal changes are observed, they must be due to energy transmitted through collisions.

Assuming no net loss or gain of particles with time (ionization equalibrium and a contained gas),

$$\frac{\partial f}{\partial t} = 0 .$$

If the gas is uniform throughout geometry-space

$$\frac{\partial f}{\partial \bar{s}} = 0 .$$

If thermal equalibrium is achieved

$$\frac{\partial f}{\partial \bar{c}} = 0 .$$

This leaves

$$\frac{df}{dt} = 0 .$$

and the function, f , is a constant with time. Only in the presence of collisions is a distribution not constant with time.

Presume the following:

(a) A contained gas.

(b) Ionization equalibrium

(c) Uniform geometric density.

(d) Electrical neutrality.

(e) Thermal equalibrium.

Under these assumptions,

$$\frac{\partial f}{\partial t} = 0 .$$

and

$$\frac{\partial f}{\partial \bar{s}} = 0 .$$

Hence, $f(\overline{s},\overline{v},t)$ becomes $f(\overline{v})$, a velocity distribution function. $f(\overline{v})$ answers the question, "What is the number density of particles in velocity-space having velocity $\overline{v}$?" $df(\overline{v})$ answers the question, "What is the number density having velocity between $\overline{v}$ and $\overline{v} + d\overline{v}$?" The number density having velocity between $\overline{v}$ and $\overline{v} + d\overline{v}$ is just the uniform average number density times the probability that a particle lies in this velocity range. The question becomes one of determining the probability function, p , but this is just the Maxwellian probability statistics.